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FACOLTÀ DI SCIENZE E TECNOLOGIE

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L'integrale di Feynman per particelle in campo elettromagnetico

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1 Motivation

Feynman proposed a formulation of quantum mechanics which is alternative to the familiar Hilbert space one. In this section, the main ideas will be roughly outlined.

The starting point is the attempt to determine the probability to find a particle at some point B, assuming that it was localized at some other point A before. As it is known from the double slit experiment and its variants, in quantum mechanics exclusive alternatives can interfere. The "exclusive alternatives" in this case are all the possible trajectories connecting the two points. Yet, a proper definition of trajectory is troublesome in quantum mechanics as it is not possible to define the position of a particle at any given time. Nevertheless, it is possible to proceed as follows.

An impenetrable surface can be interposed between the two points. The surface has some holes with detectors or other devices that can perform a measure process. Anytime the particle passes through a hole, the alternatives collapse and the particle proceeds beyond the surface from the hole, so that the position of the particle can be specified in one point. If several similar surfaces are interposed, then the position is well defined at some instants and at some points which form a sort of path. In the limit of many surfaces and many holes - which means no surfaces and no holes - the particle moves along a sort of trajectory which can be regarded as one of the mutually exclusive alternatives.

In classical mechanics, the probability that the particle reaches B would be the sum of the probabilities of the independent trajectories. In quantum mechanics, according to the previous picture, there are paths and they are allowed to interfere if all the detectors are turned off. The quantity that accounts for the interference is the phase. No path is more important than the others but each one carries a different phase. The square modulus of their sum can be interpreted as a probability. The phase will be some function of the trajectory and the correct choice is $\exp\left[(i/\hbar)S(\boldsymbol{x}(t))\right]$ where S is the classical action associated with the trajectory.

The sum of the phases is then:

$$g(\boldsymbol{x}_f, t_f; \boldsymbol{x}_i, t_i) = \sum_{paths} \exp\left[(i/\hbar)S(\boldsymbol{x}(t))\right]$$

The quantity $|g(B, t; A, 0)|^2$ is the probability that the particle reaches B after a time t. Yet, as in quantum mechanics the position is never free of uncertainty, a better definition of probability should involve a proper initial

physical state and not just an initial position. Since trajectories are expected to be a continuous set rather than a countable one, it would be interesting to be able to write an integral instead of a sum.

A more rigorous approach to the problem will follow. A quantity called propagator will be introduced and it will be shown to yield the function g which has just been investigated. In addition, the propagator, as a more abstract object, will allow the extension the previous path approach to trajectories which belong to different spaces and not just the configuration space.

2 Propagator

Given a linear differential operator $\mathcal{L}$, the Green function $g(x)$ has the following property: $\mathcal{L}g(x) = \delta(x)$

Similarily, the propagator G can be defined for the Schrodinger equation (S.E.) as a sort of Green function, with the only difference that G is an operator.

$$(H - i\hbar\partial_t)G(t, t_0) = -i\hbar\delta(t - t_0) \tag{1}$$

Given a state-vector at a certain time t_0, the propagator yields its evolution through time for $t > t_0$:

$$|\psi(t)\rangle = G(t, t_0)\,|\psi(t_0)\rangle$$

Indeed, $G(t, t_0)\,|\psi(t_0)\rangle$ is a solution of S.E. for $t \neq t_0$, since

$$(H - i\hbar\partial_t)G(t, t_0)\,|\psi(t_0)\rangle = -i\hbar\delta(t - t_0)\,|\psi(t_0)\rangle$$

Therefore, $G(t, t_0)$ is "almost" the time-evolution operator $U(t, t_0)$ and precisely $\theta(t-t_0)U(t, t_0)$ which can be directly verified, considering the Leibniz property and that $\frac{d}{dt}\theta(t - t_0) = \delta(t - t_0)$.
If the Hamiltonian is time-independent, then:

$$G(t, t_0) = \theta(t - t_0)e^{-\frac{iH}{\hbar}(t-t_0)} \tag{2}$$

The θ function will be omitted when not explicitly needed.

The propagators form a semi-group with the following properties:

- $G(t, t_0) = 0$ if $t < 0$
- $G(t_2, t_1)G(t_1, t_0) = G(t_2, t_0)$
- $G(t, t) = \mathbb{I}$

If H is time-independent, the propagator only depends on the difference of times t and t_0 and, consequently, it is invariant under time translation. With an abuse of notation, one can write $G(t, t_0) = G(t - t_0)$. Thus, from now on, t_0 will be set to zero. It must be remarked that in literature the term "Green function" is often referred to the Fourier transform of G, i.e. the energy-dependent propagator.

3 Position representation and Trotter Product

The position representation is the kernel of $G(t, t_0)$:

$$\begin{aligned}\psi(y,t) &= \langle y|\, G(t,t_0)\, |\psi(t_0)\rangle \\ &= \int dx\, \langle y|\, G(t,t_0)\, |x\rangle\, \langle x|\psi(t_0)\rangle \\ &= \int dx\, G(y,t;x,t_0)\psi(x,t_0)\end{aligned}$$

The goal is to find an explicit expression for the kernel $G(y, t; x, t_0)$.

Let us consider the simple case of a potential which only depends on the position. The Hamiltonian reads $H = p^2/2m + V(x)$. Since, in the end, time evolution will appear as a succession of events, we can split the time interval $t := t - t_0$ into N sub-intervals and, recalling the semi-group properties, re-write G as a product (the θ function will be omitted):

$$e^{-\frac{it}{\hbar}H} = \left[e^{-\frac{it}{N\hbar}H}\right]^N$$

The next step is to separate the momentum from the position i.e. the potential from the kinetic part. This is a delicate passage since V and p^2 do not commute. If two operators, say A and B, commute then $e^A e^B = e^{A+B}$ according to the Baker-Campbell-Hausdorff formula. In the present case, according to the same formula, the best we can obtain is:

$$e^{-\frac{it}{N\hbar}(T+V)} = e^{-\frac{it}{N\hbar}V} e^{-\frac{it}{N\hbar}T} + o(N^{-1})$$

where by definition an operator $R(h) = o(h)$ if $\forall\, |\psi\rangle \in \mathcal{H}\ \lim_{h\to 0} ||R(h)\, |\psi\rangle\, ||/h \to 0$. Hence, the product of N propagators reads:

$$e^{-\frac{it}{\hbar}H} = [e^{-\frac{it}{N\hbar}V} e^{-\frac{it}{N\hbar}T} + o(N^{-1})]^N$$

If N is big enough, one can hope to neglect the terms $o(N^{-1})$ in the previous expression. This possibility is offered by the Trotter formula, which will be presented in the following simplified version:

$$\forall\, |\psi\rangle \in \mathcal{H}\ \lim_{N\to\infty} (e^{A/N} e^{B/N})^N\, |\psi\rangle = e^{A+B}\, |\psi\rangle \tag{3}$$

with some operators A and B. Obviously the formula is true only under some restrictive hypotheses on A and B and there are domain issues too. The details will not be discussed here, but the main result[1] is that if $A \sim p^2/2m$, $B \sim V(x)$ (a "well-behaved" potential) and $p^2/2m + V = H$ is self-adjoint, then the Trotter product is valid. Consequently, an operator which is well defined as an Hamiltonian for ordinary quantum mechanics, should also be fine in the current description. Finally:

$$G(y,t;x) = \lim_{N\to\infty} \langle y| \left[e^{-\frac{it}{N\hbar}T} e^{-\frac{it}{N\hbar}V}\right]^N |x\rangle$$

Since the potential V only depends on the position V=V(x), V itself and its exponential are diagonal in the position basis, such as $T = \frac{\hbar^2 k^2}{2m}$ and its exponential are diagonal in the impulse basis. It is useful to insert the identity operator $\int dx\, |x\rangle \langle x|$ between every couple of consecutive terms in the previous equation.

$$G(y,t;x) = \lim_{N\to\infty} \int dx_1 ... dx_{N-1} \prod_{j=0}^{N-1} \langle x_{j+1}|\, e^{-\frac{it}{N\hbar}T} e^{-\frac{it}{N\hbar}V}\, |x_j\rangle \qquad (4)$$

Each term of the product can be rewritten this way:

$$\begin{aligned}
\langle x_{j+1}|\, e^{-\frac{it}{N\hbar}T} e^{-\frac{it}{N\hbar}V}\, |x_j\rangle &= \langle x_{j+1}|\, e^{-\frac{it}{N\hbar}T}\, |x_j\rangle\, e^{-\frac{it}{N\hbar}V(x_j)} \\
&= \langle x_{j+1}|\, e^{-\frac{it}{N\hbar}T} \left[\int dk\, |k\rangle \langle k|\right] |x_j\rangle\, e^{-\frac{it}{N\hbar}V(x_j)} \\
&= e^{-\frac{it}{N\hbar}V(x_j)} \int dk\, \langle x_{j+1}|k\rangle \langle k|x_j\rangle\, e^{-\frac{it}{N\hbar}\frac{\hbar^2 k}{2m}} \\
&= \frac{1}{2\pi\hbar} e^{-\frac{it}{N\hbar}V(x_j)} \int dk\, e^{-\frac{it}{N\hbar}\frac{\hbar^2 k^2}{2m} + \frac{ik}{\hbar}(x_{j+1}-x_j)} \qquad (5)
\end{aligned}$$

Now one can either integrate over k or just rearrange the new expression. The latter procedure will be discussed in the section about the phase-space, whereas the former approach will follow here. The integral above is Gaussian and its value is

$$\sqrt{\frac{mN}{2i\pi t\hbar}} e^{-mN(x_{j+1}-x_j)^2/it\hbar}$$

Hence, each factor in equation (4) is the product of two exponentials. The product is conveniently rewritten as the exponential of a sum. With a small rearrangement, one gets the following meaningful result:

$$\begin{aligned}
\lim_{N\to\infty} \left[\frac{m}{2i\pi\hbar(t/N)}\right]^{N/2} &\int dx_1 ... dx_{N-1} \times \\
&\times \exp\left[\frac{i(t/N)}{\hbar} \sum_{j=0}^{N-1} \left[\frac{m}{2}\left[\frac{(x_{j+1}-x_j)}{t/N}\right]^2 - V(x_j)\right]\right] \qquad (6)
\end{aligned}$$

[1] There exist more than one sufficient condition for the Trotter product

The sum in the exponent resembles the definition of a Riemann integral as N goes to infinity and t/N goes to zero.

$$\frac{t}{N}\sum_{j=0}^{N-1}\left[\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{t/N}-V(x_j)\right]\approx\int_0^t dt'\left[\frac{m}{2}\left[\frac{dx(t')}{dt'}\right]^2-V(x(t'))\right] \quad (7)$$

Nevertheless, the sum is *not* a genuine Riemann integral since the limit is taken after the integration and, in general, it is not even true that $x_j - x_{j-1}$ should go to zero as $t/N \to 0$. This issue will be one of the main topics of the next section.

The integrand in (7) is what in classical mechanics is called Lagrangian; therefore its integral with respect to time is an analogue of the classical action.

$$S(t)=\int_0^t dt'\mathcal{L}(t')$$

The discrete version of S is sometimes called "sliced action" since $S_{sliced} = \sum S(x_j, x_{j+1})$. If N is fixed, the N integrals may be thought of as a "sum" of $e^{\frac{i}{\hbar}S(x(\cdot))}$ over any possible broken line path connecting N points, while keeping the first and the last points fixed, as well as the time interval t. As N increases, seemingly-continuous successions of points appear and the sum-over-paths interpretation becomes even more reasonable. Finally, if C is the normalization constant, the following formula can be written:

$$G(x_f,t;x_i)=C\sum_{\substack{x(\cdot)\\x(0)=x_i\\x(t)=x_f}}e^{\frac{i}{\hbar}S(x(\cdot))} \quad (8)$$

This "sum" over paths can be interpreted as an integral on the "set of all paths" satisfying the conditions above: a path integral. It is important to underline that, in order to define an actual integral, a proper measure is necessary and that in the present section no "bona fide measure" is presented [27]. Thus, however simple the previous and the next expressions might be, they make sense according to (6).

$$G(x_f,t;x_i)=\int_{x_i}^{x_f}\mathcal{D}x(t)\exp\left[\frac{it}{\hbar}S(x(t))\right] \quad (9)$$

The previous formalism has been derived in one dimension but it can also apply to systems with several degrees of freedom. Let us consider, for example, a particle moving in $\mathbb{R}^3$. We write

$$G(\boldsymbol{x_f},t;\boldsymbol{x_i})=\int_{x_i}^{x_f}\mathcal{D}x(t)\mathcal{D}y(t)\mathcal{D}z(t)\exp\left[\frac{it}{\hbar}S(x(t),y(t),z(t))\right] \quad (10)$$

as a formal expression for a multi-dimensional generalization of (6). If the action can be split in a sum in which each addend depends on different set

of variables, the system is separable. The integrations over the independent coordinates can be divided and it is straightforward that:

$$S(\boldsymbol{x_1}, \boldsymbol{x_2}) = S_1(\boldsymbol{x}_1) + S_2(\boldsymbol{x}_2)$$

$$\Rightarrow G(\boldsymbol{x}_{1,f}, \boldsymbol{x}_{1,f}, t; \boldsymbol{x}_{2,i}, \boldsymbol{x}_{1,i}) = G(\boldsymbol{x}_{1,f}, t; \boldsymbol{x}_{1,i}) G(\boldsymbol{x}_{2,f}, t; \boldsymbol{x}_{2,i}).$$

Let us note that $|\langle\phi|\psi(t)\rangle|^2 = |\langle\phi| G |\psi\rangle|^2$ is the transition probability from $|\psi\rangle$ to $|\phi\rangle$ after the former has evolved for a time t. If now the two states are :

$$|\psi\rangle \mapsto |\boldsymbol{x}_i\rangle \qquad |\phi\rangle \mapsto |\boldsymbol{x}_f\rangle$$

it follows that

$$|G(\boldsymbol{x}_f, t; \boldsymbol{x}_i)|^2 \tag{11}$$

is precisely the probability that a particle localized at $\boldsymbol{x}_i$ is detected at $\boldsymbol{x}_f$ at time t: the search of such probability has been the introductory argument for the path integral. It must be remarked that if $|\psi\rangle$ is exactly $|\boldsymbol{x}_i\rangle$ the probability distribution is quite strange (38) and, in order to get a more physical result, it is better to consider an appropriate initial state (e.g. a coherent state) which is localized but which is not a position eigenvalue. If such state is $|\psi_{\boldsymbol{x}_i}\rangle$ then:

$$P(\boldsymbol{x_f}, t; \boldsymbol{x_i}) = \left| \int d^3\boldsymbol{x}\, G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) \psi_{\boldsymbol{x}_i}(\boldsymbol{x}) \right|^2$$

Finally, the composition property of propagators (as elements in a semi-group) can be rewritten in the position representation. Let us fix an intermediate time t_m such that $0 < t_m < t$.

$$\begin{aligned} G(x_f, t; x_i) &= \langle x_f| G(t, t_m) G(t_m, 0) |x_i\rangle \\ &= \langle X_f| G(t, t_m) \int dx_m \, |x_m\rangle \langle x_m| G(t_m, 0) |x_i\rangle \\ &= \int dx_m G(x_f, t; x_m, t_m) G(x_m, t_m; x_i, 0). \end{aligned}$$

The previous integral is interpreted as a sum over all possible intermediate positions a time t_m and this means that the amplitude of "events occurring in succession" multiply.

With an abuse of terminology, from now on, the kernel will be simply referred to as the "propagator".

4 The Wiener measure and the nature of Feynman paths

The trajectories appearing in the path integral can in principle assume any possible shape. Even in the $N \to \infty$ limit, there can be arbitrarily big distances between two successive vertices on the broken lines. Nevertheless, some paths are expected to be more meaningful then others (i.e. give the most significant contribution to the sum). Therefore, it is useful to study what the typical quantum path "looks like".

In order to reach this goal, it is worth considering the classical analogue of the path integral i.e. the Brownian motion, or "random walk" along with the Wiener measure. This insight sheds some light on some points.

First, it clarifies what the path integral is not: an integral in strictly mathematical terms, or at least not in a straightforward way, whereas the classical Wiener integral is. Therefore, it becomes clearer which properties are lost (or gained) after the quantization.

Second, the analogy with Brownian motion is a possible way to guess some physical properties of the path. In particular, the knowledge of the relationship between the typical length of a "jump" and its timespan will allow to compare infinitesimal quantities in the short-time limit and thus to recover Schrodinger equation from the path integral which is the ultimate way of checking its consistency.

Let us consider a particle at some point $\boldsymbol{x}_0$ in space, moving to a new position at a fixed distance ℓ but in a random direction at regular time intervals ϵ. After the first step, the probability of the particle being in an infinitesimal neighbourhood $d^3\boldsymbol{x}$ of certain position $\boldsymbol{x}$ is:

$$P(\boldsymbol{x}|\boldsymbol{x_0})d^3\boldsymbol{x} = \frac{1}{4\pi\ell^2}\delta(|\boldsymbol{x}-\boldsymbol{x_0}|-\ell)d^3\boldsymbol{x}$$

Since $P(\boldsymbol{x'}|\boldsymbol{x})$ is only a function of $\boldsymbol{x'}-\boldsymbol{x}$, the probability distribution after N steps can be expressed as a convolution product, i.e. as a conditional probability. Given the $N-1$ step distribution and a fixed $\boldsymbol{x}_0$:

$$\begin{cases} P(\boldsymbol{x}_1,1) = \frac{1}{4\pi\ell^2}\delta(|\boldsymbol{x_1}-\boldsymbol{x_0}|-\ell) \\ P(\boldsymbol{x},N) = \int d^3\boldsymbol{x'}P(\boldsymbol{x}|\boldsymbol{x'})P(\boldsymbol{x'},N-1) \end{cases}$$

By applying the Fourier transform $\mathcal{F}P(\boldsymbol{k},N) = (\mathcal{F}P(\boldsymbol{k},1))^N$ and considering that $\mathcal{F}P(\boldsymbol{k},1) = \frac{\sin k\ell}{k\ell}$, then:

$$\mathcal{F}P(\boldsymbol{k},N) = \left[\frac{\sin k\ell}{k\ell}\right]^N \approx \left[1-\frac{(k\ell)^2}{6}\right]^N \approx \exp\left[-\frac{N(\ell k)^2}{6}\right]$$

The previous approximation is justified under the hypothesis that $N \to \infty$ and $\ell \to 0$ while the exponential remains finite, i.e. $\ell^2 N$ is finite. Let $\boldsymbol{x}_0 = 0$;

after taking the anti-transform, the final result is:

$$P(\boldsymbol{x}, N) = \left[\frac{2\pi N \ell^2}{3}\right]^{-\frac{3}{2}} \exp\left[-\frac{3x^2}{2N\ell^2}\right]$$

Time can be explicitly introduced in the previous formula by calling $\epsilon N = t - t_0$. Few additional substitutions yield

$$\rho(\boldsymbol{x}, t; \boldsymbol{x_0}, t_0) := P(\boldsymbol{x}, t; \boldsymbol{x_0}, t_0) = \left[\frac{1}{4\pi D(t-t_0)}\right]^{\frac{3}{2}} \exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x_0})^2}{4D(t-t_0)}\right] \quad (12)$$

The previous probability distribution describes a diffusion process governed by a parameter $D = \ell^2/6\epsilon$, which in the case of classical Brownian motion depends on the medium. Before moving to the next passage it is useful to notice that ρ satisfies the heat equation:

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) \rho(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad (13)$$

where for simplicity we consider $t > t_0$ and $\boldsymbol{x} \neq \boldsymbol{x}_0$ Incidentally, let us observe that $\lim_{t \to t_0} \rho(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = \delta^3(\boldsymbol{x} - \boldsymbol{x_0})$.

Now that the probability of a particle moving from one region of space to another has been determined, a measure for the space of paths can be defined. Let us consider a partition of the time interval $t - t_0$, namely $\{t_0, t_1, ...t_N\}$, and an equal number of Borel sets $I_k \subset \mathbb{R}^3$. The probability that a particle travelling from $\boldsymbol{x_0}$ to $\boldsymbol{x_N}$ passes through the region I_k at time t_k (i.e. $\boldsymbol{x}(t_k) \in I_k$) is *almost*:

$$m(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) = \int_{I_1} d^3\boldsymbol{x_1} \int_{I_2} d^3\boldsymbol{x_2} ... \int_{I_{N-1}} d^3\boldsymbol{x_{N-1}} \times \\ \times \rho(\boldsymbol{x_1}, t_1; \boldsymbol{x_0}, t_0)\rho(\boldsymbol{x_2}, t_2; \boldsymbol{x_1}, t_1)...\rho(\boldsymbol{x_N}, t_N; \boldsymbol{x_{N-1}}, t_{N-1}) \quad (14)$$

It is "almost" a probability because the conditioned probability is $P(A|B) = P(A \wedge B)/P(B)$. As a consequence, to recover a proper probability one should divide the previous formula by $\rho(\boldsymbol{x}, t; \boldsymbol{x_0}, t_0)$. Alternatively, if the end-point is not fixed, i.e. if one integrates over the last variable, one gets a non-conditional probability.

As $N \to \infty$ the motion of the particle is described by a broken line trajectory just as those appearing in the path integral. The chance of big jumps drops exponentially and the big-N limit suggests that the succession of the vertices should appear as a continuous curve if the $t_j - t_{j-1} \to 0$ accordingly. Yet, there is no guarantee of differentiability.

The previous description makes it possible to define the Wiener measure in the space of paths for a Brownian particle. The basic results will be listed below.

- $\mathcal{W}(\boldsymbol{x}, t; \boldsymbol{x_0}, t_0)$ is the set of continuous curves connecting $\boldsymbol{x_0}$ and $\boldsymbol{x}$ in a time interval $t - t_0$.
- "Cylindrical sets" are defined as the sets of paths

$$\mathcal{K}(t_0, \boldsymbol{x_0}; t_1, I_1; ...; t_{N-1}, I_{N-1}; t_N, \boldsymbol{x_N}),$$

  i.e. the set of trajectories passing through I_k at t_k as described above. All $\mathcal{K}s$, their complements, their countable unions and finite intersections constitute the σ-algebra.
- A positive measure for $\mathcal{K}$ can be defined as the conditional-probability of the particle moving along one of the paths in $\mathcal{K}$. Then $\mu(\mathcal{K})$ can be defined as (14) either normalized or not.

The properties above identify a proper positive measure: the conditional Wiener measure. By choosing a set I_N for the last step instead of fixing the final point, the non-conditional Wiener measure can be found.

A functional is a function $Q : \mathcal{W}(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) \mapsto \mathbb{R}$. Q can be integrated:

$$\langle Q \rangle = \int d\mu(\boldsymbol{x}(\tau)) Q(\boldsymbol{x}(\tau)) \tag{15}$$

In practice, $\langle Q \rangle$ can be evaluated like the path integral: one integrates Q over broken lines with N vertices after setting $t_j - t_{j-1} = (t - t_0)/N = \epsilon$, and finally lets $N \to \infty$. Hence, in the discrete form, one gets:

$$d\mu(\boldsymbol{x}(t)) \mapsto \prod \frac{d^3 \boldsymbol{x_j}}{(4\pi D\epsilon)^{3/2}} \exp\left(-\frac{(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j)^2}{4D(\epsilon/N)}\right) \tag{16}$$

and

$$Q(\boldsymbol{x}(t)) \mapsto Q(\boldsymbol{x}_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_N)$$

The integral (15) is *almost* an average of the quantity Q over all possible paths. Again, it is *almost* an average because it is not normalized. This fact is clearer if one considers the functional $1 : \boldsymbol{x}(t) \mapsto 1$ i.e. if one evaluates the "volume" of the space of all paths.

$$\langle 1 \rangle = \mu(\mathcal{W}(\boldsymbol{x}, t; \boldsymbol{x_0}, 0)) = \int d\mu(\boldsymbol{x}(\tau)) = \rho(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) \neq 1$$

The previous formulas highlights certain similarities between the Wiener integral and the position representation of the propagator. Keeping this in mind, let us observe that expression (16) introduces a sort of kinetic term in the exponent; therefore, in order to make a connection with Feynman integral, it is natural to consider a functional that somehow accounts for the potential:

$$F = \exp\left[\int dt\, U(\boldsymbol{x}(t))\right]$$

for some $U(\boldsymbol{x})$. The average of F has a familiar form:

$$J = \int d\mu(\boldsymbol{x}(t)) \exp\left[-\int d\tau U(\boldsymbol{x}(\tau))\right] \tag{17}$$

or, even more explicitly:

$$J = \lim_{N\to\infty}\left[\frac{1}{4\pi D\epsilon}\right]^{\frac{3N}{2}} \int \prod_{j=1}^{N-1} d^3\boldsymbol{x}_j \exp\left[\sum_{k=0}^{N-1}\left(-\frac{(\boldsymbol{x}_{k+1}-\boldsymbol{x}_k)^2}{4D\epsilon} - U(\boldsymbol{x}_k)\right)\right] \tag{18}$$

It can be shown that the new quantity J satisfies a modified heat equation:

$$\frac{\partial}{\partial t}J = D\nabla^2 J - U(\boldsymbol{x})J \tag{19}$$

This equation resembles that of the quantum propagator (1), just like the expression for J is quite similar to G. There is indeed a correspondence:

$$D \leftrightarrow \frac{i\hbar}{m} \qquad U \leftrightarrow \frac{i}{\hbar}V \qquad \Rightarrow \qquad J \leftrightarrow G$$

(19) corresponds to S.E. up to a delta which arises immediately from allowing $t - t_0$ to be negative ($J = 0$ if $t < t_0$). Therefore, the path integral looks like an analytic continuation of the Wiener integral. One may want to regard the path integral as more than a sort of complex average of the functional $\exp\left[\frac{i}{\hbar}\int dt\mathcal{L}\right]$. The idea would be to define a complex measure for Feynman paths as

$$d\mu(\boldsymbol{x}(t)) = \mathcal{D}\boldsymbol{x}(t) \exp\left[\frac{i}{\hbar}\int d\tau \frac{m}{2}\dot{\boldsymbol{x}}^2\right]$$

maybe in its discrete form. Unfortunately, as the complexification takes place, a well-defined measure is lost: while in general it is possible to define complex measures, the complexified Wiener measure is not countably additive [17].

Finally, we should analyse the Brownian paths. It can be shown that, with respect to the Wiener measure, *almost all* Brownian trajectories are nowhere differentiable. In other words, a particle has a probability $P = 1$ of travelling along nowhere differentiable path and $P = 0$ of moving along a smooth path. This is not surprising for the random walks can be thought of as broken lines composed of an infinite number of infinitesimal segments. Therefore, an instant velocity is meaningless, whereas a mean velocity can be defined and it will vary very irregularly. In the previous discussion, it was implicit that the expression of ρ, which ultimately determines the drifting velocity, is expected to be finite and non-degenerate. This was assumed when D was fixed. The direct consequence is:

$$\boxed{\Delta t \approx |\Delta\boldsymbol{x}|^2} \tag{20}$$

In other words, finite displacements cannot occur in an infinitesimal timespan and "big jumps" are forbidden. The close resemblance to Feynman integral suggests that quantum paths should be alike. It should be said that a naive extension of (20) to quantum path may be questionable and there are other methods and approximations that yield (20) for Feynman paths under more general hypotheses and that do not involve Brownian motion. This last relation (20) is especially important and it will be used again. In particular, (20) implies that, in the path integral, the velocity is not well defined as a limit, since $\Delta x/\Delta t \to \infty$, or, equivalently, given two vertices j and $j+1$ in (6), then $(x_{j+1} - x_j)/(t/N) \to \infty$ (as anticipated, the nowhere differentiability already accounts for this).

5 Vector potential

The path integral has been obtained from the Trotter formula under the hypothesis that the potential is only dependent on the position. This excludes magnetic fields from the description. In order to introduce them, it is reasonable to simply introduce an electromagnetic potential in the Lagrangian in (8). The classical electromagnetic Lagrangian is:

$$\mathcal{L} = \frac{m}{2}\left[\frac{d\mathbf{x}}{dt}\right]^2 - V(\mathbf{x}) + \frac{e}{c}\frac{d\mathbf{x}}{dt}\mathbf{A}(\mathbf{x})$$

In the path integral formalism, its integral (the Sliced action) should be evaluated according to (7), as follows

$$\int_0^t dt' \frac{e}{c}\frac{d\mathbf{x}}{dt'}\mathbf{A}(\mathbf{x}(t')) = \frac{e}{c}\int_{\mathbf{x}(\cdot)} d\mathbf{x}\mathbf{A}(\mathbf{x}) =: \frac{e}{c}\sum_{j=0}^{N-1}(\mathbf{x}_{j+1} - \mathbf{x}_j)\,\mathbf{A}(\tilde{\mathbf{x}}(j, j+1))$$

$\tilde{\mathbf{x}}(j, j+1)$ is the point in which the vector potential is evaluated in each step and it should be some point on the segment connecting $\boldsymbol{x}_j$ and $\boldsymbol{x}_{j+1}$. This choice would be of no concern if Riemann integration theory could be applied: any choice, e.g. $\mathbf{x}_j$ or $\mathbf{x}_{j+1}$, would be appropriate. But this is a stochastic integral (Ito's theory [28]), not a Riemann integral, and the only correct choice will be $\frac{\mathbf{x}_{j+1}+\mathbf{x}_j}{2}$ as will be shown later.

The correctness of the previous guess can be verified by checking that, in an infinitesimal time interval, the propagator yields the same time evolution as the Schrodinger equation. Since the path integration describes the evolution as a succession of steps, only the first step will be kept and its timespan will be assumed infinitesimal. According to (20), also Δx is infinitesimal and its relation with Δt is fixed. Let us evaluate the vector potential at

some point $\boldsymbol{x}_f + \alpha(\boldsymbol{x}_i - \boldsymbol{x}_f)$, $\alpha \in [0,1]$.

$$\psi(\mathbf{x}_f, t) = \left(\frac{m}{2i\pi\hbar t}\right)^{3/2} \int d^3\boldsymbol{x}_i \; \psi(\mathbf{x}_i, 0) \times \\ \times \exp\left(\frac{it}{\hbar}\left[\frac{m}{2}\left[\frac{\mathbf{x}_f - \mathbf{x}_i}{t}\right]^2 - V(\mathbf{x}) + \frac{e}{c}(\mathbf{x}_f - \mathbf{x}_i)\mathbf{A}(\boldsymbol{x}_f + \alpha(\boldsymbol{x}_i - \boldsymbol{x}_f))\right]\right) \tag{21}$$

The purpose of the following passages is to get a first-order (in t) approximation of the previous expression. Let $\mathbf{x}_i - \mathbf{x}_f = \boldsymbol{\xi}$, then

$$\begin{aligned}\psi(\mathbf{x}_f, t) = \int d^3\boldsymbol{\xi} \left(\frac{m}{2i\pi\hbar t}\right)^{3/2} \exp\left(\frac{im\xi^2}{2\hbar t}\right) \times \\ \times \exp\left(-\frac{it}{\hbar}\left[V(\boldsymbol{x}_f) - \nabla_n V(\boldsymbol{x}_f)\xi_n\right] + o(t)\right) \times \\ \times \exp\left(-\frac{ie}{\hbar c}\xi_n\left[A_n(\boldsymbol{x}_f) + \alpha(\xi_k\nabla_k)A_n(\boldsymbol{x}_f)\right] + o(t)\right) \times \\ \times \left[\psi(\boldsymbol{x}_f, 0) + \xi_n\nabla_n\psi(\boldsymbol{x}_f, 0) + \frac{1}{2}\xi_m\xi_k\frac{\partial^2\psi}{\partial x_m \partial x_k}(\boldsymbol{x}_f, 0) + o(t)\right]\end{aligned}$$

The exponentials can expanded in series. Since all quantities in the following equation are evaluated in $\boldsymbol{x}_f$, the spatial dependence will be omitted.

$$\begin{aligned}\psi(t) = \left(\frac{m}{2i\pi\hbar t}\right)^{3/2} \left(1 - \frac{it}{\hbar}V\right) \int d^3\boldsymbol{\xi} \exp\left(\frac{im\xi^2}{2\hbar t}\right) \times \\ \left[1 - \frac{ie}{\hbar c}\xi_n A_n - \alpha\frac{ie}{\hbar c}\xi_n\xi_m\nabla_m A_n - \frac{e^2}{2\hbar^2 c^2}\xi_n\xi_k A_n A_k\right] \times \\ \times \left[\psi(\boldsymbol{x}_i, 0) + \xi_n\nabla_n\psi(\boldsymbol{x}_i, 0) + \frac{1}{2}\xi_m\xi_k\frac{\partial^2\psi}{\partial x_m\partial x_k}\right] + o(t)\end{aligned} \tag{22}$$

In the previous integral, all odd terms (i.e. those that are linear at least in one ξ_k) vanish and the only integrals that are left have the following forms: $\int x^2 e^{-ix^2}$ or $\int e^{-ix^2}$. After some rearrangements, one gets:

$$\begin{aligned}i\hbar\dot{\psi} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{e^2}{2mc^2}A^2 + \alpha\frac{ie\hbar}{mc}(\nabla\cdot\boldsymbol{A}) + \frac{ie\hbar}{mc}(\boldsymbol{A}\cdot\nabla) + V\right)\psi \\ &= -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{e^2}{2mc^2}A^2\psi + (\alpha - 1/2)\frac{ie\hbar}{mc}(\nabla\cdot\boldsymbol{A})\psi + \\ &\qquad + \frac{ie\hbar}{2mc}(\boldsymbol{A}\cdot\nabla)\psi + \frac{ie\hbar}{2mc}\nabla\cdot(\boldsymbol{A}\psi) + V\psi\end{aligned}$$

where $\frac{\partial\psi}{\partial t} = (\psi(t) - \psi(0))/t$ which is true for $t \to 0$. If we want the previous equation to be the Schrodinger equation, the only possible choice of α is $\alpha = 1/2$ and this fact is called "mid-point" rule.

There is no mid-point rule for the scalar potential V and any point $\tilde{\mathbf{x}}(j, j+1)$ on the segment $[\boldsymbol{x}_j, \boldsymbol{x}_{j+1}]$ would yield the correct result. The reason is that, in the series expansion of $V(\boldsymbol{x})$ about some $\tilde{\mathbf{x}}(j, j+1)$, the only term which remains is $V(\tilde{\mathbf{x}}(j, j+1))$ since it is zero order in $\epsilon = t/N$: in the sliced action, V is multiplied by ϵ and, consequently, the error deriving from changing the evaluation point (order $\epsilon^{1/2}$) is always neglectable. On the other hand, the vector potential is multiplied by $x_{j+1} - x_j$ which is of order $\epsilon^{1/2}$ and thus the error introduced by the change of evaluation point ($\epsilon^{1/2}$ again) cannot be neglected since the overall error goes as ϵ. Physically speaking, the vector potential is coupled to the "velocity" which is a highly oscillating quantity in Brownian-like paths and therefore the situation is more delicate.

Gauge invariance arises naturally in the path integral formalism. By replacing $\boldsymbol{A}$ with $\boldsymbol{A} + \boldsymbol{\nabla}\chi$ ($\chi(\boldsymbol{x})$ is a smooth function of space) we get a new integral expression with an additional term in the exponent:

$$\int_{t_i}^{t_f} \dot{\boldsymbol{x}}(t)\boldsymbol{\nabla}\chi(\boldsymbol{x}(t)) = \int_{\boldsymbol{x}(\cdot)} d\boldsymbol{x}\, \boldsymbol{\nabla}\chi(\boldsymbol{x}) = \chi(\boldsymbol{x_f}) - \chi(\mathbf{x_i}) \tag{23}$$

The previous passage seems obvious, but, since we are not dealing with Riemann integrals, a couple of things should be remarked. The first equality in (23) holds by definition:

$$\sum \epsilon \frac{\boldsymbol{x}_{j+1} - \boldsymbol{x}_j}{\epsilon} \chi\left(\frac{\boldsymbol{x_{j+1}} + \boldsymbol{x_j}}{\mathbf{2}}\right) = \sum (\boldsymbol{x}_{j+1} - \boldsymbol{x}_j)\chi\left(\frac{\boldsymbol{x_{j+1}} + \boldsymbol{x_j}}{\mathbf{2}}\right)$$

where, before taking the limit, $\dot{\boldsymbol{x}}$ is a formal notation for a finite ratio. As for the second equality in (23), which is a quite natural generalization of a fundamental property of Riemann integrals, it holds only if the integral is carried with the mid-point rule. Taking the limit in a different way (e.g. evaluating χ in the x_j points) would lead to an additional term on the right hand side of the equation.

Since all paths have the same end point and starting point, the quantity above does not depend on the specific path, and the gauge transformation has the following effect on the propagator:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) \mapsto G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) e^{\frac{ie}{\hbar c}[\chi(\boldsymbol{x_f}) - \chi(\mathbf{x_i})]}$$

Thus, the change of gauge equals to a unitary transformation:

$$\psi(\boldsymbol{x}) \mapsto e^{\frac{ie}{\hbar c}\chi(\boldsymbol{x})}\psi(\boldsymbol{x})$$

Similarly, by replacing V with $V - (1/c)\partial_t f$, where $f(t)$ is a smooth function of time, one gets

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) \mapsto G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) e^{\frac{ie}{\hbar c}[f(t_f) - f(t_i)]}$$

since

$$\int_{t_i}^{t_f} dt\, \partial_t f(t) = f(t_f) - f(t_i)$$

reasonably holds for the sliced integral. Again, $f(t_f)$ and $f(t_i)$ are the same for all paths. In the general case, let $\Lambda(\boldsymbol{x}, t)$ be a function of both space and time. The gauge transformation becomes

$$\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla\Lambda \qquad\qquad V \mapsto V - \frac{1}{c}\partial_t\Lambda$$

Thus, since

$$\int dt\ [\partial_t\Lambda(\boldsymbol{x}(t), t) + \dot{\boldsymbol{x}}(t)\nabla\Lambda(\boldsymbol{x}(t), t)] = \int dt \frac{d}{dt}\Lambda(\boldsymbol{x}(t), t) = \Lambda(\boldsymbol{x}_f, t_f) - \Lambda(\boldsymbol{x}_i, t_i)$$

one gets again

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) \mapsto G(\boldsymbol{x}_f, t; \boldsymbol{x}_i, 0) e^{\frac{ie}{\hbar c}[\Lambda(\boldsymbol{x}_f, t_f) - \Lambda(\boldsymbol{x}_i, t_i)]}.$$

6 Phase Space

As anticipated during the derivation of the position representation, if the integrals over impulses are not performed, additional N variables are kept.

$$G(x_f, t; x_i, 0) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{dk_j}{2\pi\hbar} \prod_{l=1}^{N-1} dx_l \exp\left[\frac{i}{\hbar}\sum_{q=1}^{N}\left[\frac{t}{N}\left(\frac{k_q^2}{2m} + V(x_q)\right) + k_q(x_q - x_{q-1})\right]\right] \quad (24)$$

The exponential in the previous integral can be formally written as:

$$\exp\left[\frac{i}{\hbar}\int dt\,(p\dot{x} - H(x, p))\right]$$

The previous expression recalls the Hamiltonian formalism in classical mechanics which allows defining a class of canonical variables and a way to pass from one to the other. Yet, in general, classical results cannot be extended to the path integral formalism [28].
Quantum trajectories in phase space are not smooth and not even Brownian-like. Furthermore, a classical trajectory in the phase space is completely identified by the law of motion $x(t)$ which fixes $p(t)$. However, if, before taking the limit, (24) is interpreted as a sum of broken-lines paths with N steps, between each couple of vertices x_{i-1} and x_i the momentum is redundantly specified by k_i. Thus the motion is not classical and $p(t)$ has discontinuities at each vertex. Another interpretation is possible: the time t is divided into $2N$ intervals. The motion is classical and at each vertex either the position or the momentum are alternatively specified.

7 Coherent states representation

Coherent states can be defined in a rather general fashion as a set of vectors in the Hilbert space $\mathcal{H}$ satisfying certain conditions. Namely, they must be both the images of elements from some "label-set" $\mathcal{L}$ under a strongly continuous map and they must yield a resolution of unity $\int_{\mathcal{L}} dl\, |l\rangle \langle l|$ with dl being an appropriate measure. From now on, canonical coherent states will be examined and they will be referred to simply as coherent states.

Canonical coherent states can be either defined as eigenstates of the lowering operator $a = \frac{q+ip}{\sqrt{2\hbar}}$ or as minimum uncertainty states. Some definitions are needed. The eigenstates of the harmonic oscillator can be labelled with integers $|n\rangle$. Then:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$[a, a^\dagger] = 1$$
$$a\,|0\rangle = 0$$

Let now $|z\rangle$ be a vector such that

$$\boxed{a\,|z\rangle = z\,|z\rangle}$$

and let us expand it on the $|n\rangle$ basis.

$$\begin{aligned} z\sum_n \mu_n\,|n\rangle = z\,|z\rangle = a\,|z\rangle = a\sum_n \mu_n\,|n\rangle \\ = \sum \mu_n\sqrt{n}\,|n-1\rangle = \sum \mu_{n+1}\sqrt{n+1}\,|n\rangle \end{aligned}$$

$z\mu_n = \sqrt{n+1}\mu_{n+1}$ implies that $\mu_n = \frac{z^n}{\sqrt{n!}}\mu_0$. μ_0 can be determined by imposing $1 = \langle z|z\rangle$. The result is:

$$|z\rangle = \exp\left[-\frac{|z|^2}{2}\right]\sum_n \frac{z^n}{\sqrt{n!}}\,|n\rangle$$

It follows that

$$\langle w|z\rangle = \exp\left[-\frac{|w|^2+|z|^2}{2} + w^*z\right]$$

An equivalent form will be used later:

$$\langle w|z\rangle = \exp\left[-\tfrac{1}{2}\left[w^*(w-z) - z(w^*-z^*)\right]\right] \tag{25}$$

There is another characterization of coherent states. They are harmonic oscillator ground states displaced both in space and momentum. Let T_{ξ_1}

and D_{ξ_2} be the corresponding displacement operators. It will be shown that $z = \frac{\xi_1 + i\xi_2}{\sqrt{2\hbar}}$:

$$\begin{aligned} aT_{\xi_1}D_{\xi_2}\left|0\right\rangle &= T_{\xi_1}D_{\xi_2}T_{\xi_1}^{\dagger}D_{\xi_2}^{\dagger}aT_{\xi_1}D_{\xi_2}\left|0\right\rangle \\ &= T_{\xi_1}D_{\xi_2}\left(\frac{\xi_1 + i\xi_2}{\sqrt{2\hbar}} + a\right)\left|0\right\rangle \\ &= \left[\frac{\xi_1 + i\xi_2}{\sqrt{2\hbar}}\right]T_{\xi_1}D_{\xi_2}\left|0\right\rangle \end{aligned}$$

If we define $\left|z\right\rangle = T_{\xi_1}D_{\xi_2}\left|0\right\rangle$, some properties follow directly from the observations above:

$$\left\langle z\right|q\left|z\right\rangle = \frac{\Re(z)}{\sqrt{2\hbar}} \qquad \left\langle z\right|p\left|z\right\rangle = \frac{\Im(z)}{\sqrt{2\hbar}} \qquad \Delta p\Delta q = \left\langle 0\right|q^2\left|0\right\rangle\left\langle 0\right|p^2\left|0\right\rangle = \frac{\hbar}{2}$$

Hence, the minimum uncertainty property has been proved. It is also true that any minimum uncertainty state is a coherent state.

The wavefunction associated with $\left|z\right\rangle$ can be evaluated as $T_{\xi_1}D_{\xi_2}\left|0\right\rangle$. The Schrodinger representation for $\left|0\right\rangle$ is $\pi^{1/4}\exp(-x^2/2)$. Then:

$$\psi_z(x) = \pi^{-1/4}\exp\left(\frac{i\xi_2 x}{\hbar}\right)\exp\left(-\frac{(x-\xi_1)^2}{2\hbar}\right)$$

Then, up to the irrelevant constant phase factor $\exp(i\xi_1\xi_2/\hbar)$, the wavefunction can be written as:

$$\psi_z(x) = \pi^{-\frac{1}{4}}\exp\left[-\frac{|z|^2}{2} - \frac{x^2+z^2}{2} + \sqrt{2}zx\right]$$

The coherent states allow to define a basis of vectors. Yet, their set $\mathcal{C} = \{\left|z\right\rangle : z \in \mathbb{C}\}$ is overcomplete and, consequently, the resolution is not unique. The subset $\mathcal{B} \subset \mathcal{C}$, defined as $\mathcal{B} = \{\left|w\right\rangle \in \mathcal{C} : w = m + in; m, n \in \mathbb{Z}\}$ is a possible nontrivial complete set [3]. More in general, for any complete or overcomplete set (characteristic set) $\mathcal{S}$, it is true that $\forall\left|z\right\rangle \in \mathcal{S}, \langle z|\psi\rangle = 0, \forall\left|\psi\right\rangle \in \mathcal{H} \Rightarrow \left|\psi\right\rangle = 0$. The identity operator can be expanded on this basis

$$\mathbb{I} = \int_{\mathbb{C}} \frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|$$

with $d^2z = d\Re(z)d\Im(z)$. It can be shown as follows:

$$\begin{aligned} \int_{\mathbb{C}} \frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right| &= \int_{\mathbb{C}} \frac{d^2z}{\pi}\sum_{n,m} e^{-|z|^2}\frac{z^{*n}}{\sqrt{n!}}\frac{z^m}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right| \\ &= \sum_{n,m}\frac{1}{\pi\sqrt{n!m!}}(\pi\delta_{n,m}n!)\left|n\right\rangle\left\langle m\right| \\ &= \sum_n \left|n\right\rangle\left\langle n\right| = \mathbb{I} \end{aligned}$$

Just like the position improper eigenvectors allow the definition of a representation for $\mathcal{H}$ on L^2, the coherent states allow to represent $\mathcal{H}$ on the Bargmann space. Bargmann space $\mathcal{I}$ is the Hilbert space of entire functions.

- $\mathcal{I} = \{f : \mathbb{C} \to \mathbb{C}, f \text{ entire and } \int_{\mathbb{C}} d^2z/\pi e^{-|z|^2}|f(z)|^2 < \infty\}$
- The isomorphism can be defined as follows:

$$\mathcal{H} \to \mathcal{I}$$

$$|\psi\rangle \mapsto \psi(z) = e^{\frac{|z|^2}{2}} \langle z^*|\psi\rangle$$

- The inner product is:

$$\langle\phi|\psi\rangle \mapsto \int \frac{d^2z}{\pi} e^{-|z|^2}\phi(z)^*\psi(z)$$

  or, alternatively

$$\langle\phi|\psi\rangle \mapsto \int \frac{dzdz^*}{2\pi i} e^{-|z|^2}\phi(z)^*\psi(z)$$

- An orthonormal basis for this space is $\{z^n/\sqrt{n!}\}$. Since $\langle z^*| a^\dagger |\psi\rangle = z \langle z^*|\psi\rangle$, then

$$a^\dagger f(z) = zf(z)$$

  and, in order to preserve the commutation rule $[a, a^\dagger] = 1$, then it must be true that

$$af(z) = \frac{d}{dz} f(z)$$

  Any operator which is a function of q and p is also a function of a and $a^\dagger$ and if it can be expanded is series, it can be written as $A = \sum_{i,j} a_{i,j} a^{\dagger i} a^j$ after using the commutation rules. Finally $A = \sum_{i,j} a_{i,j} z^i \left(\frac{d}{dz}\right)^j$.

The introduction of the entire representation yields a class of characteristic sets for the resolution of unity. An analytic function that vanishes on an infinite set containing an accumulation point is null: any set of this kind (e.g. any curve in $\mathbb{C}$) is a characteristic set and the overcompleteness becomes evident.

Now it is possible to proceed as it was done for the position representation. By defining $\epsilon = t/N$

$$G(z_f, t; z_i) = \langle z_f| e^{-\frac{it}{\hbar}H} |z_i\rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \frac{d^2z_k}{\pi} \prod_{j=1}^{N} \langle z_j| e^{-\frac{i\epsilon}{\hbar}H} |z_{j-1}\rangle$$

Since $N \to \infty$ and $\epsilon \to 0$, the exponential can be expanded to the first order in ϵ. Let us first define the velocity, in a discrete form, as

$$\frac{dz_j}{dt} := \frac{z_j - z_{j-1}}{\epsilon}$$

In addition, it is convenient to write

$$H(z_j^*, z_{j-1}) := \frac{\langle z_j | \, H \, | z_{j-1} \rangle}{\langle z_j | z_{j-1} \rangle}$$

and to evaluate $\langle z_j | z_{j-1} \rangle$ according to (25). Hence, before taking the limit:

$$G_N(z_f, t; z_i) = \int \prod_{k=1}^{N-1} \frac{d^2 z_k}{\pi} \prod_{j=1}^{N} \langle z_j | z_{j-1} \rangle \left[1 - \frac{i\epsilon}{\hbar} H(z_j^*, z_{j-1}) + O(\epsilon^2) \right] \quad (26)$$

$$= \int \prod_{k=1}^{N-1} \frac{d^2 z_k}{\pi} \prod_{j=1}^{N} \exp \left[-\frac{1}{2} \left[z_j^* (z_j - z_{j-1}) - z_{j-1} (z_j^* - z_{j-1}^*) \right] \right] \times$$

$$\times \exp \left[-\frac{i\epsilon}{\hbar} H(z_j^*, z_{j-1}) + O(\epsilon^2) \right]$$

$$= \int \prod_{k=1}^{N-1} \frac{d^2 z_k}{\pi} \exp \left[\epsilon \frac{i}{\hbar} \sum_{j=1}^{N} - \frac{1}{2i} \left[z_j^* \frac{dz_j}{dt} - z_{j-1} \frac{dz_j^*}{dt} \right] - H(z_j^*, z_{j-1}) + O(\epsilon^2) \right]$$

The final approximation consists in replacing the z_{j-1} terms with z_j, meaning that terms of order $\epsilon(z_j - z_{j-1})$ will be neglected. This can be questionable. Terms of this kind in the position representation can indeed be considered "small" with respect to those of the first order because of what it was said about Brownian paths. Here, instead, no Brownian-like path is available, and this can be seen as follows. As $t \to 0$, surely $\langle x' | \, G \, | x \rangle \to g \, \delta(x - x')$ but it is not true $\langle z | \, G \, | z' \rangle \to g' \delta(z - z')$ and, consequently, the paths may not even be continuous. The final formula is:

$$G(z_f, t; z_i) = \lim_{N \to \infty} \int \prod_{k=1}^{N-1} \frac{d^2 z_k}{\pi} \exp \left[i \int dt \left[\frac{i}{2} \left(z^* \frac{dz}{dt} - z \frac{dz*}{dt} \right) - \frac{1}{\hbar} H(z^*, z) \right] \right] \quad (27)$$

Another slightly different approach is possible in order to get a somehow meaningful expression for the path integral in the coherent representation without writing the explicit form of the product $\langle z_j | z_{j-1} \rangle$. Therefore, this approach can be applied to more general definitions of coherent states. The following approximation will be used:

$$\langle z_j | z_{j-1} \rangle = 1 - \langle z_j | \left(| z_j \rangle - | z_{j-1} \rangle \right) = 1 - \langle z_j | \dot{z}_j \rangle \, \epsilon \approx \exp \left[-\epsilon \, \langle z_j | \dot{z}_j \rangle \right]$$

where the "derivative" of $|z_j\rangle$ has been defined, only in a loose sense, as:

$$|\dot{z}_j\rangle := \lim_{\epsilon \to 0} \frac{|z_j\rangle - |z_{j-1}\rangle}{\epsilon}$$

It is possible to proceed further just as it was done to go from (26) to (27). Let us set $\mathcal{D}z(t) := \prod_{k=1}^{N} \frac{d^2 z_k}{\pi}$, then:

$$G(z_f, t; z_i) = \lim_{N \to \infty} \int \mathcal{D}_n z(t) \exp\left[\frac{i}{\hbar} \int dt \left[i\hbar \langle z|\dot{z}\rangle - \langle z| H |z\rangle\right]\right] \tag{28}$$

The interesting fact is that the action in the exponent looks like the following genuine quantum action:

$$\int dt \left[i\hbar\langle\psi|\dot{\psi}\rangle - \langle\psi| H |\psi\rangle\right] = \int dt \, \langle\psi| \, i\hbar\partial_t - H \, |\psi\rangle$$

By setting to zero the first variation of the quantity above, the Schrodinger equation is immediately recovered (the independent variations of $|\psi\rangle$ and $\langle\psi|$ yield the same result).

It has been shown for the position representation that the classical action and the classical trajectories play an important role in determining the propagator. In some simple cases they actually determine it completely. It is possible to wonder if anything similar happens in the coherent states representation. Let us consider the actions appearing in both of the formulations above (27) and (28) as if they were classical, i.e. functionals of some paths $z(t)$. They depend both on $z(t)$ and on its first derivative $\dot{z}(t)$ but not on higher derivatives. In particular, the absence of the second derivatives implies that the differential equations of motion for $z(t)$, obtained with the stationary action principle, will be of the first order. This fact suggests that the coherent states representation may lead to a Hamiltonian formalism, rather than to a Lagrangian formalism. Thus, $z(t)$ can be thought of as a motion in the phase space. This is not surprising since, as it has been shown for the canonical states, the real and the imaginary parts of z are proportional to the average position and momentum respectively.

Some problems arise. As mentioned before, classical trajectories are not well defined in the phase space, as far as the quantum propagator is concerned. It can be even shown that phase space paths are not even continuous. Coherent trajectories are expected to be alike, as it has been shown above. In addition, the fact that the equations of motion are of the first order means that fixing the "position" z at a certain time unambiguously determines a unique trajectory. Yet, when calculating the propagator, two "positions" (z_i and z_f) at two different times seem to be specified. Nevertheless, there is in general no classical path connecting them.

However, since it is not forbidden to find a well behaved path, the stationary phase method is still possible. The boundary conditions problem can be

overcome by looking back at the path integral. One notices that the motion is formally described by two different paths, $z(t)$ and $z^*(t)$, which are independent. Hence, the boundary conditions read $z(t_i) = z_i$ and $z^*(t_f) = z_f^*$ and are associated with first order differential equations as it should be.

Part II
Applications and examples

8 Hamiltonian spectrum

The propagator yields useful information about the spectrum. Let H be the Hamiltonian of the system under exam and let $\{|n\rangle\}$ be the complete set of its eigenvectors, so that $H|n\rangle = E_n|n\rangle$. In addition, let us define $u_n(x) := \langle x|n\rangle$. For $t > 0$

$$G(y,t;x,0) = \langle y|\,\theta(t)e^{-\frac{it}{\hbar}H}\,|x\rangle \tag{29}$$

$$= \theta(t)\sum_n \langle y|\,e^{-\frac{it}{\hbar}H}\,|n\rangle\,\langle n|x\rangle \tag{30}$$

$$= \theta(t)\sum_n u_n^*(x)u_n(y)e^{-\frac{it}{\hbar}E_n} \tag{31}$$

In order to obtain information about the spectrum, one may want to replace the time dependence with an energy dependence via Fourier transform. Yet, the θ function will limit the integration interval to $[0,+\infty]$ and, for some reasons (convergence and the positions of the poles), it is convenient to give the energy a small imaginary $i\eta$ part so that the physical result will hold as $\eta \to 0$. This means that one may use the Laplace transform instead of the Fourier transform. Formally, in the operator notation, one finds the resolvent:

$$\tilde{G}(E) = \lim_{\eta\to 0}\int_0^\infty dt\, e^{\frac{it}{\hbar}(E-H+i\eta)} = \lim_{\eta\to 0} i\hbar(E-H+i\eta)^{-1} \tag{32}$$

In the position representation:

$$\tilde{G}(x_f,E;x_i) = \lim_{\eta\to 0}\int_{-\infty}^\infty dt\, e^{\frac{i(E+i\eta)t}{\hbar}}G(x_f,t;x_i) = \lim_{\eta\to 0} i\hbar\sum_n \frac{u_n^*(x_i)u_n(x_f)}{E-E_n+i\eta} \tag{33}$$

The poles of $\tilde{G}(x_f,E;x)$ are the eigenvalues of H. It should be remarked that, before taking the $\eta \to 0$ limit, the poles lie below the real axis.

9 Free Particle

The position representation for the free propagator

$$G(t) = \exp\left[-\frac{it}{2m\hbar}p^2\right]$$

can be evaluated both directly from the propagator and from (6).

The former method is the following. As was done while deriving the position representation for the general propagator, the completeness of the momentum basis can be used.

$$G(x_f, t; x_i) = \int dk \, \langle x_f | \, G \, | k \rangle \, \langle k | x_i \rangle$$

Then

$$\sqrt{\frac{m}{2i\pi\hbar t}} \exp\left[\frac{im}{2\hbar t}(x_f - x_i)^2\right]$$

The latter method makes use of (6) with V=0. The following identity is needed:

$$\int du \frac{\sqrt{ab}}{\pi} \exp\left[-a\,(x-u)^2 - b\,(u-y)^2\right] = \sqrt{\frac{ab}{\pi(a+b)}} \exp\left[-\frac{ab}{a+b}(x-y)^2\right]$$

It can be applied to the first and the second terms in the product, getting a new similar product with $N-2$ terms. The procedure can be repeated until only one term remains. After the N integrations, both the exponential and the normalization constant become N-independent since only $(t/N)N = t$ is left. Therefore, the limit is not even necessary.

$$G(x_f, t; x_i) = \lim_{N\to\infty} \left[\frac{m}{2i\pi\hbar(t/N)}\right]^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dx_k \times \\ \times \exp\left[\frac{i(t/N)}{\hbar} \sum_{j=0}^{N-1} \left[\frac{m}{2} \frac{(x_{j+1} - x_j)^2}{t/N}\right]\right]$$

becomes

$$\boxed{G(x_f, t; x_i) = \sqrt{\frac{m}{2i\pi\hbar t}} \exp\left[\frac{im}{2\hbar t}(x_f - x_i)^2\right]}$$

It should be remarked that the classical action for the free particle is:

$$S_{classic} = \frac{m}{2} v^2 t = \frac{m}{2} \frac{(y-x)^2}{t}$$

Hence:

$$G(y, t; x) \sim e^{\frac{i}{\hbar} S_{classic}}$$

The importance of classical trajectories in path integration will be clearer in the next section.

10 Quadratic Lagrangians and classical trajectories

If the Lagrangian is at most quadratic, the exact propagator can be calculated.

$$\mathcal{L} = a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + d(t)$$

It would be possible to compute the propagator with (6), but this procedure would be rather long and difficult. It is better to get the same result from (9). Let us assume that the starting and the end-points are connected by a unique classical trajectory satisfying the Euler-Lagrange equation:

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x}\right)\mathcal{L} = 0 \tag{34}$$

Let $\bar{x}(t)$ be the classical path and let us define the new coordinate $y(t)$ such that $x(t) = y(t) + \bar{x}(t)$. To put it in words, $y(t)$ is the displacement at time t from the classical trajectory and its boundary conditions are $y(t_i) = 0$ and $y(t_f) = 0$. Since, for any given time, $\bar{x}(t)$ is fixed, then $dy = dx$ and, consequently, the elementary "path volume" does not change: $\mathcal{D}y(t) = \mathcal{D}x(t)$.

The Lagrangian under exam is quadratic and there will not be terms of higher orders than the second in the Taylor expansion of the action.

$$\begin{aligned} S[x(\cdot)] &= S[\bar{x}(\cdot)] + \frac{\delta}{\delta x(t)}S[\bar{x}(\cdot)]y(t) + \frac{1}{2}\frac{\delta^2}{\delta x^2(t)}S[\bar{x}(\cdot)]y^2(t) \\ &= S[\bar{x}(\cdot)] + \frac{1}{2}\frac{\delta^2}{\delta x^2(t)}S[\bar{x}(\cdot)]y^2(t) \end{aligned} \tag{35}$$

The meaning of the previous expression must be clarified. The action S is a functional and we need its derivative with respect to a path. Let us consider the sliced action first. A broken path is identified by the array (a vector) of its vertices $\{x_j\}$, $i = 0, 1, ..., N$. Let $\{\delta x_i\}$ be a "small" path:

$$S(\{x_j + \delta x_j\}) - S(\{x_j\}) \approx \sum_k \left.\frac{\partial S}{\partial x_k}\right|_{\{x_j\}} \delta x_k$$

Thus, the derivativative is a simple gradient. A path can be regarded a vector with infinite elements (continuum cardinality). Let $x(t)$ be a path and let $\delta x(t)$ be a "small" path. The derivative is defined as follows:

$$S[x(\cdot) + \delta x(\cdot)] - S[x(\cdot)] \approx \int \left.\frac{\delta S}{\delta x(t')}\right|_{x(\cdot)} \delta x(t')\, dt' =: \frac{\delta}{\delta x(t)}S[\bar{x}(\cdot)]\delta x(t)$$

The second derivative is obtained in the same way.

(35) implies that the integral of the sum of all terms which are linear in $y(t)$ (or in $\dot{y}(t)$) must vanish. In practice, given a Lagrangian, one may

collect all terms containing only $\bar{x}$ and its derivatives, and then calculate S_c after solving the Euler-Lagrange equation. Then, one gathers all quadratic terms containing only y and its derivatives. The result is:

$$G(x_f, t_f; x_i, t_i) = \exp\left[\frac{i}{\hbar} S_{cl}(x_f, x_i)\right] \int_0^0 \mathcal{D}y(t) \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} dt[a\dot{y}^2 + by\dot{y} + cy^2]\right]$$

Consequently, the propagator is the product of a term only depending on the classical action and a second function $F(t_f, t_i)$.

$$G(x_f, t_f; x_i, t_i) = \exp\left[(i/\hbar) S_{classic}(x_f, x_i)\right] F(t_f, t_i) \tag{36}$$

The function $F(t_f, t_i)$ is the result of a Gaussian integration since the time sliced Lagrangian has only quadratic and linear terms and can be calculated exactly. It is worth noticing that the propagators under exam only depend on the classical action and trajectory. This is of course a special case, yet, since classical mechanics must be a limit case of the quantum description, it is not unexpected that, under certain conditions, classical quantities appear. The previous result suggests the following approach.

The classic limit can be obtained as $\hbar \to 0$. Then, if $\hbar$ is small compared to $S = \int dt\, \mathcal{L}$, even small variations of S will result in large variations of $iS/\hbar$ in the exponent. Therefore, these rapid phase shifts will produce a sort of sum over random numbers on the unitary circumference in the complex plane. The mean contribution is expected to be very small. On the other hand, an important contribution should come from those groups of paths whose small variations will not affect S significantly. From a classical point of view, such paths must belong to a neighbourhood of the stationary trajectories for the action-functional, i.e. of those paths satisfying the Euler-Lagrange equation. If the action is expanded in series about the stationary trajectory and then truncated after the second order (the action will look like (35)), then the approximated propagator can be solved exactly by following the foregoing procedure. It can be demonstrated that the semi-classical propagator is (in 3 dimensions):

$$G(\boldsymbol{x_f}, t; \boldsymbol{x_i}) = \frac{1}{(2\pi i\hbar)^{3/2}} \left[\det\left(-\frac{\partial^2 S_c}{\partial \boldsymbol{x}_f \partial \boldsymbol{x}_i}\right)\right]^{\frac{1}{2}} \exp\left[\frac{i}{\hbar} S_c\right] \tag{37}$$

This is called the van Vleck propagator. This approximation is exact for quadratic Hamiltonians. A full description must account for the phase shift that occurs any time an eigenvalue of $\frac{\partial^2 S_c}{\partial \boldsymbol{x}_f \partial \boldsymbol{x}_i}$ changes sign. Furthermore, if there are many classical trajectories, G will be a sum of addends like (37). Let us point out that, despite its appearance, $\frac{\partial^2 S_c}{\partial \boldsymbol{x}_f \partial \boldsymbol{x}_i}$ is only a function of time since both $\boldsymbol{x}_f$ and S_c only depend on time once the classical trajectory is fixed.

Finally, let us observe that, if the classical trajectory is unique for every x_i and x_f (e.g. the free particle), then (36) implies:

$$|G(x_f, t_f; x_i, t_i)|^2 = |F(t_f, t_i)|^2 \tag{38}$$

Hence, (11) implies that that if the initial state is a position eigenvalue, say $|x_i\rangle$, then it the particle is equally likely to be found anywhere in space at a given instant, since the distribution $|G|^2$ does not depend on position but it only depends on time. This "probability distribution" is not normalized, and not even normalizable. This is not surprising: as a wavefunction is "squeezed" in space, it spreads in momentum space and, consequently, it is more likely to detect the particle farther. $|x_i\rangle$ is the limit case with an infinite momentum uncertainty. Nevertheless, this is not troublesome, for $|x_i\rangle$ is not a legitimate physical state.

11 Uniform electric field

The propagator of a particle in a uniform electric field belongs to the previous category with $a(t) = m/2$ and $e(t) = qE$. Indeed:

$$\mathcal{L} = T + qEx$$

Here

$$F(t_f, t_i) = \int_0^0 \mathcal{D}y(t) \exp\left[\frac{im}{2\hbar}\int_{t_i}^{t_f} dt\dot{y}^2\right]$$

is the free particle path integral with $x_f = x_i = 0$ i.e. $\sqrt{\frac{m}{2i\pi\hbar t}}$.

The classical action can be obtained by integrating the Lagrangian along the classical path $x_{cl}(t) = x_i + v_0 t + (qE/2m)t^2$, which is a solution of the Euler-Lagrange equation.

$$S_{cl} = \int_0^t (m/2)\dot{x_{cl}}^2(t') + qEx_{cl}(t')dt'$$

$$S_{cl} = \frac{(qE)^2}{3m} + Et^2 v_0 + \frac{mtv_0^2}{2} + Etx_0$$

Finally, by inserting the expression of $v_0(x_f, t; x_i)$ in the previous equation and by combining all the previous results, it is possible to write the propagator:

$$G(x_f, t, x_i) = \sqrt{\frac{m}{2i\pi\hbar t}} \exp\left(\frac{i}{\hbar}\left[\frac{m(x_f - x_i)^2}{2t} + \frac{qEt(x_f + x_i)}{2} - \frac{qEt^3}{24}\right]\right)$$

12 Harmonic Oscillator

Just like the position representation of the propagator for the free particle was obtained without explicitly performing the sum over paths, the coherent states representation for the harmonic oscillator can be evaluated in a simple way. Let us first observe that the harmonic Hamiltonian induces a "phase rotation" on coherent states.

$$H_{h.o.} = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 = \hbar\omega(a^\dagger a + 1/2)$$

with

$$H_{h.o.}\left|n\right\rangle = (\hbar\omega + 1/2)\left|n\right\rangle$$

Then (let $\hbar = 1$):

$$U_{h.o.}\left|z\right\rangle = U_{h.o.}\sum_n \frac{z^n}{\sqrt{n!}}\left|n\right\rangle = e^{-\frac{i\omega t}{2}}\sum_n \frac{z^n}{\sqrt{n!}}e^{-i\omega nt}\left|n\right\rangle = e^{-\frac{i\omega t}{2}}\left|ze^{-i\omega t}\right\rangle$$

Thus, the propagator is:

$$\begin{aligned} G(z_f,t;z_i) &= \left\langle z_f\right| G(t)\left|z_i\right\rangle = e^{\frac{i}{2}\omega t}\left\langle z_f|z_i e^{-i\omega t}\right\rangle \\ &= \exp\left[-\frac{|z_f|^2+|z_i|^2}{2} + z_f^* z_i e^{-i\omega t} - \frac{i\omega t}{2}\right] \end{aligned}$$

One can verify that (33) yields the correct spectrum:

$$\begin{aligned} G(z_f,E+i\eta;z_i) &= \lim_{\eta\to 0}\int_0^\infty dt\, e^{i(E+i\eta)t}G(z_f,t;z_i) \\ &= \lim_{\eta\to 0} e^{-\frac{|z_f|^2+|z_i|^2}{2}}\int_0^\infty dt\sum_{n=0}^\infty e^{iEt}\frac{(z_f^* z_i)^n}{n!}e^{-i\omega tn - i\omega t/2} \\ &= \lim_{\eta\to 0} e^{-\frac{|z_f|^2+|z_i|^2}{2}}\sum_{n=0}^\infty \frac{(z_i z_f^*)^n}{n!}\frac{1}{E-\omega(n-1/2)+i\eta}. \end{aligned}$$

Hence, the eigenvalues are $E_n = \omega(n+1/2)$.

The propagator in the position representation can be obtained by integrating the previous result four times:

$$\begin{aligned} G(x_f,t;x_i) = \left\langle x_f\right| G\left|x_i\right\rangle &= \int \frac{dz_f^2}{\pi}\frac{dz_i^2}{\pi}\left\langle x_f|z_f\right\rangle\left\langle z_f\right| G\left|z_i\right\rangle\left\langle z_i|x_i\right\rangle \\ &= \int \frac{dz_f^2}{\pi}\frac{dz_i^2}{\pi}\psi_{z_f}(x_f)\psi_{z_i}^*(x_i)G(z_f,t;z_i) \end{aligned}$$

Here we will only report the result which will be necessary for studying the magnetic field.

$$G(x_f,t;x_i) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}\exp\left[\frac{im\omega}{2\hbar\sin(\omega t)}((x_f^2+x_i^2)\cos(\omega t) - 2x_i x_f)\right] \tag{39}$$

13 Uniform magnetic field

The classical trajectory of a charged particle in a uniform magnetic field a is cylindrical helix, coaxial to the direction of the field. Similarly, it can be shown, in the Schrodinger picture, that in quantum mechanics the eigenstates of such a particle can be factorized in those of a free Hamiltonian and a bidimensional harmonic oscillator. Here, the corresponding propagator will be found with the phase space formalism. Since the Lagrangian depends explicitly on the vector potential, the gauge must be fixed. If $B = |\boldsymbol{B}|$ is the norm of the magnetic field, then $\boldsymbol{A}$ can be written as follows (Landau's gauge):

$$\boldsymbol{A} = \begin{bmatrix} 0 \\ Bx \\ 0 \end{bmatrix}$$

The phase space Lagrangian is

$$\mathcal{L} = \boldsymbol{p}\dot{\boldsymbol{x}} - \frac{1}{2m}\left[\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right]^2 = \boldsymbol{p}\dot{\boldsymbol{x}} - \frac{1}{2m}\left[p_x^2 + p_z^2 + \left(p_y - \frac{e}{c}Bx\right)^2\right]$$

The sliced action $S_N = \int dt\mathcal{L}$ is, according to (24):

$$S_n = \sum_{j=1}^{N}\left[\boldsymbol{p}_n(\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) - \frac{1}{2m}\left[p_{xn}^2 + p_{zn}^2 + \left(p_{yn} - \frac{e}{c}Bx_n\right)^2\right]\right]$$

The vector potential has been evaluated in $\boldsymbol{x}_n$ since in the Hamiltonian formalism, the magnetic field can be introduced with the following substitution $\boldsymbol{p}_n \mapsto \boldsymbol{p}_n - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_n)$. The propagator is then:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\frac{d^3\boldsymbol{p}_j}{(2\pi\hbar)^3}\prod_{l=1}^{N-1}d^3\boldsymbol{x}_l \exp\left[\frac{i}{\hbar}S_N\right]$$

The variables y_j and z_j only appear in S_N as $\sum_{j=1}^{N}[p_{yj}(y_j - y_{j-1}) + p_{zj}(z_j - z_{j-1})] = \sum_{j=0}^{N-1}[y_j(p_{y,j} - p_{y,j+1}) - z_j(p_{z,j} - p_{z,j+1})]$. Since $\int dk \exp[(i/\hbar)px] = (2\pi\hbar)\delta(x)$, by integrating over all y_j and z_j variables, the following term appears in the integral:

$$\prod_{j=0}^{N-1}(2\pi\hbar)^2\,\delta(p_{y,j} - p_{y,j+1})\delta(p_{z,j} - p_{z,j+1})$$

Hence

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \int\frac{dp_z dp_y}{(2\pi\hbar)^2}\prod_{j=1}^{N-1}dx_j\prod_{l=1}^{N}\frac{dp_{x,l}}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\left(S_{y,z} + S_x\right)\right]$$

with

$$S_{y,z} = p_y(y_f - y_i) + p_z(z_f - z_i) - t\frac{p_z^2}{2m}$$

$$S_x = \sum_{j=1}^{N}\left[p_{xj}(x_j - x_{j-1}) - \frac{1}{2m}\left[p_{xn}^2 + \left(p_y - \frac{e}{c}Bx_j\right)^2\right]\right]$$

S_x is the phase space Lagrangian of a one-dimensional harmonic oscillator translated of $x_0 = (c/Be)p_y$ from the origin and with frequency $\omega = Be/mc$ (Landau frequency). The integration of $\exp[(i/\hbar)S_x]$ is then (39):

$$G_{h.o.}(x_f, t; x_i) = \sqrt{\frac{m\omega}{2\pi\hbar\sin(\omega t)}}\times$$
$$\times\exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin(\omega t)}\left[((x_f - x_0)^2 + (x_i - x_0)^2)\cos(\omega t) - 2(x_f - x_0)(x_i - x_0)\right]\right]$$

The integration over p_z of $\exp[p_z(z_f - z_i) - t\frac{p_z^2}{2m}]$ yields

$$G_z(z_f, t; z_i) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(z_f - z_i)^2}{t}\right]$$

Then, finally:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \int\frac{dp_y}{2\pi\hbar}\exp\left[p_y(y_f - y_i)\right]G_{h.o.}(p_y)G_z$$

p_y is quadratic in the exponent of the integral and, consequently, the previous integral is then a Gaussian one. The terms containing p_y or, equivalently, x_0 can be written and integrated as follows:

$$\int dx_0\exp\left[-\frac{i}{\hbar}m\omega\tan\left(\frac{\omega t}{2}\right)\left(x_0 - \frac{x_f - x_i}{2} - \frac{y_f + y_i}{2\tan(\frac{\omega t}{2})}\right)^2\right] = \sqrt{\frac{\pi\hbar}{im\omega\tan(\frac{\omega t}{2})}}$$

Consequently, the propagator is:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \left[\frac{m}{2\pi i\hbar t}\right]^{3/2}\frac{\omega t}{2\sin\frac{\omega t}{2}}\exp\left[\frac{i}{\hbar}S_c\right]\exp\left[\frac{i}{\hbar}\mathcal{B}\right]$$

Where S_c is

$$S_c = \frac{m}{2}\left[\frac{(z_f - z_i)^2}{t} + \frac{\omega}{2}\cot\left(\frac{\omega t}{2}\right)\left[(x_f - x_i)^2 + (y_f - y_i)^2\right] + \omega(x_i y_f - x_f y_i)\right]$$

which is the classical action. The other term, $\mathcal{B}$, is

$$\mathcal{B} = \frac{m\omega}{2}(x_f y_f - x_i y_i)$$

and it is a boundary term which can be eliminated with a gauge transformation. The previous formula has the following form $G = \mathcal{N} \exp[(i/\hbar) S_{classic}]$. This is not surprising, since the Lagrangian, though not one-dimensional, is quadratic and, as a consequence, all the conclusions about classical paths and the propagator hold.
The way to check that S_c is actually the classical action will be sketched. The gauge does not need to be changed. In addition, since it is already known that the motion along the z axis is a free motion, it is enough to evaluate the action for the motion on the xy plane. The resulting Lagrangian is:

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 + \frac{eB}{c}x\dot{y}$$

The equations of motion are the following:

$$\ddot{y} + \omega x = 0 \quad \ddot{x} - \omega y = 0 \quad \Rightarrow \quad \dddot{x} + \dot{x} = 0 \quad \dddot{y} + \dot{y} = 0$$

The general solution is:

$$x = \frac{1}{\sin(\omega(t_f - t_i))}[(x_f - x_0)\sin(\omega(t - t_i)) - (x_a - x_0)\sin(\omega(t - t_f))] + x_0$$

and the analogue for y ($x \to y; x_i \to y_i; x_f \to y_f; x_0 \to y_0$). x_0 and y_0 are determined by the equations of motion:

$$x_0 = \frac{1}{2}\left[(x_f + x_i) + (y_f - y_i)\cos(\omega(t_f - t_i))\right]$$

$$x_0 = \frac{1}{2}\left[(y_f + y_i) - (x_f - x_i)\cos(\omega(t_f - t_i))\right]$$

This is enough to obtain the classical action S_c by integrating the Lagrangian.

It is known that the spectrum for a particle in a magnetic field is discrete if the free motion along z is not taken into account. In order to obtain this discrete spectrum, equation (33) can be used. The poles of the transformed propagator do not depend on the choice of the starting and the end points. Thus, we set $(x_i, y_i) = \mathbf{0}$ and $\boldsymbol{r} = (x_f, y_f)$.

$$G(\boldsymbol{r}, t; \mathbf{0}) = \frac{m\omega}{4\pi i\hbar \sin\frac{\omega t}{2}} \exp\left[\frac{im\omega}{4\hbar} r^2 \cot(\frac{\omega t}{2})\right]$$

By defining $x := m\omega r^2/\hbar$ and $z := \exp(i\omega t)$ the propagator can be rewritten as (we use $\sin(\alpha) = (e^{i\alpha} - e^{-i\alpha})/(2i)$):

$$G(\boldsymbol{r}, t; \mathbf{0}) = \frac{m\omega}{4\pi\hbar}\frac{1}{1-z} \exp\left[\frac{xz}{z-1}\right] \exp\left[-\frac{i\omega t}{2}\right] \exp\left[-\frac{m\omega r^2}{4\hbar}\right]$$

The previous expression includes the generating function of the Laguerre polynomials $L_n(x)$. Properties of these polynomials include:

$$\sum_{n=0}^{\infty} L_n(x) z^n = \frac{1}{1-z} \exp\left[\frac{xz}{z-1}\right]$$

$$L_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} \frac{(-1)^k}{k!} x^k \qquad\qquad L_n(0) = 1, \forall n$$

Hence:

$$G(\boldsymbol{r}, t; \mathbf{0}) = \frac{m\omega}{4\pi\hbar} \exp\left[-\frac{m\omega r^2}{4\hbar}\right] \sum_{n=0}^{\infty} L_n\left(\frac{m\omega r^2}{\hbar}\right) \exp\left[-i(n+1/2)\omega t\right]$$

The last step is to take the Laplace transform:

$$G(\boldsymbol{r}, E; \mathbf{0}) = \lim_{\eta \to 0} \frac{m\omega}{4\pi\hbar} \exp\left[-\frac{m\omega r^2}{4\hbar}\right] \sum_{n=0}^{\infty} L_n\left(\frac{m\omega r^2}{\hbar}\right) \frac{i}{(E+i\eta)/\hbar - (n+1/2)\omega}$$

Hence, the eigenvalues, also known as Landau levels are $E_n = \hbar\omega(n+1/2)$.

Part III
The Aharonov Bohm effect - a topological approach

14 Motivation

The Aharonov-Bohm effect (AB effect from now on) is a well known quantum phenomenon.

Let us consider a charged particle at some point $\boldsymbol{x}_i$ and its amplitude of being, under the quantum dynamics, at some other point $\boldsymbol{x}_f$, where it can be detected, after a certain amount of time. No force or field is applied to the particle, so that it can move freely. A solenoid producing a constant magnetic field is placed somewhere in the experimental set. Two hypotheses are needed: 1) the solenoid is infinitely long and 2) its surface is impenetrable for the particle. Hence, the magnetic field is confined in an inaccessible and finite region of space. From the classical point of view, no effect on the particle trajectory should be observed, since the field has only a local effect (Lorentz force) and it vanishes in the outer region. The quantum case is different. In the Schrodinger equation, the quantity accounting for the electromagnetic interaction is the vector potential and not the Lorentz force; therefore, a description in terms of vector potential is needed.

There are different kinds of continuous paths connecting $\boldsymbol{x}_i$ and $\boldsymbol{x}_f$ and they can be classified in terms of homotopy classes.

Paths that can be deformed into each other with continuity without letting them through the solenoid (and without "cutting" them) belong to the same homotopy class (see appendix). The need for this classification will be seen as follows. Let us consider, for example, three different paths A, B and C as shown in figure (1). These paths cannot be deformed one into the other. Let us first choose the paths A and B. Their amplitudes appearing in the path integral are $\exp((i/\hbar)(S_A + (e/c)\int_A \boldsymbol{A}))$ and $\exp((i/\hbar)(S_B + (e/c)\int_B \boldsymbol{A}))$ respectively, with $S_B$and S_A being free actions. The difference in phase is proportional to $S_B - S_A + \oint \boldsymbol{A}$. While $S_B - S_A$ clearly depends on the particular paths A and B, the quantity

$$\int_B \boldsymbol{A}d\boldsymbol{x} - \int_A \boldsymbol{A}d\boldsymbol{x} = \oint_{A,B} \boldsymbol{A}d\boldsymbol{x} = \int_{Surface(A,B)} \boldsymbol{B}d\boldsymbol{s} = \Phi$$

is only determined by the homotopy classes of A and B. This is due to the Stokes theorem. The only relevant quantity is now manifestly the flux of the magnetic field. The path C winds clockwise around the solenoid. Thus, if we consider B and C, we will find that the difference in phase is proportional to 2Φ; As for A and C, it is proportional to Φ again. Therefore, there is a difference in phase between trajectories from different classes and

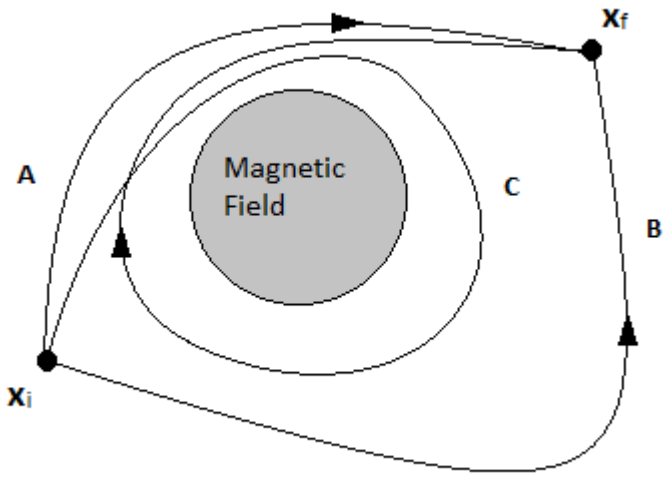

Figure 1: Path A, B and C cannot be deformed one into the other

this depends on the difference of numbers of windings around the solenoid. This difference is proportional to integer multiples of Φ. The foregoing observations imply that the total propagator should be like:

$$G \approx \sum_{windings(n)} e^{\frac{ie}{c\hbar} n\Phi} \sum e^{\frac{i}{\hbar} S} \tag{40}$$

The amplitudes of paths sharing the same number (and direction) of windings have been summed up, yielding the propagator for a single homotopy class. Afterwards the class propagators have been summed up with an appropriate phase factor determined by the rule presented above. Obviously, since only relative phases are relevant, the phase of an arbitrary class of trajectories has been set to zero and the other "winding numbers" have been fixed accordingly.

The expression (40) differs from the free propagator, meaning that the magnetic field, though confined, has a measureable effect on the particle dynamics. This can be verified experimentally with a double slit experiment as shown in picture (2). The presence of the solenoid will shift the interference figure on the screen.

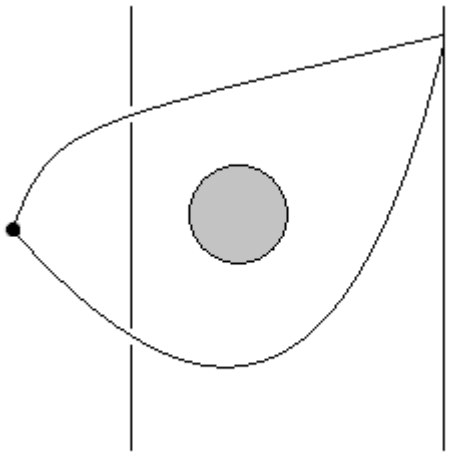

Figure 2: Double slit experiment and the AB apparatus

The foregoing discussion leads to concepts like homotopy classes and "winding numbers", suggesting a topological approach to the problem. Consequently, the following sections will deal with path integrals in multiply-connected spaces. The discussion will be more detailed than technically needed for the AB effect since the topic is interesting by itself and has a range of applications including polymer physics, path integral for spinors and systems of identical particles.

15 Path Integral on multiply connected spaces

The bidimensional physical space in which the AB effect takes place is clearly simply connected ($\mathbb{R}^2$). Instead, the insertion of an impenetrable solenoid is a mathematical idealization and introduces a "hole" in the plane. This means that the physical space can be approximated by a multiply-connected space [27]. The new topology can be used to provide a good description of the system but, wherever the formalism will lead, no ambiguity due to the multiple-connectiveness should appear in the final result. The forthcoming ideas were proposed and, in large part, developed by Schulman [27] [28].

Let us assume, from now on, that the configuration space is arcwise connected. In addition, since the space can be regarded as a differentiable manifold, it is locally isomorphic to $\mathbb{R}^2$ and is required to have every rea-

sonable local property of regularity. Thus, the space should be considered locally simply connected, locally arcwise connected and semi-locally simply connected. These local concepts will not be developed since almost no explicit use will be made of them. The required topology is summarized in the appendix.

The first observation is that the Hamiltonian alone does not provide a complete dynamics as long as the configuration space is not simply connected. This can be seen with the following simple example.

Let us consider quantum mechanics on a segment $I = [a, b]$ in which the points a and b are identified. The free Hamiltonian is proportional to $-\Delta$. Given two functions $f(x), g(x) \in L^2(I)$,

$$\begin{aligned} &\int_I dx f^*(x)(-\Delta g)(x) = \\ &= \int_I dx(-\Delta f^*)(x)g(x) - (g'(b)f(b) - g'(a)f(a)) + (f'(b)g(b) - f'(a)g(a)). \end{aligned} \tag{41}$$

Hence, unless proper boundary conditions are imposed, the free Hamiltonian is not even symmetric. At a more subtle level, even though f ang g vanish at the boundary, $-\Delta$ is essentially self-adjoint only under appropriate boundary conditions. Therefore, it is not surprising that path integrals in a multiply-connected spaces suffer from an additional ambiguity.

Paths connecting the couple of points x_i and $x_f \in \mathcal{M}$ can be divided in homotopy classes, say $\{\alpha\}$. Let us first define the "partial amplitude" $G_\alpha(x_f, t, x_i)$ for a single class α as the sum over all paths in α. If now the partial amplitudes are summed up, the old sum over all trajectories is recovered. The core idea is that, in principle, the partial amplitudes can enter the sum with class-dependent factors c_α.

$$\boxed{G(x_f, t; x_i) = \sum_\alpha c_\alpha G_\alpha(x_f, t; x_i)} \tag{42}$$

All the partial amplitudes (class propagators) satisfy Schrodinger equation individually. Let us consider the universal covering space $\mathcal{M}^*$ (it must be assumed that there exists one) and a covering projection p. All paths can be lifted via p^{-1} to M^* in a non-ambiguous way according to theorem (3)(see appendix) once a starting point $x^*_{i,\alpha} \in p^{-1}(x_i)$ is fixed. Let us choose the same starting point x^*_i for all paths of all classes. On the other hand, if also the end-points $x^*_{f\alpha} \in p^{-1}(x_f)$ were the same for paths in different classes, these paths could be deformed one into the other ($\mathcal{M}^*$ is simply connected) and p would map them down in M into the same class (see appendix). This is, of course, impossible. In other words, different homotopy classes in $\mathcal{M}$ are characterized by different end-points $x^*_{f\alpha}$ in $\mathcal{M}^*$.

Now, let us suppose that everything is well behaved so that the Hamiltonian, the Schrodinger equation and the Lagrangian can be lifted with p^{-1} to $\mathcal{M}^*$. Since $\mathcal{M}^*$ is simply connected, there is no possible ambiguity when calculating the propagator G_α from x_i^* to x_f^* for a given class. Yet, when going back to $\mathcal{M}$, there is no dynamical prescription for not summing the partial amplitudes with nontrivial coefficients. Then:

$$G^{\mathcal{M}}(x_f,t;x_i)=\sum_\alpha c_\alpha G_\alpha^{\mathcal{M}}(x_f,t;x_i)=\sum_\alpha c_\alpha G^{\mathcal{M}^*}(x_{f\alpha}^*,t;x_i^*)$$

The partial propagators are linear independent functions. This will roughly justified. Suppose that

$$\sum_\gamma K_\gamma G_\gamma(x_f,t;x_i)=0.$$

Thanks to the regularity hypothesis (semi-local simply-connectiveness), there always exists an open neighbourhood V of x_i which is simply connected, and all paths entirely contained in it must belong to the same homotopy class, say α. α partial propagator becomes δ-like as $t\to 0$. Other partial amplitudes are sums over paths which must, roughly speaking, reach out distant holes outside V: consequently, they give no contribution if $t\to 0$. This yields $K_\alpha=0$. Now let us choose a new class β. x_f can be moved backwards along a representative b of β ($\beta=[b]$) until $b\subset V$ and $x_f\in V$. The previous argument implies $K_\beta=0\ \forall\beta\in C(x_i,x_f)$.

The numbers c_α must satisfy certain conditions. In particular, c_α must be a commutative representation of the fundamental group π_1. There are at least three different ways to prove this fact according to DeWitt and Laidlaw [22], Dowker [10] and Schulman [28]. Schulman's is the most intuitive and goes as follows.

Let us attach a closed path h ($\xi:=[h]\in\pi_1(x_f)$) at x_f and let us move x_f along it. While x_f moves along ξ away and then back to the same position in M, all the points $x_{f,\alpha}^*$ in M^* move to new ξ-dependent locations $x_{f,\alpha[h]}^*$. Rigorously, this means that each path a ($[a]=\alpha\in C(x_i,x_f)$) undergoes the transformation $\alpha\to\alpha[h]$, which is clearly a 1:1 map of $C(x_i,x_f)$ onto itself. While partial amplitudes change, no physical change has been made on M and the total propagator must equal the old one up to a $[h]$-dependent phase factor.

$$\sum_\alpha c_\alpha G_\alpha(x_f,t;x_i)=e^{i\phi_\xi}\sum_\alpha c_\alpha G_{\alpha\xi}(x_f,t;x_i)=\sum_{\alpha\xi^{-1}}(e^{i\phi_\xi})(c_{\alpha\xi^{-1}})G_\alpha(x_f,t;x_i)$$

Linear independence implies $\exp(i\phi_\xi)c_{\alpha[\xi^{-1}]}=c_\alpha$. Now one can choose a new path z ($\zeta=[z]\in\pi_1(x_f)$). If the previous procedure is repeated for $[h]$ and $[z]$ in succession and then for $[h\,z]$ and it is imposed that the results are equal, finally:

$$\phi_{\xi\zeta}=\phi_\xi+\phi_\zeta.$$

This is the expected commutative representation. It can be shown that $|c_\alpha| = 1\,\forall\alpha$. Let a particular c_α be 1. Then, a small rearrangement yields:

$$G(x_f, t; x_i) = \sum_{\gamma\in\pi_1} e^{i\phi_{\alpha^{-1}\gamma}} G_\gamma(x_f, t; x_i)$$

Dowker's approach, while less intuitive, may be more general. Let us assume that $\mathcal{M}$ can be written as $\mathcal{M} = \mathcal{M}^*/\mathcal{G}$ with $\mathcal{G}$ being a properly discontinuous group of homeomorphisms (or, even better, a group of isometries). In other words, we have identified the points in $\mathcal{M}^*$ which belong to the same orbit of some $g \in \mathcal{G}$, meaning that $\mathcal{M} = \mathcal{M}^*/\sim$ with $x_1^* \sim x_2^*$ if $x_2^* = g(x_1^*)$ for some $g \in \mathcal{G}$. This identification determines a covering projection $p : \mathcal{M}^* \to \mathcal{M}^*/\mathcal{G} = \mathcal{M}$; $x^* \mapsto [x^*] =: x$. Hence, the points in a single orbit are nothing but the discrete set $p^{-1}(x),\ x \in \mathcal{M}$. Let us fix a representative x_0^* in $\mathcal{M}^*$ for each orbit. The set corresponding to a particular choice of representatives can be called $\bar{F}$ and it is, of course, not unique.

A multi-valued wavefunction $\psi(x)$ can be defined on $\mathcal{M}$, by associating with every $x \in \mathcal{M}$ a different value for each preimage of x in $\mathcal{M}^*$. These values can be regarded as images of a single-valued wavefunction on $\mathcal{M}^* = \cup_{x\in\mathcal{M}}\, p^{-1}(x)$, say $\psi_{\mathcal{M}^*}$:

$$\begin{aligned}\psi(x) \mapsto \{\psi_{\mathcal{M}^*}(g(x_0^*)); \forall g \in \mathcal{G} \text{ for some given representative } x_0^* \in p^{-1}(x)\}\\ = \{\psi_{\mathcal{M}^*}(x_0^*)\ \forall x_0^* \in p^{-1}(x)\} \qquad (43)\end{aligned}$$

If now g is fixed in the first definition of $\psi(x)$ (or, equivalently, a specific preimage of x is chosen in the second), then ψ is single-valued at a certain $x \in \mathcal{M}$. This procedure should be repeated for all $x \in \mathcal{M}$ in order to define a domain $\bar{F}$ in $\mathcal{M}^*$. Therefore, the single-valued wavefunction on $\mathcal{M}$ is identified as follows: $\psi \leftrightarrow \psi_{\mathcal{M}^*}|_{\bar{F}}$. However, once a point in $\bar{F}$ is fixed, all the other points are. This is a direct consequence of the requirement that both ψ_{M^*} and ψ are continuous, which suggests that $\bar{F}$ and $\mathcal{M}$ should be locally homeomorphic ($x_1 \to x_2 \Rightarrow x_1^* \to x_2^*$), apart from some unimportant issues at the boundarty of $\bar{F}$. There are different possible choices for $\bar{F}$ which are all "copies" of $\mathcal{M}$ in $\mathcal{M}^*$ and can be mapped one into the other by elements in $\mathcal{G}$. In particular, by fixing $\bar{F} =: \bar{F}_e$, all other domains are $\bar{F}_g = g(\bar{F}_e)$, meaning that they are in 1:1 correspondence with elements of $\mathcal{G}$.

Physics imposes that the domains $\bar{F}_g$ are equivalent.

$$\psi_{M^*}(g(x^*)) = e^{i\phi_g}\psi_{\mathcal{M}^*}(x^*) \Rightarrow e^{i\phi_{g'g}} = e^{i(\phi_g+\phi_{g'})}$$

Thus, $\{e^{i\phi_g}\}$ is an abelian representation of $\mathcal{G}$. The path integral is readily

obtained:

$$\begin{aligned}\psi_{\mathcal{M}^*}(x_f^*,t) &= \int_{\mathcal{M}^*} G(x_f^*,t;x_i^*)\psi_{\mathcal{M}^*}(x_i^*)dx_i^* \\ &= \sum_{g\in\mathcal{G}} \int_{\bar{F}_g\cong\mathcal{M}} G(x_f^*,t;g(x_i^*))\psi_{\mathcal{M}^*}(g(x_i^*))dx_i^*.\end{aligned}$$

A small rearrangement of the previous result is needed. One ought to extract the $\exp(i\phi_g)$ from the wavefunction and the propagator and remember that the conjugate of $\exp(i\phi_g)$ is $\exp(i\phi_{g^{-1}})$).

$$\begin{aligned}\psi_{\mathcal{M}^*}(x_f^*,t) &= \sum_{g\in\mathcal{G}} e^{i\phi_g} \int_{\mathcal{M}^*} G(g^{-1}(x_f^*),t;x_i^*)\psi_{\mathcal{M}^*}(x_i^*)dx_i^* \\ &= \sum_{g'\in\mathcal{G}} e^{-i\phi_{g'}} \int_{M^*} G(g'(x_f^*),t;x_i^*)\psi_{\mathcal{M}^*}(x_i^*)dx_i^*\end{aligned}$$

The isomorphism between $\mathcal{G}$ and π_1 is the final step to recover Schulman's form. Let us remark that the previous discussion implies that the homotopy classes are in a 1:1 correspondence with elements of $\mathcal{G}$ or π_1.

A trivial but important conclusion is that $c_\alpha = 1\,\forall\alpha \in C(x_i, x_f)$ is always a legitimate choice since $\{1\}$ is a commutative representation of any group.

16 Aharonov-Bohm effect on the circle

A possible model for an ideal AB apparatus is the following. The solenoid can be regarded as an infinitesimally thin cylinder which confines a finite flux. The solenoid is set in the origin of the xy plane. The appropriate magnetic field is thus $\boldsymbol{B} = \Phi\delta^2(\boldsymbol{x})\hat{z}$ with Φ being the flux. The corresponding vector potential can be derived from an analogy between the present case and an infinite wire with a current density $\boldsymbol{J} = I\delta^2(\boldsymbol{x})\hat{z}$ generating a magnetic field (Biot-Savart). The correspondence is $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$, $\boldsymbol{B} \leftrightarrow \boldsymbol{J}$ and $\Phi \leftrightarrow I$:

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\Phi}{2\pi}\frac{-y\hat{i} + x\hat{j}}{x^2+y^2} = \frac{\Phi}{2\pi}\frac{1}{r}\hat{e}_\theta \tag{44}$$

The vector potential appears in the Lagrangian as $(e/c)\boldsymbol{A}\dot{\boldsymbol{r}}$. In polar coordinates it becomes $(e\Phi/c)\dot{\theta}$ so that the Lagrangian can be written:

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{e}{c}\frac{\Phi}{2\pi}\dot{\theta} \tag{45}$$

From a classical standpoint, the last term, which accounts for the presence of the solenoid, is not physical since the Lagrangian is defined up to the total derivative of a function. The reason is that it only produces a boundary term.

Now let us consider a quantum system with the same Lagrangian (more correctly, the corresponding Hamiltonian should be considered) in a simply connected space (one should cut off some part of $\mathcal{M}$ or re-interpret the angular coordinate to get rid of multiply connectiveness). The addition of the magnetic term to the free Lagrangian as in (45) corresponds to the unitary transformation $\psi(r,\theta) \mapsto e^{(e\Phi/c)\theta}\psi(r,\theta)$, which cannot influence observable quantities and, consequently the magnetic term itself can be transformed away. In other words, the vector potential produces a gauge-dependent boundary term.

If the space is not simply connected, we cannot get rid of the magnetic term. Indeed $\int \dot{\theta} dt = \theta_f - \theta_i + 2\pi n$ is manifestly dependent on the number of windings (n) of a specific path and is not an actual boundary term.

Let us now consider the simple case of a particle confined on a circumference centred in the origin. The space under exam is $SO(2)$ (or S^1) and its covering space is $\mathbb{R}$. The projection map p is defined as

$$p : \theta \mapsto \theta - 2\pi \left[\frac{\theta}{2\pi}\right].$$

The square bracket is the integer part function. The fundamental group $\pi_1 \cong (\mathbb{Z}, +)$ is the infinite cyclic group generated by one element ($+1$ or -1). One can consider a single loop either clockwise or counterclocwise. By attaching together many loops, one gets closed paths with every possible number or windings in both directions. The elements in $\mathcal{G}$ are clearly the translations of integer multiples of 2π, meaning that $\mathcal{G} = \{T_n \text{ such that } T_n : \theta) \mapsto \theta + 2n\pi\} \cong \pi_1 \cong (\mathbb{Z}, +)$.

The preimages of θ_f are $p^{-1}(\theta_f) = \{\theta_f(n) = \theta_f + 2\pi n \text{ with } n \in \mathbb{Z}\}$ and can be obtained one from the others by applying T_n. The points $\theta_f(n)$ are the end-points for lifted paths of different homotopy classes of $SO(2)$. Then, these classes are indeed identified by these "winding numbers" n.

According to what was said in the previous section, the whole problem will be lifted in $\mathbb{R}$. For the homotopy class represented by n, the propagator reads ($I = R^2 m$):

$$\begin{aligned} G_n(\theta_f, t; \theta_i) &= G^{\mathbb{R}}(\theta_f + 2\pi n, t; \theta_i) \\ &= \int \mathcal{D}\theta(t) \exp\left[\frac{i}{\hbar}\int dt \left(\frac{I}{2}\dot{\theta}^2 + \frac{e\Phi}{2\pi c}\dot{\theta}\right)\right] \\ &= G^{\mathbb{R}}_{free}(\theta_f + 2\pi n, t; \theta_i) \exp\left[\frac{i}{\hbar}\frac{e\Phi}{2\pi c}(\theta_f - \theta_i + 2\pi n)\right] \\ &= \sqrt{\frac{I}{2i\pi\hbar t}} \exp\left[\frac{i}{\hbar}\left(\frac{(\theta_f - \theta_i + 2\pi n)^2}{t} + \frac{e\Phi}{2\pi c}(\theta_f - \theta_i + 2\pi n)\right)\right] \end{aligned}$$

Since all information has been used (and the propagator must be uniquely defined) all coefficients c_α can be set to 1. Finally, it should be observed

that each addend brings the same multiplicative phase $\exp((e\Phi/c)(\theta_f - \theta_i))$ which is a genuine boundary term that can be transformed away with a unitary map. The propagator is then:

$$G(\theta_f, t; \theta_i) = \sqrt{\frac{m}{2i\pi\hbar t}} \sum_{n\in\mathbb{Z}} \exp\left[\frac{i}{\hbar}\frac{e\Phi}{c}n\right] \exp\left[\frac{i}{\hbar}\frac{(\theta_f - \theta_i + 2\pi n)^2}{t}\right] \tag{46}$$

The previous derivation is based on the choice of a mathematical model describing the vector potential (and the magnetic field) which yields the correct dynamical description and the topological freedom on coefficients c_α has been disregarded. However, if one looks back at (46), the propagator appears as the sum of a free partial propagator multiplied by a phase. Thus, it is possible to consider the phases $\exp\left[\frac{i}{\hbar}\frac{e\Phi}{c}2\pi n\right]$ as the topological coefficients *a posteriori*. In this more topological interpretation, a coefficient (which in our case has the form $e^{i\delta n}$) represents the phase acquired by a free particle after winding n times around the singularity.

17 Aharonov-Bohm effect on the plane

If the radial dimension is restored, the configuration space $\mathcal{M}$ becomes $\mathbb{R}^2 - \{\mathbf{0}\}$. Its universal covering space $\mathcal{M}^*$ is $\mathbb{R} \times (0, \infty]$ (or it could be $\mathbb{R}^2$ itself). The introduction of the radial coordinate does not affect the relevant topology, since the multiple-connectiveness is still due to the angular coordinate in the very same way as before. The fundamental group is the same as for $SO(2)$, and so the propagator should be formally alike. Clearly, it is best to use polar coordinates. The covering projection is

$$p : (r, \theta) \mapsto \left(r, \theta - 2\pi\left[\frac{\theta}{2\pi}\right]\right)$$

and is particularly appropriate since it is a local isometry. The direct calculation of the propagator is rather lengthy, and then we will just give the formula instead and justify it later. Three references for the direct demonstration are [18] [5] [16].

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \left[\frac{m}{2\pi i\hbar t}\right] \exp\left[\frac{im}{2\hbar t}(r_f^2 + r_i^2)\right] \sum_{n=-\infty}^{\infty} \exp\left[i\frac{e\Phi}{2\pi\hbar c}2\pi n\right] \times$$
$$\times \int_{-\infty}^{\infty} d\lambda \, \exp\left[i\lambda(\theta_f - \theta_i + 2\pi n)\right] I_{|\lambda|}\left(-\frac{imr_f r_i}{\hbar t}\right) \tag{47}$$

Functions $I_\lambda(y)$ are the modified Bessel functions which are often associated with cylindrical coordinates. The first observation is that if $e\Phi/2\pi\hbar c$ is an integer, then any reference to the magnetic field disappears as if $\Phi = 0$.

Therefore, the propagator should be the free propagator in polar coordinates. The fact that winding numbers do not immediately disappear is only due to the usual coordinate singularity in polar parameterization which should not affect physics. It is possible to eliminate the winding numbers as follows. By expanding Dirac's comb function $D(\alpha) = \sum_{n=-\infty}^{\infty} \delta(n+\alpha)$ on the Fourier basis (period 1), one gets:

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n\alpha} = \sum_{n=-\infty}^{\infty} \delta(n+\alpha)$$

We apply this to (47) and we make use of the even parity of the modified Bessel functions with integer subscript $I(z)_n = I_{-n}(z)$:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \left[\frac{m}{2\pi i\hbar t}\right] \exp\left[\frac{im}{2\hbar t}(r_f^2 + r_i^2)\right] \sum_{n=-\infty}^{\infty} e^{in(\theta_f - \theta_i)} I_n\left(-\frac{imr_f r_i}{\hbar t}\right)$$

Let us call $J_n(z)$ the ordinary Bessel functions. The identities

$$\sum_{n=-\infty}^{\infty} e^{in\theta} J_n(z) = e^{iz\cos(\theta)} \qquad J_n(iz) = i^n I_n(z)$$

allow to rewrite the propagator as

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \left[\frac{m}{2\pi i\hbar t}\right] \exp\left[\frac{im}{2\hbar t}(r_f^2 + r_i^2 - 2r_i r_f \cos(\theta_i - \theta_f))\right] \tag{48}$$

which is precisely the free propagator and keeps no track of the windings. The ease with which winding numbers have been dismissed as a coordinate-dependent complication, makes us wonder how the multiply connectiveness appears in (47) instead. Had the mathematical model introduced a solenoid with a finite radius, it would have been obvious, while in (47) such information is carried by coefficients $\exp\left[i\frac{e\Phi}{2\pi\hbar c}2\pi n\right]$ and not by the n dependence of the "free" partial amplitudes (the integral part). If the coefficients are interpreted as a consequence of the introduction of the potential (44) and the c_α are set to 1, then those points in which the vector potential is not defined (the origin) must be excluded from the physical configuration space. On the other hand, if one interprets $c_\alpha = \exp\left[i\frac{e\Phi}{2\pi\hbar c}2\pi n\right]$, then the sum over paths is irreparably split into homotopy classes.

Keeping the foregoing observation in mind, since to obtain (48) we have only made use of a chain of identities, all the passages can be read backwards and one can redundantly introduce winding numbers in the polar propagator of the free particle. If we trust the topological formalism enough, each term of the sum can be interpreted as a partial amplitude (as seen before, the particle can be thought of as moving freely in $\mathcal{M}$). Then, in order to introduce multiple-connectiveness, it is enough to add the class-dependent coefficients

which must have the form $\exp(i\delta n)$. If now we impose the appropriate phase change under a 2π rotation, (47) is justified.

Finally, the interference pattern is given by

$$|G|^2 = \frac{1}{2}\sum_{j,l} G_j^* G_l + G_j G_l^*$$

In order to evaluate the pattern, it is convenient to assume that $r_i r_f m \gg \hbar t$, so that the asymptotical form of the Bessel function can be used:

$$I_{|\lambda|}(z) \approx \frac{1}{\sqrt{2\pi z}} \exp\left(z - \frac{\lambda^2 - 1/4}{2z}\right)$$

For convergence's sake, a small imaginary part can be implicitly added to the mass $m \mapsto m + i\mu$, and then one can take the limit $\mu \to 0$. After a substitution and the evaluation of a Gaussian integral, one gets:

$$G(\boldsymbol{x}_f, t; \boldsymbol{x}_i) = \sum_{n=-\infty}^{\infty} \left[\frac{m}{2\pi i\hbar t}\right] \exp\left[\frac{im}{2\hbar t}(r_f - r_i)^2\right] \times \\ \times \exp\left[i\frac{e\Phi}{2\pi\hbar c}2\pi n + \frac{imr_f r_i}{2\hbar t}(\theta_f - \theta_i + 2\pi n)^2\right] \quad (49)$$

The most important partial amplitudes are those which involve the straightest paths. The corresponding winding numbers are $n = 0, -1$ and they are referred to those the paths which do not wind at all. Thus, the probability becomes

$$P(\boldsymbol{x}_f, t; \boldsymbol{x_i}) = G_0^* G_{-1} + G_{-1}^* G_0 \\ = 2\left(\frac{m}{2\pi\hbar t}\right)^2 \cos^2\left(\pi\left(\frac{e\Phi}{2\pi\hbar c} + \frac{r_f r_i(\theta_f - \theta_i - \pi)}{\hbar t}\right)\right) \quad (50)$$

Then this is the approximate observable quantity for an idealized experimental set.

Finally, let us remark that (47) and (50) show a clear $2\pi\hbar c/e$ flux periodicity. Consequently, all the quantities of the system that depend on Φ must have the same periodicity. Thus, it is convenient to define the fundamental frequency

$$\boxed{\Phi_0 = \frac{2\pi\hbar c}{e}} \quad (51)$$

which is very important in experimental observations.

18 Experimental evidence

It was 1960, only a year after Aharonov and Bohm had published their article, when Chambers [8] came up with the results of his experiment about the AB effect. The experimental apparatus closely resembles the ideal one and it will be briefly described, without indulging in technical details.

A source of electrons S produces two coherent beams which are redirected towards an observing plane O by an "electrostatic biprism". The electrostatic biprism (efe) (which is actually unnecessary) is composed of three elements: f is a positively charged object and the two objects e are two earthed metal plates. The electrostatic field which is produced clearly bends the beams in the correct way.

On the plane O the beams meet and an interference pattern appears (4:a). It should be remarked that this picture is in interesting itself as it shows quantum interference at work. If an extended magnetic field is applied to the region a', the interference pattern is displaced on the screen but not deformed (4:b). This is not surprising since in a semiclassical picture the trajectories are curved by the Lorentz force.

Let us now introduce a localised magnetic field in the region a, instead. This is achieved by inserting a vertical whisker, i.e. a very thin piece of magnetized metal (diameter: $1\mu m$, length: $0.5mm$), which plays the role of the solenoid. In addition, the whisker is placed "in the shadow" of f, so that the electrons should not meet it on their way.

Let us apply (51) to the present case: it means that consecutive maxima are expected to occur at a "distance" Φ_0. The presence of the electric field does not indeed change the topology of the configuration space, while it will change the dynamics in some way which is not relevant.

The size of the whisker is not constant and it tapers along the z direction. Thus $d\Phi/dz$ is not neglectable. If we imagine electrons moving on different planes, we must conclude that the enclosed magnetic flux varies with height and that the single interference fringe should change accordingly without a displacement of the pattern as a whole. The interference fringes are consequently expected to tilt as it is evident in (4:c).

The main issue - as remarked by Chambers himself and by his colleague Pryce - is that since the flux varies in the whisker, a compensating radial field $H = (d\Phi/dz)/2\pi r$ must be produced in the outer region. In a semiclassical picture then the two beams move almost horizontally in a radial magnetic field. As a result, the Lorentz force should bend one beam downwards and the other upwards, thus creating a tilted interference pattern at a rate which is consistent with the observation. Nevertheless, Chambers pointed out that if the classical forces explanation was correct, there should be no fringe shift at all where the size of the whisker is constant.

The attempt to dismiss the AB effect as the result of some hidden force is still been carried out and will be quoted later.

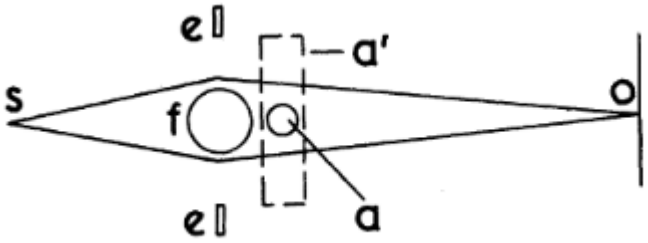

Figure 3: Chambers' experimental apparatus. From [8]

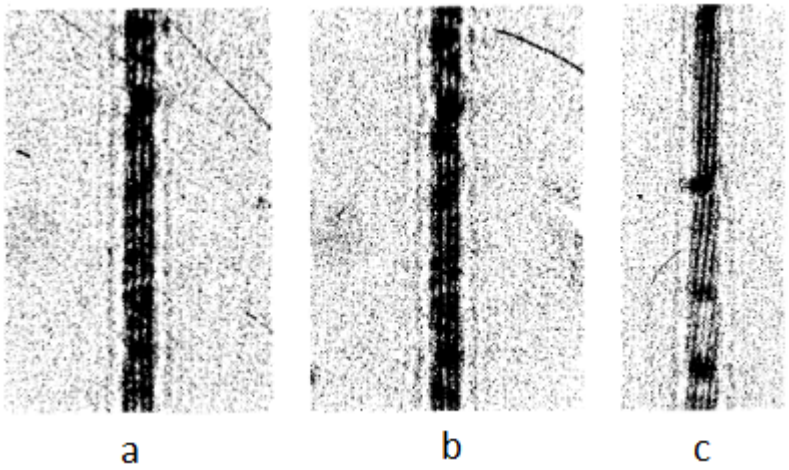

Figure 4: a) only biprism b) stray field c) magnetized whisker. From [8]

Many experiments have been carried out since Chambers'. Let us consider (again, only at a qualitative level) a particular class of them which is meant to study the resistance oscillations in mesoscopic systems and their relationship with the AB effect. The experimental apparatus that will be referred to is the one used by Webb *et al.* [30]. The goal is to measure the resistance of a small golden ring (diameter: 784 nm; size 41 nm) with two leads to be connected to an electrical circuit. A magnetic field perpendicular to the section of the ring is enclosed by the ring. If the electrons are confined inside the ring, the configuration space can be thought of as a torus or, more optimistically, as a circle. In both cases, the fundamental group π_1 is $(\mathbb{Z}, +)$, so that the topological description of this system (the class coefficients) must be similar to those that have been studied in the foregoing sections. In principle, all properties should thus be Φ_0 periodic. Yet, since in the present case electrons are not free and move in metal, further analysis is necessary. Resistance (or, equivalently, conductance) is related to the probability that an electron starting from one lead reaches the other. In vacuum, this probability oscillates as a result of the phase interference according to the AB description. This fact is still true in a medium only if the phase topological path dependence is not irreparably corrupted by random processes. It seems unavoidable: unless the size of the ring is unrealistically smaller than the mean free path, electrons will undergo several scattering processes before reaching the target. The interesting fact is that elastic scatterings can be shown to shift the phase in a deterministic way. Therefore, even in the presence of impurities, the "phase memory" is not lost until an inelastic collision occurs. The mean distance before this happens is the "coherence distance" L_ϕ and decreases as the temperature rises. Hence, in order to observe the AB effect, the temperature needs to be adequately low and that the size of system is of the same order of L_ϕ accordingly.

Picture (5) shows the result of the experiment of Webb *et al.* [30]. The resistance oscillations peak at Φ_0 as it is shown in the Fourier-transformed graph. Not surprisingly, a minor peak appears at the frequency $\Phi_0/2$ which is the first harmonic and this second peak is also important. The precise dependence of the Φ_0-periodic contributions to resistance is determined by the specific features of the paths (e.g. the impurities). Thus, if several possibilities are available (the sample is cylindrical, i.e. it is an array of rings, or the high temperature allows a nontrivial energy distribution), the overall contribution is an average of random terms which keeps no track of the Φ_0 oscillations. The $\Phi_0/2$ oscillations may survive instead because of the "weak localization" phenomenon. Basically, the probability that an electron remains where it is, is determined by the amplitude of closed paths. If there are no magnetic fields [2], such paths are reversible and they appear

[2] A condition for time reversibility is that the effective potential depends only on the

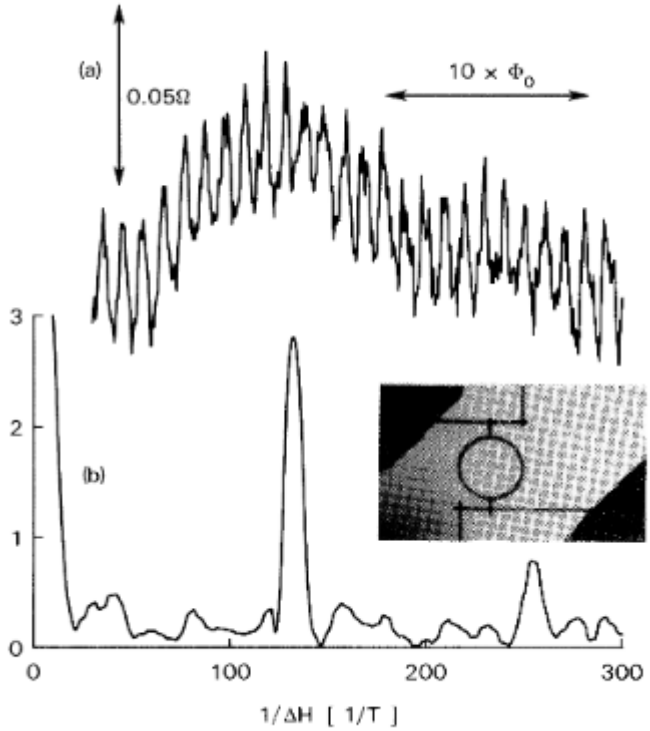

Figure 5: Magnetoresistance at 0,01 K. From [30]

in pairs (one the inverse of the other with the same phase) they produce constructive interference when $\Phi = 0$ and give a positive contribution to resistance. If the magnetic field is turned on, resistance decreases as the pairs acquire a phase factor. Let us consider those paths that wind once (if they don't wind they are not influenced by Φ and, if they wind many times, they are too long): since they enclose the flux twice as a pair, the fundamental oscillations acquire a $\Phi_0/2$ frequency. Since now the resistance dependence on Φ is completely determined (e.g. it is maximal at Φ=0 and multiple integer of $\Phi_0/2$) there is no room for statistics. Two necessary conditions for the weak localizations to occur are that the magnetic field does not strongly interact with the ring and that L_ϕ is long enough.

It must be remarked that these resistance experiments, while interesting on a phenomenological standpoint, give in general little contribution to the understanding of the true nature of the AB effect, since the magnetic field is not prevented from reaching the ring. Thus, the puzzle of the no-field interaction is not so evident.

It is worth mentioning another two more of experiments whose purpose was to further investigate whether any force is actually involved in the AB effect. The first was similar in purpose to Chambers'. Instead of attempting to create a long impenetrable solenoid, a toroidal magnet clad with a superconducting shield was placed between the source of electrons and the target surface. The toroidal shape has the advantage of limiting the field leakage. The experiment (Tonomura *et al*) successfully showed the AB effect as an appropriate displacement of the interference pattern.

In 2007 [7] further evidence was produced against a classical force-based explanation of the AB effect. Let us consider the solenoid formulation. Previous theoretical works had hinted that the two beams passing on the left and on the right could be subject to two opposite forces, one pushing forward the electrons and the other pulling them back. Hence, the relative displacement of the left and right wave packets could quantitatively account for the shift of the interference pattern and no phase shift would be needed. But, if this was the case, there would be a Φ dependence of the time of flight of the electron which was not observed when a beam was sent through the hole of a square toroidal electromagnet. However, the magnet was macroscopic whereas the AB effect is usually recorded at a considerably smaller scale. The relevance of the size might be related to the interaction time and, consequently, to the response of the solenoid to the passage of the electrons.

position. It must be remarked that this is a time reversal in a loose sense: it only applies to the particle motion and not to the whole system. In particular, it does not apply to the magnetic field. Thus, the trajectory of non-relativistic particle in a fixed magnetic field is not reversible and, in this case, it can be checked in the path integral formalism that $G(x_1, t; x_2) \neq G(x_2, t; x_1)$.

19 Interpretation

In the prevuois section a possible approach to the AB effect has been described, but there are of course other possibilities. While there seems to be general agreement on the result, its interpretation is a pretty subtle matter and is still controversial. Here is an idea of the debate.

It has been originally [1] suggested that either the particle is non-locally interacting with the confined magnetic field or that the dynamics of particle is locally influenced by the vector potential alone. Thus, the vector potential and not the magnetic field would be the fundamental quantity. This seems to be the commonly accepted explanation, consistent with the fact that it is the vector potential and not the magnetic field (Lorentz force) which appears in the Schrodinger equation. However, this may sound troublesome since the vector potential is only defined up to a gauge transformation.

It is also possible to take seriously the topological approach and regard it not just as a good tool. In this case, the fundamental quantity for the AB effect may be the flux Φ or even just the topological phase $\exp[i(e/c\hbar)\Phi]$. Bernido and Inomata [5] argued that the AB effect could be obtained from topological arguments alone since, if it was the case, topological effects should be observed in simple double slit experiments too. These topological concepts have been refined and they have also been related to gauge fields. The topological nature of the AB phase seems to be generally accepted too.

Of course much of the interpretation process relies on how much we trust the approximation, since of course infinitely long solenoids cannot be built and the magnetic field cannot be completely prevented from "leaking" in the outer region. But, much more important, the solenoid cannot be truly impenetrable and the particle cannot be prevented from tunnelling inside the solenoid at all or simply from interacting with the solenoid carriers.

It should be remarked that the AB effect does not describe a situation in which there is a potential which is not associated to a magnetic field. More recent papers (Magni, Valz-Gris) [23] suggested that the AB effect can be treated with ordinary quantum mechanics as a limit case by considering a succession of Hamiltonians H_n which converge, in a nontrivial way, towards an Hamiltonian H_∞ describing both a localized magnetic field and an infinite potential wall confining it into a forbidden region.

Finally, the classical force explanation has still supporters such as Boyer [6]. He claims that the AB effect arises from electrodynamics of pointwise objects as a relativistic effect, but he argues that this is not evident since it involves little known classical phenomena.

A Appendix: minimal homotopy theory

This brief appendix is meant to introduce the terminology and operative definitions rather than to explain the subject in a complete and consistent way.
Let $\mathcal{X}$ be a topological space and I=[0,1]

Definition 1. *A path is a continuous map* $\alpha : I \to \mathcal{X}$

The set of all paths such that $\alpha(0) = x_1$ and $\alpha(1) = x_2$ will be denoted as $C(x_1, x_2)$. There exists an associative product between paths

$$* : C(x_1, x_2) \times C(x_2, x_3) \to C(x_1, x_3) \qquad * : (a, b) \mapsto ab$$

which simply means that if the starting point of b is the end-point of a the two paths can be attached to form a new one. The parameter t is rescaled so that $ab(0) = x_1$, $ab(1/2) = x_2$ and $ab(1) = x_3$. Given a path $a(t)$ the **inverse** path is defined as $a^{-1} := a(-t)$. A path e is said to be null if $e(t) = x \, \forall t \in I$. For any other path a, $ae = ea = a$ whenever the product is well defined.

Definition 2. $\mathcal{X}$ *is arcwise connected if* $C(x, x')$ *is non-empty* $\forall x, x' \in \mathcal{X}$

Definition 3. *Two paths* a *and* $b \in C(x_1, x_2)$ *are said to be* ***homotopic*** *if there exists a continuous function*

$$h : I \times I \to \mathcal{X} \text{ such that } h(0, t) = a(t) \text{ and } h(1, t) = b(t)$$

Given $C(x_1, x_2)$, homotopy defines an equivalence relationship between paths: $a \sim b \Rightarrow [a] = [b]$. Homotopy classes will be labelled with greek letters. Let us call $\pi(x_1, x_2) = C(x_1, x_2)/\sim$. Given a point x, let us denote $\pi_1(x) = \pi(x, x)$ the quotient set of all **closed paths** starting and ending at x.

Homotopy is compatible with the product: composition, associativity and inversion do not depend of the choice of the representative. Therefore, the product can be defined for homotopy classes by using representative elements $[a][b] = [ab]$.

Theorem 1. *Given a point* $x \in \mathcal{X}$, $\pi_1(x)$ *is a group with the multiplication between homotopy classes and it is called the* ***fundamental group*** *at* ***basepoint*** x.

If $\mathcal{X}$ is arcwise connected, fundamental groups at all basepoints are isomorphic and can be simply called π_1. Given x and x', $f : \alpha \in \pi_1(x) \mapsto [c^{-1}]\alpha[c] \in \pi(x')$ is an isomorphism if $c \in \pi(x, x')$.

Definition 4. $\mathcal{X}$ *is simply connected if it is arcwise connected and* $\pi_1 = \{e\}$.

Now let $\mathcal{X}^*$ be another topological space

Definition 5. *Let $\mathcal{X}$ be connected. $p : \mathcal{X}^* \to \mathcal{X}$ is a **covering projection** if, $\forall x \in \mathcal{X}$, there exists an open set $V \subset X$ such that $x \in V$ and $p^{-1}(V) = \bigcup_i U_i$. U_i are open and disjoint sets in $\mathcal{X}^*$ and the restriction of p to U_i is a homeomorphism.*

*If X^* is connected and simply connected, p is a **universal covering**.*

Not all topological spaces have a universal covering, but, when they do, it is unique up to a homeorphism.

Theorem 2. *If p is a covering projection, it is also a local homoemorfism. In addition, $\#p^{-1}(x) = \#p^{-1}(x')\forall x, x' \in \mathcal{X}$*

Now we will use p to lift paths from $\mathcal{X}$ to $\mathcal{X}^*$

Theorem 3. *Let $a \in C(x, x')$ be a path in $\mathcal{X}$. The path $p^{-1}a : \mathcal{X} \to \mathcal{X}^*$ is uniquely defined if $p^{-1}a(0)$ is fixed.*

This follows from the fact that p is a local homeomorphism. Once one of the preimages of $a(0)$ is chosen in one of the U_i ($p(U_i) = V$ which is a neighbourhood of $a(0)$) then the homeomorphism grants injectivity so that the part of the path in V is lifted unambigously. One chooses a new point, say $a(t') \in V$, and fixes a new open set V'. Since $p^{-1}a$ must be continuous in X^* and U_i's are disjoint, there is only one possible preimage U_i' of V' which can be chosen. One proceeds this way until $a(1)$ is lifted.

On the other hand, the projection function clearly allows sending paths from $\mathcal{X}^*$ to $\mathcal{X}$ in a non ambigous way $p : C^*(x_i^*, x_f^*) \to C^(p(x_i^*), p(x_f^*))$; $a \mapsto pa$. There is a relationship between omotopy classes in $\mathcal{X}^*$ and those in $\mathcal{X}$. Given $a, b \in C^*(x_i^*, x_f^*)$ if $a \sim b$ in $\mathcal{X}^*$, then $pa \sim pb$ in $\mathcal{X}$. The converse is not true and this fact is one of the keys to path integral in multiply connected spaces. If the paths are closed, the following theorem, which clearly applies to the covering projection, holds.

Theorem 4. *Let $f : \mathcal{X}^* \to \mathcal{X}$ be a continuos map. Then there exist an homomorphism $\tilde{f} : \pi_1(x^*) \to \pi_1(f(x^*))\, \forall x^* \in \mathcal{X}^*$. If f is a homeomorphism, $\tilde{f}$ is an isomorphism.*

There is another important object.

Definition 6. *Let $\mathcal{G}$ be a group of homoeomorfism from $\mathcal{X}^*$ to itself. Its action on $\mathcal{X}^*$ is **properly discontinuos** if $\forall x^* \in \mathcal{X}$ has a neighbourhood U such that $\forall g \in \mathcal{G}$, $g \neq e$, $g(U) \cap U = \emptyset$*

To our practical purpose, pictorially, elements in $\mathcal{G}$ make single points "jump" without dragging them continuously along paths. Then an appropriate subset (somehow to be called $\mathcal{X}$) can be chosen and its images under elemnts g cover $\mathcal{X}^*$ completely without overlapping. The following theorem describes this correctly.

Theorem 5. *If $\mathcal{G}$ acts properly discontinuosly on $\mathcal{X}^*$ and $\mathcal{X}^*/\mathcal{G}$ is connected, then $p : \mathcal{X}^* \to \mathcal{X}^*/\mathcal{G}(=: \mathcal{X})$ is a covering projection.*

Theorem 6. *Let $\mathcal{X}^*$ be simply connected and let $\mathcal{G}$ acts properly discontinuosly on it, then $\mathcal{X}^*/\mathcal{G}$ is isomorphic to π_1.*

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