

1479	-2.46841	-3.29238	-6.61109	0.74621	-3.43432	-1.03865	-0.30422	2.40855	3.57606	-1.11509	3.20497	7.20177	8.1657	-0.116565	-3.1473
6841	-7.8243	5.14281	-7.19343	-1.20332	1.84345	4.16636	2.39665	0.0208252	-3.57817	-4.14458	-0.47492	0.914608	-6.33641	-3.01087	-1.8766
9238	5.14281	-4.03947	2.43619	0.455571	-2.78473	2.88738	-2.08562	7.88945	-4.85517	-5.51583	4.88178	-6.15698	3.06304	-4.25845	-1.8765
1109	-7.19343	2.43619	2.12356	-2.86321	-0.540309	-4.98338	-2.60091	5.26163	-1.38449	-0.263283	-2.83463	-3.98221	-2.21576	6.16042	4.2722
4621	-1.20332	0.455571	-2.86321	9.5065	-3.2708	2.67314	-1.98862	-2.10908	-2.58889	-1.19255	6.32787	2.66516	3.28843	4.46772	-3.6610
3432	1.84345	-2.78473	-0.540309	-3.2708	4.58452	0.494782	4.74231	-5.31854	-5.67096	-0.676868	-0.998946	7.88187	3.24483	8.09262	1.6255
3865	4.16636	2.88738	-4.98338	2.67314	0.494782	0.75548	-6.04531	0.171377	-0.348583	0.707132	-1.81468	0.103862	-0.244199	-5.99486	6.5466
0422	2.39665	-2.08562	-2.60091	-1.98862	4.74231	-6.04531	8.67484	0.572978	4.46651	-2.23802	3.16206	4.47344	-5.73083	0.68381	0.83379
0855	0.0208252	7.88945	5.26163	-2.10908	-5.31854	0.171377	0.572978	3.75263	8.33161	3.57465	1.43892	1.29786	-7.85634	5.08369	5.6460
7606	-3.57817	-4.85517	-1.38449	-2.58889	-5.67096	-0.348583	4.46651	8.33161	0.939226	-1.41935	5.91457	0.00283354	0.251408	0.857021	7.1394
1509	-4.14458	-5.51583	-0.263283	-1.19255	-0.676868	0.707132	-2.23802	3.57465	-1.41935	-2.47487	-8.46501	-2.72423	4.34547	0.031314	-0.9032
0497	-0.47492	4.88178	-2.83463	6.32787	-0.998946	-1.81468	3.16206	1.43892	5.91457	-8.46501	7.71372	-2.68265	2.2555	6.73738	-0.8071
0177	0.914608	-6.15698	-3.98221	2.66516	7.88187	0.103862	4.47344	1.29786	0.00283354	-2.72423	-2.68265	7.29527	1.967	-5.45142	0.38577
657	-6.33641	3.06304	-2.21576	3.28843	3.24483	-0.244199	-5.73083	-7.85634	0.251408	4.34547	2.2555	1.967	2.29342	-3.08668	0.94911
16565	-3.01087	-4.25845	6.16042	4.48772	8.09262	-5.99486	0.68381	5.08369	0.857021	0.031314	6.73738	-5.45142	-3.08668	-1.8623	-4.1889
4732	-1.87663	-1.87652	4.27227	-3.68105	1.62552	6.5466	0.833792	5.64605	7.1394	-0.903284	-0.807175	0.385775	0.949119	-4.18897	8.1294
1673	-0.174213	-7.20465	-8.09404	-0.878843	-2.49092	-4.51996	3.15598	-0.514373	-2.96233	-4.46197	3.81351	-0.539797	-1.41291	0.899056	3.1794
4111	-2.4841	-7.66405	3.51651	-4.6014	0.803143	2.17515	-2.09861	-0.0094157	3.37396	5.46396	0.675944	1.64131	4.10748	0.191952	-5.45142

A journey through random matrices

Luca Guido Molinari
Università degli Studi di Milano

8051	0.936109	-3.22775	-1.40967	1.04987	2.20804	2.05859	-2.74987	-1.64543	3.35408	4.29874	-5.02125	6.52605	2.01761	3.59206	0.6454
673	-4.71971	3.60494	-4.71429	6.40789	-4.49126	2.55534	6.63397	-0.945923	3.215	5.98604	-1.60141	-0.129235	2.42979	7.49154	4.0145
1479	-2.46841	-3.29238	-6.61109	0.74621	-3.43432	-1.03865	-0.30422	2.40855	3.57606	-1.11509	3.20497	7.20177	8.1657	-0.116565	-3.1473
6841	-7.8243	5.14281	-7.19343	-1.20332	1.84345	4.16636	2.39665	0.0208252	-3.57817	-4.14458	-0.47492	0.914608	-6.33641	-3.01087	-1.8766
9238	5.14281	-4.03947	2.43619	0.455571	-2.78473	2.88738	-2.08562	7.88945	-4.85517	-5.51583	4.88178	-6.15698	3.06304	-4.25845	-1.8765
1109	-7.19343	2.43619	2.12356	-2.86321	-0.540309	-4.98338	-2.60091	5.26163	-1.38449	-0.263283	-2.83463	-3.98221	-2.21576	6.16042	4.2722
621	-1.20332	0.455571	-2.86321	9.5065	-3.2708	2.67314	-1.98862	-2.10908	-2.58889	-1.19255	6.32787	2.66516	3.28843	4.46772	-3.6610
3432	1.84345	-2.78473	-0.540309	-3.2708	4.58452	0.494782	4.74231	-5.31854	-5.67096	-0.676868	-0.998946	7.88187	3.24483	8.09262	1.6255
3865	4.16636	2.88738	-4.98338	2.67314	0.494782	0.75548	-6.04531	0.171377	-0.348583	0.707132	-1.81468	0.103862	-0.244199	-5.99486	6.5466
0422	2.39665	-2.08562	-2.60091	-1.98862	4.74231	-6.04531	8.67484	0.572978	4.46651	-2.23802	3.16206	4.47344	-5.73083	0.68381	0.83379
855	0.0208252	7.88945	5.26163	-2.10908	-5.31854	0.171377	0.572978	3.75263	8.33161	3.57465	1.43892	1.29786	-7.85634	5.08369	5.6460
606	-3.57817	-4.85517	-1.38449	-2.58889	-5.67096	-0.348583	4.46651	8.33161	0.939226	-1.41935	5.91457	0.00283354	0.251408	0.857021	7.1394
1509	-4.14458	-5.51583	-0.263283	-1.19255	-0.676868	0.707132	-2.23802	3.57465	-1.41935	-2.47487	-8.46501	-2.72423	4.34547	0.031314	-0.9032
497	-0.47492	4.88178	-2.83463	6.32787	-0.998946	-1.81468	3.16206	1.43892	5.91457	-8.46501	7.71372	-2.68265	2.2555	6.73738	-0.8071
177	0.914608	-6.15698	-3.98221	2.66516	7.88187	0.103862	4.47344	1.29786	0.00283354	-2.72423	-2.68265	7.29527	1.967	-5.45142	0.38577
557	-6.33641	3.06304	-2.21576	3.28843	3.24483	-0.244199	-5.73083	-7.85634	0.251408	4.34547	2.2555	1.967	2.29342	-3.08668	0.94911
6565	-3.01087	-4.25845	6.16042	4.48772	8.09262	-5.99486	0.68381	5.08369	0.857021	0.031314	6.73738	-5.45142	-3.08668	-1.8623	-4.1889
4732	-1.87663	-1.87652	4.27227	-3.68105	1.62552	6.5466	0.833792	5.64605	7.1394	-0.903284	-0.807175	0.385775	0.949119	-4.18897	8.1294
1673	-0.174213	-7.20465	-8.09404	-0.878843	-2.49092	-4.51996	3.15598	-0.514373	-2.96233	-4.46197	3.81351	-0.539797	-1.41291	0.899056	3.1794

Gandria - 7 feb 2025



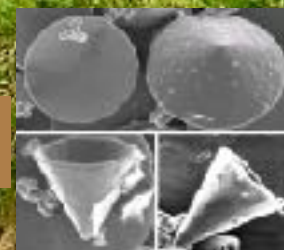
1 MATRIX MODELS
1/N
STATISTICAL MECH ON
RANDOM SURFACES

2 QUANTUM CHAOS
KICKED ROTATOR
BANDED RANDOM MATRICES



3 BLOCK MATRICES
ANDERSON LOCALISATION
A SPECTRAL DUALITY

4 NANOCONES AND PASCAL MATRICES



THE CLASSICAL GROUPS



Adolf Hurwitz

$SO(N)$ parametrization with Euler angles (Hurwitz, **1897**)

Measure invariant under the action of the group

$SU(N)$ unitary complex matrices

$Sp(2N)$ matrices with quaternions, (**1939**)



Hermann Weyl

MULTIVARIATE STATISTICS

From scalars to vectors $\mathbf{x}$: the multivariate normal distribution

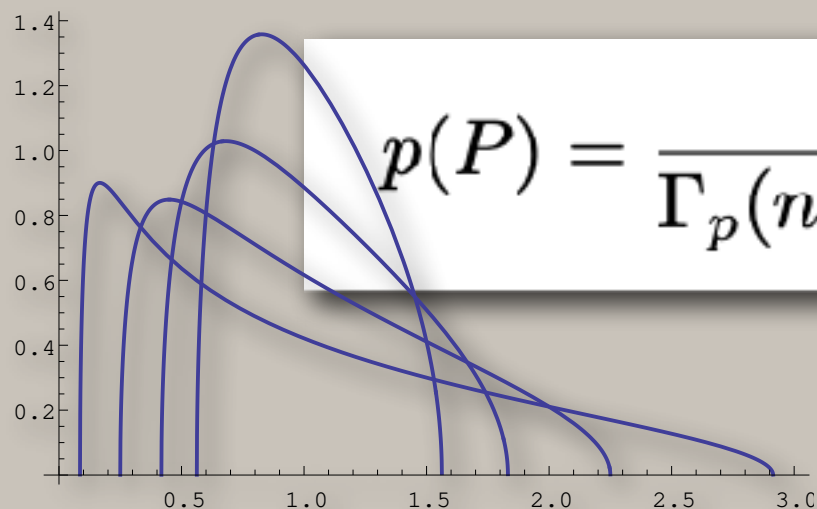
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} \sqrt{\det \Sigma}} \exp \left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

From vectors to matrices $P > 0$: the Wishart distribution (**1928**)

(the extension of χ_n^2 for $x_1^2 + \dots + x_n^2$ to $P = XX^T$)



John Wishart

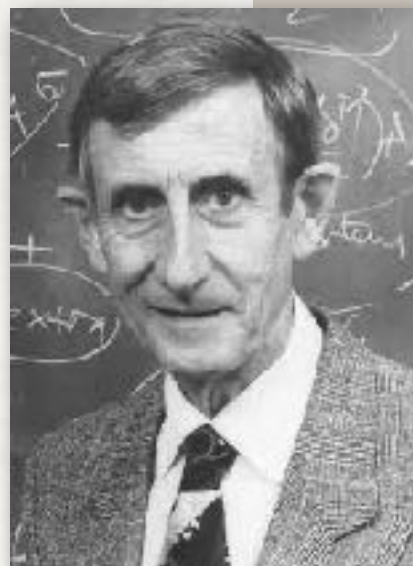
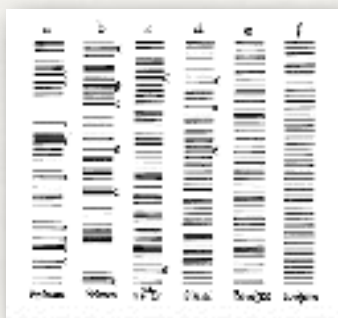


$$p(P) = \frac{1}{\Gamma_p(n/2) (2^p \det \Sigma)^{n/2}} (\det P)^{n-p-1} \exp \left[-\frac{1}{2} \text{tr}(\Sigma^{-1} P) \right]$$

The Marchenko-Pastur distribution of the eigenvalues of Wishart matrices, $\Sigma = 1$, various ratios p/n (1967)

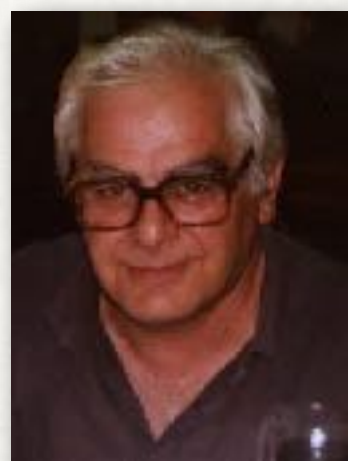
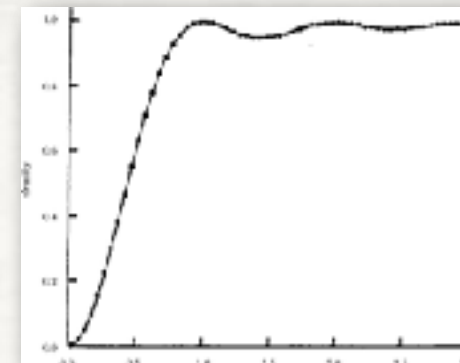


Eugene Wigner



Freeman Dyson

... a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself (RM as ensembles of Hamiltonians)



Oriol Bohigas



Edouard Brezin



Jan Ambjorn



Madan Lal Mehta



Jean B. Zuber



Anthony Zee



Leonid Pastur



Giorgio Parisi



Gernot Akemann



Peter Forrester



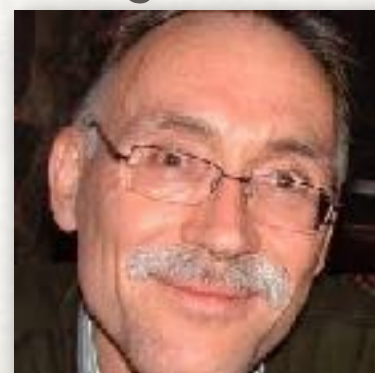
Alan Edelman



Hans Weidenmuller



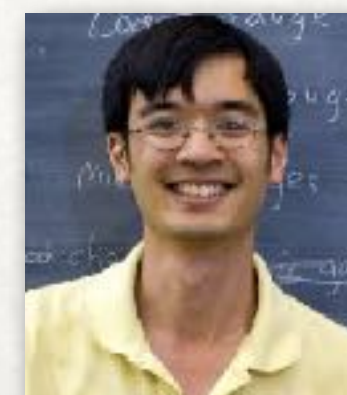
Martin Zirnbauer



Jacobus Verbaarschot



Vyacheslav Girko



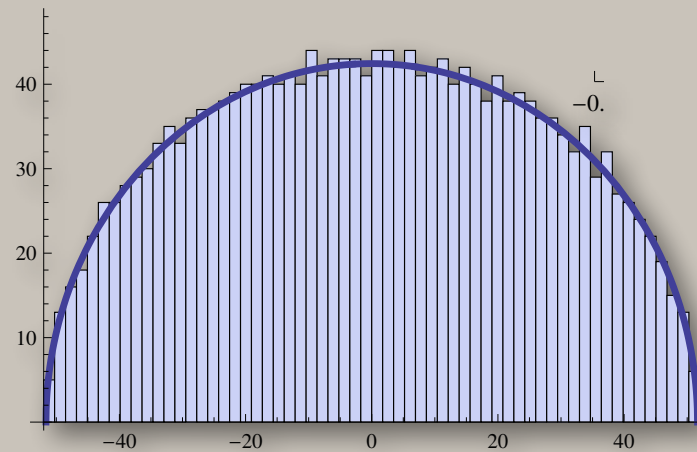
Terence Tao

GAUSSIAN RANDOM MATRICES

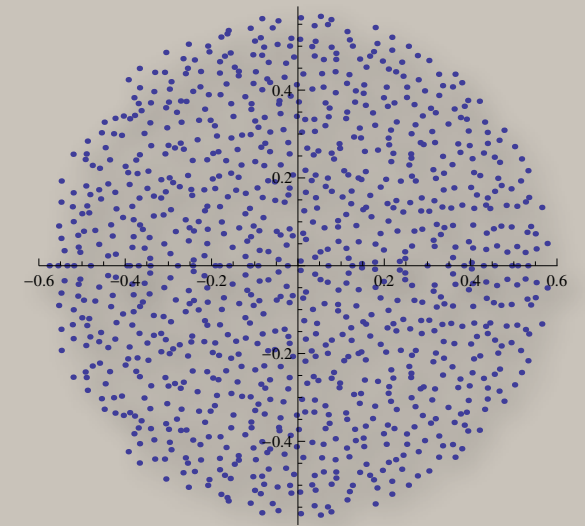
GOE: real symmetric Gaussian,
invariant for $S \longrightarrow RSR^*$, R in $SO(N)$

GUE: complex Hermitian Gaussian
invariant for $H \longrightarrow UHU^*$, U in $SU(N)$

GSE: quaternion s.a. Gaussian,
invariant for $K \longrightarrow JKJ^*$, J in $Sp(2N)$



The semicircle law



Ginibre 1965, (no symmetry)

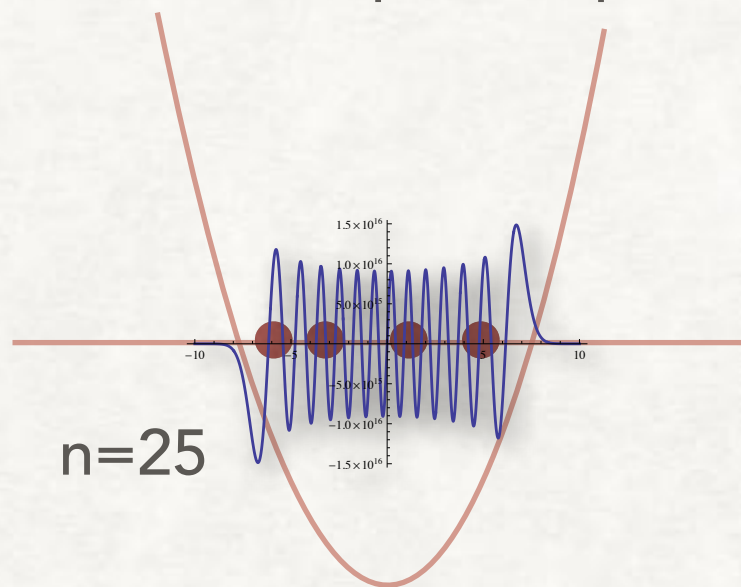
$$p(H)dH = Ce^{-\text{tr}H^2}dH = e^{-\sum_j x_j^2} \prod_{i < j} |x_i - x_j|^\beta dx_1 \dots dx_N dU_{\text{Haar}} \quad \beta = 1, 2, 4$$

Vandermonde

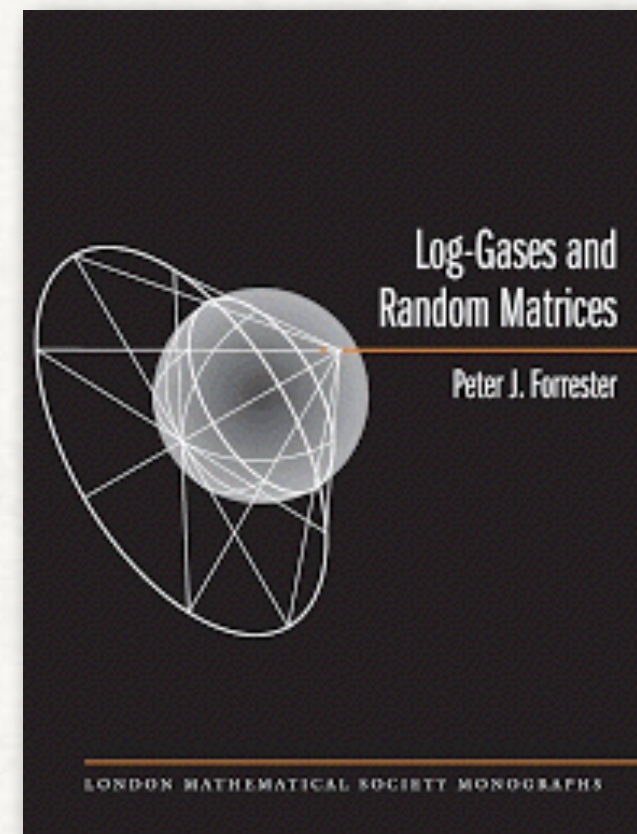
THE THREEFOLD WAY. Algebraic structure of symmetry groups
and ensembles in quantum mechanics (Freeman Dyson, 1962)

$$Z_N = \int dx_1 \dots dx_N e^{-\sum_j V(x_j) + \beta \sum_{i < j} \log |x_i - x_j|}$$

gas of particles (the eigenvalues) with log interaction (2D electrostatics)
on the line, plane, sphere (Stieltjes, Jacobi, Riesz)



$V(x) = x^2$ with interaction $\log |x-y|$
Equilibrium positions: zeros of $H_n(x)$



Saddle point (large N), Orthogonal polynomials (formally exact),
loop expansion, collective variables, ...

E.Brezin, C.Itzykson, G.Parisi, J.P.Zuber, Planar Diagrams, CMP 1978,
D. Bessis, A new method in the combinatoric of the topological expansion... CMP 1979

ORTHOGONAL POLYNOMIALS ($\beta = 2$)

$$\det \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} = \det \begin{bmatrix} p_0(x_1) & p_1(x_1) & \dots & p_{n-1}(x_1) \\ p_0(x_2) & p_1(x_2) & \dots & p_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ p_0(x_n) & p_1(x_n) & \dots & p_{n-1}(x_n) \end{bmatrix}$$

$$Z_N = \int dx_1 \dots dx_N \prod_{i>j} |x_i - x_j|^2 e^{-N \sum_j V(x_j)} = \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} \prod_k \int dx_k p_{i_r}(x_k) p_{j_s}(x_k) e^{-NV(x_k)}$$

Choose orthogonal polynomials: $\int dx e^{-NV(x)} p_i(x) p_j(x) = h_j \delta_{ij}$

$$Z_N = N! h_0 \dots h_{N-1}$$

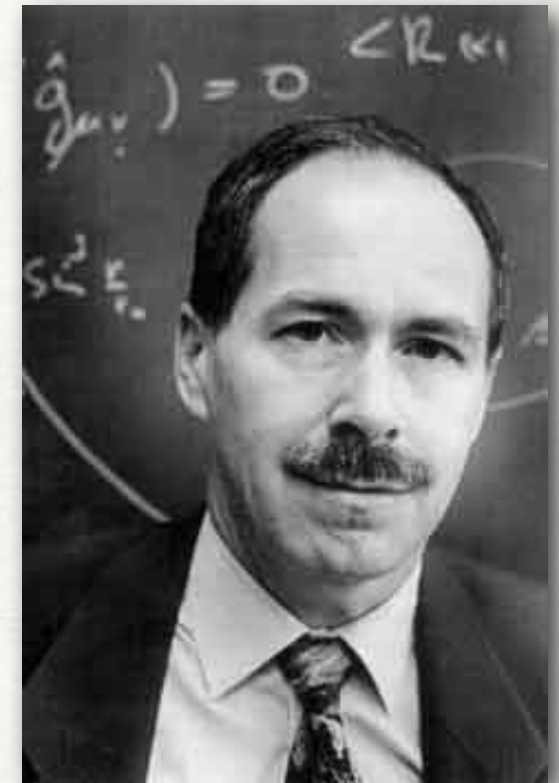
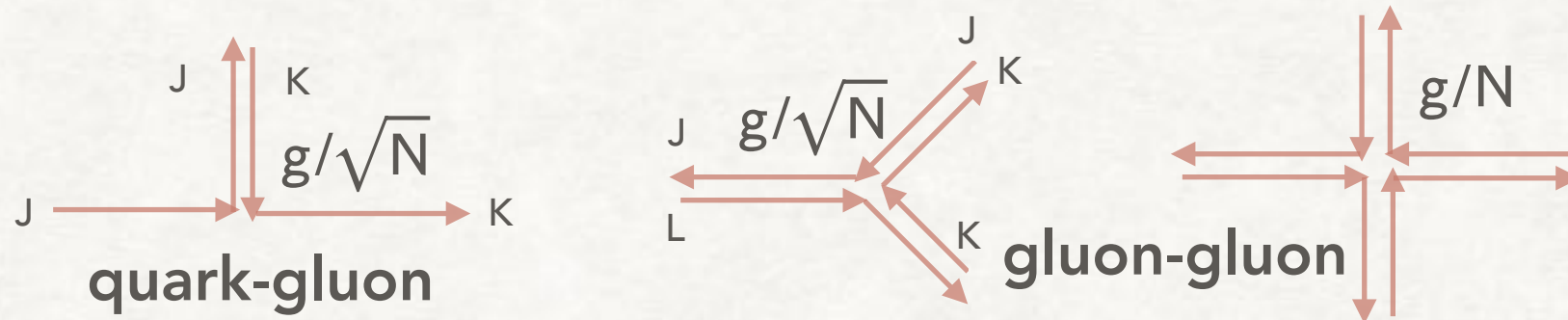
$$\rho(x) = \frac{1}{N} \langle \sum_{j=1}^N \delta(x - x_j) \rangle = e^{-V(x)} \frac{1}{N} \sum_{j=1}^N \frac{1}{h_j} p_j(x)^2 = \dots (\text{Christoffel - Darboux formula})$$

Correlation functions, level spacings, edge statistics (Tracy-Widom),
phase transitions (multicut support of density), ...

**THE STATISTICAL PROPERTIES OF THE EIGENVALUES ARE UNIVERSAL
(THEY DO NOT DEPEND ON THE DISTRIBUTION OF THE MATRIX)**

1/N EXPANSION

QCD: SU(3) local gauge theory, 3 quark states (vector field), 8 gluons (3x3 matrix field). Perturbation expansion in g gives Feynman diagrams (feasible at high energy).

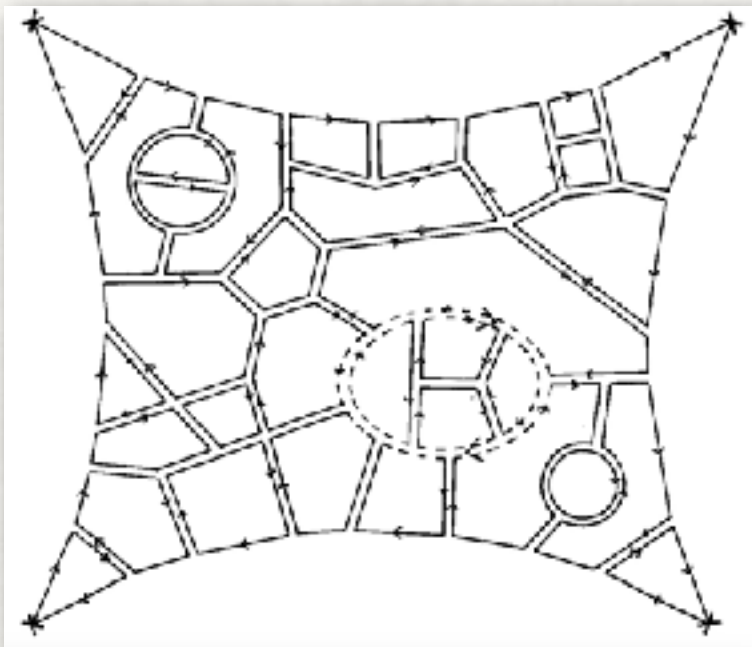


Gerard 'tHooft

SU(N) QCD

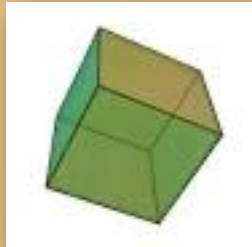
THE 1/N EXPANSION WITH g^2N FIXED CLASSIFIES DIAGRAMS ACCORDING TO THEIR EULER NUMBER

G 'tHooft, A planar diagram theory for strong interactions, Nucl. Phys. B (1974)

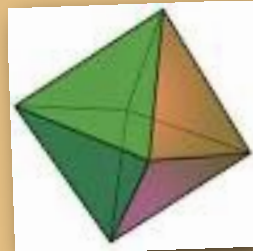


QM problem with SO(3) symmetry solved by spherical harmonics & radial equation. In SO(N) symm. the leading term in 1/N expansion of the radial solution solves an algebraic equation (in the end put $N=3$)

EULER: $\chi = 2 - 2\text{handles} - \text{holes} = \text{Faces} + \text{Vertices} - \text{Edges}$



Cube
F=6, V=8 E=12



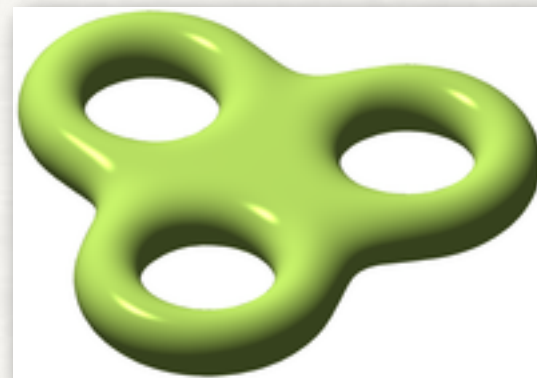
Octahedron
F=8, V=6 E=12



$\chi=2$



$\chi=0$



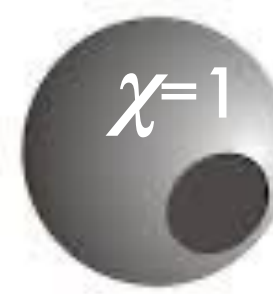
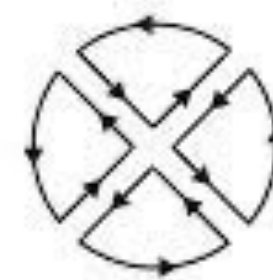
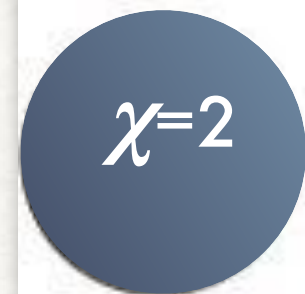
$\chi=-4$

N^2 PLANAR diagrams (sphere)

N^0 diagrams on the torus

N^{-2} diagrams on 2-torus

...



$$\left[\frac{g}{\sqrt{N}} \right]^4 \frac{g}{N} N^5$$

$$\left[\frac{g}{\sqrt{N}} \right]^4 \frac{g}{N} N^4$$

$$\left[\frac{g}{\sqrt{N}} \right]^4 N^2$$

In $d=0$ the generating functional is an ordinary matrix integral

$$Z(g, N) = \int dH e^{-N \text{tr}[H^2 + gH^4]}$$
$$dH = dx_1 \dots dx_N e^{-\sum x_j^2} \prod_{i>j} (x_i - x_j)^2 dU_{\text{Haar}}$$

$Z(g, N)$ COUNTS VACUUM DIAGRAMS

$$= N^2 Z_{\text{planar}}(g) + Z_{\text{torus}}(g) + \dots$$

STATISTICAL MECHANICS ON PLANAR
FEYNMAN DIAGRAMS — RANDOM SURFACES

$$Z_{\text{planar}}(g) = \sum_V g^V Z_V \quad \begin{array}{l} \log Z_V \text{ counts connected graphs with } V \text{ 4-vertices} \\ \text{i.e. quadrangulations (dual graphs) with } V \text{ faces} \end{array}$$

The planar series has finite radius of convergence:

$$Z_V \approx g_{\text{cr}}^{-V} \quad \text{then} \quad -\log Z_V \approx \text{Area} \log g_{\text{cr}}$$

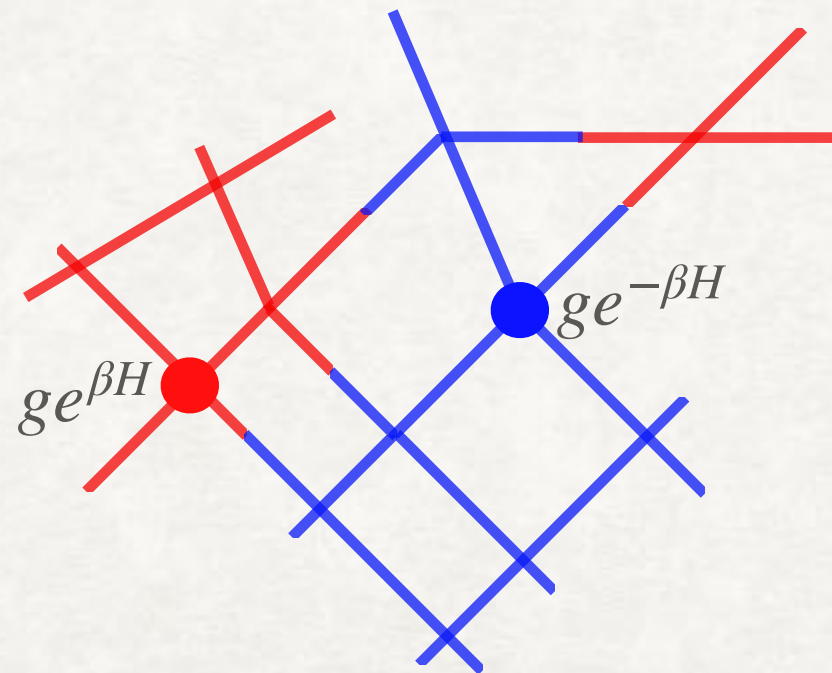
2D Ising model for ferromagnetism: Onsager (1944) $H=0$
 Spontaneous magnetization: Chen-Ning Yang (1952)

ISING MODEL ON PLANAR FEYNMAN GRAPHS $\mathcal{G}$

$$Z_{ISING} = \sum_{\mathcal{G}_{4,N}} \sum_{S_i = \pm 1} \exp \left\{ -\beta \sum_{ij} A_{ij}(\mathcal{G}_{4,N}) S_i S_j - \beta H \sum_j S_j \right\}$$

ISING MODEL ON QUARTIC GRAPHS WITH FIELD H
 IS EQUIVALENT TO THE PLANAR 2-MATRIX MODEL:

$$\int dA dB \exp \left\{ -N \text{tr} [A^2 + B^2 - 2e^{-2\beta} AB + ge^{\beta H} A^4 + ge^{-\beta H} B^4] \right\}$$



Yuri Kazakov (1986)

ISING MODEL ON RANDOM PLANAR GRAPHS

$$\int_{U(n)} dU \exp\left[\frac{1}{t} \text{tr}(AU BU^\dagger)\right] = t^{\frac{1}{2}n(n-1)} \frac{\det[\exp \frac{1}{t}(x_i y_j)]}{\Delta(x)\Delta(y)} \prod_{j=0}^{n-1} j!$$

The Harish-Chandra formula (1957)

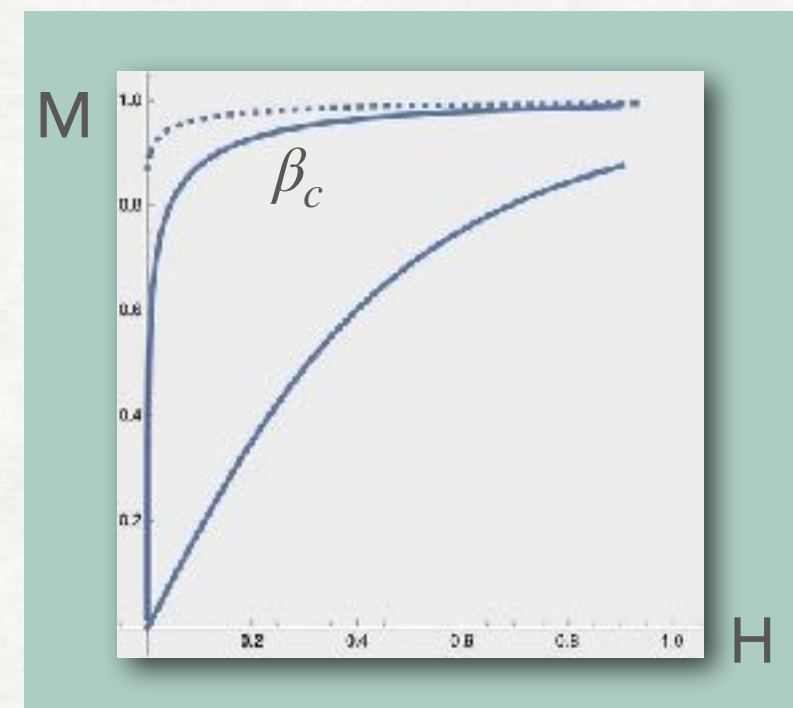
$$Z_N = \int d\mathbf{x} d\mathbf{y} \Delta(\mathbf{x}) \Delta(\mathbf{y}) \exp\left[-N \sum_j (x_j^2 + y_j^2 - 2e^{-2\beta} x_j y_j + ge^{\beta H} x_j^4 + ge^{-\beta H} y_j^4)\right]$$

Bi-orthogonal polynomials

$$\int dx dy P_j(x) Q_k(y) e^{-w(x,y)} = h_k \delta_{jk}$$

Critical limit = thermodynamic limit

critical exp	α	β	γ	δ	νd	γ_{str}
regular	0	1/8	7/4	15	2	—
random	-1	1/2	2	5	3	-1/3



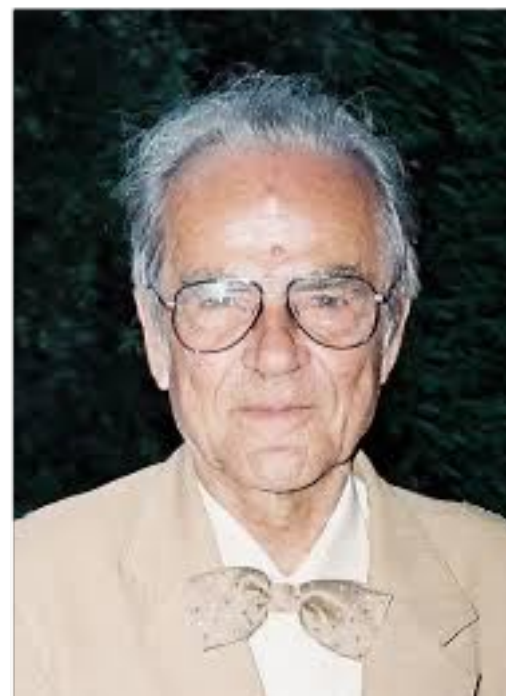
In accordance with CFT on random surfaces (Knizhnik-Polyakov-Zamolodchikov, 1988)
They satisfy the standard relations



Giulio Casati



Italo Guarneri



Boris Chirikov



Dima Shepelyansky



Felix Izrailev

1980: QUANTUM CHAOS

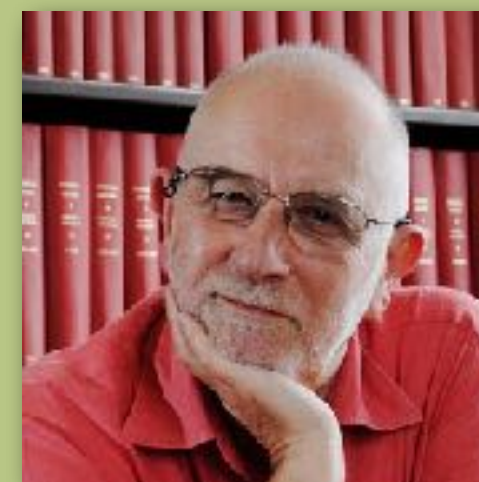
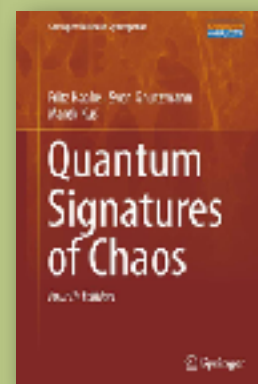
Milano - Novosibirsk



Joe Ford



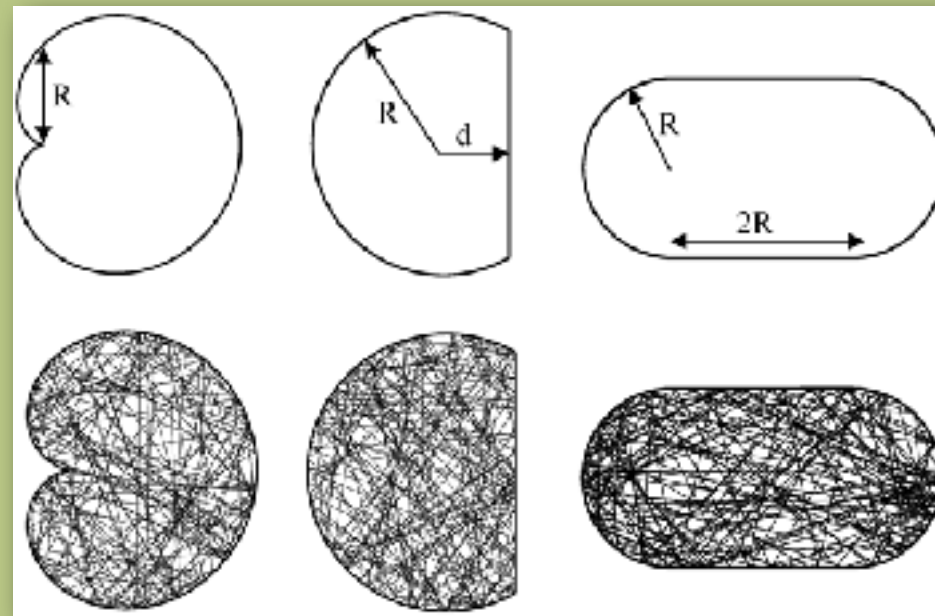
Fritz Haake



Michael Berry

QUANTUM CHAOS

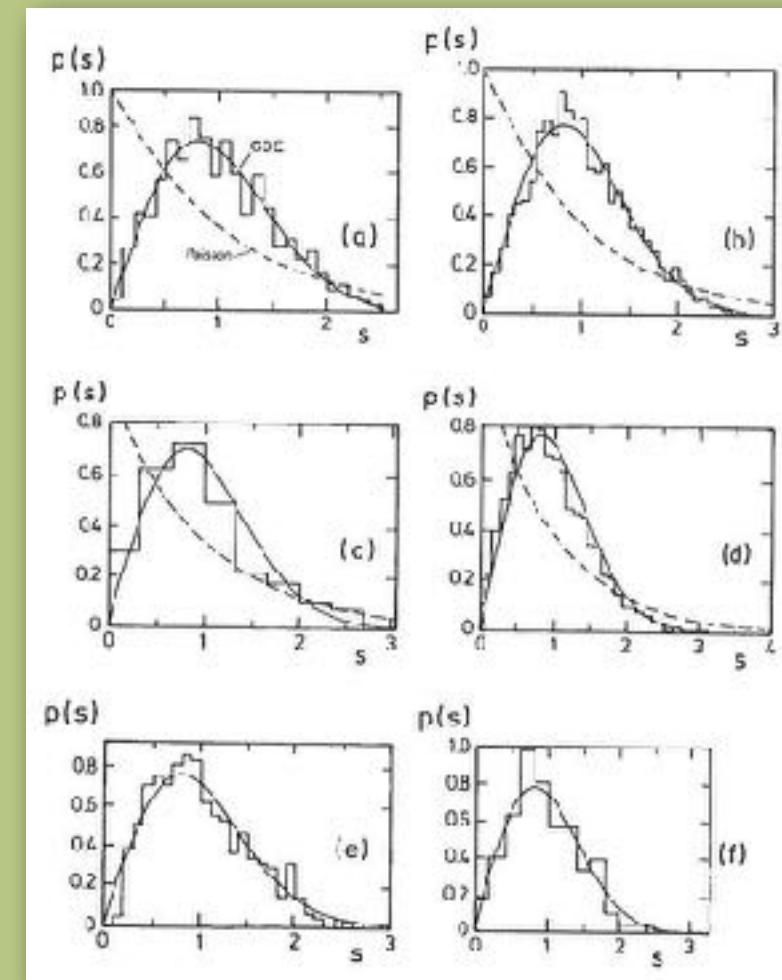
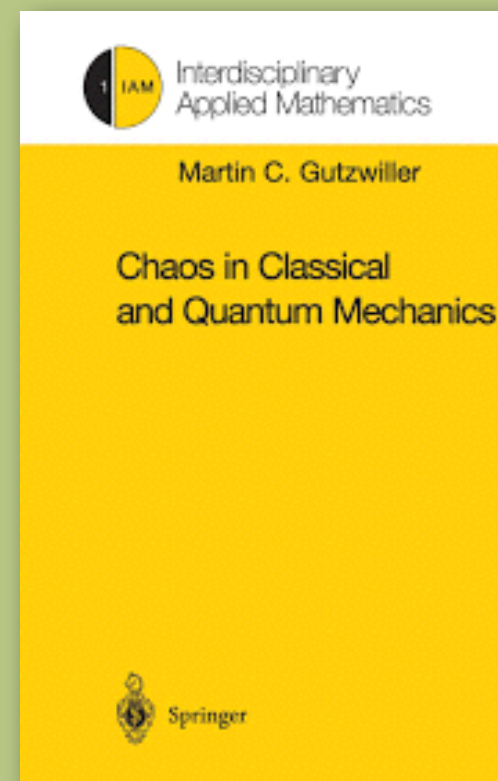
(how does classical chaos show in QM?)



Chaotic Billiards



Martin Gutzwiller



- A) Sinai billiard
- B) H atom in strong H field
- C) NO₂ molecule
- D) ...
- E) Spectrum of 3D microwave cavity
- F) Frequencies in 1/4 of Sinai stadium

THE KICKED ROTATOR

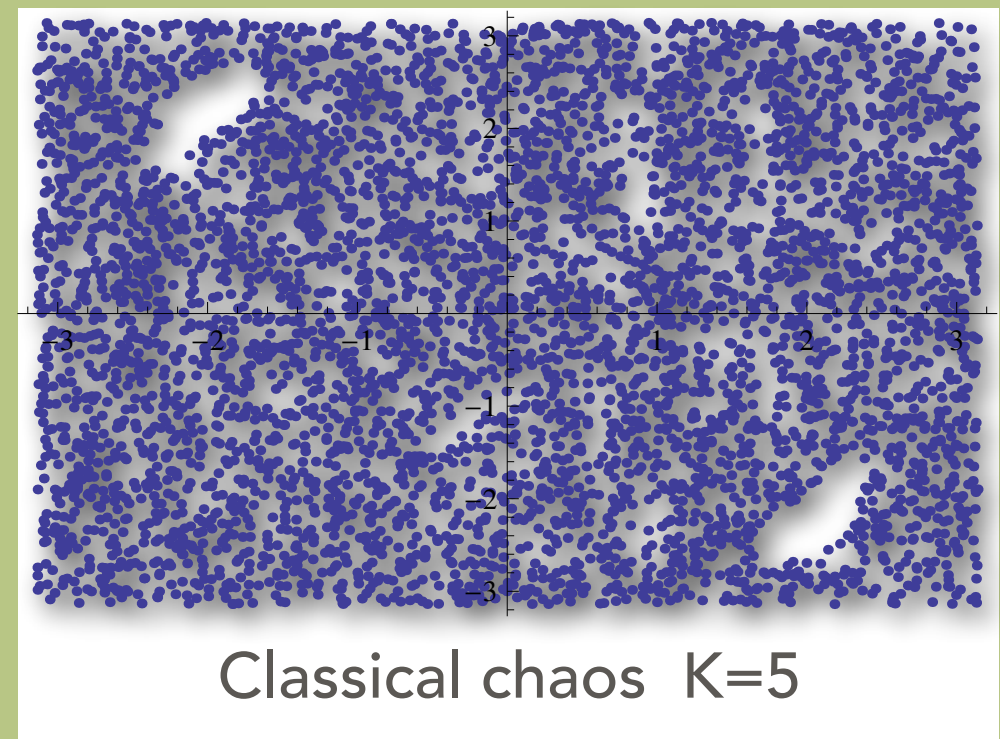
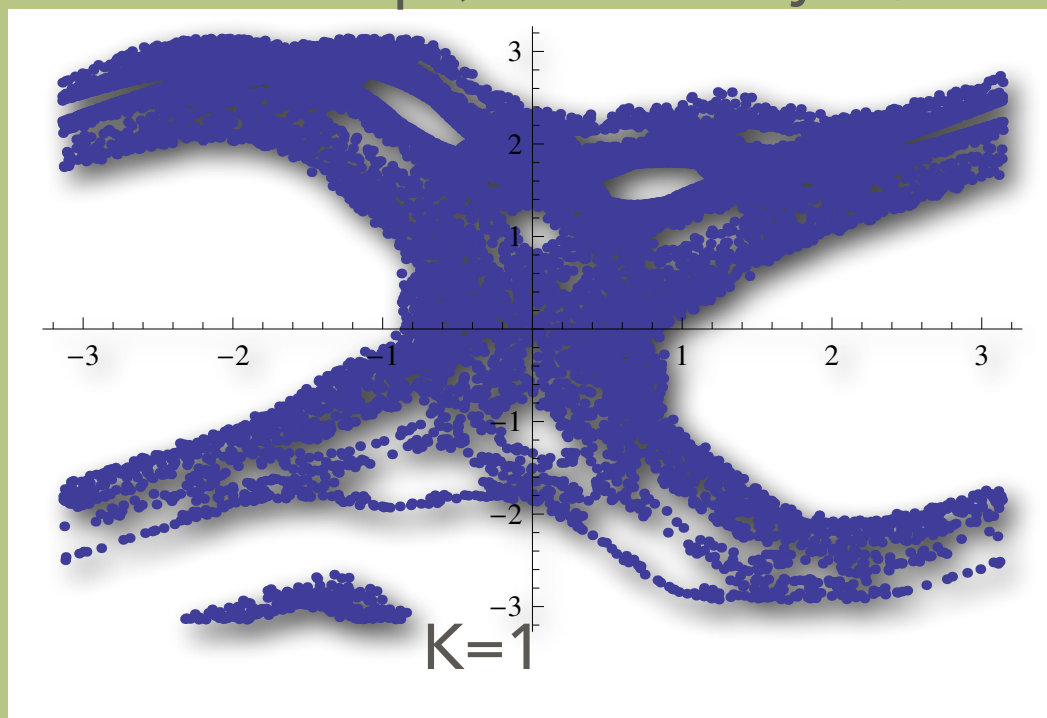
$$H = \frac{1}{2}P^2 + K \cos \theta \sum_n \delta(t - n) \quad P = -i \frac{d}{d\theta}$$

$$p' = p + K \sin \theta$$

$$\theta' = \theta + p'$$

$$U = e^{-iK \cos \theta} e^{-iP^2/2}$$

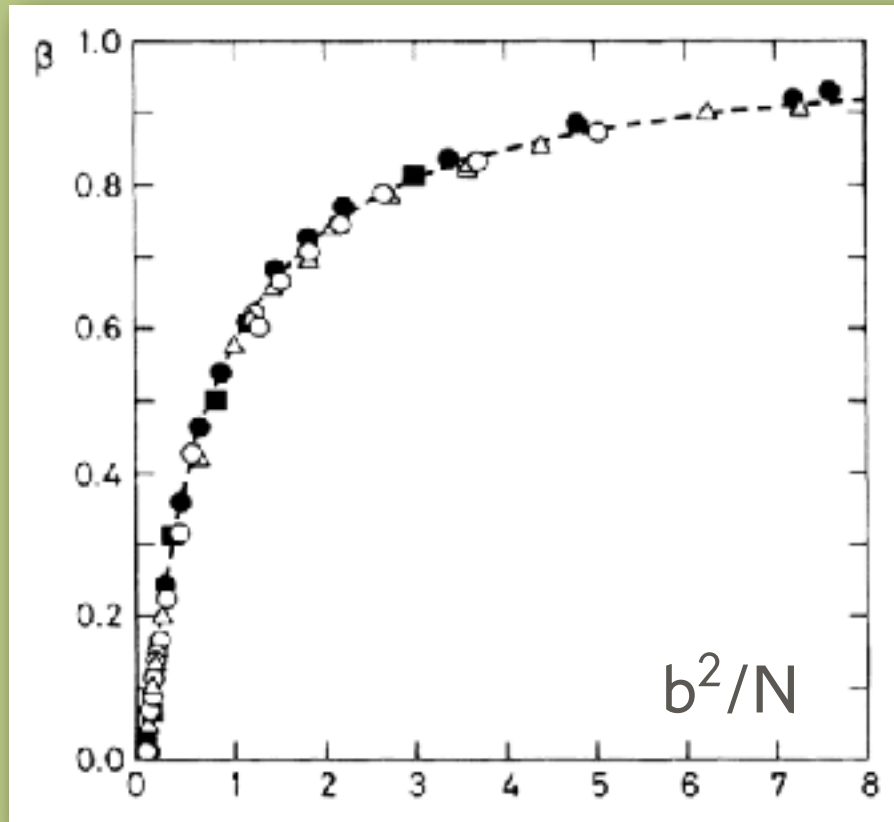
Standard Map (Chirikov-Taylor, 1979)



$K \gg 1, \langle p^2 \rangle \propto \frac{1}{2} K^2 t$. In QM the growth of energy stops

The matrix $\langle m | U | m' \rangle$ is "random" and banded with width K .

Its eigenstates are exponentially localized (dynamical localization) $\xi \approx K^2$



G.Casati, L.Molinari, F. Izrailev, Scaling properties of Band Random Matrices, 1990

Band Random Matrices

$$H = \sum_j u_j^2 \log u_j^2$$

Entropy length
of eigenvector $\mathbf{u}$

$$\beta = e^{H - H_{GOE}}$$

Scaling of the localization of eigenvectors

Determinants as Gaussian integrals with commuting and anti-commuting variables.
Average over disorder. The auxiliary variables are now coupled.
Solve in large N limit

$$\left\langle \frac{\det(M - z)}{\det(M - w)} \right\rangle \quad \left\langle \frac{\det(M - z) \det(M - z')}{\det(M - w) \det(M - w')} \right\rangle$$

Semicircle law

Scaling of localization

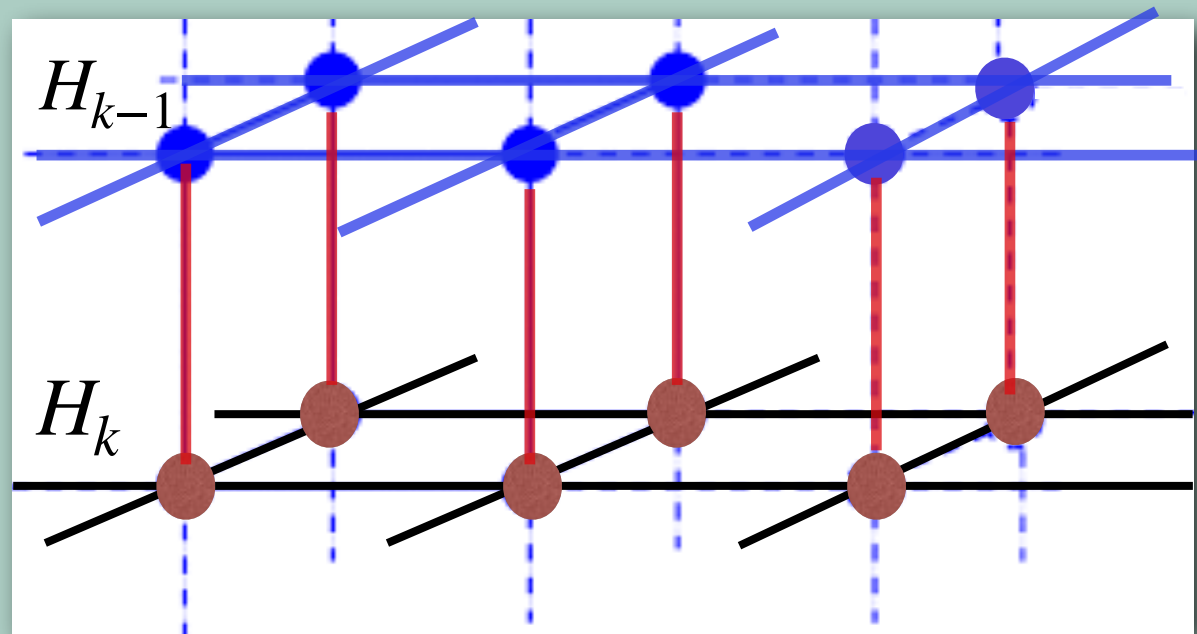


Yan Fyodorov

ANDERSON LOCALIZATION

Absence of Diffusion in Certain Random Lattices (1957)

Philip Warren Anderson



Particle in cubic lattice.

In each site a random number in $[-W, W]$

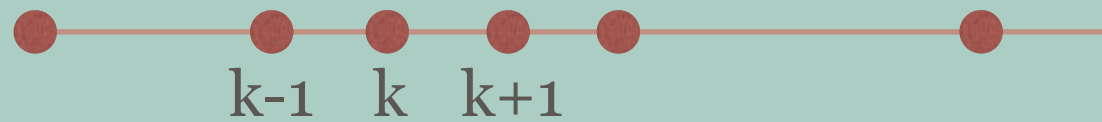
For an infinite lattice:

$W < W_c$ extended eigenvectors (metal)

$W > W_c$ exp localized eigenvector (insulator)

$$\begin{pmatrix} H_1 & 1 & & \\ 1 & H_2 & 1 & \\ & & & 1 \\ & & 1 & H_n \end{pmatrix}$$

ANDERSON LOCALIZATION IN D=1



$$u_{k+1} + u_{k-1} + \epsilon_k u_k = E u_k$$

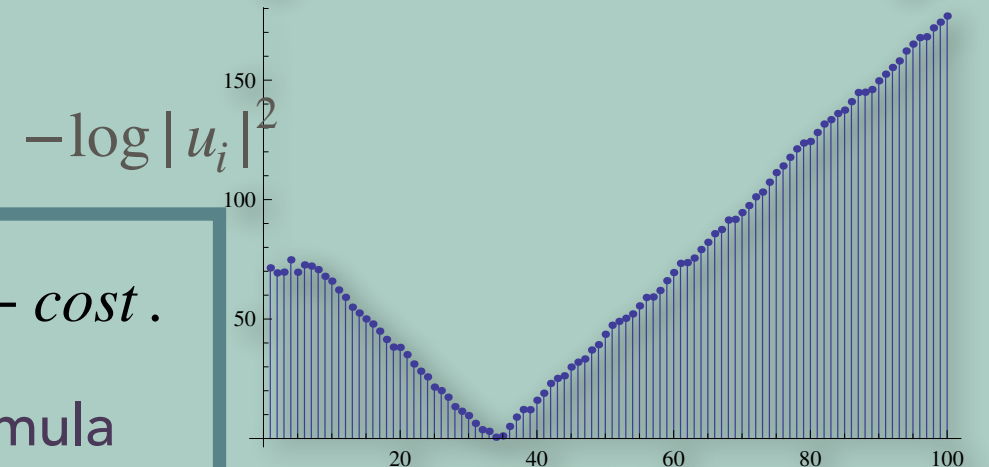
$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = T(E) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

$$u_n \approx \exp[\pm n \xi(E)]$$

$$\xi(E) = \int dE' \rho(E') \log |E - E'| + \text{const.}$$

Herbert-Jones-Thouless formula

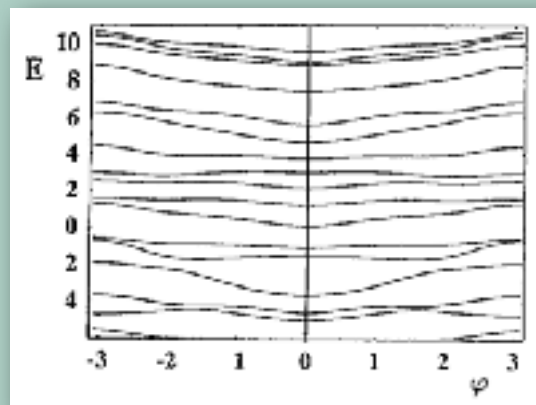
$$\begin{bmatrix} \epsilon_1 & 1 & & & \\ 1 & \epsilon_2 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & 1 & \epsilon_N \end{bmatrix}$$



LOCALISATION THROUGH ENERGY-LEVEL CURVATURES D=3

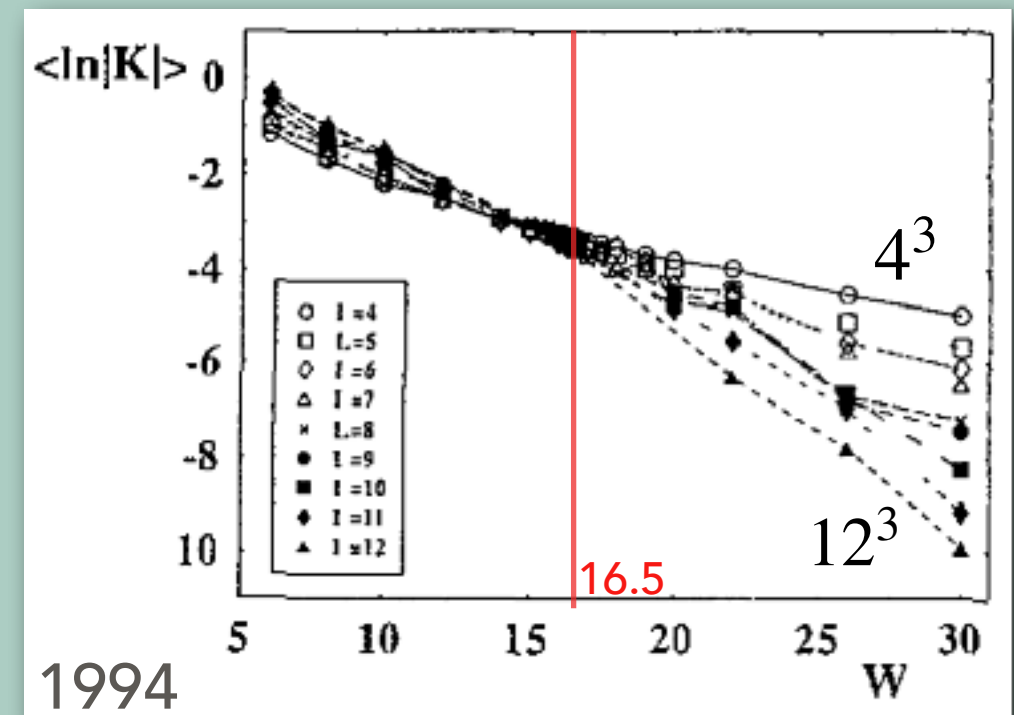


Karol Życzkowski



Motion of eigenvalues with Bloch phase

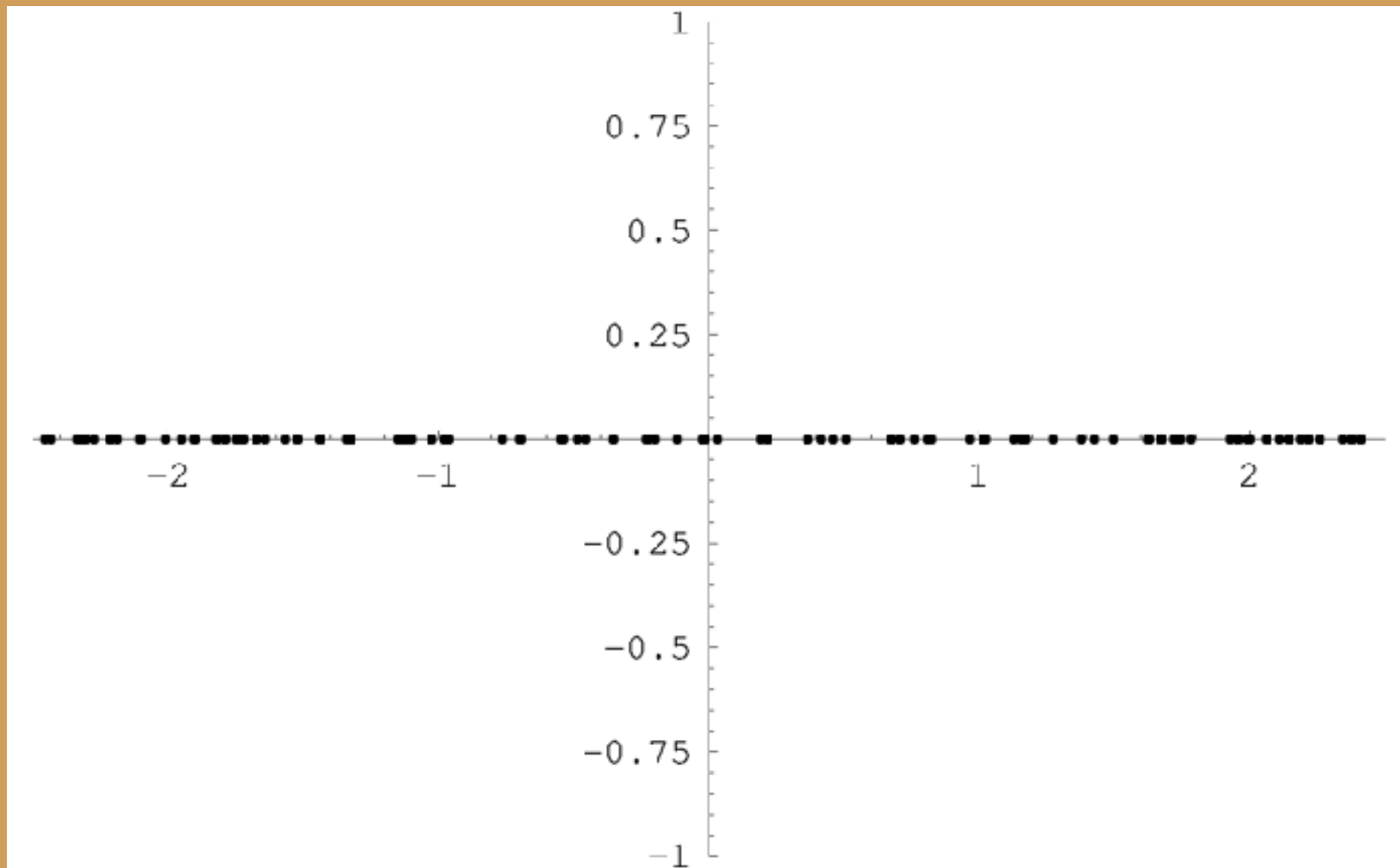
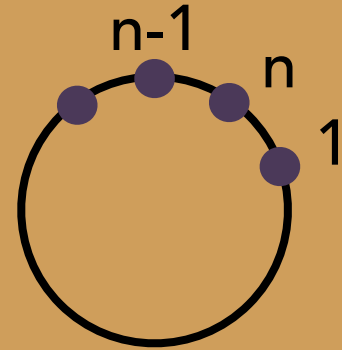
$$K = \frac{E''(0)}{\Delta} \quad \text{level curvature}$$



1994

Hatano-Nelson (PRL 1996)

$$e^{\xi}u_{k+1} + \epsilon_k u_k + e^{-\xi}u_{k-1} = Eu_k$$

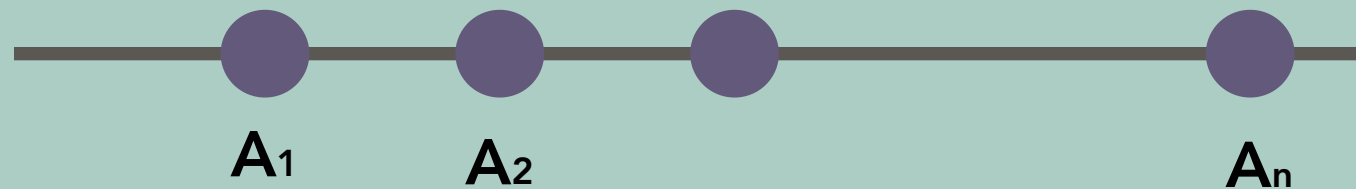


$$u_k = e^{k\xi}v_k$$

$$v_{k+1} + \epsilon_k v_k + v_{k-1} = Ev_k$$

$$v_{n+1} = e^{n\xi}v_1, \quad v_0 = e^{-n\xi}v_n$$

DIFFERENCE EQUATION & TRANSFER MATRIX



$$\underline{C_k u_{k-1} + A_k u_k + B_k u_{k+1} = E u_k, \quad k = 1 \dots n, \quad u_k \in \mathbb{C}^m}$$

A_k, B_k, C_k are
m x m matrices

$$\begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} B_k^{-1}(E - A_k) & -B_k^{-1}C_k \\ \mathbb{I}_m & 0 \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}$$

$$T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$$

The eigenvalues of $T(E)$?

SPECTRAL DUALITY: z is eigenvalue of $T(E)$ iff E is eigenvalue of $H(z)$

1997

$$T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = z \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

$$H(z) = \begin{bmatrix} A_1 & B_1 & \frac{1}{z}C_1 & \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ zB_n & & C_n & A_n \end{bmatrix}$$

THE SPECTRAL DUALITY

$$\det[z\mathbb{I}_{2m} - T(E)] = (-z)^m \frac{\det[E\mathbb{I}_{nm} - H(z)]}{\det(B_1 \cdots B_n)}$$

$$\xi_k =: \frac{1}{n} \ln |z_k|$$

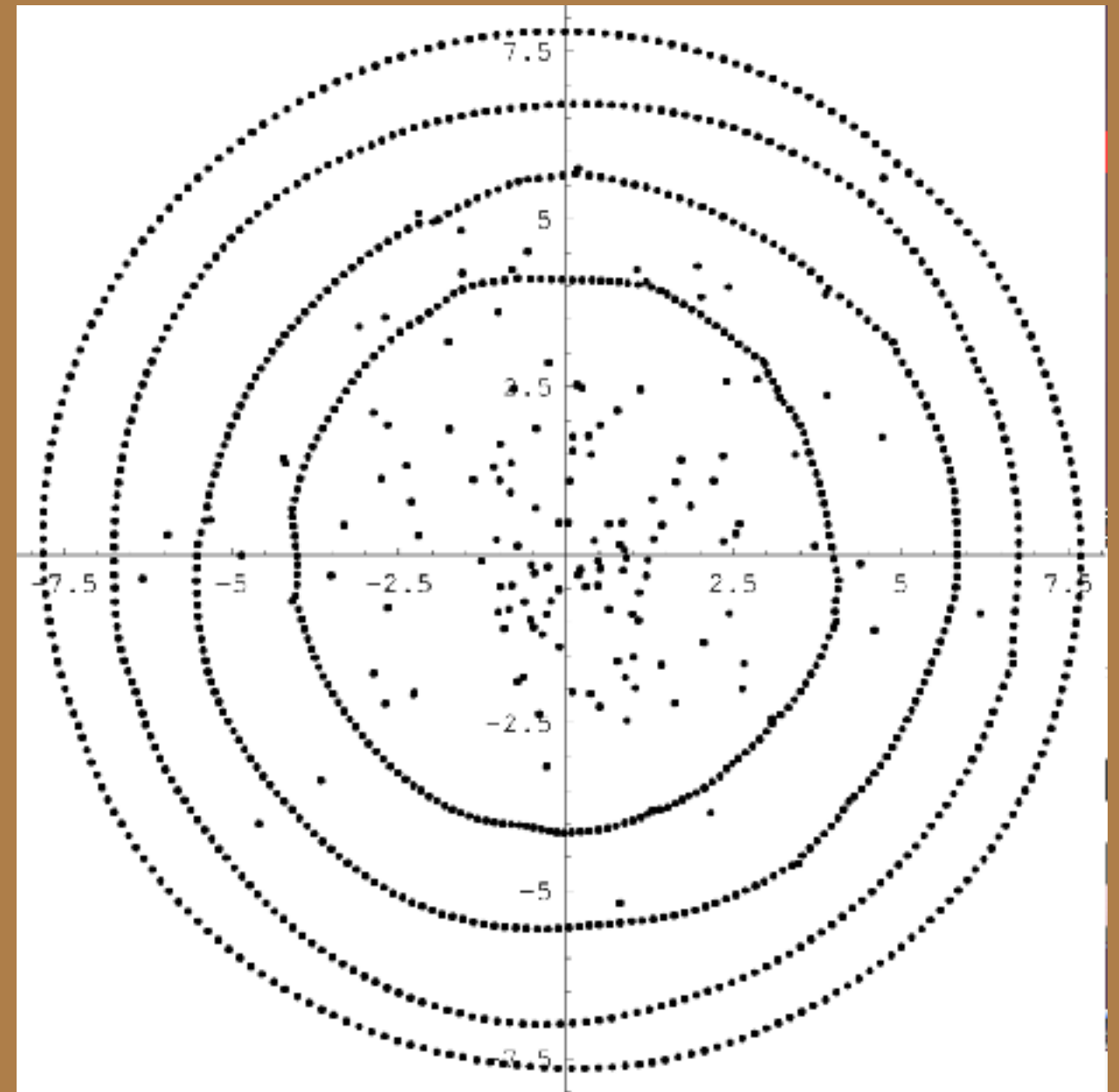
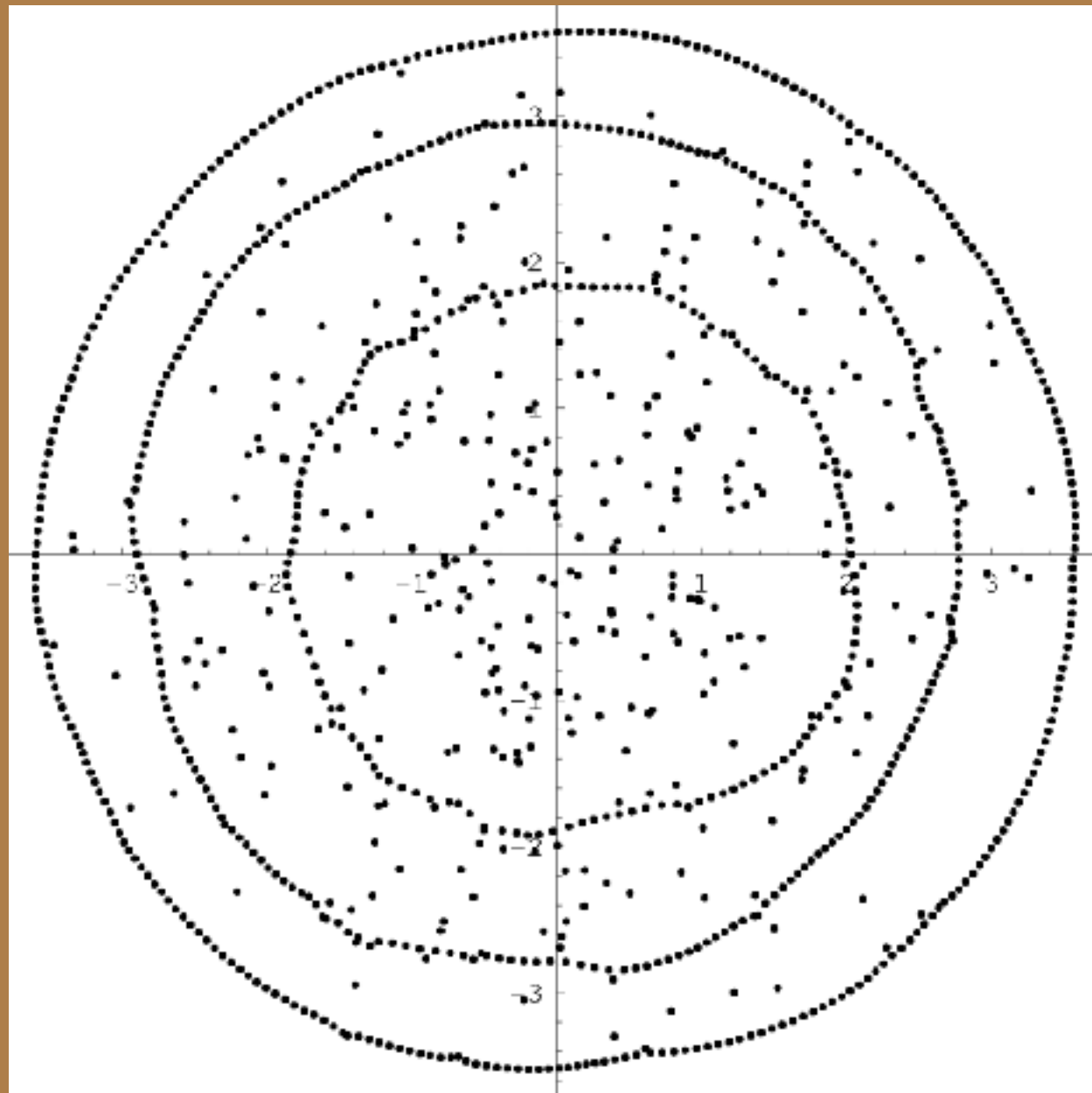
The exponents of $T(E)$

Theorem (Jensen) *If f is holomorphic and $f(0) \neq 0$, and $z_1 \dots z_n$ are its zeros in the disk of radius r , then: $\int_0^{2\pi} \frac{d\theta}{2\pi} \ln |f(re^{i\theta})| = \ln |f(0)| - \sum_k \ln(|z_k|/r)$.*

$$\sum_{k=1}^{2m} \xi_k \theta(\xi_k) = \frac{1}{n} \int_0^{2\pi} \frac{d\theta}{2\pi} \ln |\det[H(e^{i\theta}) - E]| - \frac{1}{n} \ln |\det[B_1 \cdots B_n]|$$

Exact deterministic formula, analogous to the probabilistic *Kunz-Souillard* formula

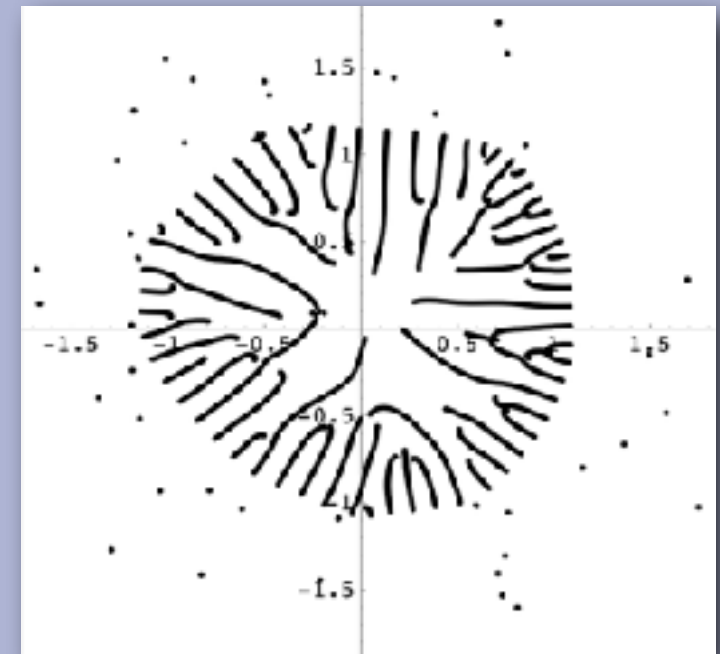
$$\det[z\mathbb{I}_{2m} - T(E)] = (-z)^m \frac{\det[E\mathbb{I}_{nm} - H(z)]}{\det(B_1 \cdots B_n)}$$



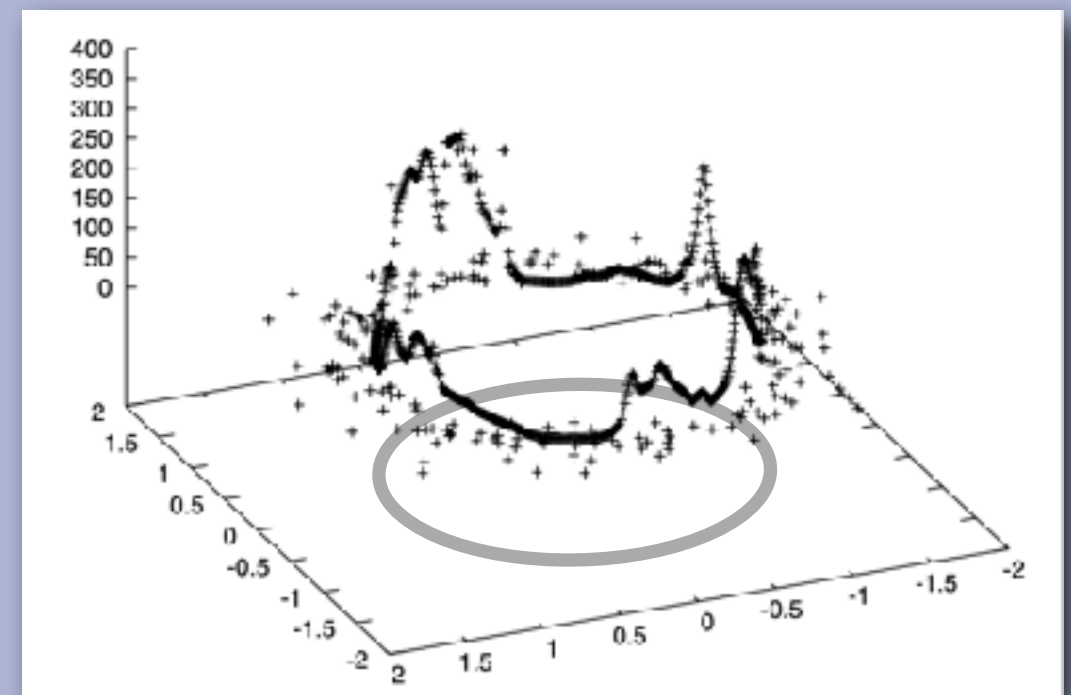
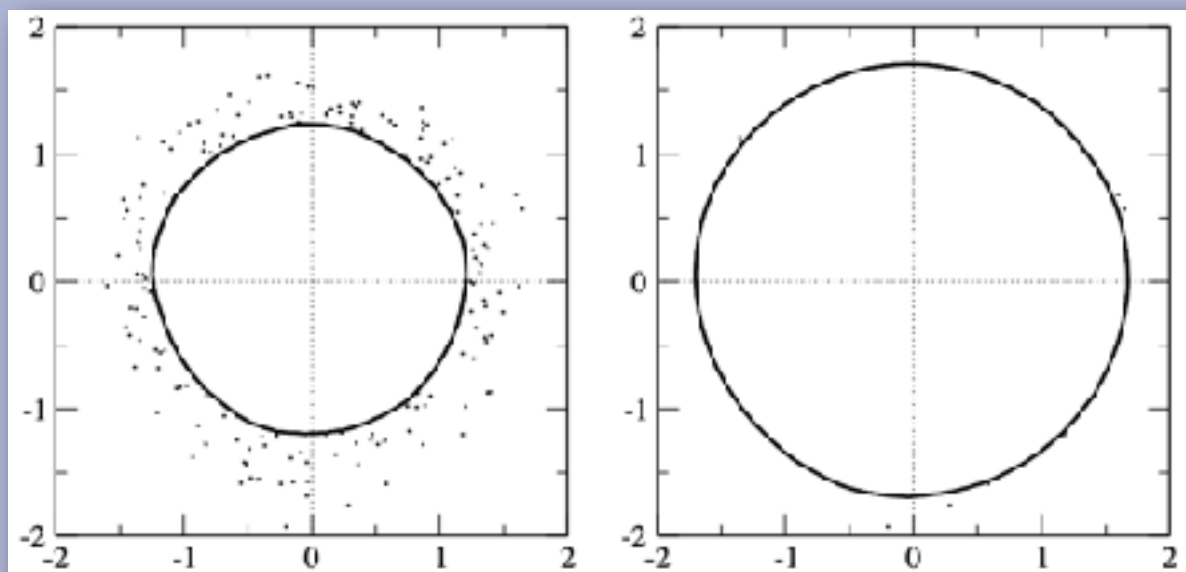
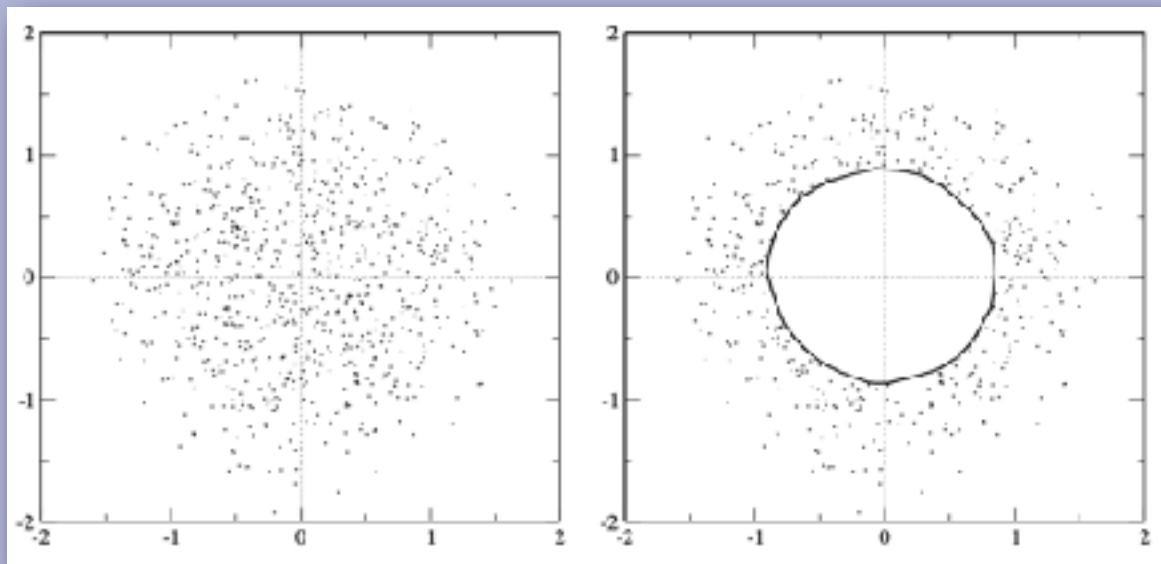
Eigenvalues occupy spectral curves, that manifest when perturbed

MOTION OF ENERGY EIGENVALUES

$$M(z^n) = \begin{bmatrix} a_1 & b_1 & z^n c_1 \\ c_2 & \ddots & \ddots \\ \ddots & \ddots & b_{n-1} \\ b_n/z^n & c_n & a_n \end{bmatrix} \quad z = e^{\xi + i\varphi}$$



motion in ξ



Eigenvalues (x,y)
and size of eigenvectors (z axis)

DETERMINISTIC EXPONENTIAL LOCALIZATION

Theorem (Demko, Moss and Smith) *Let A be a positive definite block tridiagonal matrix, with square blocks of size m , let $[a, b]$ be the smallest interval containing the spectrum of A , let $A^{-1}[i, j]$ be any matrix element in the block $(A^{-1})_{ij}$. Then:*

$$|A^{-1}[i, j]| \leq \begin{cases} C q^{|i-j|} & \text{for } |i - j| \geq 1 \\ 1/a & \text{for } i = j \end{cases}$$

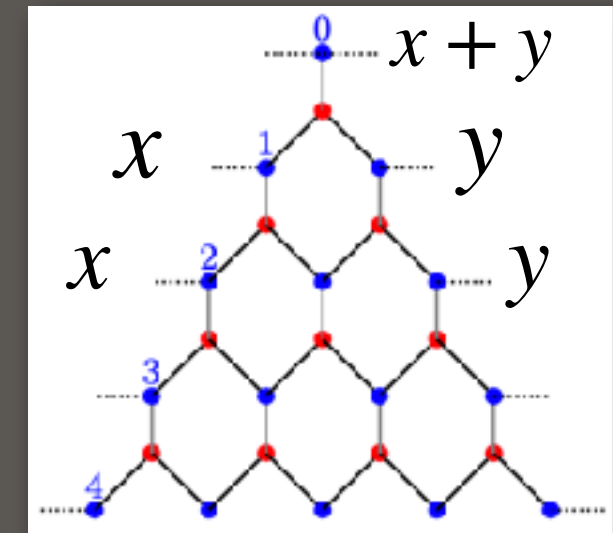
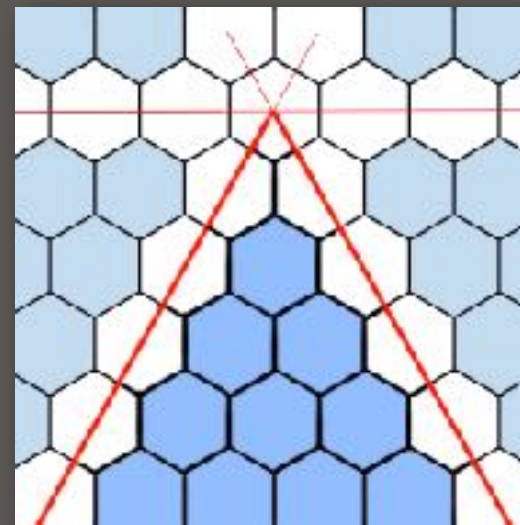
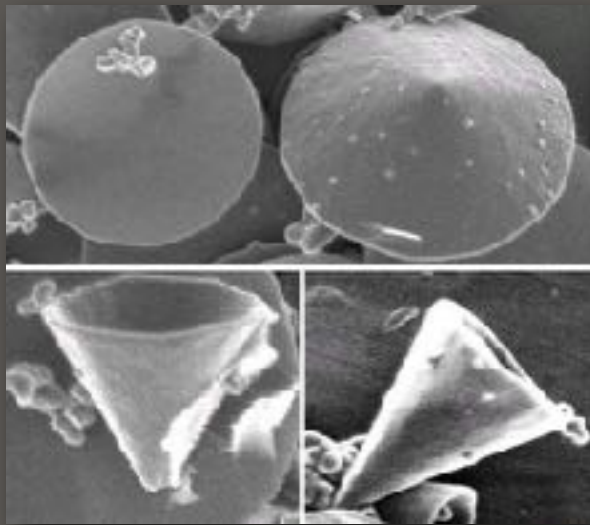
$$C = \frac{(\sqrt{b} + \sqrt{a})^2}{2ab}, \quad q = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}$$

Theorem extends with changes to a block matrix $A = A^\dagger (A A^\dagger)^{-1}$

$$(H - E)^{-1} = \begin{pmatrix} G_{11} & \cdots & G_{1n} \\ \cdots & \cdots & \cdots \\ G_{n1} & \cdots & G_{nn} \end{pmatrix} \quad T(E) = \begin{pmatrix} -G_{1n}^{-1} & -G_{1n}^{-1} G_{11} \\ G_{nn} G_{1n}^{-1} & G_{nn} G_{1n}^{-1} G_{11} - G_{n1} \end{pmatrix}$$

If $n \gg 1$ the transfer matrix $T(E)$ has m singular values larger than $Kq^{-n/2}$ and m singular values smaller than $Kq^{n/2}$
[q is evaluated for $(H - E)^\dagger (H - E)$]

GRAPHENE NANOCONES AND PASCAL MATRICES



$$H_2 = \left[\begin{array}{c|cc|cc} S_0 & 0 & 1 & & & \\ \hline 0 & 0 & 1 & y_1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ \hline & x_1 & 1 & 0 & 0 & 1 \\ \hline & 0 & 0 & 0 & 0 & 1 & y_2 \\ & 1 & & & 1 & 0 & 1 \\ & & 0 & & 1 & 0 & 1 \\ & & & 1 & 1 & 0 & 1 \\ & & & & x_2 & 1 & 0 \end{array} \right]$$

$$\det H_0 = x + y$$

$$\det H_1 = x^2 + 3xy + y^2$$

$$\det H_2 = x^3 + 9x^2y + 9xy^2 + y^3$$

$$\det H_3 = x^4 + 29x^3y + 72x^2y^2 + 29xy^3 + y^4$$

$$\det H_4 = x^5 + 99x^4y + 626x^3y^2 + 626x^2y^3 + 99xy^4 + y^5$$

$$\det H_5 = x^6 + 351x^5y + 6084x^4y^2 + 13869x^3y^3 + 6084x^2y^4 + 351xy^5 + y^6$$

Conjecture for
the determinant

from size $n^2 \times n^2$

to size $n \times n$

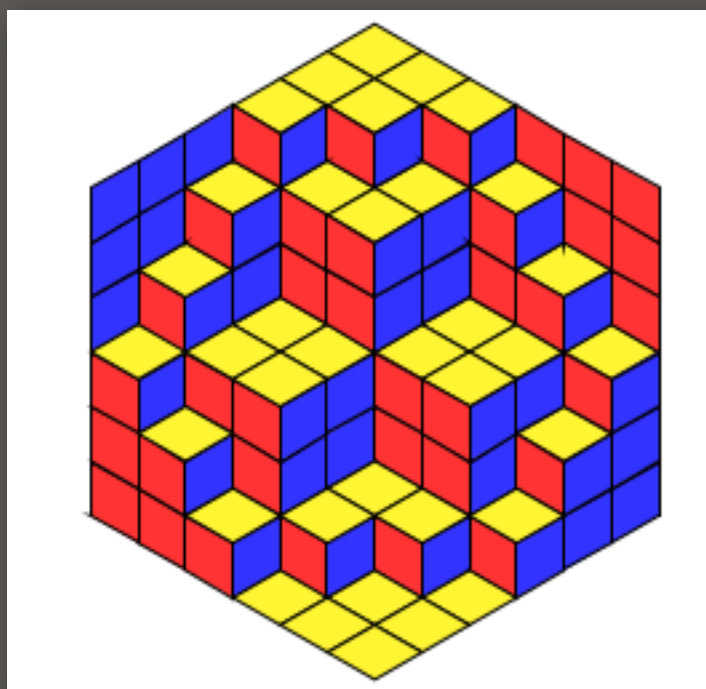
$$\left[\begin{array}{cccccc} x_n + y_n & -\binom{n}{1}y_n & \binom{n}{2}y_n & -\binom{n}{3}y_n & \dots & \pm \binom{n}{n}y_n \\ \binom{n}{1}x_n & x_{n-1} + y_{n-1} & -\binom{n-1}{1}y_{n-1} & \binom{n-1}{2}y_{n-1} & \dots & \mp \binom{n-1}{n-1}y_{n-1} \\ \binom{n}{2}x_n & \binom{n-1}{1}x_{n-1} & x_{n-2} + y_{n-2} & -\binom{n-2}{1}y_{n-2} & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & x_1 + y_1 & -\binom{1}{1}y_1 \\ \binom{n}{n}x_n & \binom{n-1}{n-1}x_{n-1} & \binom{n-2}{n-2}x_{n-2} & \dots & \binom{1}{1}x_1 & x_0 + y_0 \end{array} \right]$$

A surprising connection with the PASCAL matrix

$$L_n = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \quad L_n^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \quad Q_n = L_n L_n^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

$$\det H_n(e^{-i\theta}, e^{i\theta}) = e^{-i(n+1)\theta} \det(Q_n + e^{2i\theta})$$

computed for special angles and all n by Andrews, Ciucu, Krattenthaler, Nienhuis and Mitra, in enumerative combinatorics of plane partitions (Mac Mahon, Stanley), lozenge tilings, dense loops on a cylinder (very hard work!)



n	$\theta = 0$	$\pi/6$	$\pi/3$	$\pi/2$	$\pi/4$
2	20	$3^2\sqrt{3}$	7	0	$8\sqrt{2}$
3	132	10^2	42	2^4	70
4	1452	$25^2\sqrt{3}$	429	0	$526\sqrt{2}$
5	26741	140^2	7436	7^4	13167
6	826540	$588^2\sqrt{3}$	218348	0	$280772\sqrt{2}$

Thank you for
your attention

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coord. 45.869N 8.770E, alt. 1226m, direzione: W. Fonte: www.astrogeo.va.it



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