FEYNMAN DIAGRAMS

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In this lecture we deal with the perturbative evaluation of the one-particle Green function, also known as the *propagator* because it is proportional to the probability amplitude for the propagation of a many-body state with one particle created at x, to a many-body state with a particle being created at x'.
Particles cannot be distinguished, so in principle one cannot follow a particle trajectory: it is a many-body evolution. We shall later consider a revival of the particle concept as a quasi-particle, a many-body manifestation of a particle.

Let $H = H_0 + V$. We shall always take for granted that:
1) the Gell-Mann & Low theorem applies,
2) H_0 and its ground state $|E_0\rangle$ are suited for the T-ordered Wick theorem.
With the reduction formula, the 1-particle Green function is:

$$G_{mm'}(x,x') = \frac{1}{i}\frac{\langle E_0|\mathsf{T}S\psi_m(x)\psi^\dagger_{m'}(x')|E_0\rangle}{\langle E|S|E\rangle} \tag{1}$$

The scattering operator is available through the Dyson expansion:

$$\begin{aligned} S =& \mathsf{T}\exp\frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt\, V_{H_0}(t) \\ =& \mathsf{T}\exp\frac{1}{2i\hbar}\sum_{\mu\mu'\nu\nu'}\int_{-\infty}^{+\infty} dt\int d\mathbf{x}d\mathbf{y}\, v_{\mu\mu'\nu\nu'}(\mathbf{x},\mathbf{y})\psi^\dagger_\mu(\mathbf{x}t)\psi^\dagger_\nu(\mathbf{y}t)\psi_{\nu'}(\mathbf{y}t)\psi_{\mu'}(\mathbf{x}t) \end{aligned}$$

The four operators are at the same time. Since their relative positions are fixed, the ambiguity of equal times in a T-product is solved by writing:

$$\psi^\dagger_\mu(\mathbf{x}t^{+++})\psi^\dagger_\nu(\mathbf{y}t^{++})\psi_{\nu'}(\mathbf{y}t^{+})\psi_{\mu'}(\mathbf{x}t)$$

Now they can be freely permuted inside the T-operator, up to signs.

To gain symmetry and identify events in spacetime, another time variable t' is introduced, with a delta function. Define the “bare interaction”:

$$U^0_{\mu\mu'\nu\nu'}(x,x') = v_{\mu\mu'\nu\nu'}(\mathbf{x},\mathbf{x}')\delta(t-t')$$

where μ and μ' are the spins of the particle that exits and enters at the vertex $x=(\mathbf{x}t)$, and ν,ν' the same at the vertex x'. Note the symmetry for exchange of particles: $U^0_{\mu\mu'\nu\nu'}(x,x') = U^0_{\nu\nu'\mu\mu'}(x',x)$. Finally we write:

$$S = \mathsf{T}\exp\frac{1}{2i\hbar}\sum_{\mu\mu'\nu\nu'}\int d^4x_1 d^4x_2\, U^0_{\mu\mu'\nu\nu'}(x_1,x_2)\psi^\dagger_\mu(x_1^+)\psi^\dagger_\nu(x_2^+)\psi_{\nu'}(x_2)\psi_{\mu'}(x_1)$$

The T-expansion of S is inserted in eq.(1). It remains to evaluate a series of time-ordered ground-state averages of $4k+2$ field operators, $k=0,1,2,\ldots$. Using Wick's

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theorem, each average is a sum of total contractions, and each total contraction has a graphical representation: a Feynman diagram.

There is an enormous advantage in dealing with the complicate perturbative terms through the corresponding Feynman diagrams.

1. The Feynman diagrams of G

Let us begin with the first order term of the numerator of eq.(1):

$$\frac{1}{2i\hbar}\sum_{\mu\mu'\nu\nu'}\int d1d2\, U^0_{\mu\mu'\nu\nu'}(1,2)\langle E_0|\mathsf{T}\psi^\dagger_\mu(1^+)\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)\psi_{\mu'}(1)\psi_m(x)\psi^\dagger_{m'}(x')|E_0\rangle$$

The matrix element is evaluated via Wick's theorem, and produces 6 total contractions.

• By contracting the source operators $\psi_m(x)$ and $\psi^\dagger_{m'}(x')$ among themselves, one remains with the factors

$$\langle E_0|\mathsf{T}\psi_m(x)\psi^\dagger_{m'}(x')|E_0\rangle\ \langle E_0|\mathsf{T}\psi^\dagger_\mu(1^+)\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)\psi_{\mu'}(1)|E_0\rangle$$

The particle propagates from x' to x without interactions (first factor) while the two total contractions of V give two vacuum graphs, i.e. numbers (diagrams a and b in Fig.1). The expression, integrated with U^0, is the term $iG^0_{mm'}(x,x')\langle E_0|S^{(1)}|E_0\rangle$ of the expansion of G.

We shall prove that all diagrams with vacuum factors exactly cancel with the denominator.

• Another total contraction is

$$\langle E_0|\mathsf{T}\psi_m(x)\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)\psi^\dagger_{m'}(x')|E_0\rangle\ \langle E_0|\mathsf{T}\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)|E_0\rangle$$

where $\psi^\dagger_{m'}(x')$ is contracted with $\psi_{\nu'}(2)$ etc. i.e. the particle moves from x' to x through vertex 2. The other factor is a bubble closed in 1. The resulting expression is $i^3G^0(x,2)G^0(2,x')G^0(1,1^+)$ (diagram d in Fig.1). The total contraction with 2 and 1 exchanged (diagram c),

$$\langle E_0|\mathsf{T}\psi_m(x)\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)\psi^\dagger_{m'}(x')|E_0\rangle\ \langle E_0|\mathsf{T}\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)|E_0\rangle$$

gives the same numerical value after integration and spin summation, because of the symmetry of U^0 under exchange. Up to vertex labels, the two diagrams are the same. Only one is evaluated because this (exchange) multiplicity cancels the first order prefactor 1/2.

• The last total contractions are those that do not split. The operators are here permuted without sign changes:

$$\langle E_0|\mathsf{T}\psi_m(x)\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)\psi^\dagger_{m'}(x')|E_0\rangle$$

The contractions are next to next: $x'\to 1\to 2\to x$ (diagram e). The other possibility is $x'\to 1\to 2\to x$ (diagram f). After integrating 1,2 and spin summation, the two contributions are the same. Again: consider just one diagram and omit the prefactor 1/2.

Before stating the Feynman rules, that provide the analytic expression of each Feynman diagram, let us get rid of the denominator and of the diagrams with vacuum factors in the numerator, through the following statement.

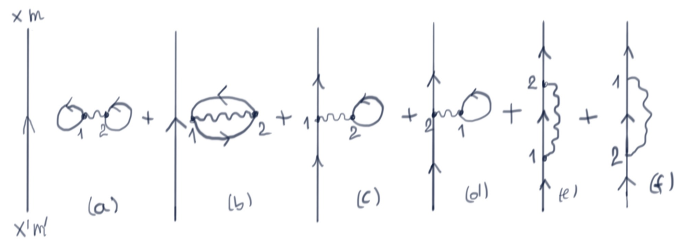


FIGURE 1. I diagrammi dello sviluppo del numeratore al primo ordine in V.

Proposition 1.1. *Let Ω be the product of creation and destruction operators at different times. The time ordered average simplifies to considering only contractions that do not contain vacuum factors.*

$$\boxed{\frac{\langle E_0|\mathsf{T}S\Omega|E_0\rangle}{\langle E_0|S|E_0\rangle} = \langle E_0|\mathsf{T}S\Omega|E_0\rangle_\star} \tag{2}$$

Proof. The S operator is a time ordered exponential in the numerator

$$\sum_{k=0}^{\infty} \frac{1}{(i\hbar)^k}\frac{1}{k!}\int dt_1 ... dt_k \langle E_0|\mathsf{T}V(t_1)...V(t_k)\Omega|E_0\rangle$$

The average is evaluated by Wick's theorem. We classify the total contractions according to the number of operators V that are wholly contracted among themselves. Such contractions produce vacuum factors.
Since in the T symbol the operators V commute, the choice of which to contract is irrelevant and gives the same vacuum terms. Then we have the factorization

$$\langle E_0|\mathsf{T}V(t_1)...V(t_k)\Omega|E_0\rangle = \sum_{\ell=0}^{k}\binom{k}{\ell}\langle E_0|\mathsf{T}V_1...V_\ell|E_0\rangle\langle E_0|\mathsf{T}V_{\ell+1}...V_k\Omega|E_0\rangle$$

This is put in the previous equation and the sums are exchanged:

$$\sum_{\ell=0}^{\infty}\frac{1}{(i\hbar)^\ell}\frac{1}{\ell!}\sum_{k=\ell}^{\infty}\frac{1}{(i\hbar)^{k-\ell}}\frac{1}{(k-\ell)!}\int dt_1...dt_\ell\langle E_0|\mathsf{T}V(t_1)...V(t_\ell)|E_0\rangle$$
$$\times\int dt_{\ell+1}...dt_k\langle E_0|\mathsf{T}V(t_{\ell+1})...V(t_k)\Omega|E_0\rangle_\star$$

With $k' = k-\ell$, the numerator is:

$$\mathrm{Num} = \sum_{\ell=0}^{\infty}\frac{1}{(i\hbar)^\ell}\frac{1}{\ell!}\int dt_1...dt_\ell\langle E_0|\mathsf{T}V(t_1)...V(t_\ell)|E_0\rangle$$
$$\times\sum_{k'=0}^{\infty}\frac{1}{(i\hbar)^{k'}}\frac{1}{k'!}\int dt_1...dt_{k'}\langle E_0|\mathsf{T}V(t_1)...V(t_{k'})\Omega|E_0\rangle_\star$$

We identify the first factor as $\langle E_0|S|E_0\rangle$, that cancels the denominator. □

2. The Feynman rules for G

$G_{mm'}(x,x') = G^0 + G^1 + G^2 + ...$ is a perturbative series where G^k is the sum of all topologically inequivalent Feynman diagrams without vacuum factors.

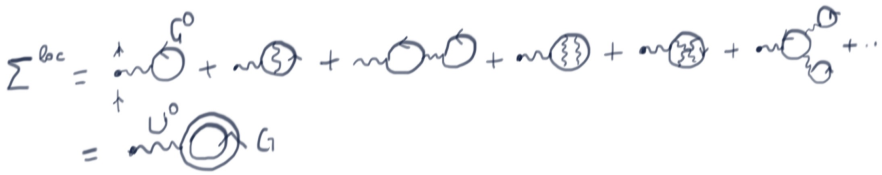
Each diagram of order k contains:
$\star$ an input point x', m' and an output point x, m;
$\star$ $2k$ internal points (trivalent vertices). In a vertex a particle line enters and exits, as well as an interaction line.
$\star$ k interaction lines U^0 that link pairs of vertices,
$\star$ $2k+1$ oriented lines iG^0, that are a total contraction of $\mathsf{T}V_1...V_k\psi\psi^\dagger$.
$\star$ The general structure of a diagram is a line oriented from x' to x, that goes though a number of vertices. The other contractions only involve Vs and produce a number L of oriented loops. The vertices in the open line and in the loops are connected by k interaction lines.

- A factor
$$\frac{1}{i}\frac{1}{(i\hbar)^k} i^{2k+1} = (\frac{i}{\hbar})^k$$
- A factor $(\pm 1)^L$ where L is the number of loops (+ refers to bosons).
This is seen by reordering the operators and factoring the total contraction in order to produce the line $x' \to 1 \to ... \to j \to x$ and partitioning the other pairs to make loops:
$$\langle E_0|\mathsf{T}(\psi_x\psi_j^\dagger)\cdots(\psi_2\psi_1^\dagger)(\psi_1\psi_{x'}^\dagger)|E_0\rangle \; \langle E_0|\mathsf{T}\prod_{i=0}^{k-j}\psi_i^\dagger\psi_i|E_0\rangle$$
$$= [i^{j-1}G^0(x,1)...G^0(j,x')] \times [\text{loop 1}] \times [\text{loop L}]$$
A loop is the total contraction $1 \to 2 \to .. \to r \to 1$ of $\langle E_0|\mathsf{T}\psi_r^\dagger\psi_r...\psi_1^\dagger\psi_1|E_0\rangle$. The contraction of the end operators is $(\pm 1)iG^0(1,r)$, while between there is the chain $iG^0(1,2)...iG^0(r-1,r)$.
- An instantaneous interaction may produce equal times in a G^0. However, since $V(t) = \frac{1}{2}\int d1d2U^0(1,2)\psi^\dagger(1^+)\psi^\dagger(2^+)\psi(2)\psi(1)$, creators have a time larger than destructors. Then it is always $G^0(\mathbf{y}t, \mathbf{y}'t^+)$.
- Omit the factors $1/k!$ and $1/2^k$ which compensate the multiplicity of the same diagram when vertex labels are introduced, due to the exchange symmetry of the potential, $U^0(1,2) = U^0(2,1)$, and the permutation symmetry of the Vs inside the T-product.

3. The self-energy

The structure of the diagrams for the propagator suggests the expression
$$G(1,2) = G^0(1,2) + \int d3\, d4\, G^0(13)\Sigma(3,4)G^0(4,2)$$
where the the self-energy Σ is the sum of all insertions that arise in the perturbation expansion, according to the Feynman rules. A subclass of insertions is the local self-energy, where the in and out points 4, 3 coincide. Inspection of the diagrams shows that

$$\Sigma^{loc}_{\mu\mu'}(x,x') = \frac{\pm i}{\hbar}\delta_4(x-x')\sum_{\nu\nu'}\int dy\, U^0_{\mu\mu',\nu\nu'}(x,y)G_{\nu'\nu}(y,y^+) \tag{3}$$



where G is the *exact* propagator. In particular, for interactions that do not depend on spin, and if $G_{\mu\nu} = \delta_{\mu\nu} G$ it is:

$$
\begin{aligned}
\Sigma^{loc}_{\mu\mu'}(x,x') =& \frac{\pm i}{\hbar}\delta_4(x-x')\delta_{\mu\mu'}(2s+1)\int dy\, v(\mathbf{x},\mathbf{y})G(\mathbf{y}t,\mathbf{y}t^+) \\
=& \pm\frac{1}{\hbar}\delta_4(x-x')\delta_{\mu\mu'}U_H(\mathbf{x})
\end{aligned} \tag{4}
$$

where $U_H(\mathbf{x}) = \int d\mathbf{y} v(\mathbf{x},\mathbf{y})n(\mathbf{y})$ is the Hartree potential with the exact particle density.

The *proper self-energy* $\Sigma^\star_{\mu\mu'}(x,x')$ is the sum of all self-energy diagrams that cannot be split in two diagrams by removal of a single G^0 line. The local self energy Σ^{loc} belongs to this set.
The non-proper self-energy diagrams may be split. When the removal is done, two points (in and out of a particle) are created and the two diagrams that are obtained, are self-energy diagrams.
One may classify diagrams as $\Sigma = \Sigma^\star + \Sigma^{(1)} + \Sigma^{(2)} + ...$ where $\Sigma^{(1)}$ are diagrams that can be split by removing a G^0 in only one way: the result are two $\Sigma^\star$ diagrams. In $\Sigma^{(2)}$ the splitting can be done in two ways: if both G^0 are removed one remains with three $\Sigma^\star$ diagrams. It is now clear that (with obvious notation. Repeated variables are integrated and summed):

$$
\begin{aligned}
\Sigma_{1,2} &= \Sigma^\star_{1,2} + \Sigma^\star_{1,3}G^0_{3,4}\Sigma^\star_{4,2} + \Sigma^\star_{1,3}G^0_{3,4}\Sigma^\star_{4,5}G^0_{5,6}\Sigma^\star_{6,2} + \dots \\
&= \Sigma^\star_{1,2} + \Sigma^\star_{1,3}G^0_{3,4}\Sigma_{4,2}
\end{aligned} \tag{5}
$$

The proper self-energy diagrams alone generate all self-energy diagrams, by solving the above integral equation.
This result is even more interesting for the propagator:

$$
\begin{aligned}
G_{1,2} &= G^0_{1,2} + G^0_{1,3}\Sigma^\star_{3,4}G^0_{4,2} + G^0_{1,3}\Sigma^\star_{3,4}G^0_{4,5}\Sigma^\star_{5,6}G^0_{6,2} + \dots \\
&= G^0_{1,2} + G^0_{1,3}\Sigma^\star_{3,4}G_{4,2} \\
&= G^0_{1,2} + G_{1,3}\Sigma^\star_{3,4}G^0_{4,2}
\end{aligned}
$$

In detail, this is the Dyson equation for the propagator:

$$
\boxed{G_{\mu\nu}(x,y) = G^0_{\mu\nu}(x,y) + \sum_{\rho\sigma}\int dx_1\, dx_2 G^0_{\mu\rho}(x,x_1)\Sigma^\star_{\rho\sigma}(x_1,x_2)G_{\sigma\nu}(x_2,y)} \tag{6}
$$

It is an integral equation. Then, even a single self-energy diagram in this equation produces an infinite sum of diagrams for the propagator.

Suppose that the exact density $n(\mathbf{x})$ is known by some method, as DFT. Then the Hartree potential can be evaluated. Such local insertions can all be summed to

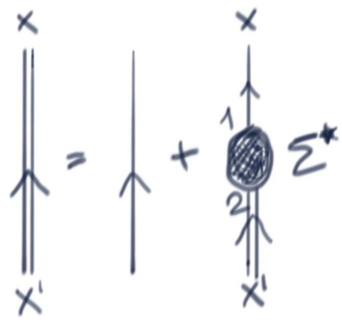


FIGURE 2. The Dyson equation for the propagator.

define the Hartree propagator (we consider the situation where spin factors):

$$G^H(x,x') = G^0(x,x') + \frac{1}{\hbar}\int dy G^0(x,y)U_H(\mathbf{y})G^H(y,x') \tag{7}$$

When such tadpole insertions are adsorbed in G^H, the Dyson equation simplifies as only the bi-local self-energy need be considered (with internal lines that are G^H):

$$G_{\mu\nu}(x,y) = G^H_{\mu\nu}(x,y) + \sum_{\rho\sigma}\int dx_1\, dx_2 G^H_{\mu\rho}(x,x_1)\Sigma^{\star biloc}_{\rho\sigma}(x_1,x_2)G_{\sigma\nu}(x_2,y) \tag{8}$$

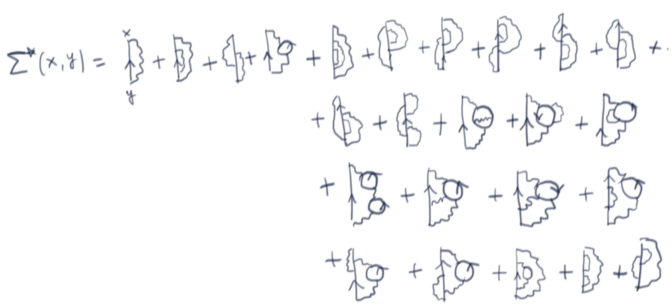


FIGURE 3. The proper self-energy graphs with local insertions accounted for by G^H, that replaces G^0. There is 1 diagram of first order, 3 of second order, then 20, 189, 2232, 31130, ... The number of diagrams grows factorially with the perturbation order.

If the Hamiltonians H, H_0 are invariant for space translations, also the propagators are, as well as the self-energy (every single Feynman diagram is translation invariant). In reciprocal space the Dyson equation becomes algebraic:

$$G_{\mu\nu}(\mathbf{k},\omega) = G^0_{\mu\nu}(\mathbf{k},\omega) + \sum_{\rho\sigma} G^0_{\mu\rho}(\mathbf{k},\omega)\Sigma^\star_{\rho\sigma}(\mathbf{k},\omega)G_{\sigma\nu}(\mathbf{k},\omega)$$

If the correlators are diagonal in the spin variables, and G^0 is that of the ideal gas, the equation gives the important expression:

$$\boxed{G(\mathbf{k},\omega) = \frac{1}{\omega - \frac{1}{\hbar}\epsilon_k - \Sigma^\star(\mathbf{k},\omega)}} \tag{9}$$