THE THERMAL FUNCTIONAL INTEGRAL FOR BOSONS

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- The generator of thermal correlators;
- Coherent states of the Bose oscillator;
- Functional representation of the thermal partition function;
- Non-interacting bosons with sources and Wick theorem;
- Interacting bosons with sources, perturbation theory.

The thermal partition function of a system of bosons at temperature T, chemical potential μ, in motion with respect to the observer with velocity $\mathbf{v}$ is:

$$Z[T,\mu,\mathbf{v}] = \operatorname{tr}\exp[-\beta(\hat{H} - \mu\hat{N} - \mathbf{v}\cdot\hat{\mathbf{P}})].$$

$\hat{H}$ is the Hamiltonian, $\hat{N}$ is the total number of bosons, $\hat{\mathbf{P}} = \sum_{\mathbf{k}} \hbar\mathbf{k}\,\hat{n}_{\mathbf{k}}$ is the total momentum. Derivatives in the parameters of the thermodinamic potential $\Omega[T,\mu,\mathbf{v}] = -k_B T\log Z$ give the thermal averages of conjugated quantities, such as the number, the entropy, the momentum:

$$N = -\frac{\partial\Omega}{\partial\mu}, \qquad S = -\frac{\partial\Omega}{\partial T}, \qquad \mathbf{P} = -\frac{\partial\Omega}{\partial\mathbf{v}}$$

Other parameters may be hidden in H, such as the volume, or external fields.

The partition function with external fields (named *sources*) coupled to operators is the *generator* of thermal averages of products of such operators. If the sources are linearly coupled to the field operators, it is the generator of Green functions. The log of it (the analogue of the gran-canonical potential) is the generator of connected Green functions.

After a presentation of the properties of the partition function with sources, we give a functional-integral representation of it, based on the coherent states of the Bose oscillators that provide the basis of second quantization.

1. The generator of thermal correlators

Consider a many-boson Hamiltonian, written in second quantization with CCR operators $\hat{b}_r^\dagger$, $\hat{b}_r$ where the index $r = 1, 2, ...$ enumerates an arbitrary 1-particle orthonormal complete basis $|r\rangle$:

$$\hat{K} = \hat{H} - \mu\hat{N} = \sum_{rs} \hat{b}_r^\dagger (h_{rs} - \mu\delta_{rs})\hat{b}_s + \frac{1}{2}\sum_{rr'ss'} \hat{b}_r^\dagger \hat{b}_s^\dagger v_{rsr's'}\hat{b}_{s'}\hat{b}_{r'}$$

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Time-ordered thermal averages can be evaluated with the help of auxiliary classical sources[1] $\phi_r(\tau)$ and $\overline{\phi_r(\tau)}$:

$$\hat K[\bar\phi,\phi] = \hat K - \hbar \sum\nolimits_r [\hat b_r \overline{\phi_r(\tau)} + \hat b_r^\dagger \phi_r(\tau)] \tag{1}$$

1.1. Generator of thermal Green functions.

The τ-evolution in presence of the sources is given by the operator $\mathscr{U}(\tau,0)$.
In the interaction picture: $\mathscr{U}(\tau,0)=\exp(-\frac{1}{\hbar}\hat K\tau)\mathscr{U}_I(\tau,0)$ where

$$\mathscr{U}_I(\tau,0) = \mathsf{T}\exp\left[\sum_r \int_0^\tau d\tau' \hat b_r(\tau')\overline{\phi_r(\tau')} + \hat b_r^\dagger(\tau')\phi_r(\tau')\right] \tag{2}$$

with the evolution of $\hat b_r$ and $\hat b_r^\dagger$ ruled by $\hat K$.

The partition function with sources is

$$\boxed{Z[\bar\phi,\phi] = \mathrm{tr}[\mathscr{U}(\hbar\beta,0)] = Z_K \langle \mathscr{U}_I(\hbar\beta,0)\rangle_K} \tag{3}$$

where $Z_K = \mathrm{tr}(e^{-\beta \hat K})$ and $\mathscr{U}(\hbar\beta,0) = e^{-\beta\hat K}\mathscr{U}_I(\hbar\beta,0)$.
The sources are encapsulated in $\mathscr{U}_I(\hbar\beta,0)$.

Sources are useful to perform functional derivatives that produce T-ordered averages of the operators they are coupled with, creation and destruction operators, in presence of the sources.

$$\begin{aligned}\frac{\delta Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}} &= \langle \frac{1}{\delta\overline{\phi_r(\tau)}} Z_K \mathscr{U}_I(\hbar\beta,0)\rangle_K \\ &= Z_K\langle \mathscr{U}_I(\hbar\beta,\tau)\hat b_r(\tau)\mathscr{U}_I(\tau,0)\rangle_K \\ &= Z_K\langle \mathsf{T}\mathscr{U}_I(\hbar\beta,0)\hat b_r(\tau)\rangle_K\end{aligned}$$

Divide by $Z[\bar\phi,\phi]$ and obtain the average of $b_r(\tau)$ in presence of the sources:

$$\langle \hat b_r(\tau)\rangle_S \equiv \frac{\langle \mathsf{T}\mathscr{U}_I(\hbar\beta,0)\hat b_r(\tau)\rangle_K}{\langle \mathscr{U}_I(\hbar\beta,0)\rangle_K} = \frac{1}{Z[\bar\phi,\phi]}\frac{\delta Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}} \tag{4}$$

Next, let us consider the double derivative

$$\begin{aligned}\frac{\delta^2 Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}\delta\phi_{r'}(\tau')} =& Z_k\langle \frac{\delta}{\delta\overline{\phi_r(\tau)}}\mathscr{U}_I(\hbar\beta,\tau')\hat b^\dagger_{r'}(\tau')\mathscr{U}_I(\tau',0)\rangle_K \\ =& Z_K\langle \mathsf{T}\mathscr{U}_I(\hbar\beta,0)\hat b_r(\tau)\hat b^\dagger_{r'}(\tau')\rangle_K\end{aligned}$$

$$\langle \mathsf{T}\hat b_r(\tau)\hat b^\dagger_{r'}(\tau')\rangle_S = \frac{1}{Z[\bar\phi,\phi]}\frac{\delta^2 Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}\delta\phi_{r'}(\tau')} \tag{5}$$

They become thermal T-ordered averages when, in the end, the sources are turned off.

[1] Here the sources couple to the field operators. Bilinears as $\sum_r f_r \hat b_r^\dagger \hat b_r$ are also useful.

1.2. **Connected Green functions.** In analogy with statistical mechanics, we introduce the functional

$$\boxed{W[\bar\phi,\phi]=\log Z[\bar\phi,\phi]} \tag{6}$$

It is the generator of connected Green functions. For example:

$$\begin{aligned}\frac{\delta^2 W[\bar\phi,\phi]}{\delta\phi(2)\delta\overline{\phi(1)}} =&\frac{\delta}{\delta\phi(2)}\left(\frac{1}{Z[\bar\phi,\phi]}\frac{\delta Z[\bar\phi,\phi]}{\delta\overline{\phi(1)}}\right)\\ =&\langle \mathsf{T}\hat b(1)\hat b^\dagger(2)\rangle_S-\langle\hat b(1)\rangle_S\langle\hat b^\dagger(2)\rangle_S\end{aligned}$$

$$\begin{aligned}\frac{\delta^3 W[\bar\phi,\phi]}{\delta\phi(1)\delta\phi(2)\delta\overline{\phi(3)}} &= \langle \mathsf{T}\hat b(3)\hat b^\dagger(2)\hat b^\dagger(1)\rangle_S-\langle \mathsf{T}\hat b(3)\hat b^\dagger(2)\rangle_S\langle\hat b^\dagger(1)\rangle_S\\ &-\langle \hat{\mathsf{T}} b(3)\hat b^\dagger(1)\rangle_S\langle\hat b^\dagger(2)\rangle_S-\langle\hat b(3)\rangle_S\langle \mathsf{T}\hat b^\dagger(2)\hat b^\dagger(1)\rangle_S+2\langle\hat b(3)\rangle_S\langle\hat b^\dagger(2)\rangle_S\langle\hat b^\dagger(1)\rangle_S\end{aligned}$$

As sources are turned off, unless a broken symmetry occurs (BEC), the terms that do not conserve the number of particles vanish.

• *The generator for non-interacting bosons.* For non-interacting bosons eq.(4) for $Z[\bar\phi,\phi]$ can be solved. The equation of motion for $b_r(\tau)$ is:

$$\begin{aligned}-\hbar\frac{\partial\hat b_r(\tau)}{\partial\tau} &= \mathscr{U}(\tau,0)^{-1}[\hat b_r,\hat K_0-\hbar{\textstyle\sum_{r'}}\phi_{r'}(\tau)\hat b^\dagger_{r'}+\overline{\phi_{r'}(\tau')}\hat b_{r'}]\mathscr{U}(\tau,0)\\ &= h_{rs}b_s(\tau)-\mu b_r(\tau)-\hbar\phi_r(\tau)\end{aligned}$$

Now take the average in presence of the sources:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau}+h_{rs}-\mu\delta_{rs}\right]\langle b_s(\tau)\rangle_S=\hbar\phi_r(\tau)$$

The equation is solved with the aid of the (thermal) Green function:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau}+h_{rs}-\mu\delta_{rs}\right]\mathscr{G}^0(s\tau,s'\tau')=-\hbar\delta_{rs'}\delta(\tau-\tau') \tag{7}$$

$$\frac{1}{Z_0[\bar\phi,\phi]}\frac{\delta Z_0[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}}=-\sum_{r'}\int_0^{\hbar\beta}d\tau'\mathscr{G}^0(r\tau,r'\tau')\phi_{r'}(\tau')$$

Finally the solution:

$$\boxed{Z_0[\bar\phi,\phi]=Z_0\exp\left[-\sum_{rr'}\iint_0^{\hbar\beta}d\tau\,d\tau'\overline{\phi_r(\tau)}\mathscr{G}^0(r\tau,r'\tau')\phi_{r'}(\tau')\right]} \tag{8}$$

With the choice of the position basis, $x=(\mathbf{x},\tau)$, it is

$$Z_0[\bar\phi,\phi]=Z_0\exp\left[-\int dxdy\overline{\phi(x)}\mathscr{G}^0(x,y)\phi(y)\right]$$

• *Wick's theorem.* The explicit expression (8) for non-interacting particles enables to express high order Green functions in terms of the 2-point Green function:

$$\mathscr{G}^0(1...n;1'...n') = (-1)^n \frac{\delta^{2n}}{\delta\bar\phi(1)...\delta\phi(n')} \exp\left[-\sum\nolimits_{jk}\bar\phi(j)\mathscr{G}^0(j,k)\phi(k)\right]$$

with $\phi = \bar\phi = 0$ in the end. Each pair of functional derivatives $-\delta^2/\delta\bar\phi(k)\delta\phi(k')$ brings down a factor $\mathscr{G}^0(k,k')$. The result of all derivatives is a product of n propagators $\mathscr{G}^0(j_1,j_1')...\mathscr{G}^0(j_n,j_n')$ where $j_1...j_n$ is a permutation of $1...n$ and $j_1'...j_n'$ is a permutation of $1'...n'$. This is the definition of a permanent:

$$\mathscr{G}^0(1...n;1'...n') = \mathsf{Per}[\mathscr{G}^0(j,j')]_{j=1...n}^{j'=1'...n'} \tag{9}$$

For example: $\mathscr{G}^0(12;34) = \mathscr{G}^0(1,3)\mathscr{G}^0(2,4) + \mathscr{G}^0(1,4)\mathscr{G}^0(2,3)$.

2. Thermal density correlators

Suppose that we are interested in thermal T-ordered correlators of the density, $\langle \mathsf{T}\hat n_K(x_1)\dots\hat n_K(x_n)\rangle_K$, where x_j stands for τ_j and position $\mathbf{x}_j$. Such correlators may be obtained from a generating functional. To this end, the Hamiltonian is modified by a term that couples the density operator $\hat n(\mathbf{x})$ to a fictitious external field $\varphi(\mathbf{x},\tau)$ that depends on τ:

$$\hat V(\tau) = \int d\mathbf{x}\,\hat n(\mathbf{x})\varphi(\mathbf{x},\tau) \tag{10}$$

In presence of time-dependent sources, the τ-evolution is given by a two-parameter propagator $\hat{\mathscr{U}}(\tau,\tau')$. In the interaction picture it factors into a source-free propagator and a residual propagator:

$$\hat{\mathscr{U}}(\tau,0) = e^{-\frac{1}{\hbar}\tau\hat K}\hat{\mathscr{U}}_I(\tau,0)$$

$$\mathscr{U}_I(\tau,0) = \mathsf{T}\exp\left[-\frac{1}{\hbar}\int_0^\tau d\tau'\int d\mathbf{x}\,\hat n_K(\mathbf{x}\tau')\varphi(\mathbf{x}\tau')\right] \tag{11}$$

The system with sources is no longer in thermal equilibrium. However, it is useful to introduce a source-dependent partition function:

$$Z[\varphi] = \operatorname{tr}\hat{\mathscr{U}}(\hbar\beta,0) = \operatorname{tr}[e^{-\beta\hat K}\hat{\mathscr{U}}_I(\hbar\beta,0)] = Z_K\langle\hat{\mathscr{U}}_I(\hbar\beta,0)\rangle_K$$

This is the generating functional of the density correlators:

$$\boxed{Z[\varphi] = Z_K\Big\langle \mathsf{T}\,\exp\Big\{-\frac{1}{\hbar}\iint_0^{\hbar\beta} d\mathbf{x}\,d\tau\,\hat n_K(\mathbf{x},\tau)\varphi(\mathbf{x},\tau)\Big\}\Big\rangle_K} \tag{12}$$

The expansion in the source-field gives the thermal correlators (in absence of the sources) i.e. $Z[\varphi]$ is the generating function of thermal density correlators:

$$\frac{Z[\varphi]}{Z_K} = 1 + \sum_{r=1}^\infty \frac{1}{r!}\left(-\frac{1}{\hbar}\right)^r \int dx_1\dots dx_r \Big\langle \mathsf{T}\hat n_K(x_1)\dots\hat n_K(x_r)\Big\rangle_K \varphi(x_r)\dots\varphi(x_1)$$

here $x = (\mathbf{x},\tau)$. Alternatively, functional derivatives of $Z[\varphi]$ in the source-field give correlators *in presence of sources* (which may be put to zero in the end). For example,

$$\hbar^2\frac{\delta^2 Z[\varphi]}{\delta\varphi(x)\delta\varphi(x')} = Z_K\Big\langle \mathsf{T}\mathscr{U}_I(\hbar\beta,0)\hat n(x)\hat n(x')\Big\rangle_K = Z[\varphi]\langle\mathsf{T}\hat n(x)\hat n(x')\rangle_\varphi$$

In analogy with the equilibrium reduction formula we define the average in presence of the sources:

$$\left\langle \mathsf{T}\hat{n}(x_1)...\hat{n}(x_n)\right\rangle_\varphi = \frac{\langle \mathsf{T}\mathscr{U}_I(\hbar\beta,0)\hat{n}(x_1)...\hat{n}(x_n)\rangle_K}{\langle \mathscr{U}_I(\hbar\beta,0)\rangle_K} = (-\hbar)^n \frac{1}{Z[\varphi]}\frac{\delta^n Z[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_n)} \tag{13}$$

With $\varphi = 0$ in the end, this evaluates the thermal correlator:

$$\langle \mathsf{T}\hat{n}(x_1)...\hat{n}(x_n)\rangle_K = (-\hbar)^n \frac{1}{Z[\varphi]}\frac{\delta^n Z[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_n)}\Big|_{\varphi=0} \tag{14}$$

2.1. **Connected correlators.**

The average density in presence of the fields is

$$\langle \hat{n}(x)\rangle_\varphi = -\hbar \frac{1}{Z[\varphi]}\frac{\delta Z[\varphi]}{\delta\varphi(x)} = -\hbar\frac{\delta W[\varphi]}{\delta\varphi(x)} \tag{15}$$

Another derivative with a factor $(-\hbar)$ gives:

$$\hbar^2 \frac{1}{Z[\varphi]}\frac{\delta^2 Z[\varphi]}{\delta\varphi(x_2)\delta\varphi(x_1)} - \hbar^2 \frac{\delta W[\varphi]}{\delta\varphi(x_2)}\frac{\delta W[\varphi]}{\delta\varphi(x_1)} = \hbar^2 \frac{\delta^2 W[\varphi]}{\delta\varphi(x_2)\delta\varphi(x_1)}$$

The left-hand side is $\langle \mathsf{T}\hat{n}(x_2)n(x_1)\rangle_\varphi - \langle \hat{n}(x_2)\rangle_\varphi \langle n(x_1)\rangle_\varphi$; the right hand side is a *connected correlator* $\langle \mathsf{T}n(x_1)n(x_2)\rangle_{c,\varphi}$.

Other derivatives give the 3 and 4-point correlators as combinations of connected ones. $\langle 1,2,3\rangle = \langle 1,2,3\rangle_c + \langle 1,2\rangle_c\langle 3\rangle + \langle 1\rangle\langle 2,3\rangle_c + \langle 1,3\rangle_c\langle 2\rangle + \langle 1\rangle\langle 2\rangle\langle 3\rangle$ etc. In general, a n-point correlator (with sources or not) is decomposable into products of connected correlators, corresponding to the possible partitions of the set $(1,2,\dots,n)$.

$W[\varphi]$ is the generator of connected density correlators:

$$W[\varphi] = \sum_r \frac{1}{r!}\left(-\frac{1}{\hbar}\right)^r \int dx_1\dots dx_r \left\langle \mathsf{T}\hat{n}(x_1)\cdots\hat{n}(x_r)\right\rangle_c \varphi(x_1)\cdots\varphi(x_r) \tag{16}$$

The connected thermal correlators are

$$\left\langle \mathsf{T}\hat{n}(x_1)\cdots\hat{n}(x_n)\right\rangle_c = (-\hbar)^n \frac{\delta^n W[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\Big|_{\varphi=0} \tag{17}$$

2.2. **The effective potential.**

Eq. (15) obtains the average density $n(x) = \langle n(x)\rangle_\varphi$ as a derivative in the source-field of the functional $W[\varphi]$, inasmuch in statistical mechanics $N = -\partial\Omega/\partial\mu$. The exchange of μ with N in statistical mechanics is done via a Legendre transformation, that changes $\Omega(T,\mu)$ to the Free energy $F(T,N) = \Omega(T,\mu(T,N)) + \mu N$. Here, the fields $n(x)$ and $\varphi(x)$ are interchanged by means of a Legendre transform.

Define the effective potential:

$$\boxed{\Gamma[n] = W[\varphi] + \frac{1}{\hbar}\int dx\, \varphi(x)n(x)} \tag{18}$$

As expected:

$$\frac{\delta\Gamma[n]}{\delta n(x)} = \int dy \frac{\delta W}{\delta\varphi(y)}\frac{\delta\varphi(y)}{\delta n(x)} + \frac{1}{\hbar}\int dy\,\frac{\delta\varphi(x)}{\delta n(x)}n(y) + \frac{1}{\hbar}\varphi(x) = \frac{1}{\hbar}\varphi(x) \tag{19}$$

A further derivative in $\varphi(y)$, the Leibnitz rule and (15) give:

$$-\delta(x-y) = \int dz \frac{\delta^2\Gamma[n]}{\delta n(x)\delta n(z)}\hbar^2 \frac{\delta^2 W[\varphi]}{\delta\varphi(z)\delta\varphi(y)}$$

$$= \int dz \frac{\delta^2\Gamma[n]}{\delta n(x)\delta n(z)} \Big\langle \mathsf{T}\delta n(z)\delta n(y)\Big\rangle_\varphi$$

3. Coherent states of the Bose oscillator

Consider a canonical pair $[\hat b, \hat b^\dagger] = 1$, irreducible on some Hilbert space (no operator exists that commutes with them, except unity). The number operator is self-adjoint and unbounded, with orthonormal eigenvectors $\hat b^\dagger \hat b|n\rangle = n\,|n\rangle$, $n = 0, 1, 2, ...$, that span the Hilbert space: $1 = \sum_n |n\rangle\langle n|$ (completeness).
The action of the operators is $\hat b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat b|n\rangle = \sqrt{n}|n-1\rangle$. The vectors are obtained from the vacuum by

$$|n\rangle = \frac{1}{\sqrt{n!}}\hat b^{\dagger n}|0\rangle$$

A coherent state is an eigenvector of the destruction operator:

$$\boxed{\hat b|\alpha\rangle = \alpha|\alpha\rangle} \tag{20}$$

The equation has a normalized solution for any complex value α:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{21}$$

The series may be summed. Insert a factor acting as unity, $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat b^\dagger} e^{-\bar\alpha\hat b}|0\rangle$, and use the formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$, that holds whenever $[A,B]$ commutes with A and B. The result is a unitary operator acting as translation in $\mathbb{C}$:

$$|\alpha\rangle = e^{\alpha\hat b^\dagger - \bar\alpha\hat b}|0\rangle$$

Two coherent states are never orthogonal:

$$\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2)+\bar\alpha\alpha'}, \quad |\langle\alpha|\alpha'\rangle| = e^{-\frac{1}{2}|\alpha-\alpha'|^2} \tag{22}$$

These useful integrals, where $d^2z = dx\,dy$, are proven in polar coordinates:

$$\int \frac{d^2z}{\pi} e^{-|z|^2}\bar z^n z^m = n!\,\delta_{nm} \tag{23}$$

$$\int \frac{d^2z}{\pi} e^{-|z|^2+\alpha\bar z} z^m = \alpha^m \tag{24}$$

The last one gives $\int \frac{d^2z}{\pi} e^{-|z|^2+\alpha\bar z}f(z) = f(\alpha)$. The function $e^{\alpha\bar z}$ is a "reproducing kernel".
The coherent states are an overcomplete set:

$$\boxed{\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1} \tag{25}$$

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \sum_{n,m}|n\rangle\langle m|\int \frac{d^2\alpha}{\pi}\langle n|\alpha\rangle\langle\alpha|m\rangle = \sum_{n,m}|n\rangle\langle m|\int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\bar\alpha^m}{\sqrt{m!}}$$

The integral is eq.(23). Then: $\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \sum_n |n\rangle\langle n| = 1.$ $\square$

One can extract countable subsets of coherent states that are still complete [1].

The Bose oscillator has two important representations, with explicit coherent states (the representations are unitarily equivalent).

3.1. **Harmonic oscillator.** In $L^2(\mathbb{R})$, on the dense subspace $\mathscr{S}(\mathbb{R})$ define:

$$(\hat{b}\varphi)(x) = \frac{1}{\sqrt{2}}[x\varphi(x) + \varphi'(x)], \qquad (\hat{b}^\dagger\varphi)(x) = \frac{1}{\sqrt{2}}[x\varphi(x) - \varphi'(x)]$$

The eigenstates of the number operator, $\hat{n} = \hat{b}^\dagger\hat{b}$ solve $nh_n(x) = -\frac{1}{2}h_n''(x) + \frac{1}{2}(x^2 - 1)h_n(x)$ are

$$h_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} e^{-\frac{1}{2}x^2} H_n(x)$$

H_n are the Hermite polynomials.

The coherent states solve $\Phi_\alpha'(x) = -(x - \sqrt{2}\alpha)\Phi_\alpha(x)$ and are shifted Gaussians

$$\Phi_\alpha(x) = \langle x|\alpha\rangle = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}(x-\alpha\sqrt{2})^2}$$

The real and imaginary parts of $\sqrt{2}\alpha$ yield the mean position and mean momentum. The square roots of variances have product $\Delta x \Delta y = \frac{1}{2}$.

3.2. **Holomorphic representation.** The Bargmann space [2] is the Hilbert space of entire functions such that $\int \frac{d^2z}{\pi} e^{-|z|^2}|f(z)|^2 < \infty$ ($d^2z = dx\,dy$) with the inner product

$$(f,g) = \int \frac{d^2z}{\pi} e^{-|z|^2}\overline{f(z)}g(z)$$

The monomials $u_n(z) = \frac{z^n}{\sqrt{n!}}$ are a complete orthonormal set[2]. The canonical operators are adjoints of each other, with action:

$$(\hat{b}f)(z) = f'(z), \qquad (\hat{b}^\dagger f)(z) = zf(z)$$

The monomials u_n are the normalized eigenfunctions of the number operator $z\frac{d}{dz}$. The coherent states are exponential functions:

$$\Phi_\alpha(z) = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}u_n(z) = e^{-\frac{1}{2}|\alpha|^2 + \alpha z}$$

4. Functional representation of the thermal Partition function

We now give a functional representation of the partition function. Sources will be added in the end.

The eigenvectors of $\hat{b}_r^\dagger\hat{b}_r$, $r = 1, 2, ...$, are the number basis $|\mathbf{n}\rangle = |n_1, n_2, ..., n_\infty\rangle$ of Fock space. An overcomplete basis are the coherent states $|\boldsymbol{\alpha}\rangle = |\alpha_1, \alpha_2, ..., \alpha_\infty\rangle$, where $\hat{b}_r|\boldsymbol{\alpha}\rangle = \alpha_r|\boldsymbol{\alpha}\rangle$. They are very convenient because in second quantization the operators are normally ordered. Then:

$$\langle\boldsymbol{\alpha}|\hat{b}_r^\dagger\hat{b}_s|\boldsymbol{\alpha}'\rangle = \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}'\rangle\,\overline{\alpha_r}\alpha_s', \quad \langle\boldsymbol{\alpha}|\hat{b}_r^\dagger\hat{b}_s^\dagger\hat{b}_{r'}\hat{b}_{s'}|\boldsymbol{\alpha}'\rangle = \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}'\rangle\,\overline{\alpha_r}\,\overline{\alpha_s}\alpha_{r'}\,\alpha_{s'}$$

We introduce the notation $\langle\boldsymbol{\alpha}|\hat{K}|\boldsymbol{\alpha}'\rangle = \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}'\rangle\,\kappa(\boldsymbol{\alpha},\boldsymbol{\alpha}')$.

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[2]The power expansion of an entire function has infinite radius of absolute convergence, and uniform convergence on any compact set in $\mathbb{C}$.

The partition function at temperature β is

$$Z = \mathrm{tr}(e^{-\beta\hat{K}}) = \int d^2\boldsymbol{\alpha}\langle\boldsymbol{\alpha}|e^{-\beta\hat{K}}|\boldsymbol{\alpha}\rangle$$

where $d^2\boldsymbol{\alpha} = \prod_{k=1}^{\infty}\frac{d^2\alpha_k}{\pi}$. In view of the next steps we introduce time labels $\tau \in [0,\hbar\beta]$ for each overcomplete resolution of the identity, and rewrite Z:

$$Z = \int d^2\boldsymbol{\alpha}_0\langle\boldsymbol{\alpha}(\hbar\beta)|e^{-\beta\hat{K}}|\boldsymbol{\alpha}(0)\rangle, \tag{26}$$

$d^2\boldsymbol{\alpha}_0 = \prod_{k=1}^{\infty}\frac{d^2\alpha_k(0)}{\pi}$ with label $\tau = 0$ and boundary condition $\boldsymbol{\alpha}(\hbar\beta) = \boldsymbol{\alpha}(0)$.

The temperature interval $[0,\hbar\beta]$ is divided into subintervals (τ_i,τ_{i+1}) with $\tau_0 = 0$ and $\tau_{N+1} = \hbar\beta$. The exponential is factored:

$$e^{-\beta\hat{K}} = e^{-\frac{1}{\hbar}(\hbar\beta-\tau_N)\hat{K}}e^{-\frac{1}{\hbar}(\tau_N-\tau_{N-1})\hat{K}}\dots e^{-\frac{1}{\hbar}(\tau_1-0)\hat{K}}$$

Between the factors, at each τ_j, an identity $\int d^2\boldsymbol{\alpha}(\tau_j)|\boldsymbol{\alpha}(\tau_j)\rangle\langle\boldsymbol{\alpha}(\tau_j)|$ is inserted.

$$Z = \int\prod_{k=0}^{N} d^2\boldsymbol{\alpha}(\tau_k)\langle\boldsymbol{\alpha}(\hbar\beta)|e^{-\frac{1}{\hbar}(\hbar\beta-\tau_N)\hat{K}}|\boldsymbol{\alpha}(\tau_N)\rangle\langle\boldsymbol{\alpha}(\tau_N)|e^{-\frac{1}{\hbar}(\tau_N-\tau_{N-1})\hat{K}}|\boldsymbol{\alpha}(\tau_{N-1})\rangle$$
$$\dots|\boldsymbol{\alpha}(\tau_1)\rangle\langle\boldsymbol{\alpha}(\tau_1)|e^{-\frac{1}{\hbar}(\tau_1-\tau_0)\hat{K}}|\boldsymbol{\alpha}(0)\rangle$$

For small intervals, a matrix element is:

$$\begin{aligned}
&\langle\boldsymbol{\alpha}(\tau_{i+1})|e^{-\frac{1}{\hbar}(\tau_{i+1}-\tau_i)\hat{K}}|\boldsymbol{\alpha}(\tau_i)\rangle\\
&\approx \langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle - \frac{1}{\hbar}(\tau_{i+1}-\tau_i)\langle\boldsymbol{\alpha}(\tau_{i+1})|\hat{K}|\boldsymbol{\alpha}(\tau_i)\rangle + \dots\\
&\approx \langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle\exp\left[-\frac{1}{\hbar}(\tau_{i+1}-\tau_i)\kappa(\boldsymbol{\alpha}(\tau_{i+1}),\boldsymbol{\alpha}(\tau_i))\right]
\end{aligned}$$

The prefactors $\langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle$ are collected:

$$\begin{aligned}
&\exp\sum_{i=0}^{N}\left[-\tfrac{1}{2}\|\boldsymbol{\alpha}(\tau_{i+1})\|^2 - \tfrac{1}{2}\|\boldsymbol{\alpha}(\tau_i)\|^2 + \overline{\boldsymbol{\alpha}(\tau_{i+1})}\cdot\boldsymbol{\alpha}(\tau_i)\right]\\
&= \exp\sum_{i=0}^{N}\left[-\|\boldsymbol{\alpha}(\tau_{i+1})\|^2 + \overline{\boldsymbol{\alpha}(\tau_{i+1})}\cdot\boldsymbol{\alpha}(\tau_i)\right]\\
&= \exp\left[-\sum_{i=0}^{N}(\tau_{i+1}-\tau_i)\overline{\boldsymbol{\alpha}(\tau_{i+1})}\cdot\frac{\boldsymbol{\alpha}(\tau_{i+1})-\boldsymbol{\alpha}(\tau_i)}{\tau_{i+1}-\tau_i}\right]
\end{aligned}$$

The full exponent is

$$-\frac{1}{\hbar}\sum_{i=0}^{N}(\tau_{i+1}-\tau_i)\left[\hbar\overline{\boldsymbol{\alpha}(\tau_{i+1})}\cdot\frac{\boldsymbol{\alpha}(\tau_{i+1})-\boldsymbol{\alpha}(\tau_i)}{\tau_{i+1}-\tau_i} + \kappa(\boldsymbol{\alpha}(\tau_{i+1}),\boldsymbol{\alpha}(\tau_i))\right]$$

In the (formal) limit $N\to\infty$ we write the action:

$$S = \int_0^{\hbar\beta}d\tau\left[\hbar\,\overline{\boldsymbol{\alpha}(\tau)}\cdot\frac{\partial\boldsymbol{\alpha}(\tau)}{\partial\tau} + \kappa(\boldsymbol{\alpha}(\tau),\boldsymbol{\alpha}(\tau))\right]$$

Now the partition function is a functional integral on an infinite set of functions $\alpha_r(\tau)$, $r = 1, 2, ...$

$$Z = \int \mathscr{D}^2 \boldsymbol{\alpha}(\tau) e^{-\frac{1}{\hbar}S} \tag{27}$$

If the Bose operators are field operators (indexed by position, spin zero), i.e. $|r\rangle = |\mathbf{x}\rangle$, the complex vector $\boldsymbol{\alpha}(\tau)$ is replaced by a complex field $\psi(\mathbf{x}, \tau)$ with periodic b.c. $\psi(\mathbf{x}, \hbar\beta) = \psi(\mathbf{x}, 0)$:

$$Z = \int \mathscr{D}^2 \psi(\mathbf{x}, \tau) e^{-\frac{1}{\hbar}S} \tag{28}$$

$$\begin{aligned} S = \int_0^{\hbar\beta} d\tau \Big[\int d\mathbf{x}\, \overline{\psi(\mathbf{x}, \tau)} \left(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) - \mu \right) \psi(\mathbf{x}, \tau) \\ + \frac{1}{2} \int d\mathbf{x} d\mathbf{y} \overline{\psi(\mathbf{x}, \tau)} \psi(\mathbf{x}, \tau)\, v(\mathbf{x} - \mathbf{y}) \overline{\psi(\mathbf{y}, \tau)} \psi(\mathbf{y}, \tau) \Big] \end{aligned} \tag{29}$$

4.1. **The ideal gas of bosons.** The convenient choice is the basis $|r\rangle = |\mathbf{k}\rangle$. Since $0 \le \tau \le \hbar\beta$, the field is Fourier expanded in Matsubara frequencies $\omega_n = \frac{2\pi n}{\hbar\beta}$:

$$\psi(\mathbf{k}, \tau) = \frac{1}{\sqrt{\hbar\beta}} \sum_n \psi(\mathbf{k}, i\omega_n) e^{-i\omega_n \tau} \tag{30}$$

The frequencies are bosonic and fulfill the boundary condition $\psi(\mathbf{k}, 0) = \psi(\mathbf{k}, \hbar\beta)$. The transformation to frequencies is unitary.
Since there is no interaction, the modes $(\mathbf{k}, i\omega_n)$ decouple (the factors $1/\pi$ for each mode are restored in the measure):

$$S_0 = \sum_{n,\mathbf{k}} \left(-i\hbar\omega_n + \epsilon_k^0 - \mu \right) |\psi(\mathbf{k}, i\omega_n)|^2$$

$$Z_0 = \prod_{\mathbf{k},n} \int \frac{d^2 z}{\pi} \exp \left[\left(i\omega_n - \frac{\epsilon_k^0 - \mu}{\hbar} \right) |z|^2 \right] = \prod_{\mathbf{k},n} \frac{1}{-i\omega_n + \frac{\epsilon_k^0 - \mu}{\hbar}}$$

The Gaussian integrals exists if $\mu < 0$. The partition function $\Omega_0 = -\frac{1}{\beta} \log Z_0$ is

$$\Omega_0 = \frac{1}{\beta} \sum_{\mathbf{k},n} \log \left(-i\omega_n + \frac{\epsilon_k^0 - \mu}{\hbar} \right) = \frac{1}{\beta} \sum_{\mathbf{k}} \log \left[1 - e^{-\beta(\epsilon_k^0 - \mu)} \right] + \text{unwanted terms}$$

A precise evaluation is done in Negele-Orland [5].

4.2. **Partition function with sources.** The partition function with sources is the generator of Green functions. For this purpose we add a source-term in the action:

$$S[\bar{\phi}, \phi] = S_0 - \hbar \int_0^{\hbar\beta} d\tau \sum_r [\overline{\alpha_r(\tau)} \phi_r(\tau) + \overline{\phi_r(\tau)} \alpha_r(\tau)]$$

The functional derivatives of $Z[\bar{\phi}, \phi]$ in the sources produce time-ordered correlation functions in presence of the sources. For example:

$$\langle \hat{b}_r(\tau) \rangle = \frac{1}{Z[\bar{\phi}, \phi]} \frac{\delta Z[\bar{\phi}, \phi]}{\delta \overline{\phi_r(\tau)}} = \frac{1}{Z[\bar{\phi}, \phi]} \int \mathscr{D}^2 \boldsymbol{\alpha}(\tau) e^{-\frac{1}{\hbar} S[\bar{\phi}, \phi]} \alpha_r(\tau)$$

When the sources are turned off one obtains the thermal average.

The time-ordering is well understood at the defining level of the functional integral. The term $-\hbar\sum_r[\hat{b}_r^\dagger\phi_r(\tau)+\overline{\phi_r(\tau)}\hat{b}_r]$ is added to K. After the slicing of the interval $[0,\hbar\beta]$, one adds the term to the weight at exponent in each subinterval

$$-\hbar(\tau_{j+1}-\tau_j)\sum_r[\overline{\alpha_r(\tau_{j+1})}\phi_r(\tau_j)+\overline{\phi_r(\tau_j)}\alpha_r(\tau_j)]$$

This explains why functional derivatives in the sources drop down fields into a time-ordered sequence.

5. Non-interacting bosons with sources

A shift of the fields in the measure does not change the measure and the functional integral. For the infinitesimal shift $\delta\overline{\boldsymbol{\alpha}}(\tau)$, the linear change in $Z[\bar{\phi},\phi]$ is:

$$0=-\frac{1}{\hbar}\int\mathscr{D}^2\boldsymbol{\alpha}\,e^{-\frac{1}{\hbar}S_0}\int_0^{\hbar\beta}d\tau\sum_r\delta\overline{\alpha_r}(\tau)\left[\hbar\frac{\partial\alpha_r(\tau)}{\partial\tau}+\sum_s h_{rs}\alpha_s(\tau)-\mu\alpha_r(\tau)-\hbar\phi_r(\tau)\right]$$

The arbitrariness of the shift gives a Ward identity for $\langle\alpha_s(\tau)\rangle$:

$$\hbar\frac{\partial\langle\alpha_r(\tau)\rangle}{\partial\tau}+\sum_s h_{rs}\langle\alpha_s(\tau)\rangle-\mu\langle\alpha_r(\tau)\rangle=\hbar\phi_r(\tau) \tag{31}$$

The equation is solved with the Green function *without* sources eq.(7):

$$\langle\alpha_r(\tau)\rangle=\frac{1}{Z_0[\bar{\phi},\phi]}\frac{\delta Z_0[\bar{\phi},\phi]}{\delta\overline{\phi_r(\tau)}}=-\int_0^{\hbar\beta}d\tau'\sum_{r'}\mathscr{G}^0(r,\tau;r',\tau')\phi_{r'}(\tau') \tag{32}$$

where we are assuming that the field is zero for $\phi=0$ (the homogeneous solution). The functional equation (32) has solution

$$\boxed{\log Z_0[\bar{\phi},\phi]=\log Z_0-\sum_{r,r'}\iint_0^{\hbar\beta}d\tau d\tau'\overline{\phi_r(\tau)}\,\mathscr{G}^0(r,\tau;r',\tau')\,\phi_{r'}(\tau')} \tag{33}$$

5.1. **Wick theorem.** With the result (33), the partition function of non interacting particles with sources is:

$$\frac{1}{Z_0}\int\mathscr{D}^2\boldsymbol{\alpha}(\tau)e^{-\frac{1}{\hbar}S_0+\sum_r\int_0^{\hbar\beta}d\tau(\bar{\phi}_r\alpha_r+\phi_r\bar{\alpha}_r)}=e^{-\sum_{rs}\iint\bar{\phi}_r(\tau)\mathscr{G}^0(r\tau;s\tau')\phi_s(\tau')}$$

Because the fields are uncoupled, it is

$$\left\langle\prod_r e^{\int_0^{\hbar\beta}d\tau\bar{\phi}_r\alpha_r}\prod_s e^{\int_0^{\hbar\beta}d\tau\bar{\alpha}_r\phi_r}\right\rangle=\prod_{rs}e^{-\iint\bar{\phi}_r(\tau)\mathscr{G}^0(r\tau;s\tau')\phi_s(\tau')d\tau d\tau'}$$

The expansion of the left side in the sources, gives a sum of products of sources with coefficients $\langle\prod\alpha_r\prod\bar{\alpha}_s\rangle_0$. The expansion of the right side in the sources gives products of propagators and sources. The equality of the two sides expresses coefficients (correlators of any order) in terms of propagators.
Every correlator in the left is either 0 or a product of propagators.
For example, in the expansions of both sides the coefficient $\bar{\phi}_{r_1}(\tau_1)\bar{\phi}_{r_2}(\tau_2)\phi_{r_3}(\tau_3)\phi_{r_4}(\tau_4)$ gives the equality:

$$\langle\mathsf{T}\hat{b}_{r_1}(\tau_1)\hat{b}_{r_2}(\tau_2)\hat{b}^\dagger_{r_3}(\tau_3)\hat{b}^\dagger_{r_4}(\tau_4)\rangle=\mathscr{G}^0(r_1\tau_1;r_3\tau_3)\mathscr{G}^0(r_2\tau_2;r_4\tau_4)+\mathscr{G}^0(r_1\tau_1;r_4\tau_4)\mathscr{G}^0(r_2\tau_2;r_3\tau_3)$$

6. Interacting bosons with sources

Ward identities originate in the same way as for non interacting particles. After a shift in the field $\alpha_r(\tau)$:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau} + \sum_s h_{rs} - \mu\delta_{rs}\right]\langle\alpha_s(\tau)\rangle + \sum_{srr'} v_{rsr's'}\langle\bar{\alpha}_s(\tau)\alpha_{r'}(\tau)\alpha_{s'}(\tau)\rangle = \hbar\phi_r(\tau)$$

A derivative in the source $\phi_v(\tau')$ gives an equation of motion for the propagator:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau} - \mu\delta_{rs} + \sum_s h_{rs}\right]\mathscr{G}(s\tau;v\tau') - \sum_{srr'} v_{rsr's'}\langle\bar{\alpha}_s(\tau)\alpha_{r'}(\tau)\alpha_{s'}(\tau)\bar{\alpha}_v(\tau')\rangle$$
$$= -\hbar\delta_{rv}\delta(\tau-\tau')$$

where the average can now be taken with no sources.

6.1. **Perturbation theory.** In presence of an interaction among bosons, the perturbative approach in the position basis is as follows.
Let $v(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau') \equiv U^0(x.x')$. The action is written as

$$S[\bar{\phi},\phi] = S_0[\bar{\phi},\phi] + \frac{1}{2}\iint dx\,dx'\;\overline{\psi(x)}\psi(x)\,U^0(x,x')\overline{\psi(x')}\psi(x')$$

A series expansion (perturbation series) is done in the functional integral:

$$Z[\bar{\phi},\phi] = \int \mathscr{D}^2\psi(x)e^{-\frac{1}{\hbar}S_0[\bar{\phi},\phi]}\sum_k\frac{(-1)^k}{k!(2\hbar)^k}\Big[\int dxdy\,\bar{\psi}(x)\bar{\psi}(y)U^0(x,y)\psi(y)\psi(x)\Big]^k$$
$$= \sum_k\frac{(-1)^k}{k!(2\hbar)^k}\left[\int dxdy\,U^0(x,y)\frac{\delta^4}{\delta\phi(x)\delta\phi(y)\delta\bar{\phi}(y)\delta\bar{\phi}(x)}\right]^k Z_0[\bar{\phi},\phi]$$

where $Z_0[\bar{\phi},\phi]$ is given in (32).
This gives the interacting partition function as a sum of vacuum graphs with lines U^0 and $\mathscr{G}^0$ (connected and unconnected).

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