LONDON VORTICES

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Let us summarise the main formulas of the theory by Ginzburg and Landau for superconductivity. The free energy and the supercurrent density are

$$F = \int d\mathbf{x} \frac{\hbar^2}{2m^\star} |(-i\boldsymbol{\nabla} + \frac{e^\star}{\hbar c}\mathbf{A})\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{\mathbf{B}^2}{8\pi} + F_n^0 \tag{1}$$

$$\mathbf{J}_S = -\frac{e^{\star 2}}{m^\star c}|\psi|^2 (\frac{\phi_0}{2\pi}\boldsymbol{\nabla}\,\alpha + \mathbf{A}) \tag{2}$$

where $\psi = |\psi|e^{i\alpha}$, $a = a'(T - T_c)$, $b > 0$ and $\phi_0 = hc/e^\star$ is the unit of magnetic flux. The single-valuedness of ψ implies this integral identity on a closed circuit:

$$\frac{m^\star c}{e^{\star 2}} \oint_C \frac{\mathbf{J}_S \cdot d\boldsymbol{\ell}}{|\psi|^2} + \int_S da\ \mathbf{n}\cdot\mathbf{B} = m\phi_0, \quad m \text{ integer} \tag{3}$$

where the surface S has boundary line C.
The (squared) coherence and penetration lengths, the GL ratio, the bulk value of the order parameter are:

$$\xi^2 = \frac{\hbar^2}{2m^\star|a|}, \quad \delta^2 = \frac{m^\star c^2 b}{4\pi e^{\star 2}|a|}, \quad \kappa = \frac{\delta}{\xi} = \frac{m^\star c}{e^\star \hbar}\sqrt{\frac{b}{2\pi}}, \quad \psi_\infty^2 = \frac{|a|}{b} \tag{4}$$

The following relations are useful:

$$\frac{H_c(T)^2}{8\pi} = \frac{a^2}{2b}, \qquad H_{c2}(T) = \kappa\sqrt{2}H_c(T) \tag{5}$$

$$\phi_0 = \frac{hc}{2e} = 2\pi\xi^2 H_{c2} \tag{6}$$

$$\xi\delta H_c = \frac{\phi_0}{2\pi\sqrt{2}} \tag{7}$$

London vortices

The study of the G.L. equations is simpler in the regime $\kappa \gg 1$, where $f = 1$ in bulk regions, and rapidly drops to zero in contact with normal regions. The free energy and the current density (2) are approximated by

$$F = \int_V d\mathbf{x} \frac{e^{\star 2}}{2m^\star c^2}\psi_\infty^2 |\frac{\phi_0}{2\pi}\boldsymbol{\nabla}\,\alpha + \mathbf{A}|^2 + a\psi_\infty^2 + \frac{b}{2}\psi_\infty^4 + \frac{\mathbf{B}^2}{8\pi} + F_n^0$$

$$\mathbf{J} = -\frac{e^{\star 2}}{m^\star c}\psi_\infty^2 (\frac{\phi_0}{2\pi}\boldsymbol{\nabla}\,\alpha + \mathbf{A}) \tag{8}$$

where the volume V *excludes regions where f differs from 1.* With the 2nd GL equation (Maxwell's equation), $\mathbf{J} = \frac{c}{4\pi}\mathrm{rot}\mathbf{B}$, the free energy becomes:

$$F = \frac{1}{8\pi}\int_V d\mathbf{x}\,(\mathbf{B}^2 + \delta^2|\mathrm{rot}\mathbf{B}|^2) - \frac{H_c^2}{8\pi} + F_{n,0} \tag{9}$$

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The same expression results in London's theory for a superfluid with uniform mass density $m^\star n_s$, velocity $\mathbf{v}_s$, supercurrent $\mathbf{J}_s = -e^\star n_s \mathbf{v_s}$. Maxwell's equation gives the kinetic energy

$$\frac{1}{2}m^\star n_s v_s^2 = \frac{1}{2}m^\star n_s \frac{c^2}{16\pi^2}\frac{|\mathrm{rot}B|^2}{e^{\star 2}n_s^2} = \frac{\delta^2}{8\pi}|\mathrm{rot}B|^2$$

The rot operator of (8) and Maxwell's equation give: rot rot $\mathbf{B} = -\frac{4\pi e^{\star 2}}{m^\star c^2}\psi_\infty^2 \mathbf{B}$

$$\mathbf{B} - \delta^2\nabla^2\mathbf{B} = 0 \qquad (10)$$

In the following we consider vortex solutions $\mathbf{B} = B(x,y)\mathbf{k}$. Then: $|\mathrm{rot}\,\mathbf{B}|^2 = (\partial_x B)^2 + (\partial_y B)^2 = -B\nabla^2 B + \frac{1}{2}\nabla^2 B^2$. The free energy per unit length is

$$\frac{F}{L} = \frac{1}{8\pi}\int_S da\, B(B - \delta^2\nabla^2 B) + \frac{\delta^2}{16\pi}\int_S da\ \nabla^2 B^2$$

where the surface S excludes the vortex cores. The first integral is zero for a solution of (10). Since $\nabla^2 = \mathrm{div}\ \mathrm{grad}$ we obtain an integral along the boundary:

$$\boxed{\frac{F}{L} = \frac{\delta^2}{16\pi}\sum_k \oint d\ell\ (\mathbf{n}\cdot\boldsymbol{\nabla})B^2} \qquad (11)$$

The sum is on all vortex cores, and the integrals are on circles of radius ξ centered in each core, with normal vector $\mathbf{n}$ pointing to the center of the core.
In a core the field B is almost constant (the supercurrents are zero inside).

1-vortex solution. For a single vortex, $B(r)$ solves (10) outside the core. In coordinates (r,θ,z) it is Bessel's equation

$$B'' + \frac{1}{r}B' - \frac{1}{\delta^2}B = 0 \qquad r > \xi \qquad (12)$$

The solution is Hankel's function $CK_0(r/\delta)$, with a constant C. The function is always positive and decreasing, with limit behaviours:

$$K_0(x) = \begin{cases} \sqrt{\frac{\pi}{2x}}\ \exp(-x) & x \gg 1 \\ -\log x + \log 2 - \gamma & x \to 0 \end{cases} \qquad (13)$$

$\log 2 - \gamma \approx 0.12$ (γ is Euler's constant). $K_0(1) = 0.421$, $K_0(2) = 0.114$. The derivative is $K_0'(x) = -K_1(x)$. Since $x = r/\delta$, the limit $x \to 0$ is achieved for $\xi/\delta \ll 1$ i.e. a type II superconductor.
To relate the constant C to a physical property, let us evaluate the flux through an annulus $\xi < r < R$, where R is arbitrary. We use $B = \delta^2\frac{1}{r}(r\frac{d}{dr}B)$:

$$\begin{aligned}\Phi(R) &= 2\pi\int_\xi^R r dr B(r) = 2\pi C\delta^2\int_{\xi/\delta}^{R/\delta} x dx \frac{1}{x}\frac{d}{dx}(-xK_1) \\ &= 2\pi C\delta[\xi K_1(\xi/\delta) - RK_1(R/\delta)]\end{aligned}$$

For $\xi \ll \delta$ we approximate $K_1(\delta/\xi) \simeq \xi/\delta$. The total flux is collected within few screening lengths δ. Then we may take $R \gg \delta$:

$$\Phi(R) \approx 2\pi\delta^2 C - C(\pi\delta)^{3/2}\sqrt{2R}\,e^{-R/\delta} \approx 2\pi\delta^2 C$$

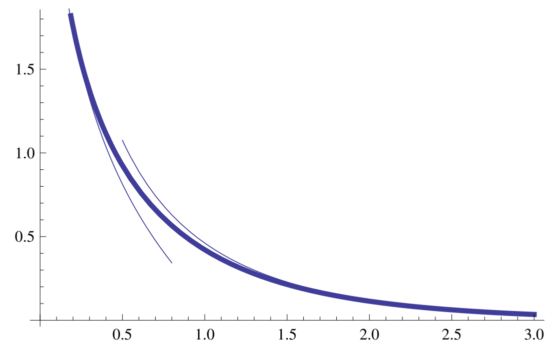

FIGURE 1. The function $K_0(x)$. The thin lines describe the limit functions in eq.(13)

We thus obtain C, and write the solution B:

$$B(r) = \frac{\Phi}{2\pi\delta^2} K_0\left(\frac{r}{\delta}\right) \tag{14}$$

The field at the core with a unit flux quantum ϕ_0, with eq.(6), is:

$$B(\xi) = \frac{2\pi\xi^2 H_{c2}}{2\pi\delta^2}\log\kappa = \frac{H_{c2}}{\kappa^2}\log\kappa = H_c\sqrt{2}\frac{\log\kappa}{\kappa}$$

The core flux is $\Phi_{core} = \pi\xi^2 B(\xi) = \pi\xi^2 H_{c2}\frac{\log\kappa}{\kappa^2} = \phi_0 \frac{\log\kappa}{2\kappa^2}$.
The lower field H_{c1} at which the flux penetrates, marking the limit of the pure diamagnetic phase, is about half of it (the term 0.12 is omitted)

$$H_{c1} = H_c \frac{\log\kappa}{\kappa\sqrt{2}} \tag{15}$$

At this value the difference of the Gibbs potentials $G_s - G_n$ becomes negative. In a type II superconductor, nothing happens as $H = H_c$.

The integral for the free energy per unit length of a vortex is evaluated on the circle $r = \xi$:

$$\frac{F_1}{L} = -\frac{\delta^2}{16\pi}\oint d\ell \, \frac{d}{dr}B^2(r) = -\frac{\delta^2}{16\pi}\frac{d}{dr}B^2(r)\Big|_{r=\xi} 2\pi\xi$$
$$= \left(\frac{\Phi}{4\pi\delta}\right)^2 K_0\left(\frac{\xi}{\delta}\right) K_1\left(\frac{\xi}{\delta}\right)\frac{\xi}{\delta}$$

In the limit $\kappa = \delta/\xi \gg 1$, it is:

$$\frac{F_1}{L} = \left(\frac{\Phi}{4\pi\delta}\right)^2 (\log\kappa + 0.12) \tag{16}$$

The constant value 0.12 will be neglected. The current density circulates around the core and fades within few London lengths:

$$\mathbf{J}(r) = \frac{c}{4\pi}\mathrm{rot}\mathbf{B} = \frac{c}{4\pi}[\mathbf{i}\partial_y B - \mathbf{j}\partial_x B] = -\frac{c}{4\pi}\frac{dB}{dr}\boldsymbol{\theta} = \frac{c\Phi}{8\pi^2\delta^3}K_1\left(\frac{r}{\delta}\right)\boldsymbol{\theta} \tag{17}$$

Example 0.1. The superconducting alloy Nb_3Sn has $T_c = 18.3K$, $\kappa \approx 40$, $\xi_0 = 3.3$nm, $\delta = 135$nm, $H_{c1} = 0.038$T. It can attain $H_{c2} = 30$T [6].
Evaluate the free energy per unit length of a vortex.

$$\xi^2 = \frac{\phi_0}{2\pi H_{c2}} = \frac{2.07\times 10^{-7}\mathrm{Oe}\cdot\mathrm{cm}^2}{2\pi\cdot 30\times 10^4\mathrm{Oe}} = 10.9\times 10^{-14}\mathrm{cm}^2, \quad \xi \approx 3.3\,\mathrm{nm}$$

With $mc^2 = 0.51$ MeV, $a_0 = \hbar^2/(me^2) = 5.29\times 10^{-2}$nm, the energy per unit length of a vortex with 1 elementary flux $\phi_0 = hc/2e$ is:

$$\epsilon = \left(\frac{hc}{8e\pi\delta}\right)^2 (\log\kappa + 0.12) = \frac{mc^2}{16}\frac{a_0}{\delta^2}(\log\kappa + 0.12) \approx 3.5\frac{\mathrm{MeV}}{\mathrm{cm}} = 5.6\times 10^{-6}\frac{\mathrm{erg}}{\mathrm{cm}}$$

The 2-vortex solution. Since the equation (10) is linear, a 2-vortex solution is the superposition a two 1-vortex solutions with flux Φ in the origin and another in position $\mathbf{R}$:

$$B(\mathbf{x}) = \frac{\Phi}{2\pi\delta^2}\left[K_0\left(\frac{|\mathbf{x}|}{\delta}\right) + K_0\left(\frac{|\mathbf{x}-\mathbf{R}|}{\delta}\right)\right] \tag{18}$$

with $R \gg \xi$. The free energy per unit length is

$$\begin{aligned}
\epsilon_2 &= \frac{\delta^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,(\mathbf{n}\cdot\boldsymbol{\nabla})(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{2}} \\
&\approx \frac{\delta^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,(\mathbf{n}\cdot\boldsymbol{\nabla})B_j^2 + 2\frac{\delta^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,(\mathbf{n}\cdot\boldsymbol{\nabla})(B_1B_2) \\
&= 2\epsilon_1 + \frac{4\delta^2}{16\pi}\left(\frac{\Phi}{2\pi\delta^2}\right)^2\oint_{|\mathbf{x}|=\xi} d\ell(-\frac{d}{dr})K_0\left(\frac{|\mathbf{x}|}{\delta}\right)K_0\left(\frac{|\mathbf{x}-\mathbf{R}|}{\delta}\right) \\
&\approx 2\epsilon_1 + \frac{\delta^2}{4\pi}\left(\frac{\Phi}{2\pi\delta^2}\right)^2\frac{1}{\delta}K_1\left(\frac{\xi}{\delta}\right)K_0\left(\frac{R}{\delta}\right)2\pi\xi
\end{aligned}$$

We neglected the contribution of B_1^2 to the hole 2 and of B_2^2 to hole 1 (i.e. $\delta/\xi = \kappa \gg 1$), and a term arising from the derivative (it is order $1/\kappa^2$ of the first one). The interaction energy per unit length among the two parallel and equal vortices at distance R is $\epsilon_{\mathrm{int}} = \epsilon_2 - 2\epsilon_1$,

$$\boxed{\epsilon_{int}(R) = \frac{\Phi^2}{8\pi^2\delta^2}K_0\left(\frac{R}{\delta}\right)} \tag{19}$$

Since K_0 is monotonically decreasing, the force per unit length between vortices is *repulsive*: $f_{12}(R) = -\epsilon'_{int}(R) > 0$.
A more accurate expression, valid for $R \gg \xi$, is [5]

$$\epsilon_{int}(R) = c^2 K_0\left(\frac{R}{\delta}\right) - \frac{d^2}{\kappa^2}K_0\left(\sqrt{2}\frac{R}{\xi}\right) \tag{20}$$

with parameters c, d. It is attractive for type I superconductors, and always attractive for vortex antivortex pairs. It is zero for $\kappa = 1/\sqrt{2}$.

Example 0.2. Show that the free energy per unit length of a single vortex with flux 2Φ is greater than the free energy of two vortices each carrying a flux Φ.

Example 0.3. (from [7]). Find the attractive force exerted on a vortex by the surface of a flat superconductor if the vortex is parallel to the surface at a distance $\ell = 50$nm and $\delta = 300$nm.
Being $\delta > \ell$, the magnetic field of the vortex reaches the surface, where $B = 0$. The situation vortex+s-surface is equivalent to two specular vortices with opposite fluxes. The interaction energy per unit length is $\epsilon = \frac{\phi^2}{8\pi^2\delta^2} K_0(\frac{2\ell}{\delta}) \approx -\frac{\phi^2}{8\pi^2\delta^2} \log \frac{2\ell}{\delta}$. The force per unit length is $F/L = -d\epsilon/d\ell = \frac{\phi^2}{8\pi^2\delta^2}\frac{1}{\ell}$.

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