A journey through random matrices

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THE CLASSICAL GROUPS

Adolf Hurwitz

SO(N) parametrization with Euler angles (Hurwitz, **1897**)
Measure invariant under the action of the group

SU(N) unitary complex matrices

Sp(2N) matrices with quaternions, **(1939)**

Hermann Weyl

MULTIVARIATE STATISTICS

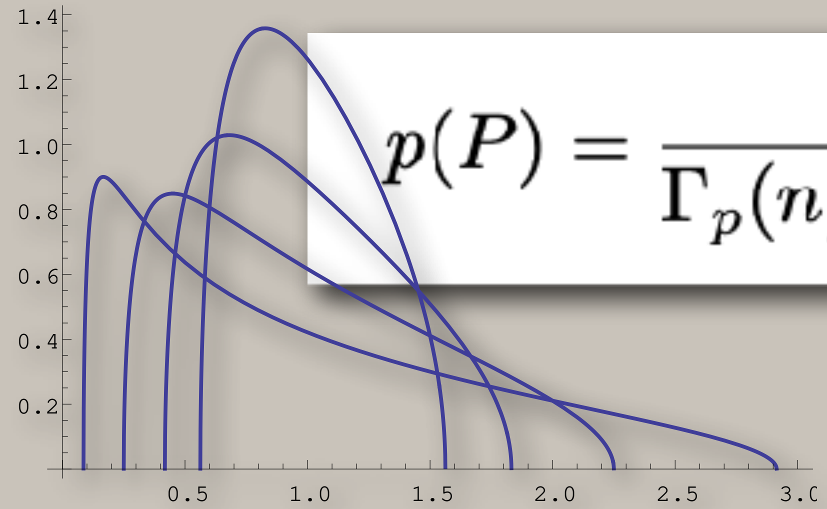
From scalars to vectors $\mathbf{x}$: the **multivariate normal distribution**

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}\sqrt{\det \Sigma}} \exp\left[-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\right]$$

John Wishart

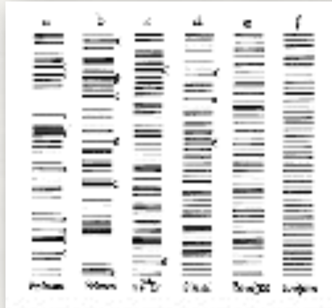
From vectors to matrices P>0: the **Wishart distribution (1928)**
(the extension of χ_n^2 for $x_1^2 + \ldots x_n^2$ to $P = XX^T$)

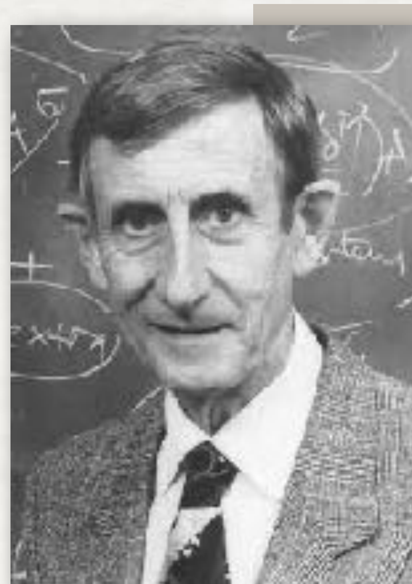
$$p(P) = \frac{1}{\Gamma_p(n/2)(2^p \det \Sigma)^{n/2}} (\det P)^{n-p-1} \exp[-\tfrac{1}{2}\mathrm{tr}(\Sigma^{-1}P)]$$



The Marchenko-Pastur distribution of the eigenvalues of Wishart matrices, $\Sigma = 1$, various ratios p/n (1967)

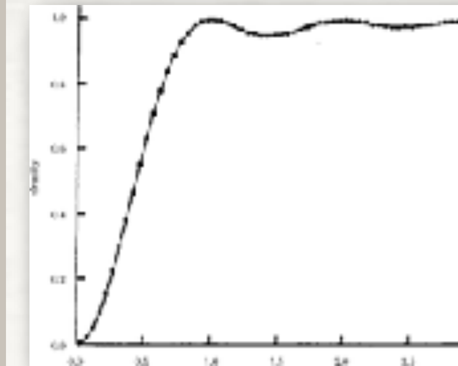
Eugene Wigner





Freeman Dyson

… a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself (RM as ensembles of Hamiltonians)



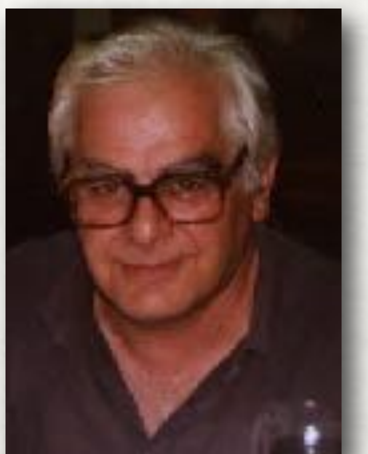
Edouard Brezin

Jan Ambjorn

Madan Lal Mehta

Jean B. Zuber

Anthony Zee



Oriol Bohigas

Leonid Pastur

Giorgio Parisi

Peter Forrester

Alan Edelman

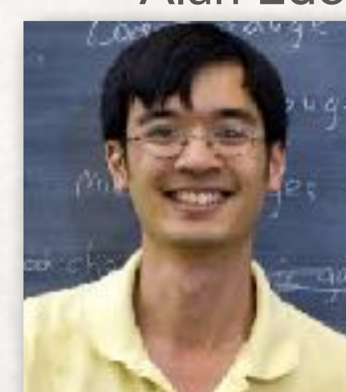
Gernot Akemann

Hans Weidenmuller

Martin Zirnbauer

Jacobus Verbaarshot

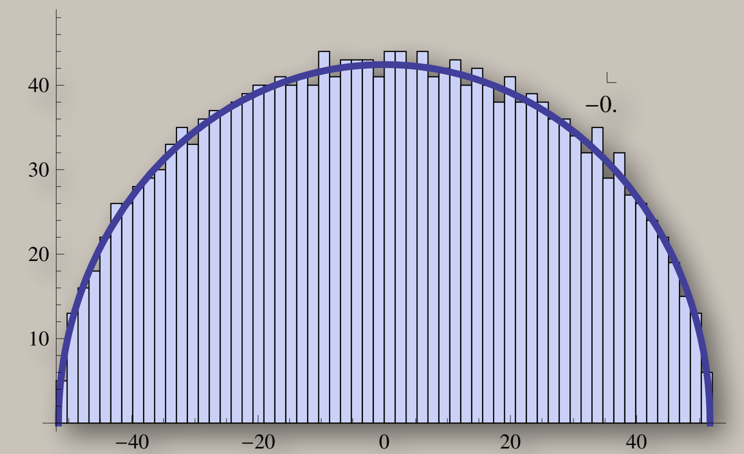
Vyacheslav Girko



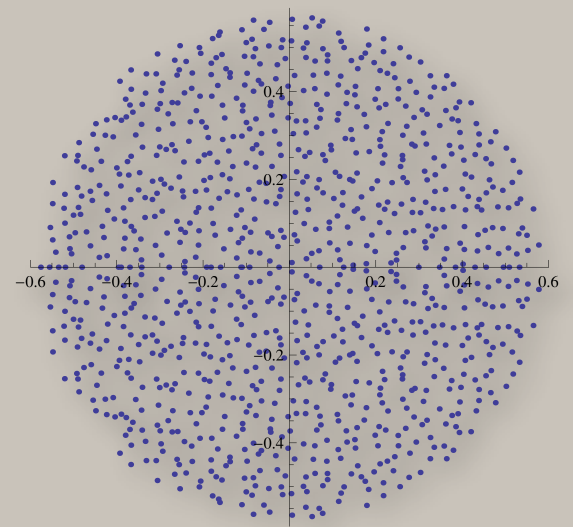
Terence Tao

GAUSSIAN RANDOM MATRICES

GOE: real symmetric Gaussian,
invariant for S ⟶ RSR*, R in SO(N)
GUE: complex Hermitian Gaussian
invariant for H ⟶ UHU*, U in SU(N)
GSE: quaternion s.a. Gaussian,
invariant for K ⟶ JKJ*, J in Sp(2N)



The semicircle law



Ginibre 1965, (no symmetry)

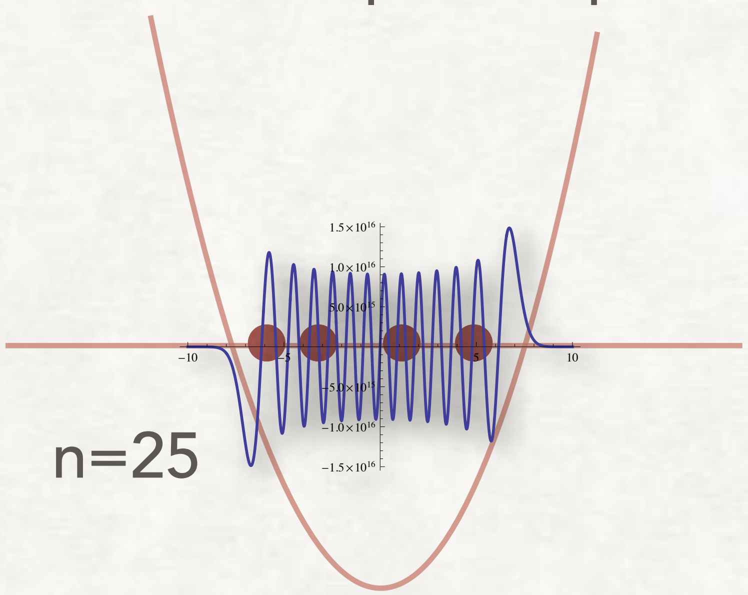
$$p(H)dH = Ce^{-\mathrm{tr}H^2}dH = e^{-\sum_j x_j^2}\prod_{i<j}|x_i - x_j|^\beta dx_1 \dots dx_N \, dU_{\mathrm{Haar}} \qquad \beta = 1,2,4$$

Vandermonde

THE THREEFOLD WAY. Algebraic structure of symmetry groups and ensembles in quantum mechanics (Freeman Dyson, 1962)

$$Z_N = \int dx_1 \ldots dx_N e^{-\sum_j V(x_j) + \beta \sum_{i<j} \log|x_i - x_j|}$$

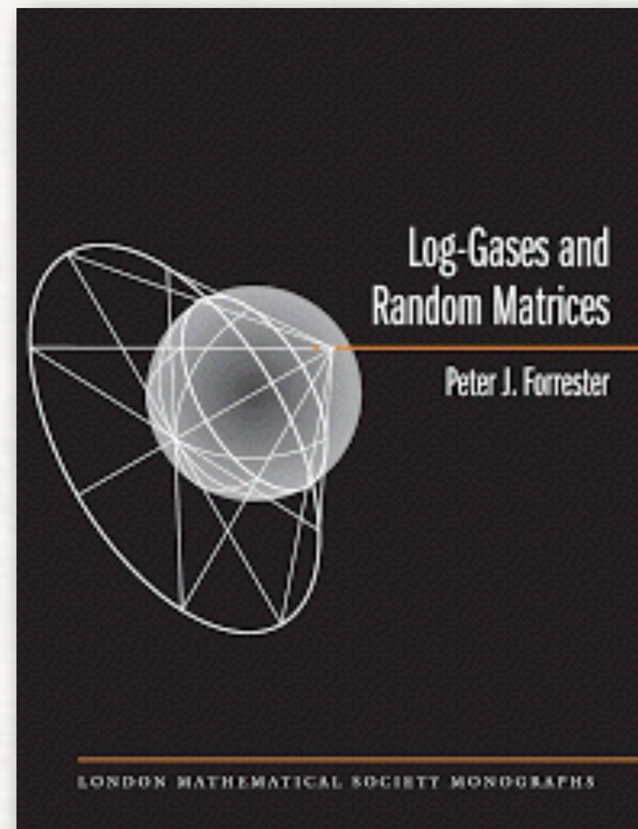
gas of particles (the eigenvalues) with log interaction (2D electrostatics) on the line, plane, sphere (Stieltjes, Jacobi, Riesz)



n=25

$V(x) = x^2$ with interaction log |x-y|

Equilibrium positions: zeros of $H_n(x)$



Saddle point (large N), Orthogonal polynomials (formally exact), loop expansion, collective variables, …

E.Brezin, C.Itzykson, G.Parisi, J.P.Zuber, Planar Diagrams, CMP 1978,
D. Bessis, A new method in the combinatoric of the topological expansion… CMP 1979

ORTHOGONAL POLYNOMIALS ($\beta = 2$)

$$\det \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} = \det \begin{bmatrix} p_0(x_1) & p_1(x_1) & \dots & p_{n-1}(x_1) \\ p_0(x_2) & p_1(x_2) & \dots & p_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ p_0(x_n) & p_1(x_n) & \dots & p_{n-1}(x_n) \end{bmatrix}$$

$$Z_N = \int dx_1 \dots dx_N \prod_{i>j} |x_i - x_j|^2 e^{-N\sum_j V(x_j)} \quad = \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} \prod_k \int dx_k p_{i_r}(x_k) p_{j_s}(x_k) e^{-NV(x_k)}$$

Choose orthogonal polynomials: $\displaystyle \int dx e^{-NV(x)} p_i(x) p_j(x) = h_j \delta_{ij}$

$$Z_N = N! h_0 \dots h_{N-1}$$

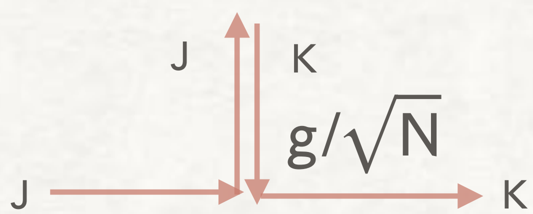
$$\rho(x) = \frac{1}{N} \langle \sum_{j=1}^{N} \delta(x - x_j) \rangle = e^{-V(x)} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{h_j} p_j(x)^2 \quad = \dots (Christoffel - Darboux \;\; formula)$$

Correlation functions, level spacings, edge statistics (Tracy-Widom), phase transitions (multicut support of density), …

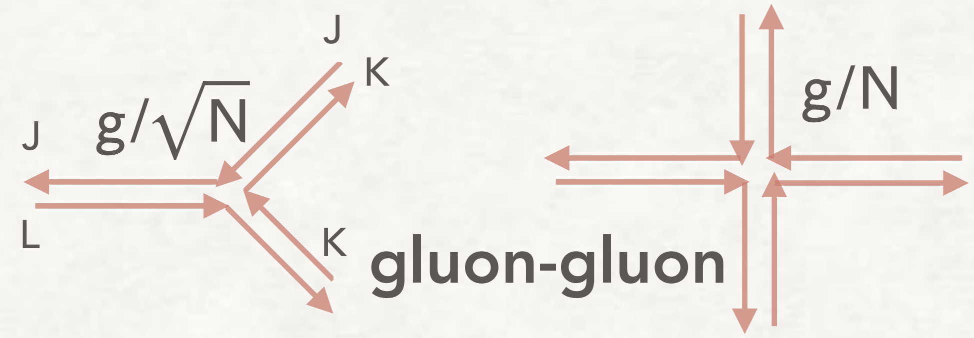
THE STATISTICAL PROPERTIES OF THE EIGENVALUES ARE UNIVERSAL (THEY DO NOT DEPEND ON THE DISTRIBUTION OF THE MATRIX)

1/N EXPANSION

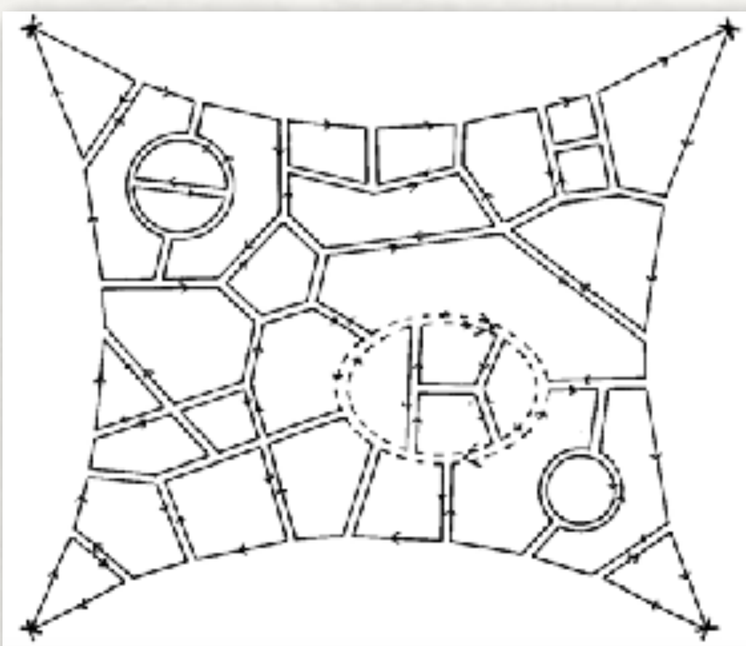
QCD: SU(3) local gauge theory, 3 quark states (vector field), 8 gluons (3x3 matrix field).
Perturbation expansion in g gives Feynman diagrams (feasible at high energy).



quark-gluon

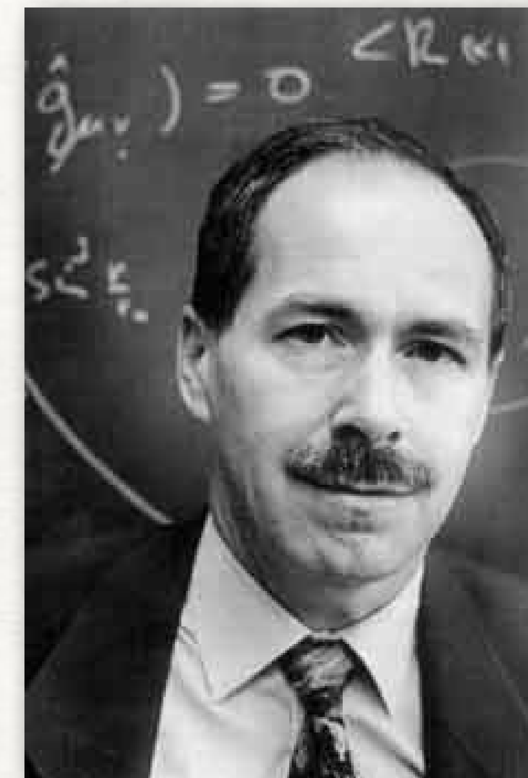


gluon-gluon



SU(N) QCD

THE 1/N EXPANSION WITH g^2N FIXED CLASSIFIES DIAGRAMS ACCORDING TO THEIR EULER NUMBER

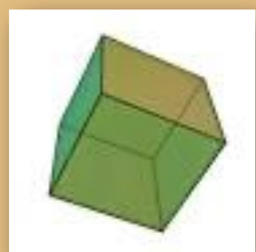


Gerard 'tHooft

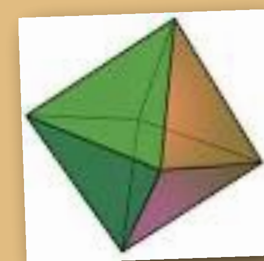
G 'tHooft, A planar diagram theory for strong interactions, Nucl. Phys. B (1974)

QM problem with SO(3) symmetry solved by spherical harmonics & radial equation.
In SO(N) symm. the leading term in 1/N expansion of the radial solution solves an algebraic equation (in the end put N=3)

EULER: $\chi = 2 - 2\text{handles} - \text{holes} = \text{Faces} + \text{Vertices} - \text{Edges}$



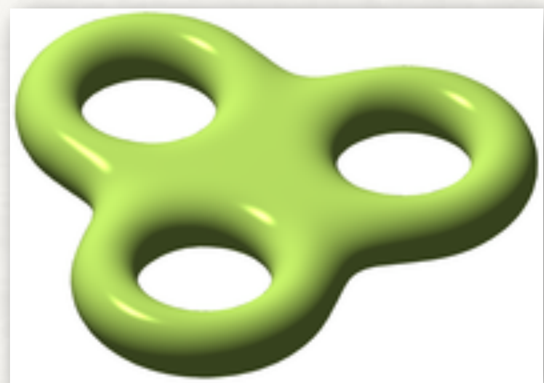
Cube
F=6, V=8 E=12



Octahedron
F=8, V=6 E=12

$\chi=2$

$\chi=0$



$\chi=-4$

$\left[\frac{g}{\sqrt{N}}\right]^4 \frac{g}{N} N^5$ $\left[\frac{g}{\sqrt{N}}\right]^4 \frac{g}{N} N^4$ $\left[\frac{g}{\sqrt{N}}\right]^4 N^2$

N^2 PLANAR diagrams (sphere)
N^0 diagrams on the torus
N^{-2} diagrams on 2-torus
…

In d=0 the generating functional is an **ordinary matrix integral**

$$Z(g,N) = \int dH\, e^{-N\,\mathrm{tr}[H^2+gH^4]}$$

$$dH = dx_1 \ldots dx_N e^{-\sum x_j^2} \prod_{i>j} (x_i - x_j)^2 dU_{Haar}$$

Z(g,N) COUNTS VACUUM DIAGRAMS

$$= N^2 Z_{planar}(g) + Z_{torus}(g) + \ldots$$

STATISTICAL MECHANICS ON PLANAR FEYNMAN DIAGRAMS — RANDOM SURFACES

$$Z_{planar}(g) = \sum_V g^V Z_V$$

$\log Z_V$ counts connected graphs with V 4-vertices
i.e. quadrangulations (dual graphs) with V faces

The planar series has finite radius of convergence:

$Z_V \approx g_{cr}^{-V}$ then $-\log Z_V \approx \mathrm{Area}\, \log g_{cr}$

2D Ising model for ferromagnetism: Onsager (1944) H=0

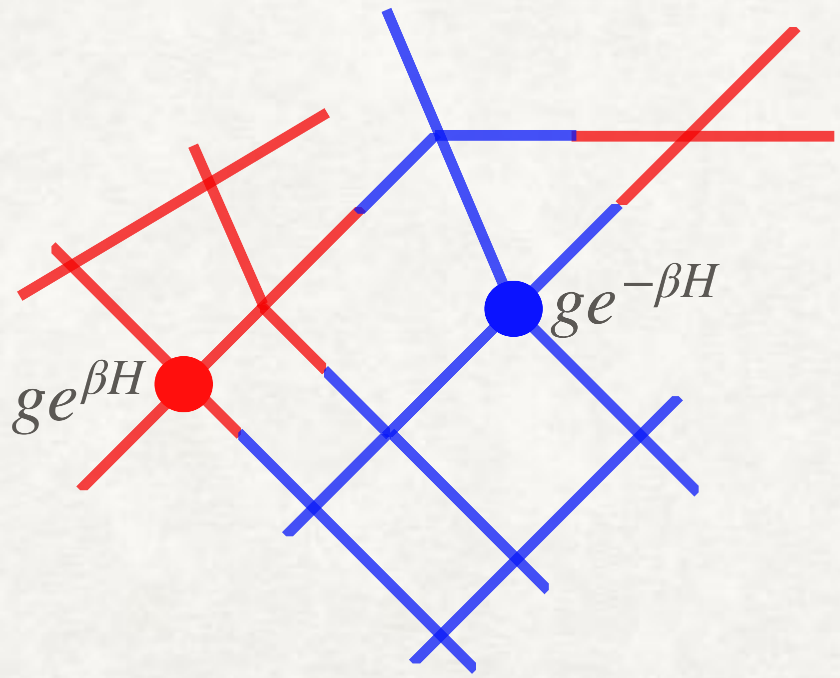
Spontaneous magnetization: Chen-Ning Yang (1952)

ISING MODEL ON PLANAR FEYNMAN GRAPHS $\mathscr{G}$

$$Z_{ISING} = \sum_{\mathscr{G}_{4,N}} \sum_{S_i=\pm 1} \exp\{-\beta \sum_{ij} A_{ij}(\mathscr{G}_{4,N}) S_i S_j - \beta H \sum_j S_j\}$$

ISING MODEL ON QUARTIC GRAPHS WITH FIELD H
IS EQUIVALENT TO THE PLANAR 2-MATRIX MODEL:

$$\int dA\, dB \exp\{-N\mathrm{tr}[A^2 + B^2 - 2e^{-2\beta} AB + g e^{\beta H} A^4 + g e^{-\beta H} B^4]\}$$



Yuri Kazakov (1986)

ISING MODEL ON RANDOM PLANAR GRAPHS

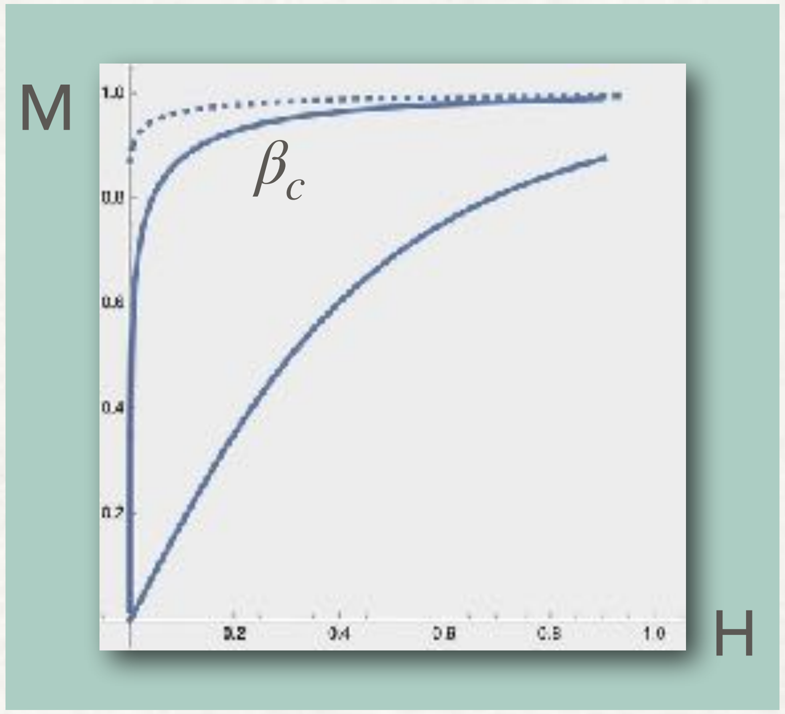
$$\int_{U(n)} dU \exp[\frac{1}{t}\mathrm{tr}(AUBU^\dagger)] = t^{\frac{1}{2}n(n-1)} \frac{\det[\exp \frac{1}{t}(x_i y_j)]}{\Delta(x)\Delta(y)} \prod_{j=0}^{n-1} j!$$

The Harish-Chandra formula (1957)

$$Z_N = \int d\mathbf{x}d\mathbf{y}\Delta(\mathbf{x})\Delta(\mathbf{y})\exp\Big[-N\sum_j (x_j^2 + y_j^2 - 2e^{-2\beta}x_j y_j + ge^{\beta H}x_j^4 + ge^{-\beta H}y_j^4)\Big]$$

Bi-orthogonal polynomials

$$\int dx\, dy P_j(x)Q_k(y)e^{-w(x,y)} = h_k\delta_{jk}$$



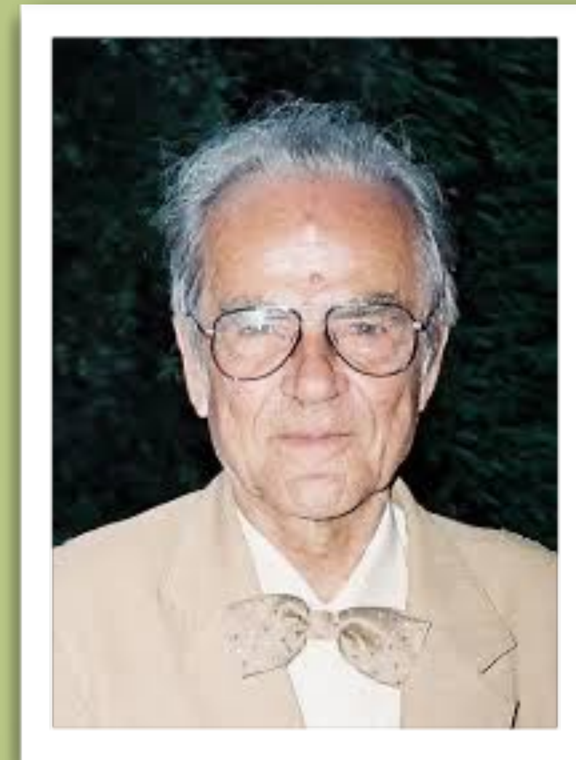
Critical limit = thermodynamic limit

critical exp	α	β	γ	δ	νd	γ_{str}
regular	0	1/8	7/4	15	2	–
random	−1	1/2	2	5	3	−1/3

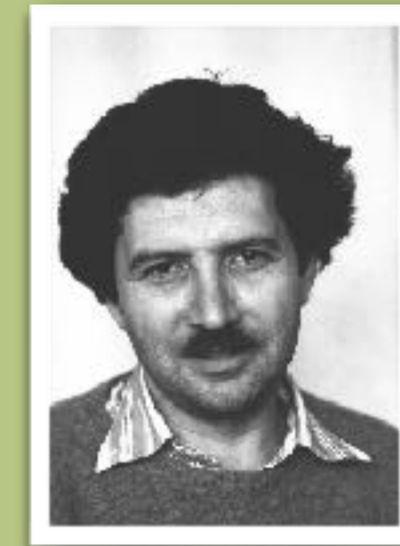
In accordance with CFT on random surfaces (Knizhnik-Polyakov-Zamolodchikov, 1988)
They satisfy the standard relations

Giulio Casati

Italo Guarneri



Boris Chirikov



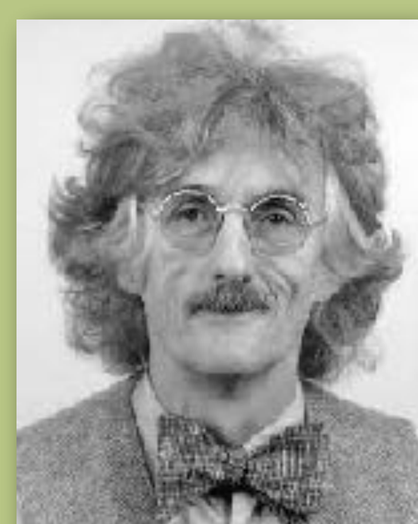
Dima Shepelyansky

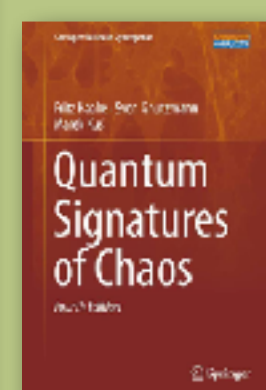
Felix Izrailev

1980: QUANTUM CHAOS

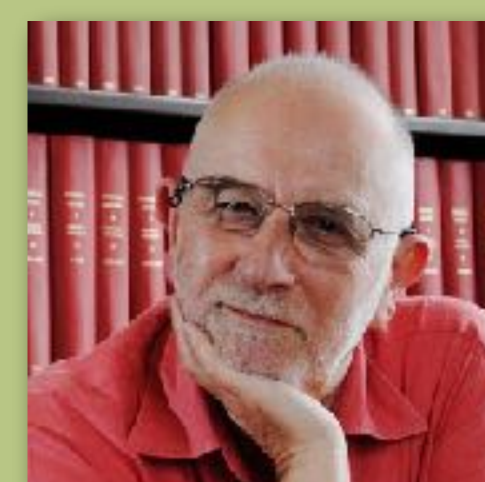
Milano - Novosibirsk

Joe Ford





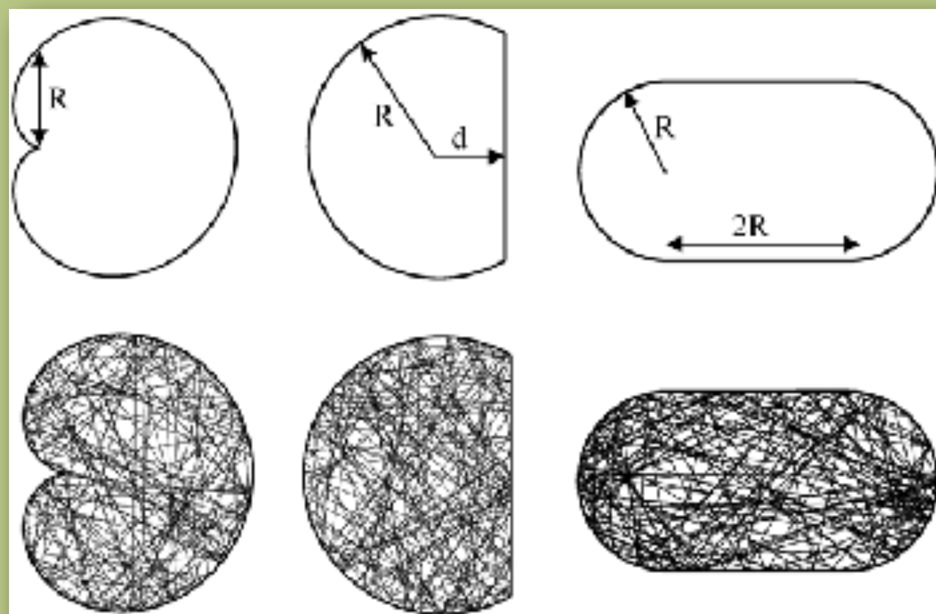
Fritz Haake



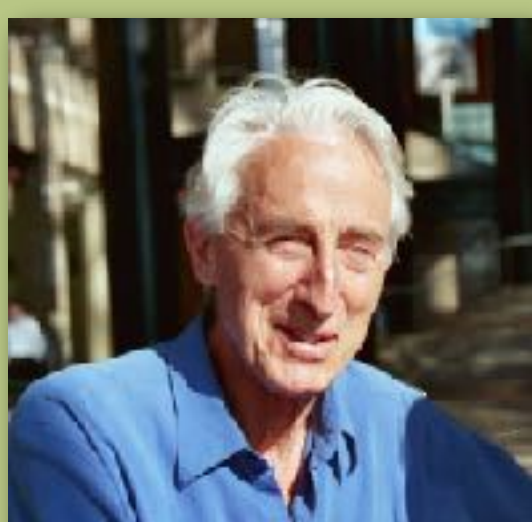
Michael Berry

QUANTUM CHAOS

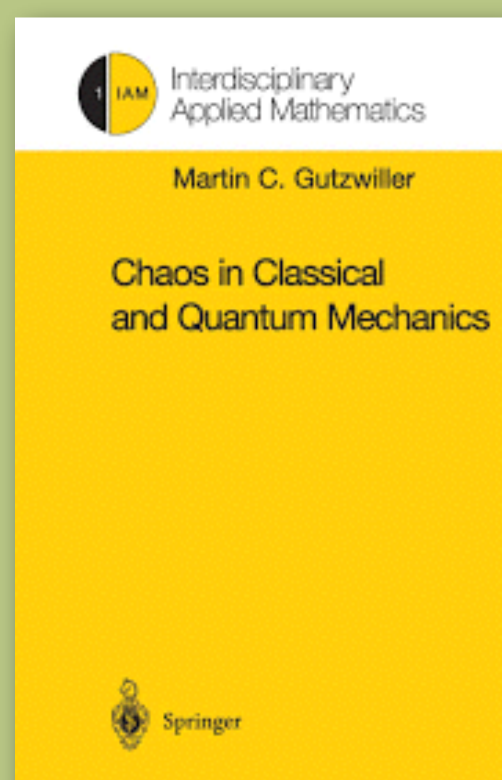
(how does classical chaos show in QM?)

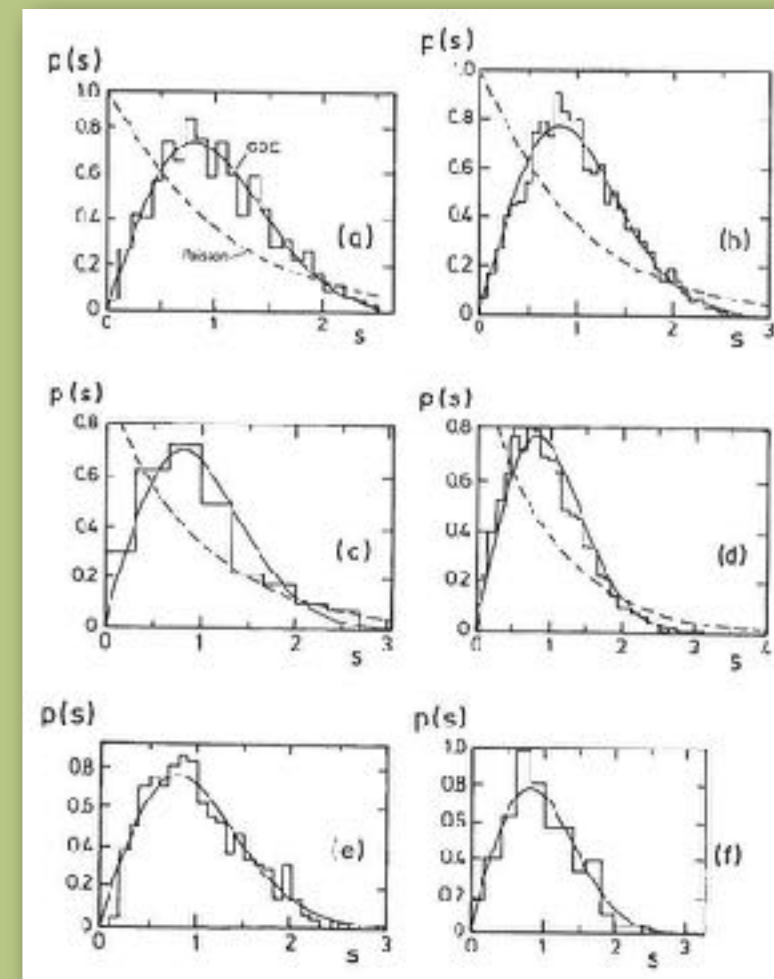


Chaotic Billiards



Martin Gutzwiller





A) Sinai billiard
B) H atom in strong H field
C) NO2 molecule
D) …
E) Spectrum of 3D microwave cavity
F) Frequencies in 1/4 of Sinai stadium

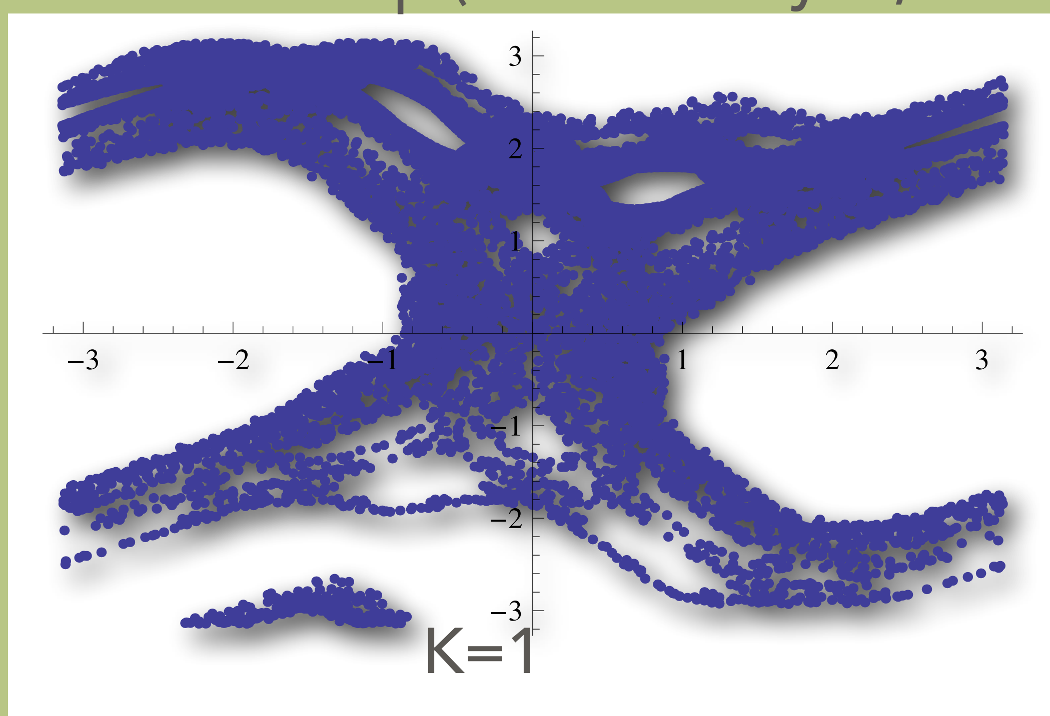
THE KICKED ROTATOR

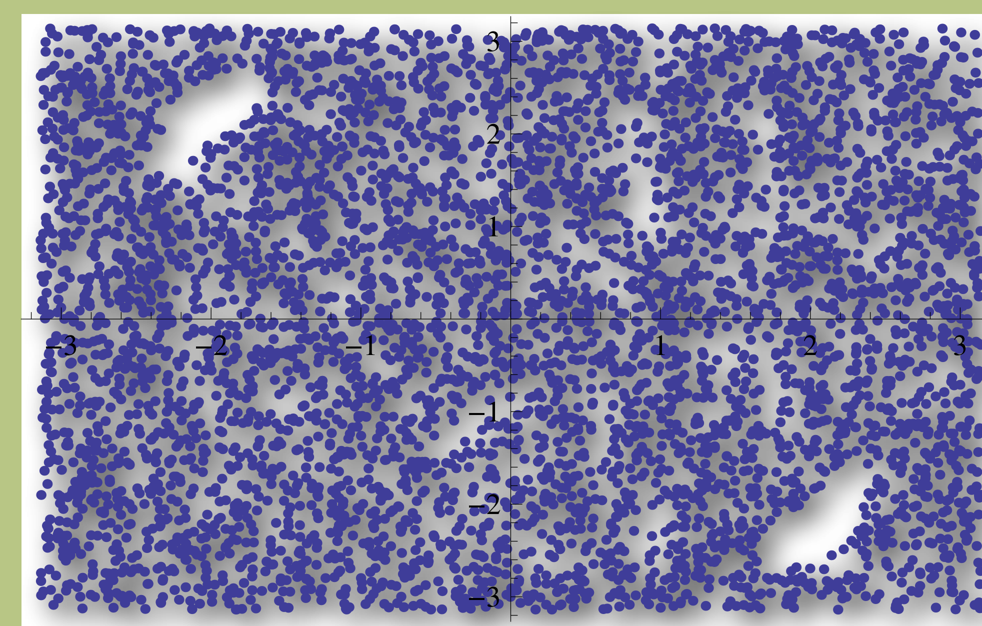
$$H = \frac{1}{2}P^2 + K\cos\theta\sum_n \delta(t-n) \qquad P = -i\frac{d}{d\theta}$$

$$p' = p + K\sin\theta$$
$$\theta' = \theta + p'$$

$$U = e^{-iK\cos\theta}e^{-iP^2/2}$$

Standard Map (Chirikov-Taylor, 1979)



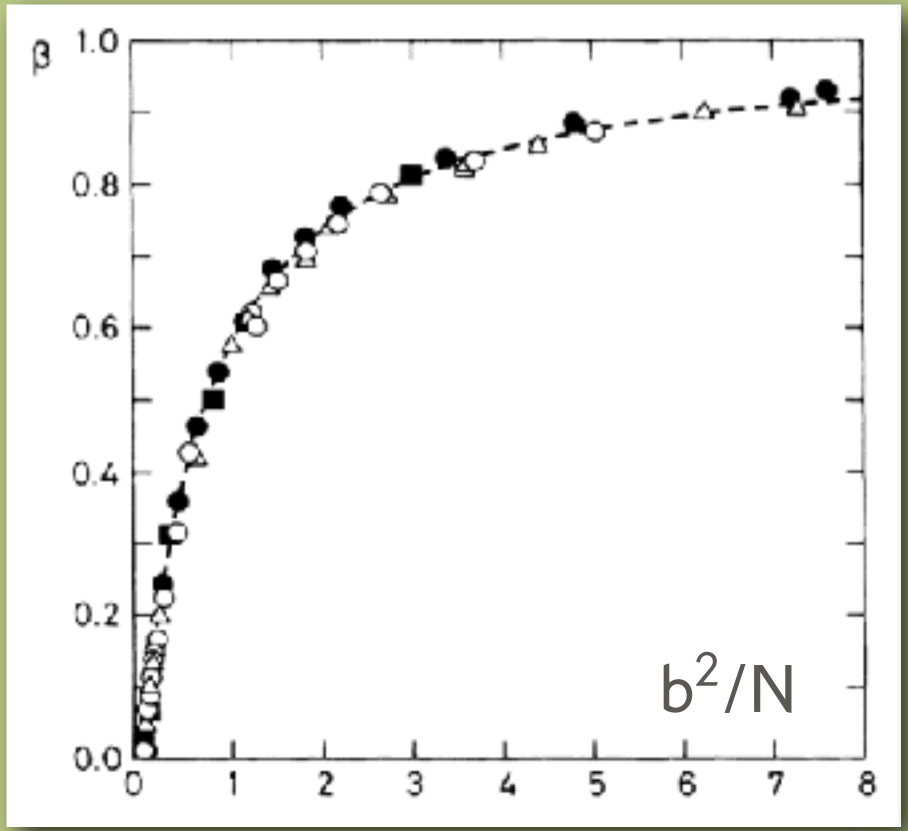


Classical chaos K=5

$K \gg 1$, $\langle p^2 \rangle \propto \frac{1}{2}K^2 t$. In QM the growth of energy stops

The matrix $\langle m | U | m' \rangle$ is ``random” and banded with width K.

Its eigenstates are exponentially localized (dynamical localization) $\xi \approx K^2$



G.Casati, L.Molinari, F. Izrailev, Scaling properties of Band Random Matrices, 1990

Band Random Matrices

$$H = \sum_j u_j^2 \log u_j^2$$

Entropy length of eigenvector $\mathbf{u}$

$$\beta = e^{H-H_{GOE}}$$

Scaling of the localization of eigenvectors

Determinants as Gaussian integrals with commuting and anti-commuting variables.

Average over disorder. The auxiliary variables are now coupled.

Solve in large N limit

Yan Fyodorov

$$\left\langle \frac{\det(M-z)}{\det(M-w)} \right\rangle \qquad \left\langle \frac{\det(M-z)\det(M-z')}{\det(M-w)\det(M-w')} \right\rangle$$

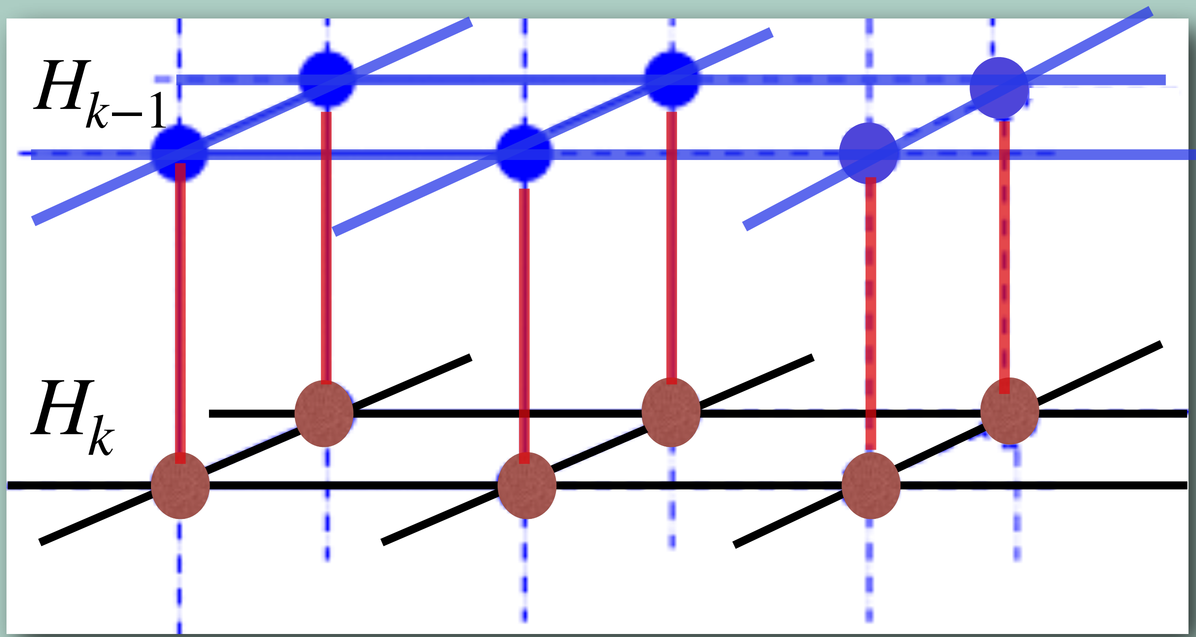
Semicircle law

Scaling of localization

ANDERSON LOCALIZATION

Absence of Diffusion in Certain Random Lattices (1957)

Philip Warren Anderson



Particle in cubic lattice.

In each site a random number in [-W,W]

For an infinite lattice:

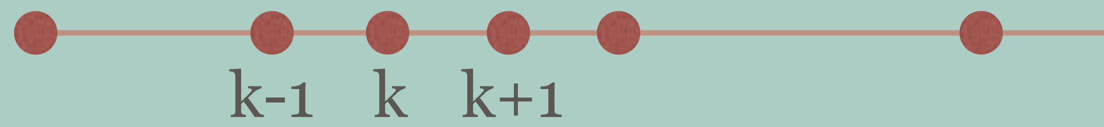
W<Wc extended eigenvectors (metal)

W>Wc exp localized eigenvector (insulator)

$$\begin{pmatrix} H_1 & 1 & & & \\ 1 & H_2 & 1 & & \\ & & & & 1 \\ & & & 1 & H_n \end{pmatrix}$$

ANDERSON LOCALIZATION IN D=1

$$\begin{bmatrix} \epsilon_1 & 1 & & \\ 1 & \epsilon_2 & & \\ & & \ddots & 1 \\ & & 1 & \epsilon_N \end{bmatrix}$$



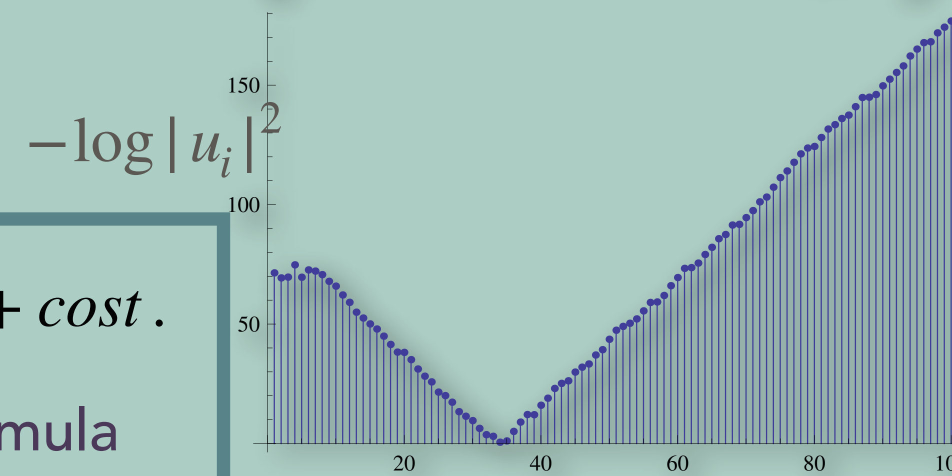
$$u_{k+1} + u_{k-1} + \epsilon_k u_k = E u_k$$

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = T(E) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

$$u_n \approx \exp[\pm n \xi(E)]$$

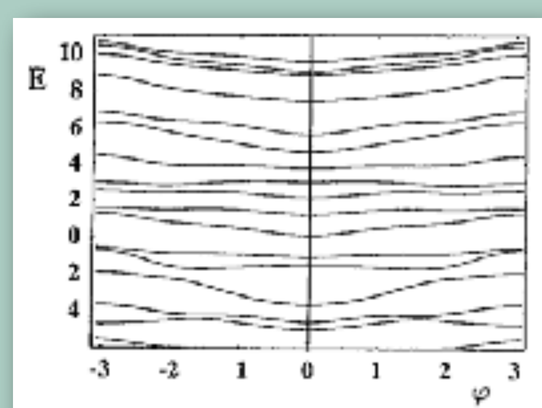
$$\xi(E) = \int dE' \rho(E') \log|E - E'| + cost\,.$$

Herbert-Jones-Thouless formula



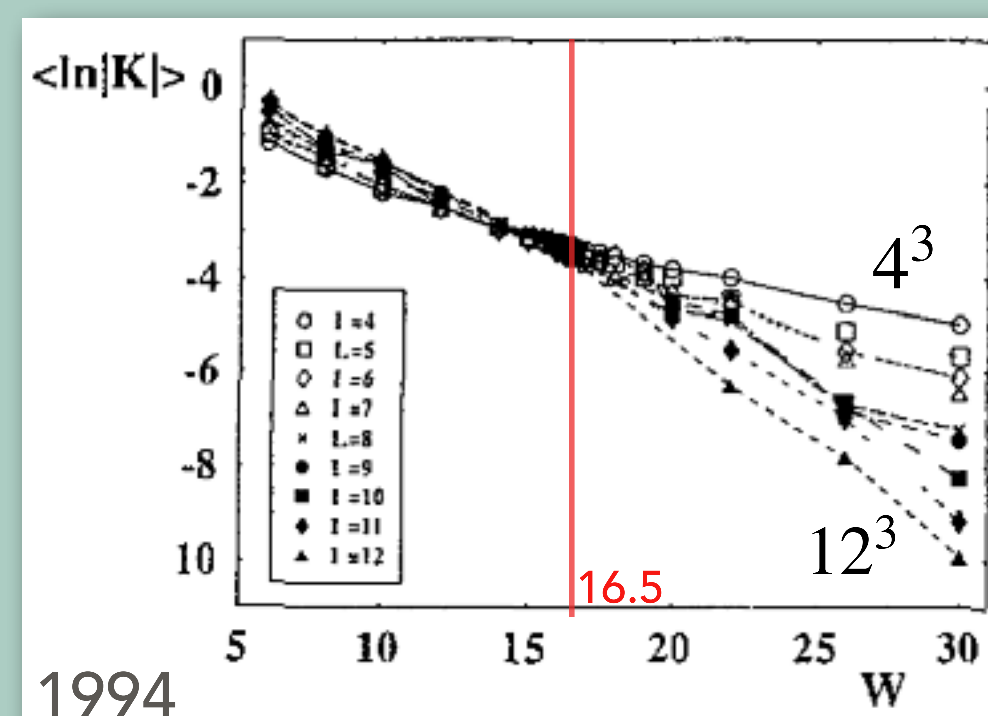
LOCALISATION THROUGH ENERGY-LEVEL CURVATURES D=3

Karol Życzkowski



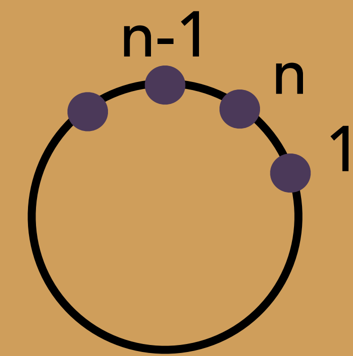
Motion of eigenvalues with Bloch phase

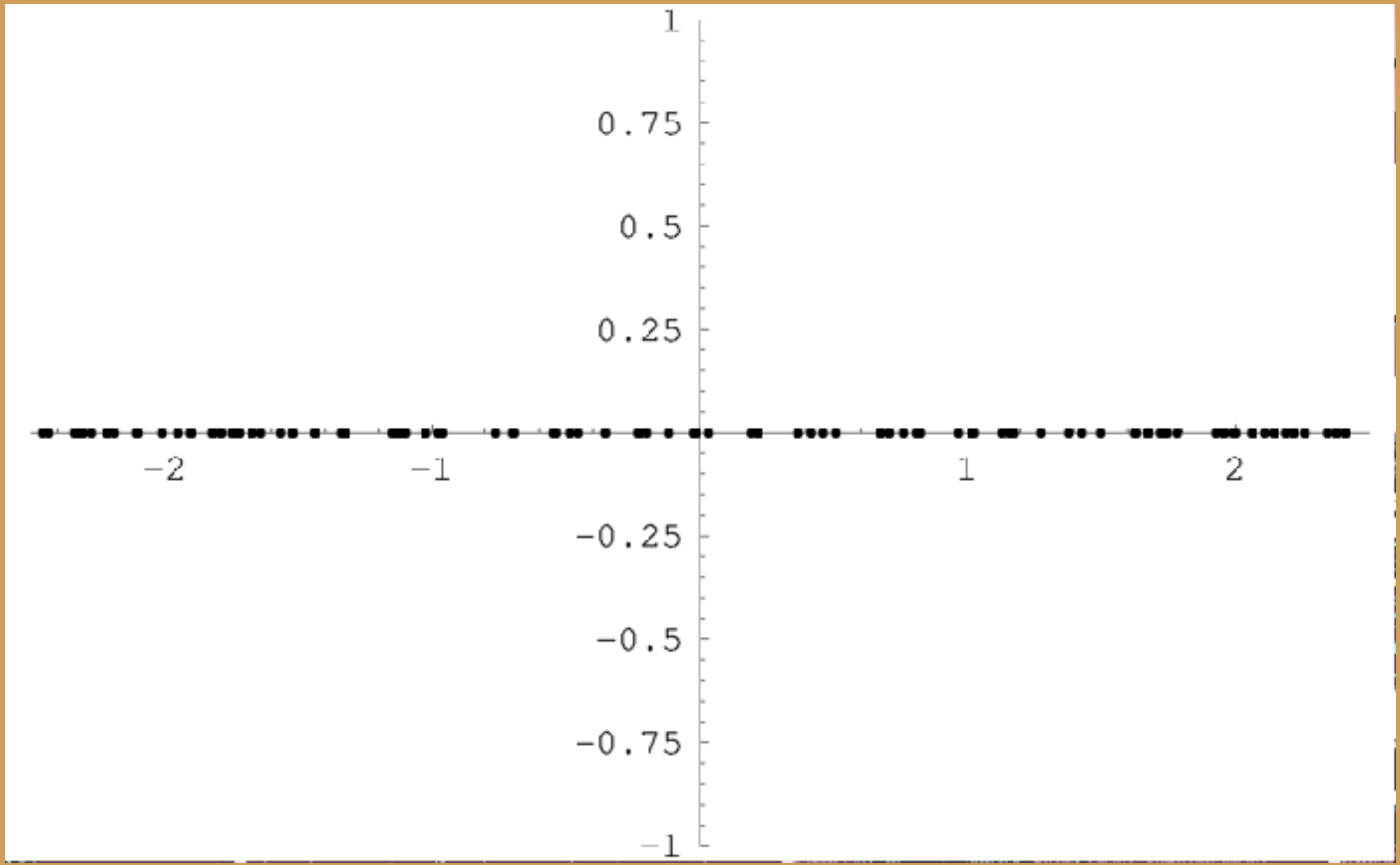
$$K = \frac{E''(0)}{\Delta}$$ level curvature



Hatano-Nelson (PRL 1996)

$$e^{\xi}u_{k+1} + \epsilon_k u_k + e^{-\xi}u_{k-1} = Eu_k$$



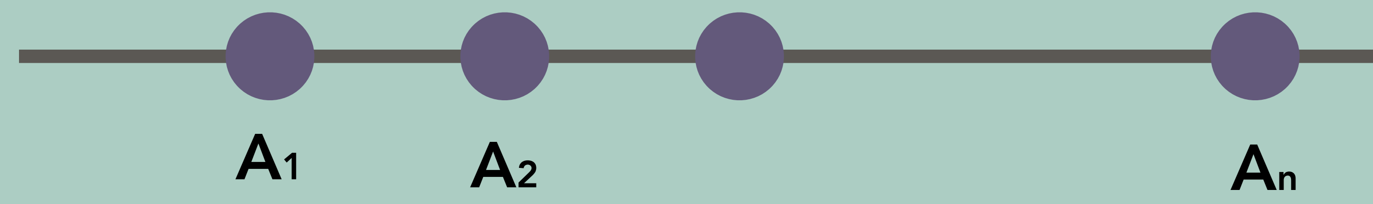


$$u_k = e^{k\xi}v_k$$

$$v_{k+1} + \epsilon_k v_k + v_{k-1} = Ev_k$$

$$v_{n+1} = e^{n\xi}v_1, \quad v_0 = e^{-n\xi}v_n$$

DIFFERENCE EQUATION & TRANSFER MATRIX



$$C_k u_{k-1} + A_k u_k + B_k u_{k+1} = E u_k \,, \qquad k = 1 \ldots n, \quad u_k \in \mathbb{C}^m$$

$$\begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} B_k^{-1}(E - A_k) & -B_k^{-1} C_k \\ \mathbb{I}_m & 0 \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}$$

$$T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$$

A_k, B_k, C_k are m x m matrices

The eigenvalues of T(E)?

SPECTRAL DUALITY: z is eigenvalue of T(E) iff E is eigenvalue of H(z)

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$$T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = z \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

$$H(z) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z} C_1 \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ z B_n & & C_n & A_n \end{bmatrix}$$

THE SPECTRAL DUALITY

$$\det[z\mathbb{I}_{2m} - T(E)] = (-z)^m \frac{\det[E\mathbb{I}_{nm} - H(z)]}{\det(B_1 \cdots B_n)}$$

$$\xi_k =: \frac{1}{n} \ln |z_k|$$

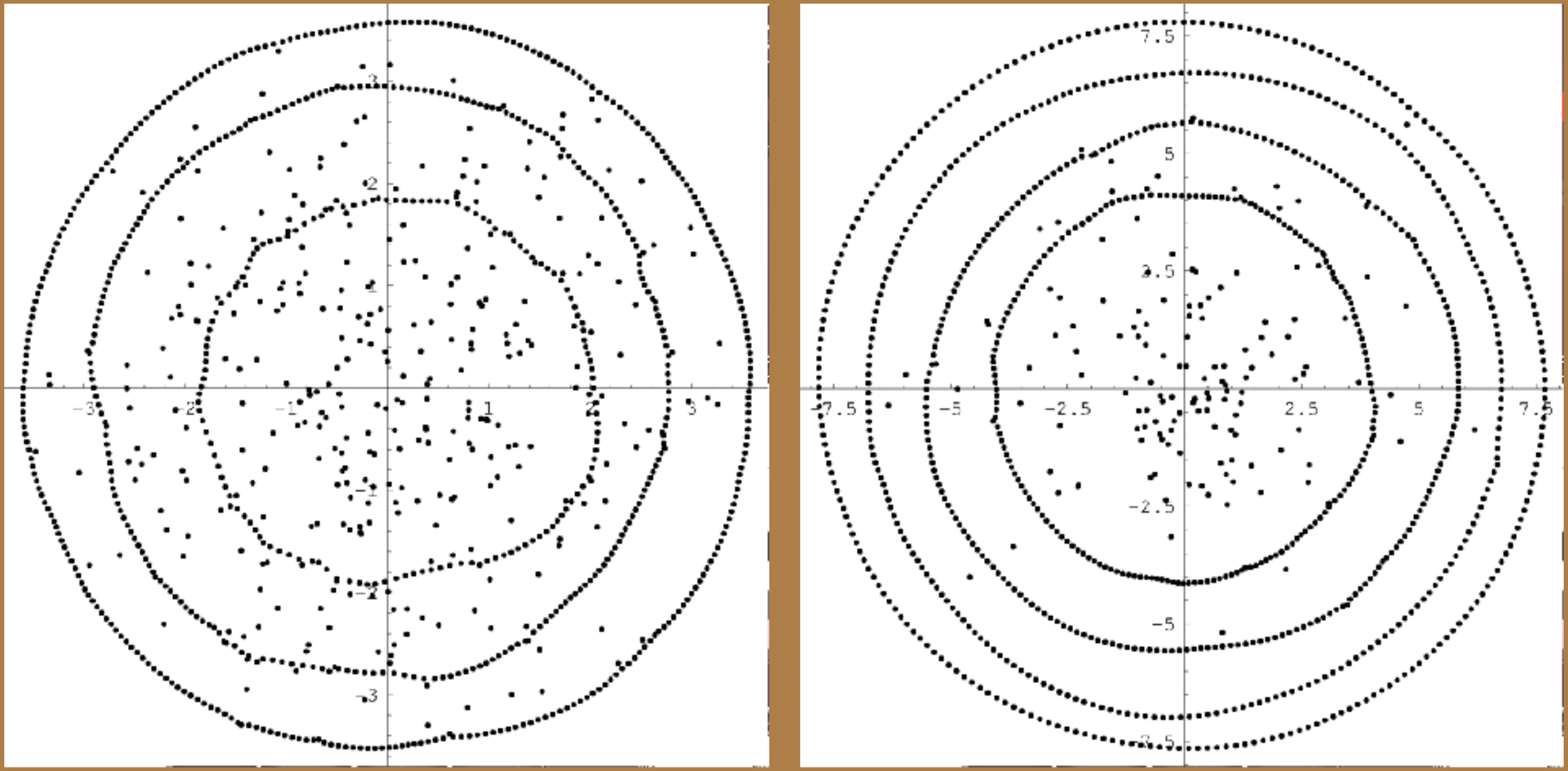
The exponents of T(E)

Theorem **(Jensen)** *If f is holomorphic and $f(0) \neq 0$, and $z_1 \dots z_n$ are its zeros in the disk of radius r, then:* $\int_0^{2\pi} \frac{d\theta}{2\pi} \ln |f(re^{i\theta})| = \ln |f(0)| - \sum_k \ln(|z_k|/r)$.

$$\sum_{k=1}^{2m} \xi_k \, \theta(\xi_k) = \frac{1}{n} \int_0^{2\pi} \frac{d\theta}{2\pi} \ln \left|\det[H(e^{i\theta}) - E]\right| - \frac{1}{n} \ln |\det [B_1 \cdots B_n]|$$

<u>Exact deterministic formula</u>, analogous to the probabilistic Kunz-Souillard formula

$$\det[z\mathbb{I}_{2m} - T(E)] = (-z)^m \frac{\det[E\mathbb{I}_{nm} - H(z)]}{\det(B_1 \cdots B_n)}$$

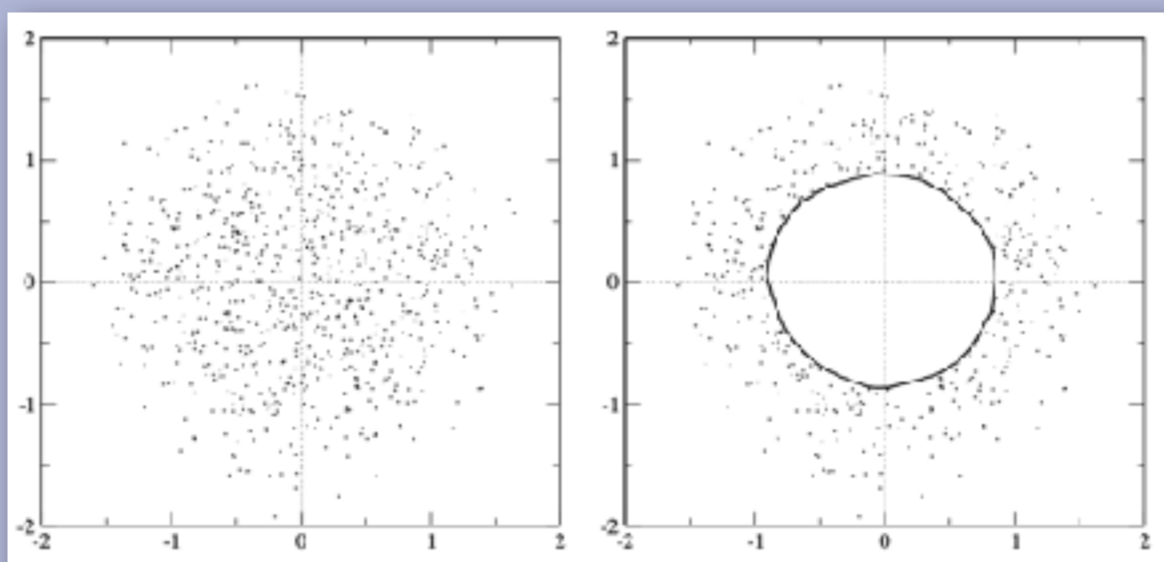


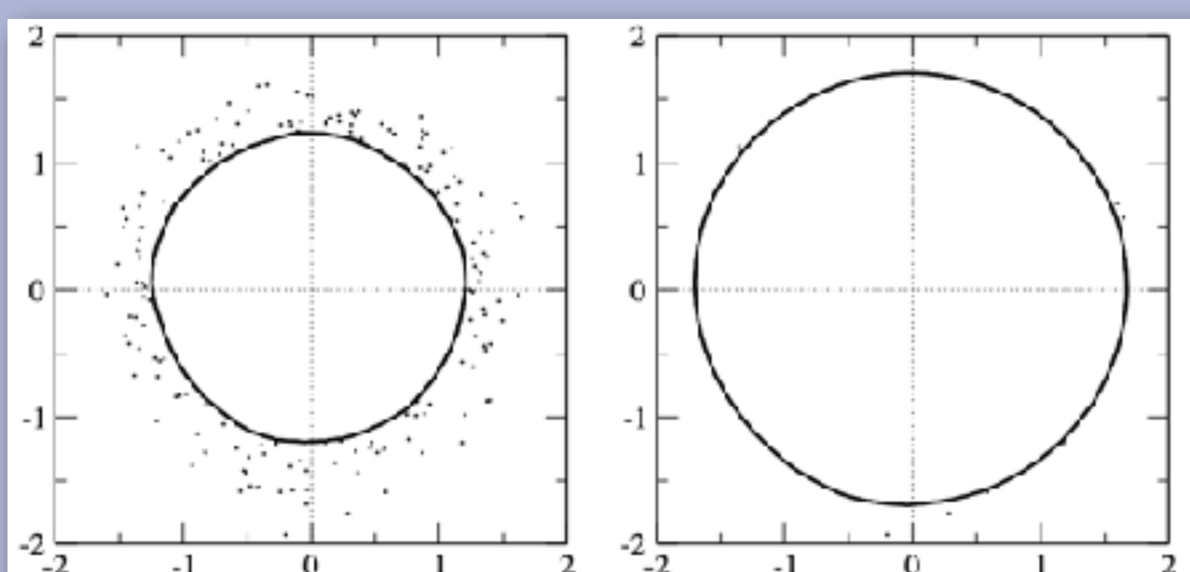
Eigenvalues occupy spectral curves, that manifest when perturbed

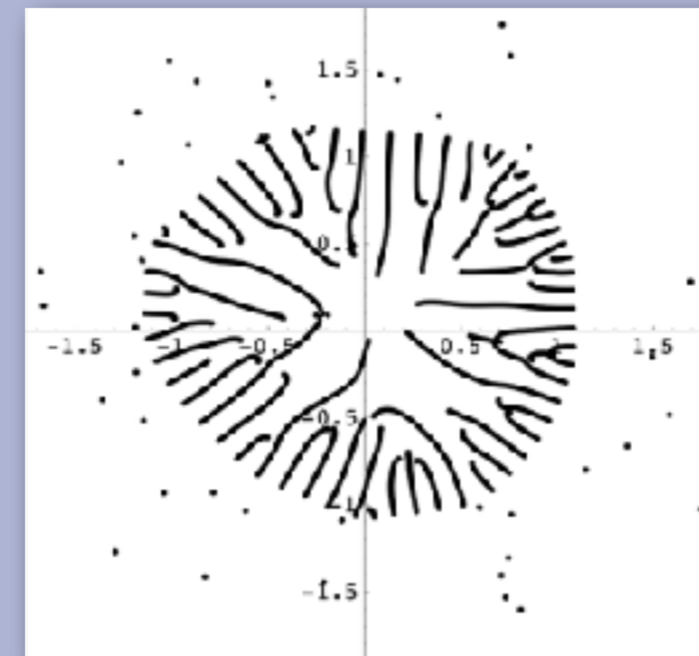
MOTION OF ENERGY EIGENVALUES

$$M(z^n) = \begin{bmatrix} a_1 & b_1 & & z^n c_1 \\ c_2 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ b_n/z^n & & c_n & a_n \end{bmatrix}$$

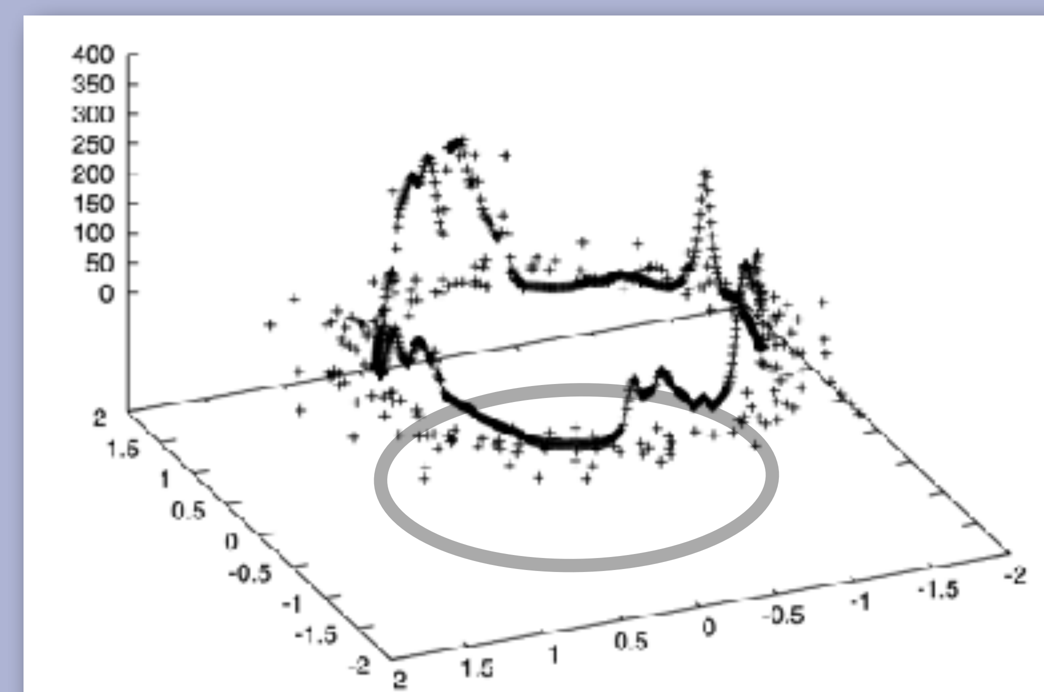
$$z = e^{\xi + i\varphi}$$







motion in ξ



Eigenvalues (x,y)
and size of eigenvectors (z axis)

DETERMINISTIC EXPONENTIAL LOCALIZATION

Theorem **(Demko, Moss and Smith)** *Let A be a positive definite block tridiagonal matrix, with square blocks of size m, let $[a,b]$ be the smallest interval containing the spectrum of A, let $A^{-1}[i,j]$ be any matrix element in the block $(A^{-1})_{ij}$. Then:*

$$\left|A^{-1}[i,j]\right| \leq \begin{cases} C\, q^{|i-j|} & \text{for } |i-j| \geq 1 \\ 1/a & \text{for } i=j \end{cases}$$

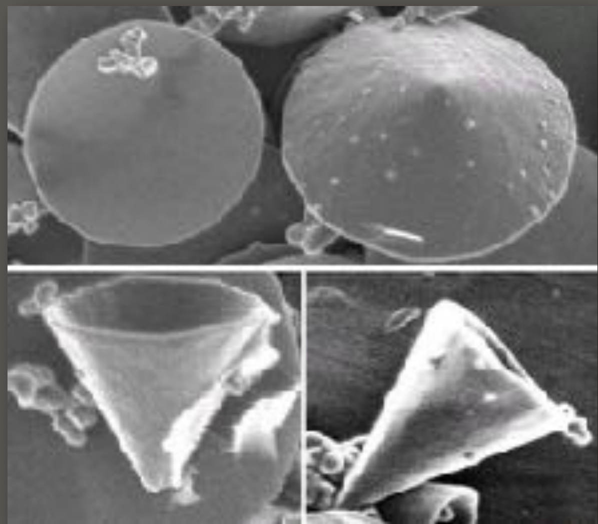
$$C = \frac{(\sqrt{b}+\sqrt{a})^2}{2ab}, \qquad q = \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}$$

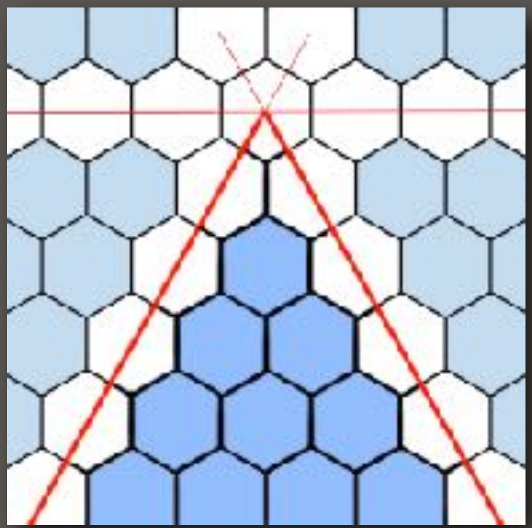
Theorem extends with changes to a block matrix $A = A^{\dagger}(AA^{\dagger})^{-1}$

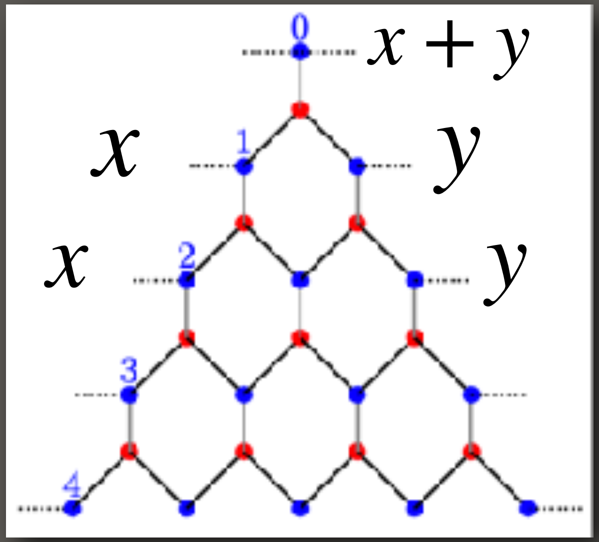
$$(\mathrm{H}-\mathrm{E})^{-1} = \begin{pmatrix} G_{11} & \dots & G_{1n} \\ \cdots & \cdots & \cdots \\ G_{n1} & \dots & G_{nn} \end{pmatrix} \qquad \mathrm{T(E)} = \begin{pmatrix} -G_{1n}^{-1} & -G_{1n}^{-1}G_{11} \\ G_{nn}G_{1n}^{-1} & G_{nn}G_{1n}^{-1}G_{11} - G_{n1} \end{pmatrix}$$

If n >>1 the transfer matrix T(E) has m singular values larger than $\mathrm{K}q^{-n/2}$ and m singular values smaller than $\mathrm{K}q^{n/2}$ [q is evaluated for $(H-E)^{\dagger}(H-E)$]

GRAPHENE NANOCONES AND PASCAL MATRICES







$$H_2 = \left[\begin{array}{c|ccc|ccccc} S_0 & 0 & 1 & & & & & & \\ \hline 0 & 0 & 1 & y_1 & 0 & 1 & & & \\ 1 & 1 & 0 & 1 & 0 & & 0 & & \\ & x_1 & 1 & 0 & 0 & & & 1 & \\ \hline & 0 & 0 & 0 & 0 & 1 & & & y_2 \\ & 1 & & & 1 & 0 & 1 & & \\ & & 0 & & & 1 & 0 & 1 & \\ & & & 1 & & & 1 & 0 & 1 \\ & & & & x_2 & & & 1 & 0 \end{array}\right]$$

$$\det H_0 = x + y$$
$$\det H_1 = x^2 + 3xy + y^2$$
$$\det H_2 = x^3 + 9x^2y + 9xy^2 + y^3$$
$$\det H_3 = x^4 + 29x^3y + 72x^2y^2 + 29xy^3 + y^4$$
$$\det H_4 = x^5 + 99x^4y + 626x^3y^2 + 626x^2y^3 + 99xy^4 + y^5$$
$$\det H_5 = x^6 + 351x^5y + 6084x^4y^2 + 13869x^3y^3 + 6084x^2y^4 + 351xy^5 + y^6$$

Conjecture for the determinant

from size $n^2 \times n^2$

to size $n \times n$

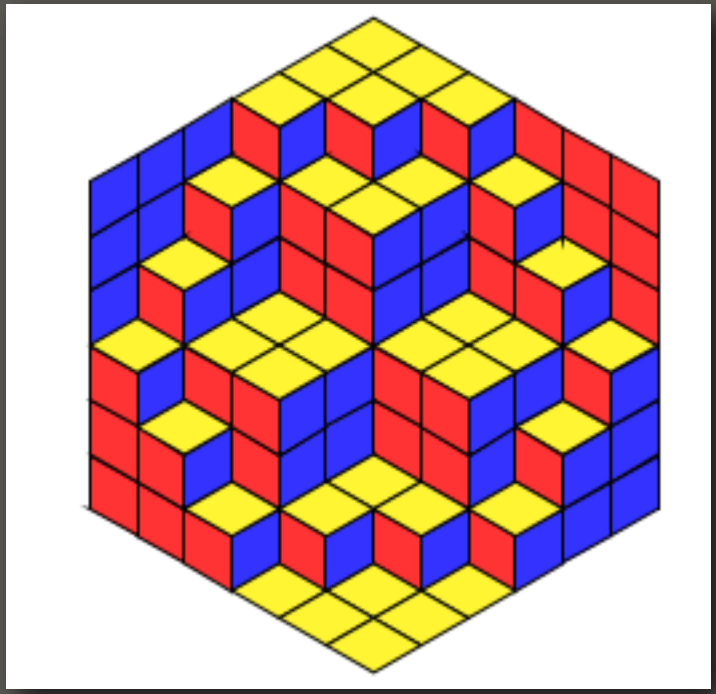
$$\begin{bmatrix} x_n + y_n & -\binom{n}{1}y_n & \binom{n}{2}y_n & -\binom{n}{3}y_n & \cdots & \pm\binom{n}{n}y_n \\ \binom{n}{1}x_n & x_{n-1}+y_{n-1} & -\binom{n-1}{1}y_{n-1} & \binom{n-1}{2}y_{n-1} & \cdots & \mp\binom{n-1}{n-1}y_{n-1} \\ \binom{n}{2}x_n & \binom{n-1}{1}x_{n-1} & x_{n-2}+y_{n-2} & -\binom{n-2}{1}y_{n-2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & x_1+y_1 & -\binom{1}{1}y_1 \\ \binom{n}{n}x_n & \binom{n-1}{n-1}x_{n-1} & \binom{n-2}{n-2}x_{n-2} & \cdots & \binom{1}{1}x_1 & x_0+y_0 \end{bmatrix}$$

A surprising connection with the PASCAL matrix

$$L_n = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \qquad L_n^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \qquad Q_n = L_n L_n^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

$$\det H_n(e^{-i\theta}, e^{i\theta}) = e^{-i(n+1)\theta} \det(Q_n + e^{2i\theta})$$

computed for special angles and all n by Andrews, Ciucu, Krattenthaler, Nienhuis and Mitra, in enumerative combinatorics of plane partitions (Mac Mahon, Stanley), lozenge tilings, dense loops on a cylinder (very hard work!)



n	$\theta = 0$	$\pi/6$	$\pi/3$	$\pi/2$	$\pi/4$
2	20	$3^2\sqrt{3}$	7	0	$8\sqrt{2}$
3	132	10^2	42	2^4	70
4	1452	$25^2\sqrt{3}$	429	0	$526\sqrt{2}$
5	26741	140^2	7436	7^4	13167
6	826540	$588^2\sqrt{3}$	218348	0	$280772\sqrt{2}$

Thank you for your attention