UNIVERSITÀ DEGLI STUDI DI MILANO
FACOLTÀ DI SCIENZE E TECNOLOGIE

Università degli Studi di Milano

FACOLTÀ DI SCIENZE E TECNOLOGIE
Corso di Laurea Magistrale in Fisica

Tesi di laurea magistrale

Fock-space operator construction of Laughlin fractional quantum Hall effect states

Relatore:
Prof. Sergio Caracciolo

Elaborato finale di:
Andrea Di Gioacchino
Matricola 864075

Correlatori:
Prof. Luca Guido Molinari
Dott. Pietro Rotondo

Anno Accademico 2015–2016

Contents

Introduction **3**
The story of the quantum Hall effect . . . 3
Experimental techniques . . . 4
Why is the QHE so important? . . . 8
Our aim . . . 9

1 The quantum Hall effect **12**
1.1 The classical Hall effect . . . 12
1.2 Landau Levels . . . 13
1.2.1 Degeneracy and filling factor . . . 15
1.2.2 The wave function of many independent electrons . . . 17
1.2.3 A qualitative explanation of the IQHE . . . 18
1.3 The fractional quantum Hall effect . . . 19
1.3.1 The Hamiltonian . . . 19
1.3.2 Exact diagonalization method . . . 22
1.3.3 Variational method: Laughlin's wave function . . . 23
1.3.4 Haldane pseudopotentials . . . 25
1.3.5 Other filling factors, plateaux formation and Composite Fermions theory . . . 29

2 Symmetric functions and Jack polynomials **32**
2.1 Partitions . . . 32
2.1.1 Squeezing . . . 33
2.1.2 Ordering . . . 33
2.2 The ring of symmetric functions . . . 34
2.2.1 Relevant basis . . . 35
2.3 Jack polynomials . . . 36
2.3.1 Properties . . . 37
2.3.2 Negative parameter Jack polynomials . . . 38
2.3.3 Jack antisymmetric functions . . . 38
2.4 Jacks recursion law . . . 39
2.4.1 Generic recursion laws . . . 40
2.4.2 Action of triangular operator . . . 40
2.4.3 Jacks recursion law . . . 42
2.4.4 Antisymmetric Jacks recursion law . . . 43
2.4.5 Remarks on recursion laws . . . 44

3 Fock-space construction of Laughlin states **45**
3.1 A long-standing problem 45
3.1.1 Bernevig's recursion law 48
3.1.2 Fermionic case 49
3.2 Beyond the recurrence relation: squeezing operator 51
3.2.1 Are we considering all the partitions? 55
3.3 Other results 57
3.3.1 Another proof of Bernevig's product rules 57
3.3.2 Symmetries and the squeezing operator 60
3.3.3 A different algorithm for the coefficients 62

4 Conclusions and future directions **65**
4.1 Our results 65
4.2 Future directions 66

Introduction

The story of the quantum Hall effect

In 1980 Von Klitzing and his team were studying the Hall effect at low temperatures (around 1 K) and high magnetic fields (around 10 T), when they observed an unexpected phenomenon: the quantization of the Hall resistance (i.e. the resistance measured along the direction perpendicular to the current flow), concurrent with a "superflow", i.e. a dissipationless current flow [1]. They coined the name *quantum Hall effect* (QHE) in order to describe this new intriguing phenomenology, which turned out to be independent of the microscopic details of the material, or of the sample type or geometry, provided that the conductor samples were almost two-dimensional.

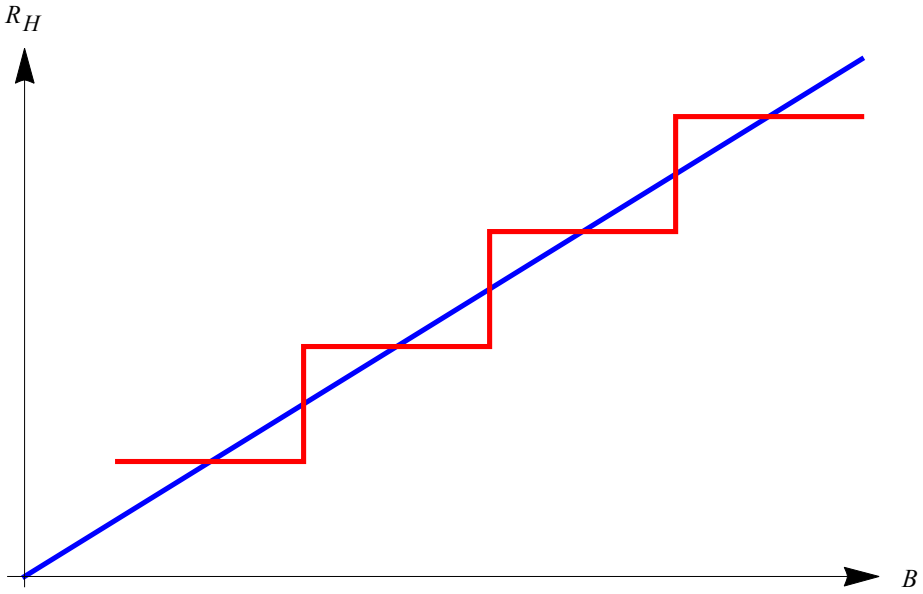

Figure 1: The blue line represents the classical expected Hall resistance R_H as function of the magnetic field B. In red there is a schematic picture of the plateaux (regions in which the Hall resistance is constant) observed at high magnetic fields (larger than 10 T) and low temperatures ($\sim$ 1 K or less). We remark that the effectively observed plateaux are neither vertically spaced by a constant value, nor all of the same width.

In particular (as shown schematically in figure 1 and with an experimental plot in figure 2) the Hall resistance was observed to be quantized at

$$R_H = \frac{h}{ne^2}, \tag{1}$$

where h is the Planck constant, e is the electron charge and n is an integer number. The quantity $h/e^2 = 25813.807\ \Omega$ is called Von Klitzing constant. This effect is today called *integer* quantum Hall effect (IQHE), in order to differentiate it from the *fractional* quantum Hall effect, described below. This quantization is quite surprising, because it does not refer to a microscopic system as many other known quantizations, but to a *macroscopic* sample.

Two years later, in 1982, Tsui, Stormer and Gossard [2] discovered the so called fractional quantum Hall effect (FQHE), i.e. the quantization of the Hall resistance at

$$R_H = \frac{h}{fe^2}, \tag{2}$$

where f is a fraction, also called *filling factor*. They have been able to obtain this result thanks to the use of higher magnetic fields and lower temperatures than those of Von Klitzing's experiment.

While the integer QHE was explained in terms of non-interacting electrons only, the FQHE was recognized to take place because of the intrinsic interacting nature of the electrons. Despite the tremendous effort done in the last 30 years in order to understand it, a systematic microscopic theory of this fascinating macroscopic quantization due to collective phenomena is yet to come.

At the very beginning, only the filling factor 1/3 was observed, but today more and more fractions have been discovered thanks to technological advance in preparing semiconductor samples. For instance, more than 30 different filling factors in the interval between 0 and 1 have been observed [3]. In figure 2 we show a measurement of the Hall and the diagonal resistance in a QHE experiment. In this figure we can easily recognize several plateaux, i.e. regions in which the Hall resistance is constant. In correspondence of these regions the longitudinal resistance suddenly drops down. In figure 3 we show an example of a recent quantum Hall effect measurement in which the diagonal resistance (i.e. the resistance measured along the direction of the current flow) is measured as a function of the magnetic field. We notice that the number of observed resistance drops is considerably greater than those observed in figure 2, as a consequence of the experimental techniques improvement.

Experimental techniques

The QHE is characteristic of two-dimensional electron systems in a transverse magnetic field. So the experimental setup used has the property of trapping electrons in two dimensions. Such systems are achieved using artificial structures, created by a process called *molecular beam epitaxy* (MBE), which allows a controllable, layer-by-layer growth, in which one type of semiconductor can be grown on top of another, approximately lattice-matched semiconductor to produce an atomically sharp interface.

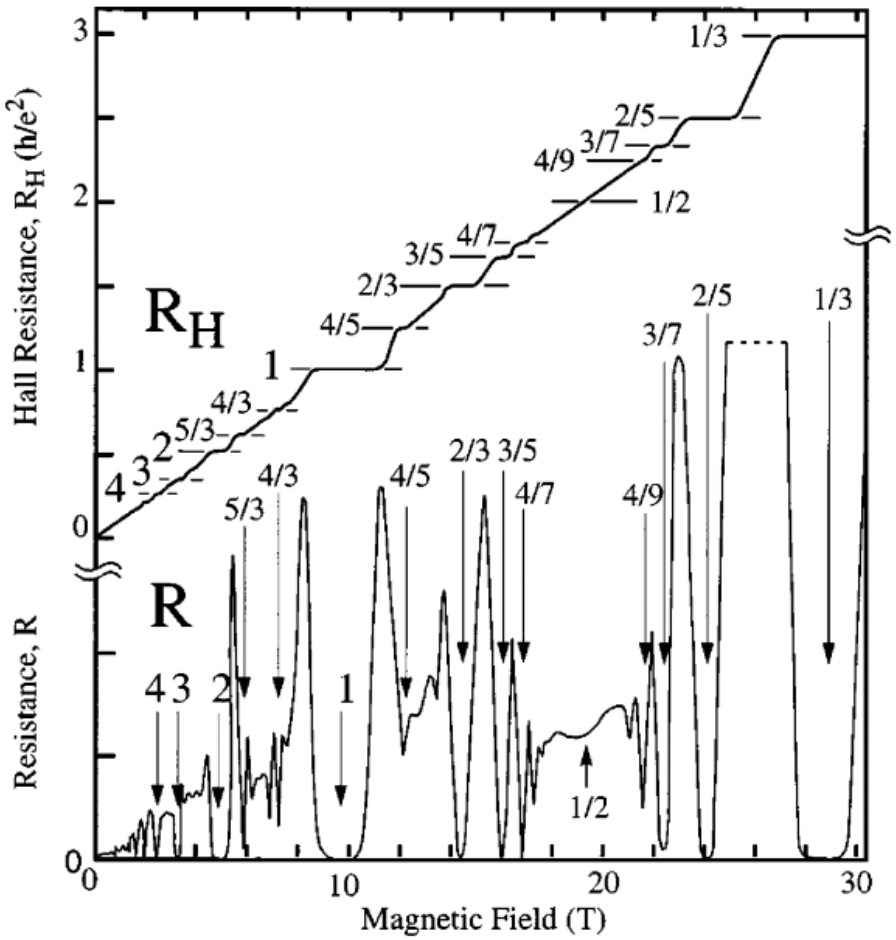

Figure 2: Overview of Hall and longitudinal resistances, R_H and R, respectively. We can clearly see the plateaux (regions in which the Hall resistance is constant) in the R_H plot, which correspond to dips in the R plot (no plateau and dip are associated with $f = 1/2$). Source: H. L. Stormer, *Rev.Mod.Phys.* **71**, 875889 (1999).

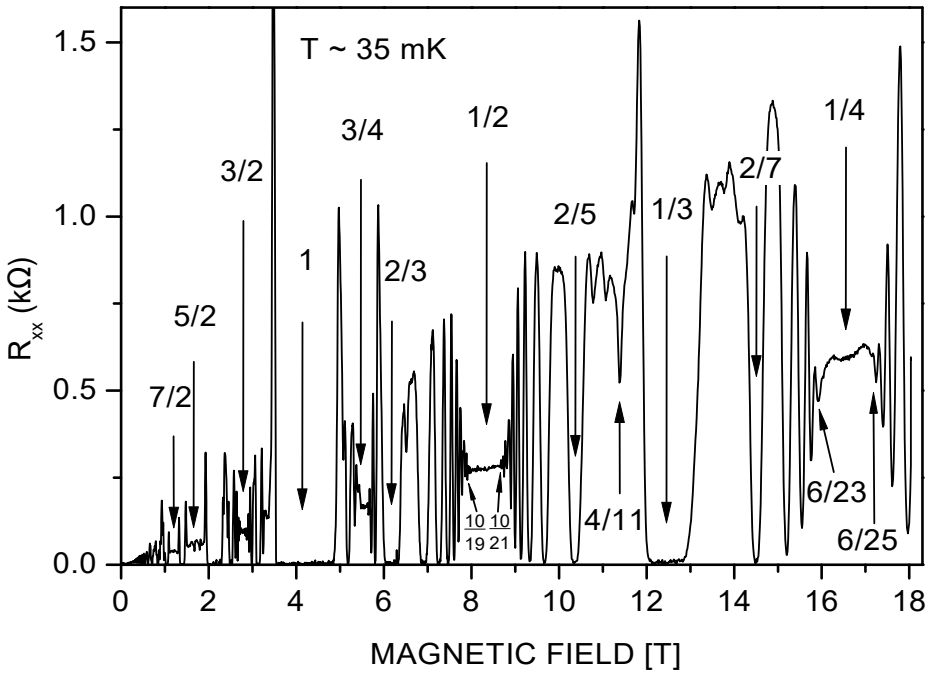

Figure 3: Longitudinal resistance as a function of the magnetic field measured in a recent quantum Hall effect experiment. A FQHE or an IQHE state is associated with each minimum. Many arrows only indicate the positions of filling factors and have no FQHE associated with them. Source: Pan, Stormer, Tsui, Pfeiffer, Baldwin and West , *Phys.Rev.Lett.* **88**, 176802 (2002).

In this way the electron mobility[1] of the resulting structure is particularly high (up to 10 millions $cm^2/(V{\cdot}s)$). An important request is that the two semiconductors must have a different energy gap. For example, two semiconductors used are AlGaAs, which has a gap of 2.2 eV and GaAs, whose gap is of 1.5 eV. An example of structure used is the "quantum well" AlGaAs–GaAs–AlGaAs: in this case the smaller gap of the central semiconductor (which typically has a width of $20 - 50$ nm) makes the realization of the quantum well possible, as shown in figure 4.

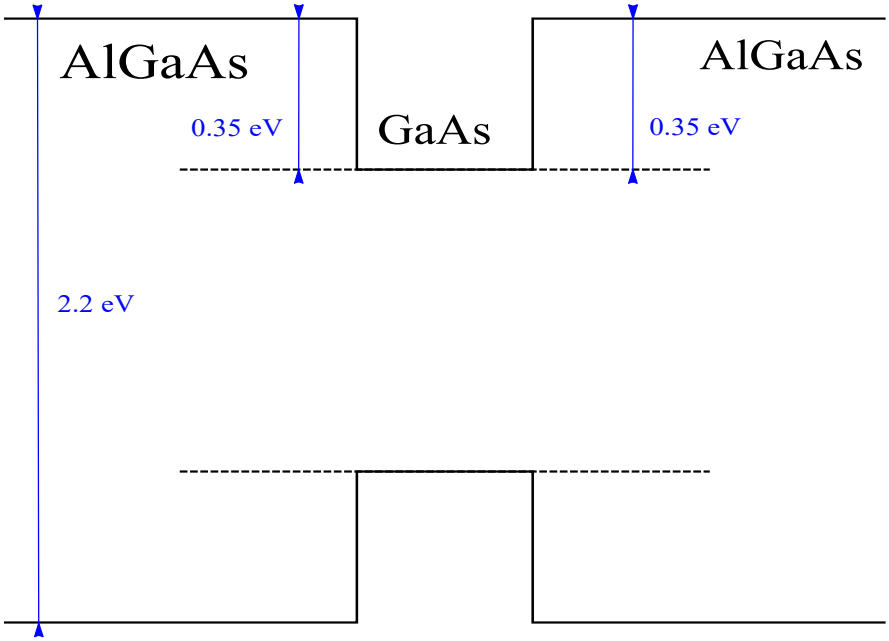

Figure 4: This is a schematic picture of the energy band diagrams of the AlGaAs–GaAs–AlGaAs (undoped) quantum well. This semiconductors structure is created growing a semiconductor on top of another. The lower and upper lines are the energy profiles of the valence and conduction bands respectively, and the square well is due to the different gap of the semiconductors: AlGaAs has a 2.2 eV gap, while GaAs gap is of 1.5 eV. Upon doping the AlGaAs regions with donors, electrons fall into the GaAs region, which width is of $20 - 50$ nm, and form a two-dimensional electron system.

If the AlGaAs semiconductors are doped, the free electrons (or holes) are trapped in the GaAs region and if the trapping energy is kept very large with respect to all the other energies, a two dimensional electron system is realized.

A good quality sample is characterized by high electron mobility. Such a sample allows the observation of more plateaux. If the electron mobility is decreased or if the temperature is increased, the QHE plateaux landscape gradually fades into the straight line typical of the classic Hall effect: in the first case the plateaux are broken because of the interaction between electrons

[1] the electron mobility is a parameter of a metal or a semiconductor that represents the resistance to the electron passage if an electric field is applied. It is defined as v_d/E, where v_d is the electron drift velocity when an electric field E is applied.

and impurities, in the second case it is the thermal energy that destructs them.

Why is the QHE so intriguing?

Since its discovery, studies about the QHE have made several physicists worthy of the Nobel prize, in 1995 and in 1998. Moreover, a very wide literature exists about the QHE in its integer or fractional form, considering books and papers (two excellent and rather recent reviews can be found in [4] and [5]). Nevertheless today, after more than 30 years from its discovery, there are still many open questions regarding the microscopic origin of the quantum Hall effect. Indeed a theory which explains all the results obtained in the years regarding this effect is lacking to date.

For the simplest series of filling factors f (referring to equation (2)), namely those of the form $1/m$ with m an odd integer, the physics of the FQHE is indeed well-understood phenomenologically thanks to the pioneering work by Laughlin and his celebrated ansatz [6], which will be considered in section 1.3.3. His approach was generalized to more complicated fractions [4, 7–9] and a huge amount of results were obtained in the years, confirming the validity of this approach based on model wave functions, which are wave functions specifically constructed in order to describe the ground state of the system for a particular series of filling factors.

The most valuable generalization of Laughlin's work is the composite fermion theory introduced by Jain in 1989 [4]. Furthermore the unexpected link between the Laughlin ansatz and the Conformal Field Theory, uncovered by Moore and Read [8, 9], allowed to put forward a new class of model wave functions, that for instance describe quite well the $f = 1/2$ state, which do not exhibits any plateau. There are two strong points that made the model wave functions approach prominent along the years:

- these wavefunctions (for instance the Laughlin series) have an impressive overlap (> 0.99) with the ground state found by exact diagonalization methods. Additionally they allow for a relatively straightforward definition of fractional quasi-particle excitations, carrying fractional charges, which have been observed experimentally [10];

- in some cases particular two-body interactions exist (built for the first time by Haldane [7]), for which the model wave functions, such as Laughlin's ones, are the exact ground state.

Nonetheless the strongly correlated nature of these wave functions does not allow to perform calculations in the thermodynamic limit, considerably limiting our understanding of this phenomenon from a microscopic point of view. Moreover, a systematic theory of the fractional quantum Hall effect, which is able to explain all the observed fractions f at once, is still missing.

On the other hand, the very fact that there are many theories used to describe this effect makes it interesting. Among them we can cite the Conformal Field Theory, Luttinger liquid theory, Chern-Simons theory and others, which came from other branches of Physics and can also be applied to explain some features of the QHE. Then the quantum Hall effect is an important testing ground for these theories, thanks to the possibility of validating the prediction of them

in QHE experiments. Moreover, the interest in the fractional quantum Hall effect is recently renewed, because the FQHE seems to be a promising hunting ground for the so called *topological phases of matter*, which have in turn promising applications in the field of quantum computation since they could allow to encode and manipulate quantum information in a manner that is resistant to error [11, 12].

Our aim

With the phrase "*Fock-space construction*" we mean the natural way in which we write a state in a second quantization framework, that is as its expansion over the *occupation number* basis, or equivalently as several creation operators applied over a vacuum. A Fock-space construction of FQHE states could give us many advantages: for example, it could allow us to calculate expectation values of some important operators as the density. It could also give us fundamental insights into the understanding of fractional excitations, since the Fock-space formalism is ideal to investigate the spectrum of a quantum theory (two examples are the BCS theory of superconductivity and also the simple harmonic oscillator).

In a general framework, the occupation number basis is obtained in the following way. We consider a basis $\{|a\rangle\}$ of an Hilbert space and we want to describe a system of n identical particles. Thus a basis is obtained inserting all the possible combination of single-particle states in

$$\mathcal{S}_{\pm} |a_1, \ldots, a_n\rangle , \tag{3}$$

where $\mathcal{S}_+$ is the symmetrization operator, $\mathcal{S}_-$ the anti-symmetrization one (we use the one or the other depending on the statistic of the particles involved). We want to pass to the occupation number basis, which is composed of state as $|n_1, n_2, \ldots\rangle$ where n_1 is the number of particle in the first state of the original single-particle basis $\{|a\rangle\}$, n_2 the number of particles in the second state and so on. We are not losing information switching between these two basis because of the action of the (anti)symmetrization operator, which make impossible for us to say which particle is in which single-particle state. Therefore the two basis are in a one-to-one correspondence. If we construct the wave functions basis of the single-particle problem, i.e. ($|x\rangle$ is the eigenstate of the position operator)

$$\psi_a(x) = \langle x|a\rangle \tag{4}$$

we can recognize in the expression $\mathcal{S}_{\pm} |a_1, \ldots, a_n\rangle$ the Slater determinats or permanents (for fermions and bosons respectively). In other words, it can be proved that

$$\langle x_1, \ldots, x_n|n_1, n_2, \ldots\rangle = \langle x_1, \ldots, x_n| \mathcal{S}_{\pm} |a_1, \ldots, a_n\rangle = \mathcal{C} \underset{Det}{\overset{Per}{}} \psi_a(x_i), \tag{5}$$

where the constant $\mathcal{C}$ is a normalization constant (it depends on our definition of $\mathcal{S}_{\pm}$).

We want now to apply this idea to the fractional quantum Hall effect, and we can start making an explicit example taken directly from the FQHE itself. The Laughlin wave function which describes the ground state of the system with

n electrons at filling factor $1/q$ is (it is obtained following the Laughlin's ansatz in section 1.3.3):

$$\psi_q(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j)^q \, e^{-\frac{1}{4}\sum_i |z_i|^2}, \tag{6}$$

where $z_j = (x_j - iy_j)/\ell$ is the complex coordinate which describes the j-th electron position and $\ell = \sqrt{\hbar c/eB}$ is called magnetic length. We want to expand this wave function over the occupation number basis, i.e. over the Slater determinants (because Laughlin's wave functions describe electrons) basis. In order to do it, we need to choose a suitable single-particle basis, which is the following (un-normalized) one[2]

$$\langle z|m\rangle = z^m e^{-\frac{1}{4}|z|^2}. \tag{7}$$

Using this basis, the exponential part of the Laughlin's wave function is trivial, but the polynomial part is very difficult to expand. For example, we can write the explicit expansion for $n = 2$, $q = 3$ (only the polynomial part is considered):

$$\psi_3(z_1, z_2) = (z_1 - z_2)^3 = \begin{vmatrix} z_1^3 & 1 \\ z_2^3 & 1 \end{vmatrix} - 3 \begin{vmatrix} z_1^2 & z_1 \\ z_2^2 & z_2 \end{vmatrix} = sl_{(3,0)} - 3sl_{(2,1)}, \tag{8}$$

where $sl_{(a,b)} = \langle z_1, z_2| \, S_- \, |a, b\rangle$ and then

$$sl_{(a,b)} = \begin{vmatrix} z_1^a & z_1^b \\ z_2^a & z_2^b \end{vmatrix}. \tag{9}$$

So this is the Fock-space picture we are searching for, but here is the problem: the number of different Slater determinats involved in this expansion, whose coefficients we have to calculate, increases as an exponential of the number of electrons. However, the coefficients of the Slater determinant expansion in equation (8) are integers, and this property holds if we increase the number of electrons. This suggests us that a combinatorial explanation of these coefficients could exist. Many authors have worked on such an expansion [13, 14], trying to understand a way to obtain the correct Slater determinants involved and their coefficients. A breakthrough (extensively explained below) has been made by Bernevig, Regnault, Haldane and their co-workers: they have proved that the polynomial parts of the Laughlin's wave functions are the anti-symmetrized version of the so-called *Jack polynomials*, which have been widely studied in the mathematics literature. This allow them to give a precise algorithm to obtain every Slater determinant involved starting from a reference state, as well as a recursion relation for the coefficients.

However, there are still many open questions, for example: is this algorithm applicable to other FQHE model wave functions, different from the Laughlin's? If so, how can we choose the reference state from which start? And again, can this path bring us to a microscopic and systematic theory of the FQHE?

An hint which make us thinking that this is an interesting path comes from a recent paper [15], whose main result is the mapping between the quantum Hall effect Hamiltonian (restricted to a particular subspace of the Hilbert space, the lowest Landau level, which is discussed in chapter 1) and a long-range repulsive

[2] this choice is motivated by physical reasons, which are explained in chapter 1.

lattice gas model in one dimension. This mapping is shown to be exact in a particular limit, called thin-torus limit. Although this limit is non-physical[3], the predicted plot of the Hall resistivity versus the magnetic field is in qualitative agreement with the one experimentally observed, for a remarkable number of filling factor. Moreover, we notice that the ground state of the long-range repulsive lattice gas model is a state in which each particle is at the same distance from its neighbours, and these states are exactly the same reference states from which Bernevig's algorithm starts.

In order to investigate this fact, in this work we develop an operatorial form of the Bernevig's algorithm, that is a Fock-space operator (i.e. composed of destruction and creation operators) that constructs Laughlin's wave functions if applied to the correct reference state. This operator is rather general, so our hope is that if it is applied to other reference states (for example those obtained in [15]), it could lead to the Fock-space construction of other model wave functions.

[3] i.e. the requests upon the parameters are almost never satisfied in the experimental setups.

Chapter 1

The quantum Hall effect

In this chapter we will review all the necessary background needed to discuss our work on the Fock-space operator construction of the fractional quantum Hall effect states. We will start from a short review of the classical Hall effect, then we will explain how a single electron behaves if it is confined in a two-dimensional plane with a transverse magnetic field. We will see that in this case the electron energy is quantized and the energy levels formed are called Landau levels. We will briefly see how this simple argument is able to explain qualitatively the integer QHE, but in order to understand the fractional QHE we need to consider the electron interactions. After a discussion on the microscopic Hamiltonian of the QHE system, we will consider the exact diagonalization and the variational methods used to study it, with particular emphasis on the Laughlin's wave functions. In the end of the chapter, Jain's Composite Fermions theory is briefly introduced for completeness.

1.1 The classical Hall effect

The local form of the Ohm law is:

$$\mathbf{J} = \sigma \mathbf{E}, \tag{1.1}$$

where σ is the conductivity tensor, $\mathbf{E}$ is the applied electric field and $\mathbf{J} = \rho q \mathbf{v}$ is the current density for particles of charge q and density ρ moving with a velocity $\mathbf{v}$. In 1879, Hall discovered that, in presence of a magnetic field, current flows in a direction perpendicular to that of the applied field, i.e. the tensor σ is not a diagonal matrix. In other words, the passage of a current induces a voltage perpendicular to the direction of the current flow. This is known as *Hall effect* and the system in which it takes place is sketched in figure 1.1.

This phenomenon has a classical origin and it is readily understood using the Lorentz equation of electrodynamics:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \tag{1.2}$$

which gives the force acting on a particle of charge q, moving with a velocity $\mathbf{v}$, in presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. This equation can be

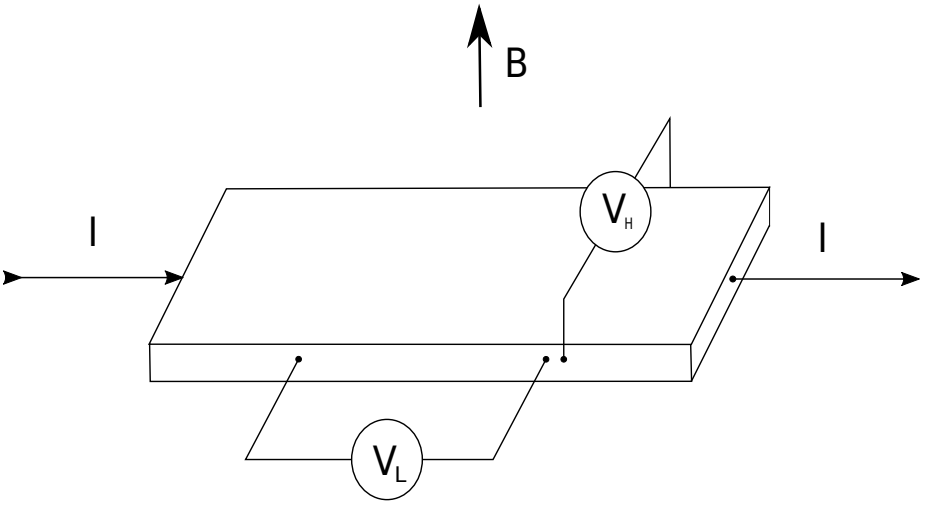

Figure 1.1: Sketch of a diagram of the classical apparatus used in Hall effect experiments. I, V_L, and V_H are the current, longitudinal voltage and the Hall voltage, respectively. The longitudinal and Hall resistances are defined as $R_L = V_L/I$ and $R_H = V_H/I$.

integrated to get the particle motion, and if we take $\mathbf{E} = E\hat{y}$ and $\mathbf{B} = B\hat{z}$, we can see that the charged particle drifts in a direction perpendicular to the plane containing the two fields, with a velocity $\mathbf{v} = c(E/B)\hat{x}$. Using this in $\mathbf{J} = q\rho\mathbf{v}$ and the inverse of Ohm law to get the Hall resistance[1], we obtain:

$$R_H = \frac{E_y}{J_x} = \frac{B}{\rho q c}. \tag{1.3}$$

This effect is routinely used to measure the density of charge carriers, and the sign of their charge (i.e. whether they are electrons or holes).

The direct proportionality between R_H and B holds until the system is not two-dimensional, the magnetic field is not too high ($B \sim$ tens of Tesla) and the temperature is not too low ($T \sim 1$ K or less).

1.2 Landau Levels

When the system is two-dimensional, the temperature is sufficiently low and the magnetic field sufficiently high, the *quantum* Hall effect takes place. In order to understand this phenomenon, we begin with the study of a simpler system, which is of fundamental importance for the quantum Hall effect: a single electron (particle of charge $-e$) moving in two dimensions in a perpendicular magnetic field. The Hamiltonian of this system is:

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2, \tag{1.4}$$

[1] the resistivity tensor is the inverse of the conductivity tensor, and in this two-dimensional system we can easily see that the Hall resistance and the Hall resistivity are the same quantity.

where m is the electron's mass, $\mathbf{p} = -i\hbar\nabla$ is the canonical momentum operator and $\mathbf{A}$ is given by

$$\nabla \times \mathbf{B} = \mathbf{A}. \tag{1.5}$$

We notice that the Schrödinger equation $H\psi = E\psi$ is gauge-invariant, i.e. it is invariant under the transformation

$$\begin{aligned} &\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \nabla\theta(\mathbf{r}); \\ &\psi(\mathbf{r}) \to \exp\left[-\frac{ie}{\hbar c}\theta(\mathbf{r})\right]\psi(\mathbf{r}). \end{aligned} \tag{1.6}$$

We consider the symmetric gauge using a suitable function θ for which

$$\mathbf{A} = \frac{\mathbf{B}\times\mathbf{r}}{2} = \frac{B}{2}(-y, x, 0) \tag{1.7}$$

holds. Inserting this in the Hamiltonian, we find:

$$H = \frac{1}{2}\hbar\omega_c\left[\left(-i\,\ell\frac{\partial}{\partial x} - \frac{y}{2\ell}\right)^2 + \left(-i\,\ell\frac{\partial}{\partial y} + \frac{x}{2\ell}\right)^2\right] \tag{1.8}$$

where $\omega_c = eB/mc$ is the *cyclotron energy* and $\ell = \sqrt{\hbar c/eB}$ is the *magnetic length*. From now on, we will measure the energies in unit of $\hbar\omega_c$ and the lengths in unit of ℓ.

With the following transformation,

$$z = x - iy = re^{i\theta}\,; \qquad \bar{z} = x + iy = re^{-i\theta} \tag{1.9}$$

and with the definition of the operators

$$\begin{aligned} b &= \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right), \\ b^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right), \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right), \\ a &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right), \end{aligned} \tag{1.10}$$

we get

$$H = \frac{1}{2} + a^\dagger a. \tag{1.11}$$

With these definitions, one can easily verify that $[a^\dagger, a] = 1$ and $[b^\dagger, b] = 1$ and all the other commutators are zero, so the Hamiltonian is equivalent to that of a one-dimensional harmonic oscillator and the energy eigenvalues are quantized at

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega_c. \tag{1.12}$$

These energy levels are called *Landau levels*.

The z component of the angular momentum is defined as

$$\begin{aligned} L &= -i\hbar\frac{\partial}{\partial\theta} \\ &= -\hbar\left(b^\dagger b - a^\dagger a\right), \end{aligned} \tag{1.13}$$

where we have used the definitions (1.9).

We notice that the energy of the particle is independent of its angular momentum, as expected from the fact that the Hamiltonian is invariant under rotation with axis $\hat{z}$ and $[H, b] = 0$, $[L, b] \neq 0$. Because $[H, L] = 0$, we choose the eigenfunctions that diagonalize the Hamiltonian and the angular momentum operator, denoting respectively with n and m the quantum numbers. This means that a state $|n, m\rangle$ has energy $(1/2 + n)\hbar\omega_c$ and angular momentum $-\hbar m$. From equation (1.13) we can see that the admitted values of m are $-n, -n+1, \ldots, 0, 1, \ldots$ if the particle is in the n-th Landau level. The application of $b^\dagger$ increases m by one unit while preserving n, whereas $a^\dagger$ simultaneously increases n and decreases m by one unit.

The state indexed by the quantum number n, m can be written as

$$|n, m\rangle = \frac{(b^\dagger)^{n+m}}{\sqrt{(n+m)!}}\frac{(a^\dagger)^n}{\sqrt{n!}}\,|0, 0\rangle\,, \tag{1.14}$$

where the state $|0, 0\rangle$ is the one that satisfies $a\,|0, 0\rangle = b\,|0, 0\rangle = 0$. Using these requests, we obtain

$$\eta_{0,0}(\mathbf{r}) = \langle \mathbf{r}|0, 0\rangle = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{4}z\bar{z}}. \tag{1.15}$$

Now we can obtain the generic n, m state simply applying the definitions of the creation operators given in equations (1.10). In particular, we are interested in the *lowest Landau level* (LLL) wave functions, i.e. the states $|0, m\rangle$, given by

$$\eta_{0,m}(\mathbf{r}) = \langle \mathbf{r}|0, m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}\eta_{0,0}(\mathbf{r}) = \frac{z^m}{\sqrt{2\pi\, 2^m\, m!}}e^{-\frac{|z|^2}{4}}. \tag{1.16}$$

In figure 1.2 there is the 3D plot of $|\eta_{0,m}|^2$ with $m = 15$ (the plots obtained for other m are similar).

1.2.1 Degeneracy and filling factor

We remark that in the LLL (but we can do an analogue reasoning for each Landau level) the eigenstate $|0, m\rangle$ has its probability density centered on the circle of radius $r = \sqrt{2m}\,\ell$, as we can see by calculating the values of $|z|^2 = r^2/\ell^2$ for which

$$\frac{\partial|\eta_{0,m}|^2}{\partial|z|^2} = 0. \tag{1.17}$$

Moreover, moving away from this circle we can see that the probability of finding the particle rapidly drops to zero (see also figure 1.2). Therefore we can think to the electron as a classical particle which moves in the circle of radius r. We consider a disk of radius R and evaluate how many particles (in this classical picture) lie inside it in a given Landau level. The largest value of m for which

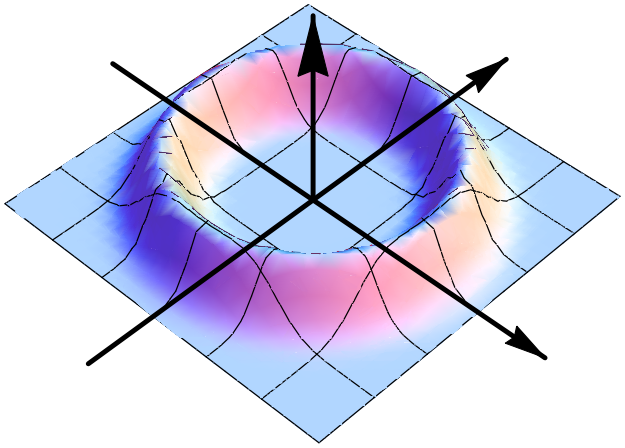

Figure 1.2: The plotted function is $|\eta_{0,m}|^2(x,y)$ with $m = 15$, so the value related to the vertical axis represents the probability density of finding an electron within the horizontal plane in the lowest Landau level with angular momentum of $-15\hbar$.

the electron is inside the disk is given by $m = R^2/2l^2$, which is thus the total number of degenerate eigenstates in the lowest Landau level that are "available" inside the disk. Therefore, the degeneracy per unit area is

$$G = \frac{R^2/2\ell^2}{\pi R^2} = \frac{1}{2\pi\ell^2} = \frac{eB}{hc} \tag{1.18}$$

(and it is the same everywhere: the calculation would have been the same if we had counted the particles from R_1 to R_2). The *filling factor* is given by

$$\nu = \frac{\rho}{G} = 2\pi\ell^2\rho, \tag{1.19}$$

where ρ is the electron density. Thus the filling factor is the number of electron per area unit over the number of degenerate state per area unit, and so it equals the number of occupied Landau levels in the case of *non-interacting* electrons at a given magnetic field.

Moreover, defining $\phi_0 = hc/e$ as the "magnetic field flux quantum", we readily see that $G = B/\phi_0$, i.e. the degeneracy per area unit is the number of

flux quantum piercing the sample. Using that in equation (1.19), we get

$$\nu = \frac{\rho}{B/\phi_0}, \tag{1.20}$$

that is another interpretation of the filling factor: it also equals the number of electrons per flux quantum.

We notice that the filling factor is inversely proportional to the magnetic field. As the magnetic field is increased, each Landau level has more and more degenerate states to allocate the fixed number of electrons, so less and less Landau levels are occupied.

1.2.2 The wave function of many independent electrons

Now we consider the wave function in the many-electron case, neglecting their interaction. If the filling factor is a non-integer number, the ground state of the system is highly degenerate. Indeed in this case we have to allocate the n highest Landau level electrons in m states at the same energy, with $m > n$. Therefore we have $\binom{m}{n}$ possible degenerate many-particle states. If $n = m$, the filling factor is integer (because also the highest Landau level is full), therefore the ground state is unique, and it is the Slater determinant composed of the single-state wave functions. As an example, we write the explicit form of the wave function ϕ_1 for the filled first Landau level:

$$\begin{aligned} \phi_1 &= \begin{vmatrix} 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ z_1^2 & z_2^2 & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] \\ &= \prod_{j<k}(z_j - z_k)\ \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right]. \end{aligned} \tag{1.21}$$

In the second line we used a well-known result. Indeed the determinant in the first line can be evaluated exactly and it is known as *Vandermonde* determinant. The simplest proof is the following. We consider a particular coordinate, for instance z_1. The determinant in the first line of (1.21) has to be a polynomial of degree $N-1$ of z_1, because in the last line of the determinant we have z_1^{N-1}. The "fundamental theorem of algebra" tell us that such a polynomial has $N-1$ zeroes, which must coincide with the position of the other particles, i.e. the other z_j, because of the form of the determinant (a determinant with two equal columns is zero). Therefore we can pull out the factor $\prod_{j\neq 1}(z_1 - z_j)$. This reasoning can be repeated for each coordinate, and using the fact that the determinant has to be anti-symmetric under coordinate exchanges, we obtain

$$V(z_1, z_2, \ldots, z_N) = \begin{vmatrix} 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ z_1^2 & z_2^2 & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = \prod_{j<k}(z_j - z_k). \tag{1.22}$$

Finally, we want to underline that the derivation presented here for the symmetric gauge, can be straightforwardly extended to other gauges (another common choice is the Landau gauge) and different geometries (disk, torus). The

choice of the gauge and of the geometry is done depending on convenience. In this dissertation all the work will be done in the symmetric gauge and plane geometry. This gauge has a remarkable property: all the wave functions in the LLL can be written as analytic function times a common exponential factor. An excellent treatise of other gauges and geometries (as well as those considered here) can be found in [4].

The only important result we need from the electron-in-a-magnetic-field problem solved in spherical geometry is the following: in this case one can see that the system has rotational invariance. Then a state with total angular momentum $L = 0$ has uniform density. Moreover, in this geometry a state with n filled Landau levels has $L = 0$ and therefore it has uniform density. There is not a similarly simple proof in the planar geometry, but the fact that a state with n filled Landau level has uniform density remains however true.

1.2.3 A qualitative explanation of the IQHE

As we can see from equation (1.3), the classical formula for the Hall resistance, for an integral filling factor $\nu = \rho\phi_0/B = n$, reads

$$R_H = \frac{h}{ne^2}. \tag{1.23}$$

If we variate the magnetic field B (or the electron density ρ), the filling factor changes continuously and so we have at this level no explanation of the plateaux formation. Now we add a fundamental ingredient: the presence of the (reducible but unavoidable) disorder, for instance caused by impurities. This has a central role in the formation of plateaux. Let us then introduce disorder in our sample: the degeneracy of Landau levels is broken and they broaden into bands, making the energy gap disappear, as shown in figure 1.3.

It has been demonstrated that [16–19] in presence of disorder and strong magnetic field, all states become localized (i.e. trapped in the potential valleys/hills caused by impurities), except those at the center of the bands, which are not affected by the introduction of disorder.

Now, in this idealized picture (we are not considering the electrons interaction) an electron in an extended (i.e. not localized) state carries a current if a potential is applied. So at fixed electrons number and magnetic field we have a definite conductance due to all the electrons in extended states below the Fermi level. Let us suppose that the Fermi level lies on an extended state. If we add an electron, it will be trapped in a localized state and so it will not contribute to increase the conductance. Even if we continue to add electrons, the conductance will not change until the Fermi level reaches the next extended level. This is the qualitative explanation of the formation of plateaux in the quantized Hall resistance.

However, given this explanation, one can suppose that

$$R_H = \frac{B}{\rho_{ex} ec}, \tag{1.24}$$

where ρ_{ex} is the density of electrons in extended states. This density is surely lower than the total density ρ, but the experiment tell us that the value of the resistance is given exactly by $R_H = h/ne^2$, as n Landau levels were carrying

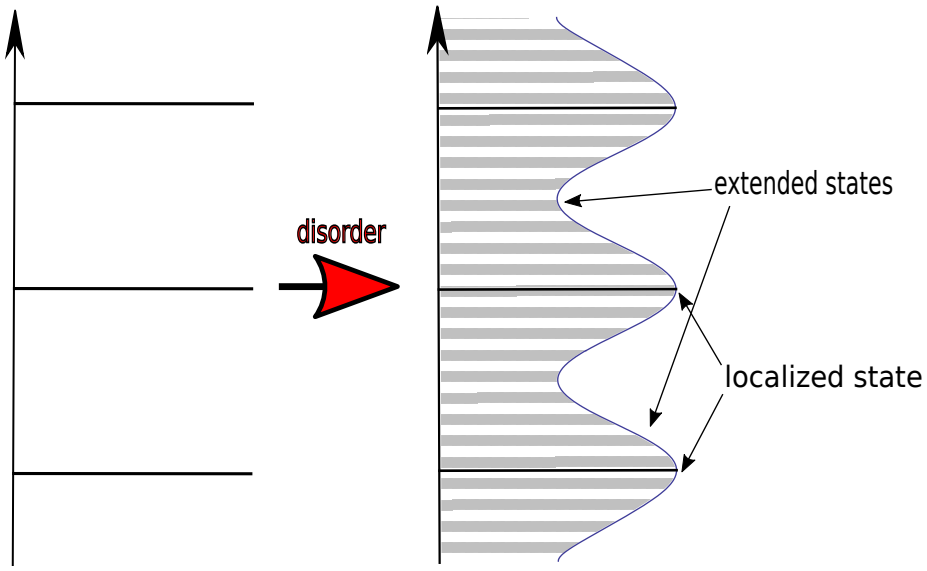

Figure 1.3: In this schematic picture, we see how the degenerate Landau level states (left) broaden into bands in the presence of disorder (right), with extended states near the center (dark lines), separated by localized states (shaded). In presence of disorder, the energy gap between Landau levels disappears.

full current, without disorder. The first solution to this enigma was provided by Laughlin [20], who showed that the Hall resistance is quantized at $R_H = h/ne^2$ as long as the Fermi level lies between the n-th and $(n+1)$-th Landau levels.

However this proof (and any other linked to the IQHE) would take us quite far from the objective of this work, but the interested reader can find it in [4,5]. In fact the QHE theory we are searching for is a microscopic theory, i.e. obtained from the Hamiltonian of the problem. The IQHE can be explained without resort to such a theory, though certainly a microscopic theory of the QHE has to explain also the IQHE. Moreover the FQHE can be explained only by taking into account the electron interactions, which is not necessary in the IQHE theory. Including Coulomb interactions is thus fundamental to move toward an explanation of the fractional quantum Hall effect. The necessary background is presented in the next section.

1.3 The fractional quantum Hall effect

1.3.1 The Hamiltonian

Our aim is to tackle the QHE from an Hamiltonian point of view. The first step is to understand the relevant terms to be included in an Hamiltonian description of interacting electrons in a two-dimensional conductor subject to a transverse magnetic field. To simplify the system we neglect the presence of disorder: as for the IQHE, the plateaux formation requires the introduction of a weak disorder, but the physics of the system can be understood also neglecting this further effect.

A good starting point is the following Hamiltonian:

$$H = \sum_j \frac{1}{2m_b}\left[-i\hbar\nabla_j + \frac{e}{c}\mathbf{A}(\mathbf{r_j})\right]^2 + \sum_j U(\mathbf{r_j}) + \sum_{j,k}\frac{e^2}{2\epsilon}\frac{1}{|\mathbf{r_j}-\mathbf{r_k}|} + g\mu\mathbf{B}\cdot\mathbf{S}, \quad (1.25)$$

where m_b is the reduced mass of the electrons, due to the band structure of the semiconductors system, U is the potential due to the host lattice, $\mathbf{S}$ is the magnetic momentum of the electrons. The FQHE is observed only with strong magnetic fields (above 10 T), suggesting that the approximation of infinite magnetic field $B \to \infty$ should be reasonable.

We can exploit some important simplifications: the first is that the periodic potential due to the host lattice is of no relevance to this problem, because the size of the electron wave packet in a magnetic field ($\sim \ell \approx 25/\sqrt{B[T]}$ nm ≈ 10 nm for typical magnetic fields) is much larger than the lattice period for typical experimental parameters (≈ 0.2 nm). Then we can neglect the term with U.

Moreover, the Zeeman splitting, i.e. the energy requested for flipping a spin, is (in Kelvin degrees)

$$E_z = 2g\mu\mathbf{B}\cdot\mathbf{S} = \frac{g}{2}\frac{m_b}{m_e}\hbar\omega_c \approx 0.3B[T]K, \quad (1.26)$$

so it is infinite in the limit of high magnetic field and also the last term of (1.25) can be neglected.

The third simplification is related to the Landau level structure: in presence of electron interactions, they broaden into bands (the degeneracy is broken). The energy gap between unperturbed levels is $\hbar\omega_c \propto B$, so it is infinite in the limit $B \to \infty$ and we can consider that the electrons cannot jump from a Landau level (or a Landau band) to the next. In experimental systems, we have

$$\hbar\omega_c = \hbar\frac{eB}{m_b c} \approx 20B[T]K \quad (1.27)$$

and for the typical electron-electron interaction

$$V_{e-e} = \frac{e^2}{\epsilon\ell} \approx 50\sqrt{B[T]}K. \quad (1.28)$$

Then the condition $\hbar\omega_c \gg V_{e-e}$, which is necessary to prevent the excitation from a Landau level to another via Coulomb interaction, is satisfied only for strong magnetic field B.

We need also low temperatures: in this way the thermal energy is weak if compared with the other energies. In our idealized picture, we take $T = 0$.

As we can immediately see, the Coulomb potential V_{e-e} is greater than the Zeeman splitting E_z for typical experimental magnetic fields (although in the limit $B \to \infty$ this is no longer true). As a consequence, FQHE states which are not fully spin-polarized are routinely observed in most experiments. However, fully spin-polarized FQHE states do occur, and in our quest for the simplest possible system we will concentrate on these states, in which the spin degrees of freedom are frozen.

Therefore from now on the electrons are considered spinless fermions, i.e. particles that obey to the Pauli exclusion principle but which have not the spin degree of freedom.

Since the Landau levels do not mix with each other, the relevant physics results from the partly filled Landau level, which is the higher one, while the electrons in the lower levels act as a charged background. This can be seen more clearly in a second-quantized form of the Hamiltonian (1.25).

The interacting part of H is

$$H_{int} = \frac{1}{2} \sum_{m_1} \sum_{m_2} \sum_{m_3} \sum_{m_4} \langle m_1, m_2 | V_{e-e} | m_3, m_4 \rangle a^\dagger_{m_1} a^\dagger_{m_1} a_{m_4} a_{m_3}, \tag{1.29}$$

where the single-particle basis used is the one which diagonalize the kinetic term, then m_i is the set of quantum numbers (n_i, L_i), where n_i is the Landau level and $-\hbar L_i$ the angular momentum of the i-th particle.

Now we suppose that the first electron involved in the interaction is in a filled Landau level. Since there is no possibility to be excited in another level, we have $n_1 = n_4$ and $n_2 = n_3$. The first level is full, then the only possibility is $L_1 = L_4$. Using the angular momentum conservation, we also obtain the condition $L_2 = L_3$. If also the other electron involved is in a full level, the term is a constant. Then we can neglect each term that corresponds to an interaction between full Landau levels. If n_2 is the partly filled level, the term takes the form $\sum_m C_m a^\dagger_m a_m$, where we have summed over m_1 using the fact that this level is full and so the operatorial part is trivial (it is always equal to 1 when applied over a state). This interaction is a background-density one. However, we have seen that a full Landau level has uniform electron density, and so each term of this kind in the Hamiltonian (1.29) is merely a constant term and we can neglect it. Therefore the only non-trivial terms in (1.29) are those referring to electrons in the higher Landau level, which is the not completely filled one.

The kinetic part of the Hamiltonian in a second-quantized form reads

$$H_{kin} = \sum_{n,L} \left(n + \frac{1}{2} \right) \hbar \omega_c a^\dagger_{n,L} a_{n,L}. \tag{1.30}$$

From this we can understand that in our strong-field limit in which the electrons cannot change their quantum number n, the kinetic part of the Hamiltonian is always a constant and then we can neglect it.

To simplify further the problem, we suppose that the lowest Landau level is the incomplete one, i.e. our filling factor is a fraction between 0 and 1. There are two reasons for this: the first is that many observed plateaux are at filling factor < 1. The second is that the system with an incomplete higher Landau level can be mapped in another with an effective potential in the LLL (as we can see from equation (1.30) using the considerations just exhibited).

The final Hamiltonian, simplified as much as possible, but containing the whole relevant physics of the FQHE in the LLL, is

$$H = P_{LLL} \sum_{j,k} \frac{e^2}{2\epsilon} \frac{1}{|\mathbf{r_j} - \mathbf{r_k}|} P_{LLL} = P_{LLL} H_{e-e} P_{LLL}, \tag{1.31}$$

where P_{LLL} is the projection operator on the LLL. We notice that if we calculate the matrix element between two states of an Hilbert space basis, we can write:

$$\langle \psi_1 | H | \psi_2 \rangle = \langle \psi_1 | P_{LLL} H_{e-e} P_{LLL} | \psi_2 \rangle = \left\langle \psi_1^{(LLL)} \right| H_{e-e} \left| \psi_2^{(LLL)} \right\rangle \tag{1.32}$$

and so the use of LLL wave functions is equivalent to the restriction of the LLL imposed over the Hamiltonian.

We want now to understand why the FQHE is so difficult to tackle from a microscopic point of view. The main difficulty is that this problem cannot be treated with perturbative methods, because we have no small adimensional parameter. This can be seen measuring the energies in unit of $e^2/\epsilon\ell$: in this case, the Hamiltonian is given by

$$H = P_{LLL}\frac{1}{2}\sum_{j,k}\frac{1}{|\mathbf{r_j}-\mathbf{r_k}|}P_{LLL}, \tag{1.33}$$

where $\mathbf{r_i}$ are measured in unit of ℓ, so there are only pure quantities. A standard many-body technique is realized turning off the interaction and taking the non-interacting ground state as starting point for an investigation of the interacting problem. However, in this case it is not practicable, because if we turn off the interaction, the resulting system has an incredible large number of degenerate ground states[2], so it is useless.

Therefore we must take advantage of other techniques to try to solve the problem: two of them that have proved to be particularly fruitful are the exact diagonalization (that can be used only for small electron number) and the variational methods.

In the next two sections we give an overview of these two methods.

1.3.2 Exact diagonalization method

The idea behind this method is that the Hamiltonian of a finite-size system, as the FQHE restricted to the LLL, can be expressed as a finite-dimensional matrix. As we have seen before, in a system of area S there are $S/2\pi\ell^2 = N_\phi$ single-particle levels (degenerate in the independent-electrons picture). Then the number of independent many-electrons state (i.e. the dimension of the space in which the system lives) is

$$N_H = \binom{N_\phi}{N_e} = \frac{N_\phi!}{N_e!\,(N_\phi - N_e)!}\,, \tag{1.34}$$

where N_e is the number of electrons. We indicate with the symbol $|m\rangle$ the m-th many-particle state of the N_H-element basis (we can concretely take, for example, the basis obtained using the Slater determinants of the single-particle states) and the Hamiltonian matrix is constructed as

$$H_{ij} = \langle i|\, H\, |j\rangle\,. \tag{1.35}$$

Using a computer program, we can diagonalize this matrix, obtaining its eigenvalues and the corresponding eigenvectors. The eigenvalues are the energies of the ground state and of the excited levels. From the eigenvector $(\xi_1, \ldots, \xi_{N_\phi})$ we can get the wave function of the corresponding eigenstate as follows:

$$\psi = \sum_i \xi_i\, |i\rangle\,. \tag{1.36}$$

[2] as an example, a toy system containing 100 electrons in 250 single particle states (within the LLL), so with a filling factor $\nu = 0.4$, has 10^{72} distinct ground state configuration, which is roughly equal to the number of atoms in the Universe.

The dimension of the matrix H increases very rapidly with N_e (as we can see using $N_\phi = N_e/\nu$ with $\nu < 1$ in equation (1.34)) and so we cannot treat with this method a system with too many electrons. As an example, a system with $\nu = N_e/N_\phi = 1/3$ and $N_e = 6$ electrons, has $N_\phi = 18$ and then $N_H = 18564$. Such a matrix can be diagonalized numerically, but for a larger electron number we are hopeless. Fortunately we can use various symmetries of the problem to block-diagonalise the Hamiltonian and so the number of electrons we can handle increases sensibly (up to 10).

Within this of approach, two important results has been obtained:

- A priori one can suppose that the ground state of a system of interacting electrons with Hamiltonian (1.33) is a Wigner crystal, i.e. a state in which each electron is localized in a regular-lattice site. It is natural to expect something similar, because the electrons could minimize the long-range repulsion of the Coulomb potential by forming a Wigner crystal . The first result following from the exact diagonalization method is that the ground state is a liquid state, i.e. a state without a long-range order. This remains true for filling factor not too small: when ν is smaller than about $1/7$, a Wigner crystal is realized as ground state;
- This liquid has the property of incompressibility, i.e. we need an infinite change in the pressure in order to modify the density of the system. This is obtained as a consequence of the important observation of a discontinuity of the energy per electron as function of the filling factor. So there is an energy gap at the filling factors of the form $\nu = p/q$ for which FQHE is observed.

As we have seen, this method cannot be used to describe systems with a large electron number. This motivated the introduction of other methods, but the importance of the exact diagonalization goes beyond the results just presented. As a matter of fact each FQHE theory proposed must pass as first examination the comparison with the exact diagonalization results. The calculation of the overlap between the exact diagonalization wave function and the one obtained from the theory proposed (for small electron number) cannot give us the certainty that this theory is correct, but it can be an useful "rule of thumb".

1.3.3 Variational method: Laughlin's wave function

Another powerful method to investigate the FQHE system is the variational ansatz. Concretely, we take a trial wave function $\psi(\alpha)$ with a set of parameters indicated by α. The value of

$$E(\alpha) = \frac{\langle\psi(\alpha)|\, H\, |\psi(\alpha)\rangle}{\langle\psi(\alpha)|\psi(\alpha)\rangle} \tag{1.37}$$

is larger than or equal to the ground-state energy, so we can find the value $\tilde{\alpha}$ which minimizes $E(\alpha)$ and in such a way we obtain our guess $\psi(\tilde{\alpha})$ for the ground state wave function.

The success of this method strongly depends on the form of the trial wave function initially chosen.

Laughlin was the first who understood how to generalize this method for the FQHE. The three following considerations allowed him to win the Nobel Prize in 1998:

1. We are studying the LLL, in which single-particle states are given by

$$\eta_{0,m}(\mathbf{r}) = \frac{z^m}{\mathcal{N}_m} e^{-\frac{|z|^2}{4}}, \tag{1.38}$$

where $\mathcal{N}_m$ is the suitable normalization coefficient, which we will neglect from now on. We will therefore obtain un-normalized wave functions. A many-electrons wave function can be expressed on the basis of Slater determinants constructed using the single particle state, so it must necessarily be of the form

$$\psi(z_1, \ldots, z_{N_e}) = \mathcal{F}_A(z_1, \ldots, z_{N_e})\, e^{-\frac{1}{4}\sum_i |z_i|^2}, \tag{1.39}$$

where $\mathcal{F}_A(z_1, \ldots, z_{N_e})$ is an anti-symmetric polynomial, due to the Pauli principle.

2. The generic term of $\mathcal{F}_A$ is $a\prod_i z_i^{m_i}$, in which the i-th electron has an angular momentum $-\hbar m_i$. Then a state of this kind has total angular momentum $-\hbar\sum_i m_i$. Now, the Hamiltonian (1.33) is invariant under full-system rotation and so the total angular momentum is a good quantum number, i.e. we can choose the basis which diagonalizes simultaneously the Hamiltonian end the total angular momentum operator. The states of this basis are the Slater determinants we are using, and so we notice that $\mathcal{F}_A$ has to be an homogeneous polynomial.

3. The Coulomb interaction tends to separate the electrons and then the wave function must be smaller when two electrons approach each other. The following step is the core of Laughlin's ansatz: inspired from this consideration, he assumed that the polynomial $\mathcal{F}_A$ is of the form

$$\mathcal{F}_A(z_1, \ldots, z_{N_e}) = \prod_{i,j} f(z_i - z_j), \tag{1.40}$$

with $f(0) = 0$. A function of this kind is called *Jastrow* function.

These three observations strongly constraint the form of the wave function, which has to be the celebrated Laughlin's ansatz:

$$\psi_q(z_1, \ldots, z_{N_e}) = \prod_{i<j} (z_i - z_j)^q\, e^{-\frac{1}{4}\sum_i |z_i|^2}, \tag{1.41}$$

where q is an odd integer and it should be the variational parameter. We notice that the maximum angular momentum an electron can have (i.e. the highest degree of its coordinate in equation (1.41)) is $M = (N_e - 1)q$. Then in the circle with radius $\sqrt{2Ml}$ there are N_e electrons, and so we can write for the filling factor

$$\nu = 2\pi\ell^2 \frac{N_e}{S} = \frac{N_e}{M} \simeq \frac{1}{q}, \tag{1.42}$$

where we used that $N_e \gg 1$. Therefore we have not variational parameters and the Laughlin's wave function ψ_q is our guess for the description of the FQHE state with filling factor $1/q$, with q odd.

This is Laughlin's great result, supported by the absence of observed plateaux at filling factor with even denominators. The so-called electron-hole symmetry,

which is revealed inserting $b_m^\dagger = a_m$ and $b_m = a_m^\dagger$ in equation (1.29) and observing that the Hamiltonian is invariant for such a transformation, allows the Laughlin wave function to take care also of filling factors of the form $1 - 1/q$. However, many fractions which are not of the forms $1/q$ and $1-1/q$ are observed in the LLL, so the Laughlin's theory is not complete.

Another reason for the great success of the Laughlin's wave function is that its overlap with the predicted wave function obtained via the exact diagonalization method (i.e. the value of $\langle \psi_{ExDiag} | \psi_{Laugh} \rangle$) is impressively good: for $\nu = 1/3$, e.g., the overlap is greater than 0.99 for electron number from 4 to 9 (we remember that the exact diagonalization method allow us to compare wave functions only for small electron number). This overlap remains good for filling factor of the form $1/q$ (or $1 - 1/q$) provided that they are greater than about $1/7$.

The obtained wave function (1.41) had great success, and it sets the basis for our theoretical understanding of the FQHE. But it is not the exact wave function, as we can intuitively see from the way we have obtained it: we have considered only the short-range effect of the Coulomb interaction, which is indeed a long-range interaction. This can be understood following Haldane [21] and constructing the model for which the wave function (1.41) is the exact ground state. This is done in the following section.

As a last observation, we notice another feature of the Laughlins ansatz. What if we take an even q? We are again describing a state for which each point of Laughlin's deduction holds, except that now the resulting wave function is symmetric under exchange of coordinates. Then we are describing hypothetical bosonic FQHE states. Of course electrons are fermions, and so there are not states of this kind, but we will see in the following that this concept of bosonic FQHE states can be useful.

1.3.4 Haldane pseudopotentials

We want to restart our study from the Hamiltonian of two particles in a magnetic field (the generalization to a larger electron number is straightforward). Then we have our Hamiltonian

$$H = \frac{1}{2m_1}(\boldsymbol{p}_1 + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_1))^2 + \frac{1}{2m_2}(\boldsymbol{p}_2 + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_2))^2 + V(\boldsymbol{r}_1, \boldsymbol{r}_2)\,. \tag{1.43}$$

We restrict our system in two dimensions[3], so $\boldsymbol{p} = (p_x, p_y)$ and $\boldsymbol{r} = (r_x, r_y)$. As we have done previously, we use the symmetric gauge, where

$$\boldsymbol{A} = \frac{\boldsymbol{B} \times \boldsymbol{r}}{2} = \frac{B}{2}(-y, x, 0)\,. \tag{1.44}$$

Following the same steps, the kinetic part of the Hamiltonian (1.43) is given by (dropping every dimensional coefficient):

$$H_k = \frac{1}{2}\left[\left(-i\frac{\partial}{\partial x_1} - \frac{y_1}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_1} - \frac{x_1}{2}\right)^2\right] + (\text{same with } 1 \leftrightarrow 2)\,. \tag{1.45}$$

[3] as always, the particles are confined in two dimensions, but we want to use the third one as the magnetic field direction.

Now the usual change of variables is done for each particle, i.e.:

$$z_n = x_n - iy_n = r_n e^{i\theta_n}\,; \qquad \bar{z}_n = x_n + iy_n = r_n e^{-i\theta_n}\,. \tag{1.46}$$

Each single-particle term in the kinetic Hamiltonian is given by:

$$H_k = \frac{1}{2}\left[-4\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}z\bar{z} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right]. \tag{1.47}$$

We can decouple the center-of-mass term of the Hamiltonian and the relative term, using the following variables:

$$Z = \frac{z_1 + z_2}{2}\,; \qquad z = z_1 - z_2\,. \tag{1.48}$$

Using these variables, we can now obtain the same form of equation (1.47) for the (now decoupled) center-of-mass and relative degrees of freedom.

Now we introduce (as we have done in equation (1.10))

$$\begin{aligned}
b_R &= \frac{1}{\sqrt{2}}\left(\frac{\bar{Z}}{2} + 2\frac{\partial}{\partial Z}\right),\\
b_R^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{Z}{2} - 2\frac{\partial}{\partial \bar{Z}}\right),\\
a_R^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\bar{Z}}{2} - 2\frac{\partial}{\partial Z}\right),\\
a_R &= \frac{1}{\sqrt{2}}\left(\frac{Z}{2} + 2\frac{\partial}{\partial \bar{Z}}\right),
\end{aligned} \tag{1.49}$$

and an analogous set of operators for the relative coordinate. With this definitions, one can easily verify that $[a, a] = 1$ and $[b, b] = 1$ if they are related to the same set of coordinate (relative or of center-of-mass), and all other commutators are zero.

So the kinetic term is given by:

$$H_k = a_R^\dagger a_R + a_r^\dagger a_r + 1. \tag{1.50}$$

As we have done for the single-particle case, we obtain the (third component of) total angular momentum $L_R = -(b_R^\dagger b_R - a_R^\dagger a_R)$ and the relative one $L_r = -(b_r^\dagger b_r - a_r^\dagger a_r)$.

We want to consider only the LLL, in which a complete set of two-particle states is given by

$$|M, m\rangle = \frac{(b_R^\dagger)^M (b_r^\dagger)^m}{\sqrt{M!m!}}|0, 0\rangle. \tag{1.51}$$

Now we consider the potential term of the Hamiltonian (1.43). We assume that $V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(\boldsymbol{r}_1 - \boldsymbol{r}_2) = V(|z|)$. In this case, the potential is isotropic[4] and so in the basis (1.51) we obtain

$$\langle M', m'|V|M, m\rangle = \delta_{MM'}\delta_{mm'}\langle m'|V|m\rangle \tag{1.52}$$

[4] it is invariant under rotations, which keep lengths unchanged.

because V acts only on the modulus of the relative coordinates, and the angular integral in $\langle m'|V(|z|)|m\rangle$ produces zero unless $m = m'$. So we can write:

$$\begin{aligned} V &= \sum_{M',m'} \sum_{M,m} |M',m'\rangle\langle M',m'|V|M,m\rangle\langle M,m| \\ &= \sum_{M,m} |M,m\rangle\langle m|V|m\rangle\langle M,m| \\ &= \sum_{M,m} V_m P_{M,m} \quad , \end{aligned} \tag{1.53}$$

where $V_m = \langle m'|V|m\rangle$ and $P_{M,m} = |M,m\rangle\langle M,m|$. The parameters V_m are called *Haldane pseudopotentials*.

If we use this equation, we can write the potential in a second quantized form, in the Hilbert space restricted to the only LLL:

$$\begin{aligned} V &= \frac{1}{2} \sum_{m_1,m_2,m_3,m_4} \langle m_1,m_2|V|m_3,m_4\rangle a^\dagger_{m_1} a^\dagger_{m_2} a_{m_4} a_{m_3} \\ &= \sum_{m_1,m_2,m_3,m_4} \sum_{M,m} \frac{V_m}{2} \langle m_1,m_2|M,m\rangle\langle M,m|m_3,m_4\rangle a^\dagger_{m_1} a^\dagger_{m_2} a_{m_4} a_{m_3}, \end{aligned} \tag{1.54}$$

where $a^\dagger_m$ is a creator of a particle with angular momentum m (in the LLL), and a_m is its destructor.

Now, the Haldane pseudopotentials can be evaluated for any given interaction, but we are mainly interested in simpler model cases, of two kinds: the first has $V_k = 1$ for all k even and $k \leq n$ with n an even integer, and all the other $V_m = 0$. The second family of models is that with $V_k = 1$ for all k odd and $k \leq m$ with m an odd integer.

The first case describes bosons, as one can see using

$$\begin{aligned} \langle z_1,z_2|m_1,m_2\rangle &= \frac{z_1^{m_1} z_2^{m_2}}{\mathcal{N}_{m_1}\mathcal{N}_{m_2}} e^{-\frac{1}{4}\left(|z_1|^2+|z_2|^2\right)}, \\ \langle z_1,z_2|M,m\rangle &= \frac{(z_1+z_2)^M (z_1-z_2)^m}{\mathcal{N}_M \mathcal{N}_m} e^{-\frac{1}{4}\left(|z_1|^2+|z_2|^2\right)}, \end{aligned} \tag{1.55}$$

where $\mathcal{N}_a$ is given by

$$\mathcal{N}_a = \sqrt{2\pi\, 2^a\, a!}. \tag{1.56}$$

Therefore in the case in which all the m are even integer numbers,

$$\langle m_1,m_2|M,m\rangle = \int \mathrm{d}z_1\, \mathrm{d}z_2\, \langle m_1,m_2|z_1,z_2\rangle\, \langle z_1,z_2|M,m\rangle \tag{1.57}$$

is symmetric for exchange of m_1 and m_2. So, only if $[a^\dagger_{m_1}, a^\dagger_{m_2}] = 0$ we can have a non-zero potential term. In the same way, one can see that the other family of models concerns fermions.

An interesting feature of this form of the FQHE Hamiltonian is the following: it has been proved that the Laughlin's wave function with filling factor $1/q$ is the exact ground state of the model with $V_k = 1$ for all k with the same parity of q and $k < q$ [5]. Therefore this property holds for fermionic and bosonic Laughlin's wave functions.

Performing the calculation of equation (1.57), we obtain the coefficients of the pseudopotentials:

$$\begin{aligned}\langle m_1, m_2|M, m\rangle = \sum_{k=0}^{M}\sum_{j=0}^{m}\binom{M}{k}\binom{m}{j}2^{-(M+m)/2}(-1)^j\cdot \\ \cdot\sqrt{\frac{(M+m-k-j)!\,(k+j)!}{M!\,m!}}\,\delta_{m_1,M+m-j-k}\;\delta_{m_2,j+k}.\end{aligned} \tag{1.58}$$

Now we can insert this coefficient in equation (1.54). We are free to decide which summation indexes to eliminate using the deltas, so there are many way to proceed, some more enlightening than others. We notice that, because of the deltas in (1.58), we have

$$\langle m_1, m_2|M, m\rangle\,\langle M, m|m_3, m_4\rangle \propto \delta_{m_1+m_2,m_3+m_4}, \tag{1.59}$$

i.e. the operator $|M, m\rangle\,\langle M, m|$ preserves the angular momentum of the two particles on which it acts. Using this delta in equation (1.54) and changing the summation indexes in the following way

$$m_1 = s + t,\;\; m_2 = s + u,\;\; m_4 = s, \tag{1.60}$$

we obtain:

$$\begin{aligned}V &= \sum_m \frac{V_m}{2}\sum_{s=0}^{\infty}\sum_{t=-s}^{\infty}\sum_{u=-s}^{\infty}\mathcal{C}\;a^\dagger_{s+t}\,a^\dagger_{s+u}\,a_{s+t+u}\,a_s \\ &= \sum_m \frac{V_m}{2}\sum_{s=0}^{\infty}\Big(\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}(\delta_{t,0}+\delta_{u,0}) + \sum_{t=-s}^{-1}\sum_{u=-s}^{-1} + \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\Big)\mathcal{C}\;a^\dagger_{s+t}\,a^\dagger_{s+u}\,a_{s+t+u}\,a_s,\end{aligned} \tag{1.61}$$

with some coefficients $\mathcal{C}$ which depends on the summation indexes. We notice that when $t + u < -s$ we always obtain zero, because there are not particles with negative angular-momentum quantum number in the LLL.

We can describe the action of the four-particle operator $a^\dagger_{s+t}\,a^\dagger_{s+u}\,a_{s+t+u}\,a_s$ by saying that it destroys two particles with angular momenta[5] s and $s + t + u$ and creates two particles with angular momenta $s + t$ and $s + u$. Therefore if we consider a one-dimensional lattice and we label its sites with integers, a particle with angular momentum m can be seen as a particle in the m-th site of this lattice. In the LLL the lattice is semi-infinite, and the first site is labelled with the number zero. Within this picture, the operator re-creates the destructed particles shifting them of t sites. In particular, if $t = 0$ or $u = 0$ the particles are respectively not moved or swapped, if t and u are positive they are *squeezed* toward each other and if t and u are negative they are moved toward the "external" of the lattice or *anti-squeezed*. Figure 1.4 illustrates the squeezing operation, while anti-squeezes are obtained following this process in the opposite direction.

We will refer to $a^\dagger_{s+t}\,a^\dagger_{s+u}\,a_{s+t+u}\,a_s$ as the *squeezing operator* (although it also anti-squeezes particles) and we will discuss it in detail in the following chapters.

[5] as we have seen, a particle in the state $|n, m\rangle$ has angular momentum $-\hbar m$. However, for brevity, we will often refer directly to m as the angular momentum of the particle.

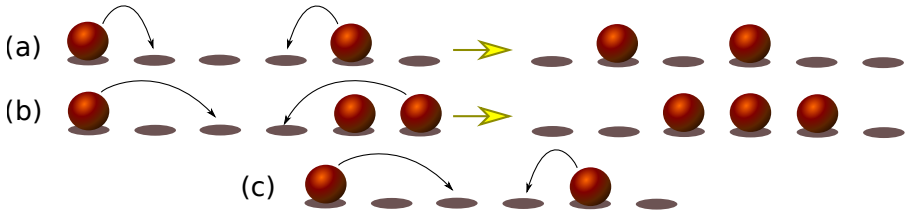

Figure 1.4: In the one-dimensional lattice represented by grey circles, each site corresponds to an angular momentum state in the LLL. (a) and (b) are two possible squeezes in which the particles are moved toward each other of one and two positions respectively, under the action of the *squeezing* operator $a^\dagger_{s+t}\, a^\dagger_{s+u}\, a_{s+t+u}\, a_s$, while (c) is a squeeze not allowed.

1.3.5 Other filling factors, plateaux formation and Composite Fermions theory

In this section we review briefly some other important facts concerning the FQHE. The topics discussed here are not used anywhere in this work, but we discuss them for completeness, including the proper references that might be used by the interested reader.

Other filling factor in Laughlin's theory

As we have seen, Laughlin's wave functions (1.41) are good approximations of the true ground states for filling factors of the form $1/q$ (and $1-1/q$, but from now on we will consider only filling factor lesser than $1/2$, because the others are obtained from the electron-hole symmetry). What for the other observed fractions? A theory due to Haldane [21] predicts the formation of plateaux at all the filling factors of the form[6]

$$\frac{2p}{2pq \pm 1}. \tag{1.62}$$

A quick look at this explanation follows. A characteristic of the Laughlin's wave functions with filling factor $1/q$ is that their product for $\prod_i (z_i - z_0)$ describes the presence of a *quasihole*, i.e. a particle with fractional charge $e^* = e/q$, with coordinates z_0. An analogous but more complicated operator applied to the wave function describes the introduction of a *quasielectron* of charge $e^* = -e/q$. In the Haldane's theory these quasi-particles are treated as bosons. The wave function of N_{qh} quasiholes formed and N_e electrons with filling factor $1/q$ is then

$$\prod_i (z_i - z_{01}) \prod_i (z_i - z_{02}) \cdots \prod_i (z_i - z_{qh}) \psi_q, \tag{1.63}$$

where ψ_q is given by (1.41). The quasiholes are charged particles in a magnetic field (it can be proved that the Laughlin state at filling factor $1/q$ is an uniform-density state, so the quasi-particles do not interact with electrons) and then the usual Landau levels are formed. Moreover, from equation (1.63) we see that the

[6] most of the observed filling factor can be written in this form (but not all of them).

highest degree of each quasihole coordinate is N_e and so there are N_e+1 possible state in the Landau level. We notice that the wave function (1.63) is invariant under exchange of quasihole coordinates, as expected from their bosonic nature. We can expect that a Laughlin state of quasiholes is realized when the filling factor is $1/2p$, where p is an integer. In this case, the relation $N_{qh} = (N_e+1)/2p$ holds.

For each electron, the highest degree of its coordinate in (1.63) is $M = (N_e - 1)q + N_{qh}$ and the filling factor of this state is $\nu = N_e/M$. Putting these together and considering $N_e \gg 1$, we get:

$$\nu = \frac{N_e}{M} = \frac{2p}{2pq+1}. \tag{1.64}$$

Using quasielectrons instead of quasihole we obtain the fractions of the form $\nu = 2p/(2pq-1)$.

Plateaux formation in Laughlin's theory

In this section we want to understand intuitively the mechanism which brings to the plateaux formation in the FQHE. The classical formula (1.3) for the Hall resistance states that at filling factor $\nu = 1/q$ we have $R_H = q\hbar/e^2$. Now we imagine to decrease the magnetic field (or equivalently increase the electron number) by a small amount. The following fraction of the form $1/q$ is not compatible with the actual filling factor, so the system creates a finite density of quasiparticle to "adjust" the filling factor according to equation (1.64). If now we insert an impurity potential, the quasiparticles localize, until their density increase to the point in which their interaction (repulsive) is stronger than the trapping potential due to the impurities.

There is an energy gap for each filling factor observed, therefore the introduction of a small impurity effect does not mixes the energy levels.

As long as the quasiparticles are localized, one can prove that the Hall resistance remains fixed to the value one can calculate using the original filling factor, without the impurity potential. Therefore a plateau at the correct value of R_H has formed, and the system passes to the next plateau (i.e. the next filling factor of the form $2p/(2pq \pm 1)$) when the quasiparticle interaction wins over the impurity potential and a Laughlin state of quasiparticle is created.

Jain's composite fermions theory

Here we present in a qualitative and intuitive way the ideas behind the composite fermions theory, mainly due to Jain. For a comprehensive explanation we suggest the excellent Jain's book [4].

A composite fermion is the bound state of an electron and an even number of flux quanta (called ϕ_0 in the previous sections). We notice that a FQHE state at filling factor $1/q$ has, as observed before in section 1.2.1, q flux quanta for each electron. We can attach some of this quanta to the electrons[7], and reformulate our theory passing from electrons to composite fermions: in this case, if each electron has $2k$ flux quanta attached, the composite fermions feel

[7] this is a delicate passage and many question about why an electron binds with a flux quanta could arise, but here we want just to give an idea of the path followed. All the answers can be found in [4].

a magnetic field $B^* = B - 2k\rho\phi_0$, i.e. there are $q - 2k$ flux quanta for each composite fermion.

The composite fermions form Landau-like levels, called Λ levels, and allow to link the FQHE states with the IQHE ones. In order to see it, we can consider the case in which $q - 2k = 1$: in this case the effective filling factor of the composite fermions is 1, i.e. they are in an integer quantum Hall state. Vice versa, if we consider composite fermions in an IQHE state with filling factor $\nu^* = n$, they feel an effective field B^* which correspond to $1/n$ flux quanta per particle. When we go back to the original electron picture, the magnetic flux per particle is given by $2k + 1/n$ and so the filling factor is

$$\nu = \frac{1}{2k + 1/n} = \frac{n}{2kn + 1}. \tag{1.65}$$

Some important features of the composite fermions theory are the following:

- it allows quantitative calculations and the predictions made with the composite fermions theory are more accurate than those obtained with other (for example Laughlin's) theories;
- the wave functions of all the states with filling factor of the form $n/(2kn+1)$ can be written easily;
- in this theory there are not a hierarchical deduction of the filling factor, differently from the Laughlin-Haldane's theory.

Even though almost all the observed filling factors can be written in the form of equation (1.65), with the recent observation of a number of new fractions indicating FQHE states, it seems that even the Composite Fermions theory is incomplete.

Chapter 2

Symmetric functions and Jack polynomials

In this chapter we discuss the mathematical background we need to develop our theory for the Fock-space operatorial construction of the FQHE states. Moreover, the partitions formalism used here proves to be useful also in the following, so particular emphasis is given to its presentation.

The main goal of this chapter is the introduction of *Jack polynomials*. They belong to a wide class of special functions, the symmetric functions. We will develop some preliminaries to define Jack polynomials and to show some of their properties.

2.1 Partitions

A partition λ is a non-increasing, definitively null sequence of integers. We call the *lenght* $l(\lambda)$ of the partition the (finite) number of its non-null entries.

Partitions are represented in various ways:

- indicating all the non-null (and possibly also part of the null) entries in decreasing order, $(\lambda_1, \dots, \lambda_{l(\lambda)}, 0, \dots, 0)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$;
- indicating all the entries in a contracted form and in decreasing order, using exponents to indicate the number of repetitions of a certain component, $(\lambda_1^{n_1}, \dots, \lambda_{l(\lambda)}^{n_{l(\lambda)}})$, again with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$;
- using the multeplicities $n_i(\lambda)$ (number of appearances of i in λ), i.e. $(n_1(\lambda), n_2(\lambda), \dots, n_i(\lambda), \dots)$.

For example, the partition[1] $(663100\dots)$ can be written also as $(6^2 3^1 1^1)$ or as $(1010020\dots)$ (see also figure 2.1).

We will refer to the sum $\sum_{i=0}^{l(\lambda)} \lambda_i$ as $|\lambda|$.

Partitions are particularly useful especially because several bases of polynomial spaces are in one to one correspondence with them. Let us consider the third representation given for partitions, in the example above $(1010020\dots)$.

[1] sometimes we do not separate the partition entries with a comma in order to make it more readable. In these cases, each digit is a partition entry.

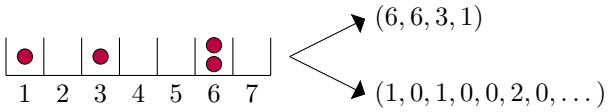

Figure 2.1: Example of different notation for partitions: the upper is the *usual* representation, the lower is the *occupation numbers* one.

This is formally equivalent to a N-particles state expressed in the occupation numbers basis. In this case, we have 1 particle in the first single-particle basis state, 1 in the third and 2 in the sixth. Within this picture, also the other representations of partitions assume physical meaning: $(6631\dots)$ lists in decreasing order which single-particle states are occupied by every single particle, and $(6^2 3^1 1^1)$ does the same in a contracted way. With this idea in mind, we can write N-particle states of the occupation number basis as $|\lambda\rangle$, with $l(\lambda) = N$. We notice that the fact that partitions are ordered sequences accounts for the permutation symmetry of identical particles. Within this picture, the one-dimensional lattice view of the LLL sketched in the end of section 1.3.4 becomes more transparent: single-particles basis states are labelled by the angular momentum eigenvalue m. For instance, the partition $(1010020\dots)$ represents a state in which one electron has vanishing angular momentum, one electron has $L_z = -2\hbar$ and two electrons have $L_z = -5\hbar$. Accordingly, (6631) is the list of electrons' angular momenta, in decreasing order. We notice that $|\lambda|$ is the total angular momentum of the system.

From now on, we will refer to the decreasing listing representation as the *usual* representation (λ_i will indicate the i-th component of λ). We will refer to the multeplicity representation as the *occupation numbers* representation.

2.1.1 Squeezing

Equipped with the partition formalism, we can now reformulate with more precision the concept of *squeezing*. Given a partition λ, two integers $0 \leq i \leq j \leq l(\lambda)$ and an integer $0 \leq s \leq \lambda_i - \lambda_j$, the *squeezing operator* R^s_{ij} acts as

$$R^s_{ij}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots) = (\lambda_1, \dots, \lambda_i - s, \dots, \lambda_j + s, \dots)^*, \tag{2.1}$$

where the $*$ means that we may have to reorder the final partition to restore the decreasing order. We notice that squeezings preserve $|\lambda|$. Two admitted squeezing and a not-admitted one are shown in figure 1.4, while figure (2.2) focuses on the usual and the occupation numbers partition representation, and on their use in order to describe squeezings.

In the right representation of figure (2.2), we can easily give a physical interpretation of squeezing operations: two particles, the i-th and j-th, are moved inward by s steps from their sites.

2.1.2 Ordering

We can establish several ordering relations on the set of partitions. Of particular importance for this work is the *squeezing induced* ordering, also called natural or dominance ordering. This is a partial ordering relation, which is defined equivalently as:

- two partitions λ and μ are such that $\lambda > \mu$ if μ can be derived from λ with a finite number of squeezings. This implies that if $|\lambda| \neq |\mu|$, λ and μ are not comparable (but also two partition with $|\lambda| = |\mu|$ can be not comparable);
- two partitions λ and μ are such that $\lambda > \mu$ if $|\lambda| = |\mu|$ and $\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i$, $\forall r > 0$.

We can introduce a total ordering relation on partitions, following Stanley [22]. This is the *reverse lexicographic* ordering: two partitions λ and μ are such that $\mu \overset{R}{\leq} \lambda$ if the first non-vanishing term $\lambda_i - \mu_i$ is positive. This is a total ordering, compatible with the dominance ordering.

In the following, we will use dominance ordering, as squeezings are closely related to some important properties of Jack polynomials.

2.2 The ring of symmetric functions

Symmetric functions generalize polynomials by removing any constraint on the number of independent variables. This generalization can be useful if we want to describe wave functions of very large many-body systems.

We call Λ_n the ring of symmetric polynomials in n independent variables, with coefficients in a generic commutative ring R (in the following, $R = \mathbb{Z}$).

We notice that Λ_n is a graded ring: indeed, $\Lambda_n = \bigoplus_{r \geq 0} \Lambda_n^r$, where Λ_n^r is the subring of symmetric homogeneous polynomials of degree r (by convention, 0 is homogeneous of every degree).

We construct a collection of surjective homomorphisms (of graded rings) in the following way: we define $\omega : \Lambda_{n+1} \to \Lambda_n$ by $\omega : f(x_1, \ldots, x_n, x_{n+1}) \to f(x_1, \ldots, x_n, 0)$ for every $f \in \Lambda_{n+1}$. The restriction of ω to Λ_n^r is again surjective, and it is a bijection if and only if $r \leq n$. These homomorphisms allow for the construction of the space Λ^r, composed by elements $f \in \Lambda^r$ which are limits of sequences $\{f_n\}_{n \geq 0}$ such that:

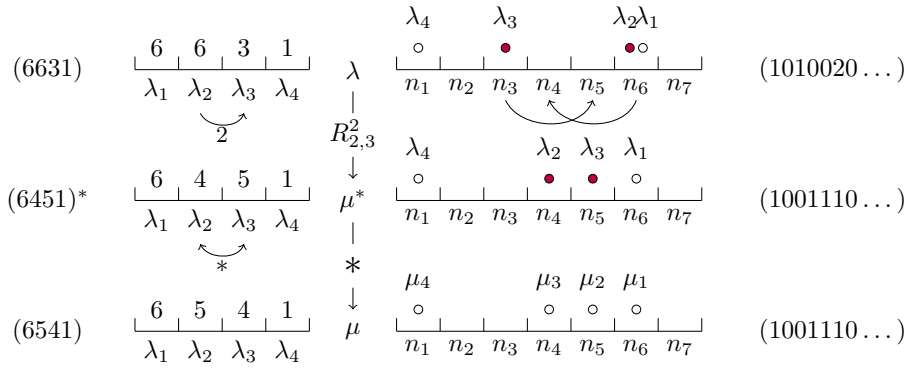

Figure 2.2: Example of squeezing. On the left, we see the squeezing in the usual representation. First, we perform the rough squeeze as defined in equation (2.1). Then, we need a reordering to restore decreasing order. On the right, we see the occupation number counterpart of the same squeezing.

- $f_n \in \Lambda_n^r$ for each $n \geq 0$;
- $f_n = \omega f_{n+1}$.

In more technical words we say that Λ^r is the inverse limit of Λ_n^r, i.e. $\Lambda^r = \lim_{\leftarrow n} \Lambda_n^r$.

Λ^r is again a ring, which we call the *ring of homogeneous symmetric functions of degree r*. Finally, we define the *ring of symmetric functions* as $\Lambda = \bigoplus_{r \geq 0} \Lambda^r$.

This is a very strict mathematical construction. We can summarize it saying that symmetric functions are the limits of sequences of regular polynomials, each one with an additional variable. Thus, symmetric functions can be regarded as polynomials in infinitely many independent variables.

In the following, we will explicitly indicate with the symbol Λ_R which ring of coefficients we take into account.

2.2.1 Relevant basis

Monomial symmetric functions

Given a partition λ, we define a monomial as $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots$. We call *monomial symmetric function* $m_\lambda = \sum_\sigma x^{\sigma(\lambda)}$ the sum of all distinct monomials obtainable from λ permuting (with the permutation σ) the variables.

For example:

$$\begin{aligned} m_{(1,2)} &= \sum_{i \neq j} x_i x_j^2, \\ m_{(1,1)} &= \sum_{i<j} x_i x_j. \end{aligned} \tag{2.2}$$

We remark that the monomial symmetric functions m_λ form a $\mathbb{Z}$-basis of $\Lambda_\mathbb{Z}$. For example, in the 3 variables case $(x_1, x_2, x_3) = (x, y, z)$,

$$x^2y^2z^2 + 4xyz + 5xy + 5xz + 5yz = m_{(2,2,2)} + 4m_{(1,1,1)} + 5m_{(1,1,0)}.$$

We notice another important fact: if we are considering N *bosons* in the LLL, apart from the usual exponential factor and for the normalization, the single-particle wave functions are of the form seen previously, i.e. z^m. In order to describe the N-particle state, the single-particle wave functions have to be symmetrized for particle exchanges, and therefore a basis for the Hilbert space of states is given precisely by symmetric homogeneous monomials of degree M (which is the total angular momentum of the system), i.e. monomial symmetric functions labelled with partition of length M. This means that we can label the elements $|\lambda\rangle$ of a basis of the N-particles LLL *bosonic* states with the partitions λ which correspond to the correct monomial function. In this case, λ_i are the angular momenta of the N particles and $|\lambda|$ is the total angular momentum of the system.

Power sum symmetric functions

If we have $\lambda = (r)$, we call m_λ the r-th power sum symmetric function p_r. For a generic partition λ, we define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. It can be shown (see [23]) that p_λ's form a $\mathbb{Q}$-basis of $\Lambda_\mathbb{Z}$.

Examples of power functions are:

$$\begin{aligned} p_1 &= m_{(1)} = \sum x_i, \\ p_2 &= m_{(2)} = \sum x_i^2, \\ p_{(2,1)} &= p_2 p_1 = \left(\sum x_i^2\right)\left(\sum x_i\right). \end{aligned} \tag{2.3}$$

Using power sum symmetric functions, we introduce a scalar product $\langle \cdot, \cdot \rangle$ on Λ such that:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda, \tag{2.4}$$

where δ is the Kroneker's delta and

$$z_\lambda = \prod_{r \geq 1} r^{n_r(\lambda)} n_r(\lambda)! = 1^{n_1(\lambda)} 2^{n_2(\lambda)} \dots n_1(\lambda)! n_2(\lambda)! \dots \tag{2.5}$$

Schur symmetric functions

We consider a partition λ and the function $D_\lambda = \det\left(x_i^{\lambda_j + n - j}\right)$, where n is the number of variables (or the length of the partition). By the properties of determinants, we know that D_λ vanishes every time $x_i = x_j$. Thus, it is divisible by the Vandermonde determinant V. *Schur functions* are defined as $S_\lambda = D_\lambda / V$. We notice that, since D_λ and V are two antisymmetric functions, S_λ is a symmetric function.

It can be shown (see [23]) that Schur symmetric functions are characterized uniquely by the following properties:

- $S_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\mu\lambda} m_\mu$;
- $\langle S_\lambda, S_\mu \rangle = 0$ for $\lambda \neq \mu$.

i.e. they are an orthogonal system with a particular "triangular" form when written on the monomial basis.

An important property of the Schur functions S_λ is that they provide a linear basis for the space of symmetric polynomials of homogeneous degree $\sum_i \lambda_i$.

2.3 Jack polynomials

Let us consider $\alpha \in \mathbb{R}$, $\alpha > 0$. We define an α-dependent scalar product $\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} \alpha^{l(\lambda)} z_\lambda$ on $\Lambda_\mathbb{R}$.

We call Jack symmetric functions (or *Jacks*) J_λ^α those uniquely characterized by the following properties:

- $J_\lambda^\alpha = m_\lambda + \sum_{\mu < \lambda} a_{\mu\lambda}(\alpha)\, m_\mu$;
- $\langle J_\lambda^\alpha, J_\mu^\alpha \rangle_\alpha = 0$ for $\lambda \neq \mu$.

A proof of the uniqueness of those functions is given in [22].

We remark that given a symmetric functions, we can always reduce it to a symmetric polynomial in N independent variables by setting $x_{N+1} = x_{N+2} = \dots = 0$. As all the properties of symmetric functions are valid for an indefinite number of independent variables, they will still hold in the polynomial case.

Therefore we can pass from Jack functions to Jack polynomials (we will use the word *Jacks* also to indicate the polynomials) with ease.

Jacks definition implies that J^α_λ are monic, i.e. the first coefficient of the expansion on the monomial basis is 1. However, other normalizations are possible. One among the most used is defined by $a_{\lambda,(1^{|\lambda|})} = |\lambda|!$. In the following we will not specify which normalization we are using, because it can be understood from the context.

2.3.1 Properties

Expansion on the monomial basis

A fundamental property of Jacks is already required in the definition

$$\begin{gathered} J^\alpha_\lambda = \sum c_{\mu\lambda} m_\mu, \\ c_{\mu\lambda} \neq 0 \Longleftrightarrow \mu \text{ can be squeezed by } \lambda. \end{gathered} \tag{2.6}$$

We can describe this property saying that Jacks' expansion on the monomial basis has non-null coefficients only for those partitions μ that we can obtain via squeezing from λ (the coefficients c are functions of the parameter α). This means that, since squeezing implies $|\mu| = |\lambda|$, Jacks are homogeneous symmetric functions of degree $|\lambda|$, and therefore they are eigenstates of the operator $\sum x_i \partial_i$ with eigenvalue $|\lambda|$.

We provide an example of such an expansion for $\lambda = (1001001)$:

$$\begin{aligned} J^\alpha_{(1001001)} = {} & c^{(1001001)}_{(1001001)} m_{(1001001)} + c^{(0110001)}_{(1001001)} m_{(0110001)} \\ & + c^{(1000110)}_{(1001001)} m_{(1000110)} + c^{(0101010)}_{(1001001)} m_{(0101010)} \\ & + c^{(0011100)}_{(1001001)} m_{(0011100)} + c^{(0003000)}_{(1001001)} m_{(0003000)}. \end{aligned}$$

Laplace-Beltrami operator

We could have defined Jack symmetric functions equivalently as the unique polynomial eigenfunctions of the so-called Laplace-Beltrami operator. This operator is of great physical relevance in relation to integrable models (to be precise, we refer to Calogero-Sutherland models, which describe N identical particles which have pair-wise inverse-square interaction in a one-dimensional system). The Laplace-Beltrami operator is defined as

$$H^\alpha_{LB} = \sum (x_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i \neq j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_i - x_j \partial_j). \tag{2.7}$$

In [22] it is proved that Jacks are eigenvectors of H^α_{LB} with eigenvalue $\left[\sum_i \left(\lambda_i^2 + \frac{\lambda_i}{\alpha}(N + 1 - 2i)\right)\right]$, i.e. the following relation hold

$$H^\alpha_{LB} J^\alpha_\lambda = \left[\sum_i \left(\lambda_i^2 + \frac{\lambda_i}{\alpha}(N + 1 - 2i) \right) \right] J^\alpha_\lambda. \tag{2.8}$$

We underline that this relation can be used as a definition for Jacks, if equation (2.6) is also required. Moreover, we notice that if $\alpha \to \infty$, $H^\infty_{LB} =$

$\sum(x_i\partial_i)^2$, which is diagonal on the monomial basis. This means that $\lim_{\alpha\to\infty} J^\alpha_\lambda = m_\lambda$, using also the fact that the eigenvalue of H^∞_{LB} corresponding to Jacks and monomials are the same.

2.3.2 Negative parameter Jack polynomials

We have defined Jacks for positive α. However, we need to consider also negative parameter Jacks for our purposes. In [24], it is shown that problems may arise only for negative rational values of α, those of the form

$$\alpha = -\frac{k+1}{r-1}. \tag{2.9}$$

For these values of the parameter, however, a criterion to select partitions whose associated Jacks are regular was found. We refer to this condition as (k, r, N)-admissibility. It describes a *generalized Pauli principle*, which prevents more than k particles in a N-particles system to occupy r consecutive sites (or single-particle states). Mathematically, this principle is formulated as $\lambda_i - \lambda_j \geq \lfloor \frac{j-i}{k} \rfloor r$ for each $i < j$ and with $\lfloor\ \rfloor$ being the floor function.

Our motivation for the introduction of negative parameter Jacks is the following. Haldane and Bernevig, in [25], first pointed out that Laughlin's wave functions (in their bosonic version, i.e. when they describe hypothetical FQHE system formed of bosons), as well as other model wave functions, are particular Jacks (see also section 3.1.1). We define bosonic Laughlin's wave functions as usual Laughlin's wave functions divided by a Vandermonde determinant, i.e. $\psi^L = \sum_{i<j}^N (z_i - z_j)^r$ with r even. Haldane and Bernevig have shown that these wave functions satisfy $\psi^L = J^{\alpha_{1,r}}_{\lambda^0(1,r)}$, where

- $\alpha_{k,r} = -\frac{k+1}{r-1}$;
- $\lambda^0(k,r)$ is the (k, r, N)-admissible partition which minimizes $|\lambda^0(k,r)|$, i.e. for $k = 1$ $(10^{r-1}10^{r-1}\dots)$.

The claim that ψ^L is a Jack is motivated in section 3.1.1, constructing H_{LB} as a sum of a constant and an operator which annihilates ψ^L.

We notice that, thank to the Jacks properties, the identification just described is the key observation for the expansion of Laughlin's wave functions on the monomial basis, i.e. the non interacting N particle basis. This is discussed in detail in section 3.1.1.

2.3.3 Jack antisymmetric functions

We have seen how Jacks can be useful for the study of bosonic systems, because of their symmetry for the exchange of coordinates. We want to define an equivalent class of antisymmetric functions to properly treat fermionic systems. We can follow the same construction of Λ to construct the ring of antisymmetric polynomials. A useful property of antisymmetric polynomials is that they are divisible by the Vandermonde determinant, and that this ratio defines a symmetric polynomial. Thus, we can build every antisymmetric function by multiplying a symmetric function with the Vandermonde determinant. An useful basis of antisymmetric functions are the *Slater determinants* sl_λ, which are nothing but

completely antisymmetrized monomials. In particular, we remark that for N fermions in the LLL, sl_λ's play the same role as m_λ's for bosons. We also notice that, as antisymmetrization (i.e. Pauli principle) prevents two variables to have the same exponent in a homogeneous polynomial (i.e. two particles to be in the same state), we can refer partitions to the *partition of minimal degree* (i.e. the state of minimal angular momentum), which is $(N-1, N-2, \ldots, 2, 1, 0)$ if N is the number of variables. Thus, we can rewrite a partition $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ for N fermions as $(\lambda_1 - (N-1), \lambda_2 - (N-2), \ldots, \lambda_N)$ without losing any kind of information.

Now we want to define antisymmetric Jacks and the simplest (and with most physical meaning) choice for this definition is (see [26])

$$S^\alpha_{\lambda'} = J^\alpha_\lambda \prod_{i<j}(z_i - z_j), \tag{2.10}$$

i.e. the multiplication for a Vandermonde determinant, with $\lambda'_i = \lambda_i + N - i$. We can justify this by the fact that Laughin's states and bosonic Laughlin's states differ only for the multiplication by a Vandermonde determinant. We are using relative angular momenta λ' as previously sketched.

In the following we will use Jacks properties that comes from the fact that they are eigenfunctions of Laplace-Beltrami operator. Therefore we have to construct an operator similar to the Laplace-Beltrami for antisymmetric Jacks. The basic idea is to use equation (2.8) (where $\left[\sum_i \left(\lambda_i^2 + \frac{\lambda_i}{\alpha}(N+1-2i)\right)\right] = E^\alpha_\lambda$):

$$\begin{aligned} E^\alpha_\lambda S^\alpha_{\lambda'} &= E^\alpha_\lambda \prod_{i<j}(z_i - z_j) J^\alpha_\lambda \\ &= \prod_{i<j}(z_i - z_j) E^\alpha_\lambda J^\alpha_\lambda = \prod_{i<j}(z_i - z_j) H^\alpha_{LB}(J^\alpha_\lambda). \end{aligned} \tag{2.11}$$

Then, by explicit calculation of $(z_i\partial_i)S^\alpha_{\lambda'}$ and $(z_i\partial_i)^2 S^\alpha_{\lambda'}$ and some manipulations (all the details are reported in [27]), we obtains that the right operator is

$$H^\alpha_{LB,\,F} = \sum_i (z_i\partial_i)^2 + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)\sum_{\substack{i,j \\ i\neq j}}\left[\frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j) - 2\frac{z_i^2 + z_j^2}{(z_i - z_j)^2}\right], \tag{2.12}$$

with eigenvalues

$$E^\alpha_{\lambda'} = \sum \lambda'_i\left[\lambda'_i - 2\left(\frac{1}{\alpha} - 1\right)i\right] + \left(\frac{1}{\alpha} - 1\right)\left[(N+1)|\lambda'| - N(N-1)\right]. \tag{2.13}$$

Within this construction, $H^\alpha_{LB,\,F}$ is diagonal on $S^\alpha_{\lambda'}$ with eigenvalue $E^\alpha_{\lambda'}$.

2.4 Jacks recursion law

The most important Jacks property for our purpose is that they admit a recursion law for the coefficients of their expansion on the monomial basis. Because of its importance in the following, we present the fundamental steps for its derivation in this section (calculation details can be found in [27]). Moreover, we present the analogue recursion law which exists also for the antisymmetric Jacks we have just constructed (this relation is due to Bernevig, see [26]).

2.4.1 Generic recursion laws

We say that an operator H acting on the symmetric/antisymmetric polynomials has a triangular action on a particular basis $\{b_\lambda\}$ if

$$Hb_\lambda = C_\lambda^\lambda b_\lambda + \sum_{\mu<\lambda} C_\mu^\lambda b_\mu, \tag{2.14}$$

with $C_\lambda^\lambda \neq 0$.

Let us consider such an operator H. We take an eigenvector f_λ of H with eigenvalue E_λ, i.e. $E_\lambda f_\lambda = Hf_\lambda$. Moreover, we suppose that:

$$f_\lambda = X_\lambda^\lambda b_\lambda + \sum_{\mu<\lambda} X_\mu^\lambda b_\mu \quad \text{with} \quad X_\lambda^\lambda \neq 0, \tag{2.15}$$

where X_μ^λ and C_μ^λ are suitable sets of coefficients. We notice that this hypothesis is redundant, as triangularity and the eigenvector relation suffice to prove it. Nevertheless, we have seen that this property is characterizing for Jacks and thus we can use it directly in our manipulations.

Then, if we plug equation (2.15) into the eigenvector relation, we obtains:

$$X_\kappa^\lambda = \frac{1}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa<\mu\leq\lambda}} X_\mu^\lambda C_\kappa^\mu, \tag{2.16}$$

whose validity is guaranteed if E_λ's are distinct for distinct λ's. This relation is recursive, and in particular we need an initial condition in order to use it. When we deal with monic Jacks, this condition is $X_\lambda^\lambda = 1$.

2.4.2 Action of triangular operator

A physical point of view suggests us that we can understand two body operators action on N particles by their action on 2 particles states (an example of this is the second-quantization form of the two body operators). In polynomial spaces, this means that we can study two body operators action on N variables basis functions using their action on 2 variable basis functions.

Now we fix a particular $\{b_\lambda\}$ as the basis: in case of symmetric polynomials we will use the symmetric monomial basis, in case of antisymmetric polynomials we will use the slater determinants one.

We suppose that H satisfies

$$\begin{aligned} Hb_{(m,n)} &= \sum_{k=0}^{(m-n)/2} F_k^{(m,n)} b_{(m-k,n+k)} \quad \text{for} \quad m > n, \\ Hb_{(m,n)} &= 0 \quad \text{for} \quad m = n. \end{aligned} \tag{2.17}$$

The first requirement is the triangularity in 2 variables. The second one allows us for a unified treatment of symmetric and antisymmetric cases. This further hypothesis is automatically satisfied in the antisymmetric case; for the symmetric one, in general it is not. However, if we take H as the Laplace-Beltrami operator and b_λ as the monomial basis, it is satisfied.

We use a "second quantization" formalism adapted to the polynomial ring to study the action of H. This construction allows for the usage of physical techniques and terminology to study polynomials. However, also an explicit calculation bring to the correct result.

Notations

We denote with Λ_1 the space of one variable polynomials (with complex coefficients and variables). We have seen how, having in mind the LLL, Λ_1 can be considered as a one particle Hilbert space. Its inner product is defined as $\langle r|s\rangle = \delta_{r,s}$, where $|r\rangle = z^r$ is the monomial basis (orthonormal for construction).

Analogously, we can see the space of k variable polynomials Λ_k as the k-particle Hilbert space associated with Λ_1, i.e. a space with a basis formed of factored states $|r_1, \ldots, r_k\rangle = |r_1\rangle \otimes \cdots \otimes |r_k\rangle = z_1^{r_1} \ldots z_k^{r_k}$ and equipped with the inner product $\langle r_1, \ldots, r_k|s_1, \ldots, s_k\rangle = \langle r_1|s_1\rangle \ldots \langle r_k|s_k\rangle = \delta_{r_1,s_1} \ldots \delta_{r_k,s_k}$.

We have now to choose the statistics of the particles: considering bosons or fermions, we have indeed to restrict Λ_k to the spaces of, respectively, symmetric polynomials Λ_k^+ or antisymmetric polynomials Λ_k^-. Within this picture, factored states $|r_1, \ldots, r_k\rangle$ and $|s_1, \ldots, s_k\rangle$ are different states only if $\{r_i\}$ is not a permutation of $\{s_i\}$. We can then label factored states by partitions λ. These partition must be such that $l(\lambda) < k$.

A basis for $\Lambda_k^\pm$ is given by, respectively, the symmetric monomials $m_\lambda = |\lambda^+\rangle$ or the Slater determinants $sl_\lambda = |\lambda^-\rangle$. We notice that, due to symmetrization and antisymmetrization procedures, the states $|\lambda^\pm\rangle$ are not normalized:

$$\left\langle \lambda^+ \middle| \lambda^+ \right\rangle = \frac{n_1(\lambda)! \ldots n_\infty(\lambda)!}{k!}, \tag{2.18}$$

$$\left\langle \lambda^- \middle| \lambda^- \right\rangle = \frac{1}{k!}. \tag{2.19}$$

We can therefore normalize them using the correct coefficients.

We introduce creation and annihilation operators $a_i^\dagger$ and a_i. In the fermionic case, their action in the occupation number picture is:

$$\begin{aligned} a_i^\dagger |n_0, n_1, \ldots, n_i, \ldots\rangle &= (-1)^{N_{SW}} |n_0, n_1, \ldots, n_i + 1, \ldots\rangle, \\ a_i |n_0, n_1, \ldots, n_i, \ldots\rangle &= \begin{cases} |n_0, n_1, \ldots, n_i - 1, \ldots\rangle & \text{for} \quad n_i \neq 0 \\ 0 & \text{for} \quad n_i = 0 \end{cases}. \end{aligned} \tag{2.20}$$

The same actions in the usual representation is:

$$\begin{aligned} a_i^\dagger |\lambda\rangle &= a_i^\dagger |\ldots, i, \ldots\rangle = (-1)^{N_{SW}} |\ldots, i + 1, \ldots\rangle, \\ a_i |\lambda\rangle &= a_i |\ldots, i, \ldots\rangle = \begin{cases} |\ldots, i - 1, \ldots\rangle & \text{for} \quad i \in \lambda \\ 0 & \text{for} \quad i \notin \lambda \end{cases}, \end{aligned} \tag{2.21}$$

The $(-1)^{N_{SW}}$ is a statistics dependent factor due to the fact that construction operators create a new particle in front of the partition. When we restore the decreasing order, we have to perform a number N_{SW} of swaps and we get the factor because of the canonical anticommutation rules. For instance

$$\begin{aligned} a_2^\dagger |6, 3, 1, 0, \ldots\rangle &= a_2^\dagger a_6^\dagger a_3^\dagger a_1^\dagger |0, 0, 0, 0, \ldots\rangle \\ &= (\pm)^2 a_6^\dagger a_3^\dagger a_2^\dagger a_1^\dagger |0, 0, 0, 0, \ldots\rangle = |6, 3, 2, 1, \ldots\rangle, \end{aligned} \tag{2.22}$$

where we have used the anticommutation rules.

We also introduce analogous creation and annihilation operators for the bosonic case which satisfy the canonical commutation rules.

Computation

Given a 2 body operator, its second quantization form is

$$H = \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m, \tag{2.23}$$

where $a^\dagger$ and a are creation and annihilation operators we have defined above. Here $\langle r,s|$ and $|m,n\rangle$ are factored states, rather than (anti)symmetrized states. This calculation is simpler if we use (anti)symmetric polynomials, and in order to do this we notice that, as H is symmetric,

$$\begin{aligned} H &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m \\ &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| S_\pm^\dagger S_\pm H S_\pm^\dagger S_\pm |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m \\ &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s|^\pm H |m,n\rangle^\pm \, a_r^\dagger a_s^\dagger a_n a_m, \end{aligned} \tag{2.24}$$

where $S_\pm$ are (anti)symmetrization operators. Therefore we can without any problem use (anti)symmetrized states in the calculation of the matrix element $\langle r,s| H |m,n\rangle$.

From now on we will drop the $\pm$ exponent of brakets, and we will consider all brakets describing (anti)symmetrized states.

After the calculation of the matrix element, we obtain:

$$\begin{aligned} H|\lambda\rangle &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m |\lambda\rangle \\ &= \sum_{\mu \le \lambda} F_k^{(m,n)} (\pm)^{N_{SW}} |\mu\rangle , \end{aligned} \tag{2.25}$$

where:

- the coefficients $F_k^{(m,n)}$ are those introduced in equation (2.17);
- in the last line, $\mu = [\lambda_1, \dots, \lambda_i - k, \dots, \lambda_j + k, \dots]$ with $\lambda_i = m$ and $\lambda_j = n$, i.e., μ is a generic partition squeezed from λ;
- the factor $(\pm)^{N_{SW}}$, where $+$ is for the bosonic case and $-$ for the antisymmetric, is caused by the reordering of μ after it is squeezed from λ.

2.4.3 Jacks recursion law

We have defined Laplace-Beltrami operator H_{LB}^α as:

$$H_{LB}^\alpha = K + \frac{1}{\alpha} V = \sum_i (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j). \tag{2.26}$$

Now we have to calculate H_{LB}^α action on monomials m_λ. We notice that K is diagonal on each m_λ with eigenvalue $\sum_i \lambda_i^2$. V is a two body operator. The

explicit calculation of its action on two variables monomials $m_{(n,p)}$, by equation (2.25), gives its action on generic monomials.

For this calculation, we suppose $n > p$:

$$\begin{aligned} Vm_{(n,p)} &= \left[\frac{x+y}{x-y}\left(x\partial x - y\partial y\right)\right]\left(x^n y^p + x^p y^n\right) \\ &= (n-p)\left[m_{(n,p)} + 2\sum_{k=1}^{(n-p)/2} m_{(n-k,p+k)}\right]. \end{aligned} \tag{2.27}$$

Keeping the notation of the precedent sections, we have $C_\kappa^\mu = F_k^{(n,p)} = \frac{2}{\alpha}(n-p)$ for $k \neq 0$ and $F_0^{(n,p)} = \frac{1}{\alpha}(n-p)$.

Now we can finally use equation (2.16) with $C_\kappa^\mu = F_k^{(n,p)} = \frac{2}{\alpha}(n-p)$ and we obtain the Jacks recursion law

$$X_\kappa^\lambda = \frac{\frac{2}{\alpha}}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \leq \lambda}} X_\mu^\lambda((\kappa_i + k) - (\kappa_j - k)), \tag{2.28}$$

where $\kappa = [\kappa_1, \ldots, \kappa_i, \ldots, \kappa_j, \ldots, \kappa_N]$, $\mu = [\kappa_1, \ldots, \kappa_i + k, \ldots, \kappa_j - k, \ldots, \kappa_N]$.

2.4.4 Antisymmetric Jacks recursion law

We have defined the antisymmetric Laplace-Beltrami operator H_F^α as:

$$\begin{aligned} H_F^\alpha &= K + \left(\frac{1}{\alpha} - 1\right) V \\ &= \sum_i (z_i\partial_i)^2 + \left(\frac{1}{\alpha} - 1\right) \sum_{\substack{i,j \\ i<j}} \left[\frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j) - 2\frac{z_i^2 + z_j^2}{(z_i - z_j)^2}\right]. \end{aligned} \tag{2.29}$$

H_F^α's action on Slaters sl_λ is calculated as for the bosonic case. K is again diagonal on sl_λ with eigenvalue $\sum_i \lambda_i^2$. Regarding the two-particle part, the calculation is similar to the bosonic one, and we obtain, supposing $n > p$ and using $k = n - p$:

$$Vsl_{(n,p)} = (n-p-2)sl_{(n,p)} + 2\sum_{l=1}^{(n-p)/2} (n-p-2l)sl_{(n-l,p+l)}. \tag{2.30}$$

Thus we have $C_\mu^\lambda = F_k^{(n,p)} = \left(\frac{1}{\alpha} - 1\right) 2(n-p-2k)$ for $k \neq 0$ and $F_0^{(n,p)} = \left(\frac{1}{\alpha} - 1\right)(n-p-2)$.

Using equation (2.16) with $C_\kappa^\mu = F_k^{(n,p)} = \left(\frac{1}{\alpha} - 1\right) 2(n-p-2k)$ we finally obtain the recursion law for antisymmetric Jacks, i.e.

$$X_\kappa^\lambda = \frac{2\left(\frac{1}{\alpha} - 1\right)}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \leq \lambda}} X_\mu^\lambda(\kappa_i - \kappa_j)(-1)^{N_{SW}}, \tag{2.31}$$

where $\kappa = [\kappa_1, \ldots, \kappa_i, \ldots, \kappa_j, \ldots, \kappa_N]$, $\mu = [\kappa_1, \ldots, \kappa_i + k, \ldots, \kappa_j - k, \ldots, \kappa_N]$ and N_{SW} is the number of swap necessary to properly reorder the partition after the squeeze.

2.4.5 Remarks on recursion laws

The first remark we want to consider concerns the validity of the obtained recursion laws. As we can see from equation (2.16), the recursion law are valid if different Jacks have different eigenvalues. Otherwise, the denominator may vanish. In [22], a lemma is given that solves the problem for positive α. It states that if two partitions generate the same eigenvalue, the two partitions are incomparable in the dominance order, thus they never belong to the same "chain of squeezings". For negative α, problems arise for the same negative rational values we have discussed in section (2.3.2). In [26] it is shown that under a limit prescription, every vanishing denominator is coupled with a vanishing numerator, such that the coefficients are still finite numbers.

There is another observation regarding the recursion laws that we want to do. They describe the decomposition of Jacks on the pure monomial basis. But we have seen how LLL is generated by multiples of monomials. This is not relevant in the following, nevertheless for completeness we write the decomposition on normalized N particles wave functions:

$$J_\lambda^\alpha = \sum_{\mu\leq\lambda} c_{\mu,\lambda} m_\mu = \sum_{\mu\leq\lambda} c_{\mu,\lambda} \frac{1}{\mathcal{N}_\mu} \mathcal{N}_\mu m_\mu = \sum_{\mu\leq\lambda} \bar{c}_{\mu,\lambda} \bar{m}_\mu, \tag{2.32}$$

where $\bar{c}_{\mu,\lambda} = c_{\mu,\lambda}/\mathcal{N}_\mu$ are the coefficients of Jacks' expansion over a differently normalized monomial basis.

Chapter 3

Fock-space construction of Laughlin states

In this chapter we will use the theory of partitions, symmetric functions and Jack polynomials in order to obtain the coefficients of the expansion of Laughlin wave functions in the free-particles basis states. We will see that we are able to pass from the recursion law obtained in the literature to a squeezing operator which, acting on particular reference states, gives the Laughlin wave functions. We will also use this operator to prove some symmetry properties of Laughlin wave functions (which have been proved in the literature in other ways).

Finally we present another algorithm which can be used in order to obtain the searched coefficients.

3.1 A long-standing problem

Laughlin's wave functions (1.41) are the basis of our theoretical understanding of the fractional quantum Hall effect. Although their explicit form as function of the electron coordinates is available, when we deal with a large number of particles it is rather difficult to use it, because of the computational difficulties in manipulating large polynomials. Another possible way to write Lauhglin's wave functions is as their expansions in Slater determinants. As we have seen, they are the anti-symmetrization of the factored single-particle states and they form a basis of the Hilbert space (which in our case is restricted to the LLL).

An example of the usefulness of such an expansion is the following: if we want to confront a model wave function (as Laughlin's) with the result of the exact diagonalization, we can profitably use the Slater determinants basis. Indeed the matrix form of the Hamiltonian is computed usually using this basis, so the overlap is easily evaluated if also the model wave function is expanded in Slater determinants.

Another reason for finding the Slater determinant expansion is the calculation of expectation values of operators. In a second quantization formalism, we write a single-particle operator as

$$O = \sum_{k,k'} \langle k| O |k'\rangle \, a_k^\dagger a_{k'}, \tag{3.1}$$

where the index k labels a single-particle basis state. For simplicity, let us take the case $\langle k| O |k'\rangle = o(k)\,\delta_{k,k'}$, where δ is the Kronecker delta. Then we have

$$O = \sum_k o(k)\; a_k^\dagger a_k \tag{3.2}$$

We have seen how Slater determinants in the LLL (i.e. for single-particle functions of the form z^m, without considering normalization and exponential factor) are labelled by partitions. Therefore given the partitions λ and μ we have the corresponding Slater determinant basis states $|\lambda\rangle$ and $|\mu\rangle$, and

$$\langle\lambda| O |\mu\rangle = \delta_{\lambda,\mu} \sum_i o(\lambda_i), \tag{3.3}$$

where λ_i are the entries in the usual representation, i.e. the angular momentum of the particles. Writing Laughlin's states $|\psi_L\rangle$ as linear combination of Slater determinants, i. e. $|\psi_L\rangle = \sum_\lambda c_\lambda |\lambda\rangle$, we can evaluate the expectation value

$$\langle\psi_L| O |\psi_L\rangle = \sum_\lambda |c_\lambda|^2 \left(\sum_i o(\lambda_i)\right). \tag{3.4}$$

The important point is that this expectation value is obtained without any integration or polynomial manipulation, when we have the expansion $|\psi_L\rangle = \sum_\lambda c_\lambda |\lambda\rangle$.

The problem is that if we want to obtain such an expansion, we have to expand the polynomial part of the Laughlin wave functions, i.e. $\prod_{i<j}(z_i - z_j)^q$, in the Slater determinant basis. In general, this requires the manipulation of large polynomials. For instance, with two particles and filling factor $\nu = 1/3$, we have:

$$\psi_{\nu=1/3}(z_1, z_2) = (z_1 - z_2)^3 = \begin{vmatrix} z_1^3 & 1 \\ z_2^3 & 1 \end{vmatrix} - 3 \begin{vmatrix} z_1^2 & z_1 \\ z_2^2 & z_2 \end{vmatrix} = sl_{(3,0)} - 3sl_{(2,1)}. \tag{3.5}$$

When we consider just three particles, the expansion becomes more difficult

$$\begin{aligned}
\psi_{\nu=1/3}(z_1, z_2, z_3) &= \left(\begin{vmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{vmatrix}\right)^3 \\
&= \begin{vmatrix} z_1^6 & z_1^3 & 1 \\ z_2^6 & z_2^3 & 1 \\ z_3^6 & z_3^3 & 1 \end{vmatrix} - 3 \begin{vmatrix} z_1^6 & z_1^2 & z_1 \\ z_2^6 & z_2^2 & z_2 \\ z_3^6 & z_3^2 & z_3 \end{vmatrix} - 3 \begin{vmatrix} z_1^5 & z_1^4 & 1 \\ z_2^5 & z_2^4 & 1 \\ z_3^5 & z_3^4 & 1 \end{vmatrix} \\
&\quad + 6 \begin{vmatrix} z_1^5 & z_1^3 & z_1 \\ z_2^5 & z_2^3 & z_2 \\ z_3^5 & z_3^3 & z_3 \end{vmatrix} - 15 \begin{vmatrix} z_1^4 & z_1^3 & z_1^2 \\ z_2^4 & z_2^3 & z_2^2 \\ z_3^4 & z_3^3 & z_3^2 \end{vmatrix} \\
&= sl_{(6,3,0)} - 3sl_{(6,2,1)} - 3sl_{(5,4,0)} + 6sl_{(5,3,1)} - 15sl_{(4,3,2)}.
\end{aligned} \tag{3.6}$$

Even using a dedicated computer program, this task becomes soon impossible to be performed if we continue increasing the particle number.

However, in these two expansions we can see that the coefficients are always *integer* numbers. This is an hint that a combinatorial interpretation of these

expansions may exists. Following this intuition, Dunne [13] found a formula for the expansion coefficients which relies on the following observations. The (polynomial part of) Laughlin wave functions are just V^{2m+1}, where V is the Vandermonde determinant and m is an integer. So, dividing by V, we obtain

$$\frac{\psi_{Laughlin}}{V} = V^{2m} = \sum_{\lambda} a_\lambda S_\lambda, \tag{3.7}$$

where we have used that a symmetric function can be expanded in the basis of Schur functions. Expanding the Laughlin wave functions in the Slater determinant basis, we can also write

$$\frac{\psi_{Laughlin}}{V} = \sum_{\lambda} b_\lambda \frac{sl_\lambda}{V} \tag{3.8}$$

and so, because $sl_\lambda/V = S_\lambda$ (using the Schur functions definition, given in section 2.2.1) we have $b_\lambda = a_\lambda$. Therefore we can expand V^{2m} in the Schur functions basis in order to obtain the coefficients. The expansion of V^{2m} in the power sum basis is rather simple to obtain, as we can see taking for instance $m = 1$:

$$\begin{aligned} V^2 &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \begin{vmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ 1 & z_2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} \end{vmatrix} \\ &= \begin{vmatrix} N & p_1 & \cdots & p_{N-1} \\ p_1 & p_2 & \cdots & p_N \\ \vdots & \vdots & \cdots & \vdots \\ p_{N-1} & p_N & \cdots & p_2 N - 1 \end{vmatrix} \\ &= \sum_{\sigma} (-1)^\sigma p_{\mu_\sigma}, \end{aligned} \tag{3.9}$$

where the sum in the last line is done over the permutation of $1, 2, \ldots, N$ and the partition μ_σ is given by

$$\mu_\sigma = \{\sigma(1) - 1, \sigma(2), \sigma(3) + 1, \ldots, \sigma(N) + N - 2\}. \tag{3.10}$$

From this expansion, the Frobenius' formula allow us to pass in the Schur function basis, and we get (the details are explained in [13]) the final formula for the coefficients

$$a_\lambda = N \sum_{\sigma \text{ with } \sigma(1)=1} (-1)^\sigma \chi^\lambda_{\mu_p} + \sum_{\sigma \text{ with } \sigma(1)\neq 1} (-1)^\sigma \chi^\lambda_{\mu_p}, \tag{3.11}$$

where the $\chi^\lambda_{\mu_p}$ are given by the Frobenius' formula, i.e.

$$p_\lambda = \sum_{\mu} \chi^\lambda_\mu S_\mu. \tag{3.12}$$

These coefficients are called characters of the symmetric group and they are always *integers*.

This algorithm has two main advantages:

- it allows us to calculate the coefficients without making any manipulation which uses polynomials;
- it makes us understand why all the coefficients of Laughlin expansion in the Slater determinant basis are integers.

However, this method also has some limits:

- in order to use equation (3.11), we have to know the characters $\chi^{\lambda}_{\mu_p}$ and an efficient algorithm for their calculation is not available;
- equation (3.11) gives us little information about the thermodynamic limit ($N \to \infty$), because we have not an explicit formula for the characters $\chi^{\lambda}_{\mu_p}$.

This last point is the most problematic: indeed *real* systems have very large N (typical electron densities in FQHE experiments are between 10^{15} and 10^{17} m^{-2}) and so we are especially interested in results valid in this limit. Therefore Dunne and other authors [14] have tried to use the just presented result in order to find expressions for the coefficients which are valid for large N, without great success.

There is also another problem: we know that Laughlin wave functions do not describe all the observed plateaux and Dunne's approach is difficult to generalize to other proposed model wave functions.

These problems have encouraged the research for other possible paths and the first big progress was made by Bernevig and Haldane [25], who recognised Laughlin wave functions to be Jack polynomials.

3.1.1 Bernevig's recursion law

Following Bernevig and Haldane [25], we show that *bosonic* Laughlin wave functions are Jack polynomials. The first step is the definition of the operator

$$D_i^{L,r} = \frac{\partial}{\partial z_i} - r \sum_{j(\neq i)} \frac{1}{z_i - z_j}. \tag{3.13}$$

Now, if ψ_r is the Laughlin wave function which describes the system at filling factor $1/r$, we can prove via direct calculation that

$$D_i^{L,r} \psi_r = 0, \tag{3.14}$$

where r is an *even* integer. Therefore the following relation holds:

$$\sum_i z_i D_i^{L,-1} z_i D_i^{L,r} \psi_r = 0. \tag{3.15}$$

Now the important remark is that $\sum_i z_i D_i^{L,-1} z_i D_i^{L,r}$ is the Laplace-Beltrami operator (2.7) with parameter $\alpha = -2/(r-1)$, minus a constant $\mathcal{E}$ (which is $rN(N-1)(N+1+3r(N-1))/12$). Therefore we can write

$$(H^{\alpha}_{LB} - \mathcal{E})\psi_r = 0 \tag{3.16}$$

and so

$$H^{\alpha}_{LB} \psi_r = \mathcal{E} \psi_r. \tag{3.17}$$

Then ψ_r is a Jack with parameter $\alpha = -2/(r-1)$, but we need to find also its labelling partition, which we will call *root* partition. We notice that the obtained class of parameters α is the same of equation (2.9) with $k = 1$. Therefore, as we have seen in section 2.3.2, there is a $(k,\, r,\, N)$-admissibility condition which the root partition has to satisfy. Using this observation, we can obtain the root partition. We know from the Laughlin wave functions, given in equation (1.41), that ψ_r describes a state of total angular momentum $M = N(N-1)r/2$. Therefore the root partition λ must have $|\lambda| = M$. Now, let us consider the partition

$$\lambda^0(1,r) = (1, \underbrace{0, 0, \ldots, 0}_{r-1\ times}, 1, \underbrace{0, 0, \ldots, 0}_{r-1\ times}, 1, \ldots), \tag{3.18}$$

which satisfies the $(k,\, r,\, N)$-admissibility condition, with $k = 1$. For this partition we have

$$|\lambda| = \sum_i r(N-i) = r\frac{1}{2}N(N-1). \tag{3.19}$$

We can easily convince ourselves that this is the *only* partition which satisfies the $(1,\, r,\, N)$-admissibility condition and has $|\lambda| = M$: if we move a 1 in the occupation number representation (3.18), in order to keep $|\lambda|$ constant we have necessarily to violate the $(1,\, r,\, N)$-admissibility condition and vice-versa.

Since we have proved that bosonic Laughlin wave functions are Jacks, we can use all the results presented in the previous chapter. In particular, we can write the expansion

$$\psi_r = J^{\alpha}_{\lambda^0_r} = \sum_{\mu \le \lambda^0_r} c_{\mu,\lambda^0_r}\, m_\mu, \tag{3.20}$$

where the sum involves only the root partition and all the partitions we can obtain from it by squeezing. We also know that there is a recursion law which allow us to write (from equation (2.28))

$$c_{\kappa,\lambda^0_r} = \frac{2/\alpha}{\rho_{\lambda^0_r} - \rho_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \le \lambda}} c_{\mu,\lambda^0_r}((\kappa_i + k) - (\kappa_j - k)) \tag{3.21}$$

where $\kappa = [\kappa_1, \ldots, \kappa_i, \ldots, \kappa_j, \ldots, \kappa_N]$, $\mu = [\kappa_1, \ldots, \kappa_i + k, \ldots, \kappa_j - k, \ldots, \kappa_N]$, i.e. κ is squeezed from μ, and

$$\rho_\lambda = \sum_i \lambda_i(\lambda_i - \frac{2}{\alpha}i). \tag{3.22}$$

The use of ρ_λ here instead of Laplace-Beltrami eigenvalues E_λ used in equation (2.28) is justified because we can readily prove that

$$\rho_\lambda - \rho_\mu = E_\lambda - E_\mu \tag{3.23}$$

if $\mu < \lambda$.

3.1.2 Fermionic case

The results just obtained are valid only for the bosonic Laughlin wave functions, but we know that the actual Laughlin wave functions are the fermionic ones.

Thus, we can use the antisymmetric Jacks theory developed in the previous chapter in order to describe such states. This approach is the same used in [26] by Thomale, Estienne, Regnault and Bernevig. The key point is that

$$\psi_{2m+1} = V\psi_{2m}, \tag{3.24}$$

where ψ_q denotes the Laughlin state with filling factor $1/q$, m is an integer and V is the Vandermonde determinant. Using the definition of antisymmetric Jacks (2.10) and the fact just proved that (r being an even number) $\psi_r = J^{\alpha(r)}_{\lambda^0_r}$, we obtain that ψ_{r+1} is an antisymmetric Jack with parameter $\alpha = -2/(r-1)$ and root partition $\lambda'^0_{r+1} = \lambda^0_r + N - i$, i.e. of the form

$$\lambda'^0_{r+1} = (1, \underbrace{0, \dots, 0}_{r\ times}, 1, \underbrace{0, \dots, 0}_{r\ times}, 1, \dots). \tag{3.25}$$

We can expand ψ_r in the Slater determinant basis (here S denotes an antisymmetric Jack)

$$\psi_r = S^\alpha_\lambda = \sum_{\mu \leq \lambda} b_{\mu\lambda} sl_\mu, \tag{3.26}$$

where again the sum concerns only the root and the partition squeezed from it. We notice that in the fermionic case each entry of the partition has to be 0 or 1 due to the Pauli principle.

Using the recursion law (2.31), we can write for the coefficient of the expansion

$$b_{\kappa\lambda} = \frac{2(\frac{1}{\alpha} - 1)}{\rho^F_\lambda - \rho^F_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \leq \lambda}} b_{\kappa\lambda}(\kappa_i - \kappa_j)(-1)^{N_{SW}} \ . \tag{3.27}$$

where $\kappa = [\kappa_1, \dots, \kappa_i, \dots, \kappa_j, \dots, \kappa_N]$, $\mu = [\kappa_1, \dots, \kappa_i + k, \dots, \kappa_j - k, \dots, \kappa_N]$, N_{SW} is the number of swap necessary to properly reorder the partition after the squeeze and $\rho^F_\lambda = \sum_i \lambda_i(\lambda_i + 2i(1 - 1/\alpha))$. We notice that N_{SW} is equal to the number of fermions that the two particles involved in the squeezing process have to pass through.

Now we present the advantages of this Jacks-based method:

- this algorithm is noticeably faster than Dunne's approach and it has been applied to increase the maximally reachable system in finite-size studies;
- in [25] and [26] it has been shown that not only the Laughlin wave functions are Jack polynomials. For instance, we can describe with Jack also the bosonic non-abelian Read-Rezayi states and their (as well as Laughlins') quasihole excitations. However, which FQHE states can be seen as Jacks and which not it is not yet well understood;
- this kind of approach has allowed the discovery of a new symmetry of the expansion coefficients, named "product rule". This symmetry appears when we consider a partition squeezed from the root that has the special property that two parts of it can be identified as partition squeezed from roots of smaller systems sizes. For example, we consider the partition (0110000110), which is squeezed from (1001001001), i.e. the root partition of the fermionic case with $\nu = 1/3$, $N = 4$. The two sub-part (011000) and (0110) can be seen as squeezed from (1001), i.e. the root of the case

$\nu = 1/3$, $N = 2$. The product rule symmetry says that the product of the two coefficients obtained from the $N = 2$ systems gives the coefficient of the $N = 4$ partition.

3.2 Beyond the recurrence relation: squeezing operator

Until now there is a considerable difference between the bosonic and the fermionic recursion laws, i.e. between equation (3.21) and (3.27). We can generalize these recursion laws and eliminate this difference introducing the *squeezing operator*.

We introduce the creation and destruction operators $a_x^\dagger$ and a_x which create and destruct a 1 in the x-th position in the occupation number representation of a partition (i.e. a particle with angular momentum x), with $[a_x, a_x^\dagger]_\pm = 1$ where the upper (lower) sign is for the bosonic (fermionic) case. These operators are canonical and so their action is

$$\begin{aligned} &a_i^\dagger \,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\,|n_1, \ldots, n_i + 1, \ldots\rangle\,; \\ &a_i \,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle \text{ if } \mathrm{n_i} > 0; \\ &a_i \,|n_1, \ldots, n_i, \ldots\rangle = 0 \text{ if } \mathrm{n_i} = 0 \end{aligned} \tag{3.28}$$

for the bosonic case, while for the fermionic case we have

$$\begin{aligned} &a_i^\dagger \,|n_1, \ldots, n_i, \ldots\rangle = (-1)^{n_1+n_2+\cdots+n_{i-1}}\,|n_1, \ldots, n_i + 1, \ldots\rangle \text{ if } \mathrm{n_i} = 0; \\ &a_i^\dagger \,|n_1, \ldots, n_i, \ldots\rangle = 0 \text{ if } \mathrm{n_i} = 1; \\ &a_i \,|n_1, \ldots, n_i, \ldots\rangle = (-1)^{n_1+n_2+\cdots+n_{i-1}}\,|n_1, \ldots, n_i - 1, \ldots\rangle \text{ if } \mathrm{n_i} = 1; \\ &a_i \,|n_1, \ldots, n_i, \ldots\rangle = 0 \text{ if } \mathrm{n_i} = 0. \end{aligned} \tag{3.29}$$

We also introduce the *squeezing operator*:

$$U = \sum_{s}^{\infty} \sum_{t=1}^{\infty} \sum_{u=t}^{\infty} (u \pm t)\, a_{s+t}^\dagger\, a_{s+u}^\dagger\, \mathcal{N}\, a_{s+t+u}\, a_s\ , \tag{3.30}$$

where the operator $\mathcal{N}$ is diagonal on basis states (Slaters or monomials) and it is given by ($\hat{n}_m = a_m^\dagger a_m$ is the number operator)

$$\mathcal{N} = \left(\sqrt{(\hat{n}_{s+u} + 1)\,(\hat{n}_{s+t} + 1)\,(\hat{n}_s + 1)\,(\hat{n}_{s+t+u} + 1)}\right)^{-1}. \tag{3.31}$$

We notice that the eigenvalue of $\mathcal{N}$ is always 1 in the fermionic case, while in the bosonic case it exactly compensates the factors which come from the creation/destruction operators, according to equations (3.29).

We have seen previously that the operatorial part $a_{s+t}^\dagger\, a_{s+u}^\dagger\, a_{s+t+u}\, a_s$ ($\mathcal{N}$ is diagonal, so it does not change the action of the operator except for a numerical factor) represent a squeeze in which the particles in sites s and $s + t + u$ are moved of t sites toward each other (in figure 3.1, 3.2, 3.3, 3.4 and 3.5 there are some example of squeezing chains obtained by repeated actions of the squeezing operator). Therefore the operator (3.30) do all the possible squeezes to a partition, weighing each squeeze with the initial or final distance (respectively for bosons or fermions) of the two particles just squeezed.

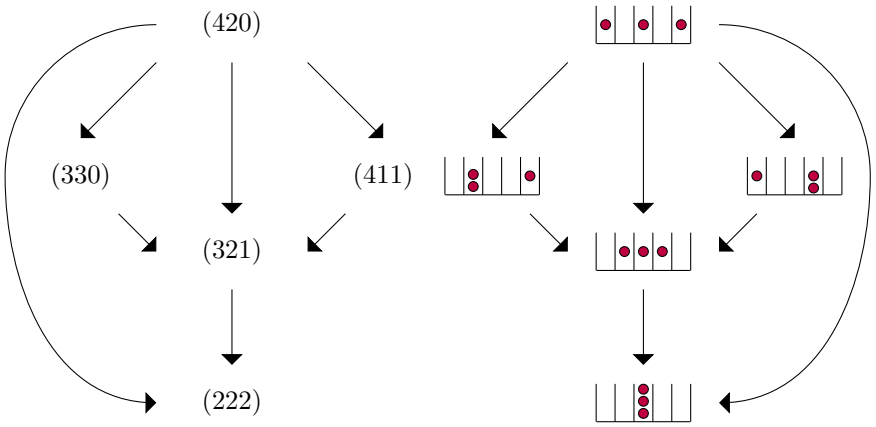

Figure 3.1: The whole graph obtained with the subsequent action of the *squeezing operator* (3.30) for $N = 3$ bosons, with filling factor $1/2$. On the left it is represented the partition graph, in which each vertex contains a state, labelled with a partition in the occupation number representation. On the right there is the same graph, with the intuitive site and particle representation.

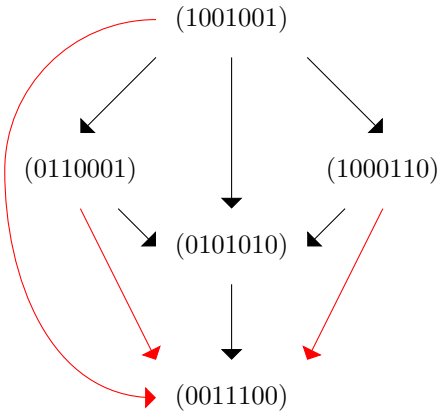

Figure 3.2: The whole graph obtained with the subsequent action of the *squeezing operator* (3.30) for $N = 3$ fermions, with filling factor $1/3$. Each vertex contains a state, labelled with a partition in the occupation number representation. The red edges correspond to squeezes that make the two fermions squeezed closer of more than one position.

Considering fermions for instance, an example of the squeezing operator action is (we recall that we can use a partition to label a Slater determinant basis state):

$$U|1,0,0,0,0,1\rangle = 3|0,1,0,0,1,0\rangle + |0,0,1,1,0,0\rangle \ . \tag{3.32}$$

Finally we introduce the "weight" operator Q^q_λ, defined by

$$Q^q_\lambda = \frac{k(q)}{\langle\lambda|\,\widetilde{Q}\,|\lambda\rangle - \widetilde{Q}} \ , \tag{3.33}$$

where $-k(q)$ is the largest *odd* integer for which it holds $-k(q) \leq q$, $1/q$ is the filling factor and $\widetilde{Q} = \sum_s (s(s - \sum_{t<s} k(q)\,\hat{n}_t))\hat{n}_s$. If we denote with ρ_ν the eigenvalue of $\widetilde{Q}$ relative to its eigenstate $|\nu\rangle$, we obtain

$$Q^q_\lambda|\mu\rangle = \frac{k(q)}{\rho_\lambda - \rho_\mu}|\mu\rangle \ , \tag{3.34}$$

where

$$\rho_\lambda(q) = \sum_{i=1}^{N} \lambda_i(\lambda_i - ik(q)). \tag{3.35}$$

Therefore Q^q_λ is a diagonal operator in the Slater determinant or permanent basis, whose purpose is to "weight" these states, hence the name *weight operator*.

We remark that, with our definition of $k(q)$, the difference $\rho_\lambda - \rho_\mu$ is a generalization of the two differences in the denominators of equations (3.21) and (3.27). Indeed $k(q)$ is exactly $2/\alpha$ for bosons and $2(\frac{1}{\alpha} - 1)$ for fermions, as we can see from the following reasoning. Let us suppose that the filling factor is $\nu = \frac{1}{q}$. We have obtained that in the bosonic case the corresponding Jack is the one with $\alpha = -\frac{2}{q-1}$, with q even. In this case, the factor $\frac{2}{\alpha}$ in equation (3.21) is therefore $-(q-1)$. On the other hand, in the fermionic case, we have seen that if we have the filling factor $\nu = \frac{1}{q}$ with q odd it holds:

$$\alpha = -\frac{2}{(q-1)-1} = -\frac{2}{q-2}. \tag{3.36}$$

In this case the factor $2(\frac{1}{\alpha} - 1)$ in equation (3.27) is simply $-q$. So in both case the factor in the recurrence relations and in ρ is the largest odd integer that most approaches q, with a minus sign.

Now, let us define the state:

$$|\psi\rangle = (\mathbb{I} - QU)^{-1}|\lambda\rangle = \big(\mathbb{I} + QU + (QU)^2 + \cdots\big)|\lambda\rangle \ , \tag{3.37}$$

where $|\lambda\rangle$ is a state of the kind described in equations (3.18) and (3.25), i.e. $|\lambda\rangle = \big|1, 0^{q-1}, 1, 0^{q-1}, \ldots\big\rangle$ for filling factor $\nu = \frac{1}{q}$.

We want to prove that the recurrence relation between the coefficients of the state (3.37) expansion in the Slater determinant/monomial basis is given by

$$b_\mu = \frac{k}{\rho_\lambda - \rho_\mu} \sum_{\substack{\theta \\ \mu < \theta \leq \lambda}} \binom{(\theta_i - \theta_j)}{(\mu_i - \mu_j)} b_\theta (\pm 1)^{N_{sw}} \ , \tag{3.38}$$

that is, the equations (3.21) and (3.27). In order to do it, we expand the state (3.37) in the Slater determinant/monomial basis

$$|\psi\rangle = \sum_{\mu} b_\mu |\mu\rangle \tag{3.39}$$

and then, multiplying for $\langle\nu|$ and using $\langle\nu|\mu\rangle = \delta_{\nu\mu}$, we obtain:

$$b_\nu = \langle\nu|(\mathbb{I} - QU)^{-1}|\lambda\rangle \ . \tag{3.40}$$

Now there are two possibility: if $\nu = \lambda$, we obtain $b_\lambda = 1$. If $\nu \neq \lambda$, $\langle\nu|\lambda\rangle = 0$ and so:

$$b_\nu = \langle\nu|QU\,(\mathbb{I} + QU + (QU)^2 + \cdots)|\lambda\rangle = \langle\nu|QU|\psi\rangle \tag{3.41}$$

We can evaluate it using that:

$$\langle\nu|QU = \left[(QU)^\dagger|\nu\rangle\right]^\dagger = [q(\nu)U^\dagger|\nu\rangle]^\dagger \tag{3.42}$$

where q(ν) is the eigenvalue of Q corresponding to the eigenstate $|\nu\rangle$. Therefore we need to calculate $U^\dagger|\nu\rangle$. $U^\dagger$ is the adjoint operator of U, namely the "*anti-squeezing" operator*:

$$U^\dagger = \sum_{s}^{\infty}\sum_{t=1}^{\infty}\sum_{u=t}^{\infty}(u \pm t)\, a^\dagger_{s+t+u}\, a^\dagger_s\, \mathcal{N}\, a_{s+t}\, a_{s+u}. \tag{3.43}$$

Applying that to $|\nu\rangle$ we obtain non-zero terms only when $s + u = \nu_j$ and $s + t = \nu_i$, with $\nu_i > \nu_j$ and ν_i, ν_j any two *filled* sites of $|\nu\rangle$. Moreover, if we are in the bosonic case, the eigenvalue of $\mathcal{N}$ exactly compensates the factors which come from the action of the creation/destruction operators, while in the fermionic case we obtain the familiar factor $(-1)^{N_{SW}}$ because of their action. We can then write the result as:

$$U^\dagger|\nu\rangle = \sum_{\theta;\nu>\theta} {(\theta_i-\theta_j) \choose (\nu_i-\nu_j)} (-1)^{N_{SW}}|\theta\rangle \ . \tag{3.44}$$

where the sum over the partition θ is another way to write the sum over index i and j (namely, i and j are biunivocally linked to the partition θ that we get after the anti-squeezing process). We can understand the difference between the bosonic and the fermionic case by considering that $t + u$ is the *final* (i.e. after the anti-squeezing process) distance between the two squeezed particles and $t - u$ is the *initial* distance between them.

Using now the adjoint of equation (3.44) in the equation (3.41), we obtain:

$$b_\nu = q(\nu) \sum_{\theta;\nu>\theta} {(\theta_i-\theta_j) \choose (\nu_i-\nu_j)} (-1)^{N_{sw}}\langle\theta|\psi\rangle \tag{3.45}$$

and then, observing that $\langle\theta|\psi\rangle = b_\theta$, we obtain the recurrence relation (3.38).

Until now we have written the weight operator Q in a unified (i.e. in a statistic-independent) fashion, but we can do the same for the squeezing operator U. In order to do this, we exchange indices t and u in the second term of equation (3.30) to get

$$U = \sum_{s}^{\infty}\sum_{t=1}^{\infty}\sum_{u=t}^{\infty} u\, a^\dagger_{s+t}\, a^\dagger_{s+u}\, \mathcal{N}\, a_{s+t+u}\, a_s \pm \sum_{s}^{\infty}\sum_{u=1}^{\infty}\sum_{t=u}^{\infty} u\, a^\dagger_{s+u}\, a^\dagger_{s+t}\, \mathcal{N}\, a_{s+t+u}\, a_s. \tag{3.46}$$

The permutation of the creation operators in the second term compensates the sign. Next, we exchange the order of the sums over t and u ($\sum_{u=1}^{\infty}\sum_{t=u}^{\infty} = \sum_{t=1}^{\infty}\sum_{u=1}^{t}$) and we get, adding the two terms ($t = u$ is double term):

$$U = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\sum_{u=1}^{\infty} u\, a_{s+t}^{\dagger}\, a_{s+u}^{\dagger}\, \mathcal{N}\, a_{s+t+u}\, a_s + \sum_{s,t=1}^{\infty} t\, a_{s+t}^{\dagger}\, a_{s+t}^{\dagger}\, \mathcal{N}\, a_{s+2t}\, a_s. \quad (3.47)$$

The second term is non-zero for bosons, and describes a sum of squeezing operators: two particles are removed from s and $s + 2t$ and placed on site $s + t$.

In conclusion, with the squeezing operator U written as in equation (3.47) and with the weight operator $Q_{\lambda}^{(q)}$ defined in equation (3.33) we can write in a unified fashion bosonic and fermionic Laughlin wave function as:

$$|\psi_L^{(q)}\rangle = \frac{1}{\mathbb{I} - Q_q U}\,|\lambda\rangle\,. \quad (3.48)$$

where the partition λ is built up using q following the rule

$$|\lambda\rangle = \left|1, 0^{q-1}, 1, 0^{q-1}, \ldots\right\rangle. \quad (3.49)$$

3.2.1 Are we considering all the partitions?

We have seen that Jacks expansion involves the root partition and those squeezed from it. In [26] the squeezing operation is defined as the movement of two particle one toward the other of *one* site. Our definition of squeeze is "more general", because the operatorial part of (3.47) moves the particles of t sites. The algorithm presented in [26], however, use the first type of squeeze only to find the involved partitions, thereafter all the types of squeeze are used in the coefficient calculations. Therefore, if the state (3.48) has the correct terms, it also has the correct coefficients, because they are univocally determined from the recursion law and the fact that we take $b_{\lambda}^{\lambda} = 1$.

In this section we prove that the operator (3.47) creates exactly the same states that can be created using only the first type of squeeze. Indeed, we just have to show that each squeeze in which the particles are shifted toward each other of more than one unit in angular momentum (or lattice position or quantum number) can be obtained composing squeezes in which the particles are shifted toward each other of only one unit (see also figure 3.5).

A squeeze of the *first type* is obtained as the action of the operator:

$$A_{s,u} = a_{s+1}^{\dagger}\, a_{s+u}^{\dagger}\, a_{s+u+1}\, a_s\ . \quad (3.50)$$

A squeeze of the *second type* is, in contrast, the result of the action of the operator:

$$A_{s,u}^{t} = a_{s+t}^{\dagger}\, a_{s+u}^{\dagger}\, a_{s+u+t}\, a_s\ , \quad (3.51)$$

with $t > 1$. In figure 3.2 we highlight with red edges some examples of second type squeezes, while the black edges represent first type squeezes.

We want to prove that we can always obtain $A_{s,u}^{t}$ with subsequent action of $A_{s,u}$. There is a better notation for squeezing operators for our current purpose. We denote with $A_{a,b}^{\ell}$ the squeezing operator which squeezes the particles in sites

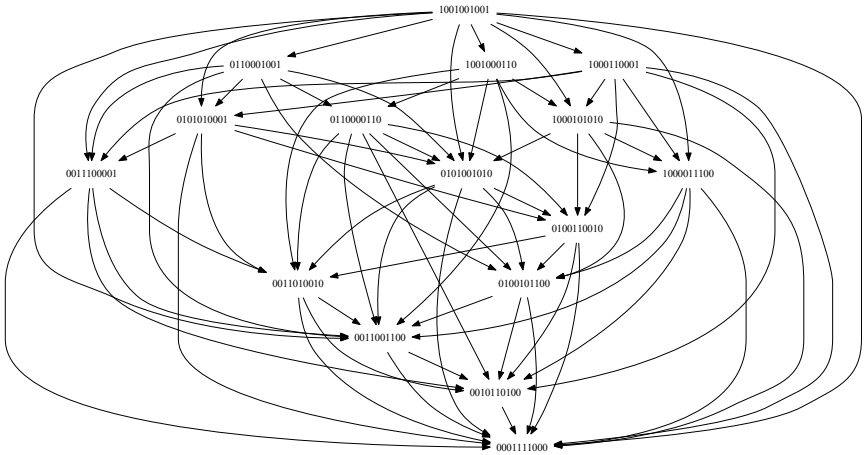

Figure 3.3: The whole graph obtained with the subsequent action of the *squeezing operator* (3.30) for $N = 4$ fermions, with filling factor 1/3. Each vertex contains a state, labelled with a partition in the occupation number representation. Each edge corresponds to a squeeze.

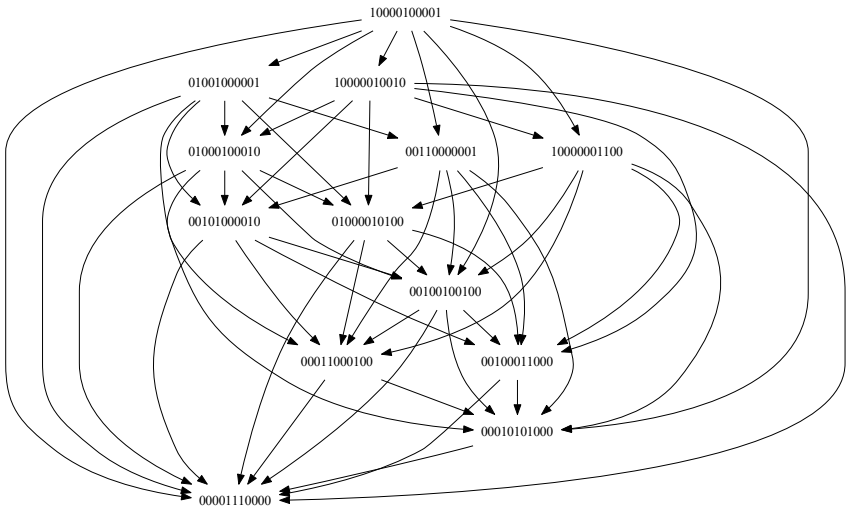

Figure 3.4: The whole graph obtained with the subsequent action of the *squeezing operator* (3.30) for $N = 3$ fermions, with filling factor 1/5. Each vertex contains a state, labelled with a partition in the occupation number representation. Each edge corresponds to a squeeze. Comparing this graph with the one in figure 3.2, we notice that if we increase the filling factor, the number of involved partitions increases considerably.

a and b, shifting them of ℓ steps toward each other (if no ℓ is specified, we imply $\ell = 1$).

We consider the state

$$|s\rangle = |\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m\rangle, \tag{3.52}$$

written in the usual representation, with $N = n+m$ particles in lattice positions $\mu_1 > \cdots > \mu_n > \nu_1 > \cdots > \nu_m$. We want to show that the action of operator $A^{\ell}_{\mu_1,\nu_m}$ can be obtained as subsequent actions of operators $A_{a,b}$, with $\nu_1 < \mu_1 - \ell < \mu_n$ and $\mu_n > \nu_m + \ell > \nu_1$. We remark that before μ_1 and after ν_m, as between $\mu_1 - \ell$ and $\nu_m + \ell$, we can have anything: this do not change in any way the proof we are giving.

Let us consider only the two more internal particles, i.e. the state $|\mu_n, \nu_1\rangle$. Without any loss in generality, we suppose that $\mu_n - (\mu_1 - \ell) > (\nu_m - \ell) - \nu_1$. If we apply the operator A_{μ_n,ν_1}, we get:

$$A_{\mu_n,\nu_1}|\mu_n, \nu_1\rangle = |\mu_n - 1, \nu_1 + 1\rangle \ . \tag{3.53}$$

We can now apply the operator A_{μ_n+1,ν_1-1} and continue doing so, until we reach the state $|\mu_n - \ell', \nu_1 + \ell'\rangle$, with $\nu_1 + \ell' = \nu_m + \ell$, and so $\ell' = (\nu_m + \ell) - \nu_1$. Therefore we have put the particle that was in ν_1 in the position where, applying $A^{\ell}_{\mu_1,\nu_m}$ to the state $|s\rangle$, would have jumped the particle that was in ν_m.

This is the trick: we shift (one at a time) μ_n and ν_1 to the position (respectively) of $\mu_n + \ell$ and $\nu_1 - \ell$. Thereafter we shift each other fermion labelled with μ of the state $|s\rangle$ to the position of the following and each other fermion labelled with ν to the position of the preceding.

Therefore we proceed. Now we manage the state

$$|s'\rangle = |\mu_1, \ldots, \mu_n + \ell', [\cdots], \nu_2, \ldots, \nu_m\rangle \tag{3.54}$$

($[\cdots]$ denotes all the particles that we have already put in their final position) exactly like the state $|s\rangle$: we consider only the more internal part, $|\mu_n + \ell', [\cdots], \nu_2\rangle$, and we move the particle with first-type squeezes, until one of them reaches its final position (which are, in this case, $\mu_n + \ell$ and ν_1).

We can iterate this process until we obtain the state (we suppose that in the previous step we used μ_1 in the squeezes and not ν_m, without any loss in generality):

$$|s^{(n)}\rangle = |\mu_1 + a, [\cdots], \nu_m\rangle \ . \tag{3.55}$$

Again we do first-type squeeze until one of the two particles reaches his final position. The other will always do the same, because of the total angular momentum (i.e. the sum of the partition entries) conservation. Indeed, the states $A^{\ell}_{\mu_1,\nu_m}|s\rangle$ and $|s\rangle$ have the same total angular momentum, like every state we can reach starting from $|s\rangle$ and doing (any type of) squeezes. So the final state too must have the same total angular momentum, and so if it has $N-1$ particles in the same position as $A^{\ell}_{\mu_1,\nu_m}|s\rangle$, the N-th is fixed by the total angular momentum conservation law.

3.3 Other results

3.3.1 Another proof of Bernevig's product rules

Here we prove Bernevig's product rules using our squeezing-operator formalism. We consider a state $|\nu\rangle = |\nu_1\nu_2\rangle$ which can be obtained via squeezing from

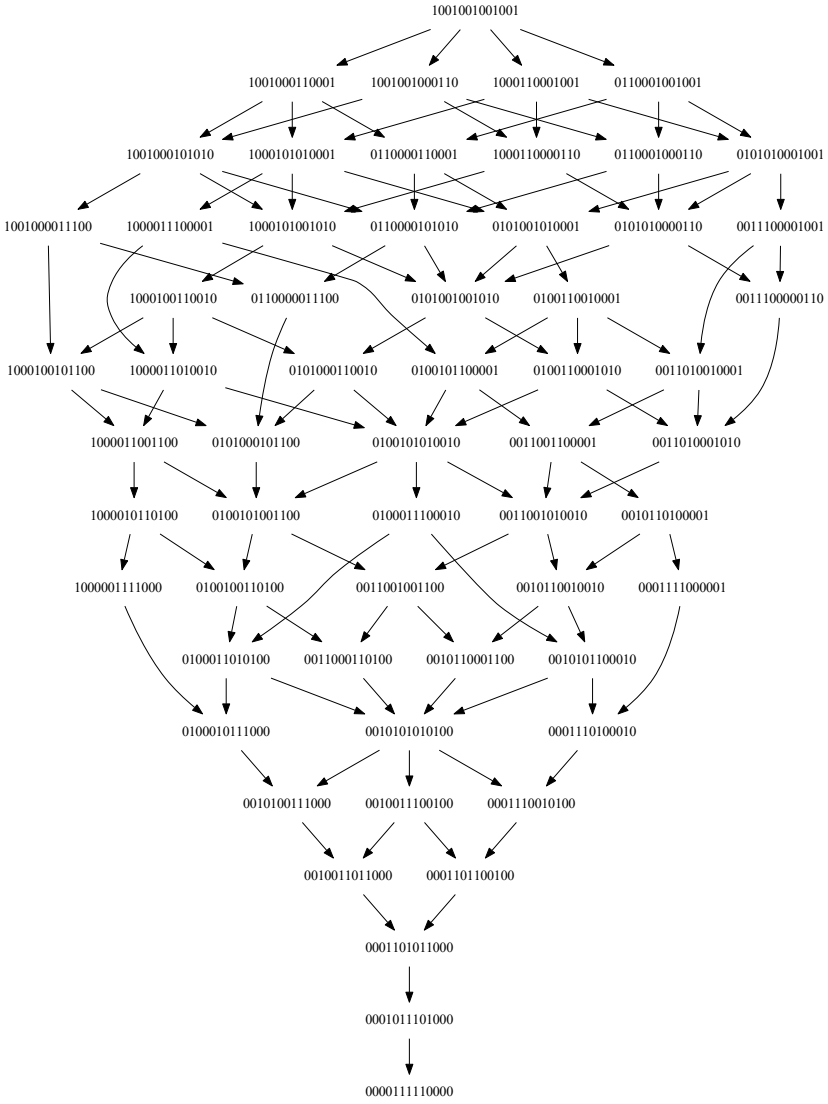

Figure 3.5: Part of the graph obtained with the subsequent action of the *squeezing operator* (3.30) for $N = 5$ fermions, with filling factor $1/3$. Each vertex contains a state, labelled with a partition in the occupation number representation. Each edge corresponds to a squeeze. In this graph only the squeezes that shift the particles toward each other of one position are considered, in order to make the graph human-readable. As proved in section 3.2.1, the partitions which are linked by these squeezes are the same we would have obtained if we had used all the possible squeezes.

$|\lambda\rangle = |\lambda_1\lambda_2\rangle$, where $|\nu_1\rangle$ is a squeezed of $|\lambda_1\rangle$, $|\nu_2\rangle$ of $|\lambda_2\rangle$. We define the operator

$$S = \mathbb{I} - QU \tag{3.56}$$

where Q is the weight operator (3.33) and U the squeezing operator (3.30). With these definitions, the coefficient b_ν of the partition $|\nu\rangle$ in Laughlin wave function expansion is given by

$$\begin{aligned} b_\nu &= \langle\nu_1\nu_2|\psi_L\rangle = \langle\nu_1\nu_2|S^{-1}|\lambda_1\lambda_2\rangle \\ &= \langle\nu_1\nu_2|S^{-1}S_1\otimes S_2\sum_{\mu_1,\mu_2}|\mu_1\mu_2\rangle\langle\mu_1\mu_2|S_1^{-1}\otimes S_2^{-1}|\lambda_1\lambda_2\rangle \\ &= \sum_{\mu_1,\mu_2}\langle\nu_1\nu_2|S^{-1}S_1S_2|\mu_1,\mu_2\rangle b_{\mu_1}b_{\mu_2}, \end{aligned} \tag{3.57}$$

where S_1 and S_2 act respectively only on the first and the second sub-partition, i.e. $S_1\otimes S_2\,|\mu_1\mu_2\rangle = (S_1\,|\mu_1\rangle)(S_2\,|\mu_2\rangle)$. We notice that with the step $\langle\mu_1\mu_2|S_1^{-1}\otimes S_2^{-1}|\lambda_1\lambda_2\rangle = b_{\mu_1}b_{\mu_2}$, the sum now involves only the partitions μ_1 and μ_2 obtainable by squeezing from λ_1 and λ_2. We explicit $S_1 = \mathbb{I} - Q_1U_1$ and same for S_2 and we obtain:

$$\begin{aligned} (3.57) &= \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|S^{-1}(\mathbb{I} - Q_1U_1 - Q_2U_2 + Q_1U_1Q_2U_2)|\mu_1,\mu_2\rangle \\ &= \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|(\mathbb{I} + O + QUO + QUQUO + \cdots)|\mu_1,\mu_2\rangle \\ &= b_{\nu_1}b_{\nu_2} + \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|O|\mu_1,\mu_2\rangle + \dots\ , \end{aligned} \tag{3.58}$$

where we have defined $O = QU - Q_1U_1 - Q_2U_2 + Q_1U_1Q_2U_2$. Now, we prove that $\sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|O|\mu_1,\mu_2\rangle = 0$ if $|\nu_1\nu_2\rangle$ is not-mixed, i.e. ν_1 can be squeezed from λ_1 and ν_2 from λ_2 (we have never used this hypothesis until this point). In this case we obtain the product rule, because for the generic term in (3.58), the following relation holds:

$$\sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|QU\cdots QU\sum_\eta|\eta\rangle\langle\eta|O|\mu_1,\mu_2\rangle \propto \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\eta|O|\mu_1,\mu_2\rangle, \tag{3.59}$$

where, if $|\nu_1\nu_2\rangle$ is not-mixed, $|\eta\rangle$ is not-mixed too. Indeed $\langle\nu_1\nu_2|QU\cdots QU|\eta\rangle = 0$ if one or more of QU mix, because a mixing squeeze increases the total angular momentum of a semi-partition (the one with lower angular momenta) and decreases that of the other (the one with higher angular momenta).

So we just need to show that $\sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|O|\mu_1,\mu_2\rangle = 0$.

$$\begin{aligned} &\sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|O|\mu_1,\mu_2\rangle \\ &= \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|QU|\mu_1,\mu_2\rangle - \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|Q_1U_1|\mu_1,\mu_2\rangle + \\ &\quad - \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|Q_2U_2|\mu_1,\mu_2\rangle + \sum_{\mu_1,\mu_2} b_{\mu_1}b_{\mu_2}\langle\nu_1\nu_2|Q_1U_1Q_2U_2|\mu_1,\mu_2\rangle. \end{aligned} \tag{3.60}$$

Now, because of the recursion law, it holds that $\sum_\mu b_\mu \langle \nu|QU|\mu\rangle = b_\nu$. We also notice that

$$\begin{aligned} U &= \sum_{s}^{\infty}\sum_{t=1}^{\infty}\sum_{u=t}^{\infty}(u-t)\, a^\dagger_{s+t}\, a^\dagger_{s+u}\, a_{s+t+u}\, a_s \\ &= \left(\sum_{s=0}^{N-1}\sum_{t=1}^{N/2}\sum_{u=t}^{N/2} + \sum_{s=N}^{\infty}\sum_{t=1}^{\infty}\sum_{u=t}^{\infty} + \sum_{s=0}^{N-1}\sum_{t=1}^{N/2}\sum_{u=N/2}^{\infty}\right)(u-t)\, a^\dagger_{s+t}\, a^\dagger_{s+u}\, a_{s+t+u}\, a_s \\ &= U_1 + U_2 + U_{mix}, \end{aligned} \tag{3.61}$$

in the case in which $l(\lambda_1) = l(\lambda_2) = l(\lambda)/2$ ($l(\lambda)$ is the length of the partition λ). The numbers 1 and 2 refers to the first and the second half of the original partition (but the same argument can be used to write $U = U_1 + U_2 + U_{mix}$ when the partition is split in any point). Using this relation, we get

$$\begin{aligned} (3.60) &= \sum_{\mu_2} b_{\nu_1} b_{\mu_2} \langle \nu_1\nu_2|QU_2|\nu_1,\mu_2\rangle + \sum_{\mu_1} b_{\mu_1} b_{\nu_2} \langle \nu_1\nu_2|QU_1|\mu_1,\nu_2\rangle - b_{\nu_1} b_{\nu_2} \\ &= b_{\nu_1} b_{\nu_2} \langle \nu_1\nu_2|Q(Q_1^{-1} + Q_2^{-1})|\nu_1\nu_2\rangle - b_{\nu_1} b_{\nu_2} \end{aligned} \tag{3.62}$$

(as intermediate step, we insert $\mathbb{I} = Q_2^{-1}Q_2$ between Q and U_2, and the same with the other term and we use again the recursion law).

Now, we assume that (we will prove this using the symmetry discussed in the next section):

$$Q|a,b\rangle = \frac{k}{\rho_{\lambda_a\lambda_b} - \rho_{ab}}|a,b\rangle = \frac{k}{\rho_{\lambda_a} - \rho_a + \rho_{\lambda_b} - \rho_b}|a,b\rangle, \tag{3.63}$$

where k is a constant. So, because $Q_1|a,b\rangle = \frac{k}{\rho_{\lambda_a} - \rho_a}|a,b\rangle$ and $Q_2|a,b\rangle = \frac{k}{\rho_{\lambda_b} - \rho_b}|a,b\rangle$, we immediately get

$$\langle \nu_1\nu_2|Q(Q_1^{-1} + Q_2^{-1})|\nu_1\nu_2\rangle = 1 \tag{3.64}$$

and the product rules are proved.

3.3.2 Symmetries and the squeezing operator

In his work, Dunne proves a Laughlin wave functions symmetry that here is obtained using our squeezing operator. This allow us to generalize this symmetry to each FQHE state for which equation (3.48) holds.

We have seen before that a state that can be labelled by a partition can also be seen as a sequence of empty/full sites (i.e. the partition in its occupation number representation). In this case, the filling factor is related to the *barycenter* of the state: indeed each state involved in the expansion of a Laughlin state at fixed filling factor, has the same barycenter.

Now we define the operator I, that reverses the state with respect to its barycenter. So, for example,

$$I|1000110\rangle = |0110001\rangle. \tag{3.65}$$

Using the usual representation, the same example reads:

$$I|621\rangle = |540\rangle. \tag{3.66}$$

Now we want to show that the Laughlin state is invariant under this operator. The squeezing operator commutes with I, because it only depends on differences of angular momenta. On the other side, the weight operator is less trivial: we need to prove that $\rho_\lambda = \sum_i \left[\lambda_i^2 + ki\lambda_i\right]$ is invariant under the action of I. Explicitly, if the filling factor is $\nu = 1/q$ it holds that (in the usual representation)

$$I|\lambda_1, \ldots, \lambda_N\rangle = |[(N-1)q - \lambda_N], \ldots [(N-1)q - \lambda_1]\rangle. \tag{3.67}$$

In this case we obtain

$$\begin{aligned} \sum_i [(N-1)q - \lambda_i] &= (N-1)^2 q^2 N - 2(N-1)q \sum_i \lambda_i + \sum_i \lambda_i^2 \\ &= \sum_i \lambda_i^2 \end{aligned} \tag{3.68}$$

where we have evaluated $\sum_i \lambda_i$ using the root partition, so

$$\sum_i \lambda_i = \sum_{l=0}^{N-1} ql = q\frac{1}{2}N(N-1). \tag{3.69}$$

We also have

$$\begin{aligned} &\sum_n [(N-1)q - \lambda_n]\,(N-n+1) \\ &\quad = N^2(N-1)q - \sum_n n(N-1)q - N\sum_n \lambda_n + \sum_n n\lambda_n + \frac{1}{2}N(N-1)q \\ &\quad = \sum_n n\lambda_n \end{aligned} \tag{3.70}$$

where $(N-n+1)$ is the position of the element $[(N-1)q - \lambda_n]$ in the inverted partition, if n was the position of λ_n in the original partition, as we can see from equation (3.67).

Now we are ready to prove the equation (3.63): using the definition of ρ given in equation (3.35), we can easily verify that

$$\rho_{\lambda_a\lambda_b} - \rho_{ab} = \rho_{\lambda_a\lambda_b} - \rho_{a\lambda_b} + \rho_{\lambda_a\lambda_b} - \rho_{\lambda_a b}. \tag{3.71}$$

Moreover, always from the equation (3.35) follows that

$$\rho_{\lambda_a\lambda_b} - \rho_{a\lambda_b} = \rho_{\lambda_a} - \rho_a. \tag{3.72}$$

The other equation, i.e.

$$\rho_{\lambda_a\lambda_b} - \rho_{\lambda_a b} = \rho_{\lambda_b} - \rho_b, \tag{3.73}$$

follows instead from the symmetry argument just given: indeed $\rho_{\lambda_a\lambda_b} = \rho_{\lambda_b\lambda_a}$ and $\rho_{\lambda_a b} = \rho_{b\lambda_a}$.

3.3.3 A different algorithm for the coefficients

In this section we want to present another possible algorithm for the calculation of coefficients of Laughlin wave functions expansion in the Slater determinant basis. Unfortunately, we have not been able until now to complete the proof, but numerical results seems to confirm the correctness of this algorithm.

A limit of the algorithm that use equation (3.27) is that in order to calculate the coefficient of a given state $|\nu\rangle$ we have to know *all* the coefficients of the states from which we can reach $|\nu\rangle$ with one squeeze. Increasing N, the coefficients calculation rapidly became an impossible task even using the most high-performance computers. If we want to find a better algorithm, we have to introduce a new identity. It come from a study made by Seidel and Chen [28] about the second-quantization form of the quantum Hall Hamiltonian and his frustration-free nature. The fundamental result we need is the following: if we define an operator

$$Q_R^m = \sum_x x^m a_{R-x} a_{R+x}, \tag{3.74}$$

and $\left|\psi_L^{(q)}\right\rangle$ is the Laughlin wave function at filling factor $\nu = \frac{1}{q}$, therefore

$$Q_R^m \left|\psi_L^{(q)}\right\rangle = 0 \quad \text{for all } R,\ m \text{ for which } 0 \le m < q \text{ and } (-1)^m = (-1)^q. \tag{3.75}$$

The proof of this result can be found in [28] and we will not be presented it in this work.

We use the fact that Laughlin wave functions are Jack polynomials, and so

$$|\psi_L\rangle = |\lambda\rangle + \sum_{\mu<\lambda} b_\mu^\lambda |\mu\rangle . \tag{3.76}$$

We suppose $q = 3$, i.e. we are at filling factor $\nu = \frac{1}{3}$. Because of equation (3.75), for each partition $\widetilde{\mu}$ it holds ($m = 1$ is the only possible m if $q = 3$):

$$\langle\widetilde{\mu}| Q_R^1 |\psi_L\rangle = 0. \tag{3.77}$$

Since the operator Q_R^1 destroys two particles, the partitions $\widetilde{\mu}$ for which the equation (3.77) is not trivial are the ones with length $N - 2$.

Now, we consider a state $|\nu\rangle$ in the expansion (3.76) of $|\psi_L\rangle$. We imagine to calculate the expectation value in equation (3.77) using $\widetilde{\mu} = \nu_{ij}$ where the subscript means that $\widetilde{\mu}$ is obtained from ν simply removing the i-th and the j-th particles, with $\nu_i > \nu_j$. For example, if $\nu = (a, b, c, d, e, f)$, we have $\nu_{2,5} = (a, c, d, f)$. We want to calculate the equation (3.77) with $\widetilde{\mu} = \nu_{ij}$, so

$$\langle\nu_{ij}| Q_R^1 |\psi_L\rangle = \sum_{\mu\le\lambda} b_\mu^\lambda \langle\nu_{ij}| Q_R^1 |\mu\rangle . \tag{3.78}$$

with $b_\lambda^\lambda = 1$. Since this equation holds for each R, we choose $R = \frac{\nu_i - \nu_j}{2}$. We notice that in this sum only the terms with the partitions μ obtainable from ν by squeezing or anti-squeezing of the i-th and the j-th particles matter. Indeed the operator Q_R^1 acts only on the two particles which it destroys, and so in order

that the expectation value does not vanish, the particles different from the i-th and the j-th must be in the same position in ν and in μ. Then

$$\langle \nu_{ij} | \, Q_R^1 \, | \psi_L \rangle = \sum_{\nu \overset{\sim}{_{ij}} \mu} b_\mu^\lambda \; \frac{\mu_i - \mu_j}{2} (-1)^{N_\mu^{SW}}, \tag{3.79}$$

where the symbol $\nu \overset{\sim}{_{ij}} \mu$ stand for "μ is equal to ν or it is obtained from ν via a single squeeze between the i-th and the j-th particle of ν" and N_μ^{SW} is the number of fermions the two particles squeezed have to pass through[1].

Using the Seidel's equation (3.75), we get

$$\sum_{\nu \overset{\sim}{_{ij}} \mu} b_\mu^\lambda \, (\mu_i - \mu_j)(-1)^{N_\mu^{sw}} = 0 \qquad \text{for each partition } \nu \leq \lambda. \tag{3.80}$$

We stress that the equation (3.80) is completely non-trivial and indeed it represents the core of the new algorithm. However, another important ingredient is missing: we conjecture that for each partition $\nu \leq \lambda$ there is always a pair of particles for which the sum in the equation (3.80) consist of only two terms, i.e. the partition ν itself and the only partition μ from which ν is obtained by a squeezing of the considered pair of particles. In that case, equation (3.80) assumes its final form:

$$b_\nu^\lambda = -b_\mu^\lambda \, \frac{\mu_i - \mu_j}{\nu_i - \nu_j} (-1)^{N_{sw}}. \tag{3.81}$$

This algorithm is considerably faster than the Bernevig's one, because we only need to know the coefficient of a partition to obtain the one of another partition. It also explains why we obtain very simple coefficients after doing an enormous number of sums using the equation (3.27).

However, the strength of the presented algorithm relies on the conjecture that for each partition μ that can be obtained from the root partition λ by subsequent squeezing, there is always (at least) one pair of particles which:

- cannot be squeezed further;
- can be anti-squeezed in only one way in order to give an "admissible" partition.

We present two examples about the case $N = 4$:

- the partition is $\mu_1 = (1, 0, 0, 0, 1, 1, 0, 0, 0, 1)$, the pair considered is the second and the third particle: this pair can not be squeezed further and the only possible anti-squeezing bring us back to the root partition. Indeed if we squeeze our pair of two units, we will get the partition $(1, 0, 1, 0, 0, 0, 0, 1, 0, 1)$ that is not admissible, i.e. can not be obtained from the root partition using squeezes. So according to the formula (3.81), the coefficient is:

$$b_{\mu_1} = -\frac{3}{1} = -3. \tag{3.82}$$

[1] in this case the factor $(-1)^{N_\mu^{SW}}$ is due to the different action of the destructor of a particle with quantum number x on a state with this particle in the position i or on another state with the particle in the position $i + j$: exactly a $(-1)^j$ factor of difference, according to equations (3.29).

- the partition is $(0,0,1,1,1,0,0,0,0,1)$, the pair considered is the first and the second particle (but also the second and the third are a possible choice). The only possible anti-squeezing of those two particles bring us to the partition $(1,0,0,0,1,1,0,0,0,1)$ which coefficient is -3. In this case, the formula (3.81) gives (the second particle passes through the third):

$$b_{\mu_2} = -(-1)b_{\mu_1}\frac{5}{1} = -15. \tag{3.83}$$

We have checked the formula (3.81) up to $N = 4$ and it seems correct. However, until now we have not been able to prove our conjecture.

So far we have considered only the case $q = 3$. However it seems quite simple to understand what happens with higher q. Let us take, for instance, $q = 5$. In this case it is obvious that our conjecture cannot hold. Indeed for each pair of particles in the root partition, we can always do at least two squeezes. But we can state now that for a generic odd q there is always a pair of particles for which the sum in equation (3.80) consist of $(q-1)/2$ terms: this conjecture is consistent with the previous one and more general. Assuming that this conjecture holds, we notice that now there are exactly $(q-1)/2$ values of m for which the equation (3.75) must holds, each of them giving a linearly independent equation to solve. So we can only compute $(q-1)/2$ coefficients at a time, solving $(q-1)/2$ *linear* equations. Thus our algorithm keep working and it remains considerably simpler (and faster) than other known algorithms.

Chapter 4

Conclusions and future directions

4.1 Our results

The study of Laughlin wave functions is of central importance for our understanding of the fractional quantum Hall effect. In particular in this work we have dealt with the Laughlin wave function expansion in the Slater determinant or monomial (respectively for the fermionic and bosonic case) basis.

In the previous chapter we presented the recent discovery that Laughlin wave functions (as other FQHE model wave functions) are Jack polynomials. Exploiting their properties, a recursion relation for the coefficients of their expansion in the single-particle state basis has been found.

With this work we have obtained an operator, called *squeezing operator*, which is written in terms of creation and destruction operators and implements this recursion law. We have proven that this operator gives us a new way to write the Laughlin wave functions, i.e. as our operator acting on a reference Slater determinant (in the fermionic case) or permanent (in the bosonic one). In this sense, we have obtained a Fock-space operator contruction of Laughlin fractional quantum Hall effect states. Moreover we have seen that this construction of Laughlin wave functions is statistics-invariant, i.e. it is the same for both bosons and fermions.

Our squeezing operator allowed us to prove other results known in the literature, i.e. the product rules and the symmetry for inversion with respect to the partition barycenter, in an original way. Our demonstrations are more general than those obtained in the literature, because now they are valid for both bosons and fermions at once. Moreover they are also valid for any FQHE state which can be written as our operator applied to a reference state: most of our proofs do not depend on which reference state we choose and the remaining are easily generalizable to other reference states.

Finally, using the fact that Laughlin wave functions are Jack polynomials we have found a new algorithm for the calculation of the coefficients of the Slater determinant expansion. This algorithm seems to be very promising because of the simplicity of the final coefficient formula, although we have not succeeded in giving a complete demonstration of it.

4.2 Future directions

With this work we have not found a universal microscopic theory of the fractional quantum Hall effect. However, our result could help the research of such a theory. Indeed in a recent work by Rotondo, Molinari, Ratti and Gherardi [15], a mapping has been found between the Hall Hamiltonian restricted to the lowest Landau level and a long-range repulsive lattice gas model in one dimension. Although this mapping is shown to be exact only in a particular limit, the so-called thin-torus limit, the results obtained are encouraging: the predicted plot of the Hall resistivity versus the magnetic field is in qualitative agreement with the one experimentally observed, for a remarkable number of filling factor, as shown in figure 4.1.

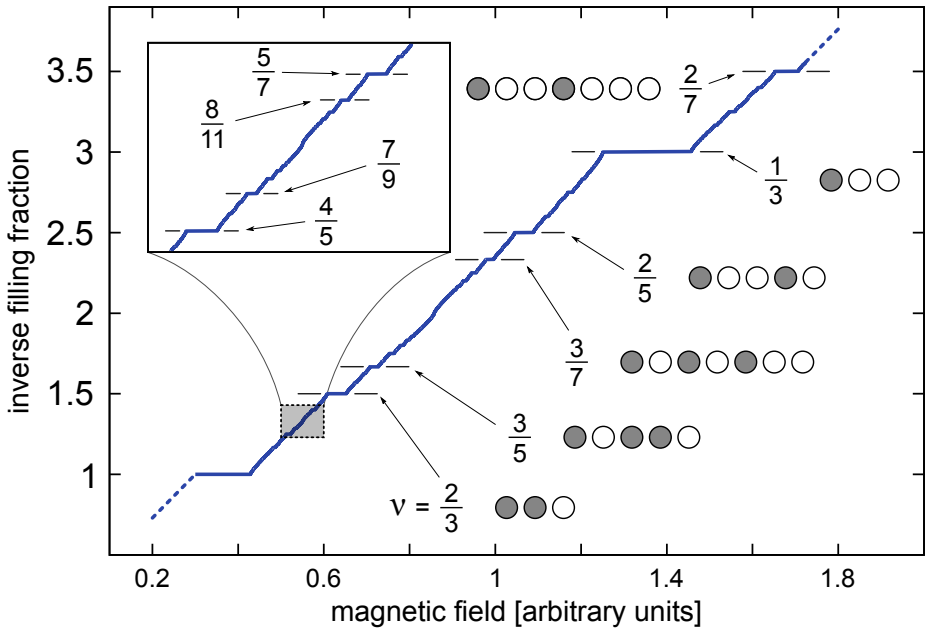

Figure 4.1: Inverse filling factor $1/\nu$ plotted against the magnetic field, as obtained in [15] by Rotondo, Molinari, Ratti and Gherardi. The most visible plateaux are highlighted with their corresponding reference partition in a empty/full site representation. This snapshot shows a qualitative agreement with the experimental measures of Hall resistivity, both for the relative widths of the plateaux and for the quasilinear trend of the landscape as a function of the magnetic field. (Inset) A portion of the staircase is magnified and some experimentally observed plateaux are marked.

The peculiar thing is that the ground state of such a lattice gas model are, for filling factor $1/q$, exactly of the form

$$|\lambda\rangle = \left|1, 0^{q-1}, 1, 0^{q-1}, \ldots\right\rangle, \tag{4.1}$$

i.e. they are the reference state used by Bernevig for Laughlin wave function construction. However, the mapping provide us reference states for a large number of filling factors, therefore the next step will be the use of our squeezing operator and these reference states in order to obtain new model wave functions.

There are other questions we can deal with using our formalism. For instance, the simplicity of the resulting coefficients could be a hint of the fact that a simpler form of the squeezing operator exists. This reasoning has taken us to the discovery of the new algorithm presented in section 3.3.3, but it is not yet clear if this algorithm can be written as an operator applied to a reference state. Therefore, the research for a definitive proof of that algorithm and its study are one of the future directions.

Bibliography

[1] K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Phys. Rev. Lett.*, 45:494–497, Aug 1980.

[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.*, 48:1559–1562, May 1982.

[3] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West. Transition from an electron solid to the sequence of fractional quantum hall states at very low landau level filling factor. *Phys. Rev. Lett.*, 88:176802, Apr 2002.

[4] Jainendra K. Jain. *Composite Fermions*. Cambridge University Press, 2007.

[5] Daijiro Yoshioka. *The Quantum Hall Effect*. Springer, 2002.

[6] R. B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395–1398, May 1983.

[7] F. D. M. Haldane. Fractional quantization of the hall effect: A hierarchy of incompressible quantum fluid states. *Phys. Rev. Lett.*, 51:605–608, Aug 1983.

[8] Gregory Moore and Nicholas Read. Nonabelions in the fractional quantum hall effect. *Nuclear Physics B*, 360(2):362 – 396, 1991.

[9] N. Read and E. Rezayi. Beyond paired quantum hall states: Parafermions and incompressible states in the first excited landau level. *Phys. Rev. B*, 59:8084–8092, Mar 1999.

[10] R de Picciotto, M Reznikov, M Heiblum, V Umansky, G Bunin, and D Mahalu. Direct observation of a fractional charge. *Physica B: Condensed Matter*, 249251:395 – 400, 1998.

[11] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80:1083–1159, Sep 2008.

[12] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003.

[13] G. Dunne. Slater Decomposition of Laughlin States. *Int. J. Mod. Phys.*, B7:4783–4813, 1993.

[14] P. Di Francesco, M. Gaudin, C. Itzykson, and F. Lesage. Laughlin's wave functions, coulomb gases and expansion of the discriminant. *International Journal of Modern Physics A*, 09(24):4257–4351, 1994.

[15] Pietro Rotondo, Luca Guido Molinari, Piergiorgio Ratti, and Marco Gherardi. Devil's staircase phase diagram of the fractional quantum hall effect in the thin-torus limit. *Phys. Rev. Lett.*, 116:256803, Jun 2016.

[16] S.V. Iordansky. On the conductivity of two dimensional electrons in a strong magnetic field. *Solid State Communications*, 43(1):1 – 3, 1982.

[17] R. F. Kazarinov and Serge Luryi. Quantum percolation and quantization of hall resistance in two-dimensional electron gas. *Phys. Rev. B*, 25:7626–7630, Jun 1982.

[18] R. E. Prange and Robert Joynt. Conduction in a strong field in two dimensions: The quantum hall effect. *Phys. Rev. B*, 25:2943–2946, Feb 1982.

[19] S. A. Trugman. Localization, percolation, and the quantum hall effect. *Phys. Rev. B*, 27:7539–7546, Jun 1983.

[20] R. B. Laughlin. Quantized hall conductivity in two dimensions. *Phys. Rev. B*, 23:5632–5633, May 1981.

[21] F. D. M. Haldane. Fractional quantization of the hall effect: A hierarchy of incompressible quantum fluid states. *Phys. Rev. Lett.*, 51:605–608, Aug 1983.

[22] Richard P Stanley. Some combinatorial properties of jack symmetric functions. *Advances in Mathematics*, 77(1):76 – 115, 1989.

[23] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 1995.

[24] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. A differential ideal of symmetric polynomials spanned by jack polynomials at beta = - (r-1)/(k+1). ArXiv.org, 2001.

[25] B. Andrei Bernevig and F. D. M. Haldane. Model fractional quantum hall states and jack polynomials. *Phys. Rev. Lett.*, 100:246802, Jun 2008.

[26] Ronny Thomale, Benoit Estienne, Nicolas Regnault, and B. Andrei Bernevig. Decomposition of fractional quantum hall model states: Product rule symmetries and approximations. *Phys. Rev. B*, 84:045127, Jul 2011.

[27] Vittorio Erba. *Slater decomposition of fractional quantum Hall effect states*. Bachelor's degree thesis, Università degli Studi di Milano, 2016.

[28] Li Chen and Alexander Seidel. Algebraic approach to the study of zero modes of haldane pseudopotentials. *Phys. Rev. B*, 91:085103, Feb 2015.