The Ginzburg-Landau theory of superconductivity (1950)

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Before any clue on a microscopic mechanism, Vitaly Ginzburg (Nobel 2003) and Lev Landau (Nobel 1962) developed a theory[1] valid near T_c. It was inspired by the recent theory of second-order phase transitions (such as ferromagnetism) developed by Landau[1], based on the existence of an order parameter.
They assumed that the superconducting phase is characterized by a complex field $\psi(\mathbf{x})$, which is zero in the normal phase. Near the transition the field is small, so that the free energy is written as a polynomial expansion with phenomenological coefficients. The thermodynamical model is quite general and effective, and was modified for describing anisotropic materials.
A microscopic derivation was obtained in 1959 by Gor'kov, based on the B.C.S. theory near the critical temperature.

I. FREE ENERGY AND G.L. EQUATIONS

For a uniform infinite superconductor in absence of magnetic field, the free energy density is a function of the order parameter ψ, independent of position, and is expanded as follows:

$$f_s[\psi,\bar\psi] = f_n + a|\psi|^2 + \frac{b}{2}|\psi|^4$$

f_n is the free energy of the normal phase, a and b are parameters that describe the material and depend on temperature. For stability, it is necessary that $b>0$.
Minimization in the order parameter gives $\psi(a+b|\psi|^2)=0$. Them either $\psi=0$ or

$$\psi_\infty^2 = -\frac{a}{b}$$

meaning that $a(T)<0$ for $T<T_c$ (the free energy is a double well in $|\psi|^2$). The difference in free energies is the condensation energy density

$$f_s(T) - f_n(T) = -\frac{a^2}{2b} = -\frac{H_c(T)^2}{8\pi}$$

Therefore:

$$\frac{a^2}{b} = \frac{H_c(0)^2}{4\pi}\left[1-\frac{T^2}{T_c^2}\right]^2 \approx \frac{H_c(0)^2}{\pi}\frac{(T_c-T)^2}{T_c^2}$$

Ginzburg and Landau make a linear expansion $a(T)=\alpha(T-T_c)$ and then choose b constant near T_c. This gives $\psi\propto\sqrt{T_c-T}$.
The jump of specific heat per unit volume at the transition is $c_s-c_n=T_c\alpha^2/b$.

In an external magnetic field $\mathbf{H}$, the order parameter ψ and the induction field $\mathbf{B}=\nabla\times\mathbf{A}$ are not uniform in the superconductor. While $\mathbf{B}$ approaches $\mathbf{H}$ near the surface and vanishes in the interior, the order parameter varies from zero near the surface to the uniform value ψ_∞ deep in the superconductor.
The free energy is postulated as

$$F_s[\psi,\overline{\psi},\mathbf{A}] = F_n[0] + \int d^3x\,\frac{1}{2m^\star}\left|\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi(\mathbf{x})\right|^2 + a|\psi(\mathbf{x})|^2 + \frac{b}{2}|\psi(\mathbf{x})|^4 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2 \qquad (1)$$

F_n is the free energy of the normal phase in absence of the field H; a and b are the parameters of the superconductor, $-e^\star$ and $m^\star$ are the charge and mass of the particles that compose the superfluid with density $|\psi(\mathbf{x})|^2$.
The integral extends to the whole space.

A. The first G.L. equation

Minimization of F_s with respect to $\overline{\psi}$ or ψ gives the first G.L. equation or its complex conjugate:

$$\boxed{\frac{1}{2m^\star}\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)^2\psi + a\psi + b|\psi|^2\psi = 0} \qquad (2)$$

In deriving the equation, an integration by parts produces a boundary term that vanishes if

$$\mathbf{n}\cdot(\mathbf{p}+\frac{e^\star}{c}\mathbf{A})\psi = 0 \qquad (3)$$

The left-multiplication of the 1st GL equation by $\overline{\psi}$ gives, with the Leibnitz rule:

$$-i\hbar\,\mathrm{div}\left[\frac{\bar\psi}{2m^\star}(\mathbf{p}+\frac{e^\star}{c}\mathbf{A})\psi\right] + \frac{1}{2m^\star}\left|\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi\right|^2 + a|\psi|^2 + b|\psi|^4 = 0$$

The separation of the real and imaginary parts gives two identities:

$$\frac{1}{2m^\star}\left|\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi\right|^2 + a|\psi|^2 + b|\psi|^4 = \frac{\hbar^2\nabla^2}{4m^*}|\psi|^2 \qquad (4)$$

$$\mathrm{div}\,\mathbf{J_s} = 0 \qquad (5)$$

where the vector field $\mathbf{J}_S$ will be soon understood as the supercurrent density:

$$\mathbf{J}_s = -\frac{e^\star}{2m^\star}\bar\psi\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi + c.c. \qquad (6)$$

[1] Among his students were: Lev Pitaevskii, Alexei Abrikosov, Igor Dzyaloshinkii, Evgenii Lifschitz, Lev Gor'kov, Isaak Khalatnikov, Roald Sagdeev, Isaak Pomeranchuk.

It is analogue of the probability current density of the Schrödinger equation. It reinforces the picture that $|\psi|^2$ is the density of some conserved fluid of particles with charge $-e^\star$ and mass $m^\star$. The net flux of $\mathbf{J}_S$ out of a closed surface is zero.
The b.c. (3) is sufficient for ensuring the physical condition that the supercurrent flows parallel to the boundary surface: $\mathbf{n}\cdot\mathbf{J}_S = 0$. Pierre Gilles de Gennes[5] stated a more general condition to ensure it:

$$\mathbf{n}\cdot\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi = \frac{i\hbar}{\lambda}\psi \tag{7}$$

where $1/\lambda = 0$ for contact with an insulator and λ is real non-zero for contact with a metal. In the second case the condition allows for the "proximity effect", whence a layer near the superconductor gains superconducting properties.

Neglecting the surface term arising from the integral of the Laplacian, eq.(4) simplifies the free energy of the superconductor:

$$F_s[\psi,\overline{\psi},\mathbf{A}] - F_n[0] = \int d^3x \left[\frac{\mathbf{B}^2}{8\pi} - \frac{b}{2}|\psi(\mathbf{x})|^4\right] \tag{8}$$

B. The second G.L. equation

It is a general statement that, at equilibrium and away from the macrosopic currents that generate $\mathbf{H}$, it is[14]:

$$\frac{\delta F}{\delta \mathbf{A}(\mathbf{x})} = 0.$$

This condition gives the second G.L. equation. Amazingly, it is a Maxwell equation with the supercurrent density:

$$\boxed{\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_s} \tag{9}$$

The boundary condition is continuity of the tangent component of $\mathbf{B}$ and $\mathbf{H}$.

With $\psi = |\psi|e^{i\theta}$ the supercurrent becomes:

$$\mathbf{J}_S = -\frac{e^{\star 2}}{m^\star c}|\psi|^2\left(\frac{\hbar c}{e^\star}\boldsymbol{\nabla}\theta + \mathbf{A}\right) \tag{10}$$

A first interesting result is now obtained: divide by $|\psi|^2$ and integrate on a closed circuit. Since the order parameter is single-valued, the phase θ can only change by an integer multiple of 2π:

$$-\frac{m^\star c}{e^{\star 2}}\oint_C \frac{\mathbf{J}_S\cdot d\boldsymbol{\ell}}{|\psi|^2} - \frac{hc}{e^\star}n = \int_\sigma \mathbf{B}\cdot\mathbf{n}\,da \tag{11}$$

If the path C is in a region where the first integral is negligible, it turns out that the flux of B through the surface σ is quantized in units of the fundamental flux

$$\phi_0 = \frac{hc}{e^\star} \tag{12}$$

C. Characteristic lengths

Deep inside a superconductor it is $\mathbf{A}=0$ and the order parameter $|\psi|$ equals ψ_∞. It is convenient to put $\psi(\mathbf{x}) = \psi_\infty f(\mathbf{x})$, where f is complex and $|f|\le 1$ (the bulk value).

The GL equation for f, after division by $|a|$, is greatly simplified in appearance:

$$\boxed{\xi^2\left(-i\boldsymbol{\nabla}+\frac{e^\star}{\hbar c}\mathbf{A}\right)^2 f - f + |f|^2 f = 0} \tag{13}$$

where ξ is the coherence length:

$$\xi(T) = \sqrt{\frac{\hbar^2}{2m^\star|a(T)|}} \tag{14}$$

It diverges at the critical temperature. It is the scale of decay of the order parameter in a s-to-n transition.

Consider the GL equation in 1D, in absence of field H with b.c. $f=1$ and $f'=0$ for $x\ll 0$:

$$-\xi^2 f'' - f + f^3 = 0$$

Multiply by $-4f'$: $\frac{d}{dx}(2\xi^2 f'^2 + 2f^2 - f^4) = 0$. Then $2\xi^2 f'^2 + 2f^2 - f^4 = C$. The b.c. give $C=1$. Noting that $f'<0$ in the transition from s to n, the square root is: $\xi\sqrt{2}f' = -(1-f^2)$. The integral is:

$$f(z) = \tanh\left(\frac{x_0 - x}{\xi\sqrt{2}}\right), \qquad x < x_0 \tag{15}$$

The identity $\mathrm{rot}\,\mathrm{rot}\mathbf{B} = \mathrm{grad}\,\mathrm{div}\mathbf{B} - \nabla^2\mathbf{B}$ and the Maxwell equation div $\mathbf{B}$=0 transsform the 2nd GL equation into $-\nabla^2\mathbf{B} = \frac{4\pi}{c}\mathrm{rot}\mathbf{J}_S$.
Now use the property $\mathrm{rot}(\lambda\mathbf{v}) = \lambda\mathrm{rot}\mathbf{v} - \mathbf{v}\times\boldsymbol{\nabla}\lambda$:

$$\begin{aligned}\nabla^2\mathbf{B} =& \frac{4\pi e^{\star 2}}{m^\star c^2}\psi_\infty^2|f|^2\mathbf{B} + \frac{4\pi}{c}\mathbf{J}_S\times\frac{\boldsymbol{\nabla}|f|^2}{|f|^2} \\ =& \frac{|f|^2}{\delta^2}\mathbf{B} + \frac{4\pi}{c}\mathbf{J}_S\times\frac{\boldsymbol{\nabla}|f|^2}{|f|^2}\end{aligned} \tag{16}$$

where $\delta(T)$ is the penetration depth:

$$\delta(T) = \sqrt{\frac{m^\star c^2 b}{4\pi e^{\star 2}|a(T)|}} \tag{17}$$

It diverges near the transition.

Consider a superconductor in half-space $x\le 0$ in presence of a uniform field H along the z-axis in $x>0$. If $\xi\ll\delta$ we approximate $|f|=1$. The field B solves $B''(x) = B(x)/\delta$ with b.c. $B(0)=H$. The solution is $B(x) = He^{x/\delta}$. i.e. the field penetrates a length δ in the superconductor.

The ratio δ/ξ is independent of temperature, and defines the important Ginzburg-Landau parameter

$$\boxed{\kappa = \frac{\delta(T)}{\xi(T)} = \frac{m^\star c}{\hbar e^\star}\sqrt{\frac{b}{2\pi}}} \tag{18}$$

	$T_c\,(*)$	$\delta\,(nm)$	$\xi\,(nm)$	H_c(mT)(*)
Cd	0.517	110	760	2.805
Al	1.175	16	1600	10.49
Sn	3.722	34	230	30.55
Pb	7.196	37	83	80.34
Nb	9.25	39	38	206

(*) Data from Springer handbook of condensed matter and materials (2005).
Nb: H_{c2}=0.42 T, H_{c1}=0.17 T (slides of A. Gurevich, Nat. High Magn. Field Lab. Florida).

D. The Gibbs potential

Consider an extended homogeneous material, in presence of a field H. The order parameter ψ as well as $\mathbf{B}$ are not uniform. The Gibbs potential, with eq.(4) is:

$$G_s = F_s - \int d\mathbf{x}\frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}$$
$$= F_n + \int d\mathbf{x}\Big[\frac{B^2}{8\pi} - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi} - \frac{b}{2}|\psi|^4\Big]$$

In the normal phase $G_n = F_n + \int d\mathbf{x}H^2/8\pi$. Insert $\psi = \psi_\infty f = (|a|/b)f$ and use the relation $a^2/2b = H_c^2(T)/8\pi$ (condensation energy density):

$$G_s - G_n = \frac{H_c^2}{8\pi}\int d\mathbf{x}\Big[\left(\frac{\mathbf{H}-\mathbf{B}}{H_c}\right)^2 - |f|^4\Big] \tag{19}$$

At $T<T_c$ and $H=H_c(T)$, two cases are interesting:
• Uniform case.
$G_s^u = G_n^u$ with the equivalent situations at the transition line: $|f|=1$ and $B=0$ or $|f|=0$ and $B=H_c$.
• Mixed case (coexistence of s and n regions).
The integral is non-zero in the transition region because both B and $|f|$ vary, and on different scales: f grows towards bulk s-region, while B decays from the value H_c in n-phase. The mixed phase occurs when $G_s^{mix} < G_s^u$, i.e. when the integral is negative.

E. A 1-dimensional problem[1]

In $d=1$ the difference of potentials per unit area is

$$\frac{\Delta G}{A} = \frac{H_c^2}{8\pi}\int_{-\infty}^{+\infty} dx\Big[\left(1-\frac{B(x)}{H_c}\right)^2 - f^4(x)\Big] \tag{20}$$

The integral defines the length L of the transition region, that depends on how B and f decay in opposite directions. If it is negative a s-n interface is favoured within the material.

Even in $d=1$ the coupled GL equations cannot be exactly solved. Consider an infinite superconducting material that is normal (n) for $x\gg 0$ with a magnetic field H_c in direction z, and superconducting (s) for $x \ll 0$.

$$f(x)\to\begin{cases}1 & x\to-\infty\\ 0 & x\to+\infty\end{cases}\qquad B(x)\to\begin{cases}0 & x\to-\infty\\ H_c & x\to+\infty\end{cases}$$

The induction field $B(x)$, with orientation z, is obtained from a vector potential $\mathbf{A}(x)=(0,A(x),0)$. The equation $\mathbf{B}=\mathrm{rot}\mathbf{A}$ is $A'(x)=B(x)$.
With $f=|f(x)|e^{i\theta(x)}$ the supercurrent is

$$\mathbf{J}_S = -\frac{e^{\star 2}}{m^\star c}\frac{|a|}{b}|f(x)|^2\Big(\frac{\hbar c}{e^\star}\theta'(x), A(x), 0\Big)$$

The x component of the 2nd GL equation gives: $\theta'(x)=0$. We set $\theta=0$. The y component and $A'=B$ give:

$$A''(x) = \frac{1}{\delta^2}f^2 A \tag{21}$$

The z component is zero. The first GL equation is

$$-\xi^2 f''(x) + \frac{e^{\star 2}A^2(x)}{2m^\star c^2|a|}f(x) - f(x) + f^3(x) = 0$$

Multiply by $2f'$ and express a total derivative:

$$\frac{d}{dx}\left[-\xi^2 f'^2 + \frac{e^{\star 2}A^2}{2m^\star c^2|a|}f^2 - f^2 + \frac{f^4}{2}\right] = f^2\frac{e^{\star 2}AA'}{m^\star c^2|a|}$$

Note that $f^2AA' = \delta^2 A'A''$ is a total derivative. Then:

$$-\xi^2 f'^2 + \frac{e^{\star 2}A^2}{2m^\star c^2|a|}f^2 - f^2 + \frac{f^4}{2} - \frac{e^{\star 2}\delta^2}{m^\star c^2|a|}\frac{A'^2}{2} = C$$

The constant $C=-1/2$ is found with the b.c. at $-\infty$. This is the final integral:

$$f'^2 - \frac{(1-f^2)^2}{2\xi^2} = \left(\frac{e^\star}{\hbar c}\right)^2(A^2f^2 - \delta^2 A'^2) \tag{22}$$

The equation can be used to eliminate A in favour of $A'=B$ in the first G.L. equation:

$$2\xi^2(f''f - f'^2) + 1 - f^4 = \frac{B^2}{H_c(T)^2} \tag{23}$$

1. Limit cases

• $\kappa \ll 1$. Let $B(x)=H_c\theta(x)$ ($\delta=0$). The function f solves (15) with $x_0=0$ because for $x>0$ B is const. ($f=0$ in (21) gives $B'=0$). $L=\int_{-\infty}^0 dx(1-f^4)>0$ (the mixed phase does not occur). The integral is $L=\int_{-\infty}^0 dx(1-f^2)(1+f^2) = -\xi\sqrt{2}\int_{-\infty}^0 dx f'(x)(1+f^2) = \xi\sqrt{2}\int_0^1 df(1+f^2) = \frac{4}{3}\xi\sqrt{2}$.

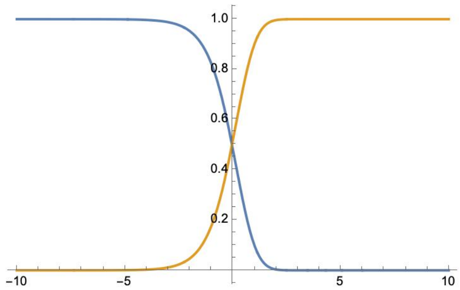

FIG. 1: The functions $f^2(x)$ (blue) and $B(x)/H_c$ (yellow) in the limit case $1-f^2 = B/H_c$. x-values are in units of δ. The function solves $f''f - f'^2 + f^2 - f^4 = 0$. To match $f(0) = 1/2$ the b.c. are $f(-10) = 1$, $f'(-10) = 8\times 10^{-7}$.

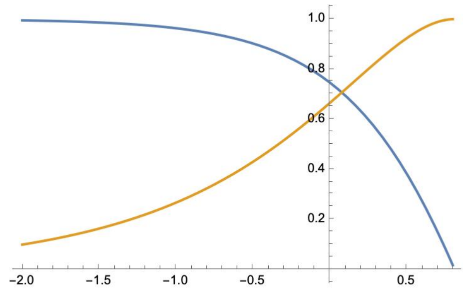

FIG. 2: The functions $f^2(x)$ (blue) and $B(x)/H_c$ (yellow) in the limit case of $\kappa \gg 1$. It is $B/H_c = \sqrt{1-f^4}$. x-values are in units of δ. The function $u = f^2$ solves $u' = -\sqrt{2}(1-u)\sqrt{1+u}$. Here we put $u(-2.8) = 0.999$

• $\kappa = 1/\sqrt{2}$. The condition $f(x)^2 = 1 - (B(x)/H_c)$ in eq.(20) makes $L = 0$. Eq. (22) simplifies: $2\xi^2 f'^2 = f^2 A^2/(\delta^2 H_c^2)$. Use $B' = (f^2/\delta^2)A$ to eliminate A. The result $\kappa = 1/\sqrt{2}$ is obtained. The functions f^2 and B/H_c are shown in fig.1.

• $\kappa \gg 1$. $L = -\frac{8}{3}(\sqrt{2}-1)\delta$ (the result is everywhere quoted but without proof. The present one is simple.). Rescale the 1D equations with $x = \delta\, x$, $B = \beta(s)H_c$, $A = \alpha(s)H_c\delta$. A dot is a derivative in s.

$$-\frac{1}{\kappa^2}\ddot{f} + \frac{1}{2}\alpha^2 f - f + f^3 = 0 \tag{24}$$

$$\frac{1}{\kappa^2}(\ddot{f}f - \dot{f}^2) + 1 - f^4 = \beta^2 \tag{25}$$

For $\kappa \gg 1$ they simplify: $\alpha = \sqrt{2}\sqrt{1-f^2}$ and $\beta = \sqrt{1-f^4}$. Since $\dot{\alpha} = \beta$ we obtain an equation for f:

$$\sqrt{2}\frac{d}{ds}\sqrt{1-f^2} = \sqrt{1-f^4}$$

The integral for L is: $\delta\int_{-\infty}^{+\infty} ds(1-\beta)^2 - f^4 = 2\delta\int_{-\infty}^{+\infty} ds(1 - f^4 - \sqrt{1-f^4})$. Let $f^2 = u$. It is $\dot{u} = -\sqrt{2}(1-u)\sqrt{1+u}$. Change ds to $du/\dot{u}$:

$$L = \sqrt{2}\delta\int_0^1 du\left(\sqrt{1+u} - \frac{1}{\sqrt{1-u}}\right) = -\frac{8}{3}\delta(\sqrt{2}-1)$$

Then $L \approx -1.1\delta$ (mixed phase occurs).

F. Mixed superconductors (1957).

Alexei Abrikosov (Nobel 2003) discovered that the linearized G.L. first equation admits a mixed solution[2], where the external field penetrates the bulk of the superconductor in quantized fluxes. *I made my derivation of the vortex lattice in 1953 but publication was postponed since Landau at first disagreed with the whole idea. Only after R. Feynman published his paper on vortices in superfluid Helium (1955) and Landau accepted the idea of vortices, did he agree with my derivation, and I published my paper[3].*

For $T < T_C$ consider the problem of lowering the intensity of a uniform magnetic field H to a value where the first solution of the GL solution appears. This highest possible value corresponds to a small field ψ, and justifies the omission of the cubic term in the GL equation. In the Landau gauge, for H aligned in the z direction, a choice is $\mathbf{A} = (0, Hx, 0)$. The linearized equation is that of a harmonic oscillator

$$\xi^2\left[-\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} + \frac{e^\star H}{\hbar c}x\right)^2 - \frac{\partial^2}{\partial z^2}\right] f - f = 0$$

Let us introduce the magnetic length $\ell^2 = \hbar c/e^\star H$ and measure lengths with this unit. The solutions have the form $f(x,y,z) = e^{iky+iqz}u(s)$ where u solves

$$-\frac{d^2u}{ds^2} + (k\ell + s)^2 u + q^2\ell^2 u = \frac{\ell^2}{\xi^2}u$$

The equation admits integrable solutions only for the discrete values of the harmonic oscillator:

$$\ell^2/\xi^2 = q^2\ell^2 + 2n + 1, \quad n = 0, 1, 2, ...$$

The largest possible H (smallest ℓ) occurs for $q = 0$ and $n = 0$. In this situation $\xi^2 = \ell^2$ i.e. there is a unit flux of the critical field in an area $2\pi\xi^2$

$$2\pi\xi^2 H_{c2} = \frac{hc}{e^\star}$$

Remembering that $H_c^2(T) = 4\pi|a|/b$, the equation is

$$\frac{H_{c2}(T)}{H_c(T)} = \kappa\sqrt{2} \tag{26}$$

The solution is translation invariant along the z-axis, and is a superposition of shifted Gaussians:

$$f(x,y) = \sum_k c_k \exp\left[iky - \frac{1}{2\ell^2}(x + k\ell^2)^2\right]$$

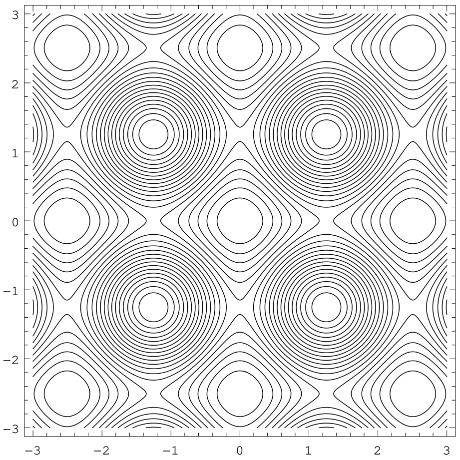

FIG. 3: Level curves of $|f(x,y)|^2$ for the square Abrikosov lattice, eq.(27) with $\xi = 1$. The central contour is a peak, while the concentric circles point to a zero, where the critical field is stronger. The phase of f changes by 2π along a circuit around a single zero. The lattice period is $\sqrt{2\pi} \approx 2.50$

It is periodic in y if $k = 2\pi p/L$:

$$f(x,y) = \sum_p c_p \exp\left[i\frac{2\pi}{L}py - \frac{1}{2\ell^2}(x + p\frac{2\pi}{L}\ell^2)^2\right]$$

It is also periodic in x, up to a phase, if c_p are all equal.

$$f(x,y+L) = f(x,y), \quad f(x + \tfrac{2\pi}{L}\ell^2, y) = f(x,y)e^{-i\frac{2\pi}{L}y}$$

The unit cell of the lattice has area $2\pi\xi^2$. It is a square if $L = \xi\sqrt{2\pi}$. Each cell is crossed by a unit flux of the critical field.
The periodic solution (up to a phase) is now rewritten:

$$\begin{aligned} f(x,y) =& e^{-\frac{x^2}{2\xi^2}}\left[1 + 2\sum_{p=1}^{\infty} e^{-\pi p^2}\cos\left(p\sqrt{2\pi}\frac{y+ix}{\xi}\right)\right] \\ =& e^{-\frac{x^2}{2\xi^2}}\theta_3\left(\sqrt{\frac{\pi}{2}}\frac{y+ix}{\xi}; e^{-\pi}\right) \end{aligned} \quad (27)$$

where $\theta_3(z,q) = 1 + \sum_{n=1}^{\infty} q^{n^2}\cos(2nz)$ is a Jacobi Theta function. The periods are $\xi\sqrt{2\pi}$ and $i\xi\sqrt{2\pi}$.

It was later discovered, by considering the quartic term, that the triangular lattice has a lower free energy[10].

The highest field allowed in a superconductor is

$$H_{c2} = \kappa\sqrt{2}H_c(T) \quad (28)$$

The relation requires $\kappa > 1/\sqrt{2}$, which defines type II superconductors.
A higher threshold $H_{c3} \approx 1.7H_{c2}$ describes the onset of superconductivity in a surface layer of width $\xi(T)$.
The lower critical field for the mixed phase is

$$H_{c1} = H_c\frac{\log\kappa}{\kappa\sqrt{2}} \quad (29)$$

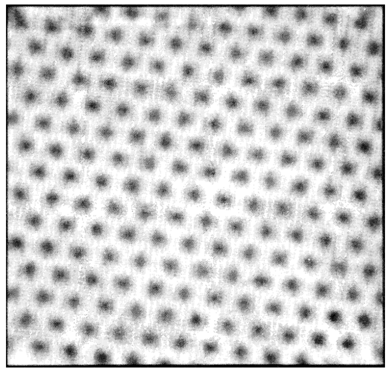

FIG. 4: STM picture of the triangular Abrikosov lattice in $NbSe_2$ ($T_c = 7.2$, $H_{c2} = 3.2$T). The width of the scan is 6000 Å. Previous observations used fine magnetic particles to mark the flux lines. The STM allows observation of the lattice in the full range of magnetic fields, and to observe the variation of the density of the states in and around a single flux line. (Hess et al.[8]).

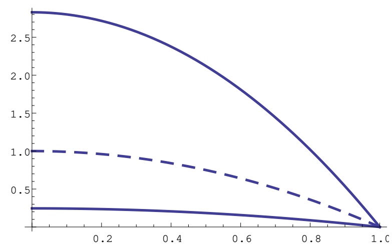

FIG. 5: The critical fields (from above) H_{c2}/H_c and H_{c1}/H_c (continuous lines), as functions of T/T_c. Below H_{c1} there is no mixed phase ($\kappa = 2$).

It is the value at which a single vortex grows in the type II superconductor when the H field is increased above the pure diamagnetic (Meissner) phase. ($G_s - G_n = 0$).

For $H_{c1} < H < H_{c2}$ the field penetrates in tubes where the phase is normal and the flux is quantized. The flux tubes form a triangular array (Abrikosov lattice). Each tube is surrounded by the superconducting phase and a thin layer of supercurrents that shield it. For $H > H_{c2}$ the normal phase occurs. For Type II nothing special happens at the value $H_c(T)$.
Type II superconductors resist fields of some tesla up a record value of 45.5T.

All chemical elements that become superconductors are Type I, with the exception of Vanadium (V, $T_c = 5.46$), Niobium (Nb, $T_c = 9.25$) and Technetium (Tc, $T_c = 7.77$) that are type II.

The Jacobi function $\theta_3(z,q)$

We recall definition and properties of one of the four entire Jacobi functions

$$\theta_3(z,q) = \sum_{n\in\mathbb{Z}} e^{i(2nz+n^2\pi\tau)} \qquad \mathrm{Im}\tau > 0 \tag{30}$$

$$= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \tag{31}$$

with $q = \exp(i\pi\tau)$. The function is doubly-periodic up to a factor, with periods π and $i\pi\tau$:

$$\theta_3(z+\pi,q) = \theta_3(z,q), \qquad \theta_3(z+\pi\tau,q) = \frac{1}{q}e^{-2iz}\theta_3(z,q)$$

A representation as infinite product:

$$\theta_3(z,q) = G\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos(2z)+q^{4n-2})$$

$$G = \prod_{n=1}^{\infty}(1-q^{2n})$$

By putting to zero the factor $n=1$ we find a zero of the function at $z_0 = \frac{\pi}{2}(1+i\tau)$, that replicates with the periodicity of the lattice.

[1] The original paper was publish in Russian in Zh. Eksp. Teor. Fiz. **20** (1950) 1064. It is reproduced in English in the volume[7] (Ch. 4). See also §45 in ref.[11].

[2] A. Abrikosov, *On the magnetic properties of superconductors of the second group*, Sov. Phys. JETP **5**(6) 1174 (1957).

[3] A. Abrikosov, *Type II superconductors and the vortex lattice*, Nobel Lecture, December 8, 2003. `https://www.nobelprize.org/uploads/2018/06/abrikosov-lecture.pdf`

[4] A. Abrikosov, *My years with Landau*, Physics Today, Jan 1973. `https://web.pa.msu.edu/people/yang/MyYearsWithLandau.pdf`

[5] P.-G. de Gennes, Superconductivity of metals and alloys, (Benjamin, 1966).

[6] V. N. Ginzburg, *Nobel Lecture: On superconductivity and superfluidity (what I have and have not managed to do) as well as on the "physical minimum" at the beginning of the XXI century*, Rev. Mod. Phys. **76** n.3 (2004) 981–998. `https://journals.aps.org/rmp/pdf/10.1103/RevModPhys.76.981`

[7] Vitaly L. Ginzburg, On superconductivity and superfluidity, a scientific autobiography, (Springer, 2009).

[8] H. F. Hess, R. B. Robinson, et al. *Scanning-Tunneling-Microscope observation of the Abrikosov flux lattice and the density of states near and inside a fluxoid*, Phys. Rev. Lett. **62** (1989) 214. `https://doi.org/10.1103/PhysRevLett.62.214`

[9] D. Hughes, *The critical current of superconductors: an historical review*, Low Temperature Physics **27** (2001) 713–722. `https://doi.org/10.1063/1.1401180`

[10] W. H. Kleiner, L. M. Roth, and S. H. Autler, *Bulk solution of Ginzburg-Landau equations for Type II superconductors: upper critical field region*, Phys. Rev. **133** A1226 (1964).

[11] E. Lifschitz and L. Pitayevski, Physique statistique - Theorie de l'état condensé, Editions MIR Moscou (traduction francaise 1990).

[12] Laurent-Patrick Lévy, Magnétisme et supraconductivité, (CNRS Editions, 1997).

[13] J. Matricon and G. Waysand, The cold wars: a history of superconductivity, (Rutgers University Press, 2003).

[14] A. V. Dmitriev and W. Nolting, *Details of the thermodynamical derivation of of the Ginzburg-Landau equations*, Supercond. Sci. Technol. **17** (2004) 443-447.

[15] V. V. Schmidt, The physics of superconductors: introduction to fundamentals and applications, (Springer 1997).

[16] Michael Tinkham, Introduction to superconductivity, (Dover reprint of McGraw-Hill 2nd Ed., 1996).