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Functional renormalization-group approach to non-equilibrium critical dynamics: calculation of the fluctuation-dissipation ratio

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Introduction

Statistical Mechanics provides a set of ideas which allow us to understand macroscopic phenomena starting from a microscopical description of the system. In the case of equilibrium systems this program is established on a firm basis. The probability distribution $P(S)$ of a macroscopic state S at equilibrium at a certain temperature $T = \beta^{-1}$ is given, in terms of the Hamiltonian of the system $\mathcal{H}(S)$, by the Boltzmann-Gibbs distribution, i.e., $P(S) \sim e^{-\beta\mathcal{H}(S)}$. This general statement has been proven very successful in the study of *phase transitions*. However, the reason for which we say that we are able to study phase transitions even if they are very complicated problems, in which more than one scale is involved, is because simplifications occur due the prominent importance of *collective behaviours* which dominates over the irrelevant effects induced by genuine microscopic physics. Thanks to this simplification, which is called *universality*, simple toy models are able to predict with great accuracy the values of observables that characterize the algebraic singularities that the system shows approaching the critical point, such as *critical exponents*. It happens that this universality characterizes also behaviors out of equilibrium whenever collective behavior dictated by a closeness to the critical point emerge.

For *non-equilibrium* systems a general description of probability distribution of the macroscopic system S, similar to Boltzmann-Gibbs one, has not been achieved. Furthermore, since they are everywhere in nature, they are even more intriguing. We observe essentially two different classes of non-equilibrium systems in nature: those which are forced by an external constant perturbation and those which are somehow prepared in a non-equilibrium initial state, which are known as *relaxation* phenomena. In this work we will focus on the last class of physical phenomena. It happens that, for example, universal behaviors shows up if a magnetic object is suddenly cooled (i.e., it undergoes a *quench*) from a high-temperature state right to the critical point of its second-order phase transition. What is more intriguing is the fact that, in this case, the relaxation dynamics never ends. During this never ending relaxation process the fluctuation-dissipation theorem does not hold at any time. It happens that an observable, the so-called fluctuation-dissipation ratio (FDR) can be used to detect if the equilibration time is infinite or not; moreover, it turns out that this is a universal quantity in the long-time limit. Since it is a universal quantity, has been studied for years with all the theoretical tools that statistical mechanics provides, such as lattice and field-theoretical approach and Monte-

Carlo simulations.

Since exact solutions have been found for only few of the problems which arise in these contexts, a method called renormalization group has been introduced for dealing with problems that have multiple scales of length. The renormalization group (RG) is not a descriptive theory of nature but a general method for constructing theories. The RG theory consists of a set of concepts and methods which can be used to understand phenomena in many different fields of physics, ranging from quantum field theory over classical statistical mechanics to non-equilibrium phenomena, either classical and quantum.

In the past twenty years, an analytic implementation of the renormalization group ideas, the so-called functional (or exact) renormalization group (fRG) has produced a number of predictions in critical equilibrium systems as well as for critical non-equilibrium one.

In this thesis we present calculation of the FDR in the long-time limit for the case of a critical quench of the *purely relaxational* model by means of the fRG technique, task that has not been achieved to date. The closest attempt is Ref. [35] in which a genuine non-equilibrium critical exponent, known as initial-slip exponent, has been calculated.

Our original contributions are the following:

- (*i*) correction of a mistake done in the calculation of Ref. [35], in order to obtain the critical initial-slip exponent, which is discussed around Fig. 4.3. In addition we clarify the physical interpretation of the equation which gives the anomalous dimension of the boundary (in time) order-parameter field. This field represents the non-equilibrium initial condition necessary to implement the dynamics induced by the temperature quench. All these results will be crucial in order to achieve our purpose, i.e. to calculate the fluctuation-dissipation ratio in the long-time limit.
- (*ii*) calculation of the critical initial slip exponent by means of a long-time analysis of the system, rather than the short-time one introduced in Ref. [35].
- (*iii*) general analytical expression, given by Eq.-(5.35), for the asymptotic value of the fluctuation-dissipation ratio in the so-called local potential approximation of the flow equation for the effective action given by the Wetterich equation (see Sec. 5.1). We have considered three increasing accuracy within the local potential approach which led us to the results for the fluctuation-dissipation ratio in Fig. 5.1.

$\phi_{m,k} \neq 0$

The chapters of this thesis are organized as follows:

1: We introduce the FDR, the main observable which we are interested in. We review the results which state that indeed it is a universal quantity in the long-time limit, for the case of purely relaxational dynamics. We discuss the general class of field theories which are used in order to describe the universal behavior of relaxational models. We review the state of the art for what concerns the calculation of the FDR in the long-time limit.

2: We introduce the central theoretical quantity for the fRG technique, i.e. the effective action. We detail its definition and properties.

3: We describe the fRG technique in its equilibrium formulation. We show how one can obtain information about the critical point, such as the static critical exponents.

4: We review the recent literature, particularly Ref. [35], which is the first analysis of the purely relaxational model with the fRG technique in which the critical initial-slip exponent has been calculated; in addition, it is our starting point for the calculation of the FDR.

5: We generalize the technique developed in Ref. [35] in order to calculate the FDR analytically.

Chapters 1, 2 and 3 are mainly introductory and short reviews of well-established background material. Chapter 4 is, instead, a review of very recent literature. The last section of Chapter 4, as well as the whole Chapter 5 present original material.

Contents

Chapter 1

Critical relaxational phenomena and the field theoretical approach

In this first chapter we give a short introduction to the main concepts and tools that are used in order to study critical relaxational phenomena, mainly from a field-theoretical perspective. Our focus is on those systems which evolve with a non-stationary dynamics. We introduce the fluctuation-dissipation ratio (FDR), which is an observable that gauge the distance from a stationary situation. We introduce the model of relaxational dynamics that we are interested in, i.e. the so-called *purely relaxational* (or A) model and its field-theoretical implementation, to characterize the dynamics of an *order parameter* field. We show how to calculate the FDR for the case of a critical quench of model A in the Gaussian solution. We review the very important concept of *universality* in the context of critical phenomena and we report the results which state that the FDR is indeed a universal observable in the renormalization group sense. In the last section we give a state of the art for what concern the determination of the fluctuation-dissipation ratio in its long-time limit, for a critical quench of the purely relaxational model.

The presentation of this chapter is inspired by Refs.[8, 6, 9] which contain all the additional details.

This chapter is organized as follows:

Sec. 1.1: We describe the relaxational dynamics of an order parameter field via a suitable Langevin equation. We define the FDR, i.e. the observable which we are mainly interested in in this thesis work.

Sec. 1.2: We introduce the purely relaxational model. We investigate the solution of the model within the Gaussian approximation, in order to set the framework and to have a first at the FDR for the case of a critical quench.

Sec. 1.3: We introduce the response function formalism, which gives a field-theoretical action starting from a Langevin equation. We discuss the implementation of the conditions which give rise to a critical quench of model A and we solve the corresponding Gaussian solution.

Sec. 1.4: After a brief recap of equilibrium critical phenomena and the concepts that arise in this context, we report the results that proved the universality of the FDR in the long-time limit.

Sec. 1.5: We review the state of the art concerning the determination of non-equilibrium universal quantities predicted by the purely relaxational model, focusing on the results for the FDR in the long-time limit.

1.1 Dynamics, the fluctuation-dissipation theorem and the fluctuation-dissipation ratio

1.1.1 Dynamics and observables

Let us consider a system in contact with a thermal bath at a given temperature T. In principle the dynamics of the system is specified by its microscopic Hamiltonian, either classical or quantum, via the evolution equations for the density matrix and phase-space density, respectively. However, this fully microscopic approach is rarely viable for actual real statistical systems. A description of the dynamics in terms of *mesoscopic* variables is, in some cases, preferable, since it focuses directly on those quantities that are expected to determine the dynamical properties at length and time scales that are much larger than the microscopic ones (at atomic or molecular level) but still small compared to macroscopic one set by the dimension of the sample. Mesoscopic variables (or observables), such as the local magnetization density in magnetic systems are obtained by averaging the corresponding microscopic quantities on mesoscopic length and time scales. This averaging is usually referred to as *coarse-graining*.

A viable approach to dynamics, which was first successfully applied to the Brownian motion, consists of a description which takes advantage of the separation between the typical time scale of fast (microscopic) and slow (mesoscopic) dynamical processes, clearly emerging in some cases. It is natural to assume that the dynamics of the mesoscopic observables can be described as the result of an effective slow deterministic drift towards a stationary state (equilibrium or not) and of a stochastic force that sums up the effect of the fast microscopic fluctuations. Of course, this description fails to reproduce the dynamics taking place at microscopic time and length scales. Hence, when the macroscopic physics is crucially related to some microscopic events such a mesoscopic description is not expected to describe the relevant physical processes. We will strictly consider systems and phenomena for which the mesoscopic description is feasible. This is the case, for example, when one considers the dynamics of the order parameter field close to a second-order phase transition at which it is known that because of the critical slowing down the dynamics occurs on a mesoscopic time scale.

Let us assume that the mesoscopic properties of the system are described by a classical field ϕ, which represents, e.y., the coarse-grained local magnetization field for the example of the magnetic system. In terms of ϕ, the effective Hamiltonian is given by $\mathcal{H}[\phi]$. The previous heuristic considerations motivate

the assumption that the dynamics of the system is described by the Langevin equation

$$\partial_t \phi(\mathbf{x}, t) = -\mathcal{D} \frac{\delta \mathcal{H}[\phi]}{\delta \phi(\mathbf{x}, t)} + \eta(\mathbf{x}, t), \tag{1.1}$$

where η is a white noise, whose first and second moment are respectively given by

$$\langle \eta(\mathbf{x}, t) \rangle = 0, \qquad \langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2\mathcal{N} \delta^{(d)}(\mathbf{x} - \mathbf{x}') \delta(t - t'), \tag{1.2}$$

where $\mathcal{D}$ and $\mathcal{N}$ are either constants or a differential operators depending on the specific model which we are considering. $\mathbf{x}$ and t represents, respectively, the d-dimensional spatial coordinates and the time.

From the Langevin equation (1.1) it is possible to derive the Fokker-Planck equation for the probability density $P[\phi(\mathbf{x}), t]$ of finding the system in a field configuration $\phi(\mathbf{x})$ at time t. The stationary distribution for which $\partial_t P[\phi(\mathbf{x}), t] = 0$ is the Gibbs one $P_G \sim e^{-\beta \mathcal{H}[\phi]}$ if and only if the Einstein relation $\mathcal{N} = \beta \mathcal{D}$ is satisfied. If the system is ergodic, in the long-time limit, $P[\phi(\mathbf{x}), t]$ will be given by P_G, independently of the initial condition of the system.

We would like to find an observable which can distinguish a stationary dynamics (translationally invariant in time) from a genuine non-equilibrium dynamics (not invariant under time translations). Local time observable such as $\langle \phi(\mathbf{x}, t) \rangle$ are not interesting in this sense because they reach an asymptotic value in the long-time limit, from which it is no longer possible to extract information about the system's dynamics. The two-times observables which we are going to consider are in the following given by:

(i) the correlation function of the mesoscopic variable $\phi(\mathbf{x})$ is given by

$$C_{\mathbf{x}-\mathbf{x}'}(t, s) = \langle \phi(\mathbf{x}, t) \phi(\mathbf{x}', s) \rangle, \tag{1.3}$$

and it is related to the relaxation of spontaneous (thermal) fluctuations of the local magnetization density field ϕ.

(ii) the response function is a way to characterize the response of the system to an external perturbation. In the following we assume that the perturbation is given by an external field $h(\mathbf{x})$ that couples linearly to the field $\phi(\mathbf{x})$ at the level of the Hamiltonian $\mathcal{H}$: $\mathcal{H}_h[\phi] = \mathcal{H}[\phi] - \int_{\mathbf{x}} \phi(\mathbf{x}) h(\mathbf{x})$. In the case of magnetic systems this external field plays the role of the magnetic field. The linear response function $R_{\mathbf{x}-\mathbf{x}'}(t, s)$ is then defined by

$$R_{\mathbf{x}-\mathbf{x}'}(t, s) = \frac{\delta \langle \phi(\mathbf{x}, t) \rangle_h}{\delta h(\mathbf{x}', s)} \Big|_{h=0}, \tag{1.4}$$

where we indicate by $\langle . \rangle_h$ the mean value over the stochastic dynamics induced by the Langevin equation (1.1) with the effective Hamiltonian $\mathcal{H}_h$. Note that because of causality $R_{\mathbf{x}-\mathbf{x}'}(t, s) = 0$ if $s > t$.

One can easily derive the following identity by using the definition of the two-time functions and Eq. (1.1):

$$2T R_{\mathbf{x}-\mathbf{x}'}(t, s) = (\partial_s - \partial_t) C_{\mathbf{x}-\mathbf{x}'}(t, s) - A_{\mathbf{x}-\mathbf{x}'}(t, s), \tag{1.5}$$

where we have defined the asymmetry function $A_{\mathbf{x}-\mathbf{x}'}(t,s)$ via

$$A_{\mathbf{x}-\mathbf{x}'}(t,s) = \langle \phi(\mathbf{x},t) D \frac{\delta \mathcal{H}[\phi]}{\delta \phi(\mathbf{x}',s)} \rangle - \langle D \frac{\delta \mathcal{H}[\phi]}{\delta \phi(\mathbf{x},t)} \phi(\mathbf{x}',s) \rangle. \tag{1.6}$$

We see that the two-time functions are not independent from each other.

1.1.2 Fluctuation-dissipation theorem and its violation: definition of fluctuation-dissipation ratio

When the system equilibrates, the relation between the two-time functions, given by Eq. (1.5), assumes a simpler form. In fact the time-translational invariance (TTI) implies that the correlation and response functions satisfy $C_{\mathbf{x}-\mathbf{x}'}(t,s) = C_{\mathbf{x}-\mathbf{x}'}(t-s,0)$ and $R_{\mathbf{x}-\mathbf{x}'}(t,s) = R_{\mathbf{x}-\mathbf{x}'}(t-s,0)$. Accordingly, $(\partial_t - \partial_s) C_{\mathbf{x}-\mathbf{x}'}(t,s) = -2\partial_s C_{\mathbf{x}-\mathbf{x}'}(t,s)$. Moreover, equilibrium is characterized by time-reversal symmetry (TRS), which implies that the correlation function of two observable $\mathcal{O}_1(t)$ and $\mathcal{O}_2(t)$ satisfies $\langle \mathcal{O}_1(t)\mathcal{O}_2(s)\rangle = \langle \mathcal{O}_1(s)\mathcal{O}_2(t)\rangle$, so that the asymmetry vanishes. Taking into account these observations, one concludes from Eq. (1.5) that

$$R_{\mathbf{x}}(t,s) = \beta \partial_s C_{\mathbf{x}}(t,s), \tag{1.7}$$

which is just one formulation of the fluctuation-dissipation theorem (FDT). Note that the FDT holds also if C is replaced by the connected correlation function, since, due to TTI, $\langle \phi_{\mathbf{x}}(t) \rangle$ is independent of time in equilibrium.

In generic dynamical situation both the assumptions, i.e. TTI and TRS, are not realistic and instead of the relationship between $R_{\mathbf{x}}$ and $C_{\mathbf{x}}$ determined by the fluctuation-dissipation theorem, given by Eq. (1.5), the following more general relation holds between these two-time functions:

$$R_{\mathbf{x}}(t,s) = X_{\mathbf{x}}(t,s) \partial_s C_{\mathbf{x}}(t,s), \tag{1.8}$$

where we have introduced the fluctuation dissipation ratio (FDR), given by

$$X_{\mathbf{x}}(t,s) = \frac{T R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)}. \tag{1.9}$$

Whenever s is larger than the equilibration time $t_{\rm eq}$ of the system, the dynamics satisfies time-reversal symmetry and time-translational invariance so that Eq. (1.7) $X_{\mathbf{x}}(t,s) = 1$.

The asymptotic value of the FDR, given by

$$X^{\infty} = \lim_{s\to\infty} \lim_{t\to\infty} X_{\mathbf{x}=0}(t,s), \tag{1.10}$$

is a suitable quantity for the description of systems with *slow dynamics*, i.e., those systems whose relaxational dynamics persists for extremely long times. In fact, $X^{\infty} = 1$ whenever $t_{eq} < \infty$, i.e., the fluctuation-dissipation ratio is equal to one if the system reaches a stationary state; $X^{\infty} \neq 1$, instead, is the signal of an ever lasting non-equilibrium dynamics.

Assuming that the system under study is translationally invariant, we consider in what follows the response function $R_{\mathbf{q}}(t,s)$ and the correlation function $C_{\mathbf{q}}(t,s)$ in Fourier space. Within it, it is natural and convenient to define the fluctuation-dissipation ratio as

$$\mathcal{X}_{\mathbf{q}}(t,s) = \frac{TR_{\mathbf{q}}(t,s)}{\partial_s C_{\mathbf{q}}(t,s)}, \tag{1.11}$$

which, however, is not the Fourier transform of $X_{\mathbf{x}}(t,s)$. Nevertheless, the following relation is verified

$$X^{\infty}_{\mathbf{x}=0} = \mathcal{X}^{\infty}_{\mathbf{q}=0} = X^{\infty}, \tag{1.12}$$

where $X^{\infty}_{\mathbf{q}=0}$ is defined from $\mathcal{X}_{\mathbf{q}}$ in analogy with X^{∞} in Eq. (1.10). For more information in this interesting very last equality we invite the interested reader to look at Ref. [8].

1.2 The purely relaxational model and its Gaussian solution

We introduce the representative of a large class of models whose purpose is to describe relaxational phenomena at a coarse-grained level in the context of unfrustrated statistical systems. The purely relaxational model (model A in the classification of Hohenberg and Halperin, given in Ref. [7]) is given by the Langevin equation presented in Eq. (1.1) in which $\mathcal{D} = D$ and $\mathcal{N} = N$, i.e., are constants. We choose the deterministic drift present in the Langevin equation to be in the class of $\mathbb{Z}_2$-invariant Hamiltonians. In this case the Hamiltonian of the system is given by the Landau-Ginzburg effective Hamiltonian, which has been a powerful tool for the study of the ferromagnetic phase transition, see Ref. [2]. The effective Hamiltonian $\mathcal{H}(\phi)$ is given, with the introduction of a magnetic field h, by

$$\mathcal{H}_{\mathrm{GL}} = \int d^d\mathbf{x} \left[\frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{r}{2}\phi^2(\mathbf{x}) + \frac{g}{4!}\phi^4(\mathbf{x}) - h(\mathbf{x})\phi(\mathbf{x}) \right], \tag{1.13}$$

where $g > 0$ in order to have a stable Hamiltonian for $r < 0$ and r is related to the temperature. In particular, within the mean field approximation (MF), r is the shift from the critical temperature T_c of the model : $r \sim (T - T_C)$, while g is an interaction term which stems from the neighbor interaction of the lattice Ising model from which the field theory, given by Eq. (1.13), arises. The first term on the r.h.s. accounts for an energy penalty for configurations that exhibit domain walls or spatial inhomogeneities.

The purely relaxational model has been proven to be very useful in order to study the following physical phenomenon. Consider an Ising ferromagnet in equilibrium at temperature T_0 greater than its critical temperature T_C of the second-order phase transition. Then, at time t_0 put the ferromagnet in contact

with a thermal bath at temperature $T < T_0$. In general the ferromagnet will undergo two distinct regimes: the first (A) is a genuine non-equilibrium one for $t < t_{\rm eq}(T)$, where $t_{\rm eq}(T)$ is the equilibration time of the system; the second (B) is a stationary regime for $t \gg t_{\rm eq}(T)$. In some cases regime (B) is never reached because $t_{\rm eq} = \infty$, for example if the ferromagnet is quenched to its critical temperature.

Let us discuss the case $g = 0$, for which it is possible to study only the regime in which $r \geq 0$. The correlation length ξ is an observable which describe how microscopic variables, such as spin and density, at different positions are related. It is a well known fact that it diverges upon approaching the critical point of a second-order phase transition as $\xi \sim r^{-\nu}$, thus describing a scale-free system at criticality. The parameter ν is one of the so-called critical exponent. Because of the divergence of the correlation length, we expect that the equilibration time $t_{\rm eq}(T_C)$ will diverge for a quench at the critical point, while it remains finite whenever the quench is at a temperature higher than the critical one, as we see in a moment from the solution of the Gaussian model, corresponding to setting $g = 0$. In momentum space the Langevin equation given by Eq. (1.1) becomes

$$\partial_t \phi_{\mathbf{q}}(t) = -D\omega_q \phi_{\mathbf{q}}(t) + \eta_{\mathbf{q}}(t), \tag{1.14}$$

where $\omega_q = (q^2 + r)$. Note that if a magnetic field linearly coupled to the order parameter field ϕ is introduced, the equation becomes

$$[\partial_t + D(q^2 + r)]\phi_{\mathbf{q}}(t) = \eta_{\mathbf{q}}(t) + Dh_{\mathbf{q}}(t). \tag{1.15}$$

Using the definition of the response function, if we consider first the evolution of $\langle \phi_{\mathbf{q}}(t) \rangle$ and then we differentiate functionally with respect to the magnetic field h, as in Eq. (1.4), we obtain

$$[\partial_t + D\omega_q] R_{0,\mathbf{q}}(t, s) = D\delta(t - s). \tag{1.16}$$

The solution is immediately given by

$$R_{0,\mathbf{q}}(t, s) = D\vartheta(t - s)e^{-D\omega_q (t-s)} = DG_{0,\mathbf{q}}(t - s), \tag{1.17}$$

where $\vartheta(t)$ is the Heaviside step-function. In the following we will refer to $G_{0,\mathbf{q}}$ as the propagator associated with the Gaussian Langevin equation in momentum space. Taking advantage of the propagator's properties we could easily write down the solution for the field with $h = 0$, which is the fundamental ingredient we need for the calculation of the correlation function:

$$\phi_{\mathbf{q}}(t) = \phi_{\mathbf{q}}(t_9) G_{0,\mathbf{q}}(t, t_0) + \int_{t_0}^{\infty} dt' G_{0,\mathbf{q}}(t, t') \eta_{\mathbf{q}}(t'), \tag{1.18}$$

where $\phi_{\mathbf{q}}(t_0)$ is the boundary field $\phi_{\mathbf{x}}(t_0)$ in Fourier space, which represent the initial condition at time $t_0 = 0$ where we do the temperature quench. From this solution, we calculate the correlation function according to the definition

$$\begin{aligned} \langle \phi_{\mathbf{q}}(t) \phi_{-\mathbf{q}}(s) \rangle =& |\phi_{\mathbf{q}}(0)|^2 G_{0,\mathbf{q}}(t, 0) G_{0,\mathbf{q}}(s, 0) + \\ &+ 2D \int_0^{\infty} dt' G_{0,\mathbf{q}}(t, t') G_{0,\mathbf{q}}(s, t'). \end{aligned} \tag{1.19}$$

We are interested in an initial microscopic state which corresponds to high-temperature macroscopic state. The initial condition is therefore itself a random variable, and the dynamical quantities which we are interested in are mediated over this initial condition and over the stochastic dynamics induced by the noise term. Let us take an initial state which is prepared with a small value of the magnetization, identified by the following properties:

$$\begin{cases} \langle \phi_0(\mathbf{x}) \rangle_{\rm ic} = a(\mathbf{x}), \\ \langle [\phi_0(\mathbf{x}) - a(\mathbf{x})][\phi_0(\mathbf{x}') - a(\mathbf{x}')] \rangle_{\rm ic} = \Delta^{-1} \delta^{(d)}(\mathbf{x} - \mathbf{x}'). \end{cases} \tag{1.20}$$

In the last equation $\langle \cdot \rangle_{\rm ic}$ means the average over the probability distribution of the initial condition and Δ may be a constant or a local differential operator in space.

Consider now the correlation of the order parameter ϕ averaged over the noise η and the initial condition, i.e., $C_{0,\mathbf{q}}(t,s) = \langle \phi_{\mathbf{q}}(t) \phi_{-\mathbf{q}}(s) \rangle_{\eta,{\rm ic}}$. We obtain

$$C_{0,\mathbf{q}}(t,s) = \frac{D}{\omega_q} \left[e^{-D\omega_q |t-s|} - e^{-D\omega_q (t+s)} \left(1 - \frac{\omega_q}{\Delta} \right) \right]. \tag{1.21}$$

Let us analyze the two relevant limiting cases:

(*i*) Note that if $\Delta = \omega_q$ then the initial condition are equilibrium one, and $C_{0,\mathbf{q}}(t,s) = C^{\rm eq}_{0,\mathbf{q}}(t-s) = \frac{e^{-D\omega_{\mathbf{q}}|t-s|}}{q^2+r}$. This last equation can indeed be directly retrieved from the Ginzburg-Landau Hamiltonian, given by Eq. (1.13), as the inverse of the double functional derivative of $\mathcal{H}$ with respect to the coarse-grained order parameter field ϕ.

(*ii*) Suppose that the system is prepared in a very defined state, which means $\Delta \gg 1$. In the limit $\Delta^{-1} \to 0$ we obtain the Dirichlet correlator, given by

$$C_{0,\mathbf{q}}(t,s) = C^{\rm D}_{0,\mathbf{q}}(t,s) = \frac{D}{\omega_{\mathbf{q}}} \left[e^{-D\omega_q |t-s|} - e^{-D\omega_q (t+s)} \right], \tag{1.22}$$

which is such that $C^{\rm D}_{0,\mathbf{q}}(t,0) = 0$, i.e., the correlation with the initial state vanishes. In this case, the differential equation which the correlation function must satisfy is given by

$$[\partial_s + D\omega_q] C_{0,\mathbf{q}}(t,s) = 2D G_{0,\mathbf{q}}(t,s) = 2R_{0,\mathbf{q}}(t,s). \tag{1.23}$$

Using Eq. (1.21) for the correlation function, the FDR is :

$$\mathcal{X}_{0,\mathbf{q}}(t,s) = \frac{R_{0,\mathbf{q}}(t,s)}{\partial_s C_{0,\mathbf{q}}(t,s)} = \frac{1}{1 + e^{2\omega_q s}(1 - \omega_q \Delta^{-1})}. \tag{1.24}$$

We see again that we can set the initial condition in such a way that $\mathcal{X}_{0,\mathbf{q}}(t,s)$ is equal to 1 at all times, i.e., $\omega_q = \Delta$. This choice would corresponds to sampling the initial condition from the equilibrium distribution of the Gaussian model described by the effective Hamiltonian given in Eq. (1.13) with $g = 0$. If, instead, the initial condition is not an equilibrium one we have two different cases:

(**1**) for $r > 0$, $X^\infty = 1$ for every value of q and for $q = 0$ we obtain $t_{eq}(T_c) \sim r^{-1} \sim \xi^2$, thus retrieving the mean field result $\nu = 1/2$.

(**2**) for $r = 0$, i.e., at the Gaussian critical point, for every $q \neq 0$ we obtain again equilibrium after an equilibration time given by $t_0^{\rm eq}(T_C) \sim q^{-2}$. For $q = 0$ the long-time limit becomes $X_0^\infty = 1/2 \neq 1$, indicating that the zero mode does not equilibrate.

We introduce the dynamic critical exponent z by means of the relation

$$t^{\rm eq}(T_C) \sim q^{-z}. \tag{1.25}$$

In the Gaussian case we have found that $z = 2$, as expected for a diffusion process. Note that upon approaching a critical point the typical time scale of dynamics of the fluctuations around the equilibrium state diverges as $\sim \xi^z$ (*critical slowing down*). This provides the natural separation between the relevant slow evolution due to the developing collective behavior and the fast one related to microscopic processes. This separation makes the mesoscopic description of the dynamics a particularly viable approach to the problem.

1.3 From a Langevin equation to a dynamical field theory

1.3.1 Construction of the dynamical action

Thanks to universality we can study in a successful way the critical regime of a ferromagnet starting from the Landau-Ginzburg effective Hamiltonian, as we have seen in the previous Section. Analogously, we expect that universal behaviors typical of non-equilibrium systems can be described in terms of a *dynamical* field theory. The procedure to obtain the dynamical action from the Langevin equation has been outlined by Janssen-De Dominicis-Peliti (JDP), see for a pedagogical introduction the Ref. [8].

The starting point is to define what kind of observables we are interested in. Let us take a generic observable $\mathcal{O}[\phi]$, function of the coarse-grained field ϕ introduced previously. Accordingly, when the dynamics of ϕ is realized by a Lengevin equation such as Eq. (1.1), we know that the interesting observable is the average of $\mathcal{O}[\phi]$, given by

$$\langle \mathcal{O} \rangle = \int [d\eta] \mathcal{O}[\phi_\eta] P_G[\eta] = \int [d\phi] \mathcal{O}[\phi] \Big\{ \int [d\eta] \delta(\phi - \phi_\eta) P_G[\eta] \Big\}, \tag{1.26}$$

where $P_G[\eta]$ is the Gaussian probability associated to the white noise η introduced previously, and ϕ_η is the solution of the Langevin equation for a given realization of the noise and for a given initial condition $\phi(t_0) = \phi_0$ assigned at time $t_0 = 0$. Recall the following properties of the Dirac delta:

$$\delta(\phi - \phi_\zeta) = \delta(\partial_t - \mathcal{F} - \zeta) \det \left[\partial_t - \frac{\delta \mathcal{F}}{\delta \phi} \right], \tag{1.27}$$

where $\mathcal{F} = -D\delta\mathcal{H}/\delta\phi$. At this point one expresses the functional Dirac delta introducing an auxiliary field $\tilde{\phi}$: $\delta(\gamma) = \int [d\tilde{\phi}] \exp\{\int dt d^d x\ \tilde{\phi}\gamma\}$. The integral of the noise can be carried out and gives

$$\langle \mathcal{O} \rangle = \int [d\phi][d\tilde{\phi}] \mathcal{O} e^{-S_{t_0}[\phi,\tilde{\phi}]}, \tag{1.28}$$

where the JDP action is given by

$$S_{t_0}[\phi,\tilde{\phi}] = \int_{t_0}^{\infty} \int d^d x \{\tilde{\phi}(\partial_t \phi - \mathcal{F}[\phi]) - D\tilde{\phi}^2\}. \tag{1.29}$$

The dynamic functional S_{t_0} is the starting point for a field theory approach to the dynamics prescribed by a Langevin equation. We remark that in Eq. (1.30) the term corresponding to the determinant is missing. To be properly evaluated, it requires a discretization of the Langevin equation and eventually its expression depends on the chosen discretization. Nevertheless the result of the computation of averages is actually independent of the particular choice which can be made in such a way to render the determinant equal to one. In turn, this implies that $R_{\mathbf{x}-\mathbf{x}'}(t,t) \sim \langle \phi(\mathbf{x},t)\zeta(x',t)\rangle = 0$, corresponding to the so-called Îto prescription in stochastic calculus.

Note that $\tilde{\phi}$ has a clear physical meaning. This is understood if we add a magnetic field to the effective Hamiltonian $\mathcal{H}$ which couples linearly to ϕ, so that $\mathcal{H}_h = \mathcal{H} - \int d^d x h\phi$. This imply that the linear response function of observable $\mathcal{O}$ with respect to the field h is given by

$$\frac{\delta\langle \mathcal{O} \rangle}{\delta h(\mathbf{x},s)} = \langle \tilde{\phi}(s) D\mathcal{O} \rangle, \tag{1.30}$$

for this reason $\tilde{\phi}(\mathbf{x},s)$ is called *response field*. In particular the response function for the order parameter reads

$$R_{\mathbf{x}-\mathbf{x}'}(t,s) = \left.\frac{\delta\langle \phi(\mathbf{x},t)\rangle_h}{\delta h(\mathbf{x}',s)}\right|_{h=0} = \beta\langle \tilde{\phi}(\mathbf{x}',s) D\phi(\mathbf{x},t)\rangle. \tag{1.31}$$

Note that, accordingly to what is done for the Langevin equation, we have to set the initial condition in order to properly describe a relaxational process. This can be done adding a boundary term $\mathcal{H}_{\mathrm{b}}[\phi_0]$ in the action, which will be discussed in the following.

1.3.2 Gaussian solution for the purely relaxational model

We turn now to the Gaussian solution of the purely relaxational model introduced in previous Section by means of the dynamical action obtained following the JDP procedure. Remember that in this case the deterministic drift given by $\mathcal{H}$ in Eq. (1.29) is chosen as the Landau-Ginzburg Hamiltonian given by Eq. (1.13). In this case the dynamical action becomes:

$$\mathcal{S}_{t_0}[\phi,\tilde{\phi}] = \int_{t_0} dt \int_q \{\tilde{\phi}\,[\partial_t + D\omega_q \phi] - D\tilde{\phi}^2\} + \mathcal{S}_{\mathrm{b}}[\phi_0], \tag{1.32}$$

where we have introduced the boundary action $\mathcal{S}_b[\phi_0]$, whose purposes is to implement the high-temperature initial state from which the relaxational dynamics starts. The boundary effective Hamiltonian is given by

$$\mathcal{S}_{\rm b}[\phi_0] = -\frac{\Delta}{2}\int d^dx\,(\phi_0 - a)^2\,. \tag{1.33}$$

To find the classical equations of motion we introduce the currents $j, \tilde{j}$ linearly coupled respectively with the fields ϕ and $\tilde{\phi}$. The classical equation of motions, which does not take into account for the effect of fluctuations, for the field ϕ and $\tilde{\phi}$ are respectively given by

$$\begin{cases} \dfrac{\delta\mathcal{S}_{t_0}}{\delta\tilde{\phi}} = 0 \to \tilde{j} + 2D\tilde{\phi} = [\partial_t + D\omega_q]\phi, \\ \dfrac{\delta\mathcal{S}_{t_0}}{\delta\phi} = 0 \to j = [-\partial_t + D\omega_q]\tilde{\phi} + \delta(t)[-\tilde{\phi}_0 + \Delta(\phi_0 - a)]. \end{cases} \tag{1.34}$$

In the last equations, fields and currents must be identified with the corresponding Fourier modes $\mathbf{q}$ and evaluated ad time t. Furthermore in order to obtain the second equation in Eq. (1.34) we have integrated by parts the term $\tilde{\phi}\partial_t\phi$ in the dynamical action S_{t_0} given by Eq. (1.32), and we have set $\tilde{\phi}(t) \to 0$ for $t \to \infty$. Accordingly we see that $\tilde{\phi}_0$ must satisfy an initial condition given by $\tilde{\phi}_0 = \Delta(\phi_0 - a)$.

We can differentiate the equation of motion for the response field, i.e., the second equation in Eq. (1.34), with respect to j, obtaining

$$[\partial_t + D\omega_q]G_{\mathbf{q}}(t,s) = \delta(t-s). \tag{1.35}$$

We can use this very last equation to write down the solution for the coarse-grained order parameter field ϕ and for the response field, respectively, as

$$\begin{cases} \phi(t) = G_{\mathbf{q}}(t,t_0)\phi_0 + \displaystyle\int_{t_0}^{\infty} G_{\mathbf{q}}(t,t')[\tilde{j}(t') + 2D\tilde{\phi}(t')]dt' = \\ \qquad = \displaystyle\int_{t_0}^{\infty} G_{\mathbf{q}}(t,t')[\tilde{j}(t') + 2D\tilde{\phi}(t') + [a - \Delta^{-1}\tilde{\phi}(t)]\delta(t)], \\ \tilde{\phi}(t) = G_{\mathbf{q}}(t,t_0)\tilde{\phi}_0 + \displaystyle\int_{t_0}^{\infty} G_{\mathbf{q}}(t',t)j(t')dt'. \end{cases} \tag{1.36}$$

Deriving the first equation in Eq. (1.36) with respect to j we obtain

$$C_{\mathbf{q}}(t,s) = G_{\mathbf{q}}(t,t_0)G_{\mathbf{q}}(s,t_0)\Delta^{-1} + 2D\int_{t_0} G_{\mathbf{q}}(t,t')G_{\mathbf{q}}(s,t')dt', \tag{1.37}$$

that is the same expression found earlier, i.e. Eq. (1.19), by means of the solution of the Langevin equation (if we choose t_0) from which the result for the fluctuation-dissipation ratio, given by Eq. (1.24) follows as before.

1.4 Universality in critical relaxational phenomena

1.4.1 second-order phase transition, a brief reminder

The main purpose of the study of critical phenomena is to extract information about the universal observables which arises in this context, such as critical exponents. These are pure numbers that characterize the particular class of field theories with given dimensionality, symmetry and range of the interaction, i.e. belonging to the same universality class. The critical exponents, such as the dynamical critical exponent z introduced in the previous Section, characterize *non analyticities* in thermodynamic quantities. To treat efficiently, i.e. taking into account the effect of fluctuations, the net-divergences which appears in a naive perturbative treatment a *regularization scheme* is mandatory. In statistical field theory we have to remind ourselves that the net-divergences that we care about are those in the infrared- and not those in the ultraviolet- regime that are naturally not present because of the cutoff Λ imposed by the lattice from which the field theory arises via suitable coarse-graining procedures.

The classical Wilsonian program for studying the critical regime is to write down the perturbative *beta-functions* for the *relevant couplings* which appear in the Landau-Ginzburg Hamiltonian and from them extract the critical exponents. We will not go into the details of perturbative renormalization group (pRG) techniques in this thesis, but we refer the reader to Refs.[5, 6]. The β-functions dictate how the couplings gets renormalized by the very effect of fluctuations. The relevant couplings are defined phenomenologically as follows. For the ferromagnetic transition we know experimentally that we have just to set two external parameters in order to fine-tune the system at criticality. These control parameters are of course the magnetic field h and the temperature T which have to be precisely fine tuned respectively at $h = 0$ and $T = T_c$, in order to hit the critical point at which the second-order phase transition occurs. The mathematical description of the system should respect this feature so that tuning only two parameters we must have a description of the critical state. In the renormalization group language we say that the Wilson-Fisher fixed point has only two relevant parameters.

The success of the Wilsonian program is to find in $d = 4 - \epsilon$ a fixed point of the perturbative β-functions of the Ginzburg-Landau action which has only two relevant parameters (t, h) and for which g is irrelevant but assume a non-zero value at the fixed point, resulting in a strongly interacting, scale free[1] field theory in the infrared which describes the Ising universality class. The critical exponents are then calculated with ϵ as the expansion parameter and, in the simplest case, at the end of the calculation which is carried at fixed order in power of ϵ, it is possible to extract the critical exponent in dimension $d = 3$ by extrapolating the solution for $\epsilon \to 1$.

[1] Remember that, experimentally, we observe magnetization clusters of *all* size at the critical point of the ferromagnetic second-order phase transition.

Let us summarize the picture of the critical points that a pRG treatment of the Landau-Ginzburg model at order ϵ gives, varying the dimensionality of the system. First of all, the Gaussian fixed point coincides with the Wilson-Fisher one in dimension larger than 4. In 4 spatial dimensions, the interaction term g is marginal, and this give rise to logarithmic deviation from the Gaussian fixed point. In dimension less than 4, g is relevant in the vicinity of the Gaussian fixed point, but, since (r, h) are always relevant, this means that the Gaussian fixed point is not anymore the critical one. This picture is presented in Fig. 1.1.

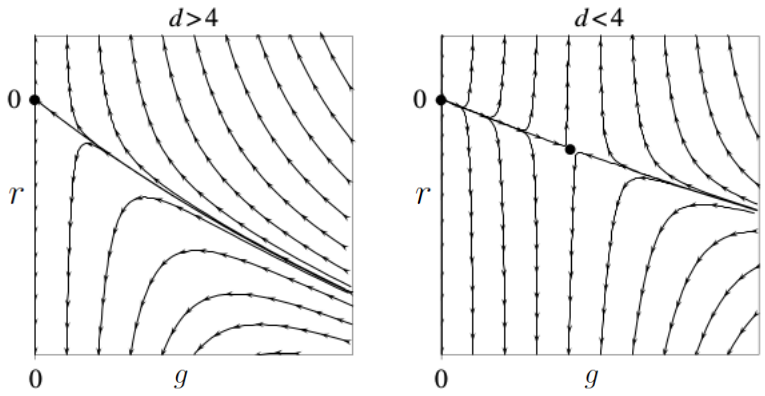


Figure 1.1: Wilsonian flow of the parameters r and g at first order in ϵ. This picture is taken from Ref. [5].

The Wilson-Fisher fixed point is characterized by *anomalous dimensions* related to the field ϕ and consequently the naive scaling analysis conducted at the level of the Landau-Ginzburg effective Hamiltonian gives wrong predictions. These anomalous dimension effects are due to the fact that the second-order phase transition problem is a multi-scale problem: the first scale is the correlation length ξ, which diverges at the critical point; the second scale is the ultraviolet cutoff $\Lambda \sim 1/a$ used to regularize the theory. The dependence of the observables from the cutoff in the vicinity of the fixed point gives rise to simple power-law dependence, thus introducing an anomalous dimension to the equilibrium correlation function of the theory. As an example of naive dimensional analysis, consider the correlation function in equilibrium situation and in the Gaussian approximation. One can see, starting from the Landau-Ginzburg Hamiltonian, that it satisfies a Laplace equation whose solution is given by

$$C(q) = \frac{1}{q^2 + r}, \tag{1.38}$$

so that one would expect by dimensional analysis to find

$$\chi = \int d^d x \; C(x) = C(q = 0) = \xi^2 \sim r^{-2\nu}. \tag{1.39}$$

This very last scaling relation is not satisfied by actual experiments, suggesting that the correct scaling form for the correlation function is, instead, given by

$$C(x) = \frac{1}{x} f(x/\xi, a/\xi). \tag{1.40}$$

It turns out that for $a/\xi < 1$, although the function f is not independent on this ratio, it nevertheless exhibits a simple power-law dependence, proportional to $(a/\xi)^\eta$, where η is a small but nonzero exponent, defining the so-called *anomalous dimension*. As a consequence, the dependence of the susceptibility on the correlation length has the form

$$\chi \sim a^\eta \xi^{2-\eta}. \tag{1.41}$$

The fundamental reason for the emergence of these anomalous dimension effects is given by the fact that critical behavior is dominated by fluctuations, and these fluctuations take place at all length scales, all the way down to the microscopic distance a. Taking into account these effects we make the following Ansatz for the two-point correlation function,

$$C(q) = |q|^{-2+\eta} \hat{C}_\pm(q\xi), \quad \xi = \xi_\pm |r|^{-\nu}. \tag{1.42}$$

Near T_c, we know that $\chi \sim \lim_{q\to 0} |q|^{-2+\eta} \hat{C}(q\xi)$; as $q \to 0$, the leading terms in the scaling function must cancel the singular prefactor, and therefore

$$\chi \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} = |r|^{-\gamma}, \text{ which implies } \gamma = \nu(2-\eta). \tag{1.43}$$

Within the Gaussian approximation, $\eta = 0$ and $\nu = 1/2$.

We conclude and summarize this subsection by emphasizing the fact that it is an experimental fact that naive dimensional analysis gives wrong results for the critical exponents. This is because of the presence of *anomalous* effects that arises from the fact that the second-order phase transition problem is a multi-scale problem.

1.4.2 Critical quench of the purely relaxational model

It is possible to fully exploit universality in order to characterize within a field-theoretical approach the non-equilibrium critical relaxation. This approach allows a systematic analysis of several aspects of these relaxation phenomena and yields analytic predictions for scaling functions, exponents, and amplitude ratios which characterize the scaling behavior of correlation and response functions at large times, within different universality classes and dynamics.

The renormalization group analysis of the purely relaxational model predicts the following behaviors after a quench to $T = T_c$ (see Ref. [8] for their detailed derivation):

(*i*) The dynamical critical exponent z which describe the diverging linear relaxation time in the vicinity of the critical point is the same in stationary and in non-equilibrium situation.

(*ii*) Only one new independent universal exponent is needed in order to fully characterize the scaling behavior of the two-time functions, i.e. the so-called *critical initial-slip* exponent θ. The resulting scaling form for the response and correlation functions are respectively given by:

$$R_{\mathbf{q}=0}(t,s) = A_R(t-s)^a(t/s)^\theta F_R(s/t), \tag{1.44}$$

$$C_{\mathbf{q}=0}(t,s) = A_C s(t-s)^a(t/s)^\theta F_C(s/t), \tag{1.45}$$

where $a = (2-\eta-z)/z$. The *non-universal* amplitudes $A_{R,C}$ are fixed by imposing that the universal scaling functions satisfy $F_{R,C}(0) = 1$. From the previous equations it follows that

$$\partial_s C_{\mathbf{q}=0}(t,s) = A_{\partial C}(t-s)^a(t/s)^\theta F_{\partial C}(s/t), \tag{1.46}$$

where, again, $F_{\partial C}(0) = 1$ and $A_{\partial C} = A_C(1-\theta)$. Note that the s/t time dependence of the two-point functions at zero momentum, given in Eqs. (1.44) and (1.45), is breaking the time-translational invariance. Furthermore the decay as a function of t is slower for larger s. This phenomenon is usually referred to as *aging*: older samples respond more slowly. The time s is called the "age" of the system or also *waiting time*, being the time elapsed since the preparation of the system.

(*iii*) The asymptotic value of the fluctuation-dissipation ratio, i.e., X^∞ is a universal observable and follows as an amplitude ratio.
For the asymptotic value of the fluctuation-dissipation ratio at $\mathbf{q} = 0$, the general scaling behaviour is given by:

$$\mathcal{X}_{\mathbf{q}=0}(t,s) = \frac{R_{\mathbf{q}=0}(t,s)}{\partial_s C_{\mathbf{q}=0}(t,s)} = \frac{A_R F_R(s/t)}{A_{\partial C} F_{\partial C}(s/t)}. \tag{1.47}$$

Note that $\mathcal{X}_{\mathbf{q}=0}$ it is a function of the ratio s/t and not of s and t separately. The asymptotic value of the FDR is given by

$$\begin{aligned} X^\infty &= \lim_{s\to\infty}\lim_{t\to\infty}\frac{R_{\mathbf{q}=0}(t,s)}{\partial_s C_{\mathbf{q}=0}(t,s)} = \\ &= \lim_{s/t\to 0}\mathcal{X}_{\mathbf{q}=\mathbf{0}}(s/t) = \frac{A_R}{A_{\partial C}} = \frac{A_R}{A_C(1-\theta)}. \end{aligned} \tag{1.48}$$

Some very important remarks:

- The critical initial slip exponent θ together with the non-universal amplitudes (A_R and A_C) of the two-time functions completely characterize the FDR. From the RG analysis it turns out that

$$\theta = -\frac{\eta_0}{z}, \tag{1.49}$$

where η_0 is the anomalous dimension of the boundary field $\phi_0 = \phi(t=0)$.

- X^∞ is a universal quantity because it is the ratio between two quantities with the same scaling dimensions (which, separately are not universal). The fact that the FDR in the aging limit depends on non-universal amplitudes like A_R and A_C rather than only on universal critical exponents makes the calculation quite more complicate than those that are made in order to predict usual universal quantities (critical exponents) which are simply connected with the eigenvalues of the linearized RG equations. In fact, for the calculations of the FDR, the determination of the two-times functions is mandatory in order to extract these non-universal amplitudes.

1.5 The fluctuation-dissipation ratio in the long-time limit: available estimates for the purely relaxational model

We present here the state of the art concerning the determination of the universal non-equilibrium quantities which characterize the purely relaxational dynamics of an unfrustrated Ising magnet.

Let us review the basics of the Monte Carlo (MC) technique, remembering that the results for universal observables obtained with this technique should be compared with the field-theoretical, either perturbative renormalization group (pRG), discussed previously, or functional renormalization group (fRG), presented in the following. In fact, the coarse-grained continuum dynamics described by the Langevin equation (1.1) is expected to be in the same universality class as lattice models with $O(N)$ symmetry, short-range interactions and spin-flip dynamics, see Ref. [13]. One of the simplest non-trivial lattice models displaying slow dynamics after a quench to the critical point is a lattice spin model in d dimension with $O(N)$ symmetry, which evolves according to a purely dissipative dynamics. In the simplest instance its Hamiltonian is given on the lattice by

$$\mathcal{H} = -\sum_{\langle \mathbf{ij} \rangle} \mathbf{s_i} \cdot \mathbf{s_j}, \tag{1.50}$$

where $\mathbf{s_i}$ is a N-component spin located at the lattice site $\mathbf{i}$, with $\mathbf{s_i}^2 = 1$. Here and in the following the symbol $\langle \mathbf{ij} \rangle$ means that the sum runs on all nearest-neighbor pairs of lattice sites. When $N = 1$ the Hamiltonian given in Eq. (1.50) describes the Ising universality class. A purely dissipative dynamics for the lattice model given by Eq. (1.50) proceeds by elementary moves that consists of attempted random changes in the direction of the spin $\mathbf{s_j}$ (spin-flip sampling). The transition rates can be arbitrarily chosen provided that the detailed-balance condition is satisfied. For analytical studies the most suited is the Glauber dynamics, explained in Ref. [13], which allows some exact solution. Given its relative simplicity, this lattice model is most studied and best understood. The physically relevant cases with finite N and $d = 2, 3$ are not analytically solvable and have been intensively investigated by means of Monte Carlo simulations.

MC: Two-dimensional Ising model. The most accurate determination of the critical exponent z is $z = 2.1665(5)$ (this number is obtained by means of equilibrium MC, since the exponent z is the same in and out of equilibrium). The most accurate value of the non-equilibrium critical exponent θ is given by $\theta = 0.383(3)$. Concerning the FDR, earlier investigations indicate $X^\infty = 0.26(1)$, and the subsequent more accurate estimate are given by $X^\infty = 0.340(5)$ and $X^\infty = 0.33(2)$. For the reference to the original works see the review in Ref. [8].

MC: Three-dimensional Ising universality class. The most accurate value for the static critical exponents are given by $\beta = 0.325(1)$, $\nu = 0.630(1)$, and $z = 2.024(6)$ recently reported in Ref. [19]. The critical initial-slip exponent θ in the review in Ref. [8], dating back to 2002, was $\theta = 0.14(1)$, but a recent and accurate estimates give $\theta = 0.135(3)$ (see Ref. [22]). Regarding the FDR a preliminary analysis, dating back to 2000 (given in Ref. [11] without demonstration of the data), gives $X^\infty \sim 0.40$, while the more recent estimate gives $X^\infty = 0.380(12)$ and $X^\infty = 0.391(12)$ (see Ref. [19]), respectively obtained by means of a heat-bath updating rule and by an application of random probing magnetic fields with small amplitude in the simulation process.

pRG: Two-loop calculation. The program of calculating the universal two-times functions, the associated non-universal amplitudes and the corresponding FDR has been worked out up to two loops in the ϵ-expansion in Ref. [17]. The analytic calculation is rather involved and therefore we summarize here only the results. The obtained exponents z and θ agree with the already known two-loop expression[2]. We give here the result for θ:

$$\theta = \frac{\epsilon}{12}\left[1 + \epsilon\left(\frac{8}{27} + \frac{2\log 2}{3}\right)\right] + O(\epsilon^3), \tag{1.51}$$

where $\epsilon = 4 - d$ in $d < 4$ while $\theta = 0$ in $d > 4$. The scaling of the response function is characterized by the non-universal amplitude (for $d < 4$)

$$A_R = 1 + \epsilon^2 \frac{3(N+2)}{8(N+8)^2} C + O(\epsilon^3), \tag{1.52}$$

where C is a numerical factor that we do not report here. In terms of $A_R, A_{\partial C}$, and θ, the long-time limit of the critical FDR is obtained via Eq. (1.48):

$$\frac{(X^\infty)^{-1}}{2} = 1 + \epsilon\frac{N+2}{4(N+8)} + \epsilon^2\frac{N+2}{(N+8)^2}\left[\frac{N+2}{8} + \frac{3(3N+14)}{4(N+8)} + c\right] + \mathcal{O}(\epsilon^3), \tag{1.53}$$

where c has an analytic expression which renders $c = -0.0415...$

fRG: local potential approximation. Recently Ref. [35] address for the first time the problem of calculating genuine universal non-equilibrium exponents using the fRG technique. We postpone to Chapter 3 the presentation of the general technique of the fRG, and to Chapter 4 the analysis of Ref. [35]. The extrapolation of the results related to the best approximation of fRG equations

[2]Note that in Ref. [22] a three-loop results for θ is given. We do not report it here since it is in strong agreement with the result reported in the very same article via MC technique.

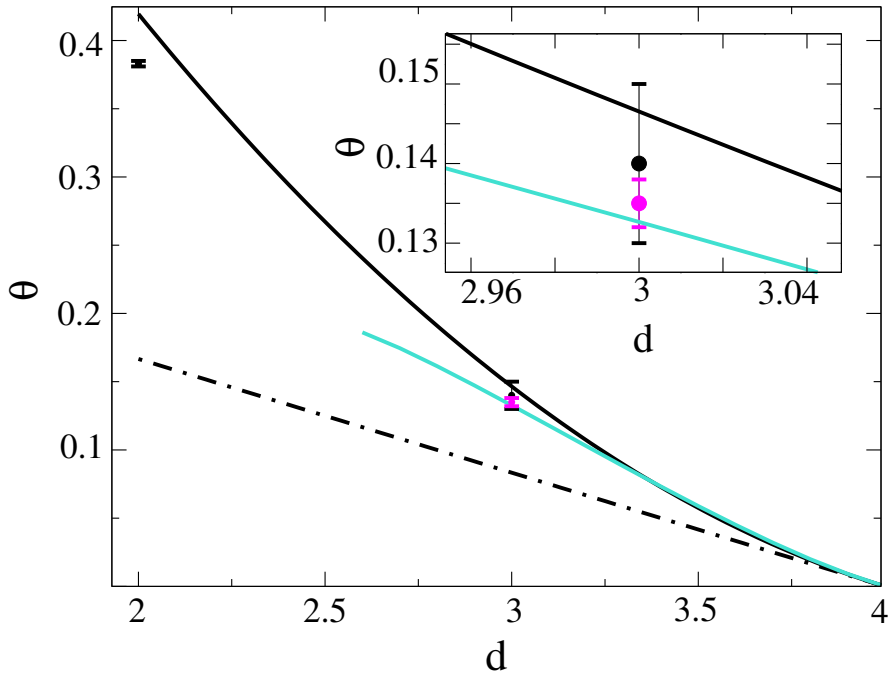


Figure 1.2: Available estimates of the critical initial-slip exponent as a function of the spatial dimensionality d. The MC results are indicted by the various symbols with error bars: the black ones are reported in Ref. [8], while the magenta one is the recent estimate of Ref. [22]. The pRG results are indicated by the black dash-dotted and solid lines, respectively at order ϵ and ϵ^2. The best fRG result to date is the solid turquoise line, obtained within a local potential approximation in Ref. [35]. Inset: magnification of the main plot for $d \simeq 3$.

for critical initial-slip exponent θ reported in Ref. [35] in $d = 3$ is given by $\theta_{fRG} \sim 0.1326$. No predictions for X^∞ using the fRG technique are given in literature to date, and providing them is the objective of this thesis.

Let us comment on all these estimates and compare them. They are reported in Figs.(1.2) and (1.3), respectively for the critical initial-slip exponent θ and for the asymptotic value of the FDR, indicated by X^∞. We can see from the figures that the recent MC predictions for θ and X^∞ (obtained in Ref. [22, 19]), indicated by magenta symbols with error bars, are not in agreement with perturbative field-theoretical calculations, indicated by black lines. In particular, for the FDR in $d = 3$ the recent MC estimates give $X^\infty_{\rm MC} = 0.380(13)$ and $0.391(2)$ which do not agree with $X^\infty_{\epsilon^2} = 0.429(6)$ (estimates for $d = 3$ of the lower solid line, i.e. of the lower Padé approximant), which has been calculated with the ϵ-expansion up to two loops.

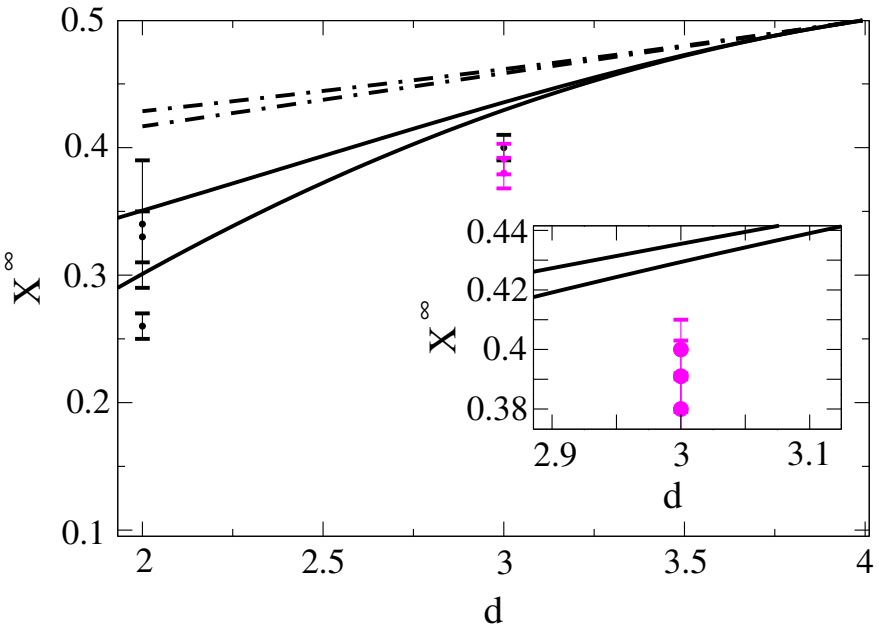


Figure 1.3: Available estimates of the asymptotic value of the FDR as a function of the spatial dimensionality of the system d. The MC results follows the same graphical picture as the results reported in Fig. 1.2. The black dash-dotted and solid lines are pRG results respectively at order ϵ and ϵ^2, for which we have plotted also the corresponding Padé approximant. No fRG results are available for X^∞ in literature. Inset: magnification of the main plot for $d \simeq 3$.

This work is meant to study how to investigate X^∞ with a different procedure from the perturbative one, the only analytic approach used so far, and, in particular, by using the functional renormalization group. We anticipate here that one of the main difficulty of fRG is that the time translational invariance of the dynamics is broken by the temperature quench.

Chapter 2

From the effective Hamiltonian to the effective action

The *bare* actions, such as the Landau-Ginzburg effective Hamiltonian or the Janssen-De Dominicis-Peliti action, do not have a direct connection with the macroscopic behaviour. The quantity that takes into account the effects of fluctuation, thus describing correctly the physics at a macroscopic level, is given by the effective action. This is indeed the quantity that the functional renormalization group technique described in the next chapter aims to calculate. This chapter is devoted to the introduction of the concept of effective action (EA). We highlight in what follows how from the knowledge of the effective action related to some specific model one has access to the full two-point functions as well as to any other Green functions of the theory.

The presentation of this chapter is inspired by Refs.[23, 6, 29] which contain all the additional details.

This chapter is organized as follows:

Sec. 2.1: We introduce the concept of EA, stressing how from the knowledge of it one has access to the full two-point functions of the theory considered. We show how from the EA, neglecting the effect of fluctuations, one obtains the mean-field approximation.

Sec. 2.2: We briefly show how the effects of fluctuation change qualitatively and quantitatively the mean-field picture, thanks to the one-loop approximation of the EA.

Sec. 2.3: We analyze the distinct scales that occur in the computation of the effective action.

2.1 The effective action

2.1.1 The Landau-Gibbs thermodynamic potential

Landau unified the mean-field approximations used to describe the second-order phase transition by means of the following ansatz for the Gibbs potential

$$\Gamma(M) = V\left(\frac{r}{2}M^2 + \frac{g}{4!}M^4 - Mh\right), \tag{2.1}$$

where $g > 0$, in order to have a potential bounded from below, M is the order parameter of the second-order phase transition and V is the volume of the system under study. We know from undergraduate thermodynamics that the minimum of the Gibbs potential (denoted by $\bar{M}(T,h) = \bar{M}(h)$) represents the equilibrium thermodynamic state at given experimental condition specified by the temperature T and the homogeneous field h. In the case of ferromagnetic systems $\bar{M}$ is the average magnetization and h is the applied magnetic field. We know that this ansatz for the Gibbs potential is qualitatively correct, in the sense that it admits a second-order phase transition from a disordered state, in which $\bar{M}$ is zero, to an ordered state, in which $\bar{M}$ acquires a non-zero value, which correspond to a symmetry-breaking state. This phase transition occurs at the critical point $r = 0, h = 0$ for the model described by Eq. (2.1).

The susceptibility χ, defined as $\chi = \partial M/\partial h$, is given by the Landau-Gibbs potential in Eq. (2.1), in terms of its derivative with respect to the order parameter M evaluated at the equilibrium configuration, i.e. $M = \bar{M}(h)$:

$$\chi^{-1}(M) = \left.\frac{\delta h}{\delta M}\right|_{M=\bar{M}(h)} = \left.\frac{\delta^2\Gamma}{\delta M^2}\right|_{M=\bar{M}(h)}. \tag{2.2}$$

We obtain from the very last equations that

$$\chi(h) = \frac{1}{r + \frac{g}{2}\bar{M}^2(h)}. \tag{2.3}$$

In this way, we have found the equation that allows us to calculate the susceptibility directly from the Landau-Gibbs potential.

In the following we will show how this very simple procedure in order to obtain information on the two-point functions of the theory, such as the susceptibility considered here, is easily generalized in order to calculate any observables starting from a field-theoretical description of the system.

2.1.2 Definition and basic properties of the effective action

Consider a field theory defined by the following partition function:

$$Z[j] = \int \mathcal{D}\phi(x)\; e^{-\mathcal{S}[\phi(x)] - \int_x j(x)\phi(x)}, \tag{2.4}$$

where we have inserted an external source $j(x)$ and $\mathcal{S}$ is the bare action (or the effective Hamiltonian $\mathcal{H}$) of the microscopic theory. From this partition

function we define the generating functional of connected Green's functions as $W[j] = \ln Z[j]$; as the name suggests, it allows the computation of all Green's functions in a very simple way: in order to calculate the connected n-point Green function it is sufficient to take the n-th functional derivatives of $W[j]$ with respect to $j(x)$ and then evaluate it in the physical configuration given by vanishing source, i.e. $j = 0$. In the following, we identify the n-th functional derivative of $W[j]$ with respect to j in a compact notation such that, for example, the first and second derivatives are given by W_j and W_{jj}. The first two physical Green functions of the theory are therefore given by:

$$\begin{cases} W_j\big|_{j=0} = \langle \phi(x) \rangle_{j=0} = \Phi_0(x), \\ W_{jj}\big|_{j=0} = \langle \phi(x)\phi(y) \rangle_{j=0} - \langle \phi(x) \rangle \langle \phi(y) \rangle_{j=0} = G_c(x,y). \end{cases} \tag{2.5}$$

The very last equation is saying that the first and the second functional derivatives of the generating functional W evaluated in the physical configuration without source, i.e. with $j = 0$, correspond respectively to the expectation value of the field and to the connected correlation function of the theory.

From $W[j]$ we define the effective action (EA) by means of a Legendre transform:

$$\Gamma[\Phi] = -W[j] + \int_x \Phi(x) j(x), \tag{2.6}$$

where $\Phi = \Phi(x) = \langle \phi(x) \rangle_j = W_j(x)$. This important quantity is the generating functional of the vertex functions (see Ref. [23]). From the very last two equations we derive the following identity

$$\Gamma_\Phi[\Phi] = j(x), \tag{2.7}$$

where again we have used the compact notation in order to indicate functional derivatives with respect to Φ in the l.h.s.. This equation shows how the *physical expectation values* $\Phi_0(x)$ of the field $\Phi(x)$ corresponds to a stationary point of the effective action,

$$\Gamma_\Phi[\Phi_0] = 0, \tag{2.8}$$

coherently with the standard thermodynamic picture implemented by the Landau-Gibbs potential, described in the previous sub-section.

For the connected two-point function of the theory we obtain:

$$G_c(x,y) = W_{jj}(x,y)\big|_{j=0} = \left(\frac{\delta j(x)}{\delta \phi(y)} \right)^{-1} \Bigg|_{j=0} = \Gamma^{-1}_{\Phi\Phi}(x,y)\big|_{\Phi=\Phi_0}, \tag{2.9}$$

which should be compared with Eq. (2.2) in order to see the continuity of the framework outlined here with the standard thermodynamic one which is used in the mean-field description made by Landau.

In full generality, the equation of motion for the two-point function of the theory is given by

$$\int d^d y \, \left[\; G_c(x,y) \Gamma_{\Phi\Phi}(y,z)|_{\Phi=\Phi_0} \; \right] = \delta^d(x-z), \tag{2.10}$$

which is obtained from Eq. (2.9).

2.1.3 The Dyson equation for the two-point function

The general equation of motion for the two-point function, given by Eq. (2.10), can be simplified in Fourier space (assuming that the physical system under study is translationally invariant in space), and it yields:

$$G_c(k) = \frac{1}{\Gamma_{\Phi\Phi}(k)|_{\Phi=\Phi_0}}. \tag{2.11}$$

Note that we can write down the following formal identity:

$$\Gamma_{\Phi\Phi}(k)|_{\Phi=\Phi_0} = G_0(k)^{-1} - \Sigma(k), \tag{2.12}$$

in which we have separated the Gaussian contribution, i.e., $G_0(k)$, from the interacting one, given by the so-called *self energy* , indicated by $\Sigma(k)$. Taking advantage of the very last equation, the connected correlation function $G_c(k)$ is now given by

$$G_c(k) = [G_0(k)^{-1} - \Sigma(k)]^{-1}. \tag{2.13}$$

This very last equation can be recast in the form of an integral equation for G_c

$$G_c = G_0 - G_0 \Sigma G_c, \tag{2.14}$$

which is called the Dyson equation for the two-point function. In the last equation we have not specified the variables and the 'product' operations between the terms in the r.h.s.. This is because Eq. (2.14) can be indeed obtained also in real space.

In what follow we will consider a non-equilibrium systems which are space-translational invariant in space, but which are not invariant under time translations, which implies that there are no simplification in the time sector and the Dyson equation will be then very useful.

In general, we can make the following statement: if one has access to the self energy Σ, and of course on the bare Hamiltonian (which gives G_0), one is able to calculate the two-point function of the theory from Eq. (2.14).

2.2 Approximations of the effective action

Note that the definition of the effective action, given by Eq. (2.6), can be rewritten as:

$$e^{-\Gamma[\Phi]+\int d^dx\ j(x)\Phi(x)} = \int \mathcal{D}\phi(x)\ e^{-\mathcal{S}[\phi]+\int d^dx\ j(x)\phi(x)}. \tag{2.15}$$

We expand the field around its minimum configuration Φ defined as $\Phi = \Phi(x) = \langle\phi(x)\rangle_j$. With the change of variable $\phi \to \Phi + \chi'$, we obtain

$$e^{-\Gamma[\Phi]} = \int \mathcal{D}\chi'(x)\ \exp\left[-S[\Phi+\chi'] + \int_x j(x)\chi'(x)\right], \tag{2.16}$$

which is the starting point for perturbation analysis at the level of effective action, in which one aim at accounting for the effects of fluctuations, encoded in the field χ', in a systematic way. In this section we review how fluctuations drastically changes the picture described by the tree level (or mean-field) approximation, detailed before.

2.2.1 Tree-level approximation of the effective action for systems in and out of equilibrium

The lowest order of approximation, called tree-level approximation, we obtain, corresponding to the saddle-point approximation applied to Eq. (2.16), one finds the following important relationship:

$$\Gamma_{tree}[\Phi] = S[\Phi]. \tag{2.17}$$

This shows that the zeroth-order approximation of the effective action, corresponding to neglecting the fluctuations, renders the bare action $\mathcal{S}$ evaluated on its minimum.

For the relevant case of the bare Landau-Ginzburg Hamiltonian Eq. (2.17) gives

$$\Gamma_{tree}[\Phi] = \mathcal{H}[\Phi] = \frac{1}{2!}\int d^dx[(\nabla\Phi)^2 + r\Phi^2] + \frac{g}{4!}\int d^dx\ \Phi^4, \tag{2.18}$$

because, in this case, $\mathcal{H} = S$. Since the equilibrium configuration Φ has to minimize the effective action it has to be homogeneous, i.e., $\Phi(x) = \Phi_0$ and we obtain the Landau-Gibbs potential introduced in the previous section, given by Eq. (2.1), with the following identification: $\Phi_0 = M$.

Now we discuss what happens in the case of model A. The bare action S_{t_0} for model A is given in Eq. (1.29). Within the tree-level approximation, using Eq. (2.17), we obtain the following expression for the effective action:

$$\Gamma_{\rm tree}[\Phi] = \vartheta(t-t_0)\int_{\mathbf{x}}\int_{t_0} dt\ \tilde{\phi}\left(\partial_t\phi - D\nabla^2\phi + DU_\phi(\phi) - D\tilde{\phi}\right), \tag{2.19}$$

where $U(\phi) = r\phi^2/2! + g\phi^4/4!$ and U_ϕ its derivative with respect to ϕ. In this case Φ is the doublet of fields $\Phi = (\phi, \tilde{\phi})$. The minimum (physical) configurations of this doublet of fields is given by $\Phi_0 = (0, \phi_m)$.

Let us retrieve the equations of motion for the Gaussian non-equilibrium two-point functions, i.e., R and C. The equation of motion for the two-point function is given by Eq. (2.10), and involves the correlation function G_c and the quantity $\Gamma_{\Phi\Phi}$. In these case both $\Gamma_{\Phi\Phi}$ and G_c are matrix, given by

$$\hat{\Gamma}_{\Phi\Phi}[\Phi] = \begin{bmatrix} 0 & \Gamma_{\tilde{\phi}\phi} \\ \Gamma_{\phi\tilde{\phi}} & \Gamma_{\phi\phi} \end{bmatrix} \text{ and } \hat{G}_c = \begin{bmatrix} C_q(t,s) & R_q(t,s) \\ R_q(s,t) & 0 \end{bmatrix}. \tag{2.20}$$

Correspondingly, the double derivative of the tree-level effective action with respect to Φ evaluated in its minimum configuration is given by

$$\hat{\Gamma}_{\text{tree},\Phi\Phi}\Big|_{\Phi_0} = \begin{bmatrix} 0 & -\partial_t - D\nabla^2 + DU_{\phi\phi}|_{\phi_0} \\ \partial_t - D\nabla^2 + DU_{\phi\phi}|_{\phi_0} & -2D \end{bmatrix}, \tag{2.21}$$

and the equations of motions are thus given by

$$\begin{cases} \left(-\partial_s + Dq^2 + DU_{\phi\phi}|_{\phi_0}\right) R_q(t,s) = \delta(t-s), \\ \left(\partial_s + Dq^2 + DU_{\phi\phi}|_{\phi_0}\right) C_q(t,s) = 2DR_q(s,t), \end{cases} \tag{2.22}$$

where the locality in time of the bare action in Eq. (2.19) implies the cancellation of the time integrals while the space integrals are properly evaluated in Fourier space. If one focuses on the high-temperature regime, in which Gaussian approximation holds, one obtains $\phi_0 = 0$ and $U_{\phi\phi}|_{\phi_0=0} = r$, thus retrieving again the Gaussian equations of motion for the response and correlation functions obtained in previous chapter, i.e., Eqs. (1.16) and (1.19).

2.2.2 One-loop approximation of the equilibrium effective action

In order to implement the one-loop approximation of the effective action we expand the bare action $\mathcal{S}[\Phi + \chi']$ present in the r.h.s. of Eq. (2.16) to the second-order in the fluctuations χ', neglecting higher-order terms. After the Gaussian integration over the fluctuation field χ' one eventually obtains:

$$\Gamma_{1-\text{loop}}[\Phi] = \mathcal{S}[\Phi] + \frac{1}{2}\text{Tr}\ln\hat{\mathcal{S}}_{\Phi\Phi}[\Phi], \tag{2.23}$$

in which the trace of the logarithm of a matrix the matrix $\hat{\mathcal{S}}_{\Phi\Phi} = \delta^2\mathcal{S}[\Phi]/\delta\Phi\delta\Phi$ is given by the sum of the logarithm of its eigenvalues. Thus, for the case of the Landau-Ginzburg Hamiltonian, the following equation for the effective action holds:

$$\begin{aligned} \Gamma_{1-\text{loop}}[\Phi] = \int d^dx \left[\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}r\Phi^2 + \frac{g}{4!}\Phi^4\right] + \\ + \text{Tr}\ln\left[-\nabla^2 + r + \frac{g}{2}\Phi^2\right]. \end{aligned} \tag{2.24}$$

Expanding the trace of the ln in the very last equation in powers of Φ around the minimum Φ_0 in the high-temperature regime, i.e. $\Phi_0 = 0$, we obtain

$$\text{Tr}\log\left[-\nabla^2 + r + \frac{g}{2}\Phi^2\right] = \text{Tr}\log[-\nabla^2 + r] + \text{Tr}\log\left[1 + \frac{1}{-\nabla^2 + r}g\frac{\Phi^2}{2}\right], \tag{2.25}$$

where the second term on the r.h.s. is explicitly given by

$$\text{Tr}\log\left[1 + \frac{1}{-\nabla^2 + r}g\frac{\Phi^2}{2}\right] = \sum_{n=1}^{\infty}\left(-\frac{g}{2}\right)^n\frac{1}{n}\text{Tr}\left(\frac{1}{-\nabla^2 + r}\Phi^2\right)^n. \tag{2.26}$$

By means of the identification $G_0 = 1/(-\nabla^2 + r)$, the effective action at 1-loop is thus given by

$$\Gamma_{1-\text{loop}}[\Phi] = \Gamma_{tree}[\Phi] + \text{Tr}\log(-\nabla^2 + r) + \sum_{n=1}^{\infty}\left(-\frac{g}{2}\right)^n\frac{1}{n}\text{Tr}(G_0\Phi^2)^n. \tag{2.27}$$

The terms summed by the series on the r.h.s. of this equation are conveniently identified with the so-called *Feynman diagrams* illustrated in Fig. 2.1. We see

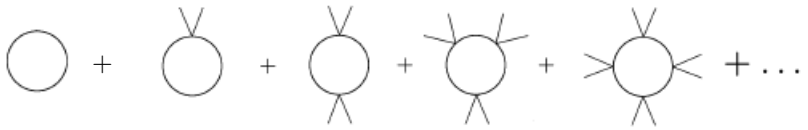


Figure 2.1: These are the diagrams which contribute, in a 1 loop approximation, to the effective action. The lines which forms the loops are gaussian correlation functions G_0, while the vertex are given by the bare coupling costant g.

that the 1-loop approximation of the effective action contains all the diagrams with one loop and all numbers of bare vertex of the Landau-Ginzburg theory. In the following we will see that the second diagram, which has two un-amputated legs, corresponds to a shift of the critical temperature. This is because it is a local contribution (the legs of the diagram are attached to the same point). The second diagram will give a shift of the fourth powers of the derivative of Γ with respect to Φ, i.e., the coupling constant g. The other contribution will give rise to renormalization of higher-order coupling constant, for example λ for the coupling of the Φ^6 term (and so on), which are irrelevant in the renormalization group sense.

2.2.3 Infrared divergences and the shift of the critical temperature

We present here how, from the analysis of the two-point function, the one-loop approximation can induce qualitative changes in our description of the critical point, such as a shift in the critical temperature with respect to the critical temperature of the Gaussian model. In the case of a constant background field $\Phi = \phi$ and via a Fourier transform one concludes that the effective action, given by Eq. (2.24), simplifies to

$$\Gamma_{1-loop}(\phi) = \frac{1}{2} r_0^2 \phi^2 + \frac{1}{4!} g_0 \phi^4 + \frac{1}{2} \int_q \log\left(q^2 + r_0 + \frac{g_0}{2}\phi^2\right), \tag{2.28}$$

note that we have add the subscript 0 in order to indicate the bare couplings.

In the absence of some sort of cutoff this integral over q is divergent because of high momenta contributions are not bounded, i.e. *ultraviolet divergences*. As it was pointed out before, one can always imagine to be considering the discretized version of the model where there is a natural momentum cut-off of the order of $\Lambda = 1/a$ where a is the lattice space. We will suppose that such a regulator exists even if not specifying it explicitly for notation simplicity. Moreover, we will show now that in some cases the system can have low momentum divergences, i.e. *infrared divergences*. These divergences are much more difficult to treat and

they will be at the heart of our concerns here, because they are actually the origin of critical behavior.

At this order, the expression for the relation between the magnetization and the external magnetic field becomes:

$$j = \frac{\partial \Gamma_{1-\text{loop}}}{\partial \phi} = r_0 \phi + \frac{1}{3!} g_0 \phi^3 + \frac{g_0}{2} \phi \int_q \frac{1}{q^2 + r_0 + g_0 \frac{\phi^2}{2}}, \tag{2.29}$$

and the susceptibility

$$\chi^{-1} = \frac{\partial j}{\partial \phi}\Big|_{j=0} = r_0 + \frac{g_0}{2} \int_q \frac{1}{q^2 + r_0}. \tag{2.30}$$

One observes that the terms that came from the one-loop correction to the mean-field approximation are suppressed by a factor of the coupling constant g_0. This property is true also at higher orders of the expansion around the mean-field approximation: each new term includes high powers of g_0.

The second-order phase transition takes place at the temperature where the magnetic susceptibility diverges. Then $r(T = T_C) = \chi^{-1}(T = T_c) = 0$. At first sight this is problematic, because at that temperature r_0 must be negative in order to compensate for the second (positive) term of the r.h.s. of Eq. (2.30), but then the integrand has a pole and is not well defined. This is an artifact of the approximation, because we can replace in that term r_0 by r because the correction would be of order g_0^2:

$$r = \frac{\partial j}{\partial \phi}\Big|_{j=0} = r_0 + \frac{g_0}{2} \int_q \frac{1}{q^2 + r}. \tag{2.31}$$

At $T = T_c$, then one finds:

$$0 = r_{0,c} + \frac{g_0}{2} \int_q \frac{1}{q^2}, \tag{2.32}$$

where we have defined $r_{0,c} = r_0(T = T_c)$. Subtracting Eq. (2.32) from Eq. (2.31) one arrives at

$$r = r_0 - r_{0,c} - \frac{g_0}{2} r \int_q \frac{1}{q^2(q^2 + r)}. \tag{2.33}$$

Depending on the dimensionality d, two different cases arise:

- If $d > 4$ the integral is dominated by the ultraviolet range of wave-numbers q but is regular when $T \sim T_C$. If the integral has a cut-off at $q \sim \Lambda$, one finds

  $$\chi^{-1} = r_0 - r_{0,c} - C g_0 r \Lambda^{d-4}, \tag{2.34}$$

  where C is a numerical constant. In this case, corrections to mean-field expression is regular and is small as long as $g_0 \Lambda^{d-4} \ll 1$. The critical exponents (such as γ) are the same as their mean-field approximation, at least in a finite neighborhood of $g_0 = 0$.

- If $d < 4$ the ultraviolet behavior is sub-leading (one can safely take the limit $\Lambda \to \infty$), but the integral is dominated (and divergent) when $T \to T_C$ (or, equivalently $r \to 0$). In that case,

$$\chi^{-1} = r_0 - r_{0,c} - rC' g_0 r^{(d-4)/2}, \tag{2.35}$$

  where C' is another numerical constant. It is clear that the mean-field approximation can only be a good approximation as long as $g_0 r^{(d-4)/2} \ll 1$. In particular, the critical regime $r \sim 0$ can not be approached by a direct expansion around the mean-field. Note that this calculations lead to a better result for the critical exponent related to the susceptibility compared to the mean-field one. In fact from Eq. (2.35) for $r \to 0$ one finds (with the identification $\chi^{-1} = r$):

$$r \sim (r_0 - r_{0,c})^{2/(d-2)} \to \chi \sim |T - T_c|^{-2/(d-2)}, \tag{2.36}$$

  where we have used the identification $r_0 - r_{0,c} = \Delta r_0 \sim T - T_c$.

Let us note at this point that the renormalization of r_c shifts towards smaller values compared to the Gaussian critical one the value of the critical temperature of the model. This is due to the fact that fluctuations tend to favor disorder rather than order, so that we have to set to lower value the temperature in order to hit the critical point.

In this section we have introduced several important concept: the physical parameter, such as r, are those defined by some physical situation (such as the vanishing of the susceptibility define r). While the bare parameter (such as r_0) which enters into the microscopic effective Hamiltonian are not directly connected with physical properties. Taking into account fluctuations the bare parameter gets *dressed* by the fluctuations, thus leading to physical parameter r different from the bare one. In the language of quantum field theory Δr_0 is the bare mass and r is the renormalized mass.

2.3 Scales of the second-order phase transitions

Before considering the strategy which allows one to avoid the difficulties of the perturbative expansion near a second-order phase transition, it is important to recognize the very origin of these difficulties. The reason is that all the wave numbers contribute significantly to the loop corrections to the mean-field if the system approaches a critical regime. It is difficult to control an approximation when many degrees of freedom with very different typical wave number scales interact in a significant way. All the momenta from the microscopic scale Λ^{-1} to the macroscopic $r^{-1/2} = \xi$ one contribute almost in the same way in the one-loop correction to mean-field approximations. The problem becomes more severe when the microscopic coupling g_0 is 'large'.

Let us analyze the distinct scales that occur in the computation of the effective action when starting from the microscopic interaction encoded in $\mathcal{H}$. We will consider the example of the Landau-Ginzburg Hamiltonian. By construction, when looking at the theory at the scale Λ we are not sensitive to the

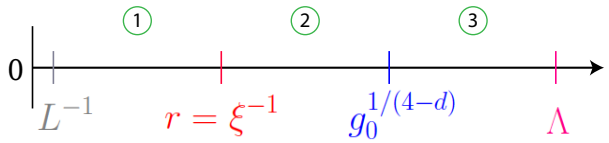


Figure 2.2: Scales in the Landau-Ginzburg model.

effects that arise from smaller momenta. Therefore, at this scale, we can identify the effective action Γ_Λ with the Hamiltonian of the system, i.e., $\Gamma_\Lambda = \mathcal{H}$. As already emphasized, we want to determine the macroscopic theory, described by the effective action Γ from the microscopic dynamics. In the course of the computation we encounter particular scales that will dominate and which will make the properties of the system change qualitatively. We will now sketch the distinct scales and outline their effects.

The relevant scales are set by Λ, the couplings in the action (mass and coupling constant) and, for finite-size system, the inverse L^{-1} of the size of the system L. For an illustration of the hierarchy of scales in momentum space see Fig. 2.2. Note that the identity $\Gamma = \mathcal{H}$ holds only at the mean-field approximation and the different scales remain being given by the bare parameters, r_0 and g_0 only at this approximation (since, as we have seen in the one-loop computation of the susceptibility the bare mass r_0 gets renormalized to r). Beyond mean-field approximation they have to be read off from the effective action. Notice also that bringing the system close to criticality requires to fine tune the bare parameters since, otherwise, the renormalized mass is of the order of the large ultraviolet scale.

With the finite scale Λ we have defined the resolution of our system in the ultraviolet. For many systems, all bare length scales are given in terms of Λ which is the fundamental ultra-violet scales and, in these cases, all microscopic scales are of order Λ, e.g., $g_0^{1/(4-d)}$.

In order to avoid considering finite-size effects we take in the following $L \to \infty$. Below Λ, the first smaller scale in momentum space is the Ginzburg scale. For the theory at hand it is defined by the coupling, $g_0^{1/(4-d)}$. As we aim at studying critical physics, the mass, which is the next distinct scale, has to be small compared to the other scales. Accordingly, it must be well separated from the Ginzburg scale. Note that only in the limit $g_0 \to \infty$ and $\Lambda \to \infty$ we can have a scale invariant theory for $T \to T_c$ for all momenta.

The behaviour of the system is qualitatively different in the regions given in Fig. 2.2. As an example, we study $\Gamma_{\phi\phi}(p)$, where we have assumed to evaluate it in a constant field configuration and hence, because of translational invariance, the correlator depends only on the absolute value of one momentum p. With reference to Fig. 2.2:

- Within region 3, $g_0^{1/(4-d)} < p \ll \Lambda$, the mass scale is negligible and, thus, the system is dominated by the scale $g_0^{1/(4-d)}$. In this domain no universal scaling behaviour related to critical physics emerges and perturbation theory works well.
- Within region 2, $r \ll p \ll g_0^{1/(4-d)}$, we find (close-to-)critical behaviour: compared to the value of the momentum, the mass is negligible, and, at the same time, the Ginzburg scale is large. As a result, the propagator exhibits a scaling according to its anomalous dimension. In the critical domain mean field fails because of the contributions of fluctuations on all scales belonging to this region add up coherently and their effects become strong. Perturbation theory re-summed by means of the renormalization group predicts the correct scaling behaviour for the two-point function with an accurate determination of η if computed at large orders (at least three loops).
- Within region 1, $p \ll r$ and, even though close to criticality, the system looks weakly correlated: the comparatively large mass suppresses deviations from mean-field and non-analyticities do not emerge. As a consequence mean-field theory (with possible perturbative corrections) works well. In a certain sense, mean-field theory is tailored for this regime. Obviously, if $r \to 0$, this region vanishes and Landau's idea of mean-field theory is no longer accurate.

Chapter 3

Introduction to the functional renormalization group

In this chapter we introduce the exact renormalization group equation for the effective action given by the functional-renormalization group (fRG) technique, based on the so-called Wetterich equation. We begin our exposition of the fRG for the case of equilibrium phenomena, leaving to the next chapter the generalization to non-equilibrium phenomena.

The presentation of this chapter is inspired by Refs.[24, 29], which contain all the additional details.

This chapter is organized as follows:

Sec. 3.1: We introduce the Wetterich equation, given by Eq. (3.9), which is an exact renormalization-group equation for the effective action.

Sec. 3.2: We analyze the infinite tower of coupled equations for the Green's functions that stems from the Wetterich equation. In this way, we introduce the fact that a truncation procedure is required, in order to be able to close this set of equations.

Sec. 3.3: We discuss the most common and simple truncation procedure, i.e., the so-called local potential approximation (LPA) of the flow equations for the Green functions. We will use this approximation in our approach to non-equilibrium phenomena exposed in chapter 4 and 5.

3.1 Exact renormalization-group equation for the effective action

3.1.1 The general idea

Critical phenomena are determined by the infrared singularities, i.e., collective behaviours, emerging from $r \to 0$, or, equivalently, $\xi \to \infty$ or $T \to T_c$. There are different ways to regularize the non-analyticities that show up in this

procedure. The first possibility is to consider a non-vanishing mass: we bring the system out of criticality, and then study the behaviour approaching the critical limit. This amounts at analyzing the change of the model with respect to a change in the mass. In a differential formulation, i.e., by taking the derivative $\frac{\partial}{\partial r}$ of the Green's functions of the theory, this leads to the Callan-Symanzik renormalization-group equation. Another possibility is to put the system in a finite box of spatial extent $L < \infty$. By means of a scaling analysis with respect to L one can thus identify the behaviour as the system approaches the critical limit, i.e., $L \to \infty$. This procedure is most conveniently applied in lattice Monte Carlo simulations.

The third way to regularize the infrared non-analyticities is realized in the functional renormalization group. The rough idea is to sum over the fluctuations existing on all wave vector k between the ultraviolet Λ and the infrared one $k = 0$, a better way than perturbatively. To this aim, we construct a family of models that interpolate smoothly and in the most convenient way between Λ and 0[1]. As we will see later, the fRG has the form of an evolution equation with respect to a momentum scale k, which we introduce as an artificial scale. The initial condition is set in the ultraviolet, corresponding to $k = \Lambda$, where $\Gamma|_\Lambda = \mathcal{H}$ and $\mathcal{H}$ is the effective Hamiltonian (or action) of the microscopic model. By slightly lowering the scale k by an amount dk we sum over fluctuations between Λ and $\Lambda - dk$. These fluctuations modify the effective action and, by iterating this step we finally reach the limit $k \to 0$, where all fluctuation on all scales have been taken into account. We are left with the (full) effective action, $\Gamma_{k=0} = \Gamma$. In summary, at a finite scale k, the running effective action Γ_k is a precursor of the effective action Γ satisfying

$$\begin{cases} \Gamma_{k=\Lambda} = \mathcal{H}, \\ \Gamma_{k=0} = \Gamma. \end{cases} \tag{3.1}$$

Due to this interpolation, the fRG equation is also known as a *flow equation*: the effective action flows through the momentum interval. The trajectory in this interpolation between $k = \Lambda$ and $k = 0$ depends on the details of the way fluctuations are summed over (i.e., on the choice of the regulator function) as we will see below. However, the limits in Eq. (3.1) are unique (see Fig. 3.1.1). This method of considering only part of the interval in Fig. 2.2 in each step relates to Wilson's idea of integrating fluctuations momentum shell by momentum shell.

In order to clarify the idea of the fRG, we construct a deformation, Z_k, of the partition function of the original microscopic model, given by Z in Eq. (2.4), by adding a term, $\Delta\mathcal{H}_k[\phi]$, to the effective Hamiltonian $\mathcal{H}$. Accordingly, Z_k is given by

$$Z_k[j] = \int \mathcal{D}\phi(x)\; e^{-\mathcal{H}[\phi] - \Delta\mathcal{H}_k[\phi] + \int_x j(x)\phi(x)}. \tag{3.2}$$

[1]Note that the fRG provides an exact equation for the effective action. However, in practical applications approximations are inevitable. Hence, the 'most convenient' way of regularization has to be chosen with respect to a given truncation scheme.

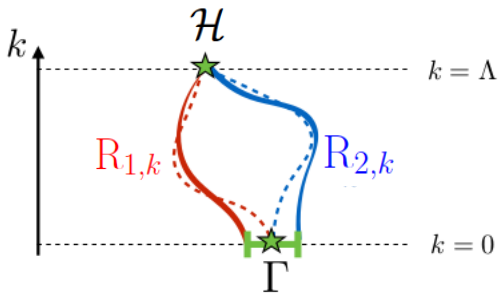


Figure 3.1: Sketch of the possible space of effective actions (horizontal axis) as a function of the cutoff scale k. In the exact fRG equation Eq. (3.9) the choice of the regulator R_k affects the path of the flow from the microscopic theory $\mathcal{H}$ at scale $k = \Lambda$ to the (full) effective action Γ but not its endpoints. This will no longer be the case when approximations are made, and the dependence of Γ on R_k becomes in turn a good indicator of the error provoked by the approximation being made. We indicate the RG flow dependence on the regulator choice, with or without (dotted lines) approximation. This image is taken from Ref. [24]

The deformation $\Delta\mathcal{H}_k$ can be chosen to be quadratic in the fields. In principle, also terms of higher-order in the fields would be possible. However, the quadratic term is the easiest one: in this case the fRG equation has one-loop structure, as we will see below. Thus, we define

$$\Delta\mathcal{H}_k[\phi] = \frac{1}{2}\int_q \phi(q)\mathrm{R}_k(q)\phi(-q), \tag{3.3}$$

where the so-called regulator R_k can be chosen arbitrarily as long as it does not contradict the condition, reported in Eq. (3.1), imposed on Γ_k. We choose the regulator as a momentum-dependent mass-like term that drives the system away from criticality when $k > 0$. In the language of quantum field theory, the term in factor of ϕ^2 in the action is the mass, and a massive particle interacts at a short range with other particles. The regulator can be seen in this case as an extra "mass term" which adds a large mass to modes of the fields with wave-vector $|q| < k$. The interactions of these particles with the others can thus be neglected: the fluctuations of these modes are now frozen. Now that we have some intuition about the regulator term $\mathrm{R}_k(q)$, let us summarize the characteristics it must have: (i) it should leave (almost) unaltered the fluctuations at scale $|q| \geq k$ in order to freeze the fluctuations at scale smaller than k. In particular, at $k = \Lambda$, *all* the fluctuations must be frozen, such that the model is described by its microscopic action, i.e., $\mathrm{R}_{k=0}(q) = 0$. (ii) The regulator term $\mathrm{R}_k(q)$ must have some regularity (at least it has to be continuous), because we will see later that its derivatives play a role in the exact differential flow equation. This properties are met by taking the regulator of a shape similar to Fig. 3.2. The fluctuation

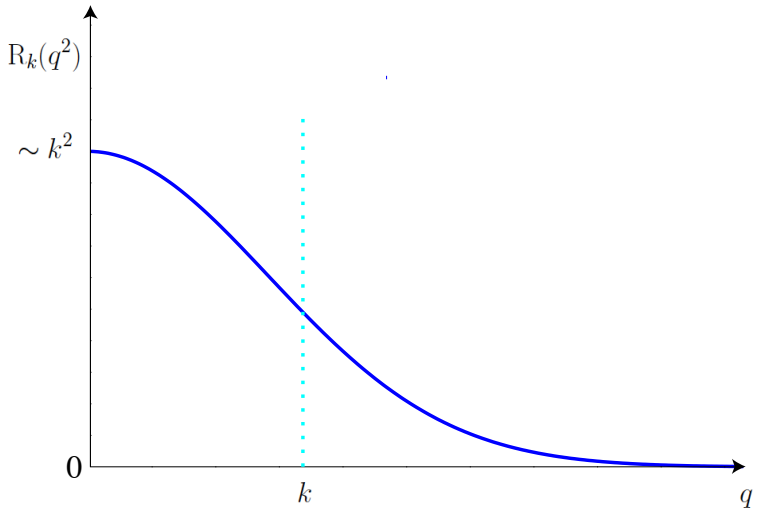


Figure 3.2: Typical shape of the cutoff function $\mathrm{R}_k(q^2)$.

modes with wavelength q of the order of k are called fast modes, in analogy with Brownian motion. Slow modes with $q \ll k$ instead are frozen by the regulator. For $k \sim \Lambda$ the regulator is of the order of Λ^2 for every q and the fluctuation are frozen. In the following we will use the so-called Litim regulator, given by

$$\mathrm{R}_k(q) = (k^2 - q^2)\vartheta(k^2 - q^2). \tag{3.4}$$

This regulator is quite convenient (especially at the lowest-order approximations) since it allows the analytical computation of the momentum integral in the renormalization-group flow. It has been also demonstrated that it is an optimized cutoff, in the sense that it reduces spurious effects, see Ref. [26].

3.1.2 The Wetterich equation

We show how to construct the modified effective action Γ_k from Z_k, given in Eq. (3.2), in such a way that it satisfies the limiting behaviour of the renormalization-group flow, given by Eq. (3.1). Starting from the modified partition function Z_k of the modified model, we define, as usual,

$$W_k = \ln Z_k, \tag{3.5}$$

and its Legendre transform

$$\Gamma_k^{Leg}[\phi] + W_k[h] = \int_x j\ \phi. \tag{3.6}$$

However, we define Γ_k using the following modified Legendre transform

$$\Gamma_k[\phi] + W_k[j] = \int_x j\phi - \Delta\mathcal{H}_k[\phi], \tag{3.7}$$

or, equivalently,

$$\Gamma_k[\phi] = \Gamma_k^{Leg}[\phi] - \Delta\mathcal{H}_k[\phi]. \tag{3.8}$$

This modification is necessary in order for Γ_k to satisfy the limiting conditions given in Eq. (3.1). Indeed, at $k = 0$ no difference has been made: $\Gamma_0 = \Gamma_0^{Leg} = \Gamma$ since $\Delta\mathcal{H}_0$ vanishes; while at $k = \Lambda$, it is easy to see that $\Gamma_\Lambda^{Leg}[\phi] \sim \mathcal{H}[\phi] + \Delta\mathcal{H}_\Lambda[\phi]$, because at this scale the mean-field approximation is almost exact, thanks to the regulator $\Delta\mathcal{H}_k$ which presents a mass $r \sim \Lambda$ large enough to suppress all fluctuations; therefore, only after subtracting $\Delta\mathcal{H}_\Lambda[\phi]$ one has $\Gamma_\Lambda \sim \mathcal{H}$, as desired.

Now we state, without proof, the cornerstone of the fRG approach, i.e., the Wetterich equation. The interested reader can find simple proof in, for example, Ref. [24]. The Wetterich equation, i.e., the exact renormalization group equation for the effective action, is given by

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\int_{x,y} \partial_k \mathrm{R}(x,y)\left(\Gamma_k^{(2)}[\phi] + \mathrm{R}_k\right)^{-1}(x,y), \tag{3.9}$$

where

$$\Gamma_k^{(2)}[\phi](x,y) = \frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)}\Big|_\phi, \tag{3.10}$$

is the propagator of the modified effective action Γ_k. $(\Gamma_k^{(2)}[\phi] + \mathrm{R}_k)^{-1}$ is the inverse of $\Gamma_k^{(2)}(x,y) + \mathrm{R}_k(x,y)$ in the sense of integral kernel linear operator. Note that:

- The Wetterich equation, like any renormalization-group equation, must be interpreted as a dynamical system: the role of time is played here by k which runs from Λ to 0. As all dynamical systems, it should be initialized with an initial condition, in this case given by $\mathcal{H} = \Gamma_\Lambda$.
- Equation (3.9) is a partial functional integro-differential equation (PDE), in the sense that $\Gamma_k[\phi] = \Gamma[k;\phi]$ depends on two variables. As a result, one has to deal with difficulties intrinsic to PDE. Even worse, the right hand side of Eq. (3.9) is *non-linear* and is *functional* in the field variable. Nevertheless, it is better posed in mathematical terms than a path-integral on the continuum.
- When the background field ϕ is uniform the (running) effective action $\Gamma_k[\phi(x) = \phi] = \Gamma_k(\phi)$ became an effective potential $U_k(\phi)$ (up to a volume factor) and the r.h.s. of Eq. (3.9) Fourier transforms into an integral over a single momentum. In $O(N = 1)$ case where $\Gamma_k^{(2)}(q,\phi)$ is a scalar, the inverse of $\Gamma_k^{(2)}(q,\phi) + \mathrm{R}_k(q)$ boils down to $1/(\Gamma_k^{(2)}(q,\phi) + \mathrm{R}_k(q))$ and

$$\partial_k U_k(\phi) = \partial_k\Gamma_k[\phi(x) = \phi] = \frac{1}{2}\int_q \frac{\partial_k \mathrm{R}_k(q)}{\Gamma_k^{(2)}(q,\phi) + \mathrm{R}_k(q)}. \tag{3.11}$$

The integrand is under control because of the presence of $\partial_k \mathrm{R}_k(q)$ in the numerator and of $\mathrm{R}_k(q)$ in the denominator: only the region where $q \sim k$ is really contributing. This implements Wilson's renormalization-group idea of momentum shell integration of fluctuations.

- If we replace, in the right hand side of Eq. (3.11), $\Gamma_k^{(2)}(q,\phi)$ by its mean-field approximation, given by $\mathcal{H}^{(2)}(\phi) + \mathrm{R}_k$, then we simplify the flow equation for the effective action as

$$\partial_k \Gamma_k = \frac{1}{2} \partial_k \mathrm{Tr} \, \ln \left(\mathcal{H}^{(2)} + \mathrm{R}_k \right). \tag{3.12}$$

As a consequence, the flow becomes a total derivative and we can (trivially) integrate between 0 and Λ:

$$\Gamma - \mathcal{H} = \frac{1}{2} \mathrm{Tr} \, \ln \left(\mathcal{H}^{(2)} \right) + \text{const.} \tag{3.13}$$

This is the equation for the one-loop effective action founded earlier. We see that the crudest approximation that we could make on the Wetterich's equation nevertheless gives exact one-loop results. This procedure could be iterated to retrieve perturbation theory to all orders, as explained in Ref. [25]. It is remarkable that substituting in Eq. (3.12) the bare inverse propagator $\mathcal{H}^{(2)} + \mathrm{R}_k$ by the full inverse functional propagator $\Gamma_k^{(2)} + \mathrm{R}_k$ turns this one-loop equation into the exact flow equation.

3.2 Exact renormalization-group equation for the n-point functions

3.2.1 Infinite hierarchy of coupled equations

In this section we show how the exact flow equation for the effective action, given by Eq. (3.9) can be used in order to obtain the flow equations for all the vertices, and thus for all the connected-correlation functions, of the theory. In order to simplify the notation, note that the Wetterich equation, given by Eq. (3.9), can be written as

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k \mathrm{R}_k(x,y) G_k(x,y), \tag{3.14}$$

where:

$$G_k(x,y) = (\Gamma_k^{(2)}[\phi] + \mathrm{R}_k)^{-1}(x,y) \tag{3.15}$$

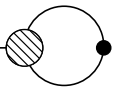
is the modified propagator in presence of the cutoff term R_k.

Starting from the Wetterich equation given in Eq. (3.14) one obtains, by functionally differantiate it with respect to the field $\phi(x)$:

$$\partial_k \Gamma_k^{(1)}(x) = \partial_k \frac{\delta \Gamma_k(\phi)}{\delta \phi(x)} = -\frac{1}{2} \mathrm{Tr} \left(\partial_k \mathrm{R}_k G_k \frac{\delta \Gamma_k^{(2)}}{\delta \phi(x)} G_k \right), \tag{3.16}$$

where we have not indicated the spatial dependences of R_k, G_k and $\Gamma_k^{(2)}$ in the r.h.s., and Tr stands for the integral over these implicit spatial dependences. It is possible to graphically represent Eq. (3.16) by means of a suitable Feynman diagram given by

$$\partial_k \Gamma_k^{(1)}[\phi] = -\frac{1}{2} \quad . \tag{3.17}$$



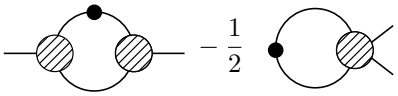
Here the straight lines inside the loop are the modified propagators G_k, given by Eq. (3.15), the black blob is the cutoff term $\partial_k \mathrm{R}_k$ while the dashed blob is the 3-particle-irreducible (3PI) vertex function, given by $\delta\Gamma_k^{(2)}/\delta\phi(x) = \Gamma_k^{(3)}(x,y,z)$. The exiting line is the only un-amputated leg of the 3PI vertex which enters into the flow of the 1PI vertex, i.e., $\Gamma_k^{(1)}$.

If we take a further derivative with respect to the field ϕ of the exact flow equation for the 1PI vertex function, given by Eq. (3.16), we obtain the following flow equation for the flow of the 2PI vertex:

$$\begin{aligned} &\partial_k \Gamma_k^{(2)}(x,y) = \partial_k \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(x)\delta\phi(y)} = \\ &= -\frac{1}{2}\mathrm{Tr}\left[\partial_k \mathrm{R}_k G_k \frac{\delta^2 \Gamma_k^{(2)}}{\delta\phi(x)\delta\phi(y)} G_k\right] + \mathrm{Tr}\left[\partial_k \mathrm{R}_k G_k \frac{\delta\Gamma_k^{(2)}}{\delta\phi(x)} G_k \frac{\delta\Gamma_k^{(2)}}{\delta\phi(y)} G_k\right], \end{aligned} \tag{3.18}$$

which has the following graphical representation, according to the convention explained above,

$$\partial_k \Gamma_k^{(2)}[\phi](x,y) = \quad - \frac{1}{2} \quad . \tag{3.19}$$



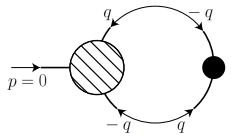
In particular, the dashed blob with four legs is the 4PI vertex function, given by $\Gamma_k^{(4)}$.

The flow equations for the n-point functions, such as Eq. (3.19), provide an infinite hierarchy of coupled equations. The coupling among these equations occurs in two ways, upwards and downwards. In fact, upwards, the equation for $\Gamma_k^{(n)}$ involves $\Gamma_k^{(n+1)}$ and $\Gamma_k^{(n+2)}$. While, downwards, all the flow equations involve $\Gamma_k^{(2)}$ which is coupled successively to all the equations above it. A number of approximation procedures have been introduced in the last twenty years in order to solve the Wetterich equation. Various approximation have been implemented in order to close the infinite tower of coupled equations: local potential approximation (LPA), that we detail in the following section, the derivative expansion (see Refs. [24, 30]) and Blaizot-Mendez-Wschebor approximation [34], as well as many others. In this work we introduce and use only the local potential approximation.

3.2.2 The case of uniform background field

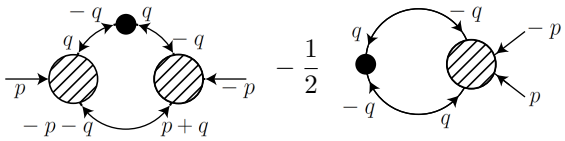
If we evaluate $\partial_k \Gamma_k^{(1)}[\phi]$ on a uniform field configuration, i.e., $\phi(x) = \phi$, the momentum is conserved at each vertex of the diagram and for each modified propagator in Eq. (3.17). Accordingly, Eq. (3.17) does not vanish only at zero momentum and in Fourier space, where $\Gamma_k^{(1)}[\phi](x) \to \Gamma_k^{(1)}(p, \phi)$, we obtain the following expression for the flow equation of the 1PI vertex function

$$\partial_k \Gamma_k^{(1)}(p = 0, \phi) = \quad . \tag{3.20}$$



Similarly the flow equation for $\partial_k \Gamma_k^{(2)}$ in Eq. (3.19) becomes, in the case of uniform background fields

$$\partial_k \Gamma_k^{(2)}(p, \phi) = \quad - \frac{1}{2} \quad , \tag{3.21}$$



i.e.,

$$\begin{aligned} \partial_k \Gamma_k^{(2)}(p, \phi) = & \int_q \partial_k \mathrm{R}_k(q) G_k^2(q, \phi) \times \\ & \times \Big[\Gamma_k^{(3)}(p, q, -p - q; \phi) G_k(q + p, \phi) \Gamma_k^{(3)}(-p, p + q, -q; \phi) + \\ & - \frac{1}{2} \Gamma_k^{(4)}(p, -p, q, -q; \phi) \Big] . \end{aligned} \tag{3.22}$$

There exist also a consistency condition, that we illustrate here, between $\Gamma_k^{(2)}(q, \phi)$ and the *effective potential*, i.e., $U_k(\phi)$: in fact we have $\Gamma_k^{(2)}(p = 0, \phi) = U_k^{(2)}(\phi) = m_k(\phi)$[2]. The identification in the very last equation, i.e., that $m_k = U_k^{(2)}$ is due to the fact that the local part of $\Gamma_k^{(2)}(q, \phi)$ is the mass term of the modified propagator G_k as one can see from Eqs.(3.11) and (3.15). Accordingly, m_k can be calculated by taking the second derivative with respect to ϕ of the flow equation for the effective potential:

$$\begin{aligned} \partial_k m_k(\phi) = \partial_k \Gamma_k^{(2)}(0, \phi) = & \int_q \partial_k \mathrm{R}_k(q) G_k^{(2)}(q, \phi) \times \\ & \times \left[G_k(q, \phi) \left(\frac{\partial \Gamma_k^{(2)}(q, \phi)}{\partial \phi} \right)^2 - \frac{1}{2} \frac{\partial^2 \Gamma_k^{(2)}(q, \phi)}{\partial \phi^2} \right] . \end{aligned} \tag{3.23}$$

[2]I.e. the effective potential could be defined as the local part of the effective action.

Equation (3.23) is the flow of the effective modified mass term in a uniform background field configuration and is coupled to the momentum-dependent part $\Gamma_k(q,\phi)$ which is coupled to the 3PI and the 4PI vertex function by Eq. (3.22).

The modified version of Eq. (3.23) for non-equilibrium phenomena will be crucial for our purpose of calculating the fluctuation dissipation ratio (FDR). In fact, the FDR is given by an amplitude ratio and in order to have access to these amplitudes one needs to calculate the two-point functions of the theory, i.e., the responses and correlation functions of the purely relaxational model. In that case, the mass $m_k(\phi)$ will acquire a time dependence induced by the temperature quench.

3.3 The local potential approximation

3.3.1 Flow equation for the effective potential

The underlying idea of the local potential approximation is that for the study of critical phenomena we are mostly interested in the long-distance physics, that is the $|q| \to 0$ region of the correlation functions with wave vector q. Thus, we keep only the lowest orders of the expansion of Γ_k in $\nabla\phi$ while we keep all orders in the field ϕ

$$\Gamma_k(\phi) = \int d^d x \left[\frac{1}{2}(\nabla\phi)^2 + U_k(\phi)\right] + O(\nabla^4). \tag{3.24}$$

All we can learn about the model at this level of approximation is contained in the effective potential $U_k(\phi)$. It should be clear that the most important information about of collective behavior and the critical behavior are hidden in the effective potential $U_{k=0}$ of the theory that, accordingly, needs to be computed as accurately as possible.

Note that if U_k is not truncated in a field expansion the RG equation of Γ_k becomes an infinite set of coupled partial differential equations for the coupling which parametrizes it. The initial condition (at scale Λ) for the effective action flow is given by the effective Hamiltonian $\mathcal{H}$ of the model. Furthermore, contrarily to perturbation theory where only the renormalizable couplings are retained in the renormalized action, all powers of the fields appear, in general, in the ansatz for Γ_k. There is no longer distinction, at this level, between relevant and irrelevant couplings (this is in fact a complication of the fRG with respect to the perturbative renormalization group technique).

Within the local potential approximation for the effective action, given by Eq. (3.24), the two-point function, obtained by differentiating $\Gamma_k[\phi]$ twice with respect to ϕ, and letting ϕ be constant, is given by

$$\Gamma_k^{(2)}(q,\phi) \xrightarrow{\text{LPA}} \Gamma_k^{(2)}(q,\phi) = q^2 + m_k(\phi). \tag{3.25}$$

The corresponding propagator is simply a massive propagator, with a ϕ-dependent mass given by $m_k(\phi)$. The flow equation for the effective potential $U_k(\phi)$, as well

as the flow for the effective modified mass $m_k(\phi)$, given in Eq. (3.23), become closed equations. In particular, the former reads:

$$k\partial_k U_k(\phi) = \frac{1}{2}\int_q \frac{k\partial_k \mathrm{R}_k}{q^2 + \mathrm{R}_k(q) + m_k(\phi)}. \tag{3.26}$$

With the choice of the Litim optimized cutoff

$$\mathrm{R}_k(q^2) = (k^2 - q^2)\vartheta(k^2 - q^2), \tag{3.27}$$

the calculation of the momentum integral in Eq. (3.26) is straightforward and one easily finds

$$k\partial_k U_k(\phi) = \frac{4a_d}{d}\frac{k^{d+2}}{k^2 + m_k(\phi)}, \tag{3.28}$$

where $a_d = 2/[\Gamma(d/2)(4\pi)^{d/2}]$ ($\Gamma(x)$ is here the gamma function) is a numerical factor stemming from the angular integral implicit in $\int_q$.

3.3.2 Analysis of critical points and the dimensionless effective potential

In order to characterize critical points it is interesting to work with the dimensionless version $\tilde{U}_k$ of U_k, in the following, since, at this point, the long-distance physics (compared to Λ^{-1}) is scaleless. This means that the potential, properly rescaled by powers of k, should be k-independent at $T = T_c$ and for sufficiently small k: it must be a fixed potential $\tilde{U}^*(\tilde{\phi})$, such that

$$k\partial_k \tilde{U}^*(\tilde{\phi}) = 0. \tag{3.29}$$

(in general we will denote by the subscript * any quantity which is evaluated at a fixed point). In our formalism, k is the analogue of the inverse running lattice spacing and, to find a fixed point, we must therefore "de-dimension" all dimensional quantities thanks to k. This is equivalent to measuring all lengths in units of the running "lattice spacing" k^{-1}. We have

$$[\Gamma_k] = k^0 \to \begin{cases} [\phi] = k^{\frac{d-2}{2}}, \\ [U_k] = k^d, \end{cases} \tag{3.30}$$

where with the simbol $[\cdot]$ we mean the dimension of the quantity inside the bracket, expressed in powers of k. We define the dimensionless variables by

$$\begin{aligned} &\tilde{x} = kx, \quad \tilde{\phi}(x) = k^{\frac{2-d}{2}}\phi(x), \\ &\tilde{U}_k(\tilde{\phi}(\tilde{x})) = k^{-d}U_k(\phi(x)) \ \text{ and } \ \tilde{m}_k(\tilde{\phi}) = k^{-2}m_k(\phi). \end{aligned} \tag{3.31}$$

In order to derive the RG equation for the dimensionless effective potential $\tilde{U}_k$ we must keep in mind that the derivative $k\partial_k$ in Eq. (3.28) is computed at fixed ϕ. We want while to compute it at fixed $\tilde{\phi}$, finding

$$k\partial_k \tilde{U}_k(\tilde{\phi}) = -d\tilde{U}_k(\tilde{\phi}) + (d-2)\tilde{U}_k^{(1)}(\tilde{\phi}) + \frac{4a_d}{d}\frac{1}{1 + \tilde{m}_k(\tilde{\phi})}. \tag{3.32}$$

We clearly see from this equation that the flow of $\tilde{U}_k$ consists of tow contributions, one that comes from the dimensions of U_k and ϕ (the first and second term on the r.h.s. of Eq. (3.32))and one that comes from the dynamics of the model (the last term on the r.h.s.). This RG equation for $\tilde{U}_k$ is a rather simple partial differential equation that can be easily integrated numerically.

The effective potential U_k has another very important property: it is the *generating functional* of the β-functions, i.e. the flow equation, for all of the couplings introduced in it. In fact, if we decide for example to expand $\tilde{U}_k$ in even powers of the field, accordingly to the $\mathbb{Z}_2$ symmetry of the theory, we obtain:

$$U_k(\phi) = \sum_{n=1}^{N_{tr}} \frac{g_{k,2n}}{2^n(2n)!} \left(\phi^2 - \phi_{m,k}^2\right)^{2n}, \tag{3.33}$$

where we have truncated the series at $n = N_{\rm tr}$ and $\phi_{m,k}$, i.e., the so-called *running homogeneous background* field is defined as

$$\frac{\delta U_k}{\delta(\phi^2)}\Big|_{\phi=\phi_{m,k}} = 0. \tag{3.34}$$

From this equation we have access to all the β-function for the couplings $g_{k,2n}$ by the following formula:

$$k\partial_k g_{k,2n} = k\partial_k \frac{2^n(2n)!}{(2n)!} \left(\frac{\partial^n}{\partial(\phi^2)^n} U_k(\phi)\right)\Big|_{\phi=\phi_{m,k}}. \tag{3.35}$$

Equations (3.33), (3.34) and (3.35) holds also for the dimensionless version of the effective potential $\tilde{U}_k(\tilde{\phi})$, if we insert in them the dimension-less version of the couplings $g_{k,2n}$ and of the fields ϕ and $\tilde{\phi}$, respectively given by $\tilde{g}_{k,2n}$, $\tilde{\phi}$ and $\tilde{\phi}_{m,k}$.

3.3.3 Critical point of the $\mathbb{Z}_2$ universality class: vanishing background field-approximation

We are now in the position to discuss the critical behaviour of the Ising model and to look for fixed points. We discuss the symmetry-breaking in the Ising model at the lowest possible level of approximation: we expand the local potential U_k around a vanishing background field, i.e., we set $\phi_{m,k} = 0$ and we choose $N_{tr} = 2$ in Eq. (3.35). Accordingly, U_k is given by

$$U_k(\phi) = \frac{r_k}{2}\phi^2 + \frac{g_k}{4!}\phi^4. \tag{3.36}$$

The β-functions for the couplings $\tilde{r}_k$ and $\tilde{g}_k$ are easily obtained via Eqs.(3.36) and (3.35), and (3.32), together with the condition $\tilde{\phi}_{m,k} = 0$. The result is given by

$$\begin{cases} k\partial_k \tilde{r}_k = -2\tilde{r}_k - \dfrac{\tilde{g}_k}{(1+\tilde{r}_k^2)^2}, \\ \\ k\partial_k \tilde{g}_k = (d-4)\tilde{g}_k + 6\dfrac{\tilde{g}_k^2}{(1+\tilde{r}_k^2)^3}, \end{cases} \tag{3.37}$$

where we absorbed the term a_d/d in Eq. (3.32), into the definition of $\tilde{g}_k$. Note that during the calculation one has to use in Eq. (3.32) the following relation:

$$m_k(\phi)|_{\phi=0} = r_k, \tag{3.38}$$

together with $U^{(1)}(\phi)|_{\phi=0} = 0$. A *fixed point* is a solution of $k\partial_k \tilde{U}_k^* = 0$. Accordingly the fixed point is a solution $(\tilde{r}^*, \tilde{g}^*)$ of Eq. (3.37) with vanishing l.h.s. in both equations. The solutions for $d \geq 3$:

$$\text{Gaussian:} \quad \begin{cases} \tilde{r}^* = 0, \\ \tilde{g}^* = 0, \end{cases} \tag{3.39}$$

$$\text{Wilson-Fisher:} \quad \begin{cases} \tilde{r}^* = -\dfrac{4-d}{16-d}, \\ \tilde{g}^* = 288\dfrac{4-d}{(16-d)^3}. \end{cases} \tag{3.40}$$

In order to test the stability properties of the Wilson-Fisher fixed point and to extract universal quantities, one has to linearize the flow equations for the dimensionless couplings around the Wilson-Fisher fixed point: for example, the largest negative eigenvalue l_{max} of the stability matrix is related to the correlation length exponent ν by:

$$\nu = -\frac{1}{l_{max}}. \tag{3.41}$$

It can be shown [24] that the critical exponent ν^{LPA} for the Ising model in spatial dimension $d = 3$ is calculated to be $\nu^{LPA} = 0.65(1)$, within the full functional LPA approach (i.e. by taking the limit $N_{\rm tr} \to \infty$ in Eq. (3.33)), that is in rather good agreement with the numerical value $\nu = 0.630(1)$ reported in Chapter 1. LPA qualitatively and quantitatively describes the critical behavior very well even if it is the the simplest approximation in the class of derivative expansions approximations (see Ref. [24] for the details on this last class of approximations and their accurate predictions of static critical exponents).

Within the vanishing background-field approximation of LPA, there is no renormalization of the derivative term of the microscopic action, and, accordingly, the exponent η vanishes, i.e., $\eta^{LPA} = 0$. We postpone the discussion of how to obtain non-vanishing anomalous dimension η to the following chapter. In some way, we have to consider contribution to the derivatives terms which comes from (spatial and temporal) non-local terms in the effective action. From ν and η all the other static exponents can be calculated thanks to well known scaling relations between exponents [6].

Chapter 4

Critical exponents of the purely relaxational model: a functional renormalization-group approach

In this chapter we introduce a recent approach for studying genuine non-equilibrium phenomena by means of the functional renormalization-group technique (fRG), exposed in its equilibrium version in the previous chapter. In Ref. [35] the case of critical relaxational model was analyzed. There, it was shown how to calculate non-equilibrium, as well as static, universal exponents; this method has been applied to the case of a critical quench of the purely relaxational model, providing the predictions for the critical initial-slip exponent θ shown in Fig. 1.2.

In this chapter we present a review of the approach introduced in Ref. [35] and, in addition, we correct a mistake done in the calculation of the dynamical exponent θ, which is discussed around Fig. 4.3. We also clarify the physical interpretation of the equation which gives the anomalous dimension of the boundary (in time) order-parameter field. We recall from Sec. 1.3.2, that this field represents the non-equilibrium initial condition necessary to implement the dynamics induced by the temperature quench. All the results presented in this chapter will be crucial in order to achieve our purpose, i.e., to calculate the fluctuation-dissipation ratio in the long-time limit, which is done in the next chapter.

The chapter is organized as follows:

Sec. 4.1: We generalize the exact renormalization-group equation for the effective action to the case of purely relaxational dynamics in the presence of the quench, which breaks time-translational invariance.

Sec. 4.2: We implement the LPA approximation within a vanishing background field approximation. This will lead to predictions of the critical initial-slip exponent θ.

Sec. 4.3: We introduce the non-vanishing background field implementation of the LPA. We show how this choice lead to a non-local contribution to the effective action, which is eventually responsible for a non-vanishing anomalous dimension, yields to more accurate predictions for θ. At the end of Sec. 4.3 we discuss our original results in this context.

4.1 Functional renormalization-group for non-equilibrium systems

Let us introduce the functional renormalization-group equation for the effective action for the case of critical non-equilibrium systems. For the purely relaxational model, the microscopic dynamical action is given by Eq. (1.29). In order to implement the temperature quench we have to add the boundary term $\mathcal{S}_b$, as described in Sec. 1.3. The resulting microscopic dynamical action is given by

$$\mathcal{S}[\phi,\tilde{\phi}] = \mathcal{S}_{\rm b}[\phi_0] + \int_{\mathbf{r}}\int_{t_0}^{\infty} dt\ \tilde{\phi}\left(\partial_t\phi + \Omega\frac{\delta\mathcal{H}}{\delta\phi} - \Omega T\tilde{\phi}\right), \tag{4.1}$$

where we assume that the quench is made at time t_0[1]. The boundary action $\mathcal{S}_b$ depends on the field $\phi(\mathbf{r},t_0)=\phi_0$.

In order to implement the fRG, as explained in the previous chapter, it is necessary to supplement the action $\mathcal{S}$ with a cutoff function $\mathrm{R}_k(q)$, and to derive the one-particle irreducible effective action $\Gamma_k[\phi,\tilde{\phi}]$ as the Legendre transform of the generating function associated to $\mathcal{S}_k$, as detailed in the previous section. The properties that the cutoff function $\mathrm{R}_k(q)$ has to satisfy in order to implement the fRG technique have been detailed in the previous section. $\mathrm{R}_k(q)$ is introduced as a quadratic term in the modified action $\mathcal{S}_k[\Phi]$, where $\Phi=(\phi,\tilde{\phi})$, given by

$$\mathcal{S}_k[\Phi] = \mathcal{S}[\Phi] + \Delta\mathcal{S}_k[\Phi], \tag{4.2}$$

with

$$\Delta\mathcal{S}_k[\Phi] = \int_{t,\mathbf{r}} \left[\Phi^t\hat{\sigma}\Phi\right]\mathrm{R}_k/2, \quad \text{where} \quad \hat{\sigma}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4.3}$$

with the Pauli matrix $\hat{\sigma}$ acting in the two-dimensional space of the variables ϕ and $\tilde{\phi}$, encoded in the doublet $\Phi=(\phi,\tilde{\phi})$.

[1] The stationary properties of this model are recovered in the limit $t_0\to\infty$, which yields a time-translational invariant action.

The flow equation for Γ_k upon varying the coarse-graining scale k is given by [35]:

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \int_{x,x'} \mathrm{Tr} \left[\vartheta(t-t_0) \hat{G}_k[\Phi](x,x') \hat{\sigma} \right] \partial_k \mathrm{R}_k(x-x'), \tag{4.4}$$

where, in order to simplify the notation, we defined $x = (\mathbf{r}, t)$, $\int_x = \int d^d r \int_{t_0}^{\infty} dt$. The matrix $\hat{G}_k$ in Eq. (4.4) is defined as

$$\hat{G}_k[\Phi](x,x') = \left[\hat{\Gamma}_k^{(2)}[\Phi] + \mathrm{R}_k \hat{\sigma} \right]^{-1} (x,x'), \tag{4.5}$$

where the inverse on the r.h.s. is taken with respect to both spatial and temporal variables, as well as to the internal matrix structure. The kernel $\hat{\Gamma}_k^{(2)}$ is the second variation of the effective action Γ_k with respect to the fields, i.e.,

$$\hat{\Gamma}_k^{(2)}[\Phi](x,x') = \begin{pmatrix} \dfrac{\delta^2 \Gamma_k}{\delta\phi(x)\phi(x')} & \dfrac{\delta^2 \Gamma_k}{\delta\tilde{\phi}(\mathbf{x})\phi(x')} \\ \dfrac{\delta^2 \Gamma_k}{\delta\phi(x)\tilde{\phi}(x')} & \dfrac{\delta^2 \Gamma_k}{\delta\tilde{\phi}(x)\tilde{\phi}(x')} \end{pmatrix}. \tag{4.6}$$

Note that now it becomes clear why $\hat{\sigma}$ enters in the cutoff function in Eq. (4.3). In fact, the only entries of $\Gamma_k^{(2)}$ in which the operator $-\nabla^2$ intervenes, responsible of the divergences which need to be cured by the cutoff, are the non-diagonal ones.

While Eq. (4.4) is exact, it is generally not possible to solve it, as we have discussed in the previous chapter. Accordingly, one has to resort to approximation scheme which render Eq. (4.4) amenable to analytic calculations. A first step in this direction is to provide an *Ansatz* for the form of the effective action Γ_k which, once inserted into Eq. (4.4), results in a set of coupled non-linear differential equations for the couplings which parametrize it. In fact, any coupling $g_{n,l}\phi^n\tilde{\phi}^l/(n!l!)$ (with n and l positive integers) appearing in Γ_k corresponds to a term of its vertex expansion, as

$$g_k^{l,n} = \left. \frac{\delta^{l+n}\Gamma_k}{\delta\tilde{\phi}^l \delta\phi^n} \right|_{\tilde{\phi}=0, \phi=\phi_m}, \tag{4.7}$$

where the derivatives of Γ_k are evaluated on some homogeneous field configurations $\tilde{\phi} = 0$ and $\phi = \phi_m$. The background field ϕ_m, is typically chosen as the minimum of the effective action Γ_k. Finally, in order to derive the RG equations for the couplings appearing in the effective potential U_k, one has to take the derivative with respect to k on both sides of the Wetterich equation in Eq. (4.4) and one finds

$$\begin{aligned} \frac{dg_k^{l,n}}{dk} = & \frac{\delta^{l+n}}{\delta\phi^n\tilde{\phi}^l} \frac{1}{2} \int_x \mathrm{Tr} \left[\vartheta(t-t_0) G_k(x,x) \frac{d\mathrm{R}_k}{dk} \sigma \right] \Big|_{\substack{\tilde{\phi}=0 \\ \phi=\phi_m}} + \\ & + \left. \frac{\delta^{l+n+1}\Gamma_k}{\delta\phi^{n+1}\tilde{\phi}^l} \right|_{\substack{\tilde{\phi}=0 \\ \phi=\phi_m}} \frac{d\phi_m}{dk}, \end{aligned} \tag{4.8}$$

from which one can evaluate the flow equation for the couplings $g_{n,l}$, once the derivative of the field ϕ_m is calculated. We conclude this section with this important remark: A very peculiar and important property of non-equilibrium system is that the effective action and the couplings which parametrize it not only depend on the scale k but also on time, as we shall see. This is due to the fact that the generalized propagator $\hat{G}_k$ which enters the r.h.s. of the Wetterich equation is generically time dependent.

4.2 The local potential approximation

The ansatz for the modified effective action Γ_k that we will consider in the following is given by

$$\Gamma_k[\phi,\tilde{\phi}] = \vartheta(t-t_0)\int_{\mathbf{x}}\int_{t_0}^{\infty} dt\ \tilde{\phi}\left(Z_k\partial_t\phi - K_k\nabla^2\phi + \frac{\partial U_k}{\partial\phi} - D_k\tilde{\phi}\right) + \\ + \Gamma_{0,k}[\phi_0,\tilde{\phi}_0], \tag{4.9}$$

where *field-independent* factors Z_k, K_k and D_k account for possible renormalizations of the derivatives and of the Markovian noise, while the generic potential $U_k(\phi)$ is a $\mathbb{Z}_2$*-symmetric local polynomial* of the order parameter ϕ.

The boundary action $\Gamma_{0,k}[\phi_0,\tilde{\phi}_0]$ which accounts for the initial conditions is given by

$$\Gamma_{0,k} = \int_{\mathbf{r}}\left(-\frac{Z_{0,k}^2}{2\tau_{0,k}}\tilde{\phi}^2 + Z_{0,k}\tilde{\phi}_0\phi_0 + Z_{0,k}h_0\tilde{\phi}_0\right). \tag{4.10}$$

Note that this form of $\Gamma_{0,k}$ is completely determined by the fact that it should reproduce the Gaussian non-equilibrium correlator, up to terms that are irrelevant in RG sense, as discussed in Ref. [35]. The factor $Z_{0,k}$ accounts for a possible renormalization of the initial response field $\tilde{\phi}_0$ [and will give rise to the critical initial slip exponent θ, through Eq. 1.49]. Finally, the presence of a non-vanishing initial field h_0 induces a non-trivial evolution of the magnetization $M(t) = \langle\phi(t)\rangle$, but it does not generate new additional critical exponents, and therefore in the rest of this work we will assume $h_0 = 0$ without loss of generality.

The coupling constants and the parameters which enters in the first line in the r.h.s. of Eq. (4.10) are the so-called bulk coupling constants and parameters, respectively. The constants which enters into $\Gamma_{0,k}$ i.e., $Z_{0,k}$ and $\tau_{0,k}$ are, instead, the boundary parameters.

The kernel $\hat{\Gamma}_k^{(2)}[\Phi] + \mathrm{R}_k\hat{\sigma}$ in Eq. (4.5) can be re-expressed formally by separating the field-independent part $\hat{G}_{0,k}^{-1}$ from the *field-dependent* part $\hat{\Sigma}_k[\Phi]$, i.e.,

$$\hat{\Gamma}_k^{(2)}[\Phi] + \mathrm{R}_k\hat{\sigma} = \hat{G}_{0,k}^{-1} - \hat{\Sigma}_k[\Phi], \tag{4.11}$$

such that, using Eq. (4.5) in the l.h.s., one obtains

$$\hat{G}_k[\Phi] = \left(\hat{G}_{0,k}^{-1} - \hat{\Sigma}_k[\Phi]\right). \tag{4.12}$$

It is important to realize that Eq. (4.12) is a Dyson equation for the modified propagator $\hat{G}_k[\Phi]$, such as Eqs. (2.14) obtained in equilibrium. As we have seen in that case, starting from $\hat{G}_{0,k}$ and $\hat{\Sigma}_{0,k}$ one has access, through $\hat{\Gamma}_k^{(2)}$, to the two-time functions of the theory.

Note that, since we assumed $\Gamma_{0,k}$ to be quadratic in the fields ϕ and $\tilde{\phi}$, its presence is completely encoded in the function $\hat{G}_{0,k}^{-1}$. Considering the quadratic part of the local ansatz for the effective action one can calculate the Gaussian generalized propagator $\hat{G}_k$ as (the details are given in Ref. [35])

$$\hat{G}_{0,k}(q,t,t') = \begin{pmatrix} C_{0,k}(q,t,t') & R_{0,k}(q,t,t') \\ R_{0,k}(q,t',t) & 0 \end{pmatrix}, \tag{4.13}$$

where $C_{0,k}, R_{0,k}$ are modified non-equilibrium Gaussian response and correlation functions, with a dispersion relation ω_q replaced by the regularized one $\omega_{k,q}$, given by:

$$\omega_{k,q} = K_k q^2 + r_k(\phi_m) + K_k(k^2 - q^2)\vartheta(k^2 - q^2), \tag{4.14}$$

where $r_k(\phi_m)$ is defined as

$$r_k(\phi_m) = \left.\frac{\delta^2 U_k}{\delta\phi\delta\phi}\right|_{\phi=\phi_{m,k}}. \tag{4.15}$$

Furthermore, taking into account the parameters K_k, D_k, Z_k the Gaussian equations of motion for the response and the correlation functions are given by,

$$(Z_k\partial_t + \omega_{k,q})R_{0,k}(q,t,s) = \delta(t-s), \tag{4.16}$$

and

$$(Z_k\partial_t + \omega_{k,q})C_{0,k}(q,t,s) = 2D_k R_{0,k}(q,t,s). \tag{4.17}$$

Noting the similarity of the previous equations with Eqs. (2.22) and (1.35) one concludes that the modified response and correlation functions are given by the following modified Gaussian two-time functions

$$R_{0,k}(q,t,s) = \vartheta(t-s)e^{-(\omega_{k,q}/Z_k)(t-s)}/Z_k, \tag{4.18}$$

$$C_{0,k}(q,t,s) = \frac{D_k}{Z_k^2(\omega_{k,q}/Z_k)}\left[e^{-(\omega_{k,q}/Z_k)|t-s|} - e^{-(\omega_{k,q}/Z_k)(t+s)}\right]. \tag{4.19}$$

For the *local* ansatz given in Eq. (4.9), the field-dependent self energy $\hat{\Sigma}_k[\Phi] = \hat{\Sigma}_k(x,x')$ reads:

$$\hat{\Sigma}_k(x,x') = \hat{\Sigma}_k(x)\delta(x-x'), \tag{4.20}$$

where the delta function $\delta(x-x') = \delta(t-t')\delta^{(d)}(\mathbf{r}-\mathbf{r}')$ appears as a consequence of the locality in space and time of the potential U_k, and where the function $\hat{\Sigma}_k(x)$ is defined as

$$\hat{\Sigma}_k(x) = -\vartheta(t-t_0)\begin{pmatrix} \tilde{\phi}(x)\dfrac{\partial^3 U_k}{\partial\phi^3}(x) & \dfrac{\partial^2 U_k}{\partial\phi^2}(x) \\ \dfrac{\partial^2 U_k}{\partial\phi^2}(x) & 0 \end{pmatrix}. \tag{4.21}$$

Equation (4.12) is more useful for our purposes if it is rephrased explicitly as a Dyson equation for the full propagator $\hat{G}_k[\Phi] = \hat{G}_k(x, x')$,

$$\hat{G}_k(x, x') = \hat{G}_{0,k}(x, x') + \int_y \hat{G}_{0,k}(x, y)\hat{\Sigma}_k(y)\hat{G}_k(y, x'), \tag{4.22}$$

with $\hat{\Sigma}_k$ and $\hat{G}_k$ are Φ-dependent.

Inserting the Dyson equation for the propagator in the Wetterich equation, the latter can be cast in the form

$$\partial_k \Gamma_k[\Phi] = \sum_{n=1}^{+\infty} \Delta\Gamma_{n,k}, \tag{4.23}$$

where the functions $\Delta\Gamma_{n,k}[\Phi]$ are defined as

$$\begin{aligned}\Delta\Gamma_{n,k}[\Phi] = \frac{1}{2}\int_{x,y_1,..,y_n} \mathrm{Tr}\Big[\hat{G}_{0,k}(x, y_1)\hat{\Sigma}_k[\Phi](y_1)\hat{G}_{0,k}(y_1, y_2) \times \\ \times\hat{\Sigma}_k[\Phi](y_2)...\hat{\Sigma}_k[\Phi](y_n)\hat{G}_{0,k}(y_n, x)\frac{d\mathrm{R}_k}{dk}\hat{\sigma}\Big].\end{aligned} \tag{4.24}$$

As a first approximation, we introduce here the ansatz for the local potential U_k expanded till the quartic power of the field ϕ, i.e.,

$$U_k(\phi) = \frac{r_k}{2}\phi^2 + \frac{g_k}{4!}\phi^4. \tag{4.25}$$

In this case, the field-dependent function $\hat{\Sigma}_k(x)$, implicitly defined in Eq. (4.12), reads

$$\hat{\Sigma}_k(t) = -\vartheta(t - t_0)g_k \begin{bmatrix} \tilde{\phi}\phi & \frac{\phi^2}{2} \\ \frac{\phi^2}{2} & 0 \end{bmatrix}. \tag{4.26}$$

Accordingly, since this $\hat{\Sigma}_k$ appears n times in the convolution which defines $\Delta\Gamma_{n,k}$, it follows that $\Delta\Gamma_{n,k}$ contains products of $2n$ possibly different fields. Because of the ansatz, also the l.h.s. of Eq. (4.23) is a polynomial of the fields, and therefore each term on the l.h.s. is uniquely matched by a term of the expansion on the r.h.s.. Accordingly, in order to derive the RG equation for the coupling of a term involving a product of $2n$ fields, it is sufficient to evaluate the corresponding $\Delta\Gamma_{n,k}$. Note that this line of argument applies also to the time-translational invariant case, and, moreover, it can be easily generalized to the case in which the potential contains powers of ϕ of higher order than those in the ansatz proposed here.

We note here that, in the following, we are going to consider the lowest-order calculations in LPA and, in doing analytical calculations, the following form for the cutoff function will be extremely useful

$$\begin{aligned}&\mathrm{R}_k(x - x') \xrightarrow{\text{Fourier}} \mathrm{R}_k(q^2)\delta(t - t'), \\ &\text{where } \mathrm{R}_k(q^2) = K_k(k^2 - q^2)\vartheta(k^2 - q^2).\end{aligned} \tag{4.27}$$

4.3 Truncation for vanishing background field

4.3.1 Derivation of the renormalization-group equations for the bulk couplings

The only non-irrelevant terms which are renormalized within this scheme are those proportional to quadratic and quartic powers of the fields ϕ and $\tilde{\phi}$, i.e., those associated with the post-quench parameter r_k, the boundary field renormalization $Z_{0,k}$ and the coupling g_k. As discussed before, the renormalization of the quadratic terms is determined by the contribution $\Delta\Gamma_{1,k}$ while the renormalization of the quartic one by the contribution $\Delta\Gamma_{2,k}$.

Let us detail the computation of $\Delta\Gamma_{1,k}$: starting from Eq. (4.24), one finds, for $n = 1$, the following expression

$$\begin{aligned}
&\Delta\Gamma_{1,k} = \\
&= \frac{1}{2}\int_{t,t',\mathbf{r},\mathbf{r}'} \mathrm{Tr}\left[\hat{G}_{0,k}(\mathbf{r}-\mathbf{r}',t,t')\,\hat{\Sigma}_k(t',\mathbf{r}')\hat{G}_{0,k}(\mathbf{r}'-\mathbf{r},t',t)\hat{\sigma}\right]\partial_k \mathrm{R}_k(\mathbf{r}-\mathbf{r}') \\
&= \frac{1}{2}\int_{t,t',\mathbf{q},\mathbf{r}'} \mathrm{Tr}\left[\hat{G}_{0,k}(q,t,t')\,\hat{\Sigma}_k(t',\mathbf{r}')\hat{G}_{0,k}(q,t',t)\hat{\sigma}\right]\partial_k \mathrm{R}_k(q^2) \\
&= k^{d+1}\frac{a_d}{d}\int_{t,t',\mathbf{r}'} \mathrm{Tr}\left[\hat{G}_{0,k}(k,t,t')\,\hat{\Sigma}_k(t',\mathbf{r}')\hat{G}_{0,k}(k,t',t)\hat{\sigma}\right], \qquad (4.28)
\end{aligned}$$

where $a_d = 2/[\Gamma(d/2)(4\pi)^{d/2}]$, with d the spatial dimensionality and $\Gamma(x)$ the gamma function. In the second equality of Eq. (4.28) one expresses $\hat{G}_{0,k}(\mathbf{r},t,t')$ in terms of its Fourier transforms $\hat{G}_{0,k}(q,t,t')$ and then calculates the integral over the spatial coordinates $\mathbf{r}$. In the third equality, instead, after performing the integration over angular variables (which generates the factor a_d), the integral over momenta q becomes trivial since the function R_k is such that (see Eq. (4.27))

$$\frac{\mathrm{dR}_k(q^2)}{\mathrm{d}k} = 2k\,\vartheta(k^2-q^2), \qquad (4.29)$$

i.e., it restricts the integration domain to $0 \leq q \leq k$, within which $\hat{G}_{0,k}(q,t,t')$ is constant and equal to $\hat{G}_{0,k}(k,t,t')$ as a consequence of the modified dispersion relation in Eq. (4.14). Similarly, $\omega_{k,q}$ is replaced by $\omega_{k,q\leq k} = \omega_{k,k} = \omega_{q=k}$ (see Eq. (4.14)). Note that, since K_k is not renormalized within this approximation, it does not contribute to Eq. (4.29) and, for simplicity, we set $K_k = 1$. Furthermore, the mass $r_k(\phi_m)$ which enters in the modified dispersion relation $\omega_{k,q}$, given by Eq. (4.14), for the case of vanishing background field configuration, i.e., $\phi_{m,k} = 0$, is simply given by r_k introduced in the ansatz for the effective potential U_k, here given by Eq. (4.25). Finally, by using the definitions (4.13)

and (4.26), one evaluates the trace in Eq. (4.28), finding

$$\begin{aligned}\Delta\Gamma_1 &= -2k^{d+1}\frac{a_d}{d}g_k\int_{\mathbf{r}'}\int_{t_0}^{+\infty}\mathrm{d}t'\,\widetilde{\phi}(\mathbf{r}',t')\phi(\mathbf{r}',t')\\ &\quad\times\int_{t_0}^{+\infty}\mathrm{d}t\,R_k(k,t',t)C_k(k,t',t)\\ &= -k^{d+1}\frac{a_d}{d}\frac{g_kD}{\omega_k^2}\int_{\mathbf{r}}\int_{t_0}^{+\infty}\mathrm{d}t'\,\widetilde{\phi}(t',\mathbf{r})\phi(t',\mathbf{r})\\ &\quad\times\left[1-f_{r,k}(t'-t_0)\right],\end{aligned}\tag{4.30}$$

where in the last equality the integral over time t was calculated. Furthermore, $\omega_k=\omega_{q=k}$ and $D_k=D$ in vanishing background field configuration. $f_{r,k}(t)$ in Eq. (4.30) is given by

$$f_{r,k}(t)=e^{-2\omega_k t}\left[1+2\omega_k t\left(1-\frac{\omega_k}{D\tau_{0,k}}\right)\right],\tag{4.31}$$

and it corresponds to the time-dependent part of the result of the integration over t in the first equality of Eq. (4.30). Note that the terms proportional to ϕ^2 contained in $\hat{\Sigma}_k$, do not appear in the final result (as required by causality, see Ref. [6]) since they are multiplied by a factor $\vartheta(t-t')\vartheta(t'-t)=0$.

The flow for r_k can be simply obtained by comparison, as explained above, thus yielding the following β-function

$$\frac{dr_k(t)}{dk}=-k^{d+1}\frac{a_d}{d}\frac{g_kD}{(k^2+r_k)^2}[1-f_{\tau,k}(t-t_0)].\tag{4.32}$$

We consider now the renormalization of the quartic term, which can be read off from $\Delta\Gamma_{2,k}$. Note that, in this case, one has to deal with the fact that $\Delta\Gamma_{2,k}$ depends on two times, contrary to $\Delta\Gamma_{1,k}$ which is local in time. A localization procedure is used [2], which eventually yields

$$\begin{aligned}\Delta\Gamma_2 =&\frac{3}{2}k^{d+1}\frac{a_d}{d}\frac{g_k^2D^2}{\omega_k^4}\int_{\mathbf{r}}\int_{t_0}^{+\infty}dt\tilde{\phi}^2(t)\phi^2(t)[1-f_{D,k}(t-t_0)]\\ &+k^{d+1}\frac{a_d}{d}\frac{g_k^2D^2}{\omega_k^3}\int_{\mathbf{r}}\int_{t_0}^{+\infty}dt\tilde{\phi}(t)\phi^3(t)[1-f_{g,k}(t-t_0)].\end{aligned}\tag{4.35}$$

[2]The localization procedure, that we introduce for simplicity for the case of a non-local effective action which is quadratic in the fields, proceeds as follows. Starting from a non-local Γ_k (which arises in the r.h.s. of the Wetterich equation) of the type

$$\Gamma_k=\int_t\tilde{\phi}(t)\phi(t)\hat{\Sigma}_{1,k}(t)+\int_{t,t'}\tilde{\phi}(t)\phi(t')\hat{\Sigma}_{2,k}(t,t'),\tag{4.33}$$

following [35] we make the following expansion $\phi(t')=\phi(t)+\dot{\phi}(t)(t'-t)+\mathcal{O}((t-t')^2)$, so that one obtains

$$\Gamma_k\sim\int_t\tilde{\phi}(t)\phi(t)\left(\hat{\Sigma}_{1,k}(t)+\int_{t'}\hat{\Sigma}_{2,k}(t,t')\right)+\int_t\tilde{\phi}(t)\dot{\phi}(t)\int_{t'}(t'-t)\hat{\Sigma}_{2,k}(t,t').\tag{4.34}$$

The first contribution on the r.h.s. of this last equation, proportional to $\tilde{\phi}(t)\phi(t)$, is local in time. The contribution proportional to $\tilde{\phi}(t)\dot{\phi}(t)$, instead, is responsible for the emergence of a power-law behaviour of Z_k near $k\sim 0$: $Z_k=k^{-\eta_Z}$.

where $f_{g,k}$ and $f_{D,k}$ are, similarly to $f_{r,k}$, exponentially decaying functions upon increasing the time t and therefore they do not contribute to the renormalization of the couplings at long times. These integrals provide a contribution proportional to $\tilde{\phi}^2\phi^2$; however, this operator is irrelevant for $d > 2$ and it can be neglected, since our truncation includes only relevant couplings. On the other hand, the term proportional to $\tilde{\phi}\phi^3$ renormalizes the relevant coupling g_k. In fact, by applying the Wetterich equation for the coupling constants, given in Eq. (4.8), specialized to the case of the quartic coupling $g_k \sim \Gamma_k^{(1,3)}|_{\Phi=0}$, one finds

$$\frac{dg_k}{dk} = 6k^{d+1}\frac{a_d}{d}\frac{g_k^2 D}{(k^2+r_k)^3}\left[1 - f_{g,k}(t)\right], \tag{4.36}$$

where the analytic expression of $f_{g,k}$, unimportant to our purposes, can be found in Ref. [35].

4.3.2 Analysis of the interacting fixed point

Since $f_{r,k}(t)$ and $f_{g,k}(t)$ vanish exponentially fast upon increasing the time t, their contributions to the renormalization of the time-independent parameter r_k and g_k can be neglected if we are interested only in the limit $t \to \infty$ and not on its dynamics.

In order to study the flow of r_k and g_k it is convenient to introduce the corresponding dimensionless quantities

$$\tilde{r}_k = r_k/k^2, \quad \tilde{g}_k = g_k D k^{d-4} a_d/d. \tag{4.37}$$

Their flow equations in the long-time limit [where the functions $f_{r,k}(t)$ and $f_{g,k}(t)$ vanishes exponentially fast, see Eq. (4.31)] are given by

$$\begin{aligned} k\frac{d\tilde{r}_k}{dk} &= -2\tilde{r}_k - \frac{\tilde{g}_k}{(1+\tilde{r}_k)^2}, \\ k\frac{d\tilde{g}_k}{dk} &= \tilde{g}_k\left[(4-d) + 6\frac{\tilde{g}_k}{(1+\tilde{r}_k)^3}\right]. \end{aligned} \tag{4.38}$$

These equations can be evaluated with arbitrary d because of their non-perturbative nature. To retrieve known results in ϵ expansion, let us evaluate them with $d = 4 - \epsilon$. If we do this one can easily see that Eq. (4.38) admits two fixed points: the Gaussian one $(\tilde{r}_G^*, \tilde{g}_G^*) = (0, 0)$ and the Wilson-Fisher one, which at the leading order in ϵ, reads $(\tilde{r}_{WF}^*, \tilde{g}_{WF}^*) = (-\epsilon/12, \epsilon/6) + O(\epsilon^2)$. By linearizing the dimensionless fRG equation one finds that the Gaussian fixed-point is stable only for $\epsilon < 0$, while the Wilson-Fisher one is stable only for $\epsilon > 0$. The latter has an unstable direction (related to temperature), and from the inverse of the negative eigenvalue of the associated stability matrix, one derives the critical exponent ν, which reads $\nu = 1/2 + \epsilon/12 + O(\epsilon^2)$, i.e., it is the same as in equilibrium [5]. As a check we compare the previous flow Eqs.(4.38) for the bulk coupling, with those obtained earlier in the context of the equilibrium theory, given by Eq. (3.37), finding that they actually coincide.

Since $\tau_{0,k}$ does not receive any correction from the renormalization, its flow equation is simply determined by its canonical dimension and thus

$$k\partial_k \tilde{\tau}_{0,k} = -2\tilde{\tau}_{0,k}. \tag{4.39}$$

Accordingly, $\tilde{\tau}_{0,k}$ has only one stable fixed point $\tilde{\tau}_0^* = +\infty$, in the infrared regime (i.e., for $k \to 0$). Close to this fixed point, any possible term in the boundary action Γ_0 (except for $\tilde{\phi}_0\phi_0$, as we show in the next section) is irrelevant for $d > 2$, and therefore the ansatz given by Eq. (4.10) is consistent.

4.3.3 Analysis of initial-slip exponent θ in the short-time limit

In order to find the critical initial-slip exponent we could[3] focus, as explained in Ref. [35], on the short-time limit of the β-function of the mass in the disordered state r_k, which is given in Eq. (4.32). At short times, the function $f_{r,k}(t)$ provides contributions containing fields of the temporal boundary, thus renormalizing the boundary action $\Gamma_{0,k}$. In fact, the formal identity

$$\int_{t_0}^{+\infty} dt\ g(t)e^{-c(t-t_0)} = \sum_{n=0}^{\infty} \frac{1}{c^{n+1}} \frac{d^n g}{dt^n}\Big|_{t=t_0}, \tag{4.40}$$

with $c > 0$ and $g(t)$ an arbitrary smooth function, can be used in order to express the part of the integral involving $f_r(t)$ on the r.h.s. of Eq. (4.30), as

$$\int_{t_0}^{+\infty} dt\ \tilde{\phi}(t)\phi(t) f_{r,k}(t-t_0) = \sum_{n=0}^{\infty} \frac{c_{n,k}(\tau_{0,k})}{(2\omega_k)^{n+1}} Z_{0,n,k} \frac{d^n}{dt^n} \left[\tilde{\phi}(t)\phi(t)\right]\Big|_{t=t_0}, \tag{4.41}$$

with

$$c_{n,k}(\tau_{0,k}) = (n+2) - \frac{(n+1)\omega_k}{D\tau_{0,k}}, \tag{4.42}$$

where additional numerical factors $Z_{0,n,k}$ have been introduced in order to account for possible renormalization of the boundary operators, with $Z_{0,n,k=1}$ in the non-renormalized theory. Most of the terms in the sum renormalizes irrelevant (in the renormalization group sense) operators which were not included into the original ansatz for the boundary action and therefore one can neglect them. The only relevant term corresponds to $n = 0$: again from comparison the β-function for the coupling $Z_{0,k} = Z_{0,0,k}$ it has been obtained , i.e.,

$$\frac{dZ_{0,k}}{dk} = k^{d+1} \frac{a_d}{d} \frac{g_k D}{(k^2 + r_k)^2} \left[1 - \frac{k^2 + r_k}{2D\tau_{0,k}}\right] Z_{0,k}. \tag{4.43}$$

The anomalous dimension η_0 is defined by

$$\eta_0 = -\frac{k}{Z_{0,k}} \frac{dZ_{0,k}}{dk}. \tag{4.44}$$

[3] We will show that this is not the only way to predict θ within the LPA of the fRG equation.

Accordingly, its value at the Wilson-Fisher fixed point is given by

$$\eta_0^* = -\frac{\tilde{g}^*_{WF}}{(1+\tilde{r}^*_{WF})^3}, \tag{4.45}$$

where we have used Eqs. (4.37), which define the dimensionless couplings $\tilde{r}_k, \tilde{g}_k$ and we have evaluate them at the WF fixed point, taking into account that $\tilde{\tau}_0^* = +\infty$ as shown in Sec. 4.3.2.

Being this approximation a local approximation in space and time, no bulk anomalous dimension arise for $t \neq 0$, thus we find $\eta_K^* = \eta_Z^* = \eta_D^* = 0$. Accordingly, using the scaling equation

$$z = 2 - \eta_K + \eta_Z, \tag{4.46}$$

one finds $z = 2$ for the case of the vanishing background field implementation of the LPA.

The initial-slip exponent θ is obtained by the scaling relation given by Eq. (1.49), and is therefore given by

$$\theta = -\frac{\eta_0^*}{z} = \frac{g^*_{WF}}{2(1+\tilde{r}^*_{WF})^3}. \tag{4.47}$$

We note that, using the coordinates of the WF fixed point at order ϵ given in Eq. (3.40), one finds

$$\theta = \frac{\epsilon}{12} + \mathcal{O}(\epsilon^2), \tag{4.48}$$

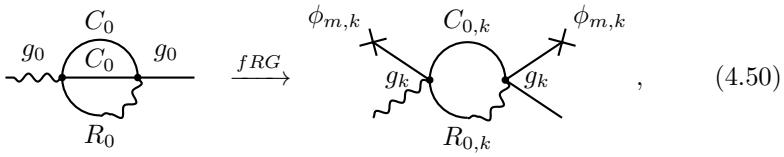
which agrees up to the first order in ϵ with the result obtained in the standard ϵ-expansion [18], reported in Eq. (1.51).

4.4 Truncation for non-vanishing background field

In this section we discuss the results of an improved ansatz for the potential U_k in the effective action, namely

$$U_k = \frac{g_k}{4!}(\phi^2 - \phi^2_{m,k})^2 + \frac{\lambda_k}{6!}(\phi^2 - \phi^2_{m,k})^3. \tag{4.49}$$

Because it corresponds to an expansion around a finite homogeneous value $\phi_{m,k}$ this choice has the advantage to capture the leading divergences of two loops corrections in a calculation which is technically carried at one-loop, as typical of background field methods, and thus it allows us to calculate, for instance, the renormalization of the factors Z_k, K_k and D_k. In fact, the presence of a background field $\phi_{m,k}$ reduces two loop-diagrams to one-loop ones in which an internal classical line (corresponding to a correlation function $C_{0,k}$, where has been replaced by the insertion of two expectation values $\phi_{m,k}$, as shown in Figs. 4.1 and 4.2 where straight lines stand for the field ϕ while wavy lines for the response field $\tilde{\phi}$. For instance, the renormalization of Z_k and K_k comes



(4.50)

Figure 4.1: Diagram S, which gives contribution to Z_k and K_k. The wavy lines indicate response fields $\tilde{\phi}$, while the solid lines indicate the order parmeter fields ϕ. The diagram on the left is the one which is encountered in perturbative two-loops calculation (see Ref. [17]), while the diagram on the right is the non-local contribution added to the two-time functions by the non-vanishing background field LPA, corresponding to replacing one of the two C_0 on the left with two $\phi_{m,k}$ on the right.

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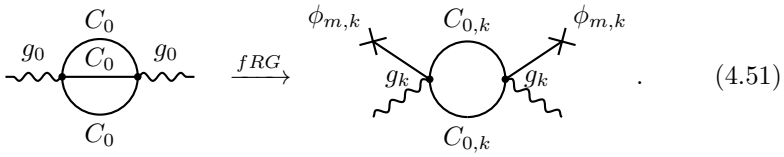

(4.51)

Figure 4.2: Diagram (D), which gives contribution to D_k. Note that with wavy lines we have indicated response field $\tilde{\phi}$, while with the solid lines we have indicated the order parmeter fiele ϕ. The diagram on the left is the one which is encountered in perturbative two-loops calculation (see Ref. [17]), while the diagram on the right is the non-local contribution added to the noise term in the effective action by the non-vanishing background field LPA, corresponding to replacing one of the two C_0 on the left with two $\phi_{m,k}$ on the right.

,

from the diagram S in Fig. 4.1 while the renormalization of the noise strength D_k comes from the diagram D in Fig. 4.2.

Furthermore, we added a sextic interaction λ_k, which is marginal for $d = 3$ and therefore it is expected to contribute with sizable corrections to the value of the critical exponents only upon approaching $d = 3$. In fact, upon including this term, the effective action contains all the non-irrelevant operators in $d = 3$.

4.4.1 Analysis of bulk flow equations

We report here the result for the $\beta-$functions of the bulk couplings, referring to Ref. [35] for further details.

Let us point out, however, that there are two differences with respect to the

case of vanishing background field approximation discussed in Sec. 4.3. First of all, since the factor K_k is renormalized within the ansatz discussed here, the derivative with respect to k of the regulator $\mathrm{R}_k(q)$, defined in Eq. (4.27), has also to account for the dependence of the renormalization factor K_k on k, as

$$\partial_k \mathrm{R}_k(q) = \frac{K_k}{k}\vartheta(k^2-q^2)[2k^2 - \eta_K(k^2-q^2)], \tag{4.52}$$

where we made use of the definition of η_K given in Eq. (4.61). In fact, since the factor K_k depends on k within this approximation, the derivative with respect to k produces a contribution proportional to η_K.

Then we cast the equation for the function $\hat{G}_k$ into a Dyson equation, as Eq. (4.22). By taking the second variation of the effective action Γ_k in Eq. (4.9) evaluated with $(\tilde{\phi}, \phi) = (0, \phi_{m,k})$. the field-dependent function $\hat{\Sigma}_k$ defined as (we assume $t_0 = 0$ for simplicity):

$$\hat{\Sigma}_k(x) = -g_k \vartheta(t) \begin{pmatrix} \tilde{\rho}(x) & \rho(x) - \rho_{m,k} \\ \rho(x) - \rho_{m,k} & 0 \end{pmatrix}, \tag{4.53}$$

where we define

$$\rho = \frac{\phi^2}{2}, \quad \tilde{\rho} = \tilde{\phi}\phi, \quad \rho_{m,k} = \frac{\phi_{m,k}^2}{2}. \tag{4.54}$$

$\hat{G}_{0,k}$ is defined according to Eqs.(4.13) and (4.14) but with the post-quench parameter $r_k(\phi_{m,k}) = \delta^2 U_k / \delta\phi\delta\phi|_{\phi=\phi_{m,k}}$ given by

$$m_k = \frac{2}{3}\rho_{m,k} g_k. \tag{4.55}$$

The use of $\mathbb{Z}_2$ invariants $\rho = \phi^2/2$ and $\tilde{\rho} = \phi\tilde{\phi}$ is customary in the context of fRG and it helps simplifying the notation in what follows.

The form of $\hat{\Sigma}_k$ in Eq. (4.53) allows us to express the r.h.s. of the fRG equation (4.23) as a power series of $\rho - \rho_{m,k}$: this provides, together with the vertex expansion given by Eq. (4.7), a way to unambiguously identify the renormalization of the terms appearing in the potential U_k in Eq. (4.49). In fact, $\rho_{m,k}$ and the couplings g_k and λ_k are identified as

$$0 = \frac{dU_k}{d\rho}\Big|_{\rho=\rho_{m,k}}, \quad \frac{g_k}{3} = \frac{d^2U_k}{d\rho^2}\Big|_{\rho=\rho_{m,k}}, \quad \frac{\lambda_k}{15} = \frac{d^3U_k}{d\rho^3}\Big|_{\rho=\rho_{m,k}}, \tag{4.56}$$

where the first condition actually defines $\rho_{m,k}$ as the minimum of the potential. In terms of the effective action Γ_k the previous equations become

$$\frac{\delta\Gamma_k}{\delta\tilde{\rho}}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}} = 0, \quad \frac{g_k}{3} = \frac{\delta^2\Gamma_k}{\delta\tilde{\rho}\delta\rho}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}}, \quad \frac{\lambda_k}{15} = \frac{\delta^3\Gamma_k}{\delta\tilde{\rho}\delta\rho^2}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}}. \tag{4.57}$$

By taking a total derivative with respect to k of each equality one finds

$$\frac{\delta}{\delta\tilde{\rho}}\frac{\partial\Gamma_k}{\partial k}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}} + \frac{\delta^2\Gamma_k}{\delta\tilde{\rho}\delta\rho}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}} \frac{d\rho_{m,k}}{dk} = 0, \tag{4.58}$$

$$\frac{1}{3}\frac{dg_k}{dk} = \frac{\delta^2}{\delta\tilde{\rho}\delta\rho}\frac{\partial\Gamma_k}{\partial k}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}} + \frac{\delta^3\Gamma_k}{\delta\tilde{\rho}\delta^3\rho}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}}\frac{d\rho_{m,k}}{dk}, \tag{4.59}$$

$$\frac{1}{15}\frac{d\lambda_k}{dk} = \frac{\delta^3}{\delta\tilde{\rho}\delta\rho^2}\frac{\partial\Gamma_k}{\partial k}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}} + \frac{\delta^4\Gamma_k}{\delta\tilde{\rho}\delta^3\rho}\Big|_{\tilde{\rho}=0,\rho=\rho_{m,k}}\frac{d\rho_{m,k}}{dk}, \tag{4.60}$$

which, after replacing $\partial\Gamma_k/\partial k$ with the fRG equation thanks to the Dyson equation, together with the LPA ansatz for Γ_k given by Eqs.(4.9) and (4.49), render the flow equations for $\rho_{m,k}, g_k$ and λ_k.

The flow equations for the coefficients K_k, Z_k, D_k can be conveniently expressed in terms of the corresponding anomalous dimensions η_D, η_Z and η_K, defined as :

$$\eta_D = -\frac{k}{D_k}\frac{dD_k}{dk}, \quad \eta_Z = -\frac{k}{Z_k}\frac{dZ_k}{dk}, \quad \eta_K = -\frac{k}{K_k}\frac{dK_k}{dk}. \tag{4.61}$$

First of all, we note that $\eta_D = \eta_Z$: this is a consequence of detailed balance (see [6]), which characterizes the equilibrium dynamics of model A. In fact, while the short-time dynamics after the quench violates detailed balance inasmuch time-translational invariance is broken, in the long-time limit detailed balance is restored.

Following Ref. [35], we introduce dimensionless couplings as

$$\tilde{m}_k = \frac{1}{3}\frac{\rho^2_{m,k}g_k}{K_k k^2},\ \tilde{g}_k = \frac{a_d}{d}\frac{D_k}{Z_k K_k^2}\frac{\tilde{g}_k}{k^{4-d}},\ \tilde{\lambda}_k = \frac{a_d}{d}\frac{D_k^2}{Z_k^2 K_k^3}\frac{\lambda_k}{5k^{6-2d}}. \tag{4.62}$$

Let us note that the explicit form of the flow equations for the bulk coupling comes from a calculation analogous to the one discussed in the previous section. In particular, the flow of m_k takes contributions from both the flow equation for $\rho_{m,k}$ and g_k. Similarly, the renormalization of $Z_{0,k}$ is determined by the contribution localized at $t = 0$ of the coefficient of the quadratic term $\tilde{\phi}\phi$ in the effective action, given by Eq. (4.9), equipped with the potential U_k given in Eq. (4.49).

The results for the flow equations of the dimensionless bulk couplings, given in Ref. [35], are the following:

$$\begin{aligned} k\frac{d\tilde{m}_k}{dk} &= (-2+\eta_k)\tilde{m}_k + \left(1-\frac{\eta_K}{d+2}\right)\times \\ &\times \frac{2\tilde{g}_k}{(1+\tilde{m}_k)^2}\left[1+\frac{3}{2}\left(\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right)^2 + \frac{3\tilde{m}_k}{1+\tilde{m}_k}\left(1+\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right)^2\right], \end{aligned} \tag{4.63}$$

$$\begin{aligned} k\frac{d\tilde{g}_k}{dk} =& g_k\left[d-4+2\eta_K+\left(1-\frac{\eta_K}{d+2}\right)\frac{6\tilde{g}_k}{(1+\tilde{m}_k)^3}\left(1+\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right)^2\right] + \\ & \left(1-\frac{\eta_K}{d+2}\right)\frac{\tilde{\lambda}_k}{(1+\tilde{m}_k)^2}\left(-2+3\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right), \end{aligned} \tag{4.64}$$

$$k\frac{d\tilde{\lambda}_k}{dk} = \tilde{\lambda}_k\left[2d-6+3\eta_K+30\left(1-\frac{\eta_K}{d+2}\right)\frac{\tilde{g}_k}{(1+\tilde{m}_k)^3}\left(1+\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right)\right]+$$
$$-18\left(1-\frac{\eta_K}{d+2}\right)\frac{\tilde{g}_k^2}{(1+\tilde{m}_k)^4}\left(1+\frac{\tilde{m}_k\tilde{\lambda}_k}{\tilde{g}_k^2}\right). \tag{4.65}$$

The anomalous dimensions which emerge in the vicinity of the critical point turn out to be, instead (see Ref. [35] for details)

$$\eta_K = \frac{3\tilde{m}^*\tilde{g}^*}{(1+\tilde{m}^*)^4}\left(1+\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)^2, \tag{4.66}$$

$$\eta_Z = \eta_D = \left(1-\frac{\eta_K}{d+2}\right)\frac{9\tilde{m}^*\tilde{g}^*}{2(1+\tilde{m}^*)^4}\left(1+\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)^2. \tag{4.67}$$

The take-home message is that, in the case of the expansion for U_k given by Eq. (4.49) bulk anomalous dimensions arise, and they are explicitly given by Eqs. (4.66) and (4.67).

4.4.2 Analysis of the critical initial-slip exponent: correction to Ref. [35]

In this section we reconsider the calculation of θ done in Ref. [35] by accounting for non-vanishing background field in the implementation of the LPA.

The equations which define the anomalous dimension η_0 of the boundary field ϕ_0 are given by:

$$\begin{cases}\eta_0 = -k\frac{dZ_0}{dk}, \\ k\frac{dZ_0}{dk} = \left(k\frac{d}{dk}r_k\right)\Big|_{t=0},\end{cases} \tag{4.68}$$

where r_k is the mass in the high-temperature state, defined as before in Eq. (4.25), which is given by

$$r_k = \frac{\delta^2 U_k[\phi]}{\delta\phi^2}\Big|_{\phi=\phi_{phys}=0} = -\frac{g_k}{3!}\phi_{m,k}^2 + \frac{\lambda_k}{5!}\phi_{m,k}^4. \tag{4.69}$$

This last equation has been obtained by using the ansatz for U_k given by Eq. (4.49).

However, we point out that in Ref. [35] the term proportional to λ_k in Eq. (4.69) has not been included into the calculation of the β-function for the mass r_k. This mistake affects the predictions of the critical-initial slip exponent, obtained through a short-time limit of the flow of the mass r_k. Note that in the case $\lambda_k = 0$ the analysis done in Ref. [35] is correct, since the term in Eq. (4.69) proportional to λ_k is vanishing in this approximation. Their result is given by

$$\eta_0 = -\left(1-\frac{\eta}{d+2}\right)\frac{\tilde{g}^*}{(1+\tilde{m}^*)^3}\left[1+\frac{9\tilde{m}^*}{2(1+\tilde{m}^*)}\right]. \tag{4.70}$$

The correct flow equation for η_0 with the sextic ansatz given by Eq. (4.49) is obtained through the analysis of the short-time behaviour of the β-function of r_k, as explained in the previous Sec. 4.3.3. The result for the case of non-vanishing background field $\phi_{m,k} \neq 0$, according to the definitions of η_0 and r_k given, respectively, in Eqs. (4.68) and (4.69), is given by[4]

$$\begin{aligned}\eta_0 = &-\left(1-\frac{\eta_K}{d+2}\right)\frac{\tilde{g}^*}{(1+\tilde{m}^*)^3}\times \\ &\times\left[\left(1+\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)\left(1-\frac{3}{2}\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)-\left(1-\frac{3}{2}\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}+\right. \\ &+\frac{9\tilde{m}^*}{2(1+\tilde{m}^*)}\left(1+\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)\left(1-\frac{11}{4}\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)+ \\ &\left.+\frac{27}{2}\frac{\tilde{m}^{*2}}{(1+\tilde{m}^{*2})^2}\left(1+\frac{\tilde{m}^*\tilde{\lambda}^*}{\tilde{g}^{*2}}\right)^3\right].\end{aligned} \tag{4.71}$$

where all the fixed-point values $\tilde{g}^*, \tilde{m}^*$ and $\tilde{\lambda}^*$ are those of the Wilson-Fisher fixed point.

The procedure to follow in order to obtain the prediction of the critical initial-slip exponent θ in generic dimension d is the following:

1. In order to find the Wilson-Fisher fixed point one should set to zero the l.h.s. of Eqs. (4.63),(4.64),4.65) and solve the equations obtained for fixed d, obtaining the coordinates $(\tilde{m}^*_{\rm WF}(d), \tilde{g}^*_{\rm WF}(d), \tilde{\lambda}^*_{\rm WF}(d))$.
2. With the knowledge of the coordinates of the fixed point one should calculate the anomalous dimension η_K, η_D using Eqs. (4.66),(4.67) and then η_0 using Eq. (4.71).
3. With the knowledge of η_K, η_Z and η_0 one is in the position to determine the dynamical critical exponent z and subsequently the critical initial-slip exponent θ, according to

$$z = 2 - \eta_K + \eta_Z, \quad \theta = -\frac{\eta_0}{z}. \tag{4.72}$$

The result of this analysis is given in Fig. 4.3. There, we have added to the lines reported in Fig. 1.2 three red lines, which constitute one of the two main results of the present study and which correspond to three approximations of increasing accuracy. In particular, from the less to the more accurate, they are obtained as follows:

(i) The red dot-dashed line has been obtained with a vanishing background, as explained in the previous section. Remember that, in this case, we have obtained the coordinate of the fixed point $(\tilde{r}^*_{\rm WF}(d), \tilde{g}^*_{\rm WF}(d))$ using

[4]The details of the calculation which leads to Eq. (4.71) can be found in Ref. [35], remembering Eqs. (4.68) and (4.69).

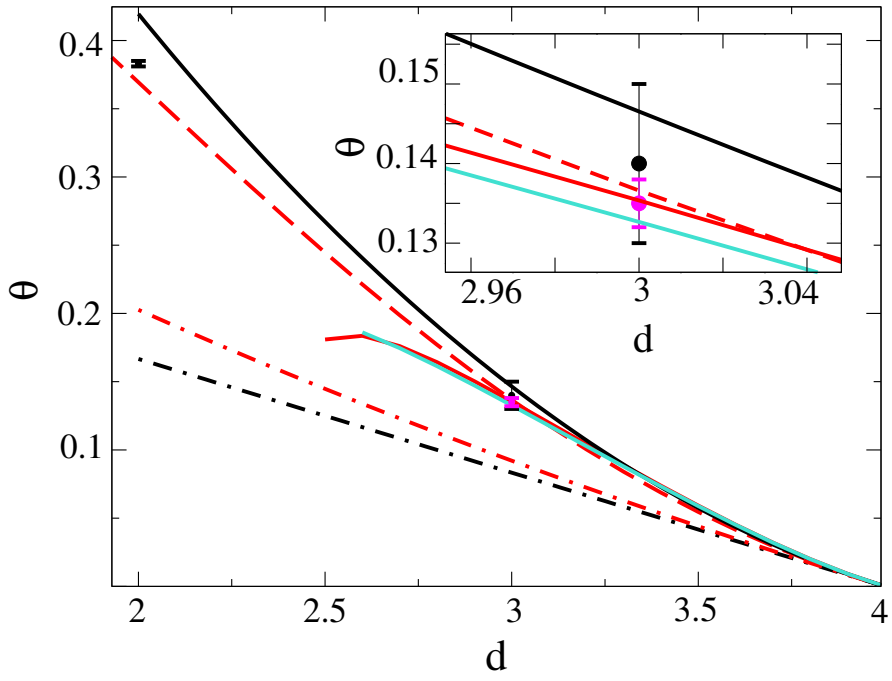

Figure 4.3: Estimates of the Critical initial-slip exponent θ as a function of the spatial dimensionality d of the system. The black lines are the predictions based on perturbative renormalization-group calculations and the dots with error bars corresponds to the results of Monte Carlo simulations, while the turquoise line is the prediction of Ref. [35] based on an incomplete fRG analysis. These results were already presented in Fig. 1.2. The additional three red lines indicate our results for θ obtained on the basis of the fRG technique in LPA approximation: the dot-dashed line is the result of the vanishing background field approximation (see Sec. 4.3), while the dashed and the solid red lines are the predictions of the non-vanishing background field approximation (see Sec. 4.4), respectively for the case of vanishing and non-vanishing sextic coupling λ_k in the ansatz for the effective potential U_k given by Eq. (4.49). Note that the solid red line is the correction to the calculation done in Ref. [35], reported as the solid turquoise line. Inset: magnification of the main plot for $d \simeq 3$.

Eq. (4.38) for fixed d, and then we have calculated θ using Eqs. (4.45) and (4.47).

(ii) The red dashed line has been obtained with the non-vanishing background field approximation outlined here, but setting $\tilde{\lambda}^*_{\rm WF} = 0$, i.e., considering the ansatz for U_k given in Eq. (4.49) with $\lambda_k = 0$. η_0 in this case is given by Eq. (4.70).

(iii) The solid red line, instead, is our best approximation, which has been found via an expansion of U_k around its ordered minimum configuration, including the term of 6th order in the power of the fields, i.e., here we have consistently considered $\lambda_k \neq 0$ which led us to Eq. (4.71). This result has to be compared with that represented by the turquoise solid line corresponding to the prediction presented in Ref. [35], which, however is not fully correct, as it does not account for all the terms proportional to λ_k in Eq. (4.69). The difference between these two predictions is of about the 2%, in $d = 3$, but the prediction presented in this work (solid red line) turns out to be in better agreement with the recent MC data of Ref. [19] than the prediction of Ref. [35], as shown by the inset.

As pointed out in Ref. [35], the fRG results that can be obtained with the sextic ansatz for the effective potential are in remarkable agreement with the two-loop expansion for $d \gtrsim 3.2$, while increasing discrepancies emerge at smaller values of d. In particular, for $d \leq 3$ additional stable fixed points appears in the solution of Eqs.(4.63),(4.64) and (4.65) beyond the Wilson-Fisher one, while for $d \leq 2.5$ the latter disappears. This is not surprising, since for $d < 3$ new relevant terms are allowed, and therefore the sextic potential is no longer an appropriate ansatz and additional terms have to be introduced. In particular, the number of relevant operators diverges as d approaches 2: one should indeed recall that in $d = 2$ any term of the form $\tilde{\phi}\phi^{2n+1}$, with positive integer n, is relevant in the RG sense and therefore the correct truncation for the effective action requires considering a full functional ansatz for the potential, beyond the polynomial expansion used in this work (see Ref. [24] where it is shown how to deal with this issue within th standard equilibrium approach to fRG).

We mention here that, in our opinion, a full-functional LPA approach in non-equilibrium situations is non-trivial because of the different time-dependence which affects in a different manner the different bulk couplings. This is shown by the fact that the functions $f_{r,k}, f_{g,k}$ and $f_{D,k}$ do not have the same time dependencies, as explicitely shown in Ref. [35]. The reason for this complication is merely that higher-order couplings gets renormalized by higher orders $\Delta\Gamma_n$. In doing the calculations one should retain all the time-dependent terms present in the β-function of the disordered mass r_k in order to properly obtain the short-time limit of it, with the method exposed in this chapter, and, consequently, θ.

Chapter 5

Calculation of the fluctuation-dissipation ratio: a functional renormalization-group approach

This chapter, which contains original material, presents the calculation of the two-time functions, i.e., the response R and the correlation C functions, and the calculation of the asymptotic value X^∞ of the fluctuation-dissipation ratio, introduced in Chapter 1, with a functional renormalization-group (fRG) approach. The analysis is conducted within the local potential approximation (LPA) of the fRG technique, introduced in Sec. 3.3 for the case of equilibrium systems and in Sec. 4.2 for the case of non-equilibrium ones.

This chapter contains the following original results:

- Calculation of the critical initial-slip exponent θ by means of a long-time analysis of the system, rather than the short-time one introduced in Ref. [35].
- General analytical expression, given by Eq. (5.35), of the asymptotic value of the fluctuation-dissipation ratio within the LPA approximation of the flow equation for the effective action. We have considered three approximation of increasing accuracy within the local potential approach which led us to the results for the fluctuation-dissipation ratio summarized in Fig. 5.1.

The chapter is organized as follows:

Sec. 5.1: We discuss how to obtain within the LPA approximation the full two-time functions R and C and, with this information, how to calculate the asymptotic value of the FDR.

Sec. 5.2: We detail our strategy for the implementation of LPA with vanishing background field, introduced in Sec. 4.3.

Sec. 5.3: We detail our strategy for the case of non-vanishing background field, introduced in Sec. 4.4.

Sec. 5.4: We discuss our predictions for the asymptotic value of the fluctuation-dissipation ratio, comparing them with those available in the literature.

5.1 Two-time quantities in the local potential approximation

The first step in order to calculate the fluctuation-dissipation ratio is the calculation of the two-time functions, as discussed in Sec. 1.4.2. Let us focus first on the response function R. From the knowledge of the effective action Γ we know that using Eq. (2.10) one has access on the two-point functions of the theory. Taking into account Eqs. (4.6) and (4.13), one concludes that the equation of motion for the response function R for the case of a homogeneous system in space but with a dynamics which breaks time-translational invariance is given by

$$\int_{t'} \Gamma_q^{(1,1)}[\Phi](t,t')\Big|_{\Phi_{phys}} R_q(t',s) = \delta(s-t), \tag{5.1}$$

where

$$\Gamma^{(1,1)} = \frac{\delta^2\Gamma}{\delta\phi\delta\tilde{\phi}} \quad \text{and} \quad \Phi_{phys} = (\tilde{\phi}_{phys} = 0, \phi_{phys}). \tag{5.2}$$

In the l.h.s. of Eq. (5.1), $\Gamma^{(1,1)}$ is evaluated on a physical configuration of fields given by Φ_{phys}. In particular, in the following we choose a high-temperature, i.e., $\phi_{phys} = 0$. Remember from Sec. 1.4.2 that in order to calculate X^∞ we are interested in the $q = 0$ mode of the two-point functions, because this is the only mode which does not reach equilibrium in the aging limit for a quench to the critical point, as explained in Sec. 1.2. In the following, we no longer indicate the subscript q, but we assume that the the quantities are all evaluated at zero momenta, i.e., with $q = 0$.

Our functional renormalization-group approach to the problem of solving Eq. (5.1) is discussed within the LPA approximation for the case of equilibrium and non-equilibrium systems in chapters 3 and 4, respectively. We recall here that the LPA approximations means that we retain only contributions to the effective action which fit into the following Ansatz:

$$\Gamma_k = \int_{\mathbf{r}} \int_{t_0}^{\infty} dt\ \tilde{\phi} \left(Z_k \partial_t \phi - K_k \nabla^2 \phi + \frac{\delta U_k(\phi)}{\delta\phi} - D_k \tilde{\phi} \right), \tag{5.3}$$

where U_k is the local (in space and time) potential, and $\tilde{\phi}, \phi$ are real fields which depends on the space and time coordinates, i.e., $\tilde{\phi} = \tilde{\phi}(\mathbf{r}, t)$ and $\phi = \phi(\mathbf{r}, t)$.

Consider the case, discussed in Sec. 4.2, in which we expand the potential U_k around a vanishing background field. Let us write again this Ansatz here:

$$U_k(\phi) = \frac{r_k}{2}\phi^2 + \frac{g_k}{4!}\phi^4. \tag{5.4}$$

The first thing to realize in order to simplify Eq. (5.1) is that $\Gamma_k^{(1,1)}$ within the LPA approximation, when evaluated in the physical configuration of the fields Φ_{phys}, is given by

$$\Gamma_k^{(1,1)}|_{\Phi_{phys}}(t,s) = [\partial_t + r_k(t)]\,\delta(t-s), \quad \text{with} \quad r_k = \frac{\delta^2 U_k(\phi)}{\delta\phi^2}\Big|_{\Phi_{phys}}, \tag{5.5}$$

where, according to the fact that within the Ansatz proposed for U_k given in Eq. (5.4), no anomalous dimension related to Z_k arises, as discussed in Sec. 4.3, i.e., $Z_k = 1$. In this case, the equation of motion for the running response function R_k is given by

$$[\partial_t + r_k(t)]\,R_k(t,s) = \delta(t-s). \tag{5.6}$$

From similar considerations one obtains the following equation of motion for the running correlation function C_k

$$[\partial_t + r_k(t)]\,C_k(t,s) = 2DR(t,s), \tag{5.7}$$

where we have taken advantage of the simplification $D_k = D = 1$ discussed in Sec. 4.3 for the case of the vanishing background field approximation of the effective potential U_k.

Remember from Sec. 4.3.1, in particular from Eq. (4.32), that the modified mass $r_k(t)$ becomes time-dependent as soon as the Wetterich equation makes it flow from scale $k = \Lambda$, where it is given by $r_{k=\Lambda} = r_\Lambda$, to scale $k = 0$, where it is given by $r_{k=0} = r$, because of the explicit time dependence contained in the Gaussian non-equilibrium propagators which enters in the r.h.s. of the Wetterich equation. From the β-function of the mass r_k we can find r simply by integrating over the momenta k from a scale $k = \Lambda$ to $k = 0$. Remember that this β-function should be equipped with an initial condition for r_Λ. The choice of r_Λ is dictated by the fact that, in order to implement the critical quench, one should obtain a physical mass $r(t)$ which vanishes in the long time limit; in this way, in fact, in the long-time limit one recovers a scale-free theory which can describe the system at criticality.

Given $r(t)$ we can solve the equation of motion for the physical response function R (at $q = 0$):

$$[\partial_t + r(t)]\,R(t,s) = \delta(t-s), \tag{5.8}$$

as

$$R(t,s) = \theta(t-s)e^{-\int_s^t r(t)}. \tag{5.9}$$

Similarly, for C one obtains

$$[\partial_t + r(t)]\,C(t,s) = 2DR(t,s), \tag{5.10}$$

whose solution is given by

$$C(t,s) = 2D\int_0^\infty d\tau\; R(t,\tau)R(s,\tau), \tag{5.11}$$

where we have set $t_0 = 0$.

Let us summarize what we have found so far: Once $r(t)$ has been calculated, the response and correlation functions are completely fixed within the LPA approximation. For the case of vanishing background field approximation of U_k, the two-time functions are in fact given by Eqs. (5.9) and (5.11).

Now that we know how to calculate the two-time functions R and C in a local potential approximation, let us now analyze the strategy which we will follow for the determination of the asymptotic value of the fluctuation-dissipation ratio, i.e., of X^∞. From the general renormalization-group analysis, presented in Sec. 1.4.2, we know that the fluctuation-dissipation ratio in the long-time limit is given by

$$X^\infty = \frac{A_R}{A_C(1-\theta)}, \tag{5.12}$$

where the non-universal amplitudes A_R and A_C have been defined in Sec. 1.4.2.

We have seen in Chap. 4 that the critical initial-slip exponent θ can be retrieved from the analysis of the β-function of the anomalous dimension η_0 of the boundary field ϕ_0. We will show in the following that this is not the only way to obtain predictions for θ as one could understand now: in fact, from the explicit form of Eq. (5.9) in the long-time limit one recovers the general scaling function for R given by Eq. (1.44), and, by comparison, one can obtain θ.

However, the task that remain to us is the calculation of the ratio A_R/A_C. Note that from scaling relations given in Eqs.(1.44) and (1.45) we realize that simplifications occurs for the calculation of the amplitude ratio A_R/A_C in the case of LPA approximation:

$$\frac{A_R}{A_C} = \lim_{s\to\infty}\lim_{t\to\infty}\frac{sR(t,s)}{C(t,s)} = \lim_{s\to\infty}\frac{s}{2\int_0^s d\tau[R(s,\tau)]^2}, \tag{5.13}$$

where in last equality we have used Eqs. (5.9) and (5.11) and the simplification that can be done to re-express the response function, i.e., that

$$R(t,s) = R(t,\tau)R(\tau,s), \quad \text{with} \quad t>\tau>s, \tag{5.14}$$

which holds in LPA approximation as one can prove easily from Eq. (5.9).

As we will prove in the following, the last limit in Eq. (5.13) can be evaluated analytically within the LPA approximation. The result is given by:

$$\frac{A_R}{A_C} = \frac{1-2\theta}{2}, \tag{5.15}$$

so that the formula for X^∞, given by Eq. (5.12), simplifies in LPA approximation to

$$X^\infty = \frac{1}{2} - \frac{\theta}{2(1-\theta)}. \tag{5.16}$$

In the following section we detail the calculation of $r(t)$ for the case of a critical quench and the proof of Eq. (5.16) for the Ansatz given by Eq. (5.4) for

the local effective potential U_k. This case is simpler than that of non-vanishing background field, in which the Ansatz for the effective potential U_k is given by Eq. (4.49), mainly because of the presence of the anomalous dimension which has to be taken into account properly. We treat the case of non-vanishing background field approximations in Sec. 5.3 below.

Let us conclude this section with the following remark: the formula given in Eq. (5.16) for the long-time limit of the fluctuation-dissipation ratio holds within the LPA approximation for the case of vanishing and non-vanishing background field approximations of the effective potential U_k. This can be understood now on the basis that even in the improved Ansatz given by Eq. (4.49), Γ_k is still local in time in a pure LPA truncation (by the very definition of it given by Eq. (5.3)). Accordingly, similar simplifications to the one which lead to Eqs. (5.9),(5.11), and (5.16) can be used, as we show in Sec. 5.3.

5.2 The case of vanishing background field

5.2.1 Calculation of the time-dependent mass

Let us now discuss how we can retrieve $r(t)$ within the $\phi_m = 0$ approximation for U_k, given in Eq. (5.4). The flow equation for the mass $r_k(t)$ is given by Eq. (4.32), which we report here:

$$\frac{dr_k(t)}{dk} = -k^{d+1}\frac{a_d}{d}\frac{g_k D}{(k^2+r_k)^2}\left[1 - f_{r,k}(t)\right]. \tag{5.17}$$

Let us list the three prescriptions that we use in order to obtain the physical mass, i.e., $r(t) = r_{k=0}(t)$, from Eq. (5.17) for the case of a critical quench.

1. A priori, in the r.h.s. of the last equation we should allow a time dependence of the variables r_k and g_k. Below we use the following approximation (that in Ref. [31] leads to good predictions in a different context): we neglect the implicit time dependence on the r.h.s. of Eq. (5.17), i.e., we replace the mass $r_k(t)$ and the coupling constant $g_k(t)$ with their long-time limits $r_k(t\to\infty) = r_k$ and $g_k(t\to\infty) = g_k$.
2. Our choice in order to integrate Eq. (5.17) over k is to approximate the r.h.s. by evaluating the couplings r_k and g_k at the Wilson-Fisher fixed point, found in Sec. 4.3.2, through the analysis of the β function in the long-time limit, i.e., we set in the r.h.s. of Eq. (5.17)

$$\tilde{r}^*_{WF} = r_k/k^2, \quad \text{and} \quad \tilde{g}^*_{WF} = g_k D k^{d-4} a_d/d. \tag{5.18}$$

Once these equations are taken into account, Eq. (5.17) becomes

$$\begin{aligned}\frac{dr_k(t)}{dk} = &-k\frac{\tilde{g}^*_{WF}}{(1+\tilde{r}^*_{WF})^2}\times \\ &\times\left\{1 - e^{-2k^2 t(1+\tau^*_{WF})}\left[1 + 2k^2 t(1+\tau^*_{WF})\right]\right\},\end{aligned} \tag{5.19}$$

where we have evaluated $f_{r,k}(t)$, given by Eq. (4.31), at the Wilson-Fisher fixed point. Note that the coordinates of the Wilson-Fisher fixed point in the parameters space given by (r^*_{WF}, g^*_{WF}) depend parametrically on the spatial dimensionality of the system d via Eq. (4.38).

3. Let us now integrate the β-function for r_k, obtaining

$$\int_{\Lambda}^{0} \frac{dr_k(t)}{dk} dk = r(t) - r_{\Lambda}. \tag{5.20}$$

The bare parameter r_Λ has to be properly fine tuned in order to implement the critical temperature quench: the physical mass $r(t)$ has to satisfy $\lim_{t\to\infty} r(t) = 0$ in order to obtain a scale-free propagator in the long-time limit. In formula:

$$r_\Lambda : \lim_{t\to\infty} r(t) = 0. \tag{5.21}$$

Evaluating the integral $\int_\Lambda^0$ on the l.h.s. of Eq. (5.20) by means of Eq. (5.19) and the prescription given in Eq. (5.21) for the bare parameter r_Λ one obtains the following result for the physical mass $r(t)$:

$$r(t) = -\frac{1}{t} \frac{\tilde{g}^*_{WF}}{2(1+\tilde{r}^*_{WF})^3} \left\{ 1 - e^{-2\Lambda^2(1+\tilde{r}^*_{WF})t} \left[1 + \Lambda^2(1+\tilde{r}^*_{WF})t \right] \right\}. \tag{5.22}$$

We note here that the precise value of the parameter r_Λ corresponds to a negative shift of the critical temperature, accordingly to what we have seen for the case of the one-loop approximation in the end of Sec. 2.2.2.

In particular, the mass given by Eq. (5.22) in the aging (or long-time) regime, (i.e. $t\Lambda^2 \gg 1$, thus neglecting the exponential in the round brackets), simplifies to

$$r_{ag}(t) = -(1/t)\Theta_{LPA}, \tag{5.23}$$

where Θ_{LPA} is a numerical factor given by

$$\Theta_{LPA} = \frac{\tilde{g}^*_{WF}}{2(1+\tilde{r}^*_{WF})^3}. \tag{5.24}$$

We have anticipated, using the notation Θ_{LPA} that the response function is, in fact, given in the aging limit, i.e., $t \gg s \gg \Lambda^{-2}$, by

$$R_{ag}(t,s) = \vartheta(t-s) \left(\frac{t}{s} \right)^{\Theta_{LPA}}, \tag{5.25}$$

where we have used Eq. (5.9) and we replaced the mass $r(t)$ with its aging limit given by Eq. (5.23).

The amplitude A_R and the critical exponent θ are obtained through a comparison with the scaling function for R given in Eq. (1.44). This leads to

$$A_R = 1, \quad \theta = \Theta_{LPA}. \tag{5.26}$$

Note that our results for Θ_{LPA} agrees with the ones obtained in Ref. [35], in particular with Eqs. (4.45) and (4.47). Furthermore, the prediction for the

universal amplitude A_R does not differ from the Gaussian one, as one can see from Eq. (1.17). This is typical of local approximations, since, as one can see from Eq. (1.52), only non-local contributions (which arises in two-loops calculations, i.e., at order ϵ^2 in perturbative calculations) shift the value of A_R from 1.

Let us highlight a difference from the previous short-time analysis, used in Ref.[35] and detailed in Sec. 4.3.3, in order to calculate θ: we do not need to investigate the renormalization of the boundary term Z_0, nor we have to invoke the scaling relation $\theta = -\eta_0/z$, in order to compute the critical initial-slip exponent θ. This happens because we have shown how one can compute the response function in the aging limit, and thus obtain θ by means of a comparison with its scaling form given by Eq. (1.44).

Now we are able to calculate the correlation function C, from Eq. (5.11). Unlike the response function R, C depends on the whole history of dynamics from the time in which we do the quench, i.e., $t_0 = 0$. In fact, assuming $t > s$ we obtain from Eq. (5.11) that

$$C(t,s) = 2R(t,s)\int_0^s d\tau\ [R(s,\tau)]^2, \tag{5.27}$$

where we have used Eq. (5.14).

Note that replacing R with R_{ag} is allowed only if the observation times t, s are much longer than Λ^{-2}, so it seems not permitted in the r.h.s. of the last equation, because the integration interval is $\tau \in [0, s]$. However, in the aging limit, simplification occurs and the exponential decaying part of time-dependent mass in Eq. (5.22), which depends on the microscopic scale $\Lambda \sim 1/a$ (where a is the spacing of the underlying lattice model) does not add any contribution, as proven in Appendix A.

Even more simplifications occur in the ϵ-expansion version of Eq. (5.27), as we discuss in the next section.

5.2.2 Comparison with the results of the first-order ϵ-expansion

In order to retrieve known results in ϵ-expansion we do an expansion in $\epsilon = 4 - d$ of the response function inside the integral in Eq. (5.27), so that we obtain

$$\begin{aligned}\int_0^s d\tau R^2(s,\tau) &= \int_0^s d\tau \left[1 - 2\int_\tau^s dt\ r(t) + \mathcal{O}(\epsilon^2)\right] \\ &= s\,(1 + \tilde{g}^*_{WF}) + \mathcal{O}(\epsilon^2), \quad \text{if} \quad \Lambda^2 s \gg 1,' \\ &= s(1+2\theta) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{5.28}$$

where in the second equality we have expanded $r(t)$ given by Eq. (5.22) up to the first order in ϵ and in the last equality we have used the fact that at this order

$\theta = g^*_{WF}/2 + \mathcal{O}(\epsilon^2)$: in fact we found in Sec. 4.2.2 that the Wilson-Fisher fixed point is given by $(\tilde{r}^*_{WF}, \tilde{g}^*_{WF}) = (-\epsilon/12, \epsilon/6)$ at first order in the ϵ-expansion and from Eq. (4.48) we know that $\theta = \epsilon/12 + \mathcal{O}(\epsilon)$.

Inserting Eq. (5.28) in Eq. (5.27), one finds

$$C_{ag}(t,s) = 2(1+2\theta)s\left(\frac{t}{s}\right)^{\theta}, \tag{5.29}$$

where we have used the result for the response function in the aging limit given by Eq. (5.23). By comparison of Eq. (5.29) with the scaling formula for C in the aging limit, given by Eq. (1.45), we obtain the following non-universal amplitude for the correlation function:

$$A_C = 2(1+2\theta) + \mathcal{O}(\epsilon^2). \tag{5.30}$$

We are now in the position to retrieve the FDR at first order in ϵ, since we have calculated the amplitudes, i.e., A_R and A_C, and the critical initial-slip exponent θ. The result follows from Eq. (5.12) and it is given by:

$$X^{\infty} = \frac{1}{2}\left(1 - \frac{\epsilon}{12}\right) + \mathcal{O}(\epsilon^2), \tag{5.31}$$

which perfectly agrees with the one-loop perturbative predictions, reported in Eq. (1.53) (note that the comparison should be done with $N = 1$, in order to evaluate the perturbative result for the Ising universality class discussed here).

5.2.3 Calculation of the fluctuation-dissipation ratio in the long-time limit

While the aging limit for the response function is easily taken in Eq. (5.9) via Eq. (5.23) finding Eq. (5.25), as we have seen in previous sections, this procedure is more subtle for the correlation function given by Eq. (5.27), and the details are given in Appendix A. The major result of the analysis reported in Appendix A is that, in order to compute the integral in Eq. (5.27) in the aging limit, one can replace the time-dependent mass $r(t)$ given in Eq. (5.22) with its aging limit, i.e., with

$$r(t) \to r_{ag}(t) = -\frac{\theta_{LPA}}{t}. \tag{5.32}$$

Note that this is telling us that the microscopic details, such as $\Lambda \sim 1/a$, do not contribute to the correlation function in the aging limit.

In this case, the integral in Eq. (5.27), can be computed analytically and gives

$$C_{ag}(t,s) = \frac{2}{1-2\theta}s\left(\frac{t}{s}\right)^{\theta}. \tag{5.33}$$

By comparison of this result with the scaling form for the correlation function, given by Eq. (1.45), one concludes that

$$A_C = \frac{2}{1-2\theta}. \tag{5.34}$$

The fluctuation-dissipation ratio in the aging limit, given by Eq. (5.12), is therefore given by

$$X^\infty = \frac{1}{2} - \frac{\theta}{2(1-\theta)}, \tag{5.35}$$

where in the last equality we have used the results obtained in LPA approximation for the non-universal amplitudes of the two-time functions given by Eqs. (5.26) and (5.33). Note here that if Eq. (5.35) is expanded in power of ϵ one retrieve Eq. (5.31) at first order in ϵ, as it should.

5.3 The case of non-vanishing background field

In this section we detail our result for the asymptotic value of the FDR by means of a non-vanishing homogeneous background field approximation, introduced in Sec. 4.4, in which the Ansatz for U_k is given by Eq. (4.49).

Remember from the discussion in Sec. 4.4 that this Ansatz adds contribution non-local in time in the self energy in $\Gamma_k^{(1,1)}$ and in the noise D_k, thus allowing us to retrieve the anomalous dimensions η_D, η_Z and η_K as one can see from Eqs. (4.67) and (4.66).

We follow the same strategy as that outlined in Sec. 5.1 in order to determine R and C: we make an Ansatz for the effective action of the form given by Eq. (4.9), where the running parameter Z_k, K_k, D_k are field and time-independent. The only time-dependent quantity in the Ansatz is given by the mass r_k (i.e., by the local part of the self energy). Remember from Sec. 4.4.2 that, in this approximation, the physical mass for a system in its high-temperature regime is given by Eq. (4.69).

5.3.1 Two-time quantities in the presence of non-vanishing anomalous dimension

The main difference with respect to the case of vanishing background field approximation, described in Sec. 5.2, is the presence of the parameters Z_k, D_k and K_k, which are expected to result into anomalous dimensions. Taking them into account, the equations of motion for the two-time functions at scale k (evaluated at zero momentum) and in the disordered phase, previously given by Eqs. (5.6) and (5.7), are now given by

$$\left[Z_k \partial_t + r_k(t)\right] R_k(t,s) = \delta(t-s), \tag{5.36}$$

$$\left[Z_k \partial_t + r_k(t)\right] C_k(t,s) = 2D_k R_k(t,s). \tag{5.37}$$

Since Z_k is independent of time, in the LPA approximation we are dealing with (as described in Sec. 4.4) we can manipulate the equations of motion in order to find

$$\left(\partial_t + \frac{r_k(t)}{Z_k}\right)\tilde{R}_k(t,s) = \delta(t-s), \tag{5.38}$$

$$\left(\partial_t + \frac{r_k}{Z_k}\right)\tilde{C}_k(t,s) = 2\frac{D_k}{Z_k}\tilde{R}_k(t,s), \tag{5.39}$$

where we have defined the *reduced* two-time functions:

$$\tilde{R}_k(t,s) = Z_k R_k(t,s), \quad \text{and} \quad \tilde{C}_k(t,s) = Z_k C_k(t,s). \tag{5.40}$$

The solutions of Eqs. (5.38) and (5.39) for the reduced running two-time functions are respectively given by

$$\tilde{R}_k(t,s) = \theta(t-s)e^{-\int_s^t d\tau\ \frac{r_k(\tau)}{Z_k}}, \tag{5.41}$$

$$\tilde{C}_k(t,s) = 2\frac{D_k}{Z_k}\int_0^\infty d\tau\ \tilde{R}_k(t,\tau)\tilde{R}_k(s,\tau). \tag{5.42}$$

Now we can see that the amplitude ratio at scale k is given by

$$\begin{aligned}\frac{A_{R,k}}{A_{R,k}} &= \lim_{s\to\infty}\lim_{t\to\infty}\frac{sR_k(t,s)}{C_k(t,s)} = \frac{s\tilde{R}_k(t,s)}{\tilde{C}_k(t,s)} = \\ &= \lim_{s\to\infty}\lim_{t\to\infty}\frac{s\tilde{R}_k(t,s)}{2\frac{D_k}{Z_k}\int_0^s d\tau\ \tilde{R}_k(t,\tau)\tilde{R}_k(s,\tau)} \\ &= \lim_{s\to\infty}\frac{s}{2\frac{D_k}{Z_k}\int_0^s d\tau\ [R_k(s,\tau)]^2},\end{aligned} \tag{5.43}$$

where we have Eq. (5.14) in the last equality. We see that the ratio between the two-time functions in this approximation is again just given as a function of s[1]. This last result for the amplitude ratio should be compared with the one obtained in the previous Section, i.e. Eq. (5.13), where we have used the so-called $\phi_m = 0$ approximation for the effective potential U_k.

Let us discuss in more detail the analysis which we have make so far to derive Eq. (5.43):

- The overall constant Z_k which appears in the definition of the reduced two-time functions defined in Eqs.(5.36) and (5.37) does not appear in the final formula for the amplitude ratio being a ratio of the two. Moreover, the ratio D_k/Z_k is equal to one since their β-functions and initial condition are the same (see Ref. [35] for the explicit form of these β-function and remember that the initial condition are set to be $Z_k = D_k = 1$).

[1]This is true also for the case of the one-loop perturbative calculation and the reason is that in both cases the self energy is local in time.

- The only quantity which has to be computed in order to determine the response and correlation functions at scale $k = 0$ is the β-function for the *reduced* mass, given by r_k/Z_k, as one can see from Eqs. (5.41) and (5.42). We use the adjective reduced in order to distinguish the mass r_k from the one which plays the same role in the equation for the reduced two-time functions, i.e., r_k/Z_k.

5.3.2 Flow equation for the reduced mass

We have seen that in order to solve the equation of motion for the reduced response function, given by Eq. (5.41), one should compute the reduced mass, given by r_k/Z_k.

In order to simplify the flow equation for the reduced mass r_k/Z_k one can proceed as follows. The exact relation which links the flow equation for the mass and the one for the reduced mass:

$$Z_k \partial_k \left(\frac{r_k(t)}{Z_k} \right) = \partial_k r_k(t) - r_k(t) \frac{1}{Z_k} \partial_k Z_k, \tag{5.44}$$

in which the r.h.s. is merely the derivative of the l.h.s. Note that the second term on the r.h.s. gives $\sim -r_k(t)\eta_Z$ [by the very definition of the anomalous dimension related to the parameter Z_k, given by Eq. (4.61)] in the infrared regime, i.e., $k \to 0$. Furthermore it amounts to a renormalization of the critical temperature, being in the r.h.s. of the flow equation of r_k/Z_k [2]. From now on we drop this term: this correspond to a specific choice of the bare mass such that in the limit of long times the mass at scale $k = 0$ vanishes. With this simplification, one obtains the following flow equation for the reduced mass

$$\partial_k \left(\frac{r_k(t)}{Z_k} \right) = \frac{1}{Z_k} \partial_k r_k(t). \tag{5.45}$$

This equation implies that what is needed in order to determine the flow equation for the reduced running mass r_k/Z_k is simply the flow equation for the running mass r_k, which is defined in Eq. (4.69), properly rescaled by Z_k. We will not go into the full details of the calculation here, referring the interested reader to Appendix B.

We summarize here the main steps which one should follow in order to determine the flow equation for the physical mass $r_k(t)$.

1. The physical mass r_k depends upon the bulk parameters $\phi^2_{m,k}$, g_k and λ_k. This means that the flow equation for r_k depends on those bulk parameters listed above, defined in Eqs. (4.56) and (4.4.1); which are given by Eqs. (4.63,4.64) and (4.65).
2. Once the flow equations for the bulk couplings are obtained (remember that for the case of g_k and λ_k a localization procedure is necessary, since

[2] Consistently with our truncation procedure on the local Ansatz for the effective action (in which we retain only the explicit time-dependent contribution given by the Gaussian correlation functions which appears in the r.h.s. of the Wetterich equation, see Sec. 5.1).

the terms $\Delta\Gamma_{k,2}$ and $\Delta\Gamma_{k,3}$ [defined in Eq. (4.24)] depend respectively on two and three times. The integration over time yields eventually the functions $f_{r,k}$, $f_{g,k}$, and $f_{\lambda,k}$, in a similar way to what happened in Eqs.(5.22) and (4.31).

3. The final step is to integrate the β-function for the mass with the prescriptions, given in Sec. 5.1, appropriate for a critical quench of the model.

In the end, one finds an equation for the reduced mass r_k/Z_k at scale $k = 0$ which is very similar to the one obtained in the previous section, i.e., Eq. (5.22). We report here the result in a compact notation:

$$\left.\frac{r_k}{Z_k}\right|_{k=0} = -\frac{\theta}{t}\left[1 - f_r(\Lambda^z t)\right], \tag{5.46}$$

where the precise form of f_r can be found in Appendix B (for the case of the Ansatz given in Eq. (4.49) with $\lambda_k = 0$). The precise form of θ, as we proof in Appendix B, coincides with the one obtained in Sec. 4.4.2 by means of the relation $\theta = -\eta_0/z$, i.e., it allows to properly take into account the effect of the anomalous dimensions (remember that $z = 2 - \eta_Z + \eta_K$).

Since the reduced physical mass is given by Eq. (5.46) we can use the result of Appendix A, which states that the asymptotic fluctuation-dissipation ratio in this case is given by Eq. (5.35).

5.4 Predictions of the asymptotic value of the fluctuation-dissipation ratio

In this section we analyze our results for the asymptotic value X^∞ of the fluctuation-dissipation ratio. We remark here the important finding which is encoded in Eq. (5.35): in order to calculate X^∞ one simply insert in it the prediction of the critical initial-slip exponent θ, discussed in Secs. 4.3 and 4.4, given respectively by Eqs. (4.47), (4.71) and (4.72).

As one can see from Fig. 4.3, we have found three approximation for θ (reported in red) of increasing accuracy. The details of previous findings have been given in Fig. 1.2. The corresponding three approximations for the asymptotic fluctuation-dissipation ratio X^∞ are reported in Fig.5.1 with red lines. Note that, in order to facilitate the comparison, we have used the same line style for the same approximation used (the red solid line in Fig. 4.3 corresponds to the red solid line in Fig. 5.1 and so on.). Previously available results presented in the literature and reported also in Fig. 5.1 have been already discussed in Fig. 1.3.

Let us list our predictions here:

(i) The dot-dashed line is the one which has been found for the expansion of the effective potential U_k around a vanishing background field configuration given by Eq. (5.4).

(*ii*) The dashed line is obtained through the determination of θ given by the expansion of U_k around a non-vanishing homogeneous background field ϕ_m, given by Eq. (4.49), where we set $\lambda_k = 0$.

(*iii*) The solid red line is the best approximation obtained in this thesis and it relies on the ansatz given in Eq. (4.49).

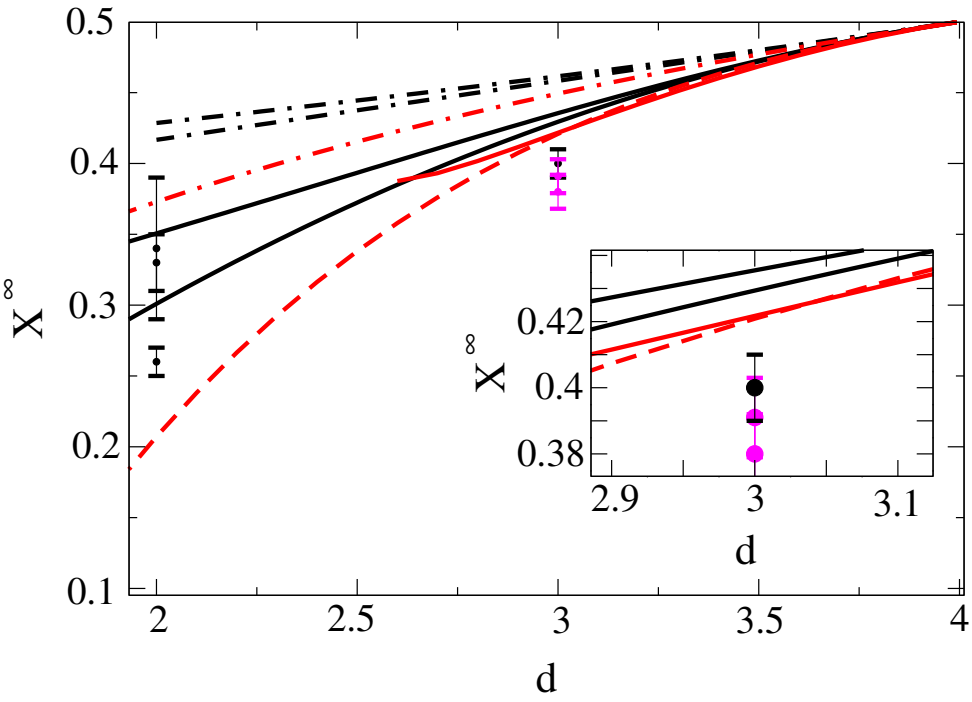

Figure 5.1: X^∞ as a function of the spatial dimensionality d of the system. Red lines correspond to our predictions within LPA approximation of the fRG technique. The dot-dashed line is obtained by means of a vanishing background field approximation of the effective potential U_k given in Eq. (5.4). The dashed and the solid red lines are, instead, obtained by means of the non-vanishing background field approximation for U_k given in Eq. (4.49), respectively, for the case of $\lambda_k = 0$ and $\lambda_k \neq 0$. Black and magenta lines and symbols with error bars are those reported in the literature so far and they have presented in detail in Fig. 1.3 (they are a combination of one and two-loops perturbative calculation, and of the results of Monte Carlo simulation). Inset: magnification of the main plot for $d \simeq 3$.

The remarkable systematic improvements of fRG results with respect to the previous perturbative renormalization-group (pRG) if compared to the recent

MC predictions (reported Fig. 5.1 with magenta symbols with error bars) needs to be explained. This improvement, in our opinion, is due to two peculiarities which concern the non-perturbative nature of the fRG technique. In fact,

(1) The β-functions for the bulk couplings (either given by Eq. (4.38) for $\phi_m = 0$ or by Eq. (4.63),(4.64) and (4.65) for $\phi_m \neq 0$) can be evaluated for arbitrary dimension d. The Wilson-Fisher fixed point is more accurately found[3] than the one obtained through the analytical continuation $\epsilon \to 1$ in perturbative calculation. This reflects into a more precise determination of θ and therefore of X^∞, given respectively by Eqs. (5.24) and (4.71),(4.72) and (5.16).

(2) The amplitude of the response function is $A_R = 1$ in LPA approximation and therefore gives the same result as the one obtained by means of pRG calculation at order ϵ, given in Eq. (1.52), where we set to zero the contribution proportional to ϵ^2. The fRG calculation in LPA underestimate this contribution if compared to the pRG results[4] at order ϵ^2. Instead we see that the fRG permits a more precise determination of the non-universal amplitude of the correlation function, given by A_C, if compared to the perturbative renormalization-group result at order ϵ. For a comparison of analytical formula between pRG at order ϵ and fRG in LPA approximation see Tab. 5.1[5].

	pRG at order ϵ	**fRG in LPA**
θ	$\frac{\epsilon}{12} + \mathcal{O}(\epsilon^2)$	$\frac{\tilde{g}^*_{WF}}{2(1+\tilde{r}^*_{WF})^3}$
A_R	$1 + \mathcal{O}(\epsilon^2)$	1
A_C	$2(1+2\theta) + \mathcal{O}(\epsilon^2)$	$\frac{2}{1-2\theta}$
X^∞	$\frac{1}{2}(1-\theta) + \mathcal{O}(\epsilon^2)$	$\frac{1}{2} - \frac{\theta}{2(1-\theta)}$

Table 5.1: Difference between perturbative treatment and non-perturbative calculation in LPA approximation.

[3] This is an à posteriori statement, in the sense that we do not have a rigorous proof of it.

[4] This consideration is due to the fact that the contribution at order ϵ^2 for the amplitude A_R arises once that non-local in time contribution are taken into account by the two-loop calculation for the determination of the self energy .

[5] Note that the prediction for θ in the fRG column is the result for the vanishing background approximation of U_k detailed in Sec. 4.3. We report here this result since the comparison with the pRG prediction at order ϵ is easier. Our final formula for θ, given by a non-vanishing background field approximation for U_k reported in Eq. (4.49), is given by Eqs. (4.71) and (4.72).

Conclusions and future perspective

In this thesis we have calculated the asymptotic value of the fluctuation-dissipation ratio for the purely relaxational model of critical dynamics by means of the functional renormalization-group technique, extending its domain of application compared to that already explored in the literature. We have solved the problem in a local potential approximation for the flow equation of the modified effective action. Our primary result is the fact that the asymptotic value of the fluctuation-dissipation ratio, which is a universal observable in the renormalization-group sense, within the local potential approximation discussed in this work is simply given by

$$X^{\infty} = \frac{1}{2} - \frac{\theta}{2(1-\theta)}, \tag{5.47}$$

where θ is the critical initial-slip exponent. It is remarkable that this equation is valid for all the approximation that can be done for a local potential U_k.

Since the proof of Eq. (5.47) relies on the calculation of the two-time functions some remarkable facts have been found established, i.e.:

- In this work, instead of relying on the short-time expansion (given in Ref. [35] and reported in Sec. 4.3), we have shown how one can obtain θ simply by comparing the result of the calculation of the two-time functions in the aging limit with their general scaling form given by Eqs. (1.44) and (1.45).
- We clarify the physical interpretation of the equation which gives the anomalous dimension of the boundary (in time) order-parameter field and the physical mass for a high-temperature state, see Sec. 4.4.
- The definition of the physical mass, and of η_0, which arises from the discussion presented in Sec. 4.4 led us to highlight a mistake in the calculation that has been reported in Ref. [35], in order to obtain the result for η_0 and for the critical initial-slip exponent θ, since they are simply related by $\theta = -\eta_0/z$. The calculation that we have done in order to properly fix this mistake has leads us to a correction of the order of the 2% prediction of θ given in Ref. [35]. The prediction presented in this work turns out to be in better agreement with the recent MC data of Ref. [19] than the prediction of Ref. [35].

For what concerns possible improvement of this work they can take, in our perspective, three possible different directions: (A) the first is the natural improvement of the calculation done in this thesis. In fact we have discussed how to deal with an ansatz for the effective potential truncated at the third order in an expansion in powers of $(\phi^2 - \phi_m^2)$. The natural improvement is to add to the ansatz given by Eq. (4.49) higher-order terms in powers of $(\phi^2 - \phi_m^2)$, i.e., for example, $g_{8,k}(\phi^2 - \phi_m^2)^4$, and to properly calculate θ generalizing in a straightforward way the procedure outlined in Chapter 4 and 5 in order to take into account the new bulk coupling constant, such as $g_{8,k}$. (B) The second direction of research is to go beyond the local potential approximation in order to take into account non-local contribution in the self-energy and in the noise term. One thing that is expected to qualitatively change once this non-local contribution have been taken into account is the fact that the non-universal amplitude A_R of the response function will be no longer equal to 1, since as one can see from the two-loops perturbative calculation done in Ref. [17], these non-local contributions adds a term proportional to ϵ^2 to A_R. A careful analysis of the mode-coupling approximation, which has been done in the context of glassy materials by Bouchaud, Cugliandolo et al. in Ref. [36], will give some fruitful insight in order to generalize the local potential approximation in such a way to take into account non-local contributions in time at the level of the ansatz which has to be done for the effective action Γ_k in order to close the infinite hierarchy of equations generated by the Wetterich equation. (C) The third direction of research is to apply the framework discussed here for the analysis of X^∞ and θ to other models of relaxational dynamics (for a complete review of these model see, e.g, Ref. [8]).

Appendices

Appendix A

Aging limit of the correlation function in local potential approximation

A.1 Proof of Eq. (5.16)

In this appendix we detail the calculations that led us to our final formula for X^∞, given by Eq. (5.16). Remember from Sec. 5.1, that we have to compute the limit in Eq. 5.13 for the case of a critical quench of the model. This amount to study the following quantity:

$$\lim_{s\to\infty} \int_0^s dt' \; R(s,t')^2, \tag{A.1}$$

where

$$R(s,t') = e^{-\int_{t'}^{s} dt'' \; r(t'')}. \tag{A.2}$$

The LPA prediction for the mass $r(t)$ for the case of a critical quench of the model is given by Eq. (5.22), which can be rewritten as

$$r(t) = -\frac{\theta}{t}\left[1 - f_r(\Lambda^z t)\right], \tag{A.3}$$

where f_r is a function which decays exponentially fast as t grows (as one can see explicitly from Eq. (5.22)). Let us simplify the integral in Eq. (A.1), taking advantage of Eqs. (A.2) and (A.3):

$$\int_0^s dt' \; R(s,t')^2 = \int_0^s dt' \left(\frac{s}{t'}\right)^{2\theta} \exp\left(-2\theta \int_{t'}^{s} \frac{dt''}{t''} f_r(\Lambda^z t'')\right). \tag{A.4}$$

With the change of variables $t' = s\tau$ we obtain

$$\int_0^s dt' \; R(s,t')^2 = s \int_0^1 d\tau \; \tau^{-2\theta} \exp\left(-2\theta \int_{s\tau}^{s} \frac{dt''}{t''} f_r(\Lambda^z t'')\right). \tag{A.5}$$

Now setting $t'' = xs$ we obtain

$$\int_0^s dt' R(s,t')^2 = s \int_0^1 d\tau\ \tau^{-2\theta} \exp\left(-2\theta \int_\tau^1 \frac{dx}{x} f_r(\Lambda^z x s)\right). \tag{A.6}$$

Remember that the aging limit is reached when $A = \Lambda^z s \to \infty$. Breaking the integral in τ we obtain

$$\begin{aligned}\int_0^s dt' R(s,t')^2 = s \int_{A^{-1}}^1 d\tau\ \tau^{-2\theta} \exp\left(-2\theta \int_\tau^1 \frac{dx}{x} f_r(Ax)\right) + \\ + s \int_0^{A^{-1}} d\tau\ \tau^{-2\theta} \exp\left(-2\theta \int_\tau^1 \frac{dx}{x} f_r(Ax)\right).\end{aligned} \tag{A.7}$$

The integral in the first line is convergent and the limit $A \to \infty$ can be safely taken. This integral is given in the aging limit by

$$\int_0^1 d\tau\ \tau^{-2\theta} = \frac{1}{1-2\theta}. \tag{A.8}$$

Let us analyze the integral in the last line of Eq. A.7. Again we break the integral in x and we obtain for it the following expression

$$\exp\left(-2\theta \int_{A^{-1}}^1 \frac{dx}{x} f_r(Ax)\right) \int_0^{\Lambda^{-1}} d\tau\ \tau^{-2\theta} \exp\left(-2\theta \int_\tau^{A^{-1}} \frac{dx}{x} f_r(Ax)\right). \tag{A.9}$$

With the change of variable $y = Ax$ last equation simplify to

$$\exp\left(-2\theta \int_1^A \frac{dy}{y} f_r(y)\right) \int_0^{\Lambda^{-1}} d\tau\ \tau^{-2\theta} \exp\left(-2\theta \int_\tau^{A^{-1}} \frac{dx}{x} f_r(Ax)\right). \tag{A.10}$$

The exponential in front of the integral in τ is a convergent quantity, so let us focus on the integral in τ itself. With the change of variable $u = A\tau$ we obtain

$$A^{-1+2\theta} \int_0^1 du\ u^{-2\theta} \exp\left(-2\theta \int_{u/A}^{1/A} \frac{dx}{x} f_r(Ax)\right), \tag{A.11}$$

and as long as $-1 + 2\theta < 0$ this integral gives zero contribution in the aging limit $A = \Lambda^z s \to \infty$ to the limit given by Eq. (A.2).

In summary we have obtained

$$\lim_{s\to\infty} \int_0^s dt' R(s,t')^2 \sim \frac{s}{1-2\theta}, \tag{A.12}$$

which when it is inserted in Eq. 5.13 gives

$$\frac{A_R}{A_C} = \frac{1-2\theta}{2}, \tag{A.13}$$

and in the end we obtain from Eq. (5.12)

$$X^\infty = \frac{1}{2} - \frac{\theta}{2(1-\theta)}. \tag{A.14}$$

As promised we have obtained Eq. 5.16 reported in the main text.

Appendix B

Flow equations in the presence of anomalous dimension

In this appendix we give a rather detailed description of the calculations necessary in order to find the flow equation of the disordered mass, given by Eq. (4.69). Since the inclusion of the term proportional to λ_k only makes the calculations a bit more lengthy but do not adds any relevant complications we detail here the calculations for an Ansatz given by Eq. (4.49) in which we set $\lambda_k = 0$. In this case the physical mass is given by :

$$r_k = \frac{\delta^2 U_k(\phi)}{\delta\phi^2}\Big|_{\phi=\phi_{phys}=0} = -\frac{g_k}{3!}\phi_{m,k}^2 = -\frac{1}{2}m_k, \tag{B.1}$$

where m_k is the mass in the ordered phase, i.e.

$$m_k = \frac{\delta^2 U_k(\phi)}{\delta\phi^2}\Big|_{\phi=\phi_{m,k}}. \tag{B.2}$$

This simplification as we have said does not hide any relevant complications, and allows us to detail in the simplest way how the effect of anomalous dimensions lead us to a form of the physical mass (at scale $k = 0$) given by Eq. (A.3), in which θ is precisely given by $\theta = -\eta_0/z$ obtained with the short-time expansion introduced in Ref. [35], i.e., it allows to properly take into account the effect of the anomalous dimensions (remember that $z = 2 - \eta_Z + \eta_K$).

B.0.1 Calculation of the β-function for $m_k(t)$ in the presence of anomalous dimension

From the definition of the mass in the ordered phase, given in Eq. (4.55), we know that its flow equation with respect to the scale k introduced by the cutoff R_k is given by

$$k\partial_k m_k = k\partial_k\left(\frac{2}{3}\rho_{m,k}g_k\right) = \frac{2}{3}g_k k\partial_k\rho_{m,k} + \frac{2}{3}\rho_{m,k}k\partial_k g_k, \tag{B.3}$$

where $\rho_{m,k} = \phi^2_{m,k}/2$.

We would like to note that this last equation follows from the more general flow equation for the self energy in the ordered phase:

$$k\partial_k \Sigma_{m,k} = \left(k\partial_k \Gamma_k^{(1,1)}\right)\Big|_{\phi_{m,k}} + \left(\frac{\delta}{\delta\rho}\Gamma_k^{(1,1)}\right)\Big|_{\phi_{m,k}} k\partial_k \rho_{m,k}, \tag{B.4}$$

where

$$\Sigma_{m,k} = \Gamma_k^{(1,1)}\Big|_{\phi_{m,k}}. \tag{B.5}$$

$\partial_k \Sigma_{m,k}$, as we have seen in chap. 4.3, contains a contribution non local in time which has to be properly localized. This contribution is the first one in the r.h.s. of Eq. (B.4), which is given explicitly by

$$k\partial_k \Gamma_k^{(1,1)} = -\frac{1}{2}\int G_k \frac{\delta^2 \Gamma_k^{(2)}}{\delta\tilde{\phi}\delta\phi} G_k \dot{\mathrm{R}}_k \sigma + \int G_k \frac{\delta\Gamma_k^{(2)}}{\delta\tilde{\phi}} G_k \frac{\delta\Gamma_k^{(2)}}{\delta\phi} G_k \dot{\mathrm{R}}_k \sigma, \tag{B.6}$$

where the second term in the r.h.s. is the one which depends on two times and have used the sort-hand notation in which the integral $\int$ denotes the integrals over momenta and time, and the trace over the internal matrix structure.. We remind here that one should be careful because, in order to be consistent with the local potential approximation, we should extract the local in time part of $\Sigma_{m,k}$. We represent the localization procedure by a proper operator $\mathcal{L}$, in such a way to write

$$\mathcal{L}(\Sigma_{m,k}) = m_k. \tag{B.7}$$

Note that the procedure which defines the operator $\mathcal{L}$ has been already introduced in Ref. [35] in order to find the flow of the mass m_k and it has been presented in this work in Sec. 4.2.1. In what follows we detail the action of $\mathcal{L}$.

Evaluating Eq. (B.6) in the ordered homogeneous (in time and space) background configuration $\Phi_k = (\tilde{\phi}_b, \phi_b) = (0, \phi_{m,k})$ we obtain

$$\begin{aligned} k\partial_k \Gamma_k^{(1,1)}\Big|_{\Phi} = &-\frac{1}{2} U^{(4)}(\phi)\Big|_{\phi_m} \int G_k^0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0 \dot{\mathrm{R}}_k \sigma + \\ &+ \left(U^{(3)}(\phi)\Big|_{\phi_m}\right)^2 \int G_k^0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} G_k^0 \dot{\mathrm{R}}_k \sigma, \end{aligned} \tag{B.8}$$

where the second term is the one which depends on two times. Remember that

$$G_k^0(t,s) = \begin{bmatrix} C_{0,k}(t,s) & R_{0,k}(t,s) \\ R_{0,k}(s,t) & 0 \end{bmatrix}, \tag{B.9}$$

is the inverse of the Hessian $(\Gamma_k^{(2)} + \mathrm{R}_k\sigma)$ obtained from the Ansatz for Γ_k and taking into account the presence of the cutoff term R_k in the modified action. Because of the cutoff function R_k we have to evaluate G_k^0 with the modified dispersion relation $\omega_k = K_k k^2 + m_k$. The two point function are thus given by

$$R_k(t,s) = \frac{1}{Z_k}\theta(t-s) e^{-\frac{\omega_k}{Z_k}(t-s)}, \tag{B.10}$$

$$C_k(t,s) = \frac{D_k}{Z_k^2}\frac{1}{(\omega_k/Z_k)}\left[e^{-(\omega_k/Z_k)|t-s|} - e^{-(\omega_k/Z_k)(t+s)}\right]. \tag{B.11}$$

In this way we have unpackaged the flow equation for the mass in the ordered configuration.

In what follows we take into account all the terms which has to be properly computed in order to explicitly compute the r.h.s. of Eq. (B.3) via Eqs. (B.4) and (B.7).

For what concerns the flow of the minimum $\rho_{m,k}$ we find from Eq. (4.64)

$$\begin{aligned} k\partial_k\rho_m &= \frac{-3}{g_k\phi_{m,k}}\left(k\partial_k\frac{\delta U(\phi)}{\delta\phi}\right)\Big|_{\phi_{m,k}} \\ &= -\frac{3}{g_k\phi_{m,k}}\left(k\partial_k\frac{\delta\Gamma_k}{\delta\tilde{\phi}}\right)\Big|_{\phi_{m,k}}, \end{aligned} \tag{B.12}$$

and

$$k\partial_k\frac{\delta\Gamma_k}{\delta\tilde{\phi}} = -\frac{1}{2}\int G_k\frac{\delta\Gamma_k^{(2)}}{\delta\tilde{\phi}}G_k\dot{\mathrm{R}}_k\sigma. \tag{B.13}$$

Evaluating the very last equation in the constant in time background field Φ_k we obtain

$$\left(k\partial_k\frac{\delta\Gamma_k}{\delta\tilde{\phi}}\right)\Big|_{\Phi_k} = -\frac{1}{2}U_k^{(3)}(\phi)\Big|_{\phi_{m,k}}\int G_k^0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}G_k^0\dot{\mathrm{R}}_k\sigma. \tag{B.14}$$

Taking into account Eqs.(B.8,B.12,B.14) the formula for $k\partial_k\Sigma_{m,k}$ given by Eq. (B.4) simplifies to

$$\begin{aligned} &k\partial_k\Sigma_{m,k} = \\ &= \left[-\frac{1}{2}U_k^{(4)}(\phi)\Big|_{\phi_m} + \frac{3}{2}\left(\frac{[U_k^{(3)}(\phi)|_{\phi_{m,k}}]^2}{g_k\phi_{m,k}^2}\right)\Big|_{\phi_m}\right]\int G_k^0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}G_k^0\dot{\mathrm{R}}_k + \\ &+\left(U_k^{(3)}(\phi)\Big|_{\phi_m}\right)^2\int G_k^0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}G_k^0\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}G_k^0\dot{\mathrm{R}}_k\sigma. \end{aligned} \tag{B.15}$$

Evaluating the factors in front of the integrals using the Ansatz for the local potential given by $U_k = g_k(\phi - \phi_{m,k}^2)^2$, we obtain

$$\begin{aligned} k\partial_k\Sigma_{m,k} = g_k\Bigg[&\int G_k^0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}G_k^0\dot{\mathrm{R}}_k\sigma + \\ &+3m_k\int G_k^0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}G_k^0\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}G_k^0\dot{\mathrm{R}}_k\sigma\Bigg], \end{aligned} \tag{B.16}$$

which is the β-function for the self energy in the ordered phase, obtained from Eq. (B.4). In the next section we will do the integrals in Eq. (B.16) and we analyze the contribution of Eq. (B.4) to the flow of its local part, i.e., m_k (see Eq. (B.7)).

B.0.2 Integrals and projection on the local ansatz for the effective action

At this point we evaluate the trace and the integrals which appears in Eq. (B.16):
(i) The first term inside the square braket is given by

$$\int G_k^0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0 \dot{\mathrm{R}}_k \sigma = \mathrm{Tr} \int_{q,t_1} G_k^0(t_1,t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0(t,t_1) \dot{\mathrm{R}}_k \sigma. \quad \text{(B.17)}$$

(ii) The second term in the square bracket is given by

$$\begin{aligned} &\int G_k^0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} G_k^0 \dot{\mathrm{R}}_k \sigma \\ &= \mathrm{Tr} \int_{q,t_1} G_k^0(t_1,t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0(t,s) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} G_k^0(s,t_1) \dot{\mathrm{R}}_k \sigma. \end{aligned} \quad \text{(B.18)}$$

We evaluate the trace of the (i) and (ii) terms given above as
(i) The first term gives

$$\mathrm{Tr}\left(G_k^0(t_1,t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0(t,t_1) \sigma \right) = 2 R_{0,k}(t,t_1) C_{0,k}(t,t_1). \quad \text{(B.19)}$$

(ii) The second term gives

$$\begin{aligned} &\mathrm{Tr}\left(G_k^0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G_k^0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} G_k^0 \sigma \right) = C_{0,k}(t,t_1) R_{0,k}(t,s) R_{0,k}(s,t_1) + \\ &+ R_{0,k}(t,s) C_{0,k}(s,t_1) R_{0,k}(t,t_1) + C_{0,k}(t,s) R_{0,k}(t,t_1) R_{0,k}(t_1,s). \end{aligned} \quad \text{(B.20)}$$

At this point one should properly compute the integrals over momentum and times in Eqs.(B.17) and (B.18). Equation (B.17) imply that we have to integrate Eq. (B.19) over momentum q and over the t_1 time coordinate. Also Eq. (B.20) needs to be integrated over the momentum q as shown in Eq. (B.20). Furthermore since it represents the non-local in time contribution needs to be properly projected on its local part: Eq. (B.18) needs to be integrated over the time s, according to the localization procedure exposed in Ref. [35] (reviewed in Sec. 4.3). It is this very extra-integration over time that it is implemented by the localization operator $\mathcal{L}$ which we have introduced before.

The integral over q thanks to the Litim cutoff is easily computed and it is given by

$$\int_q k \partial_k \mathrm{R}_k = 2 \frac{a_d}{d} K_k k^{d+2} \left(1 - \frac{\eta_k}{d+2} \right). \quad \text{(B.21)}$$

The integrals over time of the contributions (i) and (ii) are respectively given by (taking into account the localization procedure for the term (ii))

$$2 \int_0^t R_k(t,t_1) C_k(t,t_1) dt_1 = \frac{D_k}{Z_k \omega_k^2} \left(1 - e^{-(\omega_k/Z_k)t} (1 + 2 \frac{\omega_k}{Z_k} t) \right), \quad \text{(B.22)}$$

and

$$\int_0^\infty ds \int_0^\infty dt_1 \Big(C_k(t,t_1)R_k(t,s)R_k(s,t_1) + R_k(t,s)C_k(s,t_1)R_k(t,t_1) + \\ + C_k(t,s)R_k(t,t_1)R_k(t_1,s) \Big) = \\ = \frac{D_k}{Z_k \omega_k^3} \left\{ 1 - e^{-2\omega_k t/Z_k} \left[1 + 2\frac{\omega_k}{Z_k} t \left(1 + 2\frac{\omega_k}{Z_k} t \right) \right] \right\}. \tag{B.23}$$

We are now ready to give the final result for the flow over k of the local part of the self energy, i.e. $\mathcal{L}(\Sigma_{m,k})(t) = m_k(t)$:

$$\begin{aligned} k\partial_k m_k(t) &= 2k^{d+2} K_k \frac{a_d}{d} \frac{g_k D_k}{Z_k \omega_k^2} \left(1 - \frac{\eta_k}{d+2} \right) \times \\ &\times \left\{ \left[1 - e^{-(\omega_k/Z_k)t} \left(1 + 2\frac{\omega_k}{Z_k} t \right) \right] + \right. \\ &\left. + 3\frac{m_k}{\omega_k} \left[1 - e^{-2\omega_k/Z_k} \left(1 + 2\frac{\omega_k}{Z_k} t \left(1 + 2\frac{\omega_k}{Z_k} t \right) \right) \right] \right\} = \\ &= 2\frac{k^{d-2} K_k}{(1 + \frac{m_k}{k^2 Z_k})^2} \frac{a_d}{d} \frac{g_k D_k}{Z_k K_k^2} \left(1 - \frac{\eta_k}{d+2} \right) \times \\ &\times \left\{ \left[1 - e^{-2t(k^2 K_k/Z_k)\left(1 + \frac{m_k}{Z_k k^2}\right)} \left(1 + 2t(k^2 K_k/Z_k) \left(1 + \frac{m_k}{Z_k k^2} \right) \right) \right] + \right. \\ &+ 3\frac{m_k}{K_k k^2 (1 + \frac{m_k}{k^2 Z_k})} \left[1 - e^{-2t(k^2 K_k/Z_k)(1 + \frac{m_k}{Z_k k^2})} \times \right. \\ &\left. \left. \times \left(1 + 2t \left(k^2 K_k/Z_k \right) \left(1 + \frac{m_k}{Z_k k^2} \right) \left[1 + 2t \left(k^2 K_k/Z_k \right) \left(1 + \frac{m_k}{Z_k k^2} \right) \right] \right) \right] \right\}, \end{aligned} \tag{B.24}$$

where in the last equality we have simply used $\omega_k = K_k k^2 (1 + m_k/k^2 K_k)$.

B.0.3 Evaluation of the flow equation of the mass for the case of a critical quench

Passing now to dimensionless variables for the flowing parameters g_k, m_k, i.e. using the first two equations reported in Eq. (4.62), we obtain

$$\begin{aligned} k\partial_k m_k(t) &= 2K_k k^2 \frac{\tilde{g}_k}{(1 + \tilde{m}_k)^2} \left(1 - \frac{\eta}{d+2} \right) \times \\ &\times \left[(1 - f_{1,k}(t)) + 3\frac{\tilde{m}_k}{1 + \tilde{m}_k} (1 - f_{2,k}(t)) \right]. \end{aligned} \tag{B.25}$$

Note that in the long-time limit (in which the functions $f_{1,k}$ and $f_{2,k}$ vanish exponentially) we obtain the same β-function for the mass found in Ref. [35] in the $\lambda_k = 0$ approximation.

Remember that in order to solve the equation of motion for the two-time function in the presence of anomalous dimensions (given by Eqs.(5.38) and (5.39)) we have introduced the reduced mass given by r_k/Z_k. In Sec. 5.3.2 we have proven that the renormalization-group flow of the reduced mass is simply related to the flow of the physical mass r_k by Eq. (5.45). Taking advantage of that relation we obtain the β-function of the reduced mass in the ordered phase m_k as

$$\begin{aligned} k\partial_k \frac{m_k(t)}{Z_k} = 2\frac{K_k k^2}{Z_k}\frac{\tilde{g}_k}{(1+\tilde{m}_k)^2}\left(1-\frac{\eta}{d+2}\right)\times \\ \times\left[(1-f_{1,k}(t))+3\frac{\tilde{m}_k}{1+\tilde{m}_k}(1-f_{2,k}(t))\right] \end{aligned} \tag{B.26}$$

Now we want to integrate this equation in the vicinity of the WF fixed point. In order to do so we choose, as explained in Sec. 5.2, to evaluate the coupling constant *and* the parameters Z_k and K_k at the WF fixed point, i.e.,

$$\begin{aligned} &\tilde{g}_k \to \tilde{g}^*, \quad \tilde{m}_k \to \tilde{m}^*, \\ &K_k = k^{-\eta_K} \quad \text{and} \quad Z_k = k^{-\eta_Z}. \end{aligned} \tag{B.27}$$

Evaluating Eq. (B.26) with the prescription given by Eq. (B.27), one obtains

$$\begin{aligned} k\partial_k \frac{m_k(t)}{Z_k}\Big|_{WF} = 2k^z\frac{\tilde{g}^*}{(1+\tilde{m}^*)^2}\left(1-\frac{\eta_K}{d+2}\right)\times \\ \times\left[(1-f^*_{1,k}(t))+3\frac{\tilde{m}^*}{1+\tilde{m}^*}(1-f^*_{2,k}(t))\right]. \end{aligned} \tag{B.28}$$

Now we integrate this equation over the scale k between the arbitrary scale Λ at which one defines the bare Hamiltonian (and its bare parameters) and we precisely fine tune bare mass in order to have a physical mass at scale $k = 0$ which vanishes in the long-time limit. This choice of the bare parameter corresponds to set the system at the critical temperature of its second-order phase transition. The result is given by

$$\begin{aligned} \frac{m_k(t)}{Z_k}\Big|_{k=0} = \left(1-\frac{\eta_K}{d+2}\right)\frac{2}{z}\frac{\tilde{g}^*}{(1+\tilde{m}^*)^3 t}\times \\ \times\left[(1-F^*_{1,k}(t))+\frac{3\tilde{m}^*}{2(1+\tilde{m}^*)}\left[(3-F^*_{2,k}(t))\right]\right. . \end{aligned} \tag{B.29}$$

Now we should find the reduced physical mass in the disordered phase in order to solve the equations of motion for the two-point functions in the presence of anomalous dimensions effects. Note that the simple relation between the physical mass in the disordered phase and the ordered mass is given in Eq. (B.1), which holds in the $\lambda_k = 0$ approximation which we are dealing with here. This means that the reduced physical mass is given in terms of the reduced mass in

the ordered phase simply by

$$\frac{r_k(t)}{Z_k}\Big|_{k=0} = -\frac{1}{2}\frac{m_k(t)}{Z_k}\Big|_{k=0} = -\left(1-\frac{\eta_K}{d+2}\right)\frac{1}{z}\frac{\tilde{g}^*}{(1+\tilde{m}^*)^3 t}\times \\ \times\left\{\left[1-F^*_{1,k}(t)\right]+\frac{3\tilde{m}^*}{2(1+\tilde{m}^*)}\left[3-F^*_{2,k}(t)\right]\right\}. \tag{B.30}$$

Some final remarks:

1. Equation (B.30), if it is evaluated in the long-time limit will give a form of $r_{ag}(t)$ of the type given in Eq. (5.23), from which one can extract θ as explained in Chap. 5 and realize that it is the same result found by means of the short-time limit approach given in Ref. [35] (for comparison see Eq. (4.70) and (4.72)).
2. The proof given in Appendix A, which holds strictly speaking only for the form of the mass given in Eq. (A.3), can be generalized in a straightforward way in order to take into account the more general structure given by Eq. (B.30).

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