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Facoltà di Scienze Matematiche, Fisiche, Naturali
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Integrals of Motion in the 2 Dimensional Ising Model and Lattice-Conformal Dictionary

Alessandro Nigro
Matr. 606589
Codice P.A.C.S.: 11.25.Hf

Relatore:
Prof. Sergio Caracciolo

Correlatore:
Dott. Antonio Rago

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dedicato a tutti coloro che mi hanno sempre sostenuto

Contents

1. Conformal Invariance in Field Theory

In this chapter we will present a brief overview of Conformal Field Theory fundamentals, with particular attention to its connection to Statistical Mechanics and Quantum Field Theory, indeed CFT is a powerful tool capable in many situations of shedding light on connections between SM and QFT.
We will be particularly interested in unitary CFTs, and more specifically in Minimal Models since they have been proved to be in correspondence with a wide class of solvable models in SM, the simplest of which is the Ising Model.
Infact at the end of the chapter we will show how all we have said about CFT perfectly applies to the Ising Model, which will be presented both as a Statistical Mechanical model and as a Fermionic Field Theory (we will not talk about its bosonic representation which is obtained as a particular limit of the $\lambda\phi^4$ theory).
Throughout this chapter the main reference (unless otherwise stated) will be [1].

1.1 The Conformal Group and Its Algebra

The conformal group is the group of coordinate transformations under which the metric tensor gets rescaled by a position-dependent Ω factor. That is, if $x \to x'$, then $g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$, where $g'_{\mu\nu} = \Omega(x) g_{\mu\nu}(x)$.
Now, if we define $v \cdot w = g_{\mu\nu} v^\mu w^\nu$ we are led to conclude that the angle between two vectors is preserved:

$$\cos\theta = \frac{v \cdot w}{\sqrt{v^2 w^2}} \to \frac{\Omega v \cdot w}{\Omega\sqrt{v^2 w^2}} \ .$$

Some properties of such transformations are more easily derived through their infinitesimal form:

$$x'^\mu = x^\mu + \epsilon^\mu(x) \qquad , \qquad |\epsilon(x)| \ll 1 \tag{1.1}$$

Now,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu \epsilon^\mu \tag{1.2}$$

so that to first order in ϵ:

$$\delta g_{\mu\nu} = -2\partial_{(\mu}\epsilon_{\nu)} \tag{1.3}$$

The requirement that the transformation be conformal is readily seen to sound as:

$$2\partial_{(\beta}\epsilon_{\alpha)} = \Psi g_{\alpha\beta} \tag{1.4}$$

which is known as the Conformal Killing Equation. By taking the trace of both sides of (1.4) we see that $\Psi = \frac{2}{D}\partial\cdot\epsilon$ (D is the underlying space dimension) and then

$$\Omega = 1 + \Psi \tag{1.5}$$

By taking the 4-divergence of (1.4) it follows that:

$$\Box\epsilon_\beta + \frac{(D-2)}{2}\partial_\beta\Psi = 0 \quad \text{, where } \Box = g_{\mu\nu}\partial^\mu\partial^\nu \tag{1.6}$$

and repeating the same trick we conclude:

$$(D-1)\Box\Psi = 0 \qquad \rightarrow \qquad \Box\Psi = 0 \quad , \quad D > 1 \tag{1.7}$$

Now, applying ∂_λ to (1.6) and simmetrizing we obtain, after using (1.7):

$$(D-2)\partial_\lambda\partial_\beta\Psi = 0 \tag{1.8}$$

This, for $D > 2$ implies that $\Psi \sim \partial\cdot\epsilon$ is at most quadratic in x so we have only a finite set of generators for conformal transformations.

For $D = 2$ the last equation is not necessarily true so that we can define a Global Conformal Group (and algebra) like in higher dimensions, but beside this structure we have a Local Conformal Algebra which happens to be ∞-dimensional (a property which makes 2 dimensional conformal field theories exactly solvable as we will see).

In the case $D > 2$ we can write:

$$\epsilon^\mu = \begin{cases} a^\mu & \text{(traslations)} \\ \omega^\mu{}_\nu x^\nu & \text{(rotations, } \omega \text{ skew symmetric)} \\ \lambda x^\mu & \text{(dilatations)} \\ b^\mu x^2 - 2x^\mu b\cdot x & \text{(special conformal)} \end{cases} \tag{1.9}$$

So that by counting the parameters we have that the dimension of the $D > 2$ conformal group is $\frac{(D+1)(D+2)}{2}$.

The finite transformations are all obvious but the special conformal ones for which:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu \quad \rightarrow \quad \delta\Big(\frac{x^\mu}{x^2}\Big) = b^\mu \tag{1.10}$$

so that by integrating the equation still holds, and keeping higher order terms we find:

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + b^2x^2 + 2b\cdot x} \tag{1.11}$$

1.1.1 $D = 2$ *Conformal Algebra*

In this case we adopt complex coordinates $z, \overline{z}$ in the $\mathbb{R}^2$ plane:

$$\begin{cases} z = x + iy \\ \overline{z} = x - iy \end{cases} \tag{1.12}$$

In these coordinates the Conformal Killing Equation (1.4) takes the form of Cauchy-Riemann Equations so that $\epsilon(z)$ is holomorphic.
Under an holomorphic transformation $w = f(z)$ we have:

$$ds^2 = dzd\overline{z} \quad \rightarrow \quad \left|\frac{\partial f}{\partial z}\right|^2 dzd\overline{z} \quad , \quad \Omega = \left|\frac{\partial f}{\partial z}\right|^2 \tag{1.13}$$

If we write the infinitesimal version of this transformation $z \rightarrow z + \epsilon(z)$, and expand $f(z)$ as:

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n \tag{1.14}$$

It makes sense to take as a basis for such transformations $\epsilon_n = -z^{n+1}$, so that:

$$\delta z = [l_n, z] = -z^{n+1} \qquad n \in \mathbb{Z} \tag{1.15}$$

where the l_n are the generators of the infinitesimal conformal transformations $z \rightarrow z + \epsilon_n(z)$, satisfying:

$$[l_n, l_m] = (n - m)l_{n+m} \tag{1.16}$$

It is readily understood that we have analogous relations for antiholomorphic transformations with their generators satisfying

$$[l_m, \overline{l}_n] = 0 \qquad \forall \quad n, m \in \mathbb{Z} \tag{1.17}$$

This is the classical local conformal algebra, which is, as we will soon see, the $c = 0$ case of a Virasoro Algebra.
If we furthermore impose a regularity requirement on the Local Conformal Algebra so that changes of coordinates are well defined as $z \rightarrow 0, \infty$, that is on the whole Riemann Sphere $\mathbf{S}^2 \cup \infty$, we can easily realize that only the $\mathbf{SL}(2, \mathbb{C})/\mathbb{Z}_2$ subgroup, generated by l_0, l_1, l_{-1} and their antiholomorphic counterparts, survives as a global transformation generator.
Specifically we have that $l_{-1}, \overline{l}_{-1}$ generate left and right traslations, $l_1, \overline{l}_1$ special conformal, whereas $l_0 + \overline{l}_0$ generates the scalings and $i(l_0 - \overline{l}_0)$ the rotations.
We also notice that since the 2D conformal algebra is a direct sum of two disjoint algebras we have that conformal transformations are factorized into holomorphic and antiholomorphic parts.

1.2 Stress Tensor, Symmetries and Conservation Laws

Let us consider now a classical field theory with an action

$$S = \int d^D x \sqrt{g} \mathcal{L}(\phi, \bigtriangledown^c \phi)$$

built on some fields and their derivatives respecting principles such as locality, causality and invariance under some group of transformations. We know from Nöether's theorem that to each invariance of the lagrangian we can associate a conserved current J.
We want to produce an expression for J, restricting our consideration to consequences of conformal invariance. As it might be noticed we are momentarily releasing the requirement of working in a flat space for the sake of generality.
We define the Stress Tensor variationally through:

$$\delta S = -\frac{1}{2} \int d^D x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{1.18}$$

now, under a conformal transformation $x \to x + \epsilon$ we have, using (1.4):

$$\delta g_{ab} = -2 \bigtriangledown_{(a} \epsilon_{b)} = -\frac{2}{D} \bigtriangledown^d \epsilon_d g_{ab} \tag{1.19}$$

so that

$$\delta S = \int d^D x \sqrt{g} T^{ab} \bigtriangledown_{(a} \epsilon_{b)} \tag{1.20}$$

This tells us that T^{ab} is a symmetric tensor. Now if we pose $J^a = T^{ab} \epsilon_b$ we have:

$$\delta S = \int d^D x \sqrt{g} (\bigtriangledown^a J_a - \epsilon_b \bigtriangledown_a T^{ab}) \tag{1.21}$$

and if $J \to 0$ quickly enough as $|x| \to \infty$ we conclude from traslation invariance that the Stress Tensor is conserved.

$$\bigtriangledown_a T^{ab} = 0 \tag{1.22}$$

Now, if we consider (1.20) and use (1.4) dilatation invariance tells us that:

$$T^a{}_a = T = 0 \tag{1.23}$$

so that T^{ab} is traceless as a consequence of scaling invariance.
Special Conformal invariance tells us nothing more about properties of T^{ab}.
Moreover, the properties so far estabilished are sufficient to conclude that the special conformal current is already conserved as a consequence of scaling plus traslation invariance, so that the special conformal symmetry comes as a gift of scaling invariance. This is not a coincidence but is a well known property of statistical mechanical models near a second order phase transition. The interpretation of T^{ab} itself as a current and the possibility to build conformal currents from it gives us a natural way to implement conformal symmetry at the quantum level.

1.3 Conformal Field Theories

We will now define a conformal field theory as a field theory satisfying the following properties:

1. There exists a set of fields $\{A_i\}$,which is usually infinite, and $\partial_\mu A_i \in \{A_i\}$

2. There exists another set of fields $\{\phi_j\} \subset \{A_i\}$, called *Quasi Primary* which transform as tensor densities under conformal transformations, that is:

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/D} \phi_j(x') \tag{1.24}$$

   where Δ_j is the dimension of ϕ_j, and $\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-D/2}$.
   As a consequence we have a covariance property of the correlation functions under conformal transformations.In the sense that:

$$\left\langle \phi_1(x_1) \dots \phi_n(x_n) \right\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/D}_{x=x_1} \dots \left|\frac{\partial x'}{\partial x}\right|^{\Delta_n/D}_{x=x_n} \left\langle \phi_1(x'_1) \dots \phi_n(x'_n) \right\rangle \tag{1.25}$$

3. All the remaining fields in the family $\{A_i\}$ are expressible as linear combinations of quasi primary fields and their derivatives.

4. There exists a vacuum invariant under global conformal transformations (in $D = 2$ this means $\mathbf{SL}(2, \mathbb{C})$ invariance).

This definition is valid only for scalar fields, we will see how in 2 dimensions, due to the decoupling of holomorphic and antiholomorphic parts we will be able to include also fields with a spin s.

1.3.1 Form of the Correlators in D Dimensions

The second property exposed above has the virtue of completely determining 2 and 3 point functions for quasi primary fields in a conformal field theory, while higher order correlators will be completely fixed for $D = 2$, as we will see, by the requirement of Minimality for the representation of the conformal group acting on the Hilbert space of the theory.
Before going any further it will be convenient to spend some words about the conformal invariants upon which a correlator may depend; so we fix N points $x_1 \dots x_N$ in our D-dimensional space and look for a traslational invariant to find out that the allowed dependence must be of the form $(x_i - x_j)$, if we now look for a rotational and traslational invariant we are forced to choose $r_{ij} = |x_i - x_j|$, again adding scaling invariance to the other two requirements we are left with the invariant $\frac{r_{ij}}{r_{kl}}$.
Special conformal invariance is a more complicated matter but it can be settled by noting that

the following relation holds:

$$\left|x_i' - x_j'\right|^2 = \frac{\left|x_i - x_j\right|^2}{(1 + b^2 x_i^2 + 2b \cdot x_i)(1 + b^2 x_j^2 + 2b \cdot x_j)} \tag{1.26}$$

so that the quantity:

$$\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}} \tag{1.27}$$

is invariant under all the global conformal group, such a quantity is called a *cross-ratio*; the number of indipendent cross ratios is $N(N-3)/2$.
By using the covariance properties of the correlators under conformal transformations it is straightforward to show that 2 and 3 point functions are of the form:

$$\left\langle \phi(x_1)\phi(x_2) \right\rangle = \begin{cases} \frac{C_{12}}{r_{12}^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{1.28}$$

$$\left\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \right\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.29}$$

Higher order correlators begin to have a dependence on arbitrary functions of the independent cross-ratios, these functions, as we will see, can be determined for $D = 2$ as solutions of differential equations due to the existence of so-called null states in the Hilbert space.
For example, by considering that for $N = 4$ we have only 2 independent cross-ratios, and imposing all the constraints coming from global conformal invariance we can show that:

$$\left\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \right\rangle = F\Big(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}}\Big) \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\sum_{i=1}^4 \Delta_i/3} \tag{1.30}$$

1.4 $D = 2$ Conformal Field Theories

The $D = 2$ situation shows some little difference from higher dimensional cases because of the factoring of holomorphic and antiholomorphic transformations that takes place in the conformal group.
For this reason it's more convenient to choose two indipendent conformal dimensions h and $\overline{h}$ such that under a transformation of the group $z \to f(z)$ a primary field changes as:

$$\phi(z, \overline{z}) \to \Big(\frac{\partial f}{\partial z}\Big)^h \Big(\frac{\overline{\partial f}}{\overline{\partial z}}\Big)^{\overline{h}} \phi(f, \overline{f}) \tag{1.31}$$

We recover the previous definitions in the special case in which $h = \overline{h} = \Delta_\phi/2$, this situation corresponds to a spinless field since we will be quite soon able to show that $s = h - \overline{h}$ (this implies some more knowledge about the algebraic structure of the Hilber Space, which we will

soon provide).
Now, under an infinitesimal transformation $z \to z + \epsilon(z)$ it is simple to show that:

$$\phi(z,\overline{z}) \to \phi(z,\overline{x}) + ((h\partial\epsilon + \epsilon\partial) + (\overline{h\partial}\overline{\epsilon} + \overline{\epsilon}\overline{\partial}))\phi(z,\overline{z}) \tag{1.32}$$

that is

$$\delta_{\epsilon\overline{\epsilon}}\phi(z,\overline{z}) = ((h\partial\epsilon + \epsilon\partial) + (\overline{h\partial}\overline{\epsilon} + \overline{\epsilon}\overline{\partial}))\phi(z,\overline{z}) \tag{1.33}$$

If we now consider that

$$\delta_\epsilon\phi := [Q,\phi] \tag{1.34}$$

where Q is some conformal charge yet to be explicitly identified (but already introduced at a classical level), we can with ease derive a set of differential equations for correlation functions by setting $\epsilon = z^n$, which for $n = 0, 1, 2$ is enough to completely determine 2 and 3 point functions as above. For example the 2 point function satisfies:

$$\begin{aligned}((h_1\partial_1\epsilon(z_1) + \epsilon(z_1)\partial_1) + (\overline{h}_1\overline{\partial}_1\overline{\epsilon}(\overline{z}_1) + \overline{\epsilon}(\overline{z}_1)\overline{\partial}_1) + (h_2\partial_2\epsilon(z_2) + \\ +\epsilon(z_2)\partial_2) + (\overline{h}_2\overline{\partial}_2\overline{\epsilon}(\overline{z}_2) + \overline{\epsilon}(\overline{z}_2)\overline{\partial}_2))\big\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\big\rangle = 0\end{aligned} \tag{1.35}$$

without further comments we shall write the 2 and 3 points correlators as:

$$\big\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\big\rangle = \begin{cases} \frac{C_{12}}{z_{12}^{2h}\overline{z}_{12}^{2\overline{h}}} & h_1 = h_2 = h,\ \overline{h}_1 = \overline{h}_2 = \overline{h} \\ 0 & \text{otherwise} \end{cases} \tag{1.36}$$

$$\begin{aligned}&\big\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\big\rangle = \\ &= \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3+h_1}z_{13}^{h_3+h_1-h_2}\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3}\overline{z}_{23}^{\overline{h}_2+\overline{h}_3+\overline{h}_1}\overline{z}_{13}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}}\end{aligned} \tag{1.37}$$

1.4.1 Quantization for $D = 2$ and the OPE

To truly quantize our field theory we need operator charges to use as generators of continuous transformations for the system, such as traslations, rotations and time evolution.
We may want to start from an underlying manifold that is somehow more familiar such as the infinite cylinder, with a spatial periodic coordinate $\sigma_1 \in [0, 2\pi]$, and a time $\sigma_0 \in \mathbb{R}$ (this is the case of String Theory's world sheet, or more simply of a field theory where we have compactified a dimension in order to mitigate the possible infrared divergences); we can now map this manifold (which has a trivial Euclidean metric) to the complex plane through the exponential map:

$$z = e^{\sigma_0 + i\sigma_1} \tag{1.38}$$

Now the remote past is represented by the origin of the complex plane, and the future is the infinity point; equal time sections become circles of constant radius about the origin and so time

flows radially outward from $z = 0$.
What suggests us that we are going in the right way is the fact that now the generator of time evolution for the system is indeed the dilatation generator of the complex plane; spatial traslations on the cylinder are generated in the plane by the rotation generator.
In this context it is natural to look at the stress tensor as the fundamental object of our theory since as we showed previously a theory which is invariant under conformal transformations possesses a current J^a, built from the stress tensor, whose divergence is closely related to the stress tensor's divergence and its trace, as it can be seen by combining (1.20) and (1.21), so that since T^{ab} is conserved and traceless, J^a is also conserved. In complex coordinates (1.12) the conservation law for the stress tensor reads:

$$\overline{\partial} T_{zz} = 0 \quad \partial T_{\overline{z}\overline{z}} = 0 \tag{1.39}$$

since the tracelessness condition translates into:

$$T_{z\overline{z}} = T_{\overline{z}z} = 0 \tag{1.40}$$

It is then natural to define the charges as:

$$Q_{\epsilon\overline{\epsilon}} = \frac{1}{2\pi i} \oint (T(z)\epsilon(z)dz + \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})d\overline{z}) \tag{1.41}$$

where our contour of integration is an equal time slice and is counterclockwise oriented for both z and $\overline{z}$.
Now that we have a charge, we can look at it as originating (1.33) through the commutator (1.34). The problem now is to define what we mean by commutator in this context; since our charge is given as a slice integral of a locally defined expression which we may want to make sense as an insertion inside a Path Integral beside other fields evaluated at other points, we should remember that Green Functions are generally guaranteed to converge for time ordered products of operators so that we are led to introduce a Radial Ordering:

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases} \tag{1.42}$$

In this way we will define the equal time commutator as:

$$[\oint_{E.T.slice} dz A(z), B(w)] := \oint dz R(A(z)B(w)) \tag{1.43}$$

where in the righthand side the contour of integration is a small circle around w obtained as a difference of two contours centered about the origin and avoiding w so that in one case $|z| < |w|$ and in the other case $|z| > |w|$. From now on the radial ordering symbol will be omitted.

With our new definition of $Q_{\epsilon\overline{\epsilon}}$ (1.41) and the explicit expression for the variation of a primary field $\phi(z,\overline{z})$ (1.33) it is immediate to infere that the product between $T(z)$ and $\phi(w,\overline{w})$ must have the following short distance singular behaviour, which we will call OPE (Operator Product Expansion) between T and ϕ:

$$T(z)\phi(w,\overline{w}) = \frac{h\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\phi(w,\overline{w})}{(z-w)} + \dots \tag{1.44}$$

with a similar expression for the antiholomorphic part.
The OPE is not a peculiarity of Conformal Field Theories, it is quite common to find short distance singularities as two operators approach one another as insertions of a path integral, in general if we have two operators A,B approaching one another and a complete set of local operators O_i, we can write:

$$A(x)B(y) = \sum_i C_i(x-y)O_i(y) \tag{1.45}$$

and if all the operators in this expression have definite scaling properties, the functions C_i are constrained to behave as:

$$C_i \sim \frac{1}{|x-y|^{\Delta_A+\Delta_B-\Delta_{O_i}}} \tag{1.46}$$

This is infact the case of CFTs where in addition the OPE defines an associative algebra that is fully characterized by the algebraic structure of the Hilbert space that will allow us to exactly express the OPE between two primary fields as a sum over conformal families whose coefficients will be related to the 3 point function's coefficients C_{ijk} in a very simple way.

1.4.2 Conformal Ward Identities

We have already met some forms of Ward Identities, although we did not mention it when we produced differential equations for correlation functions as a consequence of (1.33), now we know that the natural way to obtain infinitesimal transfromations is to commute the fields with the right charges so that:

$$\delta_\epsilon(\phi_1(x_1,)\dots\phi_n(x_n)) = [Q_\epsilon, \phi_1(x_1)\dots\phi_n(x_n)] = \sum_{k=1}^{n}(\phi_1(x_1))\dots[Q_\epsilon,\phi_k(x_k)]\dots\phi_n(x_n)) \tag{1.47}$$

When we defined the charge in (1.41), we could as well have pointed out that $Q_{\epsilon\overline{\epsilon}}$ can be decomposed as a sum of two separate holomorphic and antiholomorphic charges (obviously defined):

$$Q_{\epsilon\overline{\epsilon}} = Q_\epsilon + Q_{\overline{\epsilon}} \tag{1.48}$$

So that now the meaning of our equations is made clear and we can proceed to put (1.47) inside an expectation value and to substitute the espression for Q_ϵ, and then use the OPE (1.44) to

obtain:

$$\begin{aligned}&\Big\langle \oint \frac{dz}{2\pi i}\epsilon(z)T(z)\phi_1(w_1,\overline{w}_1)\dots\phi_n(w_n,\overline{w}_n)\Big\rangle = \\ &= \sum_{k=1}^{n}\oint \frac{dz}{2\pi i}\epsilon(z)\Big(\frac{h_k}{(z-w_k)^2}+\frac{\partial_{w_k}}{(z-w_k)}\Big)\big\langle\phi_1(w_1,\overline{w}_1)\dots\phi_n(w_n,\overline{w}_n)\big\rangle\end{aligned} \tag{1.49}$$

or, in a non integrated version:

$$\big\langle T(z)\phi_1(w_1,\overline{w}_1)\dots\phi_n(w_n,\overline{w}_n)\big\rangle = \sum_{k=1}^{n}\Big(\frac{h_k}{(z-w_k)^2}+\frac{\partial_{w_k}}{(z-w_k)}\Big)\big\langle\phi_1(w_1,\overline{w}_1)\dots\phi_n(w_n,\overline{w}_n)\big\rangle \tag{1.50}$$

These equations can be a very powerful tool to compute the form of correlators.

1.4.3 TT OPE and Central Charge

In general primary fields always have transformation laws of the type shown in (1.31) and (1.33), from the first of these two equations we can notice that the field ϕ has a transformation law that can be interpreted as a tensor's, with the field having h lower z indexes and $\overline{h}$ lower $\overline{z}$ indexes, so that its infinitesimal variation is such that it could be derived as the most general expression, linear in ϵ , with $(h+1)$ lower z indexes and $\overline{h}+1$ lower $\overline{z}$ indexes. The right coefficients can be then chosen to agree with the OPE. If we now want to exctract information about the $T(z)T(w)$ OPE, we must proceed in a way that is quite similar. First of all we notice that $T(z)$ is an object with two lower z indexes, so that in analogy with the above situation we may risk to pose $h=2$; the most general form for the variation of $T(z)$ is then:

$$\delta_\epsilon T(z) = \alpha\epsilon(z)\partial T(z) + \beta\partial\epsilon(z)T(z) + \gamma\partial^3\epsilon(z) \tag{1.51}$$

Which implies an OPE of the form:

$$T(z)T(w) = \frac{6\gamma}{(z-w)^4} + \frac{\beta T(w)}{(z-w)^2} + \frac{\alpha\partial T(w)}{(z-w)} \tag{1.52}$$

now if we require that $\beta = h = 2$, $\alpha = 1$ (in analogy with the primary case) and that the 2 point function be normalized as:

$$\big\langle T(z)T(w)\big\rangle = \frac{c}{2}\frac{1}{(z-w)^4} \tag{1.53}$$

so that $\gamma = \frac{c}{12}$, we obtain the following infinitesimal transformation law:

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\partial^3\epsilon(z) \tag{1.54}$$

and and OPE of the form:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1.55}$$

With a twin equation for the antiholomorphic part.
The choice of the γ constant has been done simply for convenience, since it will influence the structure of the Virasoro Algebra (soon to be introduced) which in turn will fix the normalization of the stress tensor's two point function. The constant c will be called *Central Charge* and we will see that the requirement of a unitary theory will constrain c to be greater or equal to 0. We finally want to say something about the finite transformation law for $T(z)$ under $z \to f(z)$, which takes the form:

$$T(z) \to (\partial f)^2 T(f) + \frac{c}{12} S(f, z) \tag{1.56}$$

where

$$S(f, z) = \frac{\partial f \partial^3 f - 3/2(\partial^2 f)^2}{(\partial f)^2} \tag{1.57}$$

is called the Swartzian derivative. It is interesting that the Swartzian derivative of a global conformal transformation vanishes, implying an infinitesimal transformation law identical to (1.33), this could also have been inferred from the fact that $\epsilon(z)$ is at most quadratic in z for $\mathbf{SL}(2, \mathbb{C})$ mappings. This fact is the statement that $T(z)$ is a *Quasi Primary Field*.
$S(f, z)$ will be also very useful in measuring the shift of the vacuum energy due to the change of geometry of the background manifold for the theory.

1.5 Algebraic Structure of the Hilbert Space

Our goal is to build the Hilbert space of a conformal field theory starting from the tools already in our hands, that is the globally invariant vacuum $|0\rangle$, the stress tensor, the primary fields of the theory and the OPE.

1.5.1 Mode Expansions and the Virasoro Algebra

We begin by defining a mode expansion for a generic scaling operator $A(z)$ of dimension h through:

$$A(z) = \sum_{n \in Z - h} \frac{A_n}{z^{n+h}} \tag{1.58}$$

so that the modes A_n have scaling dimension n. The inverse relation is given by:

$$A_n = \oint \frac{dz}{2\pi i} z^{h+n-1} A(z) \tag{1.59}$$

In particular for the stress tensor we have $h = 2$ and we conventionally call the modes L_n.
Using (1.59) and (1.55), we now want to try to evaluate the commutator of two modes; a problem arises about how to define the commutator of two contour integrations$[\oint dz, \oint dw]$, this is done by fixing w and performing the z integration on a small circle around w, the w contour is then

taken to be a circle around $w = 0$. So that we readily find the commutation relations of the *Virasoro Algebra*:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,\,0} \tag{1.60}$$

and the analogous antiholomorphic relation; we point out that the full algebra is a direct sum of the two holomorphic and antiholomorphic algebras so that

$$[L_m, \overline{L}_n] = 0 \tag{1.61}$$

The first remarkable property of this algebra is that it admits a subalgebra generated by L_0, L_1, L_{-1} that does not involve the central charge, so that the $\mathbf{SL}(2,\mathbb{C})$ subalgebra of the local conformal algebra still determines an exact symmetry for the theory even if we have a central charge term.

We will now provide the concept of adjoint, which is essential to our algebraic structure.

If we consider an in state created by the insertion of a primary operator A at $z = 0$ and call it $A(0)\big|0\big\rangle = \big|h\big\rangle$, we are naturally led to consider a similar out state which has an insertion at the ∞ point; we want such a state to be considered as the adjoint of $A(0)\big|0\big\rangle$.

The right thing to do is to consider the $\mathbf{SL}(2,\mathbb{C})$ mapping $z \to 1/z$ which takes ∞ to the origin and to proceed by defining the adjoint as the transformed field under such a mapping:

$$A^\dagger(z,\overline{z}) = \frac{1}{z^{2h}\overline{z}^{2\overline{h}}} A(\frac{1}{z}, \frac{1}{\overline{z}}) \tag{1.62}$$

So that in the limit $z \to \infty$ we can write:

$$\big\langle h\big| = \lim_{z\to\infty} \big\langle 0\big| z^{2h}\overline{z}^{2\overline{h}} A(z,\overline{z}) \tag{1.63}$$

As a consequence of this definition it can be easily shown that the stress tensor's modes of expansion satisfy:

$$L_n^\dagger = L_{-n} \tag{1.64}$$

So that we also have that $\mathbf{SL}(2,\mathbb{C})$ is a subalgebra of the Virasoro Algebra stable under the adjoint operation.

Let us now consider the effects of regularity requirements imposed on the state representation of the stress tensor. If we require regularity as $z \to 0$ for the state $T(z)\big|0\big\rangle$ we obtain the condition:

$$L_n\big|0\big\rangle = 0 \text{ for } n \geq -1 \tag{1.65}$$

and, taking the adjoint:

$$\big\langle 0\big| L_n \text{ for } n \leq 1 \tag{1.66}$$

So that we find out once more that the vacuum must be $\mathbf{SL}(2,\mathbb{C})$ invariant. Now by inserting the mode expansions for $T(z)$ into $\big\langle 0\big|T(z)T(w)\big|0\big\rangle$ we could easily show that the two point function really has the form of (1.53), and we could also compute mechanically although in a laborious way all higher order correlators.

1.5.2 Highest Weight States and Descendants

If we consider once more the state $\phi(z)\big|0\big\rangle$ (where ϕ is a primary field) as $z \to 0$ and call it $\big|h\big\rangle$, the usual problem arises to impose regularity conditions on it, which tells us:

$$\phi_n\big|0\big\rangle = 0 \text{ for } n \geq -h+1 \tag{1.67}$$

Now, if we consider the commutator:

$$[L_n, \phi(z)] = h(n+1)z^n\phi(z) + z^{n+1}\partial\phi(z) \tag{1.68}$$

we discover that it is 0 for $n > 0$ and $z = 0$, while for $n = z = 0$ we find the very useful relation:

$$[L_0, \phi(0)]\big|0\big\rangle = h\phi(0)\big|0\big\rangle \tag{1.69}$$

so we discover that:

$$L_0\big|h\big\rangle = h\big|h\big\rangle \tag{1.70}$$

And still another gift comes from (1.68) and (1.59):

$$[L_n, \phi_m] = (n(h-1) - m)\phi_{m+n}, \tag{1.71}$$

so that for $n = 0$ we find:

$$[L_0, \phi_m] = -m\phi_m \tag{1.72}$$

This is interesting because it tells us that we can also identify $\big|h\big\rangle$ as:

$$\big|h\big\rangle = \phi_{-h}\big|0\big\rangle \tag{1.73}$$

We will now define the notion of *Descendant* at level N as a state generated by a primary operator (called a *Highest Weight State*), on which we let act a string of operators $L_{-n_1} \dots L_{-n_k}$ ($n_i > 0$) with $\sum_i n_i = N$. Such a state has a conformal weight $h + N$, and the number of these states amounts to $P(N)$, that is the number of partitions of N into positive integer parts.
The action of the L_n is made more manifest by the commutation relation:

$$[L_n, L_0] = nL_n \tag{1.74}$$

That implies:

$$L_0L_n\big|h\big\rangle = (h-n)L_n\big|h\big\rangle \tag{1.75}$$

So we have learned that the L_n act as rising operators for the conformal weight for $n < 0$, and in the case $n > 0$ using (1.67), (1.71) and (1.73) we deduce:

$$L_n\big|h\big\rangle = 0 \text{ for } n > 0 \tag{1.76}$$

Some constraints due to unitarity come from the relation:

$$\langle h|L_{-n}^{\dagger}L_{-n}|h\rangle = (2nh + c/12(n^3 - n))\langle h|h\rangle \tag{1.77}$$

Which for n large, united to the requirement of a positive definite norm requires $c > 0$, while for $n = 1$ it implies $h \geq 0$.
Summing up we have learned that a unitary representation of the Virasoro Algebra must have:

$$h \geq 0 \text{ and } c > 0 \tag{1.78}$$

In a short while we will see how an even more restricted range of values for c and h can be picked out from a more detailed analisys, these will characterize the Minimal Unitary Models.
Now, as an interesting example of descendant state it would be worth to consider $T(0)|0\rangle$.
If we notice that:

$$L_{-2} = \oint \frac{dz}{2\pi i}\frac{T(z)}{z} \sim T(0) \tag{1.79}$$

we immediately arrive at the relation:

$$T(0)|0\rangle = L_{-2}|0\rangle \tag{1.80}$$

so that the stress tensor is a level 2 descendant of the identity operator, incidentally this gives us still another explanation of why T is not a primary field, and also gives us a criterion to recognize a quasi primary operator: it is not a level 1 descendant of the previous level.

1.5.3 Correlators of Descendants and Fusion Rules

It is interesting to consider more complicated correlators formed not just by primary fields, but also from descendants. Let us consider for simplicity a correlator involving only one descendant of the form:

$$\langle \phi_1(w_1, \overline{w}_1) \dots \phi_{n-1}(w_{n-1}, \overline{w}_{n-1}) L_{-k_1} \dots L_{-k_l} \phi_n(z, \overline{z})\rangle \tag{1.81}$$

By using the commutator (1.68) to annihilate L_{-k_s} on the out vacuum we can show it to take the form:

$$\begin{aligned}\langle \phi_1(w_1, \overline{w}_1) \dots \phi_{n-1}(w_{n-1}, \overline{w}_{n-1}) L_{-k_1} \dots L_{-k_l} \phi_n(z, \overline{z})\rangle = \\ \mathcal{L}_{-k_1} \dots \mathcal{L}_{-k_l} \langle \phi_1(w_1, \overline{w}_1) \dots \phi_{n-1}(w_{n-1}, \overline{w}_{n-1}) \phi_n(z, \overline{z})\rangle\end{aligned} \tag{1.82}$$

where:

$$\mathcal{L}_{-k} = -\sum_{i=1}^{n-1}\Big(\frac{(1-k)h_i}{(w_i - z)^k} + \frac{\partial_{w_i}}{(w_i - z)^{k-1}}\Big) \tag{1.83}$$

So that the above correlator is completely determined in terms of primary fields correlators.
The same could in principle be carried out for more complicated correlators so that the only

fundamental correlators for the theory are those of primary fields.
Now, if we turn our attention to the OPE structure we can use the primary-descendant structure of the Hilbert space to group it as (you can check that the scaling dimensions in the sum terms are right):

$$\phi_i(z,\overline{z})\phi_j(w,\overline{w}) = \sum_{p(k\overline{k})} C_{ijp}^{(k\overline{k})} z^{h_p-h_i-h_j+\sum_l k_l} \overline{z}^{\overline{h}_p-\overline{h}_i-\overline{h}_j+\sum_l \overline{k}_l} \phi_p^{(k\overline{k})}(w,\overline{w}) \tag{1.84}$$

where by $\phi_p^{(k\overline{k})}(w,\overline{w})$ we mean the descendant at level $(k\overline{k})$ of ϕ_p with respect to the two Virasoro algebras of the theory. If we now consider an arbitrary 3 point function, we have shown that it can be computed in terms of the three point function for the primaries so that:

$$\left\langle \phi_i\phi_j\phi_k^{(l,0)} \right\rangle = C_{ijk}\mathcal{L}_{-l} \frac{1}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k+h_i} z_{ik}^{h_k+h_i-h_j} \overline{z}_{ij}^{\overline{h}_i+\overline{h}_j-\overline{h}_k} \overline{z}_{jk}^{\overline{h}_j+\overline{h}_k+\overline{h}_i} \overline{z}_{ik}^{\overline{h}_k+\overline{h}_i-\overline{h}_j}} \tag{1.85}$$

If on the other hand we use the OPE we find:

$$\left\langle \phi_i\phi_j\phi_k^{(l,0)} \right\rangle = \sum_{p(s\overline{s})} C_{ijp}^{(s\overline{s})} z^{h_p-h_i-h_j+\sum_l s_l} \overline{z}^{\overline{h}_p-\overline{h}_i-\overline{h}_j+\sum_l \overline{s}_l} \left\langle \phi_p^{(s\overline{s})}(w,\overline{w})\phi_k^{(l,0)} \right\rangle \tag{1.86}$$

And remembering that 2 point functions of operators of different conformal weight must vanish:

$$\left\langle \phi_i\phi_j\phi_k^{(l,0)} \right\rangle = \sum_{(s\overline{s})} C_{ijk}^{(s\overline{s})} z^{h_k-h_i-h_j+\sum_l s_l} \overline{z}^{\overline{h}_k-\overline{h}_i-\overline{h}_j+\sum_l \overline{s}_l} \left\langle \phi_k^{(s\overline{s})}(w,\overline{w})\phi_k^{(l,0)} \right\rangle \tag{1.87}$$

Now, equating the two expressions we see that both of them must be null when C_{ijk} is null so that also the second must be proportional to it and we are led to write:

$$C_{ijk}^{(s\overline{s})} = C_{ijk}\beta_{ij}^{k(s)}\overline{\beta}_{ij}^{k(\overline{s})} \tag{1.88}$$

The functions $\beta_{ij}^{k(s)}$ are in principle determinable, but to our goals it suffices to notice that the 3 point functions' non null coefficients determine which conformal families are allowed in the OPE between two fields; most often the null coefficients are determinable on the basis of some symmetry principle for the field theory under consideration, as is the case for example in the Ising Model, where the symmetries are, as we will see, Duality and Spin Reversal. So we will say that the null C_{ijk} determine the so called *Fusion Rules* for the model which we can simbolically write as:

$$[\phi_i] \times [\phi_j] = \sum_p C_{ijp}[\phi_p] \tag{1.89}$$

1.5.4 Verma Modules, Kac Determinant and Unitarity

We have seen how a highest weight state $\big|h\big\rangle$ has descendants which can be characterized by their eigenvalue N under the L_0 operator, these states can also be organized into an hierarchy as the level of descendance N increases. In this way we could hope to build a state representation of the Virasoro Algebra, but a problem arises about wether at a given level of descendance all states are linearly independent or not. In the latter case there must exist a linear combination of states which we must quotient out of the Hilbert Space by requiring it to be equivalent to the null vector. This procedure is called dimensional reduction; before dimensional reduction the set composed of a highest weight state and its descendants $\mathcal{V}(c,h)$ is called a *Verma Module*, and after dimensional reduction it constitutes an irreducible representation of the Virasoro Algebra. The Hilbert Space is then built as a direct sum over Verma Modules:

$$\mathcal{H} = \bigoplus_{(h,\overline{h})} \mathcal{V}(c,h) \otimes \mathcal{V}(c,\overline{h}) \tag{1.90}$$

Let us now spend some words about how to determine the existence of null states in a Verma Module.

If, for a fixed N, we consider the Gram Matrix (i.e. the matrix of all possible inner products) for the $P(N)$ states of the form $L_{-n_1}\dots L_{-n_k}\big|h\big\rangle$ with $\sum_i n_i = N$, it happens that such a matrix has a vanishing determinant (The *Kac Determinant*) if the vectors are not all linearly independent, and its null eigenvectors expanded on the $P(N)$ states at a given level of descendance N give the linear combination that must vanish.

Let us give 2 instructive examples, for $N=1$, normalizing $\big\langle h\big|h\big\rangle = 1$ (i.e. the 2 point functions are normalized to 1), we have:

$$\big\langle h\big|L_1L_{-1}\big|h\big\rangle = 2h \tag{1.91}$$

which for $h=0$ simply states what we already know, that is $L_{-1}\big|0\big\rangle = 0$. For $N=2$, taking as a basis $\{L_{-2}, L_{-1}^2\}$, we easily build the gram matrix:

$$\begin{pmatrix} 4h+c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} \tag{1.92}$$

and taking the determinant we find:

$$2h(16h^2 + (2c-10)h + c) \tag{1.93}$$

Which can be trivially null for $h=0$, corresponding to the fact that $L_{-1}^2\big|0\big\rangle = 0$, or it could have nontrivial zeroes for special values of h that can be determined obviously as a function of c.

The corresponding null vector can be shown to be:

$$\big(L_{-2} - \frac{3}{4h+2}L_{-1}^2\big)\big|h\big\rangle = 0 \tag{1.94}$$

In general the occurence of a null state at level n will imply that at a level $N > n$ there will be $P(N-n)$ null states.

This will prove very useful because it will provide us with more constraints for the N point correlators, since for example they will be annihilated by a differential operator as a consequence of (1.94):

$$(\mathcal{L}_{-2} - \frac{3}{4h+2}\mathcal{L}_{-1}^2)\big\langle\phi(z,\overline{z})\dots\big\rangle = 0 \tag{1.95}$$

Now let us spend some words about unitarity. In the last section we have learned that unitary representations of the Virasoro Algebra can occur only for values of h and c in the range given by (1.78), this is surely true, but we can say more. A detailed analisys of the Kac Determinant, whose goal is to dermine the existence of imaginary norm states, can show that although for $h \geq 0,\ c \geq 1$ there is nothing that prevents us form having unitary representations, in the region $h \geq 0,\ 0 \leq c < 1$ unitary prepresentations of the Virasoro Algebra may occur only at discrete values of the central charge indicized by an integer $m \geq 2$ and for a set of fields depending on two more integers $1 \leq p \leq m-1,\ 1 \leq q \leq p$.

Explicitly we have:

$$c = 1 - \frac{6}{m(m+1)} \quad m = 3, 4, \dots \tag{1.96}$$

$$h_{p,\,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad 1 \leq p \leq m-1,\ 1 \leq q \leq p \tag{1.97}$$

These are called *Minimal Unitary Models* $\mathcal{M}_{m,m+1}$ and it has been shown, by comparison of critical exponents, that the operator content of each one falls within the universality class of a critical statistical mechanical model; for example $m = 3$ is the Ising Model, $m = 4$ the Tricritical Ising Model, m=5 the 3-States Potts Model and $m = 6$ the Tricritical 3-States Potts Model.

If we release the requirement of unitarity we can still find finite operator content theories as above Indicized by two integers, these are just the *Minimal Models* $\mathcal{M}_{m,\,n}$ (without "unitary"), and their central charge satisfies:

$$c = 1 - \frac{6(m-n)^2}{mn} \tag{1.98}$$

where m and n are coprime integers.

Going back to unitary models we can notice that the conformal weights (1.97) possess the symmetry $p \to m-p,\ q \to m+1-q$ so that we can extend the range of q to $1 \leq q \leq m$ so that we obtain $m(m-1)/2$ couples of primary fields with equal conformal weight.

The model $\mathcal{M}_{3,4}$ for examples possesses 3 couples of primary fields of weight $h_{1,1} = h_{2,3} = 0$ (identity operator), $h_{1,2} = h_{2,2} = 1/16$ and $h_{2,1} = h_{1,3} = 1/2$.

1.5.5 CFT on the Torus and Modular Invariance

We are now interested in finding a way to define a CFT on a manifold of higher genus such as the torus. Such a manifold can be obtained by twisting a finite length cylinder so that we get periodic boundary conditions in both directions (peculiar mixes of periodic and antiperiodic boundary conditions produce non orientable underlying manifolds such as Klein bottles).
If we adopt complex coordinates on the cylinder $z = x + iy$, and take x to be periodic under $x \to x + 1$, we are still left with the need to introduce a new direction (i.e. a complex number τ) which has to be periodically identified, so that $z = z + \tau$.
Another thing we cannot do without are the two generators of space and time traslations i.e. the Hamiltonian H and the momentum P, these are obtained considering the exponential mapping $w = exp(2\pi iz)$ from the complex plane, where the Hamiltonian is simply the generator of dilatations $L_0 + \overline{L}_0$ and the momentum the generator of rotations $i(L_0 - \overline{L}_0)$.
In order to obtain $(L_0)_{Cyl}$ we need to consider the Swartzian derivative (1.57) of the exponential mapping which gives the following transformation law for the Stress Tensor:

$$T_{Cyl}(z) = -4\pi^2\Big(w^2T(w) - \frac{c}{24}\Big) \tag{1.99}$$

where w is the coordinate in the complex plane.
This tells us that only L_0 is changed by the mapping so that:

$$(L_0)_{Cyl} = L_0 - \frac{c}{24} \tag{1.100}$$

This gives us the following expressions for the generators on the cylinder:

$$H = L_0 + \overline{L}_0 - \frac{c}{24} - \frac{\overline{c}}{24} \tag{1.101}$$

$$P = i\Big(L_0 - \overline{L}_0 - \frac{c - \overline{c}}{24}\Big) \tag{1.102}$$

It is very important to notice that the mapping between complex plane and cylinder has caused a shift in the vacuum energy of the system, called *Casimir Energy*, this has simply the effect of changing the normalization of the functional integral.
Furthermore combining the two periodic conditions tells us that $z = z + \tau + 1$ so that it would have been the same to choose $\tau + 1$ (more generally $\tau + k$ with $k \in \mathbb{Z}$) or τ as the periodic direction, this is the hint of a more rich underlying structure that has been identified with the modular group $SL(2, \mathbb{Z})$.
The theory under consideration must be invariant under redefinitions of the modular parameter τ of the form:

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2} \tag{1.103}$$

Such a group of transformations is generated by:

$$\begin{cases} \tau \to \tau + 1 \\ \tau \to -\frac{1}{\tau} \end{cases} \tag{1.104}$$

If we now define the *Virasoro Characters* as:

$$\chi_{c,h}(q) = Tr_{\mathcal{V}(c,h)} q^{L_0 - \frac{c}{24}} = \sum_{N=0}^{\infty} d_N q^{h+N-\frac{c}{24}} \quad q = e^{2\pi i \tau}, \tag{1.105}$$

where d_N is the degeneracy of the N^{th} level of descendance inside the Verma module $\mathcal{V}(c,h)$, we have that as a consequence of modular invariance, under modular transformations, the Virasoro Characters transform among themselves in a unitary representation of the modular group, so that we get:

$$\chi_{c,h}(\tau + 1) = \sum_{h'} \mathcal{T}_{h,h'} \chi_{c,h'}(\tau) \tag{1.106}$$

$$\chi_{c,h}\Big(-\frac{1}{\tau}\Big) = \sum_{h'} \mathcal{S}_{h,h'} \chi_{c,h'}(\tau) \tag{1.107}$$

With all these concepts in hand we are now ready to build a partition function starting form the expression:

$$Z(\tau) = Tr e^{2\pi(i\Re\mathrm{e}\tau P - \Im\mathrm{m}\tau H)} \tag{1.108}$$

which, introduced the modular parameter $q = exp(2\pi i\tau)$, and for $\overline{c} = c$ takes the form:

$$Z(q) = (q\overline{q})^{-\frac{c}{24}} Tr q^{L_0} \overline{q}^{\overline{L}_0} \tag{1.109}$$

Where the trace is taken over all the Hilbert Space.
It is now possible (this is due to the Hilbert Space structure) to decompose the partition function into a bilinear form in the Virasoro Characters:

$$Z(q) = \sum_{h,\overline{h}} \mathcal{N}_{h,\overline{h}} \chi_{c,h}(q) \chi_{c,\overline{h}}(\overline{q}) \tag{1.110}$$

In this last expression $\mathcal{N}_{h,\overline{h}}$ is an integer that numbers the multiplicity of occurrence of $\mathcal{V}(c,h) \otimes \mathcal{V}(c,\overline{h})$ in the Hilbert Space; modular invariance of the partition function is warranted by the unitarity of the representation of the modular group carried by the Virasoro characters.
In non chiral (i.e. whose content is only of spinless fields) theories only tensor products of Verma Modules with $h = \overline{h}$ may occur, this situation corresponds to a *diagonal theory* whose partition function is simply:

$$Z(q) = \sum_{h} \big|\chi_{c,h}(q)\big|^2 \tag{1.111}$$

An example of such a theory is $\mathcal{M}_{3,4}$ which thanks to its diagonal form can be identified with the Critical Ising Model. There are also examples of non diagonal chiral theories as is the case for example of $\mathcal{M}_{5,6}$ which corresponds to the 3 States Potts Model.

1.6 Some Identifications

1.6.1 $\mathcal{M}_{3,4}$ as the Ising Model

Let us first of all recall some common lore about the Ising Model.
This model is defined on a square lattice trough the Hamiltonian:

$$H = J \sum_{<i,j>} \sigma_i \sigma_j + h \sum_i \sigma_i \tag{1.112}$$

where $< i, j >$ denotes the sum over nearest neighbor sites, and the variables σ_i must belong to the set $\{-1, 1\}$; h represents an external magnetic field, which to our purposes will be considered equal to 0 (the $h \neq 0$ situation will be achievable in the context of CFT perturbation theory).
This theory presents a 2^{nd} order phase transition separating an ordered phase ($\langle \sum_i \sigma_i \rangle \neq 0$) from a disordered phase ($\langle \sum_i \sigma_i \rangle = 0$).
In the disordered phase we have a finite correlation length ξ and 2 point functions fall off exponentially as $\langle \sigma_n \sigma_0 \rangle \sim exp(-n/\xi)$, while at the critical point ξ diverges and correlators fall of with power law (a manifest signal of scaling invariance):

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{n^{d-2+\eta}} \tag{1.113}$$

We have also a lattice interaction energy $\epsilon_i = \frac{1}{2d} \sum_k \sigma_i \sigma_{i+k}$ (here k denotes the unity displacement in one of the $2d$ directions) which at criticality behaves as:

$$\langle \epsilon_n \epsilon_0 \rangle \sim \frac{1}{n^{2(d-1/\nu)}} \tag{1.114}$$

where ν is related to the correlation length's divergence (as $T \to T_c$) expressed in terms of the reduced temperature $t = (T - T_c)/T_c$ so that $\xi \sim t^{-\nu}$.
Now, for $d = 2$ the Ising Model has been solved by Onsager and Kaufmann and its critical exponents have been shown to be $\eta = 1/4$ and $\nu = 1$, therefore if we consider $n \sim r$ (i.e. the continuum limit) we readily see from correlation functions that at criticality σ and ϵ behave as *spinless* fields of conformal dimensions $\overline{h}_\sigma = h_\sigma = 1/16$ and $\overline{h}_\epsilon = h_\epsilon = 1/2$.
These conformal weights tell us that we must consider the $\mathcal{M}_{3,4}$ model as the correct critical continuum version of the Ising Model and the presence of spinless fields confirms what we already

knew form the requirement of modular invariance, i.e. that we must consider the diagonal theory with partition function:

$$Z(q) = \left|\chi_{1/2,0}(q)\right|^2 + \left|\chi_{1/2,1/16}(q)\right|^2 + \left|\chi_{1/2,1/2}(q)\right|^2 \tag{1.115}$$

We finally want to spend some words about symmetries in the Ising Model, from the Hamiltonian (1.112) is is manifest that the *spin reversal* is a symmetry of the model, furthermore Kramers and Wannier have shown [18] that the model possesses a *duality* symmetry which exchanges order and disorder parameters ($\sigma \leftrightarrow \mu$) and reverses energy ($\epsilon \to -\epsilon$).
These considerations tell us that 3 point correlation functions involving an odd number of σ or an odd number of ϵ (but no σ or μ together with ϵ) must vanish so that we are left with the following fusion rules for the model:

$$\begin{cases} \epsilon \times \epsilon = 1 \\ \sigma \times \sigma = 1 + \epsilon \\ \sigma \times \epsilon = \sigma \end{cases}$$

Another very useful consequence of duality symmetry for the Ising Model is that by comparison of the partition function written in terms of the order and disorder parameter it is possible to find an equation that fixes the temperature for the phase transition:

$$e^{-2\beta_c} = \tanh(\beta_c) \tag{1.116}$$

where $\beta_c = 1/T_c$.

1.6.2 Massless Fermion as $\mathcal{M}_{3,4}$

Let us consider now a massless free fermion with an action

$$S = \frac{1}{8\pi} \int d^2x \overline{\Psi} \not{\partial} \Psi \tag{1.117}$$

with

$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \tag{1.118}$$

$$\not{\partial} = \sigma_x \partial_x + \sigma_y \partial_y = \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix} \tag{1.119}$$

$$\overline{\Psi} = \Psi^\dagger \sigma_x = \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix} \tag{1.120}$$

where σ_i are Pauli's sigma matrices. Using this considerations we can write the action as a sum of an holomorphic plus an antiholomorphic part:

$$S = \frac{1}{8\pi}\int d^2z(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}) \tag{1.121}$$

This action yields the following equations of motion:

$$\begin{cases} \overline{\partial}\psi = 0 \\ \partial\overline{\psi} = 0 \end{cases} \tag{1.122}$$

which can be used to compute the Stress Tensor using the formula (which holds for a lagrangian depending only on a field and its derivative):

$$T^{\mu\nu} = \frac{\delta\mathcal{L}(\phi,\partial\phi)}{\delta(\partial_\mu\phi)}\phi^\nu - g^{\mu\nu}\mathcal{L} \tag{1.123}$$

so that we find, after normal ordering the expression:

$$\begin{cases} T(z) = \frac{1}{2} : \psi(z)\partial\psi(z) : \\ \overline{T}(\overline{z}) = \frac{1}{2} : \overline{\psi}(\overline{z})\overline{\partial\psi}(\overline{z}) : \end{cases} \tag{1.124}$$

The propagator can be computed by first putting the theory on a cylinder, decomposing into positive and negative frequency parts and then by considering the mapping (1.38), it is then just a matter of summing up a geometric series to reach the result:

$$\begin{cases} \langle\psi(z)\psi(w)\rangle = -\frac{1}{z-w} \\ \langle\overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle = -\frac{1}{\overline{z}-\overline{w}} \end{cases} \tag{1.125}$$

This result could also be quickly reached if we had used the identity:

$$\overline{\partial}\frac{1}{z} = \delta_2(z,\overline{z}) \tag{1.126}$$

We now want to compute the central charge for this theory by analizing the TT OPE, all we have to do is to remember we are dealing with Grassmann Variables so that for example $\psi^2(z) = 0$, $\psi(z)\psi(w) = -\psi(w)\psi(z)$, and then use Wick's Theorem for Fermionic Fields.

We readily get dropping regular terms:

$$\begin{aligned} T(z)T(w) &= \frac{1}{4} : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) := \\ &= \frac{1}{4}\Big[\frac{\partial\psi(z)\partial\psi(w)}{z-w} - \frac{\partial\psi(z)\partial\psi(w)}{(z-w)^2} - \frac{\psi(z)\partial\psi(w)}{(z-w)^2} - \frac{2\psi(z)\psi(w)}{(z-w)^3} + \\ &+ \frac{2}{(z-w)^4} - \frac{1}{(z-w)^4}\Big] = \frac{1}{4}\Big[\frac{1}{(z-w)^4} + \frac{(\partial\psi(w))^2}{z-w} - \\ &- \frac{\partial\psi(w)\psi(w) + (z-w)\partial^2\psi(w)\psi(w)}{(z-w)^2} + \frac{\psi(w)\partial\psi(w) + (z-w)(\partial\psi(w))^2}{(z-w)^2} - \\ &- 2\frac{\psi^2(w) + \partial\psi(w)\psi(w)(z-w) + 1/2(z-w)^2\partial^2\psi(w)\psi(w)}{(z-w)^3}\Big] = \\ &= \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}(1/2\psi(w)\partial\psi(w)) + \frac{1}{z-w}\big[\partial(1/2\psi(w)\partial\psi(w))\big] \end{aligned} \tag{1.127}$$

This tells us, by comparison with (1.55), that we have a central charge $c = 1/2$, repeating the same calculation for the antiholomorphic part we easily discover that $c = \overline{c}$.
In close analogy we determine the conformal weight of the ψ operators by the $T\psi$ OPE:

$$\frac{1}{2} : \psi(z)\partial\psi(z) : \psi(w) = \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \tag{1.128}$$

So that ψ and $\overline{\psi}$ are respectively $(1/2, 0)$ and $(0, 1/2)$ Primary Fields with a spin $s = h - \overline{h} = 1/2$, in close analogy to the operator content of the $\mathcal{M}_{3,4}$ model.
To proceed further we need to introduce the mode expansion for the field ψ as:

$$i\psi(z) = \sum \frac{\psi_n}{z^{n+1/2}} \tag{1.129}$$

where the i factor was put for mere convenience, and the summation set will depend on the boundary conditions we will consider as we shall shortly see. The anticommutation relations for the modes are constrained by the $\psi\psi$ short distance expansion (i.e. the propagator) to be:

$$\{\psi_n, \psi_m\} = \delta_{n+m,0} \tag{1.130}$$

We can now conceive to impose 2 different kinds of boundary conditions as $z \to e^{2\pi i}z$, *Periodic* and *Antiperiodic*, and these will impose respectively $n \in \mathbb{Z} + 1/2$ and $n \in \mathbb{Z}$.
The change of BC will be achieved by the insertion of Primary Operators (called *Twist Fields*) on the incoming and outgoing vacuum.
Such operators will have the following short distance expansion with ψ:

$$\psi(z)\sigma(w) = (z-w)^{-1/2}\mu(w) + \dots \tag{1.131}$$

From dimensional analisys σ and μ will have the same conformal weight.
The in-state $\sigma(0)\big|0\big\rangle = \big|h_\sigma\big\rangle$ will be possibly annihilated by the ψ_n for $n > 0$, this is insured as

long as $h_\sigma < 1$; since we do not know h_σ we could take this as an ansatz and hope to find results that are consistent.
With this anstatz we can calculate the 2 point function in the antiperiodic sector:

$$\langle h_\sigma | \psi(z)\psi(w) | h_\sigma \rangle = -\frac{1}{2}\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w} \tag{1.132}$$

This result can be used to determine the expectation value of the stress tensor in the antiperiodic sector.
Now, if $z = w + \epsilon$ we can easily show that:

$$\langle h_\sigma | \psi(z)\partial\psi(w) | h_\sigma \rangle = -\frac{1}{\epsilon^2} + \frac{1}{8w^2} \tag{1.133}$$

which is simply the statement that

$$\langle h_\sigma | T(z) | h_\sigma \rangle = \frac{1}{16}\frac{1}{w^2} \tag{1.134}$$

Let us now consider the $T\sigma$ OPE:

$$T(z)\sigma(0)|0\rangle = \frac{h_\sigma \sigma(0)}{z^2}|0\rangle + \dots \tag{1.135}$$

which, normalizing $\langle h_\sigma | h_\sigma \rangle$ to 1, enables us to conclude:

$$\langle h_\sigma | T(z) | h_\sigma \rangle = \frac{h_\sigma}{z^2} \tag{1.136}$$

so that we have the impressive result

$$h_\sigma = h_\mu = \frac{1}{16} \tag{1.137}$$

This tells us that we have found the $h = 1/16$ Primary Field corresponding to the other Primary Field present in $\mathcal{M}_{3,4}$, so that the identification of the two theories is complete.
We want to point out that since $\mathcal{M}_{3,4}$ falls into the universality class (i.e. critical exponents are exactly the same) of the *Ising Model* we have also discovered that such a model (at the critical point and at zero external field) has a field content of a free massless Fermion; later on we will see how the off critical and zero external field situation (what we will call the thermic perturbation) will have a description in terms of a massive free Fermion, this in turn will mean breakdown of conformal symmetry.
Furthermore the presence of a *Dual* Twist Field μ is compatible with the description of the Ising Model which infact possesses a *Duality* Symmetry ($\sigma \leftrightarrow \mu$).

2. Boundary and Integrability in $D = 2$

In this chapter we will deal with CFT defined on manifolds with boundary, in order to discover how the existence of a boundary modifies the structure of the Hilbert Space of the theory, and as a consequence of the partition function itself; furthermore we will be led to identify a set of scaling operators that naturally lives on the boundary, which will be in 1 on 1 correspondence with all the possible choices of boundary conditions for the underlying Statistical Mechanical model.
We will also introduce CFT Perturbation Theory, and we will content ourselves with providing an example of relevant perturbation. This in order to introduce the concept of CFT breaking and of Integrable Deformation of a CFT.
Integrable Deformations of CFT will thus provide a particular example of a wider class of Field Theories, which are infact the Integrable Field Theories, such theories are very interesting since they provide highly nontrivial examples of completely solvable QFT, and suggest a different approach to scattering theory, based on considering the S matrix itself as the fundamental object of interest.

2.1 CFTs with Boundary

The problem of considering CFTs on manifolds with boundary arises naturally in Statistical Mechanics when we consider a theory with some definite boundary conditions and consider its critical behaviour; another important connection (but anyway far from our goals) is open String Theory.
References for this section can be found in Cardy's works [3][4][5][6].

2.1.1 Boundary Conditions and Correlators

Two dimensional manifolds with boundary are classified by their topology, and in particular simply connected manifolds can be mapped to the complex upper half plane.

This is a good reason to consider the upper half plane from the start as our main example. In order to preserve some sort of conformal invariance we are forced to require that under infinitesimal transfromations $z \to z + \epsilon(z)$ the real axis be mapped into itself, this is because the distance of the image of a point lying on the boundary from the boundary itself would define a local scale for the system and the shortest of such local scales would thus define a global scale which would force correlators to decay exponentially as their distance from the boundary grows. This requirement is easily seen to mean that ϵ must be real:

$$\epsilon(\overline{z}) = \overline{\epsilon}(z) \tag{2.1}$$

Another important requirement is that boundary conditions on fields must be conformally invariant so that the scaling properties of primary operators tell us that we must consider homogeneous B.C. such as:

$$\phi|_B = 0 \tag{2.2}$$

If we now want to go further we are forced to consider the antiholomorphic dependence of the fields to be constrained to $\overline{z} = z^*$.
And following this line of thought we constrain also the stress tensor to satisfy:

$$T(z^*) = \overline{T}(z) \tag{2.3}$$

which in turn implies that $T = \overline{T}$ on the real axis so that there is no energy or momentum flux across the boundary.
This procedure can be interpreted conversion of the antiholomorphic degrees of freedom into holomorphic degrees of freedom on the lower half plane.
We are now ready to consider the conformal ward identities for this system, since $\delta_{\epsilon,\overline{\epsilon}} = \delta_\epsilon + \delta_{\overline{\epsilon}}$, considering a closed contour C (and its complex coniugated $\overline{C}$) lying in the upper half plane we have:

$$\begin{aligned}
&\delta_{\epsilon,\overline{\epsilon}}\big\langle \phi_1(z_1.\overline{z}_1), \dots, \phi_1(z_k, \overline{z}_k)\big\rangle = \\
&- \oint_C \frac{dw}{2\pi i}\epsilon(w)\big\langle T(w)\phi_1(z_1, z_1^*), \dots, \phi_1(z_k, z_k^*)\big\rangle + \\
&- \oint_{\overline{C}} \frac{dw^*}{2\pi i}\overline{\epsilon}(w^*)\big\langle \overline{T}(w^*)\phi_1(z_1, z_1^*), \dots, \phi_1(z_k, z_k^*)\big\rangle
\end{aligned} \tag{2.4}$$

Now we can deform C and $\overline{C}$ to follow very closely the real axis so that the two pieces of contours along the real axis cancel each other and we are left with a single contour C' no longer contained in the upper half plane encircling the insertions at $\{z_1, \dots, z_k, z_1^*, \dots, z_k^*\}$, so that:

$$\delta_{\epsilon,\epsilon^*}\big\langle \phi_1(z_1, z_1^*), \dots, \phi_1(z_k, z_k^*)\big\rangle = - \oint_{C'} \frac{dw}{2\pi i}\epsilon(w)\big\langle T(w)\phi_1(z_1.z_1^*), \dots, \phi_1(z_k, z_k^*)\big\rangle \tag{2.5}$$

which for $\phi(z,\overline{z}) = \phi(z)\overline{\phi}(\overline{z})$ means that a k points correlator on the complex upper half plane satisfies the same differential equation of a $2k$ points correlator on the whole plane, modulus the fact that we must impose the constraint $\overline{z} = z^*$.
As an example we can consider the 1 point function of a primary field of conformal weights $h = \overline{h}$; assuming that in the bulk ($|z| \to \infty$) the expression vanishes we are readily led to ($y = \Im z$):

$$\left\langle \phi(z,\overline{z}) \right\rangle_\beta = \frac{A_\phi^\beta}{2y^{2h}} \tag{2.6}$$

where A_ϕ^β is an amplitude depending on the field ϕ and on the boundary condition labeled by β.

2.1.2 Boundary States and Operators

In the framework of the previous section arises the problem to consider the limit of an operator insertion at z as the point z tends to the boundary, and even more to consider the correlation of insertions living on the boundary (we are interested mainly in the first situation).
This is done by considering a Primary field as a product of holomorphic and antiholomorphic parts, constrained so that $z = z^*$ $h = \overline{h}$, and then introducing the following short distance expansion (OPE):

$$\phi(z,\overline{z}) = \phi(z)\phi(z^*) \sim \sum_i (2iy)^{h_i - 2h} C_{\phi\psi_i}^\beta \psi_i(x) \tag{2.7}$$

Where again β labels a boundary condition, x, y are respectively the real and imaginary parts of z and finally the $\{\psi_i(x)\}$ are a family of boundary fields of scaling dimension h_i which we normalize as:

$$\left\langle \psi_i(x_1)\psi_j(x_2) \right\rangle_\beta = \delta_{i,j}(x_1 - x_2)^{-2h_i} \tag{2.8}$$

In particular taking the expectation value of (2.7) and considering (2.6) we discover:

$$C_{\phi,\mathbf{1}}^\beta = A_\phi^\beta \tag{2.9}$$

and

$$C_{\phi,\psi_i}^\beta \left\langle \psi_i(x) \right\rangle_\beta = 0 \ , \ \psi_i \neq 1 \tag{2.10}$$

Further information about the state representation of boundary operators can be obtained by putting the theory on a finite length cylinder of dimensions L, T (which can be mapped through the exponential map to an anulus in the complex plane).
After compactifying one dimension we first of all need to remember that in analogy with the previous section on the boundary we must have $T(z^*) = \overline{T}(z)$ so that:

$$\sum_{n\in\mathbb{Z}} \frac{L_n}{z^{*n+2}} |\alpha\rangle = \sum_{m\in\mathbb{Z}} \frac{\overline{L}_m}{z^{m+2}} |\alpha\rangle \tag{2.11}$$

which taking as inner boundary S^1, so that $z^* = 1/z$, implies after renaming the summation index m:

$$(z^2 L_n - z^{-2}\overline{L}_{-n})\big|\alpha\big\rangle = 0 \tag{2.12}$$

This equation must hold for all $z \in S^1$ and in particular for $z = 1$, so that we get the constraint:

$$(L_n - \overline{L}_{-n})\big|\alpha\big\rangle = 0 \tag{2.13}$$

This constraint has the immediate effect of reducing the Hilbert Space of States of the theory to (remember that only half of the conformal generators survive):

$$\mathcal{H}_{Boundary} = \bigoplus_h \mathcal{V}(c,h) \tag{2.14}$$

which is of course embedded into the bigger Hilbert Space (1.90), and infact we can solve the above constraint inside this wider space as:

$$\big|j\big\rangle\big\rangle = \sum_N \big|j,N\big\rangle \otimes U\overline{\big|j,N\big\rangle} \tag{2.15}$$

where $\big|j\big\rangle\big\rangle$ are the so called Ishibashi States, $\big|j,N\big\rangle$ is a state belonging to the dimensionally reduced verma module $\mathcal{V}(c,j)$, N labels a state inside a given level of descendance, and finally U is a antiunitary operator satisfying:

$$U\overline{\big|j,0\big\rangle} = \overline{\big|j,0\big\rangle}^* \tag{2.16}$$

$$[\overline{L}_n, U] = 0 \tag{2.17}$$

the second equation in particular tells us that U can be expressed as a function of the L_n.
We want to point out that the states $\big|j\big\rangle\big\rangle$ defined as above are orthogonal but have an infinite norm:

$$\big\langle\big\langle j\big|j\big\rangle\big\rangle = \sum_{NM} \big\langle j,N\big| \otimes \overline{\big\langle i,N\big|}U^\dagger U\overline{\big|i,M\big\rangle} \otimes \big|j,M\big\rangle = \sum_{NM} \delta_{NM} = \infty \tag{2.18}$$

This problem can be fixed by changing the normalization of $\big|j,N\big\rangle$ for example to $1/N$.
With the help of these Isibashi states we can now generate all the possible boundary states by linear combination.

2.1.3 Partition Function and Modular Invariance

Now that we have shed some light on the underlying Hilbert Space structure we can observe that the choice of two different quantization schemes allows us to express the partition function in two important ways.

Taking time as the periodic direction we find out that the hamiltonian H depends on the boundary condidtions $H = H_{\alpha\beta}$, and local conformal invariance implies that its spectrum falls into irreducible representations of the Virasoro algebra so that recalling the Hilbert Space structure we are led to write:

$$Z_{\alpha\beta}(q) = \sum_h n^h_{\alpha\beta}\chi_{c,h}(q) \tag{2.19}$$

Where $n^h_{\alpha\beta}$ is the number of times the representation h occurs in the spectrum of the Hamiltonian, and the modular parameter q has been identified as:

$$q = e^{2\pi i\tau} \ , \ \tau = iT/2L \tag{2.20}$$

Now, from the previous chapter we know that under a modular transformation $\tau \to -1/\tau$ the characters transform according to a unitary representation of the modular group, furthermore such a transformation formally exchanges the roles of the compactified dimensions T, L, so that calling $\tilde{q}$ the transformed of q under the modular inversion we can write:

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}S_{ij}\chi_{c,j}(\tilde{q}) \tag{2.21}$$

On the other hand taking space as the compactified dimension the Hamiltonian does not depend on the boundary condidtions and has the simple expression already introduced in the previous chapter, so that the partition function takes the form:

$$\begin{aligned} Z_{\alpha\beta}(q) =& \big\langle \alpha \big| \tilde{q}^{L_0 - \frac{c}{24}} \big| \beta \big\rangle = \sum_{ij} \big\langle \alpha \big| i \big\rangle \big\rangle \big\langle \big\langle i \big| (\tilde{q}^{1/2})^{L_0 + \overline{L}_0 - \frac{c}{12}} \big| j \big\rangle \big\rangle \big\langle \big\langle j \big| \beta \big\rangle = \\ =& \sum_j \big\langle \alpha \big| i \big\rangle \big\rangle \big\langle \big\langle i \big| \beta \big\rangle \chi_{c,j}(\tilde{q}) \end{aligned} \tag{2.22}$$

Comparison of the two expressions yields:

$$\sum_i S_{ij} n^i_{\alpha\beta} = \big\langle \alpha \big| j \big\rangle \big\rangle \big\langle \big\langle j \big| \beta \big\rangle \tag{2.23}$$

If we now consider a boundary state $\big|\tilde{0}\big\rangle$ such that the only representation that occurs in the Hamiltonian $H_{\tilde{0}\tilde{0}}$ is the identity, that is to say $n^i_{\tilde{0}\tilde{0}} = \delta_{i,0}$, it immediately follows from (2.23) that $S_{0,j} = \big|\big\langle \big\langle j \big| \tilde{0} \big\rangle \big\rangle\big|^2$, so that since $S_{0,j} > 0$ because of unitarity we have:

$$\big|\tilde{0}\big\rangle = \sum_j \sqrt{S_{0,j}} \big| j \big\rangle \big\rangle \tag{2.24}$$

and similarly requiring that only the representation l propagates in $H_{\tilde{0}\tilde{l}}$ (i.e. $n^i_{\tilde{0}\tilde{l}} = \delta_{i,l}$), we get:

$$\big|\tilde{l}\big\rangle = \sum_j \frac{S_{l,j}}{\sqrt{S_{0,j}}} \big| j \big\rangle \big\rangle \tag{2.25}$$

This is an important result because using the Ishibashi states and modulare invariance of the partition function we have been able to build a complete correspondence between bulk and boundary states.
In particular by taking for $\big|\alpha\big\rangle$ and $\big|\beta\big\rangle$ two states as above we easily get the result:

$$\sum_i S_{i,j} n^i_{\tilde{k}\tilde{l}} = \frac{S_{k,i} S_{l,j}}{S_{0,j}} \tag{2.26}$$

which, once we know the modular S matrix tells us all we need to know about the operator content of theories with certain boundary conditions.

2.2 Perturbation Theory

So far we have dealt only with critical theories, but it's also interesting to approach off critical theories by studying perturbations to the conformally invariant action inside a Path Integral formalism.
This will lead us to distinguish a new class of field theories which shares an important property with the Conformal Field Theories, that is the existence of an infinite set of conserved currents which will make the theory in principle completely solvable (*Integrable*).
Further details and rigorous proofs of what follows can be found in [8][12][7][2].

2.2.1 Breakdown of Conformal Symmetry

If we now consider some CFT with a conformally invariant action S_{CFT} and perturb this critical fixed point with some linear combination of relevant operators so that:

$$S = S_{CFT} + \sum_i \lambda_i \int d^2 z \phi_i(z, \overline{z}) \tag{2.27}$$

we have that in general the perturbed action loses scaling invariance so that the stress tensor stops being a traceless object, but anyway since the action is given as a sum of an invariant plus a symmetry breaking term, we are still able to analyze the pattern of such a breaking.
The perturbed action may then flow under the Renormalization Group transformations to another fixed point, which might be another CFT (for example the $\phi_{1,3}$ perturbation of $\mathcal{M}_{p,p-1}$ flows to $\mathcal{M}_{p-1,p-2}$), or simply a (noncritical) massive field theory.
Let us now consider for the sake of simplicity the effect of a perturbation with a single relevant primary field ϕ, recalling (1.20) and (1.19), under $z \to z + \epsilon(z)$ we find the following expression for the variation of the action:

$$\delta S = \frac{1}{2} \int d^2 z (\partial \cdot \epsilon) \Theta(z, \overline{z}) \tag{2.28}$$

where Θ denotes the trace of the stress tensor which happens to be also its spin 0 part.
On the other hand since $\delta S_{CFT} = 0$ using (1.33) we get:

$$\delta S = \lambda \int d^2 z \delta\phi(z,\overline{z}) = \lambda \int d^2 z (h-1)(\partial \cdot \epsilon)\phi(z,\overline{z}) \tag{2.29}$$

so that we find the following expression for the trace component of the stress tensor:

$$\Theta(z,\overline{z}) = 2\lambda(h-1)\phi(z,\overline{z}) \tag{2.30}$$

This example concretely shows how a perturbation may have the effect of breaking conformal invariance, since as we already know the tracelessness of the stress tensor is strictly related to scaling invariance of the theory.
In this context it is also possible to compute correlation functions perturbatively as a series involving only the conformal correlators, this is done by expanding the interaction term as a power series:

$$\begin{aligned} \big\langle X \big\rangle &= \int \mathcal{D}[\phi] X e^{S_{CFT} + \lambda \int d^2 w \phi(w,\overline{w})} = \\ &= \big\langle X \big\rangle_{CFT} + \sum_{k=1}^{\infty} \lambda^k \int d^2 w_1 \ldots \int d^2 w_k \big\langle X \phi(w_1,\overline{w}_1) \ldots \phi(w_k,\overline{w}_k) \big\rangle_{CFT} \end{aligned} \tag{2.31}$$

where X denotes an insertion in the Path Integral.The integrals appearing in the last term above are not always finite, and may need an appropriate regularization procedure to make sense.

2.2.2 Deformations of CFT and Integrability

A Conformal Field Theory has an infinite set of conserved currents , for example every expression built purely on the holomorphic or antiholomorphic part of the stress tensor and its derivatives does define a conserved current since the dependence of such objects is purely on the z or $\overline{z}$ variables.
If we wanted to be quantistically rigorous we had to normal order such objects (when there happen to be products of operators that diverge as their points of insertion approach each other), following either the usual normal ordering prescription or the Conformal Normal Ordering prescription, which is simply obtained by arranging all the operator modes in increasing order after substituting their mode expansions.
It is possible to show that the two prescriptions are equivalent since they both reproduce Wick's theorem, and that it's possible to pass from one prescription from the other.
The existence of this infinite set of conserved currents is the reason that makes a CFT a solvable theory, that is we can in principle determine everything about it, all the theories sharing this property are called *Integrable*.

It can happen that a perturbation of a CFT defines an integrable theory, in this case the deformed theory's conservation laws can be interpreted as deformations of conformal conservation laws. The first trivial example is the stress tensor itself, which as long as we do not lose traslational invariance remains conserved:

$$\overline{\partial}T(z,\overline{z}) = -\frac{1}{4}\partial\Theta(z,\overline{z}) \tag{2.32}$$

where from (2.30) we see what is already ovious, i.e. the fact that as $\lambda \to 0$ the conservation law reduces to the statement that T is purely holomorphic. The corresponding integral of motion is the momentum:

$$P = \oint (dzT + d\overline{z}\Theta) \tag{2.33}$$

In general the study of the deformation of a conservation law is carried out as follows.
Let $J(z,\overline{z})$ be a conserved current for the Conformal Action of dimension $(s,0)$, the statement that J is conserved must be interpreted as an operator statement, that is to say it holds weighted on the conformal measure:

$$\left\langle X\overline{\partial}J_{z,\dots}\right\rangle_{CFT} + \left\langle X\partial J_{\overline{z},\dots}\right\rangle_{CFT} = 0 \tag{2.34}$$

Let ϕ be the perturbing field, we define the OPE of J and ϕ as:

$$J(z)\phi(w,\overline{w}) = \sum_k \frac{A^{(k)}(w,\overline{w})}{(z-w)^k} \tag{2.35}$$

where the modes $A^{(k)}(z,\overline{z})$ have scaling dimension $(s+h-k,\overline{h})$, and only a finite number of $A^{(k)}(z,\overline{z})$ with $k>0$ can exist because otherwise for k greater than some $\tilde{k}$ they would have a negative scaling dimension.
We will now substitute this OPE inside the path integral, and carry the calculation out only to first order in λ, although we could in principle carry it out completely since only a finite number of terms are involved.
Now, to first order in λ we have:

$$\left\langle XJ(z,\overline{z})\right\rangle = \left\langle XJ(z)\right\rangle_{CFT} + \lambda\int d^2w\left\langle XJ(z)\phi(w,\overline{w})\right\rangle_{CFT} \tag{2.36}$$

And applying $\overline{\partial}$ to (2.36) we finally get:

$$\overline{\partial}\left\langle XJ(z,\overline{z})\right\rangle = \lambda\overline{\partial}\int d^2w\left\langle XJ(z)\phi(w,\overline{w})\right\rangle_{CFT} \tag{2.37}$$

and regulating the integral term with a step function cut off we get:

$$\begin{aligned}
&\overline{\partial}\int d^2w\big\langle XJ(z)\phi(w,\overline{w})\big\rangle_{CFT} = \overline{\partial}\lim_{a\to 0}\int d^2w H(|z-w|^2-a^2)\big\langle XJ(z)\phi(w,\overline{w})\big\rangle_{CFT} = \\
&= \overline{\partial}\lim_{a\to 0}\sum_k\int d^2w\frac{H(|z-w|^2-a^2)}{(z-w)^k}\big\langle XA^{(k)}(w,\overline{w})\big\rangle_{CFT} = \\
&= \lim_{a\to 0}\sum_k\int d^2w(z-w)\frac{\delta(|z-w|^2-a^2)}{(z-w)^k}\big\langle XA^{(k)}(w,\overline{w})\big\rangle_{CFT} = \\
&= \lim_{a\to 0}\sum_k\int_0^\infty d\rho\rho\int_0^{2\pi}d\theta e^{(1-k)\theta}\rho^{1-k}\delta(\rho^2-a^2)\big\langle XA^{(k)}(z-\rho e^{i\theta},\overline{z}-\rho e^{-i\theta})\big\rangle_{CFT} = \\
&= \lim_{a\to 0}\sum_k\int_0^\infty d\rho\int_0^{2\pi}d\theta e^{(1-k)\theta}\rho^{2-k}\frac{\delta(\rho-a)}{2a}\big\langle XA^{(k)}(z-\rho e^{i\theta},\overline{z}-\rho e^{-i\theta})\big\rangle_{CFT} = \\
&= \sum_k\lim_{a\to 0}\frac{1}{2}\int_0^{2\pi}d\theta e^{(1-k)\theta}a^{1-k}\big\langle XA^{(k)}(z-ae^{i\theta},\overline{z}-ae^{-i\theta})\big\rangle_{CFT} = \\
&= \sum_k\pi\delta_{k,1}\big\langle XA^{(k)}(z,\overline{z})\big\rangle_{CFT} = \pi\big\langle XA^{(1)}(z,\overline{z})\big\rangle_{CFT}
\end{aligned} \tag{2.38}$$

This equation tells us that the Conformal conservation law is spoiled by the perturbation already at the first order unless $A^{(1)}(z,\overline{z})$ is a total z derivative.
This first order result allows us to achieve useful information about the $\phi_{1,3}$ (Thermal) perturbation of the Ising Model, if we consider infact the holomorphic fermion (of dimensions (1/2,0)) itself as a current which is conserved in virtue of the equations of motion, considering that $A^{(1)}(z,\overline{z})$ has scaling dimensions $(0,1/2)$ we see that the only possible choice is:

$$A^{(1)} = \overline{\psi} \tag{2.39}$$

and all the other expressions for $k > 1$ must vanish since they have negative scaling dimensions, so that our first order calculation is an exact calculation that tells us (we give also the antiholomorphic twin equation):

$$\begin{cases} \overline{\partial}\psi = \pi\lambda\overline{\psi} \\ \partial\overline{\psi} = \pi\lambda\psi \end{cases} \tag{2.40}$$

and, using (1.119) we readily obtain:

$$(\not{\partial} - \pi\lambda)\Psi(z,\overline{z}) = 0 \tag{2.41}$$

Which is simply the massive Dirac Equation, so that the off critical Ising Model in zero magnetic field is described by a massive fermion theory. This situation provides also an example of situation where the conformal conservation law is broken by the perturbation.
A non trivial example of deformable conservarion law is that of $J(z) =: T^2(z) :$, under a $\phi_{1,3}$

perturbation, whose conservation is ensured by the existence of a third level null vector in $\mathcal{V}(1/2, h_{1,3})$. As a consequence we have a spin 3 conserved charge.
Indeed A.B. Zamolodchikov has shown [8] that the $\phi_{1,3}$ perturbation Ising Model possesses an infinite set of conserved currents of the form:

$$\begin{aligned} X^{(k+1)}_{2n+k} &= zX^{(k)}_{2n+k+1} + \overline{z}X^{(k)}_{2n+k-1} \\ n \in \mathbb{Z},\ k &= -1, 0, 1, 2, \ldots \end{aligned} \tag{2.42}$$

satisfying:

$$\overline{\partial} X^{(k)}_{2n+k+1} = \partial X^{(k)}_{2n+k-1} \tag{2.43}$$

with the corresponding $s = 2n + k$ conserved charges:

$$P^{(k)}_{2n+k} = \oint (X^{(k)}_{2n+k+1} dz + X^{(k)}_{2n+k-1} d\overline{z}) \tag{2.44}$$

Where the first term in the recurrence relation (2.42) is given by:

$$X^{(0)}_{2n+1} = zT_{2n+2} + \overline{z}T_{2n} \tag{2.45}$$

with:

$$\begin{aligned} T_{2n} &= \lambda^{1-2n} : \partial^{n-1}\psi\partial^n\psi : \quad ,\ n = 1, 2, 3, \ldots \\ T_0 &= \lambda : \overline{\psi}\psi : \sim \Theta \\ T_2 &\ \sim T \\ T_{-2n} &= \overline{T}_{2n} \end{aligned} \tag{2.46}$$

We notice that $P^{(-1)}_{2n-1}$ corresponds to the integrals of motion of the conformal family of the holomorphic part of the stress tensor.

2.3 Consequences of Integrabiliy on Scattering Theory

If we consider an Integrable field theory with an infinite set of integrals of motion P_s labelled by their spin s, we discover that Intergability constrains the $n-$particle S matrix to be factorized into $n(n-1)/2$ $2-$particle amplitudes and to satisfy the Yang Baxter Equation.
Such an equation can be considered as a starting point of view in the search for Integrable Models, indeed in an axiomatic approach one looks for self consistent solutions (i.e. S matrices) of the equations describing Integrable Theories, and then tries to gain insight on the integrals of motion. It is in this framework that it has been shown that the magnetic perturbation of the Ising Model is described by a highly nontrivial scattering theory containing 8 particles.
Detailed information about the subject can be found in [7][2][8].

2.3.1 Unitary Equations and Crossing Symmetry

Let us consider a scattering process described in momentum space $A_i(p_i)A_j(p_j) \to A_k(p_k)A_l(p_l)$. Lorentz invariance constrains the S matrix to be a function of the 3 Mandelstam variables s, t, u which are infact Lorentz scalars:

$$\begin{aligned} s &= (p_i + p_j)^2 \\ t &= (p_i - p_k)^2 \\ u &= (p_i - p_l)^2 \\ s + t + u &= \sum_{b=1}^{4} m_b^2 \end{aligned} \tag{2.47}$$

The constraint satisfied by these variables tells us at a first glance that we need only consider 2 of them, furthermore momentum conservation forces us to discard another one so that we are left with only one independent variable.
Let us now introduce the rapidity variable θ which has the virtue of parametrizing the on mass-shell condition:

$$p_i = m_i \begin{pmatrix} \cosh\theta_i \\ \sinh\theta_i \end{pmatrix} \tag{2.48}$$

in this parametrization the s variable can be written as:

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta_{ij} \quad \theta_{ij} = \theta_i - \theta_j \tag{2.49}$$

this tells us that in scattering processes only the relative rapidity θ_{ij} is important, and that the function $s(\theta_{ij})$ is periodic of period $2\pi i$ so that the complex plane (in the variable θ_{ij}) is foliated into strips and as a consequence the S matrix will be characterized by its analytic structure inside one of these strips.
We will now define the S matrix elements through:

$$\big|A_i(\theta_1)A_j(\theta_2)\big\rangle_{in} = S_{ij}^{kl}(\theta_{12})\big|A_k(\theta_2)A_l(\theta_1)\big\rangle_{out} \tag{2.50}$$

Unitarity requirement can be translated into:

$$\sum_{n,m} S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l \tag{2.51}$$

If we want also a crossing symmetry that allows us to equate the S matrix elements of the two processes:

$$\begin{aligned} A_i(p_i)A_j(p_j) &\to A_k(p_k)A_l(p_l) \\ A_i(p_i)A_k(p_k) &\to A_j(p_j)A_l(p_l) \end{aligned} \tag{2.52}$$

the result can be heuristically achieved by observing that the Mandlestam variables $s(\theta)$ and $t(\theta)$ differ as functions of θ only in the sign of the term $2m_i m_j \cosh\theta$, so that we can write $s(\theta) = t(i\pi - \theta)$ (this is true if $m_j = m_k$ so this justifies the exchange $j \leftrightarrow k$ in the S matrix) which suggests us:

$$S_{ik}^{lj}(\theta) = S_{ij}^{kl}(i\pi - \theta) \tag{2.53}$$

More constraints could be imposed on the S matrix if we required it to be invariant under Parity and Time Reversal symmetries:

$$\begin{aligned} S_{ij}^{kl}(\theta) &= S_{ji}^{kl}(\theta) \quad P \\ S_{ij}^{kl}(\theta) &= S_{kl}^{ij}(\theta) \quad T \end{aligned} \tag{2.54}$$

2.3.2 Consequences of Integrability

It is common lore in quantum field theory that to a conserved current is associated a charge that generates a group of symmetries for the system, so that Integrability tells us that the theory under consideration has an $\infty-$dimensional symmetry.
The presence of such an infinite number of constraints on scattering processes has the effect of forbidding particle production, so that only elastic scattering processes may occur, and furthermore the sets of initial and final momenta are forced to coincide. If we label the charges by their spin s and call them P_s we can consider the evolutor:

$$U_s(a) = e^{iaP_s} \tag{2.55}$$

where we observe that U_1 produces a traslation of a on the fields in configuration space, while in general higher s evolutors shift plane waves by a momentum dependent phase, so that by acting appropriately with a combination of such operators we can arbitrarily shift the points of interaction in a generic process without altering the scattering amplitude (let's recall that the P_s generate symmetries of the action, and that the S matrix is a functional of the action).
As a consequence only $2-$particle scatterings are fundamental, and the $n-$particles S matrix must be factorized into $n(n-1)/2$ $2-$particle amplitudes.
Another consequence comes from equating 2 different ways of factoring $3-$particles processes, which yields the Yang-Baxter Equation:

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 k_3}^{j_1 j_3}(\theta_{13}) S_{k_1 i_3}^{j_2 k_3}(\theta_{23}) = S_{i_1 i_3}^{k_1 k_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) S_{i_1 k_3}^{k_2 j_3}(\theta_{23}) \tag{2.56}$$

where summation over repeated indexes is understood.

2.3.3 Analytic Structure of S and Bootstrap Principle

As we already mentioned when we introduced it, the S matrix has an analytic structure which is completely specified once we have knowledge of its poles in an analitycity strip of width $2\pi i$,

the S matrix becomes then a meromorphic function in the complex plane.
Let us consider the S matrix in the neighborhood of one of those singularities:

$$S_{ij}^{kl}(\theta) \sim \frac{iR_{ij}^{n}}{\theta - iu_{ij}^{n}} \tag{2.57}$$

Such poles represent resonances in scattering processes, which can typically be identified with bound states (n labels the bound state).
The bootstrap principle consists in identifying the bound states themselves as asymptotic states so that (2.49) tells us:

$$m_n^2 = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^n \tag{2.58}$$

This equation provides a constraint on the location of the poles since the cyclic permutations of u_{ij}^n must be the angles of a triangle of sides m_i, m_j, m_n, so that:

$$u_{ij}^n + u_{in}^j + u_{jn}^i = 2\pi \tag{2.59}$$

Now, if we consider a theory with a non degenerate mass spectrum, or anyway where mass degeneracy is resolved by the different eigenvalues under the P_s, we have that the S matrix is diagonal (i.e. the upper indexes are redundant) and that the bootstrap principle can be expressed as:

$$S_{i\bar{l}}(\theta) = S_{ij}(\theta + i\overline{u}_{j\ l}^{k})S_{ik}(\theta - i\overline{u}_{l\ k}^{j}) \tag{2.60}$$

with $\overline{u}_{j\ l}^{k} = \pi - u_{j\ l}^{k}$. In this situation the general solution of the unitarity and crossing symmetry equations (2.51) and (2.53) can be expressed as an arbitrary product of functions $s_x(\theta)$ [7], where:

$$s_x(\theta) = \frac{\sinh((\theta + i\pi x)/2)}{\sinh((\theta - i\pi x)/2)} \tag{2.61}$$

with the $s_x(\theta)$ satisfying the following properties:

$$\begin{aligned} &s_x(\theta) = s_x(\theta + 2\pi i) \\ &s_x(\theta)s_x(-\theta) = s_{-x}(\theta) = 1 \\ &s_x(\theta) = s_{x+2}(\theta) = s_{-x}(-\theta) \\ &s_0(\theta) = -s_1(\theta) = 1 \\ &s_x(i\pi - \theta) = -s_{1-x}(\theta) \end{aligned} \tag{2.62}$$

Thanks to the periodicity property it is possible to choose the range for the x variable to be the interval $[-1, 1]$. We notice that the s_x have a simple pole at $\theta = i\pi x$ and a simple zero at $\theta = -i\pi x$.
We still have to solve the boostrap equation, this is a very delicate job, since we must look for

solutions that are consistent with Integrability and with a set of Bootstrap Fusion Rules we have to choose as an ansatz (characterizing the model).
If we consider an asymptotic state $\big|A_a(\theta)\big\rangle$ that is an eigenstate of the P_s with eigenvalue $\omega_s^a(\theta)$ we have that Lorentz invariance constrains ω to have the form:

$$\omega_s^a(\theta) = \chi_s^a e^{s\theta} \tag{2.63}$$

where $\chi_1^a = m_a$; furthermore locality imposes for a many particle state:

$$P_s\big|A_{a_1}(\theta_1)\dots A_{a_k}(\theta_k)\big\rangle = (\omega_s^{a_1}(\theta_1) + \dots + \omega_s^{a_k}(\theta_k))\big|A_{a_1}(\theta_1)\dots A_{a_k}(\theta_k)\big\rangle \tag{2.64}$$

Now, for some resonant values of the rapidity 2−particle states fuse together to give another asymptotic state:

$$\lim_{\epsilon\to 0}\epsilon\big|A_a(\theta + i\overline{u}_{ac}^b - \frac{\epsilon}{2})A_b(\theta - i\overline{u}_{bc}^a + \frac{\epsilon}{2})\big\rangle = \big|A_{\overline{c}}(\theta)\big\rangle \tag{2.65}$$

and, applying the P_s we get the following set of equations for the χ:

$$\chi_s^a e^{is\overline{u}_{ac}^b} + \chi_s^b e^{is\overline{u}_{bc}^a} = \chi_s^{\overline{c}} \tag{2.66}$$

such equation can be solved only after we have chosen some Bootstrap Fusion Rule:

$$A_i \times A_j = \sum_k N_{ij}^k A_k \tag{2.67}$$

where $N_{ij}^k \in \{0, 1\}$.
A rather simple example of solution of these equations is given by a system containing only one particle with the fusion rule:

$$A \times A \to A \tag{2.68}$$

so that $\overline{u}_{aa}^a = \pi/3$, and (2.66) becomes:

$$\cos(\frac{s\pi}{3}) = \frac{1}{2} \tag{2.69}$$

which implies consistency with an infinite set of charges of spin:

$$s = 1, 5, 7, 12, 13, 18, \dots \tag{2.70}$$

This is a rather curios situation since the particle A appears as a bound state of itself.

3. Integrable Lattice Models

In this chapter we are going to give a definition of Lattice Integrable Models, this definition is formulated in a way that is a bit different from Integrability for continuum Field Theories but nonetheless shows many formal analogies from the point of view of the equations, in which the transfer matrix $\mathbf{T}$ and the S matrix play similar roles.
Next we are going to introduce the anisotropic Ising Model, first as an example, and then in its general formulation in terms of $\mathbf{A}_n$ Models.
In this framework we are going to study the off critical model (bulk thermal perturbation), and to solve the Functional equation by applying Thermodinamical Bethe Ansatz (TBA) techniques. Such a solution will be one of the main objects of interest in the next chapter, where we will compute its continuum limit and use it in a way that will shed some light on the relationship of the model to CFT and in particular $\mathcal{M}_{3,4}$.

3.1 Lattice Integrability

In the last chapter we defined an Integrable Field Theory as a theory possessing an ∞-dimensional symmetry, and as a consequence we had that the S matrix satisfied a set of functional equations. The definition of Integrability for a statistical mechanical model defined on a lattice, is on the other hand a bit different, since it is formulated in terms of the properties of the transfer matrix T, which is a functional of the Hamiltonian of the system (like the S matrix) that allows us (in the cases where it is known to exist) to write the partition function as:

$$Z = \mathrm{Tr}\ T^N \tag{3.1}$$

Furthermore since T is a functional of the Hamiltonian H, the Hamiltonian itself can be thought as a functional of the transfer matrix, indeed there must exist a wide class of lattice models which can be described more conveniently in terms of the properties of T, which will reflect properties of the hamiltonian H.
Anyway it is not right to think that every lattice model possesses a transfer matrix description, since its existence is based on the requirement of locality for the Hamiltonian, which is surely the case of nearest neighbor interactions on a lattice, but on the other hand the study and

classification of models possessing a transfer matrix suggests itself as a general approach with the capability of shedding light on the mechanisms that make some theories solvable.
Lattice Integrable Models are models characterized by requirement that their transfer matrices must form a *one parameter commuting family* in a so-called Spectral Parameter u (which is related in a convenient way to the coupling constants of the model) such that:

$$[\mathbf{T}(u), \mathbf{T}(v)] = 0 \ \forall u, v \in \mathbb{C} \tag{3.2}$$

this requirement implies that the eigenvectors of T do not depend on the spectral parameter u. Furthermore it is possible to expand $\mathbf{T}(u)$ as a series about some point u_0, and the coefficients of such an expansion shall be a set of matrices $\mathbf{I}_n$ which we shall call the *lattice integrals of motion*. Moreover we have that the commuting property (3.2) implies that such integral of motion must all commute with each other.
We will see in the next chapter how it is possible to compute the eigenvalues of these integrals of motion and follow them in the continuum limit.

3.2 The 2D Anisotropic Ising Model

Although the $D = 2$ Ising Model has been solved over the years by several people and in different ways, for example by Onsager(1944) and Onsager-Kaufmann (1949), we will present a more recent approach [9] that has the virtue of explicitly showing the algebraic structure mentioned in the previous section.
We will consider the model on a square lattice, but in a different way from the previous solutions we will rotate the lattice of $\pi/4$ and consider the lattice of dimensions L, L' where L is the number of columns and L' is the number of faces in a column, we will also impose periodic boundary conditions in one of the two directions.
We will define the Hamiltonian by introducing 2 distinct coupling constants (this is why the model is anisotropic):

$$-\beta\mathcal{H} = J \sum_{<i,l>}^{even} \sigma_i\sigma_l + K \sum_{<i,l>}^{odd} \sigma_i\sigma_l \tag{3.3}$$

where β is understood as the inverse of the temperature, $\sigma \in \{-1, 1\}$, and the two sums run over nearest neighboor sites in the left and right zigzagging columns that compose a column of faces, so that each spin appears exactly twice in each sum.

Let us define the partition function through:

$$\begin{aligned} Z_{L,L'} = &\sum_{\{\sigma\}} e^{-\beta\mathcal{H}} = \sum_{\{\sigma\}} \prod_{<i,l>}^{even} e^{J\sigma_i\sigma_l} \prod_{<i,l>}^{odd} e^{K\sigma_i\sigma_l} = \\ &= \sum_{\{\sigma\}} \prod_{<i,l>}^{even} (\cosh(J) + \sigma_i\sigma_l \sinh(J)) \prod_{<i,l>}^{odd} (\cosh(K) + \sigma_i\sigma_l \sinh(K)) \end{aligned} \tag{3.4}$$

Where we have used the well known relation:

$$e^{\alpha a} = \cosh(\alpha) + a\sinh(\alpha) \quad a^2 = 1 \tag{3.5}$$

now, if we observe that the following identity holds:

$$\begin{aligned} &(\cosh(J) + \sigma\sigma'\sinh(J)) = (\cosh(J) + \sinh(J))\delta_{\sigma\sigma'} + \\ &+ (\cosh(J) - \sinh(J))\delta_{\sigma-\sigma'} = e^J\delta_{\sigma\sigma'} + e^{-J}\delta_{\sigma-\sigma'} \end{aligned} \tag{3.6}$$

and collect in each of the two products the factors containing some fixed spin τ, we are naturally led to identify the so called *Boltzmann Weights* (the subscripts R, L stand for left and right):

$$W_L\left(\begin{array}{cc|c} \tau & \sigma' & J \\ & \sigma & \end{array}\right) = (e^J\delta_{\sigma\tau} + e^{-J}\delta_{\tau-\sigma})(e^J\delta_{\sigma'\tau} + e^{-J}\delta_{\tau-\sigma'}) \tag{3.7}$$

$$W_R\left(\begin{array}{cc|c} \sigma' & \tau & K \\ \sigma & & \end{array}\right) = (e^K\delta_{\sigma\tau} + e^{-K}\delta_{\tau-\sigma})(e^K\delta_{\sigma'\tau} + e^{-K}\delta_{\tau-\sigma'}) \tag{3.8}$$

It is in terms of these weights that we are going to build the transfer matrix, this is done by observing that in terms of W_R and W_L the partition function takes the form:

$$Z_{L,L'} = \sum_{\{\sigma\}} \prod_{i=1}^{L'} \sum_{\{\tau\}} \prod_{\alpha=1}^{L} W_L\left(\begin{array}{cc|c} \tau^{(\alpha)} & \overline{\sigma}^{(\alpha,i+1)} & J \\ & \sigma^{(\alpha,i)} & \end{array}\right) W_R\left(\begin{array}{cc|c} \overline{\sigma}^{(\alpha,i+1)} & \tau^{(\alpha+1)} & K \\ \sigma^{(\alpha,i)} & & \end{array}\right) \tag{3.9}$$

This reads like:

$$Z_{L,L'} = \mathrm{Tr}\mathbf{T}^{L'} \tag{3.10}$$

with the transfer matrix element defined (for free boundaries) as:

$$T_{\{\sigma\}\{\overline{\sigma}\}} = \sum_{\{\tau\}} \prod_{\alpha=1}^{L} W_L\left(\begin{array}{cc|c} \tau^{(\alpha)} & \overline{\sigma}^{(\alpha,i+1)} & J \\ & \sigma^{(\alpha,i)} & \end{array}\right) W_R\left(\begin{array}{cc|c} \overline{\sigma}^{(\alpha,i+1)} & \tau^{(\alpha+1)} & K \\ \sigma^{(\alpha,i)} & & \end{array}\right) \tag{3.11}$$

whereas for fixed boundary conditions we have a slightly modified definition:

$$T_{\{\sigma\}\{\overline{\sigma}\}} = \sum_{\{\tau\}} W_R \left(\begin{matrix} 1 & \\ & \tau^1 \end{matrix} \middle| K \right) \left[\prod_{\alpha=1}^{L-1} W_L \left(\begin{matrix} \tau^{(\alpha)} & \\ & \sigma^{(\alpha,i)} \end{matrix} \middle| J \right) \cdot \right.$$
$$\left. \cdot W_R \left(\begin{matrix} \overline{\sigma}^{(\alpha,i+1)} & \\ & \tau^{(\alpha+1)} \end{matrix} \middle| K \right) \right] W_L \left(\begin{matrix} \tau^L & \omega \\ & \omega \end{matrix} \middle| J \right) \tag{3.12}$$

here the subscripts $\{\sigma\}\{\overline{\sigma}\}$ stand for the streak spin configurations, and if we consider a cylindrical geometry we must require $\sigma^{(\alpha,i)} = \sigma^{(\alpha,i+L)}$.
If we now consider for example fixed $(+,+)$ boundary conditions we have that $\omega = 1$ and since the $L-1$ σ in a streak can take only two values the transfer matrix has dimensions $2^{L-1} \times 2^{L-1}$, the same holds for the fixed $(+,-)$ boundary conditions but with $\omega = -1$, while for (F,F) the transfer matrix has dimensions $2^L \times 2^L$.
Now, in Baxter's book [19] it has been shown that for this model the requirement of criticality can be translated into a constraint for the coupling constants J and K which has the clearly recognizable form of a *Duality* relation, so that at the critical point (i.e. at the fixed temperature $T = T_c$) we have a new kind of duality symmetry for the anisotropic Ising Model which concerns the coupling constants.
This constraint is expressed by:

$$\sinh(2J)\sinh(2K) = 1 \tag{3.13}$$

Such a constraint can be conveniently parametrized in terms of a so called *Spectral Parameter* u in the following way:

$$\begin{cases} \sinh(2J) = \cot(2u) \\ \sinh(2K) = \tan(2u) \end{cases} \tag{3.14}$$

So that the Boltzmann Weights simplify to:

$$W_L \left(\begin{matrix} \tau & \sigma' \\ & \sigma \end{matrix} \middle| u \right) = (\cot(u)\delta_{\sigma\sigma'} + \delta_{\sigma-\sigma'})\delta_{\sigma\tau} + (\tan(u)\delta_{\sigma\sigma'} + \delta_{\sigma-\sigma'})\delta_{\sigma-\tau} \tag{3.15}$$

$$W_R \left(\begin{matrix} \sigma' & \\ \sigma & \tau \end{matrix} \middle| u \right) = (\tan(\frac{\pi}{4} - u)\delta_{\sigma\sigma'} + \delta_{\sigma-\sigma'})\delta_{\sigma\tau} + (\cot(\frac{\pi}{4} - u)\delta_{\sigma\sigma'} + \delta_{\sigma-\sigma'})\delta_{\sigma-\tau} \tag{3.16}$$

We readily observe that form the point of view of the Spectral Parameter the weights W_L and W_R possess the manifest symmetry:

$$\begin{cases} u \to \lambda - u \\ W_L \left(\begin{matrix} \tau & \sigma' \\ & \sigma \end{matrix} \middle| u \right) \longleftrightarrow W_R \left(\begin{matrix} \sigma' & \\ \sigma & \tau \end{matrix} \middle| u \right) \end{cases} \tag{3.17}$$

Which corresponds to exchanging the coupling constants $J \longleftrightarrow K$ or from a different point of view to reflection symmetry of the theory along the lattice direction which is orthogonal to our streak of spins.
Where we shall call the parameter λ *crossing parameter* and we shall refer to the symmetry above as the *crossing symmetry* of the Boltzmann weights. For this model we have $\lambda = \pi/4$.
Furthermore W_L and W_R are both periodic functions of period π, this apparently innocent observation is important in what follows since it is going to provide us with a criterion to restrict an invariance that this model seems to suggest.
It is indeed manifest that the Boltzmann weights description of this lattice model is invariant under multiplication by arbitrary functions of u; this invariance is too wide, and breaks the discrete traslational invariance $u \to u + \pi$ that the model possesses; in order to prevent such a damage we will require such arbitrary functions to satisfy the same periodicity properties of the Boltzmann Weights, so that we find out that:

$$\begin{cases} W_L \to f(u) W_L \\ W_R \to h(u) W_R \\ Z_{L,L'} \to (h(u) f(u))^{LL'} Z_{L,L'} \end{cases} \tag{3.18}$$

Now, taking into account such an arbitrariness we take:

$$\begin{cases} W_L \to \cos(2u)/\sqrt{2} W_L \\ W_R \to \sin(2u)/\sqrt{2} W_R \\ Z_{L,L'} \to (\sin(4u)/4)^{LL'} Z_{L,L'} \end{cases} \tag{3.19}$$

and the Boltzmann Weights become:

$$W_L \left(\begin{matrix} \tau & \sigma' \\ & \sigma \end{matrix} \middle| \, u \right) = (\cos^2(u)\delta_{\sigma\sigma'} + \frac{\sin(2u)}{\sqrt{2}}\delta_{\sigma-\sigma'})\delta_{\sigma\tau} + (\sin^2(u)\delta_{\sigma\sigma'} + \frac{\sin(2u)}{\sqrt{2}}\delta_{\sigma-\sigma'})\delta_{\sigma-\tau} \tag{3.20}$$

$$\begin{aligned} W_R \left(\begin{matrix} \sigma' & \\ \sigma & \tau \end{matrix} \middle| \, u \right) =& (\sqrt{2}\cos^2(\frac{\pi}{4} - u)\delta_{\sigma\sigma'} + \frac{\cos(2u)}{\sqrt{2}}\delta_{\sigma-\sigma'})\delta_{\sigma\tau} + \\ &+ (\sqrt{2}\sin^2(\frac{\pi}{4} - u)\delta_{\sigma\sigma'} + \frac{\cos(2u)}{\sqrt{2}}\delta_{\sigma-\sigma'})\delta_{\sigma-\tau} \end{aligned} \tag{3.21}$$

Now, it is possible to show [19][9] that the transfer matrices built from these weights form a commuting family in the spectral parameter u, and that, as a consequence of (3.17), they possess the symmetry:

$$\mathbf{T}(u) = \mathbf{T}(\frac{\pi}{4} - u) \tag{3.22}$$

Furthermore $\mathbf{T}(u+\pi/4)$ is proportional to the inverse of $\mathbf{T}(u)$, so that we are led to a very simple Functional equation taking the form:

$$\mathbf{T}(u)\mathbf{T}(u+\pi/4) = G_L(u)\mathbf{1} \tag{3.23}$$

where $G_L(u)$ is a function of period 2λ depending on the choice of boundary conditions, which always happens to be proportional to a real polynomial in the variable e^{4iu}.
As a consequence of the commuting property we have that the same equation holds for the eigenvalues $\Lambda(u)$ of $\mathbf{T}(u)$ The choice of gauge for the weights has been made in order to avoid $G_L(u)$ having singularities in the complex plane.
An example of $G_L(u)$ is given, for free boundaries, by the following expression [9]:

$$G_L(u) = \frac{\cos^{2(2L+1)}(2u) - \sin^{2(2L+1)}(2u)}{\cos(4\mathrm{u})} \tag{3.24}$$

we will now show how such an expression can be found to be proportional to a polynomial of degree $4L$ in the variable $z = e^{4iu}$.
First of all we notice that the following identities hold:

$$\begin{cases} \cos^2(2u) = \frac{1}{4}\Big(z + \frac{1}{z} + 2\Big) \\ \sin^2(2u) = -\frac{1}{4}\Big(z + \frac{1}{z} - 2\Big) \\ \cos(4u) = \frac{1}{2}\Big(z + \frac{1}{z}\Big) \end{cases} \tag{3.25}$$

Now, using these identities we find:

$$\begin{aligned} G_L(z) &= 4^{-2L}\frac{\Big[\Big(z+\frac{1}{z}+2\Big)^{2L+1} + \Big(z+\frac{1}{z}-2\Big)^{2L+1}\Big]}{2\Big(z+\frac{1}{z}\Big)} = \\ &= \frac{4^{-2L}\Big[\sum_{k=0}^{2L+1}\Big(z+\frac{1}{z}\Big)^k 2^{2L+1-k}\binom{2L+1}{k} - \sum_{k=0}^{2L+1}(-1)^k\Big(z+\frac{1}{z}\Big)^k 2^{2L+1-k}\binom{2L+1}{k}\Big]}{2\Big(z+\frac{1}{z}\Big)} = \\ &= 4^{-2L}\sum_{k=0}^{2L+1}\binom{2L+1}{k}2^{2L-k}\Big(z+\frac{1}{z}\Big)^{k-1}(1+(-1)^{k+1}) = \\ &= 4^{-2L}\sum_{l=0}^{L}\binom{2L+1}{2l+1}4^{L-l}\Big(z+\frac{1}{z}\Big)^{2l} = 4^{-2L}\sum_{l=0}^{L}\binom{2L+1}{2l+1}4^{L-l}\sum_{m=0}^{2l}\binom{2l}{m}z^{2(m-l)} \end{aligned} \tag{3.26}$$

So that collecting z^{-2L} we finally arrive at the expression:

$$G_L(z) = 4^{-2L}z^{-2L}\sum_{l=0}^{L}\sum_{m=0}^{2l}\binom{2L+1}{2l+1}\binom{2l}{m}z^{2(m+L-l)}4^{L-l} \tag{3.27}$$

The polynomial part of the last expression clearly has $4L$ roots in the complex plane organized in complex coniugated pairs, an by inspection it is easy to see that it has no real roots (it's a sum of quadratic terms with non negative coefficients), and furthermore since $G_L(z) = G_L(-z)$ if $\overline{z}$ is a solution also $-\overline{z}$ happens to be one.
Now we are authorized to group the solutions of opposite sign into one factor, allowing us to express $G_L(z)$ as the following product:

$$G_L(z) = 4^{-2L} z^{-2L} \prod_{r=1}^{2L} (z^2 - \gamma_r^2(L)) \tag{3.28}$$

where $\gamma_r(L)$ are the independent half of the roots indicized by an integer r.
In order to proceed further we might be tempted to look for an analytic expression for the γ_r (which has been obtained in [9]), but anyway it proves sufficient to our goals to convince ourselves that $G_L(u)$ has only imaginary zeroes, this can be done for example by numerical analisys.
Furthermore we observe that as a consequence of equation (3.22) the eigenvalues of the transfer matrix satisfy:

$$\Lambda(u) = \Lambda(\lambda - u) \tag{3.29}$$

which is the statement that the zeroes of $G_L(z)$ possess another symmetry in the z plane, since $z(\lambda - u) = -1/z(u)$ relates the zeroes inside S^1 to those outside of it, so that we are led to conclude that we have $2L$ roots inside S^1 and $2L$ roots ouside, organized in pairs $\{\gamma_r, -1/\gamma_r\}$.
So that now if $\tilde{z}$ is a zero also $-\tilde{z}$, $\tilde{z}^*$ and $-1/\tilde{z}$ have to be zeroes.
All this implies from the point of view of the variable u that the roots with positive imaginary part are mapped into the line $u(x) = \lambda/2 + ix$, while roots with negative imaginary part go into $u(x) = 3\lambda/2 + ix$ with $x \in \mathbb{R}$.
And by observing that $G_L(u + \lambda) = G_L(u)$ we have the whole pattern of zeroes which repeats with period λ, and furthermore the eigenvalues are themselves periodic functions satisfying $\Lambda(u) = \Lambda(u + 2\lambda)$.
This last observation is the key to building all the possible solutions of the Functional equation, the eigenvalues $\Lambda(u)$ are infact fully characterized by their pattern of zeroes, which must be compatible with the Functional equation, so that the roots must be organized into lines, and in such a way that traslating the pattern of λ and superposing it to the pattern itself, we get the same roots of $G_L(u)$ with the same multiplicity. This is achieved by considering only the roots lying on the line $u(x) = \lambda/2 + ix$, and choosing an arbitrary number of zeroes out of the set $\cup_{r=1}^{2L} \{\gamma_r(L)\}$ we replace each of them with a root which has the real part shifted by λ, obtaining two complementary patterns of zeroes organized on two parrallel lines of distance λ.

After all these considerations let us consider the Functional equation in the z variable:

$$\Lambda(z)\Lambda(-z) = 4^{-2L} z^{-2L} \prod_{r=1}^{2L} (z^2 - \gamma_r^2(L)) \tag{3.30}$$

The solution is now readily built by piecing up all our previous considerations; firstly we need a choice that places some part of the roots on the positive imaginary axis and the complementary on the negative part, this is done by assigning to each $\gamma_r(L)$ a + or a − sign that is by the choice of a succession $\{\mu_r\}_{r=1}^{2L}$ with $\mu_r \in \{0,1\}$ and secondly since $z(u+\lambda) = -z(u)$ in order to reproduce the z^{-2L} factor of $G_L(u)$ we need $\Lambda(u)$ to be proportional to $(iz)^{-L}$, so that we can finally write:

$$\Lambda(z) = \pm(4iz)^{-L} \prod_{r=1}^{2L} (z + \mu_r \gamma_r(L)) \tag{3.31}$$

where we observe that an overall sign ambiguity is left to be solved by the choice of normalization as $z \to 1$ which must be $\Lambda(1) = 1$ (in order to achieve $\mathbf{T}(0) = \mathbf{1}$).
Now, by grouping together the couples of terms considered $\{\gamma_r, -1/\gamma_r\}$, and requiring that their μ coefficients be the same, we are led to write:

$$\begin{aligned}
\Lambda(z) &= \pm (4iz)^{-L} \prod_{r=1}^{L} (z + \mu_r \gamma_r(L)) \Big(z - \frac{\mu_r}{\gamma_r(L)} \Big) = \\
&= \pm (4iz)^{-L} \prod_{r=1}^{L} z \Big(z - \frac{1}{z} + \mu_r \Big(\gamma_r(L) - \frac{1}{\gamma_r(L)} \Big) \Big) = \\
&= \pm \prod_{r=1}^{L} \mu_r 2^{-L} \prod_{r=1}^{L} \Big(\mu_r \frac{z^2-1}{2iz} + \frac{1}{2i} \Big(\gamma_r(L) - \frac{1}{\gamma_r(L)} \Big) \Big)
\end{aligned} \tag{3.32}$$

and requiring appropriate normalization tells us:

$$\prod_{r=1}^{L} \mu_r = \mp \tag{3.33}$$

now if we parametrize the γ_r as:

$$\gamma_r = e^{i\omega_r} \tag{3.34}$$

and notice that

$$\frac{z^2-1}{2iz} = \sin(4u) \tag{3.35}$$

we finally arrive at the expression:

$$\Lambda(u) = 2^{-L} \prod_{r=1}^{L} (\mu_r \sin(4u) + \sin(\omega_r)) \tag{3.36}$$

which provides us the solution of the Functional equation, and furthermore considering all the possible choices of the overall sign mentioned above, and the constraint (3.33) we find out that we have 2^L independent eigenvalues so that we have the right dimensionality for $\mathbf{T}$ with free boundary conditions.
This method of solution can be applied with very slight modifications to the other choices of boundary conditions, since it is essentially based on the quasi-polynomial nature of $G_L(z)$ and its symmetries, the only laborious task being the explicit calculation of G itself from the transfer matrix, a task that has anyway been carried out in [9].

3.3 $\mathbf{A}_n$ *Models*

In order to reach a generalization of the previous section we are going to spend some words to introduce the $\mathbf{A}_n$ Models which are Ising-like square lattice models where to each lattice site j is assigned a height variable $a_j \in \{1, 2, \dots n\}$, with $n \geq 3$ (n is an integer that characterizes the model).
The local height variables a_j being constrained to satisfy the adjacency rules (for $< i, j >$ nearest neighbors):

$$\begin{cases} 0 < a_i \leq n \\ |a_i - a_j| \leq 1 \\ 1 < a_i + a_j < 2n + 1 \end{cases} \tag{3.37}$$

Such models possess a transfer matrix description (and it is indeed the fundamental approach to the model) and such a transfer matrix happens to be built out of *Boltzmann Weights* W which are given in terms of four-height interactions around each plaquette of the lattice (known infact as Interaction Round a Face or IRF), depending on a spectral parameter u.
Here the analogy to the previous section is not between the W and the W_L or W_R introduced previously but with the product of the two factors built up so that the σ in the two terms coincide, so that what we are now calling a Boltzmann Weight in the previous language would look as:

$$W\left(\begin{array}{cc|c} \sigma & \overline{\tau} & \\ \tau & \overline{\sigma} & u \end{array}\right) = W_L\left(\begin{array}{cc|c} \tau & \overline{\sigma} & \\ & \sigma & u \end{array}\right) W_R\left(\begin{array}{cc|c} \overline{\sigma} & & \\ \sigma & \overline{\tau} & u \end{array}\right) \tag{3.38}$$

Where it is understood that the above analogy holds only for the specific example introduced in the previous section, and has only the goal of making contact with what has been said so far.
In this new framework we will define three-height interactions (somewhat analogous to W_L and W_R) only for boundary interactions, and we will call them B_L and B_R; for a complete review of the most general case it is advisable to look at [15].

It is also possible to identify a crossing parameter λ taking the general form:

$$\lambda = \frac{\pi}{n+1} \tag{3.39}$$

so that the transfer matrix element can be defined on a lattice of width N [15] as:

$$\begin{aligned}\left\langle a_1 \dots a_{N+1}\right|\mathbf{T}(u)\left|b_1 \dots b_{N+1}\right\rangle = \sum_{c_1 \dots c_{N+1}} B_L \left(\begin{matrix} b_1 & \\ a_1 & \end{matrix} \middle| \begin{matrix} c_1 \\ \end{matrix} \middle| \lambda - u \right) \cdot \\ \cdot \left[\prod_{j=1}^{N} W \left(\begin{matrix} c_j & c_{j+1} \\ a_j & a_{j+1} \end{matrix} \middle| u \right) W \left(\begin{matrix} b_j & b_{j+1} \\ c_j & c_{j+1} \end{matrix} \middle| \lambda - u \right) \right] B_R \left(\begin{matrix} & b_{N+1} \\ c_{N+1} & \end{matrix} \middle| \begin{matrix} \\ a_{N+1} \end{matrix} \middle| u \right)\end{aligned} \tag{3.40}$$

we now observe that for $n = 3$ we have $\lambda = \pi/4$ which is, as pointed out in the previous section, the crossing parameter for the Anisotropic Ising Model, this is not a coincidence since the peculiar choice of the rotated lattice used to solve it is a consequence of the adjacency conditions (3.37) which, choosing as boundary condition a sequence of heights having alternately $a = 2$ and $a = 1, 3$, causes an even (i.e. with all the $a = 2$) and an odd sublattice to decouple; the remaining 2 state model on the odd sublattice is infact isomorphic to the Ising Model, and the only way to draw the links between the sites avoiding the even sublattice is to draw them rotated at an angle of $\pi/4$.

It has been shown [14][15][16] that the $\mathbf{A}_n$ models are subject to a second order phase transition, and that their operator content falls within the universality class of a Unitary Conformal Field Theory, more specifically of a Minimal Model of Central Charge

$$c = 1 - \frac{6}{n(n+1)} \tag{3.41}$$

Anyway we want to point out that in general the $\mathbf{A}_n$ Models can be studied even in the off critical region, since the Boltzmann Weights are introduced in a way that contemplates the presence of a temperature-like variable q, which plays the role of a *modular* parameter for the Elliptic θ functions which are the elementary tools that are used to build the Boltzmann Weights. Such a modular parameter is related to the reduced temperature $t = (T - T_c)/T$, that controls the criticality of the theory by the relation

$$t = q^2 \tag{3.42}$$

Now, as a consequence of the adjacency rules we have that the only non zero Boltzmann Weights are [15]:

$$\begin{aligned} W \left(\begin{matrix} a \pm 1 & a \\ a & a \mp 1 \end{matrix} \middle| u \right) &= \frac{\theta_1(\lambda - u, q)}{\theta_1(\lambda, q)} \\ W \left(\begin{matrix} a & a \pm 1 \\ a \mp 1 & a \end{matrix} \middle| u \right) &= \frac{\sqrt{\theta_1((a-1)\lambda, q)\theta_1((a+1)\lambda, q)}}{\theta_1(a\lambda, q)} \frac{\theta_1(u, q)}{\theta_1(\lambda, q)} \end{aligned} \tag{3.43}$$

$$W\left(\begin{array}{cc} a & a\pm 1 \\ a\pm 1 & a \end{array}\middle| u\right) = \frac{\theta_1(a\lambda \pm u, q)}{\theta_1(a\lambda, q))}$$

where we notice that we have introduced the Elliptic Theta Function θ_1, which is defined (together with the other 3 elliptic functions) through the formula:

$$\begin{cases} \theta_1(u,q) = 2q^{1/4}\sin(u)\prod_{n=1}^{\infty}(1-2q^{2n}\cos(2u)+q^{4n})(1-q^{2n}) \\ \theta_2(u,q) = 2q^{1/4}\cos(u)\prod_{n=1}^{\infty}(1+2q^{2n}\cos(2u)+q^{4n})(1-q^{2n}) \\ \theta_3(u,q) = \prod_{n=1}^{\infty}(1+2q^{2n-1}\cos(2u)+q^{2(2n-1)})(1-q^{2n}) \\ \theta_4(u,q) = \prod_{n=1}^{\infty}(1-2q^{2n-1}\cos(2u)+q^{2(2n-1)})(1-q^{2n}) \end{cases} \tag{3.44}$$

an exhaustive overlook of all the lore about these functions can be found in [20], for the time being we will content ourselves with pointing out that the function θ_1 (and similarly θ_2) enjoys the following quasiperiodicity properties which will prove useful in the following sections:

$$\begin{cases} \theta_1(u+\pi,q) = -\theta_1(u,q) \\ \theta_1(u - i\log(q), q) = -\frac{e^{-2iu}\theta_1(u,q)}{q} \end{cases} \tag{3.45}$$

As a consequence of their definition the Boltzmann Weights enjoy rotation and relection symmetries:

$$W\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle| u\right) = W\left(\begin{array}{cc} d & a \\ c & b \end{array}\middle| u\right) = W\left(\begin{array}{cc} b & c \\ a & d \end{array}\middle| u\right) = W\left(\begin{array}{cc} b & a \\ c & d \end{array}\middle| u\right) \tag{3.46}$$

crossing symmetry:

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array}\middle| u\right) = \sqrt{\frac{\theta_1(a\lambda,q)\theta_1(c\lambda,q)}{\theta_1(b\lambda,q)\theta_1(d\lambda,q)}}W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \lambda - u\right) \tag{3.47}$$

and height reversal:

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array}\middle| u\right) = W\left(\begin{array}{cc} L+1-d & L+1-c \\ L+1-a & L+1-b \end{array}\middle| u\right) \tag{3.48}$$

And finally we define the non zero boundary weights B_L and B_R as:

$$B_L\left(\begin{array}{cc} a & a\mp 1 \\ a & \end{array}\middle| u\right) = \sqrt{\frac{\theta_1((a\mp 1)\lambda,q)}{\theta_1(a\lambda,q)}}\frac{\theta_1(u\mp\xi_L(a),q)\theta_1(u\pm a\lambda\pm\xi_L(a),q)}{\theta_1^2(\lambda,q)} \tag{3.49}$$

$$B_R\left(\begin{array}{cc|c} a\mp 1 & a & u \\ & a & \end{array}\right) = \sqrt{\frac{\theta_1((a\mp 1)\lambda,q)}{\theta_1(a\lambda,q)}}\frac{\theta_1(u\mp\xi_R(a),q)\theta_1(u\pm a\lambda\pm\xi_R(a),q)}{\theta_1^2(\lambda,q)} \tag{3.50}$$

where $\xi_L(a)$ and $\xi_R(a)$ are arbitrary parameters.

3.4 The $\mathbf{A}_3$ Model

The $\mathbf{A}_3$ Model is obtained from the $\mathbf{A}_n$ Models with $n = 3$ and has a crossing parameter $\lambda = \pi/4$.

If we consider the transfer matrix as defined in (3.40) through the Boltzmann Weights (3.43) and boundary weights (3.49), it is possible to find out that, as a consequence of the quasiperiodicity properties (3.45) of the elliptic functions, the matrix itself (which following the notation of [15] from now on we shall call $\mathbf{D}(u)$ instead of $\mathbf{T}(u)$) enjoys the following periodicities:

$$\mathbf{D}(u+\pi) = \mathbf{D}(u) \tag{3.51}$$

$$\mathbf{D}(u - i\log(q)) = \mathbf{D}(u) \tag{3.52}$$

As a consequence we have that $\mathbf{D}$ is a doubly periodic function which is completely defined by its analytic properties inside a rectangle that we may take as:

$$(-\frac{\lambda}{2},\frac{7}{2}\lambda)\times i(\frac{1}{2}\log(q),-\frac{1}{2}\log(q)) \tag{3.53}$$

Furthermore, by using the definition of $\mathbf{D}$ it is possible to show that it satisfies a Functional equation of the form:

$$\begin{aligned}\mathbf{D}(u)\mathbf{D}(u+\lambda) &= \Big(\mathbf{1}+\mathbf{d}(u)\Big)\underbrace{\left[\frac{\theta_1(u-\lambda)\theta_1(u+\lambda)}{\theta_1(\lambda)^2}\right]^{2N}}_{F_N(u)}\underbrace{\frac{\theta_1(2u-2\lambda)\theta_1(2u+2\lambda)}{\theta_1(2u-\lambda)\theta_1(2u+\lambda)}\,\alpha_0^1(u)\beta_0^1(u)}_{S(u,\xi_L,\xi_R)} \\ &= \Big(\mathbf{1}+\mathbf{d}(u)\Big)\;F_N(u)\;S(u,\xi_L,\xi_R) \qquad (3.54)\\ &= \frac{1}{\theta_1(2u-\lambda)\theta_1(2u+\lambda)}\Bigg\{\left[\frac{\theta_1(u-\lambda)\theta_1(u+\lambda)}{\theta_1(\lambda)^2}\right]^{2N}\theta_1(2u-2\lambda)\cdot \\ &\quad\cdot\theta_1(2u+2\lambda)\alpha_0^1(u)\beta_0^1(u)+(-1)^N\left[\frac{\theta_1(u)\theta_1(u-2\lambda)}{\theta_1(\lambda)^2}\right]^{2N}\theta_1(2u)^2\frac{\alpha_1^3(u)}{\beta_{-2}^1(u)}\Bigg\}\end{aligned}$$

where $\mathbf{d}$ is a matrix proportional to the identity that takes the form:

$$\mathbf{d}(u) = \mathbf{1}\left[\frac{\theta_1(u)\theta_1(u-2\lambda)}{\theta_1(u-\lambda)\theta_1(u+\lambda)}\right]^{2N}(-1)^N\frac{\theta_1(2u)^2}{\theta_1(2u-2\lambda)\theta_1(2u+2\lambda)}\,\frac{\alpha_1^3(u)}{\alpha_0^1(u)\beta_0^1(u)\beta_{-2}^1(u)}. \tag{3.55}$$

Such a matrix satisfies a functional equation which for obvious reasons is called the Inversion Equation:

$$\mathbf{d}(u)\mathbf{d}(u+\lambda) = \mathbf{1} \tag{3.56}$$

As a consequence of the simple form of the $\mathbf{d}$ matrix (which for more complicated models is not diagonal but is expressed in terms of $\mathbf{D}$ itself), we have that the Functional equation written in terms of the eigenvalues D of $\mathbf{D}$ is independent of the eigenvalue under consideration.
We still have to define the $\alpha_k^r(u)$ and the $\beta_k^r(u)$ that appear in $\mathbf{D}$'s definition, this is readily done through:

$$\alpha_k^r(u,q,a_L,a_R) = \theta_k^r(u-\xi_L(a_L),q)\theta_k^r(u+\xi_L(a_L),q)\theta_k^r(u-\xi_R(a_R),q)\theta_k^r(u+\xi_R(a_R),q) \tag{3.57}$$

$$\begin{aligned}\beta_k^r(u,q,a_L,a_R) =& \theta_{k-a_L}^r(u-\xi_L(a_L),q)\theta_{k+a_L}^r(u+\xi_L(a_L),q)\cdot \\ &\cdot \theta_{k-a_L}^r(u-\xi_R(a_R),q)\theta_{k+a_R}^r(u+\xi_R(a_R),q)\end{aligned} \tag{3.58}$$

where

$$\theta_k^r(u,q) = \frac{\prod_{j=0}^{r-1}\theta_1(u+k\lambda-j\lambda,q)}{\theta_1(\lambda,q)} \tag{3.59}$$

3.4.1 Boundary Conditions and CFT Operator Content

As pointed out when introducing the $\mathbf{A}_n$ Models in (3.41), it happens that the critical $\mathbf{A}_3$ model falls within the universality class of a unitary $c = 1/2$ Conformal Field Theory which is infact the $\mathcal{M}_{3,4}$ Minimal Model.
In general each value of u should correspond to some deformed theory whose Continuum Scaling Limit arises from a perturbation of $\mathcal{M}_{3,4}$, in particular the isotropic theory is achieved when $u = \lambda/2$ as shown in [15] and essentially the reason resides in the crossing symmetry of the Boltzmann Weights.
The criticality of the theory is controlled by the modular parameter $q = \sqrt{t}$, the critical regime being reached for $q \to 0^+$ while the high temperature massive phase is approached as $q \to 1^-$.
So that, appealing to the first section of chapter 2, we are naturally led to a correspondence between the boundary conditions of the lattice model and the operator content of the underlying CFT.
Such a correspondence can be achieved by fixing one boundary and letting the opposite boundary assume in sequence all the three allowed configurations, the only unclear matter being wether there should be any strong prescription in the choice of the fixed boundary.
By fixing the left boundary in a $\{1,2,1,2,1,\ldots\}$ configuration which we shall call a $(+)$ boundary, naming the $\{3,2,3,2,\ldots\}$ configuration $(-)$ and the $\{2,\pm1,2,\ldots\}$ (F) (which stands for free

boundary), we postulate the following correspondence with the CFT operator content:

$$\begin{array}{cccc} \text{spin} & \text{height} & \text{boundary cond.} & (r,s) & \Delta_{r,s} \\ + & 1,2,1 & (+,+) & (1,1) & 0 \\ - & 3,2,3 & (+,-) & (2,1) & \frac{1}{2} \\ F & 2,2\pm 1,2 & (+,F) & (1,2) & \frac{1}{16} \end{array} \tag{3.60}$$

we shall see in the next chapter that such a correspondence, at least for $(+,+)$ and $(+,-)$ boundaries shall prove to be correct.
The choice of a $(+)$ left boundary implies in terms of boundary weights the following conditions[15]:

$$a=1, \qquad \xi_L=-\frac{\lambda}{2}, \qquad B_L\left(\begin{array}{cc|} 1 & 1 \\ 1 & 2 \end{array} u,\xi_L\right)=\sqrt{\frac{\theta_1(2\lambda)}{\theta_1(\lambda)}}\,\frac{\theta_1(u+\xi_L)\,\theta_1(u-\lambda-\xi_L)}{\theta_1(\lambda)^2} \tag{3.61}$$

The right boundary will then be able to assume the following conditions:

$$\begin{aligned} +: &\quad a=1, \quad \xi_R=-\tfrac{\lambda}{2}, \quad B_R\left(\begin{array}{cc|} 2 & 1 \\ & 1 \end{array} u,\xi_R\right)=\sqrt{\frac{\theta_1(2\lambda)}{\theta_1(\lambda)}}\,\frac{\theta_1(u+\xi_R)\,\theta_1(u-\lambda-\xi_R)}{\theta_1(\lambda)^2} \\ -: &\quad a=3, \quad \xi_R=\tfrac{5}{2}\lambda, \quad B_R\left(\begin{array}{cc|} 2 & 3 \\ & 3 \end{array} u,\xi_R\right)=\sqrt{\frac{\theta_1(2\lambda)}{\theta_1(3\lambda)}}\,\frac{\theta_1(u-\xi_R)\,\theta_1(u+3\lambda+\xi_R)}{\theta_1(\lambda)^2} \\ F: &\quad a=2, \quad \xi_R=??, \quad B_R\left(\begin{array}{cc|} 2\pm 1 & 2 \\ & 2 \end{array} u,\xi_R\right)=\sqrt{\frac{\theta_1((2\pm 1)\lambda)}{\theta_1(2\lambda)}}\,\frac{\theta_4(u\pm\xi_R)\,\theta_4(u\mp 2\lambda\mp\xi_R)}{\theta_4(\lambda)^2} \end{aligned} \tag{3.62}$$

where the question mark in place of the value of ξ_R for (F) boundary has been placed because of the fact that we presumed it to be arbitrary (specifically to assume the value 0), but the results of our analisys, while supporting the arbitrariness of ξ_R anyway seem to suggest that there might be something wrong with the operator content assignment, so that the doubt is cast on the $(+,F)$ sector.
A possible solution of this problem (anyway so far *not* yet solved) may reside in the fact that the conformal *fusion rules* might give a prescription requiring the left boundary to be fixed in a configuration different from $(+)$ (possibly (F)).
Finally we point out that the adjacency rules constrain the number of faces in a row so that the $(+,+)$ and $(+,-)$ boundaries must have $N=2L$ while in the $(+,F)$ case we must have $N=2L+1$ (L represents the number of faces in a row).

3.4.2 $D(u)$'s Zeroes Structure

Let us now call $D(u)$ the eigenvalues of $\mathbf{D}(u)$ and let us define $\mathcal{F}(u)$ through:

$$\mathcal{F}(u) = \Big(1 + d(u)\Big) F_N(u) S(u, \xi_L, \xi_R) \tag{3.63}$$

so that we can write the Functional equation as:

$$D(u, q) D(u + \lambda, q) = \mathcal{F}(u, q) \tag{3.64}$$

where the homogeneous term $\mathcal{F}(u)$ which explicitly depends on the choice of boundary conditions is a doubly periodic function on the cell (3.53).
Such an equation shares many properties with the particular case (3.23), and indeed if we fix $(+,+)$ or $(+,-)$ boundaries it is possible to show that in the limit $q \to 0$, apart from a gauge dependent factor the two homogeneous terms $\mathcal{F}(u)$ and $G_L(u)$ are equal and thus share the same analytic structure, which being both analytic functions means that they have the same zeroes.
Furthermore $\mathcal{F}(u)$ and $G_L(u)$ share the π periodicity property, and the eigenvalues $D(u)$ are subject to crossing symmetry:

$$D(u) = D(\lambda - u) \tag{3.65}$$

specifically if we consider $(+,+)$ or $(+,-)$ boundaries we can enforce the periodicity of $\mathcal{F}(u)$ by asserting that its zeroes are organized on the same lines as $G_L(u)$, so that they repeat with a periodicity of λ (this result comes from numerical computation of $\mathcal{F}(u)$'s zeroes).
This is sufficient to state that the topological structure of the zeroes of $\mathcal{F}(u)$ must be characterized in the same way as that of $G_L(u)$ even in the off critical region, the only real differences being the exact position of the zeroes, and the fact that the cell periodicity for $q \neq 0$ reduces in the limit $q \to 0$ to a strip periodicity (since $\log(q) \to -\infty$).
Anyway it is interesting to point out that the task of numerically computing $\mathcal{F}(u)$'s zeroes is indeed possible thanks to the fact that we are can restrict our search to a compact region, while in the critical limit, although the region is no longer compact we can count on the quasi polynomial structure of $G_L(u)$ which makes the problem once again solvable.
Furthermore we point out that as a consequence of the periodicity properties of the elliptic θ functions $\mathcal{F}(u)$ is the same for both $(+,+)$ and $(+,-)$ boundaries so that the solutions corresponding to the different sectors will have just a different combinatorial characterization of the zeroes.
Now, in the $(+,+)$ and $(+,-)$ sectors $\mathcal{F}(u)$ has $N/2 = L$ zeroes on the line $u = \lambda/2 + ix$ with $x \in (0, -1/2 \log q)$ (let's recall we have complex coniugation symmetry), while in the $(+,F)$ sector we have $(N-1)/2 = L$ zeroes.

In all the sectors the zeroes are organized in topological structures called *1-strings* and *2-strings*. 1-strings are just single zeroes of real part $\lambda/2$ and imaginary part $0 < v_k < -1/2 \log q$ such that:

$$D(\lambda/2 \pm iv_k) = 0 \quad k = 1, \ldots, m \tag{3.66}$$

where m denotes the number of 1-strings, while 2 strings are couples of zeroes sharing the same imaginary part, while their real part takes the values $\lambda/2 \pm \lambda$, and we shall call their number n. Now, the number of all zeroes configurations sharing the same number m of 1-strings and satisfying the constraint

$$n + m = L \tag{3.67}$$

is simply given by the corresponding binomial coefficient, so that the total number of configurations is:

$$\sum_{m=0}^{L} \binom{L}{m} = 2^L \tag{3.68}$$

furthermore, if we want to count all the configurations with m even we have (when L is also even):

$$\sum_{k=0}^{L/2} \binom{L}{2k} = 2^{L-1} \tag{3.69}$$

and similarly we have 2^{L-1} odd-m configurations, so that in order to get transfer matrices with the right dimensionality it is natural to put a constraint on the number of 1-strings in the $(+,+)$ and $(+,-)$ sectors which sounds as:

$$\begin{aligned} &\text{B.C.} \qquad m \\ &(+,+) \rightarrow \text{ even} \\ &(+,-) \rightarrow \text{ odd} \end{aligned} \tag{3.70}$$

while in the $(+,F)$ sector there is no need of constraining the 1-string content since all the configurations already suffice to generate a transfer matrix with the right 2^L dimensionality.
For all choices of boundary conditions anyway it is possible to give a characterization of the state (i.e. transfer matrix eigenvalue) in terms of a decreasing sequence of numbers $\{I_1, I_2, \ldots, I_m\}$ called *topological numbers* which express the position of 1-strings related to the position of 2-strings.
Each I_k tells us how many 2-strings the k-th 1-string has to go through in order to reach its position in the pattern of zeroes starting from the configuration where all the 2-strings are heaped on the bottom, so that ordering the imaginary part of the 1-strings v_k into an increasing sequence $\{v_k\}_{k=1}^m$ we have that the topological numbers $\{I_k\}$ must necessarily arrange into a

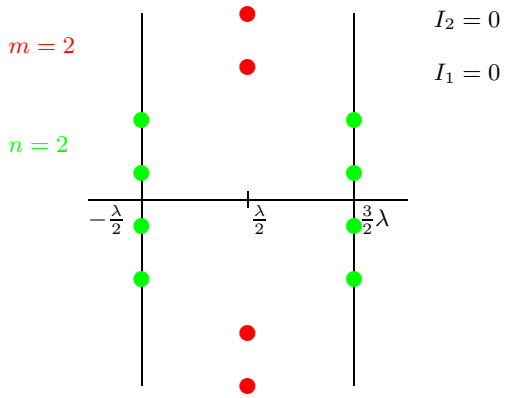

Figure 3.1 Example of the structure of zeroes labeled by the topological number {0,0}

decreasing succession.
Clearly, the I_k have to satisfy the following constraint:

$$I_k \leq n \ \forall k \tag{3.71}$$

Such a characterization of the eigenvalues in terms of 1-strings and 2-strings also happens to give us a natural criterion for ordering the states, first of all we order the states by their increasing m value, the ordering between equal m states is done so that the state with all the 2-strings at the bottom of the tower comes first, and then each time a 2-string is pushed over a 1-string the "energy" increases by one "unit".

Such an energy might be called more appropriately be called a *combinatorial* energy which we shall denote by E.
It is possible to show that the combinatorial energy of a generic state is given by:

$$E = E_{base} + \sum_{k=1}^{m} I_k \tag{3.72}$$

where the energy of the base is given for $(+,+)$ and $(+,-)$ boundaries by:

$$E_{base} = \frac{m^2}{2} \tag{3.73}$$

while for $(+,F)$ boundaries it should take the form:

$$E_{base} = \frac{1}{16} + \frac{m(m+1)}{2} \tag{3.74}$$

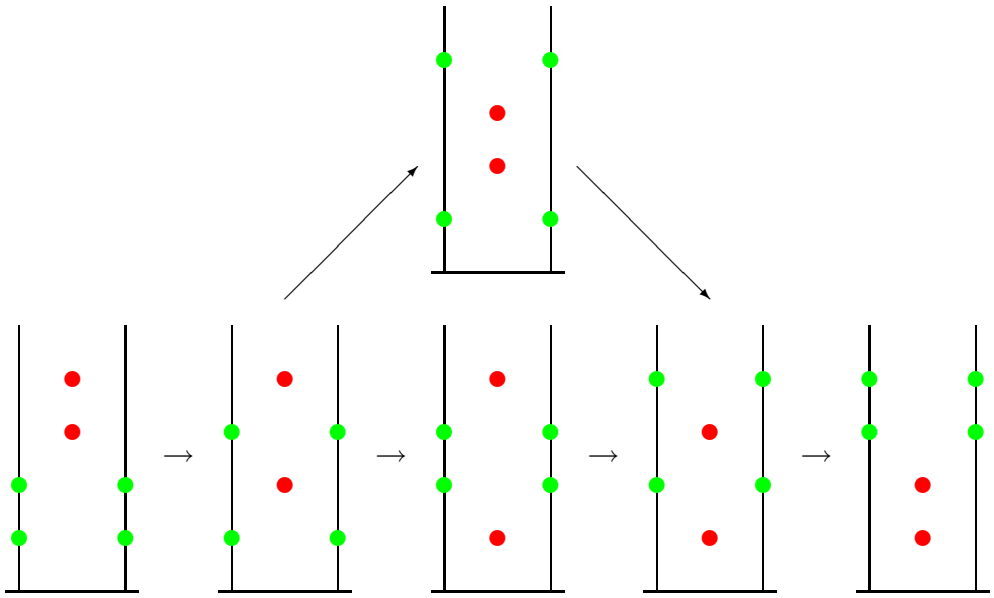

Figure 3.2 An example of the tower of states built on the base state with $m = 2$ of Fig.3.1, the numbers represent the increment of energy with respect to the base. (Figure shown with Courtesy of Prof.P.Pearce)

The only difficult part being a rigorous justification of the formulas given for E_{base}.
About the matter it is interesting to notice that since $\{\}$ and $\{0\}$ are the ground states respectively of the $(+,+)$ and $(+,-)$ sectors, the base energy in those cases takes respectively the values 0 and 1/2 which happen to be the supposed conformal weights of the undelying CFT operator content; in analogy for the $(+,F)$ sector we have $\{\}$ as a ground state and the base energy takes the value 1/16.
Such observations partially convince us that we are dealing with the right expression for the energy, since at the isotropic point and in the continuum limit the energy of the corresponding CFT is indeed given by:

$$E = L_0 - \frac{c}{24} \tag{3.75}$$

which not considering the vacuum shift $\frac{c}{24}$ is in agreement with the combinatorial energy at least for the ground states, while the agreement for the excited states shall become more and more precise as $N \to \infty$ and going to the critical point $q \to 0$.
In the next chapter we shall have more to say about how to take the continuum limit of lattice states.

3.4.3 Solution of the Functional Equation: the TBA

First of all let us recall the form of the Functional equation (3.64), we then define an x coordinate in the following way:

$$u = \frac{\lambda}{2} + \frac{i}{4}x, \qquad D_1(x) \stackrel{\text{def}}{=} D(u) \tag{3.76}$$

where we are going to solve (3.64) for the following values of x:

$$x \in (2\log q, -2\log q) \tag{3.77}$$

for convenience we will rewrite (3.64) after applying a traslation:

$$u \to u - \frac{\lambda}{2}: \qquad D(u - \frac{\lambda}{2})\ D(u + \frac{\lambda}{2}) = \mathcal{F}(u - \frac{\lambda}{2}) \tag{3.78}$$

we then use (3.76) to write (3.64) in the following form:

$$u = \frac{\lambda}{2} + \frac{i}{4}x, \quad D_1(x + i\frac{\pi}{2})\ D_1(x - i\frac{\pi}{2}) = \mathcal{F}(u - \frac{\lambda}{2}) = \mathcal{F}(\frac{i}{4}x) \stackrel{\text{def}}{=} \mathcal{F}_1(x) \tag{3.79}$$

At this point we could be tempted to follow the solution method used in [16] and try to Fourier-expand the logaritmic derivative of (3.79), anyway before being allowed to do so, we have to remove the zeroes of $D_1(x)$ in order to deal with an analytic function for which a Fourier expansion does make sense.
Now, if we consider what has been said in the previous section about the position of the zeroes it is manifest that inside the rectangle $|\mathrm{Im}(x)| < \pi,\ |\mathrm{Re}(x)| < -2\log q$ the function $D_1(x)$ has roots only for real x which are infact due to the presence of 1-strings.
In order to reach our result we define the function

$$p(x, v_k) = \frac{\theta_1(\frac{i}{2}(x - 4v_k), t)}{\theta_2(\frac{i}{2}(x - 4v_k), t)} = \frac{\theta_1(\frac{i}{2}(x - 4v_k), q^2)}{\theta_2(\frac{i}{2}(x - 4v_k), q^2)} \tag{3.80}$$

where t is the reduced temperature already introduced previously.
We observe that the p function happens to satisfy the same equation as $D_1(x)$:

$$p(x + i\frac{\pi}{2}, v_k)\ p(x - i\frac{\pi}{2}, v_k) = 1 \tag{3.81}$$

furthermore we observe that p can be used to collect all the zeroes of D_1 through the product:

$$\prod_{k=1}^{m} p(x, v_k)p(x, -v_k) \tag{3.82}$$

so that we can surely assert that the function

$$D_{\mathrm{ANZ}}(x) \stackrel{\text{def}}{=} \frac{D_1(x)}{\prod_{k=1}^{m} p(x, v_k)p(x, -v_k)} = D_1(x) \prod_{k=1}^{m} \frac{\theta_2(\frac{i}{2}(x - 4v_k), t)}{\theta_1(\frac{i}{2}(x - 4v_k), t)}\ \frac{\theta_2(\frac{i}{2}(x + 4v_k), t)}{\theta_1(\frac{i}{2}(x + 4v_k), t)} \tag{3.83}$$

does not have zeroes inside the rectangle $|\mathrm{Im}(x)| < \pi,\ |\mathrm{Re}(x)| < -2\log q$ (ANZ stands for analytic and not zero).
We then observe that as a consequence of (3.81)D_{ANZ} still satisfies the Functional equation:

$$D_{\mathrm{ANZ}}(x+i\frac{\pi}{2})\ D_{\mathrm{ANZ}}(x-i\frac{\pi}{2}) = \mathcal{F}_1(x) \tag{3.84}$$

so that now we can safely consider the logarithmic derivative:

$$[\log D_{\mathrm{ANZ}}(x+i\frac{\pi}{2})]' + [\log D_{\mathrm{ANZ}}(x-i\frac{\pi}{2})]' = [\log \mathcal{F}_1(x)]' \tag{3.85}$$

and, considering the Fourier expansion of $[\log(D_{ANZ})]'$ we have

$$[\log(D_{ANZ}(x))]' = \sum_{k=-\infty}^{\infty} C_k e^{\frac{ik\pi x}{2\log q}} \tag{3.86}$$

$$C_k = -\frac{1}{4\log q}\int_{2\log q}^{-2\log q} [\log(D_{ANZ}(x))]' e^{-\frac{ik\pi x}{2\log q}} dx \tag{3.87}$$

and similarly calling F_k the Fourier coefficients of $[\log \mathcal{F}_1(x)]'$ we arrive at the following equation for the coefficients:

$$\left[e^{-\frac{k\pi^2}{4\log q}} + e^{\frac{k\pi^2}{4\log q}}\right] C_k = F_k \tag{3.88}$$

which tells us that:

$$[\log(D_{ANZ}(x))]' = \sum_{k=-\infty}^{\infty} \frac{e^{\frac{ik\pi x}{2\log q}}}{e^{-\frac{k\pi^2}{4\log q}} + e^{\frac{k\pi^2}{4\log q}}} F_k \tag{3.89}$$

so that substituting the integral expression for F_k we get:

$$[\log(D_{ANZ}(x))]' = -\frac{1}{4\log q}\int_{2\log q}^{-2\log q} [\log \mathcal{F}_1(y)]' dy \sum_{k=-\infty}^{\infty} \frac{e^{\frac{ik\pi(x-y)}{2\log q}}}{e^{-\frac{k\pi^2}{4\log q}} + e^{\frac{k\pi^2}{4\log q}}} \tag{3.90}$$

now, naming the summation term as:

$$k(x-y) = -\frac{1}{4\log q}\sum_{k=-\infty}^{\infty} \frac{e^{\frac{ik\pi(x-y)}{2\log q}}}{e^{-\frac{k\pi^2}{4\log q}} + e^{\frac{k\pi^2}{4\log q}}} \tag{3.91}$$

we are led to write

$$[\log(D_{ANZ}(x))]' = k * [\log \mathcal{F}_1]'(x) \tag{3.92}$$

which, by using the properties of convolution to move the $(x-y)$ argument from the kernel k to $[\log \mathcal{F}_1]'$, can be integrated to yield:

$$\log(D_{ANZ}(x)) = k * \log \mathcal{F}_1(x) + D \tag{3.93}$$

where D is some integration constant which we will not need to know.
Now, restoring the zeroes we arrive at the solution describing the eigenvalues on the line $u = \lambda/2 + ix/4$:

$$\log D_1(x) = \sum_{k=1}^{m} \log[p(x, v_k)p(x, -v_k)] + k * \log \mathcal{F}_1 + D \tag{3.94}$$

The convolution kernel $k(x)$ has been computed in [16] in terms of Elliptic θ functions to have the expression:

$$k(x, q) = \frac{\theta_2(0, q^4)\theta_3(0, q^4)\theta_3(ix, q^4)}{2\pi\theta_2(ix, q^4)} \tag{3.95}$$

Finally if we recall $\mathcal{F}(u)$'s definition we can write:

$$\log \mathcal{F}_1(x) = \log \mathcal{F}(\frac{i}{4}x) = \log\left(1 + d(\frac{i}{4}x)\right) + \log F_N(\frac{i}{4}x) + \log S(\frac{i}{4}x, \xi_L, \xi_R) \tag{3.96}$$

so that remembering the expression for $F_N(\frac{i}{4}x)$ we observe that its logarithm gives a contribution proportional to N and thus can be identified with a *Bulk Energy* term (which diverges in the thermodynamic limit $N \to \infty$), while $S(\frac{i}{4}x, \xi_L, \xi_R)$ gives a contribution independent of N which anyway diverges as we approach the critical regime ($q \to 0$) and can thus be identified with a *Surface Energy*.
Now, since in the next chapter we are going to deal with the continuum limit of $\log D(u)$ (which consists both of $N \to \infty$ and $q \to 0$), it is natural to conclude by defining a subtracted Energy so as to give rise to meaningful quantities in the continuum limit:

$$\log D_{\text{finite}}(x) \stackrel{\text{def}}{=} \log D_1(x) - k * \log\left[F_N(\frac{i}{4}x)S(\frac{i}{4}x, \xi_L, \xi_R)\right] \tag{3.97}$$

and, more explicitly:

$$\log D_{\text{finite}}(x) = \sum_{k=1}^{m} \log[p(x, v_k)p(x, -v_k)] + k * \log\left(1 + d(\frac{i}{4}x)\right) \tag{3.98}$$

Where we observe that the constant of integration D has finally disappeared in the subtraction, corresponding to a shift in the vacuum energy.

4. Integrals Of Motion: Lattice-Conformal Correspondence

4.1 Scaling Limit of the $\mathbf{A}_3$ Model

Taking the *Continuum Scaling Limit* of a statistical mechanical model means considering its critical behaviour in the thermodynamical limit.
Such a double limit ($N \to \infty$,$t \to 0$) can in principle be computed along infinite paths, however it is possible to show [16] that only one prescription allows us to obtain a nontrivial (i.e. not 0 and not ∞) limit for the subtracted Energy $\log D_{finite}$, and such a prescription has the form:

$$N \to \infty, \quad t \to 0 \quad | \quad \mu = N\ |t|^{\nu} = \text{fixed} \tag{4.1}$$

where $\nu = 1$ is the critical exponent of the correlation length $\xi \sim 1/t^{\nu}$ of the model and the variables t and μ are unerstood as dimensionless.
If we now introduce a lattice spacing a, we can consider the lattice length to be $R = Na$; furthermore $a^{-}1$ has mass dimension and so does the quantity $4|t|^{\nu}/a$, and in order to obtain a finite and dimensionless μ we may think the scaling limit to be obtained as:

$$R = \lim_{N\to\infty,\ a\to 0} N\ a\ , \qquad m = \lim_{t\to 0,\ a\to 0} \frac{4|t|^{\nu}}{a} \tag{4.2}$$

So that now as a consequence of (4.1) the only surviving parameter will be μ wich will gain a physical meaning through the equation:

$$\mu = \frac{1}{4} mR \tag{4.3}$$

We shall denote such a way of computing the limit as csl (standing for continuum scaling limit).
Now, as we approach the critical behaviour the periodicity rectangle becomes so wide that we can consider it a strip (an analiticity strip infact), and the 1-strings characterizing a solution of the Functional equation shall be located at a remote distance from the real axis (this happens at least for the first excited states, but in principle for t sufficiently small we shall always find ourselves in such a situation).

As a 1-string configuration is pushed remotely far from the real axis we observe the appearing of a tower of 2-strings under the 1-string pattern. This from a point of view of the combinatorial energy E amounts to a divergence of the ground state energy, a divergence which we are going to control by applying a uniform traslation to the u plane

$$x \to x + \log N \tag{4.4}$$

so that the 1-string pattern $\{v_k\}$ in th upper half plane is kept at a finite distance from the real axis.

Now, by considering such prescriptions we can define the scaling limit of the $\log \mathbf{D}_{finite}$ as follows:

$$\log \hat{D}(x) = \lim_{\text{csl}} \ \log D_{\text{finite}}(x + \log N) \tag{4.5}$$

Let us now consider the scaling limit of the convolution term

$$k * \log\left(1 + d(\frac{i}{4}x)\right) = \int_{2\log q}^{-2\log q} k(x-y) \log\Big(1 + d\Big(\frac{i}{4}y\Big)\Big) dy \tag{4.6}$$

which using the properties of convolution to move $(x-y)$ to the d term can be rewritten as:

$$k * \log\Big(1 + d\Big(\frac{i}{4}x\Big)\Big) = \int_{2\log q}^{-2\log q} k(y) \log\Big(1 + d\Big(\frac{i}{4}(x-y)\Big)\Big) dy \tag{4.7}$$

and if we now consider that the functions under the integral sign are all periodic and that the integration domain is a whole period, we can apply the traslation (4.4) (which leaves the value of the integral unchanged) and use again the property of convolution used above to get:

$$k * \log\Big(1 + d\Big(\frac{i}{4}(x + \log N)\Big)\Big) = \int_{2\log q}^{-2\log q} k(y) \log\Big(1 + d\Big(\frac{i}{4}(y + \log N)\Big)\Big) dy \tag{4.8}$$

Now if we consider that $q \to 0$ and $N \to \infty$ with:

$$q = \sqrt{\frac{\mu}{N}} \tag{4.9}$$

and use the result of [16] which states that:

$$\lim_{q \to 0} k(x, q) = \frac{1}{2\pi \cosh x} \tag{4.10}$$

after defining

$$\hat{d}(x) = \lim_{\text{csl}} \ d\Big(\frac{i}{4}(x + \log N)\Big) \tag{4.11}$$

we reach the following expression for the scaling limit of the convolution term:

$$\lim_{csl} k * \log\Big(1 + d\Big(\frac{i}{4}(x + \log N)\Big)\Big) = \int_{-\infty}^{+\infty} \frac{1}{2\pi \cosh(x-y)} \log(1 + \hat{d}(y)) dy \tag{4.12}$$

Where $\hat{d}(x)$ could in principle be computed analytically although this will not reveal strictly necessary to reach our result since it will prove sufficient to truncate the scaling limit for some N large enough.
Anyway we will provide an analytic calculation for the scaling limit of the 1-string term appearing in the subtracted energy which takes the form:

$$\sum_{k=1}^{m} \log[p(x, v_k)p(x, -v_k)] \tag{4.13}$$

First of all let us notice that the 1-strings positions v_k, as a consequence of the prescription chosen to compute the scaling limit, must keep a finite distance from the upper boundary of the periodicity rectangle, so that we may write the following scaling law:

$$4v_k + \log\left(\frac{\mu}{N}\right) \sim y_k \tag{4.14}$$

now let us consider the scaling limit of the p functions defined in (3.80), and in order to do so we observe that they can appear in the summation term only with one of the two following arguments in the u slot of the θ functions:

$$\begin{cases} x - 4v_k \sim x - y_k + \log\mu - \log N \\ x + 4v_k \sim x + y_k - \log\mu + \log N \end{cases} \tag{4.15}$$

so that after applying the uniform traslation (4.4) we see that the argument of the θ functions in $p(x, v_k)$ is limited as $N \to \infty$, while the argument of the θ in $p(x, -v_k)$ happens to be divergent as $\log N^2$.
Before proceeding we need also to remember that the elliptic functions θ_1 and θ_2 apart from their product definition given in the previous chapter admit a series expansion in the modular parameter which reads [20]:

$$\begin{cases} \theta_1(z, a) = 2a^{1/4} \displaystyle\sum_{k=0}^{\infty} (-1)^k a^{k(k+1)} \sin((2k+1)z) \quad |a| < 1 \\ \theta_2(z, a) = 2a^{1/4} \displaystyle\sum_{k=0}^{\infty} a^{k(k+1)} \cos((2k+1)z) \quad |a| < 1 \end{cases} \tag{4.16}$$

so that if the argument z stays limited it is possible to write:

$$\lim_{a\to 0} \frac{\theta_1(z, a)}{\theta_2(z, a)} = \tan z \tag{4.17}$$

this is infact the case of $p(x, v_k)$ for which without further problems we readily find:

$$\lim_{csl} p(x + \log N, v_k) = \tan\left(\frac{i}{2}(x - y_k + \log\mu)\right) = i \tanh\left(\frac{x - y_k + \log\mu}{2}\right) \tag{4.18}$$

Let us now consider the scaling limit of $p(x,-v_k)$, in this situation the argument of the θ functions is divergent, so that after calling $x+y_k-\log\mu=\overline{x}$ we calculate:

$$
\begin{aligned}
&\lim_{csl} p(x+\log N,-v_k)=\lim_{N\to\infty}\frac{\theta_1(\frac{i}{2}(\overline{x}+\log N^2),\frac{\mu}{N})}{\theta_2(\frac{i}{2}(\overline{x}+\log N^2),\frac{\mu}{N})}=\\
&=\lim_{N\to\infty}\frac{\sum_{k=0}^{\infty}(-1)^k\Big(\frac{\mu}{N}\Big)^{k(k+1)}\sin(\frac{(2k+1)i}{2}(\overline{x}+\log N^2))}{\sum_{k=0}^{\infty}\Big(\frac{\mu}{N}\Big)^{k(k+1)}\cos(\frac{(2k+1)i}{2}(\overline{x}+\log N^2))}=\\
&=\lim_{N\to\infty}\frac{\sum_{k=0}^{\infty}(-1)^k i\Big(\frac{\mu}{N}\Big)^{k(k+1)}(e^{\frac{2k+1}{2}(\overline{x}+\log N^2)}-e^{-\frac{2k+1}{2}(\overline{x}+\log N^2)})}{\sum_{k=0}^{\infty}\Big(\frac{\mu}{N}\Big)^{k(k+1)}(e^{\frac{2k+1}{2}(\overline{x}+\log N^2)}+e^{-\frac{2k+1}{2}(\overline{x}+\log N^2)})}=\\
&=\lim_{N\to\infty}\frac{\sum_{k=0}^{\infty}(-1)^k i\Big(\frac{\mu}{N}\Big)^{k(k+1)}(N^{2k+1}e^{\frac{2k+1}{2}\overline{x}}-N^{-(2k+1)}e^{-\frac{2k+1}{2}\overline{x}})}{\sum_{k=0}^{\infty}\Big(\frac{\mu}{N}\Big)^{k(k+1)}(N^{2k+1}e^{\frac{2k+1}{2}\overline{x}}+N^{-(2k+1)}e^{-\frac{2k+1}{2}\overline{x}})}=\\
&=\lim_{N\to\infty}\frac{\sum_{k=0}^{\infty}(-1)^k i\mu^{k(k+1)}N^{1+k(1-k)}(e^{\frac{2k+1}{2}\overline{x}}-N^{-2(2k+1)}e^{-\frac{2k+1}{2}\overline{x}})}{\sum_{k=0}^{\infty}\mu^{k(k+1)}N^{1+k(1-k)}(e^{\frac{2k+1}{2}\overline{x}}+N^{-2(2k+1)}e^{-\frac{2k+1}{2}\overline{x}})}=\\
&\lim_{N\to\infty}\frac{i\Big[N(e^{\frac{\overline{x}}{2}}-N^{-2}e^{-\frac{\overline{x}}{2}})-\mu^2N(e^{\frac{3\overline{x}}{2}}-N^{-2}e^{-\frac{3\overline{x}}{2}})\Big]}{\Big[N(e^{\frac{\overline{x}}{2}}+N^{-2}e^{-\frac{\overline{x}}{2}})+\mu^2N(e^{\frac{3\overline{x}}{2}}+N^{-2}e^{-\frac{3\overline{x}}{2}})\Big]}=\\
&=i\frac{e^{\frac{\overline{x}}{2}}-\mu^2e^{\frac{3\overline{x}}{2}}}{e^{\frac{\overline{x}}{2}}+\mu^2e^{\frac{3\overline{x}}{2}}}=i\frac{e^{\frac{\overline{x}}{2}}-e^{\frac{3\overline{x}}{2}+2\log\mu}}{e^{\frac{\overline{x}}{2}}+e^{\frac{3\overline{x}}{2}+2\log\mu}}=i\frac{1-e^{x+y_k+\log\mu}}{1+e^{x+y_k+\log\mu}}=-i\tanh\Big(\frac{x+y_k+\log\mu}{2}\Big)
\end{aligned}
\tag{4.19}
$$

and thanks to this result we can write the scaling limit of the 1-string term of $\log\mathbf{D}_{finite}$ as:

$$
\begin{aligned}
&\lim_{csl}\sum_{k=1}^{m}\log[p(x+\log N,v_k)p(x+\log N,-v_k)]=\\
&=\sum_{k=1}^{m}\log\Big[\tanh\Big(\frac{x-y_k+\log\mu}{2}\Big)\tanh\Big(\frac{x+y_k+\log\mu}{2}\Big)\Big]
\end{aligned}
\tag{4.20}
$$

so that finally piecing up all our results we have the following result for the continuum scaling limit of $\log D_{finite}$:

$$\log \hat{D}(x) = \sum_{k=1}^{m} \log \left[\tanh \frac{x + \log \mu - y_k}{2} \tanh \frac{x + \log \mu + y_k}{2} \right] + \int_{-\infty}^{+\infty} dy \frac{\log(1 + \hat{d}(y))}{2\pi \cosh(x - y)} \quad (4.21)$$

4.1.1 $\log \hat{D}$ *Expansion and Integrals of Motion*

We are now at the crucial point of our work, which follows the main lines of [10] .
We are about to show that $\log \hat{D}(x)$ (which is related to the free energy density in the scaling limit) admits an expansion of the form

$$\log \hat{D}(x) = -\sum_{n=1}^{\infty} C_n I_{2n-1}(\mu) e^{(2n-1)x} \quad , \quad x < 0 \quad (4.22)$$

this will be done by explicitly computing the expression for the products $C_n I_{2n-1}(\mu)$.
First of all let us introduce the following expansions:

$$\begin{aligned} \log \left(\frac{1-t}{1+t} \right) &= \log(1-t) - \log(1+t) = \\ &= \sum_{n=1}^{\infty} \frac{t^n}{n} ((-1)^n - 1) = -2 \sum_{k=1}^{\infty} \frac{t^{2k-1}}{2k-1} \end{aligned} \quad (4.23)$$

$$\begin{aligned} \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{1 + e^{2x}} = 2e^x \sum_{n=0}^{\infty} (-1)^n e^{2nx} = \\ = 2e^x \sum_{k=1}^{\infty} (-1)^{k-1} e^{(2k-2)x} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{(2k-1)x} \end{aligned} \quad (4.24)$$

let us now use (4.23) to expand the 1-string part in (4.21).
By calling $\tilde{y}_k = y_k - \log \mu$ we have:

$$\begin{aligned} &\log \tanh \frac{x + \log \mu - y_k}{2} = i\pi + \log \tanh \frac{\tilde{y}_k - x}{2} = \\ &= i\pi + \log \left[\frac{1 - e^{x - \tilde{y}_k}}{1 + e^{x - \tilde{y}_k}} \right] = i\pi - 2 \sum_{n=1}^{\infty} e^{(2n-1)x} \frac{e^{-(2n-1)\tilde{y}_k}}{2n-1} = \\ &= i\pi - 2 \sum_{n=1}^{\infty} e^{(2n-1)x} \frac{e^{-(2n-1)(y_k - \log \mu)}}{2n-1} \end{aligned} \quad (4.25)$$

and similarly

$$\log \tanh \frac{x + \log \mu + y_k}{2} = i\pi - 2 \sum_{n=1}^{\infty} e^{(2n-1)x} \frac{e^{-(2n-1)(-y_k - \log \mu)}}{2n-1} \quad (4.26)$$

so that the 1-string term admits the following expansion:

$$\begin{aligned}&\sum_{k=1}^{m}\log\left[\tanh\frac{x+\log\mu-y_k}{2}\tanh\frac{x+\log\mu+y_k}{2}\right]=\\&=2m\pi i-\sum_{n=1}^{\infty}e^{(2n-1)x}(-1)^k\sum_{k=1}^{m}\frac{4}{2n-1}e^{(2n-1)\log\mu}\cosh((2n-1)y_k)\end{aligned}\tag{4.27}$$

let us now consider the convolution term and use (4.24) to obtain:

$$\begin{aligned}&\int_{-\infty}^{+\infty}dy\frac{\log(1+\hat{d}(y))}{2\pi\cosh(x-y)}=\int_{-\infty}^{+\infty}dy\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}e^{(2k-1)(x-y)}\log(1+\hat{d}(y))=\\&=-\sum_{k=1}^{\infty}e^{(2k-1)x}\int_{-\infty}^{+\infty}\frac{dy}{\pi}e^{-(2k-1)y}\log(1+\hat{d}(y))\end{aligned}\tag{4.28}$$

so that piecing up and dropping the uninteresting πi terms we arrive at the following result for C_nI_{2n-1}:

$$C_nI_{2n-1}(\mu)=\sum_{k=1}^{m}\frac{4}{2n-1}e^{(2n-1)\log\mu}\cosh((2n-1)y_k)+(-1)^k\int_{-\infty}^{+\infty}\frac{dy}{\pi}e^{-(2k-1)y}\log(1+\hat{d}(y))\tag{4.29}$$

Where it is worth pointing out that the constants C_n are independent of the choice of boundary conditions, and the μ dependence of the I_{2n-1} is interpreted so that μ generates a *Renormalization Flux* from the U.V. critical theory ($\mu\to 0$) to the I.R. high temperature theory ($\mu\to\infty$). Now following the spirit of what has been said in section 3.1 we shall consider the I_{2n-1} as the eigenvalues of an infinite set of commuting *integrals of motion*, and we shall be interested in following their renormalization flow from the U.V. to the I.R. .

It is in the U.V. limit that we shall finally be able to make contact with CFT since the I_{2n-1} flow into the eigenvalues of a set of operators $\mathbf{I}_{2n-1}$ which represent an infinite family of commuting integrals of motion that as we shall see arise naturally in CFT (the first of them is indeed the energy $\mathbf{I}_1=L_0-c/24$) and in particular allow us to identify the U.V. scaling limit of the $\mathbf{A}_3$ model with the Minimal CFT $\mathcal{M}_{3,4}$.

The I.R. fixed point should instead correspond to a one particle massive quantum field theory of which we might be able to observe the zero-momentum massive eccitations.

It is also important to emphasize that in this framework to every lattice state $\{I_1,\ldots,I_k\}$ (here th I_k refer to the topological numbers introduced in the last chapter and not to the coefficients of the expansion!) corresponds the succession of its coefficients (4.29), and since the coefficients are themselves in correspondence with the eigenvalues of a set of CFT operators we have that each lattice state is mapped directly into a CFT state being an eigenvector of all the $\mathbf{I}_{2n-1}$.

4.2 Integrals of Motion in CFT

As explained in chapter 2 Conformal Field Theories are integrable, since they possess an infinite set of independent integrals of motion. Furthermore it is possible [13] to build a *commuting* family of such operators by quantizing the integrals of motion of the classical Sine-Gordon equation.
A general expression for these integral of motion is up to now unknown, but anyway they can be obtained constructively as polynomials the L_n by requiring

$$[\mathbf{I}_{2n-1}, \mathbf{I}_{2l-1}] = 0 \quad \forall\, l, n = 1, 2, \ldots \tag{4.30}$$

and that the $\mathbf{I}_{2n-1}$ have conformal dimensions:

$$(h, \overline{h}) = (2n-1, 0) \tag{4.31}$$

so that $\mathbf{I}_{2n-1}$ has spin $2n-1$.
An expression of the first few of them can be found in [13][12][10] and is given by:

$$\mathbf{I}_1 = L_0 - \frac{c}{24} \tag{4.32}$$

$$\mathbf{I}_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} \tag{4.33}$$

$$\begin{aligned}\mathbf{I}_5 = &\sum_{m,n,p\in\mathbb{Z}} \delta_{m+n+p,\ 0} : L_m L_n L_p : + \frac{3}{2}\sum_{n=1}^{\infty} L_{1-2n}L_{2n-1} + \\ &+ \sum_{n=1}^{\infty}\left(\frac{11+c}{6}n^2 - \frac{c}{4} - 1\right)L_nL_{-n} - \frac{c+4}{8}L_0^2 + \frac{(c+2)(3c+20)}{576}L_0 + \\ &- \frac{c(3c+14)(7c+68)}{290304}\end{aligned} \tag{4.34}$$

where the : : denotes *Conformal Normal Ordering* which can be obtained by arranging all the L_n in an increasing sequence with respect to n.
Our goal now is to diagonalize such operators in order to compare their eigenvalues and eigenvectors with the integrals of motion from TBA in order to build a map of lattice states to CFT states.
The diagonalization has to be carried out inside a Verma Module, and since $\mathbf{I}_1$ already rises the degeneracy at different levels of descendance, we are interested in building a matrix representation of our $\mathbf{I}_{2n-1}$ at a given level of descendance K.
In general if we choose a level of descendance K and build all the strings of L_{-n_i} operators with $n_i > 0$ and $\sum n_i = K$ we have that due to the presence of null states the dimensionality of the

spanned space is most of the times reduced.
We have chosen to build the matrix representation of the $\mathbf{I}_{2n-1}$ using a sovracomplete set of states, and then after diagonalization we have applied dimensional reduction so that some of the eigenstates have been reduced to null states, and the corresponding eigenvalues have been discarded.
In the procedure of diagonalization it has proved useful to notice that if $\big|h+K\big\rangle$ is some K-th descendant of the highest weight $\big|h\big\rangle$ the following identity holds:

$$L_n\big|h+K\big\rangle = 0 \quad \forall\ n > K \tag{4.35}$$

so that the infinite summation term in $\mathbf{I}_3$, when acting on some descendant at level K, can be truncated as follows:

$$\sum_{n=1}^{\infty} L_{-n}L_n\big|h+K\big\rangle = \sum_{n=1}^{K} L_{-n}L_n\big|h+K\big\rangle \tag{4.36}$$

there is an analogous problem with the infinite summations appearing in $\mathbf{I}_5$, although the only real problem is represented by the following term:

$$\sum_{m,n,p\in\mathbb{Z}} \delta_{m+n+p,\ 0} : L_mL_nL_p : \tag{4.37}$$

which we shall decompose as follows:

$$\begin{aligned}\sum_{m,n,p\in\mathbb{Z}} \delta_{m+n+p,\ 0} : L_mL_nL_p :=& \sum_{p} \delta_{3p,\ 0}L_p^3 + 3\sum_{0\neq n\neq p\neq 0} \delta_{2n+p,\ 0} : L_n^2L_p : + \\ &+ \sum_{m\neq n\neq p} \delta_{m+n+p,\ 0} : L_mL_nL_p :\end{aligned} \tag{4.38}$$

so that now we can work out the 3 pieces separately:

$$\sum_{p} \delta_{3p,\ 0}L_p^3 = L_0^3 \tag{4.39}$$

$$\begin{aligned}&\sum_{0\neq n\neq p\neq 0} \delta_{2n+p,\ 0} : L_n^2L_p := \sum_{n>p\neq 0} \delta_{2n+p,\ 0}L_pL_n^2 + \sum_{0\neq n<p\neq 0} \delta_{2n+p,\ 0}L_n^2L_p^2 = \\ &= \sum_{n>p\neq 0} [\delta_{2n+p,\ 0}L_pL_n^2 + \delta_{2p+n,\ 0}L_p^2L_n] = \sum_{n>0} L_{-2n}L_n^2 + \sum_{p<0} L_p^2L_{-2p} = \\ &= \sum_{n>0}(L_{-2n}L_n^2 + L_{-n}^2L_{2n})\end{aligned} \tag{4.40}$$

$$\begin{aligned}&\sum_{m\neq n\neq p} \delta_{m+n+p,\ 0} : L_mL_nL_p := 3\sum_{n\neq p}^{n,p\neq 0} \delta_{n+p,\ 0} : L_0L_nL_p : + \sum_{m\neq n\neq p}^{m,n,p\neq 0} \delta_{m+n+p,\ 0} : L_mL_nL_p := \\ &= 3[\sum_{n>p\neq 0} \delta_{n+p,\ 0} : L_0L_nL_p : + \sum_{n<p\neq 0} \delta_{n+p,\ 0} : L_0L_nL_p :]+\end{aligned} \tag{4.41}$$

$$+\sum_{0<m\neq n\neq p}^{n,p\neq 0}[\delta_{m+n+p,\ 0}:L_mL_nL_p:+\delta_{-m+n+p,\ 0}:L_{-m}L_nL_p:]=3!\sum_{n>0}L_{-n}L_0L_n+$$

$$+\sum_{0<m\neq p\neq n>0}^{p\neq 0}[\delta_{m+n+p,\ 0}:L_mL_nL_p:+\delta_{m-n+p,\ 0}:L_mL_{-n}L_p:+\delta_{-m+n+p,\ 0}:L_{-m}L_nL_p:+$$

$$+\delta_{-m-n+p,\ 0}:L_{-m}L_{-n}L_p:]=3!\sum_{n>0}L_{-n}L_0L_n+3\sum_{m\neq n\neq p}^{m,n,p>0}\delta_{m+n-p,\ 0}[:L_mL_nL_{-p}:+$$

$$+:L_{-m}L_{-n}L_p:]=3!\sum_{n>0}L_{-n}L_0L_n+3!\sum_{0<m<n}(L_{-m-n}L_mL_n+L_{-n}L_{-m}L_{n+m})$$

so that piecing up and using the previous considerations we arrive at the following truncation:

$$\begin{aligned}&\sum_{m,n,p\in\mathbb{Z}}\delta_{m+n+p,\ 0}:L_mL_nL_p:\big|h+K\big\rangle=\\&=3!\left[\sum_{n=1}^{K}L_{-n}L_0L_n+\sum_{n=2}^{K}\sum_{m=1}^{n-1}(L_{-m-n}L_mL_n+L_{-n}L_{-m}L_{n+m})\right]\big|h+K\big\rangle\end{aligned}\tag{4.42}$$

which allows us a safe algebraic computation of the matrix representation of $\mathbf{I}_5$ simply using the commutation rules of the Virasoro Algebra.

We present in the following pages some tables (4.1, 4.2, 4.3) with the results of the diagonalization in the case of the Ising Model $\mathcal{M}_{3,4}$.

$h=0$ $c=1/2$ Sector		
Eigenstate	I_3	I_5
$\|0\rangle$	0.00425347	-0.00190877
$L_{-2}\|0\rangle$	4.08759	7.57014
$L_{-3}\|0\rangle$	18.3793	97.0072
$-1.30208L_{-4}\|0\rangle+1.5625L_{-2}^2\|0\rangle$	22.1709	104.517
$-1.73958L_{-4}\|0\rangle-1.15972L_{-2}^2\|0\rangle$	50.1709	521.601
$-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle$	53.9626	529.111
$-12.1875L_{-5}\|0\rangle-12.1875L_{-3}L_{-2}\|0\rangle$	106.463	1832.5
$-19.8914L_{-6}\|0\rangle-23.8697L_{-4}L_{-2}\|0\rangle+22.8751L_{-3}^2\|0\rangle$	68.2543	618.548
$-16.6631L_{-6}\|0\rangle+33.3262L_{-4}L_{-2}\|0\rangle-4.16577L_{-3}^2\|0\rangle$	110.254	1840.01
$-124.859L_{-6}\|0\rangle-149.831L_{-4}L_{-2}\|0\rangle-31.2148L_{-3}^2\|0\rangle$	194.254	4997.92

Table 4.1 Eigenstates and eigenvalues of $\mathbf{I}_3$ and $\mathbf{I}_5$ in the h=0 sector

4.3 TBA Results Vs CFT

In this section we shall partially carry out the U.V. mapping of states between the $\mathbf{A}_3$ lattice model and the $\mathcal{M}_{3,4}$ CFT.

$h = 1/2$ $c = 1/2/$ Sector		
Eigenstate	I_3	I_5
$\|1/2\rangle$	0.150087	0.02291242
$L_{-1}\|1/2\rangle$	3.94175	7.53911
$L_{-2}\|1/2\rangle$	18.2334	96.9762
$L_{-3}\|1/2\rangle$	50.0251	521.569
$L_{-4}\|1/2\rangle + 0.5L_{-3}L_{-1}\|1/2\rangle$	106.317	1832.46
$L_{-4}\|1/2\rangle - 0.8333L_{-3}L_{-1}\|1/2\rangle$	22.3168	104.548
$L_{-5}\|1/2\rangle + 0.666667L_{-4}L_{-1}\|1/2\rangle$	194.108	4997.89
$L_{-5}\|1/2\rangle - 0.857143L_{-4}L_{-1}\|1/2\rangle$	54.1084	529.142
$L_{-6}\|1/2\rangle + 0.75L_{-5}L_{-1}\|1/2\rangle + 0.222222L_{-4}L_{-2}\|1/2\rangle$	320.4	11522.3
$L_{-6}\|1/2\rangle$	203.375	5374.68
$L_{-6}\|1/2\rangle - 1.25L_{-5}L_{-1}\|1/2\rangle + 0.222222L_{-4}L_{-2}\|1/2\rangle$	110.4	1840.04
$L_{-6}\|1/2\rangle + 0.75L_{-5}L_{-1}\|1/2\rangle - 0.962963L_{-4}L_{-2}\|1/2\rangle$	67.6751	618.579

Table 4.2 Eigenstates and eigenvalues of $\mathbf{I}_3$ and $\mathbf{I}_5$ in the h=1/2 sector

The correspondence will be achieved by comparing the ratios of the eigenvalues of $\mathbf{I}_3$ and $\mathbf{I}_5$ for couples of CFT states with corresponding ratios from TBA, and using the fact that the constants C_n must simplify in the ratio, so that we are dealing with the same observable considered in CFT.

We present in the following pages a table (4.4) showing some ratios computed from CFT in the $h = 0$ sector and from TBA in the $(+,+)$ sector with $N = 100$ and $\mu = 10^{-8}$, and a similar table (4.5) for ratios from CFT in the $h = 1/2$ sector and from TBA in the $(+,-)$ sector with the same parameters as the previous one.

It is surely worth to point out that the agreement of the TBA data with CFT is quite remarkable in both sectors, being accurate most of the times up to 3 significative digits of precision.

It is also necessary to remark that we have not been able to follow the I.R. flow of the ratios until they reached full thermalization because of the numerical approximation of the convolution term, and also because the reduced tempertature t has the following N, μ dependence:

$$t = \frac{\mu}{N} \tag{4.43}$$

so that for fixed N, since $|t| < 1$ we have that μ has to stay in the range $(0, N)$, and the full I.R. thermalization can be reached only for N large enough, a situation where the numerical computation of the zeroes of the righthand-side of the Functional equation becomes a very onerous task. Anyhow it is apparent enough, since the first 2 digits in the I.R. flow are stable, that the ratios we have computed all flow into the value 1. Anyway this does not complete the analisys of the I.R. massive theory's spectrum, since we expect that in analogy to what has been achieved in [11] for the energy $\mathbf{I}_1$ in the $\mathcal{M}_{4,5}$ model, the ratios of the higher charges for more excited states of the spectrum should flow into values different from 1.

$h = 1/16$ $c = 1/2$ Sector		
Eigenstate	I_3	I_5
$\|1/16\rangle$	-0.00486111	0.00197035
$L_{-1}\|1/16\rangle$	1.16181	0.995026
$L_{-2}\|1/16\rangle$	9.32847	31.7797
$L_{-3}\|1/16\rangle - 1.5L_{-2}L_{-1}\|1/16\rangle$	10.4951	32.7728
$L_{-3}\|1/16\rangle + 2L_{-2}L_{-1}\|1/16\rangle$	31.4951	241.314
$L_{-4}\|1/16\rangle - 1.6L_{-3}L_{-1}\|1/16\rangle$	32.6618	242.308
$L_{-4}\|1/16\rangle + 2.66667L_{-3}L_{-1}\|1/16\rangle$	74.6618	1016.89
$L_{-5}\|1/16\rangle + 3L_{-4}L_{-1}\|1/16\rangle - 2L_{-3}L_{-2}\|1/16\rangle$	40.8285	273.092
$L_{-5}\|1/16\rangle - 3.22222L_{-4}L_{-1}\|1/16\rangle + 0.666667L_{-3}L_{-2}$	75.8285	1017.88
$L_{-5}\|1/16\rangle + 3L_{-4}L_{-1}\|1/16\rangle + 0.666667L_{-3}L_{-2}\|1/16\rangle$	145.828	3103.3
$L_{-6}\|1/16\rangle - 2.10526L_{-5}L_{-1}+$ $+3.36842L_{-4}L_{-2}\|1/16\rangle - 1.89474L_3^2\|1/16\rangle$	41.9951	274.085
$L_{-6}\|1/16\rangle + 1.58435L_{-5}L_{-1}\|1/16\rangle+$ $-0.469438L_{-4}L_{-2}\|1/16\rangle - 0.577017L_{-3}^2\|1/16\rangle$	83.9951	1048.67
$L_{-6}\|1/16\rangle - 4.8L_{-5}L_{-1}\|1/16\rangle+$ $+0.8L_{-3}^2\|1/16\rangle$	146.995	3104.29
$L_{-6}\|1/16\rangle + 3.36232L_{-5}L_{-1}\|1/16\rangle+$ $+0.927536L_{-4}L_{-2}\|1/16\rangle + 0.057971L_{-3}^2\|1/16\rangle$	251.995	7722.
$L_{-6}\|1/16\rangle + 2.04395L_{-5}L_{-1}\|1/16\rangle+$ $-0.56346L_{-4}L_{-2}\|1/16\rangle - 0.664095L_{-3}^2\|1/16\rangle$	87.4371	1251.62

Table 4.3 Eigenstates and eigenvalues of $\mathbf{I}_3$ and $\mathbf{I}_5$ in the h=1/16 sector

In the final pages of the chapter we shall present some plots of the renormalization group flow for some of the ratios appearing in the tables above, along the horizontal axis we have reported $\log \mu$.

The results could have been even more close to the CFT predictions if we had used an analytic computation (which is just a technical matter) of the scaling limit convolution term (4.12), this has become manifest in the value of the ratio involving the $(+,+)$ ground state $\{\}$, which from CFT is known to be very small, and furthermore since the vacuum value of $C_n I_{2n-1}$ does not involve any 1-string term we can surely lay blame for the imprecision on the convolution term, which has the only fault of being numerical and not analytic.

And finally we present two tables (4.6,4.7) showing the mapping of lattice states to CFT states obtained from the TBA for the $(+,+)$ and $(+,-)$ sectors.

CFT			TBA		
State1/State2	$I_3^{(1)}/I_3^{(2)}$	$I_5^{(1)}/I_5^{(2)}$	State1/State2	$I_3^{(1)}/I_3^{(2)}$	$I_5^{(1)}/I_5^{(2)}$
$\|0\rangle/L_{-2}\|0\rangle$	0.00104	0.000252	$\{\}/\{0,0\}$	0.	0.
$L_{-2}\|0\rangle/L_{-3}\|0\rangle$	0.222402	0.0780369	$\{0,0\}/\{1,0\}$	0.222006	0.077929
$L_{-3}\|0\rangle/-1.73958L_{-4}\|0\rangle+$ $-1.15972L_{-2}^2\|0\rangle$	0.366334	0.18598	$\{1,0\}/\{2,0\}$	0.365746	0.185533
$-1.73958L_{-4}\|0\rangle-1.15972L_{-2}^2\|0\rangle/$ $-1.30208L_{-4}\|0\rangle+1.5625L_{-2}^2\|0\rangle$	2.26292	4.99059	$\{2,0\}/\{1,1\}$	2.26682	5.00318
$-1.73958L_{-4}\|0\rangle-1.15972L_{-2}^2\|0\rangle/$ $-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle$	0.929735	0.985806	$\{2,0\}/\{2,1\}$	0.929886	0.985863
$-1.73958L_{-4}\|0\rangle-1.15972L_{-2}^2\|0\rangle/$ $-12.1875L_{-5}\|0\rangle-12.1875L_{-3}L_{-2}\|0\rangle$	2.12201	3.51322	$\{2,0\}/\{3,0\}$	2.12621	3.52458
$-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle/$ $-12.1875L_{-5}\|0\rangle-12.1875L_{-3}L_{-2}\|0\rangle$	0.506867	0.288737	$\{2,1\}/\{3,0\}$	0.505782	0.28779
$-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle/$ $-19.8914L_{-6}\|0\rangle-23.8697L_{-4}L_{-2}\|0\rangle+$ $+22.8751L_{-3}^2\|0\rangle$	0.790611	0.855408	$\{2,1\}/\{2,2\}$	0.790765	0.855684
$-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle/$ $-16.6631L_{-6}\|0\rangle+33.3262L_{-4}L_{-2}\|0\rangle+$ $-4.16577L_{-3}^2\|0\rangle$	0.489439	0.287559	$\{2,1\}/\{3,1\}$	0.48846	0.286624
$-3.19444L_{-5}\|0\rangle+4.25926L_{-3}L_{-2}\|0\rangle/$ $-124.859L_{-6}\|0\rangle-149.831L_{-4}L_{-2}\|0\rangle+$ $-31.2148L_{-3}^2\|0\rangle$	0.277794	0.105866	$\{2,1\}/\{4,0\}$	0.276521	0.105093

Table 4.4 CFT ratios vs. TBA ratios for the $h=0$ $(+,+)$ sector for $N=100$, $\mu=10^{-8}$

4.4 Achievements and Future Developments

In this work we have been able to recover the results already known for the universality class of the Ising model by showing that such a model, interpreted as the $\mathbf{A}_3$ Lattice Integrable Model, falls within the universality class of the $\mathcal{M}_{3,4}$ Conformal Field Theory and shares with it the most important consequence of integrability, that is the existence of an infinite set of integrals of motion which we have been able to compute numerically in both cases at least for $\mathbf{I}_3$ and $\mathbf{I}_5$. Anyway there are still many questions that have to be further investigated, the first of which concerns the operator content of the one particle massive quantum field theory corresponding to the I.R. flow. In particular we want to reach for the model $\mathbf{A}_3$ the same results obtained in [11] for the $\mathbf{A}_4$ model and the energy $\mathbf{I}_1$, the first of which is the observation of the zero-momentum massive excitations and secondarily the exploration of new sectors of the theory such as $-1<q<0$ (which should correspond to the low temperature regime that by virtue of duality symmetry should be completely analogous to $0<q<1$) or the imaginary values of q with $|q|<1$.

CFT			TBA		
State1/State2	$I_3^{(1)}/I_3^{(2)}$	$I_5^{(1)}/I_5^{(2)}$	State1/State2	$I_3^{(1)}/I_3^{(2)}$	$I_5^{(1)}/I_5^{(2)}$
$\|1/2\rangle/L_{-1}\|1/2\rangle$	0.03807	0.00386	$\{0\}/\{1\}$	0.0370191	0.00411191
$L_{-1}\|1/2\rangle/L_{-2}\|1/2\rangle$	0.2161	0.0777	$\{1\}/\{2\}$	0.215791	0.0776347
$L_{-2}\|1/2\rangle/L_{-3}\|1/2\rangle$	0.364485	0.185932	$\{2\}/\{3\}$	0.363902	0.185485
$L_{-3}\|1/2\rangle/L_{-4}\|1/2\rangle+$ $+0.5L_{-3}L_{-1}\|1/2\rangle$	0.470528	0.284628	$\{3\}/\{4\}$	0.469596	0.28371
$L_{-4}\|1/2\rangle+0.5L_{-3}L_{-1}\|1/2\rangle/$ $L_{-4}\|1/2\rangle-0.833333L_{-3}L_{-1}\|1/2\rangle$	4.76399	17.5275	$\{4\}/\{0,0,0\}$	4.78175	17.6286
$L_{-4}\|1/2\rangle+0.5L_{-3}L_{-1}\|1/2\rangle/$ $L_{-5}\|1/2\rangle+0.666667L_{-4}L_{-1}\|1/2\rangle$	0.547721	0.366647	$\{4\}/\{5\}$	0.546382	0.365169
$L_{-4}\|1/2\rangle+0.5L_{-3}L_{-1}\|1/2\rangle/$ $L_{-5}\|1/2\rangle-0.857143L_{-4}L_{-1}\|1/2\rangle$	1.96489	3.46308	$\{4\}/\{1,0,0\}$	1.96913	3.47449
$L_{-5}\|1/2\rangle+0.666667L_{-4}L_{-1}\|1/2\rangle/$ $L_{-6}\|1/2\rangle+0.75L_{-5}L_{-1}\|1/2\rangle+$ $+0.222222L_{-4}L_{-2}\|1/2\rangle$	0.60583	0.433758	$\{5\}/\{6\}$	0.604063	0.431655
$L_{-5}\|1/2\rangle+0.666667L_{-4}L_{-1}\|1/2\rangle/$ $L_{-6}\|1/2\rangle-1.25L_{-5}L_{-1}\|1/2\rangle+$ $+0.222222L_{-4}L_{-2}\|1/2\rangle$	1.75822	2.71619	$\{5\}/\{2,0,0\}$	1.76281	2.72727
$L_{-5}\|1/2\rangle+0.666667L_{-4}L_{-1}\|1/2\rangle/$ $L_{-6}\|1/2\rangle+0.75L_{-5}L_{-1}\|1/2\rangle+$ $-0.962963L_{-4}L_{-2}\|1/2\rangle$	2.86823	8.07963	$\{5\}/\{1,1,0\}$	2.72727	8.14169
$L_{-6}\|1/2\rangle-1.25L_{-5}L_{-1}\|1/2\rangle+$ $+0.222222L_{-4}L_{-2}\|1/2\rangle/L_{-6}\|1/2\rangle$	0.54284	0.342353	$\{2,0,0\}/\{3,0,0\}$	0.555666	0.366118

Table 4.5 CFT ratios vs. TBA ratios for the $h = 1/2$ $(+,-)$ sector

And here we reach the second point, since in order to reach the I.R. thermalization for the integrals of motion we have to improve the precision of our program.

This can be done in two ways, the first is to reduce the presence of numerical computation at the essential, and this means to carry out the exact computation of the convolution term appearing in $\log \hat{\mathbf{D}}$; while the second way is to optimize the routine that finds the zeroes of the righthand-side term of the Functional equation, in order to be able to run the program for greater N and thus to let μ assume a wider range of values.

Another problem to solve is that concerning the identification of the correspondence of the $h = 1/16$ sector of CFT with some choice of boundary condition for the lattice model.

CFT State		Lattice State
$\|0\rangle$	$\longrightarrow$	$\{\}$
$L_{-2}\|0\rangle$	$\longrightarrow$	$\{0,0\}$
$L_{-3}\|0\rangle$	$\longrightarrow$	$\{1,0\}$
$-1.30208L_{-4}\|0\rangle + 1.5625L_{-2}^2\|0\rangle$	$\longrightarrow$	$\{1,1\}$
$-1.73958L_{-4}\|0\rangle - 1.15972L_{-2}^2\|0\rangle$	$\longrightarrow$	$\{2,0\}$
$-3.19444L_{-5}\|0\rangle + 4.25926L_{-3}L_{-2}\|0\rangle$	$\longrightarrow$	$\{2,1\}$
$-12.1875L_{-5}\|0\rangle - 12.1875L_{-3}L_{-2}\|0\rangle$	$\longrightarrow$	$\{3,0\}$
$-19.8914L_{-6}\|0\rangle - 23.8697L_{-4}L_{-2}\|0\rangle + 22.8751L_{-3}^2\|0\rangle$	$\longrightarrow$	$\{2,2\}$
$-16.6631L_{-6}\|0\rangle + 33.3262L_{-4}L_{-2}\|0\rangle - 4.16577L_{-3}^2\|0\rangle$	$\longrightarrow$	$\{3,1\}$
$-124.859L_{-6}\|0\rangle - 149.831L_{-4}L_{-2}\|0\rangle - 31.2148L_{-3}^2\|0\rangle$	$\longrightarrow$	$\{4,0\}$

Table 4.6 U.V. state correspondence CFT$\longrightarrow$TBA for the $(+,+)$ sector

CFT State		Lattice State
$\|1/2\rangle$	$\longrightarrow$	$\{0\}$
$L_{-1}\|1/2\rangle$	$\longrightarrow$	$\{1\}$
$L_{-2}\|1/2\rangle$	$\longrightarrow$	$\{2\}$
$L_{-3}\|1/2\rangle$	$\longrightarrow$	$\{3\}$
$L_{-4}\|1/2\rangle + 0.5L_{-3}L_{-1}\|1/2\rangle$	$\longrightarrow$	$\{4\}$
$L_{-4}\|1/2\rangle - 0.8333L_{-3}L_{-1}\|1/2\rangle$	$\longrightarrow$	$\{0,0,0\}$
$L_{-5}\|1/2\rangle + 0.666667L_{-4}L_{-1}\|1/2\rangle$	$\longrightarrow$	$\{5\}$
$L_{-5}\|1/2\rangle - 0.857143L_{-4}L_{-1}\|1/2\rangle$	$\longrightarrow$	$\{1,0,0\}$
$L_{-6}\|1/2\rangle + 0.75L_{-5}L_{-1}\|1/2\rangle + 0.222222L_{-4}L_{-2}\|1/2\rangle$	$\longrightarrow$	$\{6\}$
$L_{-6}\|1/2\rangle$	$\longrightarrow$	$\{3,0,0\}$
$L_{-6}\|1/2\rangle - 1.25L_{-5}L_{-1}\|1/2\rangle + 0.222222L_{-4}L_{-2}\|1/2\rangle$	$\longrightarrow$	$\{2,0,0\}$
$L_{-6}\|1/2\rangle + 0.75L_{-5}L_{-1}\|1/2\rangle - 0.962963L_{-4}L_{-2}\|1/2\rangle$	$\longrightarrow$	$\{1,1,0\}$

Table 4.7 U.V. state correspondence CFT$\longrightarrow$TBA for the $(+,-)$ sector

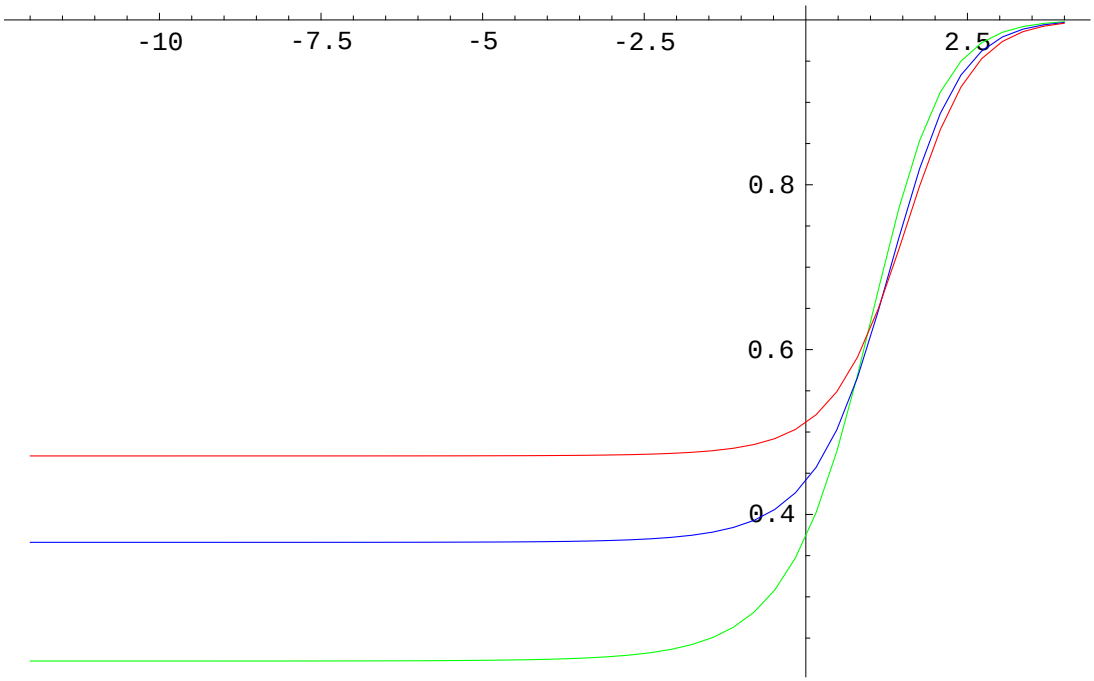

Figure 4.1 Flow of the I_3 ratios vs. $\log \mu$ with $N = 200$ for the following couples of states: $\{0,0\}/\{1,0\}, \{1,0\}/\{2,0\}, \{2,0\}/\{3,0\}$

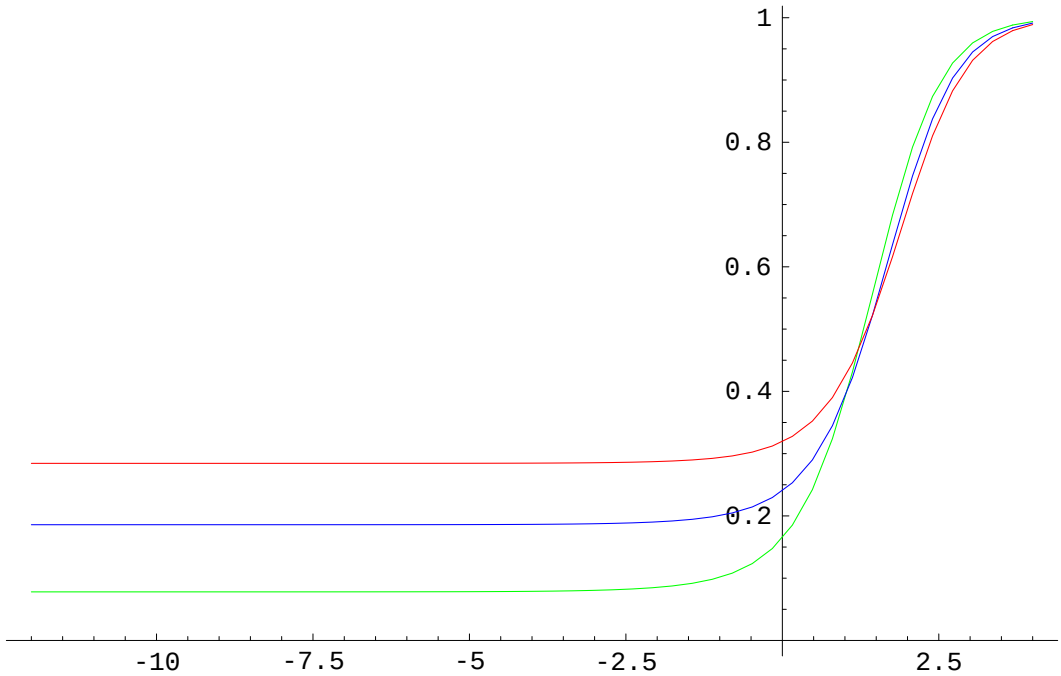

Figure 4.2 Flow of the I_5 ratios vs. $\log \mu$ with $N = 200$ for the following couples of states: $\{0,0\}/\{1,0\}, \{1,0\}/\{2,0\}, \{2,0\}/\{3,0\}$

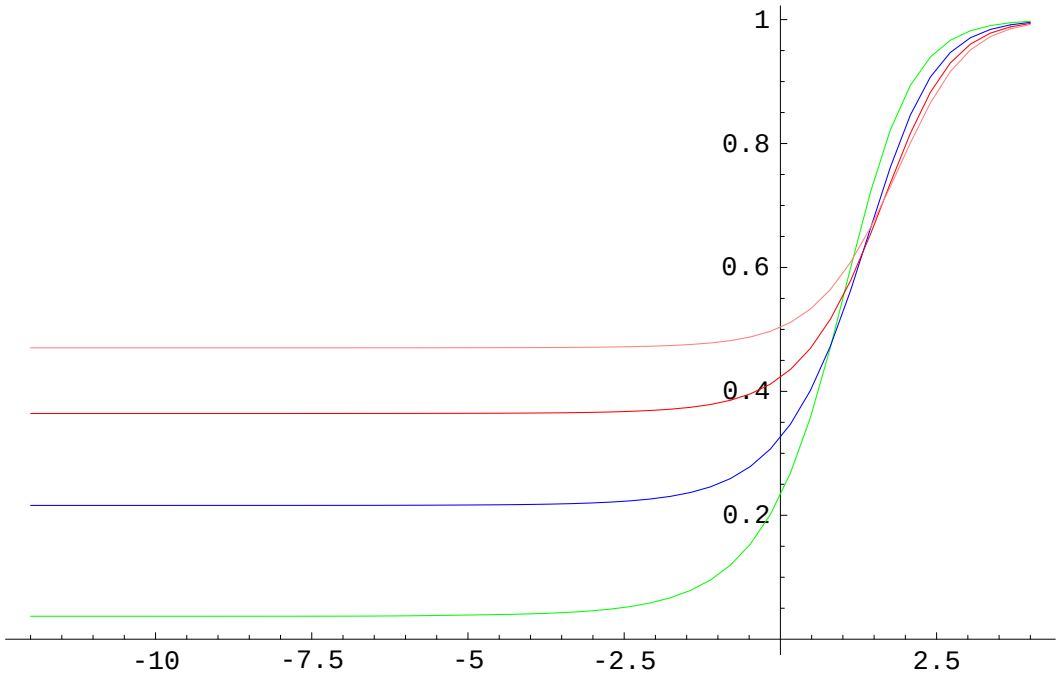

Figure 4.3 Flow of the I_3 ratios vs. $\log \mu$ with $N = 200$ for the following couples of states: $\{0\}/\{1\}, \{1\}/\{2\}, \{2\}/\{3\}, \{3\}/\{4\}$

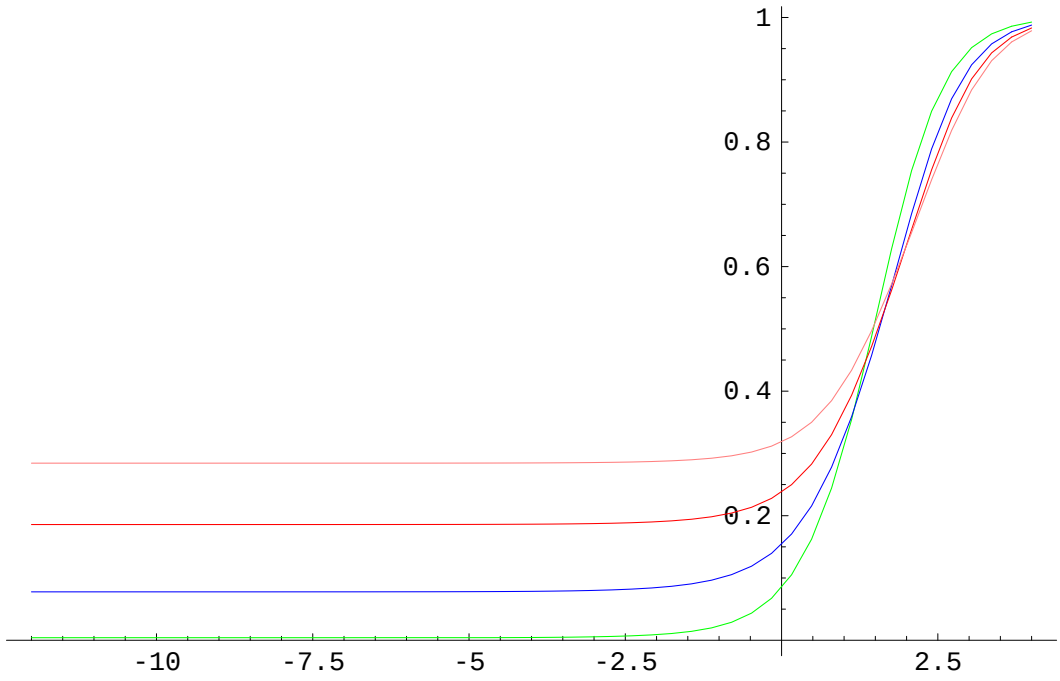

Figure 4.4 Flow of the I_5 ratios vs. $\log \mu$ with $N = 200$ for the following couples of states: $\{0\}/\{1\}, \{1\}/\{2\}, \{2\}/\{3\}, \{3\}/\{4\}$

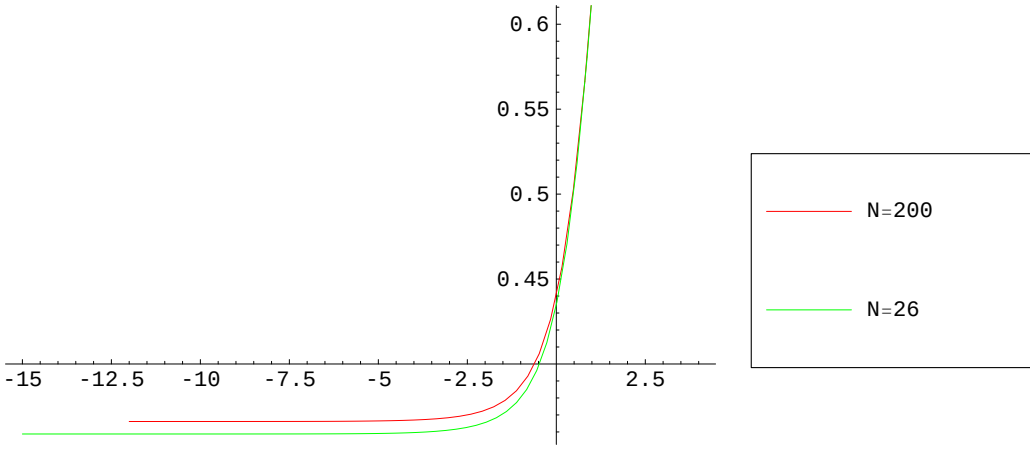

Figure 4.5 Comparative plot showing 2 flows for different $N = 26, 200$ of the I_3 ratio vs. $\log \mu$ for the couple of states $\{1, 0\}/\{2, 0\}$

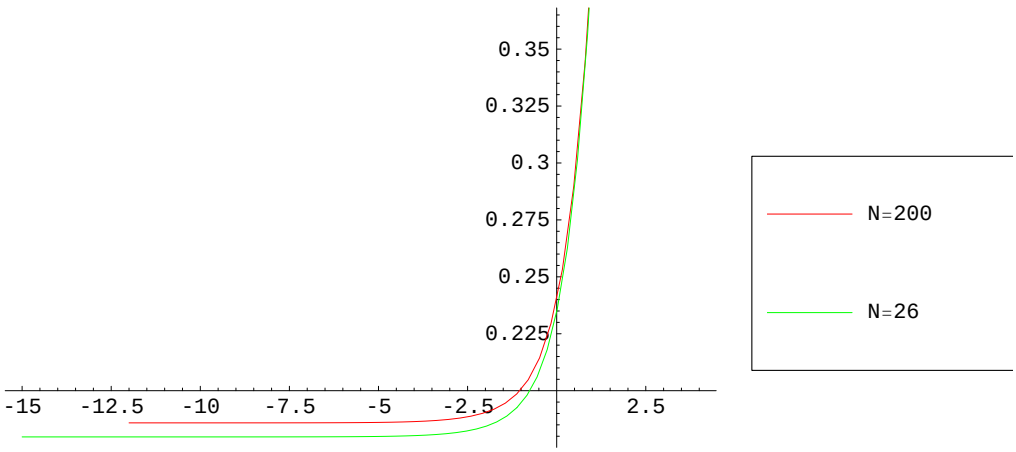

Figure 4.6 Comparative plot showing 2 flows for different $N = 26, 200$ of the I_5 ratio vs. $\log \mu$ for the couple of states $\{1, 0\}/\{2, 0\}$

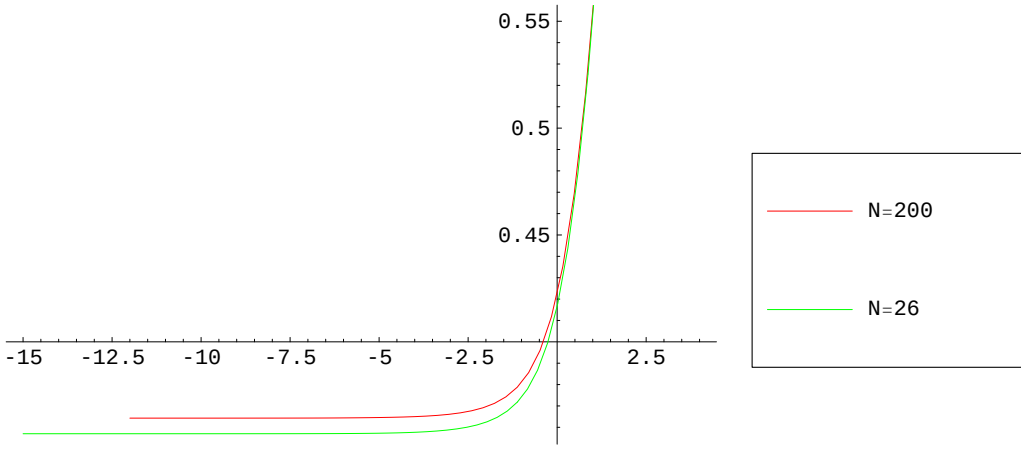

Figure 4.7 Comparative plot showing 2 flows for different $N = 26, 200$ of the I_3 ratio vs. $\log \mu$ for the couple of states $\{2\}/\{3\}$

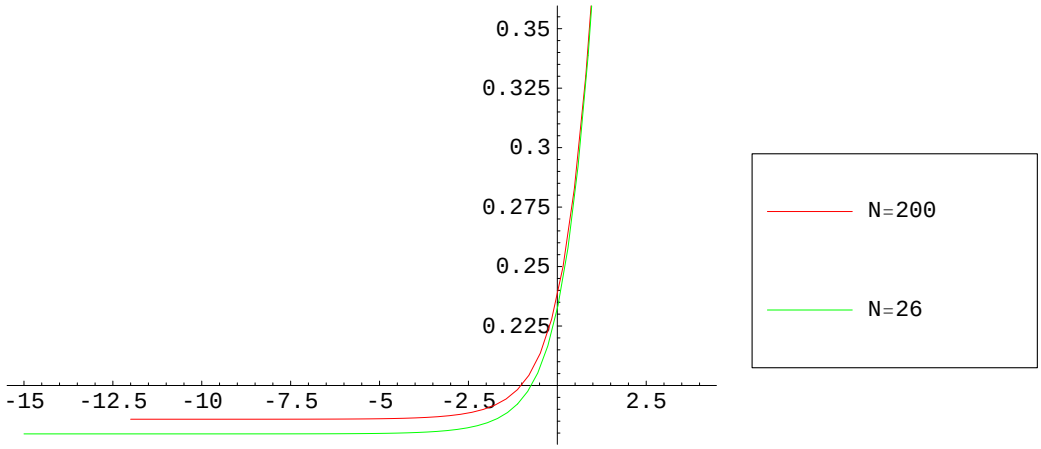

Figure 4.8 Comparative plot showing 2 flows for different $N = 26, 200$ of the I_5 ratio vs. $\log \mu$ for the couple of states $\{2\}/\{3\}$

5. Riassunto in Italiano

5.1 Capitolo 1

In questo capitolo si introduce il concetto di invarianza confrome in teoria dei campi.
Viene introdotto il gruppo conforme in generale come il gruppo di trasformazioni che riscalano la metrica di un fattore $\Omega > 0$, dopodichè si considerano trasformazioni infinitesime e si scopre che il campo vettoriale ϵ^μ che genera tali trasformazioni soddisfa l'Equazione di Killing Conforme:

$$\partial_{(\beta}\epsilon_{\alpha)} = \frac{1}{D}\partial \cdot \epsilon g_{\alpha\beta} \tag{5.1}$$

A questo punto si considerando l'Equazione di Killing Conforme al variare della dimensione D della varietá sottostante si scopre che per $D > 2$ il gruppo ha un numero finito di generatori, mentre invece per $D = 2$ si devono distinguere una invarianza conforme globale (che é essenzialmente $\mathbf{SL}(2, \mathbb{C})$) la quale è finito-dimensionale e una invarianza conforme locale che invece é infinito-dimensionale.
A questo punto si definisce una teoria di campo conforme come una teoria di campo in cui sono presenti dei campi ϕ_j detti *primari* i quali trasformano come densitá tensoriali sotto trasformazioni conformi, e l'esponente h_j con cui compare lo Jacobiano nella legge di trasformazione definisce il *peso conforme* del nostro campo primario:

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{h_j} \phi_j(x') \tag{5.2}$$

Un altra proprietá delle teorie conformi é che le funzioni di correlazione degli operatori primari devono trasformarsi in modo covariante sotto trasforamzioni del gruppo conforme:

$$\left\langle \phi_1(x_1) \dots \phi_n(x_n) \right\rangle = \left|\frac{\partial x'}{\partial x}\right|^{h_1}_{x=x_1} \dots \left|\frac{\partial x'}{\partial x}\right|^{h_n}_{x=x_n} \left\langle \phi_1(x'_1) \dots \phi_n(x'_n) \right\rangle \tag{5.3}$$

La struttura operatoriale di una tale teoria deve inoltre essere organizzata in famiglie conformi che sono insiemi di operatori che contengono un solo operatore primario e sicuramente tutte le sue derivate, mentre tutti gli altri operatori della famiglia sono ottenibili come combinationi lineari di operatori primari e loro derivate.
Infine dal punto di vista dello spazio di Hilbert è necessario richiedere l'esistenza di uno stato

di vuoto invariante sotto trasformazioni conformi globali.
Ora, come conseguenza della sua definizione una teoria conforme è tale che le sue funzioni di correlazione sono tutte determinate fino alla funzione a 3 punti in modo univoco, mentre i correlatori di ordine superiore sono determinati a meno di di una funzione incognita dei cosiddetti *cross ratios* che sono i piu generali oggetti invarianti sotto tutto il gruppo conforme globale che si possono costruire dalle coordinate.
Per esempio la funzione a due punti di due operatori primari di pesi conformi $h_1,\ h_2$ è:

$$\big\langle \phi(x_1)\phi(x_2)\big\rangle = \begin{cases} \frac{C_{12}}{r_{12}^{2h}} & h_1 = h_2 = h \\ 0 & h_1 \neq h_2 \end{cases} \tag{5.4}$$

Per le teorie conformi in 2 dimensioni si puo ottenere molto di più, e questo è dovuto alla presenza di una invarianza locale infinito-dimensionale.
In due dimensioni il gruppo conforme si presenta come prodotto tensoriale di una parte cosiddetta olomorfa e di un altra detta antiolomorfa e la definizione di campo primario viene quindi data rispetto alle proprietá di trasfromazione sotto al gruppo di trasformazioni olomorfe ($z \to f(z)$) e antiolomorfe ($\overline{z} \to \overline{f}(\overline{z})$); accade quindi che un operatore primario sia etichettato da due pesi conformi $h,\ \overline{h}$, cosicchè sotto una trasformazione del gruppo si ha:

$$\phi(z,\overline{z}) \to \Big(\frac{\partial f}{\partial z}\Big)^h \Big(\frac{\overline{\partial f}}{\overline{\partial}\overline{z}}\Big)^{\overline{h}} \phi(f,\overline{f}) \tag{5.5}$$

In questo schema generale diventa interessante cercare un operatore che possa essere utilizzato per costruire la carica che genera la simmetria conforme, e questo operatore é il *tensore energia-impulso* $T^{\mu\nu}$.
Mediante il tensore energia impulso e il vettore di Killing conforme corrispondente alla trasformazione è quindi possibile costruire una corrente $J^a = T^{ab}\epsilon_b$ che risulta automaticamente conservata per le isometrie (per cui $\partial \cdot \epsilon = 0$):

$$\partial_a J^a = T^c_c \partial \cdot \epsilon \tag{5.6}$$

e che invece risulta essere conservata per le isometrie conformi solamente se si aggiunge la richiesta che il tensore energia impulso sia privo di traccia, cosicché concludiamo che il tensore energia impulso associato a una teoria di campo conformememnte invariante deve evere traccia nulla.
Se ora consideriamo la carica Q associata alle correnti sopra menzionate e la trasportiamo in seconda quantizzazione otteniamo un operatore che possiamo usare per genereare le trasformazioni sui campi, e confrontando la legge di trasformazione infinitesima di un campo primario ϕ con quella che si ottiene usando la carica Q si arriva a stabilire che il prodotto tra un campo primario e il tensore nergia impulso (o meglio la componente $T_{zz} = T$ in coordinate complesse), inteso come inserzione operatoriale in un integrale funzionale deve necessariamente avere una

certa espansione a corta distanza che viene denominata OPE (operator product expansion) e che caratterizza completamente le leggi di trasformazione di un operatore primario.

$$T(z)\phi(w,\overline{w}) = \frac{h\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\phi(w,\overline{w})}{(z-w)} + \dots \tag{5.7}$$

A questo punto é legittimo chiedersi cosa debba succedere se applichiamo in sequenza due trasformazioni conformi, la risposta a questa domanda si trova nell'analisi dell'espansione a corta distanza del prodotto del tensore energia impulso con se stesso:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{5.8}$$

Quello che si trova in questo caso é che il tensore energia impulso non trasforma come un campo primario e nella sua espansione a corta distanza (OPE) si ha un ulteriore termine dominante che compare moltiplicato da una costante numerica che caratterizza la teoria conforme che stiamo considerando, ovverosia la *carica centrale* c. Da questa espansione concludiamo inoltre che il $T(z)$ é un campo con dimensione conforme $h = 2$.
A questo punto ci si puo chiedere come sia fatto lo spazio di Hilbert di una teoria conforme, e la risposta si trova ancora una volta nel tensore energia impulso per il quale si definisce una espansione in modi operatoriali L_n (essenzialmente una serie di Laurent):

$$T(z) = \sum_{n\in Z} \frac{L_n}{z^{n+2}} \tag{5.9}$$

Gli L_n soddisfano delle relazioni di commutazione che costituiscono una estensione centrale delle relazioni di commutazione dei generatori del gruppo conforme locale in 2 dimensioni:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,\,0} \tag{5.10}$$

L'algebra costituita da questi modi operatoriali viene detta Algebra Di Virasoro ed è di primaria importanza in teoria delle Stringhe e meccanica statistica.
Sono proprio i modi operatoriali di Virasoro gli oggetti che utilizzeremo per generare tutto lo spazio di Hilbert a partire dallo stato di vuoto.
È interessante osservare che i generatori della parte olomorfa del gruppo conforme globale soddisfano un'algebra (generata da L_0, L_1, L_{-1}) che si osserva essere sottoalgebra dell'algebra di Virasoro cosicchè l'invarianza globale dello stato di vuoto fa si che questo sia annichilato da tutti i modi operatoriali che generano l'invarianza globale, e inlotre è annichilato anche dagli L_n per $n > 0$.
Possiamo osservare che grazie alle loro relazioni di commutazione modi L_n nel loro insieme agiscono in modo simile agli operatori L^+, L^- che compaiono nella teoria delle rappresentazioni dei

momenti ancolari dei momenti angolari.
In tale modo dato uno stato di partenza detto *highest weight* generato dall'inserzione di un operatore primario sul vuoto $\big|h\big\rangle = \phi(0)\big|0\big\rangle$, si puo generare una gerarchia di discendenti $L_{-n_1}\dots L_{-n_k}\big|h\big\rangle$ che hanno come autovalore rispetto a L_0 (che é un operatore autoaggiunto) il valore $h + \sum_i n_i$.
L'insieme di uno stato highest weight e di tutti i suoi discendenti costituisce un cosiddetto *Modulo di Verma* $\mathcal{V}(c,h)$ e lo spazio di Hilbert é costruito come una somma diretta sui prodotti tensoriali di moduli di Verma corrispondenti agli operatori primari presenti nella teoria:

$$\mathcal{H} = \bigoplus_{h,\overline{h}} \mathcal{V}(c,h) \otimes \mathcal{V}(c,\overline{h}) \tag{5.11}$$

A questo punto ci si chiede in quali casi sia possibile costruire delle rappresentazioni unitarie su un siffatto spazio di Hilbert e la risposta a questa domanda sta nella richiesta che nello spazio di Hilbert non ci debbano essere stati a norma nulla, e qualora ve ne fossero per ottenere uno spazio in cui si abbia una rappresentazione unitaria dovremmo eliminare questi stati tramite un quoziente ritenendoli equivalenti al vettore nullo.
Di fatto uno studio molto dettagliato delle condizioni sotto cui una teoria conforme puó essere unitaria ci porta a individuare un set discreto di modelli $\mathcal{M}_{p,p+1}$, con carica centrale $c = 1 - 6/p(p+1)$ e con pesi conformi $h_{r,s}$ indicizzati da coppie di interi, detti *modelli minimali* per i quali vi sono un numero finito di operatori primari e per cui l'OPE costituisce un algebra associativa chiusa caratterizzata da delle *regole di fusione* tra famiglie conformi:

$$\phi_i \times \phi_j = \sum_k C_{ijk}\phi_k \tag{5.12}$$

dove la somma corre su tutti gli operatori primari.
La struttura dei vettori nulli all'interno dei moduli di Verma per questi modelli fa si che sia possibile scrivere delle equazioni differenziali per tutte le funzioni di correlazione degli operatori primari cosicché si rivela possibile almeno in linea di principio risolvere completamente la teoria.
A questo punto ci si pone il problema di formulare una teoria conforme in modo tale che le proprietà fino ad ora descritte siano indipendenti dalle proprietà topologiche della varietà sottostante, in particolare dal momento che una varietà bidimensionale è completamente caratterizzata dal suo genere topologico (le varietà dello stesso genere topologico sono tra loro globalmente diffeomorfe), si cerca un modo per scrivere la funzione di partizione della teoria tale per cui il risultato non dipenda dalla topologia dello spazio sottostante.
Affrontando il problema nel caso piu semplice che è il toro si scopre che la funzione di partizione, calcolata come traccia su tutto lo spazio di Hilbert di un operatore unitario costruito dai modi zero dell'algebra di Virasoro, viene a dipendere da un parametro di natura geometrica detto

parametro modulare q il quale caratterizza la direzione in cui la varietà mappata sul piano complesso deve essere identificata periodicamente.
La funzione di partizione deve dunque essere invariante sotto trasformazioni che ridefiniscono i parametri modulari in modo compatibile con le periodicitá del manifold ovverosia il gruppo $SL(2,\mathbb{Z})$. In particolare la funzione di partizione sará costruita attraverso combinazioni lineari di prodotti di oggetti detti *caratteri dell'algebra di Virasoro*:

$$\chi_{c,h}(q) = Tr_{\mathcal{V}(c,h)} q^{L_0 - \frac{c}{24}} = \sum_{N=0}^{\infty} d_N q^{h+N-\frac{c}{24}} \quad q = e^{2\pi i \tau} \tag{5.13}$$

In questo linguaggio la funzione di partizione sará una forma sesquilineare nei caratteri dell'algebra di Virasoro:

$$Z(q) = \sum_{h,\overline{h}} \mathcal{N}_{h,\overline{h}} \chi_{c,h}(q) \chi_{c,\overline{h}}(\overline{q}) \tag{5.14}$$

E l'invarianza modulare sarà garantita dal fatto che i caratteri si traformano tra di loro sotto $q \to \tilde{q}$ secondo una rappresentazione unitaria finitodimensionale del gruppo modulare:

$$\chi_{c,h}(\tilde{q}) = \sum_{h'} \mathcal{S}_{h,h'} \chi_{c,h'}(q) \tag{5.15}$$

In questo contesto si è interessati a studiare il primo modello minimale unitario $\mathcal{M}_{3,4}$ della serie discreta sopra menzionata, che contiene solamente 3 operatori primari i cui pesi conformi sono tali che le proprietá di riscalamento (che è una trasformazione conforme!) delle loro funzioni di correlazione permettono di identificarli con il contenuto operatoriale del *modello di Ising* definito su reticolo e ben noto in meccanica statistica. Questa identificazione non è un caso isolato, infatti in meccanica statistica ogni sistema soggetto a una transizione di fase di secondo ordine ha una lunghezza di correlazione divergente, un tale sistema diventa invariante per trasfromazioni di scala, e siccome é possibile dimostrare che ampliando l'invarianza sotto rototraslazioni (isometrie) agli scaling si ottiene automaticamente una invarianza conforme, abbiamo che la simmetria conforme caratterizza siffatti modelli di meccanica statistica al punto critico.
Inoltre una analisi accurata porta a scoprire che la teoria di campo corrispondente a un fermione non massivo in 2 dimensioni ha lo stesso contenuto operatoriale del modello di Ising e conseguentemente anche del primo modello minimale unitario.

5.2 Capitolo 2

In questo capitolo ci occupiamo di 2 argomenti un po' di versi tra loro, il primo dei quali sono le teorie conformi con bordo.
Supponiamo dunque per qualche ragione (di solito un problema di Open String Theory, o di

meccanica statistica con condizioni al bordo) di essere interessati a definire una teoria conforme su una varietà con bordo.
Quello che salta subito all'occhio è che non si deve consentire alle trasformazioni conformi di mappare il bordo nel bulk e viceversa, questo evidentemente perchè la distanza più grande a cui venisse mappato un punto dal bordo nel bulk definirebbe una scala di lunghezza per la teoria, evidentemente rompendo l'invarianza sotto scaling e di conseguenza anche la possibilità di formulare una teoria conforme con bordo.
Un tipico esempio di varietà con bordo è rappresentato dal semipiano complesso superiore, e il vincolo menzionato sopra si traduce in coordinate complesse nella richiesta che il campo di Killing conforme assuma sul bordo unicamente valori reali:

$$\epsilon(\overline{z}) = \overline{\epsilon}(z) \tag{5.16}$$

Più in generale questo su può tradurre dicendo che il vettore di Killing conforme non può avere componeneti ortogonali al bordo.
Come ulteriore richiesta per garantire una sorta di invarianza conforme residua si ha la necessità che le condizioni al bordo per i campi stessi presenti nella teoria debbano essere conformemente invarianti e questo ha l'effetto di selezionare solamente condizioni al bordo omogenee come:

$$\phi|_B = 0 \tag{5.17}$$

Queste richieste hanno delle conseguenze immediate sul calcolo delle funzioni di correlazione, ovverosia che se analizziamo le proprietà di trasformazione dei correlatori sotto trasformazioni conformi infinitesime ci rendiamo conto che una funzione a n punti in una teoria con bordo soddisfa le stesse equationi di una funzione a $2n$ punti in una teoria senza bordo, pur di richiedere il vincolo $\overline{z} = z^*$.
Ne segue immediatamente che per esempio la funzione a 1 punto contrariamente alle teorie senza bordo assume un valore di aspettazione non nullo e decade con legge di scala man mano che ci allontaniamo dal bordo della varietá, in modo tale che a una distanza infinita dal bordo (cioé nel bulk) assuma un valore nullo:

$$\left\langle \phi(z,\overline{z}) \right\rangle_\beta = \frac{A_\phi^\beta}{2y^{2h}} \quad , y = \mathrm{Im} z \tag{5.18}$$

dove nella formula sopra β etichetta le condizioni al bordo e A_ϕ^β è una ampiezza caratteristica dell'operatore in considerazione.
In questo contesto emerge in modo naturale l'esistenza di una famiglia di operatori che ha come varietà di supporto il bordo del nostro manifold; tali operatori hanno delle funzioni di correlazione che soddisfano leggi di scala e pertanto ha senso associarvi dei pesi conformi che si

trovano essere doppie rispetto alle corrispondenti dimensioni di bulk.
La soluzione del vincolo di bordo menzionato sopra ha come immediata conseguenza quella di ridurre lo spazio di Hilbert della teoria eliminando il settore antiolomorfo, in modo tale che ora lo spazio é solamente una somma diretta sui moduli di Verma (e non su loro prodotti tensoriali):

$$\mathcal{H} = \bigoplus_h \mathcal{V}(c,h) \tag{5.19}$$

Inoltre si scopre che come conseguenza del vincolo gli operatori di bordo (nella loro rappresentazione in termine di stati) sono tutti esprimibili mediante combinazioni lineari dei cosiddetti *stati di Ishibashi* i quali sono in corrispondenza biunivoca con gli operatori della teoria di bulk.
Come conseguenza della struttura dello spazio di Hilbert si ha che la funzione di partizione per un modello minimale unitario definito su una varietá con bordo (per esempio un cilindro), deve essere costruita come una forma lineare nei caratteri dell'algebra di Virasoro e i coefficienti di questa combinazione lineare saranno legati alla scelta di condizioni al bordo in modo tale che per ogni scelta dei bordi vi sia un solo contenuto operatoriale in grado di propagare nella teoria.
Si ha quindi per la funzione di partizione:

$$Z_{\alpha\beta}(q) = \sum_h n^h_{\alpha\beta}\chi_{c,h}(q) \tag{5.20}$$

Dove α e β etichettano le condizioni al bordo del cilindro e $hn^h_{\alpha\beta}$ è un coefficiente che a seconda delle condizioni al bordo seleziona in modo opportuno il contenuto operatoriale della teoria.
La seconda parte del capitolo parte da alcune nozioni di base sulla teoria delle perturbazioni per una teoria conforme per arrivare a parlare più in generale di teorie di campo integrabili.
Se consideriamo una teoria di campo conforme si può immaginare di perturbare la sua azione per esempio aggiungendo un'espressione locale in uno degli operatori primari moltiplicata per una costante di accoppiamento:

$$S = S_{CFT} + \lambda \int d^2 z \phi(z,\overline{z}) \tag{5.21}$$

a questo punto si puo applicare una trasformazione conforme e calcolare la variazione dell'azione sotto di essa, che sarà semplicemente data dalla variazione del termine di interazione.
Dalla definizione generale del tensore energia impulso è quindi possibile dedurre che la variazione dell'azione sotto una trasformazione conforme sarà proporzionale a una espressione locale nella traccia del tensore energia impulso (che chiameremo Θ) cosicchè uguagliando le due espressioni ottenute per la variazione dell'azione si deduce che come conseguenza della perturbazione la traccia del tensore energia impulso è proporzionale all'operatore primario con cui abbiamo perturbato:

$$\Theta(z,\overline{z}) = 2\lambda(h-1)\phi(z,\overline{z}) \tag{5.22}$$

Questo è un segnale chiaro che la simmetria conforme è stata rotta.
Questo sicuramente implicherà che molte delle infinite correnti conservate non saranno piu tali, ma ad ogni modo potrebbe succedere che in alcuni casi una legge di conservazione continui a valere anche dopo la perturbazione.
Un esempio di corrente che sicuramente rimane conservata è il tensore energia impulso, per il quale nonostante venga a cadere la condizione di traccia nulla, continua a valere la legge di conservazione della quantità di moto ed energia:

$$\overline{\partial} T(z,\overline{z}) = -\frac{1}{4}\partial\Theta(z,\overline{z}) \tag{5.23}$$

È interessante notare che la legge di conservazione in questo caso non implica piu che tale tensore possegga solamente le componenti a spin 2 poiché ora possiede anche la componente a spin 0 (cioé la traccia Θ).
Un esempio invece di teoria per cui si abbia una corrente la cui legge di conservazione venga rotta da una perturbazione è il primo modello minimale unitario $\mathcal{M}_{3,4}$ (ovverosia il modello di Ising), per il quale se consideriamo come corrente conservata la parte olomorfa del fermione stesso e perturbiamo con il cosiddetto operatore termico scopriamo che la legge di conservazione della teoria conforme diventa semplicemente l'equazione di Dirac massiva.
Esistono tuttavia per questa teoria un insieme infinito di correnti che sotto la perturbazione rimangono conservate, cosicchè la perturbazione termica continua a possedere infiniti integrali del moto.
Chiameremo i modelli derivanti da una generica perturbazione di una teoria conforme per i quali continua a esistere un set infinito di correnti conservate *deformazioni integrabili* di una teoria conforme.
Più in generale si pone quindi il problema di studiare teorie di campo che possiedono una simmetria infinito-dimensionale che chiameremo *modelli integrabili*.
Nell'approccio a tali modelli in genere si perde di vista la descrizione in termini di una Hamiltoniana o di una azione e la matrice di scattering S diventa l'oggetto di fondamentale interesse. Sfruttando l'esistenza di una simmetria infinito-dimensionale è possibile dimostrare che nei processi di scattering a n particelle la matrice S si fattorizza in $n(n-1)/2$ ampiezze di scattering a 2 particelle e che inoltre sono proibiti i processi di decadimento e di annichilazione cosicché lo scattering diventa completamente elastico. Inoltre le ampiezze di scattering a 2 particelle devono soddisfare l'equazione di Yang Baxter che è una equazione funzionale che viene costruita come conseguenza della possibilità di spostare in modo arbitrario i punti di interazione in un processo d'urto.
Di fondamentale importanza per questo approccio è inoltre il *principio di bootstrap* che essenzialmente consiste nella possibilitá di considerare gli stati legati derivanti dai processi di scattering

come stati asintotici presenti nella teoria, l'equazione di bootstrap per la matrice S che essenzialmente afferma quanto appena detto deve però essere risolta con una procedura autoconsistente che si fonda su un ansatz che impone delle *regole di fusione di bootstrap* che essenzialmente affermano quali stati legati possono nascere da un processo di scattering tra due particelle presenti nella teoria. La soluzione dell'equazione di bootstrap deve inoltre essere fatta in modo consistente con i principi di località , tipicamente di Lorentz-Invarianza e in ogni caso consistente con eventuali simmetrie che vogliamo richiedere dalla teoria di scattering.

5.3 Capitolo 3

In questo capitolo si definiscono i modelli integrabili su reticolo come modelli per i quali esiste una descrizione in termini di una matrice trasferimento $\mathbf{T}$ dipendente da un *parametro spettrale* u rispetto al quale forma una *famiglia commutante a un parametro*:

$$[\mathbf{T}(u), \mathbf{T}(v)] = 0 \ , \ \forall u, v \tag{5.24}$$

È interessante osservare che questi modelli di meccanica statistica condividono con le teorie di campo integrabili un tipo di approccio in cui la matrice di trasferimento $\mathbf{T}$ gioca un ruolo analogo alla matrice di scattering S. Per esempio l'equazione di Yang Baxter soddisfatta dalla matrice S ha un corrispettivo nella descrizione mediante matrice di trasferimento che sta nascosto nell'affermazione che le $\mathbf{T}(u)$ formano una famiglia a un parametro di matrici commutanti.
Si può mostrare che delle matrici di trasferimento che soddisfino le proprietà sopra esposte possono essere costruite mediante somme e prodotti a partire da degli oggetti elementari detti *pesi di Boltzmann* i quali devono possedere per $u \to \lambda - u$ una invarianza detta *simmetria di crossing* (da cui λ prende il nome di *parametro di crossing*).
L'equazione di Yang Baxter in questo caso assume proprio la forma di una relazione locale tra i pesi di Boltzmann che ha una forma molto simile all'omonima equazione della teoria dei campi integrabile.
Come per le teorie di campo integrabili vogliamo evidenziare il fatto che la descrizione del modello in termini di una Hamiltoniana passa decisamente in secondo piano (se vogliamo questo è simile a considerare una teoria di campo efficace in cui la descrizione microscopia dei processi diventa non necessaria), anche se in alcuni casi come il modello di Ising siamo in grado di costruire a partire da essa proprio i pesi di Boltzmann e in seguito la matrice di trasferimento.
Inoltre di nuovo in analogia con le teorie di campo integrabili abbiamo che la transer matrix $\mathbf{T}(u)$, o ancora meglio il suo logaritmo, può essere espansa in una serie i cui coefficienti sono delle matrici $\mathbf{I}_n$ che rappresentano gli integrali del moto della teoria su reticolo; vedremo nel

prossimo capitolo come sia possibile calcolare gli autovalori di questi integrali del moto e seguirli nel limite continuo.
A questo punto si introduce il modello di Ising anisotropo (definito su un reticolo quadrato ruotato di $\pi/4$) per il quale a partire dall'Hamiltoniana

$$\mathcal{H} = -\frac{1}{\beta}\Big[J \sum_{<i,l>}^{even} \sigma_i\sigma_l + K \sum_{<i,l>}^{odd} \sigma_i\sigma_l\Big] \tag{5.25}$$

si costruiscono esplicitamente i pesi di Boltzmann e si mostra la forma che deve assumere la matrice di trasferimento.
Dopodichè si presenta il calcolo degli autovalori $\Lambda(u)$ di $\mathbf{T}(u)$ per il modello critico e per una scelta particolare di condizione al contorno.
La solutione si basa sul fatto che la matrice $\mathbf{T}$ possiede una simmetria di crossing:

$$\mathbf{T}(u) = \mathbf{T}(\lambda - u) \tag{5.26}$$

e soddisfa una equazione funzionale del tipo:

$$\mathbf{T}(u)\mathbf{T}(u+\lambda) = G(u)\mathbf{1} \tag{5.27}$$

e dal momento che le $\mathbf{T}$ formano una famiglia commutante la stessa equazione funzionale vale anche per gli autovalori Λ di $\mathbf{T}$:

$$\Lambda(u)\Lambda(u+\lambda) = G(u) \tag{5.28}$$

La soluzione di tale equazione si fonda essenzialmente sul fatto che per il modello in considerazione G è una espressione quasi polinomiale nella variabile $z = e^{4iu}$, che specificamente assume la forma:

$$G_L(z) = 4^{-2L} z^{-2L} \sum_{l=0}^{L} \sum_{m=0}^{2l} \binom{2L+1}{2l+1} \binom{2l}{m} z^{2(m+L-l)} 4^{L-l} \tag{5.29}$$

dove L è un intero che caratterizza le dimensioni del reticolo.
A questo punto una attenta analisi delle proprietà di simmetria degli zeri del termine omogeneo dell'equazione funzionale (che per la sua forma potremmo chiamare identità di inversione) consente di raggiungere una soluzione analitica basata unicamente su una disposizione combinatoria degli zeri di $\Lambda(u)$ ottenuta a partire dagli zeri di $G(u)$.
Questa soluzione ha la forma:

$$\Lambda(u) = 2^{-L} \prod_{r=1}^{L} (\mu_r \sin(4u) + \sin(\omega_r)) \tag{5.30}$$

dove $\{\mu_r\}_{r=1}^{L}$ è una successione a valori in $\{-1, 1\}$ che caratterizza l'autovalore da un punto di vista combinatorio e $\omega_r = -i \log \gamma_r$, dove i $\{\gamma_r\}_{r=1}^{L}$ sono gli L zeri indipendenti di $G(u)$.

Dopo questo esempio passiamo a una descrizione più generale del modello appena presentato in termini di modelli $\mathbf{A}_n$ che sono modelli definiti a partire dai loro pesi di Boltzmann per i quali a ogni sito reticolare viene associata una variabile altezza $1 < h < n$ la quale è vincolata a differire di ± 1 dal valore assunto in uno dei siti primi vicini.
Per questi modelli gli unici pesi di Boltzmann diversi da zero sono quelli per cui le configurazioni delle altezze da cui dipendono rispettano le regole di adiacenza sopra citate. Le espressioni analitiche dei pesi di Boltzmann sono ottenute mediante funzioni θ ellittiche di parametro modulare q; tale parametro controlla la criticità della teoria che diventa critica per $q \to 0$ e va in regime di alta temperatura per $q \to 1$, inoltre q é legata alla temperatura ridotta t attraverso la relazione $t = q^2$.
Dunque in termini di questi pesi di Boltzmann è possibile costruire la matrice di trasferimento che soddisfa una certa equazione funzionale del tipo già incontrato nell'esempio precedente per la quale abbiamo un parametro di crossing $\lambda = \pi/(K+1)$.
Ora, per $K = 3$ abbiamo che il modello in esame si riduce proprio al modello di Ising, questo è dovuto alle regole di adiacenza che una volta fissate certe condizioni al bordo fanno si che si disaccoppino due reticoli uno dei quali è banale dal momento che la variabile altezza è identicamente 2, mentre l'altro è un reticolo con una variabile che puo assumere solamente i valori $h = 1, 3$ e che è evidentemente isomorfo al reticolo di Ising.
A questo punto usando questo formalismo siamo in grado di scrivere l'equazione funzionale che descrive il modello di Ising anche nel regime off-critico. La sua soluzione passa attraverso la conoscenza degli zeri di $G(u)$ (che è una funzione doppiamente periodica sul rettangolo $(-\pi/8, 7\pi/8) \times i(\frac{1}{2}\log q, -\frac{1}{2}\log q)$), zeri che sono organizzati lungo rette parallele all'asse immaginario e che si ripetono con una periodicità di $\pi/2$, a questo punto viste le proprietà di periodicità degli zeri risulta risulta necessario che le soluzioni abbiano zeri organizzati in strutture chiamate *1-stringhe* e *2-stringhe*.
Le 1-stringhe sono zeri singoli situati su una retta di parte immaginaria costante pari a $\pi/8$ mentre le 2-stringhe sono coppie di zeri aventi la stessa parte immaginaria e parte reale pari a $\pi/8,\ 3\pi/8$.
Le soluzioni $D(u)$ dell'equazione funzionale sono dunque generate a partire dagli zeri di $G(u)$ lungo la retta di parte reale costante pari a $\pi/8$, scegliendo un numero n di 1-stringhe e, detto L il numero degli zeri di $G(u)$, $L - n$ 2-stringhe disposte secondo una configurazione individuata da una sequenza $\{I_k\}$ di numeri topologici che assieme a n caratterizzano completamente l'autovalore della matrice di trasferimento.
In particolare per condizioni al bordo $(+,+)$ (tutti gli spin su) dovremo richiedere che n sia pari, mentre invece per bordi $(+,-)$ (spin su sul bordo sinistro e spin giu sul bordo destro) n deve essere dispari.

Vale inoltre la pena sottolineare che la funzione $G(u)$ non dipende dalla scelta delle condizioni al bordo (cioè è la stessa) per bordi di tipo $(+,+)$ e $(+,-)$.
A questo punto forti della nostra completa conoscenza degli zeri degli autovalori della matrice di trasferimento, procediamo con il *TBA* (che sta per ansatz termodinamico di Bethe) che consiste essenzialmente nel risolvere l'equazione funzionale lungo la retta $u = \pi/8 + ix/4$ supponendo per qualche ragione di conoscere la disposizione esatta degli zeri degli autovalori.
Passando attraverso uno sviluppo in serie di Fourier della derivata logaritmica dell'equazione funzionale, da cui si è stata sottratta la parte non analitica dovuta agli zeri degli autovalori, si riesce ad esprimere la soluzione in forma chiusa facendo uso di un kernel di convoluzione costruito mediante le funzioni ellittiche.
A questo punto abbiamo in mano una soluzione analitica della nostra equazione funzionale e possiamo osservare che essa ha dei contributi che divergono nel limite termodinamico (dimensione del reticolo $N \to \infty$) e nel regime critico ($q \to 0$); dal momento che saremo interessati al *limite continuo di scala* (in cui $N \to \infty$,$q \to 0$ simultaneamente) sottrarremo questi contributi e definiremo quindi $D_{finite}(x)$, che invece avrá un limite di scala finito, come:

$$\log D_{\text{finite}}(x) = \sum_{k=1}^{m} \log[p(x, v_k)p(x, -v_k)] + k * \log\left(1 + d(\frac{i}{4}x)\right) \tag{5.31}$$

dove $d(\frac{i}{4}x)$ è una espressione nota dal modello in termini di funzioni θ ellittiche, e $k(x)$ è un kernel definito come:

$$k(x) = -\frac{1}{4\log q} \sum_{k=-\infty}^{\infty} \frac{e^{\frac{ik\pi x}{2\log q}}}{e^{-\frac{k\pi^2}{4\log q}} + e^{\frac{k\pi^2}{4\log q}}} \tag{5.32}$$

e che può anch'esso essere espresso in termini di funzioni θ ellittiche.

5.4 Capitolo 4

In questo capitolo si parte esattamente dalla fine del capitolo precedente, ovverosia dal fatto che siamo riusciti a risolvere l'equazione funzionale determinando quindi gli autovalori della matrice di trasferimento per un reticolo di dimensione finita N.
Se ora consideriamo gli autovalori sottratti $D_{finite}(x)$ siamo interessati ad ottenere una espressione analitica per il loro limite di scala.
E possibile mostrare che il limite di scala è ottenibile per $N \to \infty$, $q \to 0$ secondo la prescrizione $N|t|^\nu = \mu$ dove $t = q^2$ è la temperatura ridotta, $\nu = 1$ è l'esponente critico della lunghezza di correlazione del modello (di Ising), mentre μ è un parametro adimensionale che ci tornerà presto utile.
A questo punto osserviamo che per $N \to \infty$ gli zeri degli autovalori della matrice di trasferimento tendono a mantenere una distanza finita dal bordo del rettangolo di analiticità $(-\frac{\lambda}{2}, \frac{7}{2}\lambda)\times$

$i(\frac{1}{2}\log q, -\frac{1}{2}\log q)$, ma poichè $q \to 0$ il bordo del rettangolo tende a spostarsi verso $i\infty$, cosicchè siamo costretti per mantenere gli zeri in una regione limitata del piano ad applicare la traslazione uniforme $x \to x + \log N$. Con queste prescrizioni siamo quindi in grado di eseguire un calcolo analitico del limite di scala, che per convenienza effettuiamo su $\log D_{finite}(x)$ (che ha le dimensioni di un energia),e che chiameremo $\log \hat{D}(x)$.
Quello che si ottiene è un risultato che è composto di due termini, il primo dei quali è il termine di 1-stringhe che corrisponde alle eccitazioni di energia rispetto allo stato di vuoto del settore $(+,+)$, mentre il secondo termine è invece un termine di convoluzione che rappresenta l'energia del ground state del settore $(+,+)$:

$$\log \hat{D}(x) = \sum_{k=1}^{m} \log \left[\tanh \frac{x + \log \mu - y_k}{2} \tanh \frac{x + \log \mu + y_k}{2} \right] + \int_{-\infty}^{+\infty} dy \frac{\log(1 + \hat{d}(y))}{2\pi \cosh(x - y)} \quad (5.33)$$

dove si capisce che $\hat{d}(x)$ è il limite di scala del termine $d(\frac{i}{4}x)$ presente in $\log D_{finite}(x)$.
A questo punto siamo interessati a fare una espansione di $\log \hat{D}(x)$ in serie rispetto alle funzioni $\{e^{(2n-1)x}\}_{n=1}^{\infty}$:

$$\log \hat{D}(x) = -\sum_{n=1}^{\infty} C_n I_{2n-1}(\mu) e^{(2n-1)x} \quad (5.34)$$

Tale espansione si fa in modo analitico e detti $C_n I_{2n-1}(\mu)$ i suoi coefficienti si trova:

$$C_n I_{2n-1}(\mu) = \sum_{k=1}^{m} \frac{4}{2n-1} e^{(2n-1)\log \mu} \cosh((2n-1)y_k) + (-1)^n \int_{-\infty}^{+\infty} \frac{dy}{\pi} e^{-(2n-1)y} \log(1 + \hat{d}(y)) \quad (5.35)$$

Siamo ora interessati a studiare il comportamento degli $I_{2n-1}(\mu)$ al variare di $\mu \in (0, \infty)$.
È importante ricordare che i coefficienti C_n sono delle costanti che non dipendono dalle condizioni al bordo e del resto sfruttando il fatto che sono delle costanti diventa naturale studiare, invece dei coefficienti $I_{2n-1}(\mu)$ che nel nostro modello non sono direttametne osservabili, i rapporti degli I_{2n-1} per diverse coppie di stati (ciascuno individuato dai suoi numeri topologici$\{I_k\}$).
Quello che si trova è che per $\mu \to 0$ (detto limite U.V.) i rapporti tra gli $I_{2n-1}(\mu)$ fluiscono fino a raggiungere un punto fisso.
Per il limite $\mu \to \infty$ (detto limite I.R.) è evidente che la stabilizzazione completa del flusso è quasi raggiunta, ma poichè nel calcolo del termine di convoluzione in $\log \hat{D}$ si è impiegata una approssimazione numerica che tronca il limite di scala a un certo valore fissato $\overline{N}$, si ha che siccome $|q| < 1$, $\mu \in (0, \overline{N})$, μ non può assumere valori sufficientemente grandi da osservare una termalizzazione completa.
Ora, quello che accade è che dallo studio delle teorie conformi, che per quanto detto in precedenza sappiamo essere teorie integrabili, sappiamo che esse possiedono una infinità di integrali del moto $\mathbf{I}_{2n-1}$ tra loro commutanti e di spin $2n-1$ dei quali si conosce la forma esatta solamente

per quelli di ordine piu basso, ma che possono comunque essere costruiti in maniera iterativa a partire dai generatori dell'algebra di Virasoro.
Se noi a questo punto diagonalizziamo questi operatori $\mathbf{I}_{2n-1}$ all'interno dei moduli di Verma corrispondenti ai diversi settori di un certo modello minimale unitario, siamo in grado di conoscere i loro autovalori che non a caso chiameremo I_{2n-1}.
Quello che accade è che se in particolare consideriamo il modello minimale più semplice $\mathcal{M}_{3,4}$ e diagonalizziamo $\mathbf{I}_3$ e $\mathbf{I}_5$ nei 3 settori della teoria di peso conforme 0,1/2 e 1/16 scopriamo che i rapporti dei loro autovalori sono in corrispondenza biunivoca con i rapporti calcolati in precedenza dal modello su reticolo ne limite U.V. .
Questo ci permette di identificare in modo chiaro e univoco il contenuto operatoriale del nostro modello su reticolo corrispondente alla scelta delle condizioni al bordo $(+,+)$ (a cui corrisponde l'operatore identitá della teoria conforme) e condizioni al bordo $(+,-)$ (a cui corrisponde un operatore primario di peso conforme 1/2).
Inoltre diventa possibile creare una corrispondenza diretta tra stati della teoria su reticolo, individuati dai propri numeri topologici, e stati dello spazio di Hilbert della teoria conforme.
Per quanto riguarda il flusso I.R. appare abbastanza evidente che i rapporti da noi calcolati fluiscaono nel valore 1 indipendentemente dal settore considerato. Tuttavia una attenta lettura di [11] suggerisce che questo possa essere vero solamente per gli stati meno eccitati quello che probabilmente si potrebbe ottenere da una analisi piu accurata dello spettro della matrice di trasferimento è che ci saranno gruppi di stati dello stesso settore che andranno a fluire nel limite I.R. nello stesso stato.
Inoltre si congettura che la teoria di campo corrispondente al limite I.R. sia una teoria massiva a una particella e che lo spettro di stati nel continuo ottenibile dal flusso I.R. corrisponda alle eccitazioni massive a momento nullo (in pratica alle particelle ferme).

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