Quantum Field Theory Extension

MSc Physics and Astronomy, Theoretical Physics track

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1 Course guide

Quantum Field Theory (QFT) is the mathematical framework that describes the behaviour of subatomic particles as well as quasi-particles in condensed matter systems. It is built upon the combination of classical field theory, quantum mechanics, and special relativity. In particular, QFT is the language of the Standard Model of particle physics, one of the most successful physical theories ever constructed by humankind.

In this course, we discuss a number of topics in Quantum Field Theory, extending the topics covered in the preceding course. In particular, we study how infinities arises in QFTs and how they can be tamed using the renormalization procedure, and present the quantization of the photon field and its consequences for the description of the interactions between matter and gauge fields.

Embedding. The present course is part of the joint UvA and VU Master course (MSc) in Physics and Astronomy, in particular within the *Theoretical Physics* track. The course takes place during the first four teaching weeks of January, and consists on 4 hours of lectures plus 2 hours of tutorial work per week.

This *Quantum Field Theory extension* course is, as its name indicates, the natural continuation of the *Quantum Field Theory* course, and indeed here we will assume that all students have already followed it. The topics covered in the present extension are of general interest for all theoretical physics students, and those interested in further pursuing a deeper study of QFT after this course are encouraged to enroll in the *Particles and Fields* and/or the *Advanced Quantum Field theory* courses. Both courses continue this exploration of the rich nature of QFTs from various point of view, the first more focused towards phenomenological applications and the second concentrating on the more formal aspects of the theory.

The topics covered on this extension have been chosen to minimize the overlap between previous courses that the students have followed (in particular the QFT course) and future courses that the students might choose to follow (specifically aQFT and P&F). Therefore students interested in achieving a deep understanding of QFT in general should follow this course in addition to the previous QFT course and the two subsequent courses, since while some degree of repetition is unavoidable, we have made an effort to keep it to a minimum.

Course assessment. The assessment of the course will be based on homework problems, and therefore there will be no final exam. The completion of all the homework assignments is a requisite to pass the course. The exact dates for the homework problems submission will be announced during the first week of lectures.

The two homework problem sets are included in Apps. D and D of these notes, respectively.

Course outline. This course, and the corresponding lecture notes, are organized as follows. In the first part of the course, Sect. 2, we introduce calculations in quantum theory beyond the Born approximation, showing how ultraviolet divergences arise in loop diagrams for the specific case of $\lambda\phi^4$ theory. We also demonstrate how in renormalizable quantum field theories these divergences can be absorbed into a redefinition of the Lagrangian bare (unphysical) parameters. We will introduce a useful language to relate the behaviour of quantum field theories at energy different scales, called the renormalization group equation. In the second part of the course, we move in Sect. 3 to discuss of the quantization of Quantum Electrodynamics (QED), the quantum field theory of the electromagnetic interactions, deriving in particular in canonical quantization of the photon field and deriving the Feynman rules of the theory, both when the photon is coupled to scalars

and to fermions. In this part we will also present for completeness a review of the classical symmetries of the Abelian gauge theory, and complete the discussion by performing calculations of a number of simple scattering processes in scalar QED.

In addition, to make these lecture notes more self-consistent, we have added two extra appendices, App. A with a brief review of some results in QFT that the students should have already encountered before, and App. B for a list of useful mathematical expressions and identities that are required at some point during the course.

Course materials. The main resource for this course are the lecture notes, that are available via my personal webpage:

http://juanrojo.com/teaching

These lecture notes will be updated as the course goes on. These notes are however not meant to be the only study resource, but rather they represent a guide to help the student to navigate within the course material and when needed consult additional references. In addition, the lecture notes have not been proof-read or cross-checked by other people: the students are encouraged to let me know of mistakes and typos that they might find. These lecture notes are naturally complemented by the corresponding lecture notes of Daniel Baumann's *Quantum Field Theory* course, also available online.

The topics covered in this course are inherited to three main textbooks, namely:

- *Quantum Field Theory*, Mark Srednicki, Cambridge University Press.

  This textbook is freely accessible online as a .pdf file:

  https://www.physics.utoronto.ca/~luke/PHY2403F/References.html

- *Quantum Field Theory and the Standard Model*, Matthew D. Schwartz, Cambridge University Press.

  More information about this textbook can be found here:

  http://users.physics.harvard.edu/~schwartz/teaching

- *An introduction to Quantum Field Theory*, Michael E. Peskin and Daniel V. Schroeder, Westview Press. A classic QFT textbook. The solutions for the exercises in some of the earlier chapters of the book can be found here:

  http://homerreid.dyndns.org/physics/peskin/index.shtml

For the interested students, other related online lectures notes that they might consider to also study are the following ones:

- David Tong's lecture notes on Quantum Field Theory:

  http://www.damtp.cam.ac.uk/user/tong/qft.html

- Sidney Coleman's Harvard Quantum Field Theory course:

https://arxiv.org/pdf/1110.5013.pdf

- Michael Luke's QFT lecture notes

https://www.physics.utoronto.ca/~luke/PHY2403F/References.html

Finally, let me mention that a potentially useful overview of several QFT textbooks by Flip Tanedo can also be found here:

https://fliptomato.wordpress.com/2006/12/30/
from-griffiths-to-peskin-a-lit-review-for-beginners/

Learning objectives. At the end of the course, the student will be able to:

- Understand the physical origin of the infinities that arise in calculations of scattering processes in Quantum Field Theory beyond the Born approximation.
- Calculate finite one-loop processes in QFT by removing these infinities using the renormalization method in the case of the scalar $\lambda\phi^4$ theory.
- Relate physical phenomena at different scales by using the renormalization group flows.
- Quantize electromagnetism and perform calculations involving spin-1 photon fields.
- Understand what are the implications of electromagnetism's classical symmetries at the QFT level.
- Become familiar with simple interaction processes in Quantum Electrodynamics.

Acknowledgments

I am very grateful to Alejandra Castro for providing me with her course notes - the present lecture notes have grown out of Alejandra's excellent notes. I am also grateful to Daniel Baumann, Diego Hofmann, Eric Laenen, and Wouter Waalewijn for useful discussions and feedback when preparing this course.

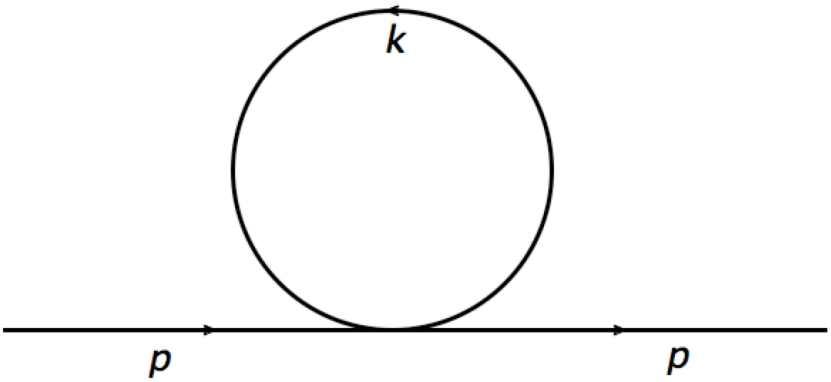


Figure 2.1. One of the loop diagrams that arise in calculations using $\lambda\phi^4$ theory beyond the Born approximation, with k^μ being the four-momentum running in the loop.

2 Renormalization in QFTs: $\lambda\phi^4$ theory at one loop

Beyond the Born approximation (the so-called *leading order* in the perturbative expansion), the calculation of scattering amplitudes in generic QFT involves the presence of *loops* of virtual particles. These particles will be in general different from those present in the initial and final states of the scattering. A crucial aspect of these *virtual diagrams* is that in many cases they are *divergent*, that is, it is not possible to associate finite result to them, which is obviously problematic. In this first part of the course we will present how infinities arise in calculations in Quantum Field Theories beyond the Born level, and illustrate how they can be dealt with by a suitable redefinition of the parameters of the Lagrangian by means of the *renormalization process*.

Here we will focus on the $\lambda\phi^4$ scalar field theory, defined by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4\,, \tag{2.1}$$

where λ is a dimensionless coupling constant and m is the mass of the (real) scalar field. The choice of $\lambda\phi^4$ theory to illustrate the renormalization paradigm is motivated by its simplicity, which yet still capturing the most relevant physical aspects of the problem.

How can infinite results appear in QFT calculations? Let us consider for example the diagram in Fig. 2.1, which arises in loop calculations of $\lambda\phi^4$ theory, in particular for the first non-trivial correction to the free-field propagator. Using the Feynman rules of the theory, we find that the matrix element associated to this specific amplitude $\mathcal{M}$ is given by

$$i\mathcal{M} = i\lambda\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2-i\epsilon}\,, \tag{2.2}$$

where k^μ is the four-momentum running in the loop. It is easy to convince yourselves that this integral is *divergent*. To show this, in the region where momenta are much larger than the mass, $k^\mu \gg m$, we have that

$$\mathcal{M} \sim \lambda\int d^4k\frac{1}{k^2} \sim \int k\,dk \to \infty\,, \tag{2.3}$$

where we have used that in Euclidean space $d^4k = k^3 dk d\Omega_4$. So we find that the one-loop amplitude show in Fig. (2.1), is not only divergent, but also *quadratically* divergent, irrespective of the specific value of

the coupling constant λ (that is, even for a vanishingly small λ). This kind of divergent integral appears in almost every attempt to compute matrix elements beyond leading order in perturbation theory. The problem was so severe that by the late 1930s the founding fathers of quantum mechanics such as Dirac, Bohr and Oppenheimer were about to give up QFT as a fundamental theory of Nature because of these divergent integrals.

How we can deal with these divergences and make physical sense out of the QFT predictions beyond the Born approximation? The crucial point here is that it is possible to reorganize the calculation in way that all divergences are absorbed into a redefinition of the (unobservable) parameters of the theory Lagrangian, in a way that all physical observables are finite. This program is generically known as *renormalization*, and will be the main topic of this first part of the course. But first of all, let us gain a bit more of physical insight about the origin of these infinities by studying the famous *Casimir effect*.

2.1 Infinities in QFTs: the Casimir effect

Let us illustrate how infinities can appear in QFT calculations with a simple example, namely the zero-point energy of a free (non-interacting) scalar field ϕ. The Hamiltonian for this scalar field in terms of the creation and annihilation operators $a_k^\dagger$ and a_k can be written as

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(a_k^\dagger a_k + \omega\left(\vec{k}\right)V\right) , \tag{2.4}$$

where V is the volume where the field is confined, and by virtue of the dispersion relation we have

$$\omega\left(\vec{k}\right) = \sqrt{\vec{k}^2 + m^2} , \tag{2.5}$$

with m being the particle mass. We can use Eq. (2.4) to compute the *vacuum energy*, defined as the expectation value of the Hamiltonian in the ground state, finding

$$E_0 \equiv \langle 0|H|0\rangle = V\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\omega(k) \sim V\int \frac{d^3k}{(2\pi)^3}\frac{k}{2} \sim V\int k^3 dk \to \infty , \tag{2.6}$$

so even a simple non-interacting scalar QFT leads to an infinite zero-point energy E_0. However, this is *per se* not a problem, since zero-point energies are not physical observables, and only differences in energy can be accessed experimentally. Still this seems to be a rather remarkable effects, so maybe this infinite vacuum energy can have some observable effects after all?

To verify that indeed this vacuum energy leads to observable effects, let us consider the following setup: a box of size L, where inside we place a moving plate at position l. We impose Dirichlet boundary conditions for the scalar field both at the sides of the box, $\phi(x = 0) = \phi(x = L) = 0$ and at the plate, $\phi(x = l)$. In other words, we have a perfectly reflective plate. The setup is illustrated in Fig. 2.2. The total energy of the vacuum state in this configuration will then be the sum of the contributions from two regions in the box, namely

$$E_0(l) = E_0(x \leq l) + E_0(x \geq (L - l)) . \tag{2.7}$$

Now, as one moves the plate inside the box, that is, as one changes the value l, from general considerations

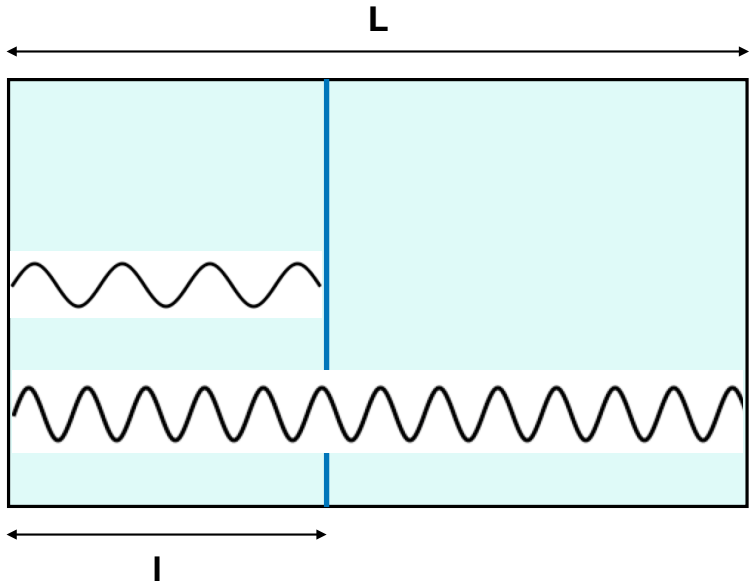


Figure 2.2. Setup for the Casimir effect. A box of length L contains a perfectly reflective plate at a distance l from one side of the box. We impose Dirichlet boundary conditions both at the sides of the box, $\phi(x=0)=\phi(x=L)=0$ and at the plate $\phi(x=l)$.

there should appear a force exerted on the plate given by the variation of the energy with respect to l,

$$F_c = -\frac{dE_0(l)}{dl}\,, \tag{2.8}$$

pushing the system to a minimum of the energy. This force is the so-called *Casimir force*, predicted by the Dutch physicist Hendrik Casimir in 1948 and observed experimentally 10 years later. The Casimir force is therefore a real physical effect, and a direct consequence of the (infinite) zero-point energy of a quantum field theory. Note that the fact that the vacuum energy is infinite is not necessary a problem: any offset in E_0 will be canceled out when we take the derivate: only variations of E_0 with respect to l are physically meaningful, while the actual value of E_0 is not.

Let us now compute this Casimir force, where for the sake of simplicity we will restrict ourselves to a one dimensional box. In this case, in the expression for the vacuum energy Eq. (2.6) one replaces the integral over k by a finite sum over the field modes ω_n that satisfy the Dirichlet boundary conditions, namely

$$E_0(a) = \langle 0|H|0\rangle = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\pi n}{a}\,, \tag{2.9}$$

where a is the length of the region of the box that contributes to E_0, and we have used that $\omega_n = \pi n/a$. Therefore, the total vacuum energy taking into account the two regions of the box from Fig. 2.2 is

$$E_0 = E_0(l) + E_0(L-l) = \left(\frac{1}{l}+\frac{1}{L-l}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n\,, \tag{2.10}$$

which is infinite, as in the continuum case. Now, as we have discussed above, while in principle there is nothing wrong with E_0 being infinite (since it is not an observable quantity), the Casimir force Eq. (2.8),

being experimentally observable, must be finite. Computing this force, we find

$$F_C = -\frac{dE_0}{dl} = \left(\frac{1}{l^2} - \frac{1}{(L-l)^2}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n\,, \tag{2.11}$$

which is still infinite, unless $l = L/2$, where for which the Casimir force vanishes, $F_c = 0$. So it seems that just by putting a plate inside an empty box and nothing else we can obtain an infinite repulsive force, which of course makes no physical sense.

How can we understand this result? To begin with, one notes that the Dirichlet boundary conditions for the scalar field at $x = 0, l$, and L that force the quantization of its modes arise from the interaction between the field and the plates, which are themselves not infinitely thin and continuous but rather compose by finite-size atoms. So the problem here is that the free theory is too simplistic to give realistic predictions: a full theory should account for the interactions of the field ϕ with the atoms in the plate. In particular, field modes with very small wavelengths (that is, very high energies) will go straight through the plates and will not interact with the atoms that compose it. Therefore, assuming that *all* field modes will be quantized due to the plate is an inconsistent assumption.

In order to take this physical requirement into account, namely the the high energy modes should not contribute to the Casimir force, what we can do is to regulate the sum in Eq. (2.10) by suppressing the contribution of the high frequency modes as follows:

$$E(a,\Lambda) = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n(a)e^{-\omega_n(a)/\pi\Lambda}\,, \quad \omega_n(a) = \frac{\pi n}{a}\,, \tag{2.12}$$

where Λ is for the time being an arbitrary parameter with the same units as the frequencies ω_n, whose purpose will be revealed in short. The strategy then is to compute the zero-point energy $E_0(l)$ and the Casimir force $F_c(l)$ as a function of Λ and then at the end of the calculation take the limit of $\Lambda \to \infty$, that is, take the limit where the high-energy modes are not suppressed.

Adopting this strategy, what one gets is

$$E(a,\Lambda) = \frac{\pi}{2a}\sum_{n=1}^{\infty} n e^{-\epsilon n}\,, \qquad \epsilon = \frac{1}{a\Lambda}\,, \tag{2.13}$$

which can be expressed in terms of an infinite sum with a finite result,

$$E(a,\Lambda) = -\frac{\pi}{2a}\frac{\partial}{\partial\epsilon}\left(\sum_{n=1}^{\infty} e^{-\epsilon n}\right) = -\frac{\pi}{2a}\frac{\partial}{\partial\epsilon}\left(\frac{1}{e^{\epsilon}-1}\right) = \frac{\pi}{2a}\left(\frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}\left(\epsilon^2\right)\right)\,, \tag{2.14}$$

where terms suppressed when $\epsilon \to 0$ can be neglected, since this corresponds to the physical limit where $\Lambda \to \infty$. Combining now the contributions from the two parts of the box, one finds

$$E_0(l,\Lambda) = \frac{\pi}{2l}\left(l^2\Lambda^2 - \frac{1}{12} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right)\,, \tag{2.15}$$

$$E_0(L-l,\Lambda) = \frac{\pi}{2(L-l)}\left((L-l)^2\Lambda^2 - \frac{1}{12} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right)\,, \tag{2.16}$$

so that their sum is

$$E_0 = E_0(l) + E_0(L-l) = \frac{\pi}{2} L\Lambda^2 - \frac{\pi}{24l} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) , \tag{2.17}$$

where in the last step we have assumed that the plate is not too close to the center of the box, so that the approximation $L \gg l$ holds.

Therefore we now find that the zero-point energy is still infinite in the $\Lambda \to \infty$ limit, but the Casimir force $F_c(l)$ is instead *finite*, since

$$F_c = -\frac{dE_0}{dl} = -\frac{\pi}{24l^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) , \tag{2.18}$$

a result which is found to be in excellent agreement with the experiment. So what we have learned with this calculation? That observable quantities are independent on the specific details of how we treat the modes with the highest energy, in order words, on the details of the interaction between the scalar field ϕ and the plate. This means that once we properly regulate the UV physics, then the IR behaviour of the theory is well-defined, as one could have expected on general grounds.

The exponential suppression of the high energy modes of scalar field implemented by Eq. (2.12) is not the only possible way to regulate the ultraviolet divergences. One can check that other regulation strategies lead to the same result for the Casimir force, including for instance:

- A hard cut-off:

$$E(a) = \frac{1}{2} \sum_n \omega_n \Theta(\pi\Lambda - \omega_n) , \tag{2.19}$$

- A Gaussian cut-off:

$$E(a) = \frac{1}{2} \sum_n \omega_n \exp\left(-\frac{\omega_n^2}{\pi^2\Lambda^2}\right) , \tag{2.20}$$

- A ξ-function suppression:

$$E(a) = \frac{1}{2} \sum_n \omega_n \left(\frac{\omega_n}{\Lambda}\right)^{-\xi} . \tag{2.21}$$

While each of this regulators gives a different result for the (unobservable) vacuum energy, the physically measurable Casimir force is always the same. More generally, it is possible to show that any regulation strategy that has the form

$$E(a) = \sum_n \omega_n f\left(\frac{n}{a\Lambda}\right) , \tag{2.22}$$

and that satisfies that $f(0) = 1$ and

$$\lim_{x\to\infty} \left(x\partial^{(i)} f(x)\right) = 0 \tag{2.23}$$

will lead to the same result for the IR physics. In particular, note that the condition $f(0) = 1$ ensures that the regulator of the high energy divergences does not affect the infrared behaviour of the theory.

In summary, we have learned from this exercise that the Casimir force is independent of the regulator and that it can be understood as a purely infrared (as opposed to ultraviolet) effect: the IR physics is effectively *decoupled* from the behaviour of the theory in the ultraviolet. In this respect, the infinite result that we obtained in Eq. (2.11) was a consequence of applying the theory beyond its regime of validity, in particular by neglecting the interactions between the scalar field and the plate.

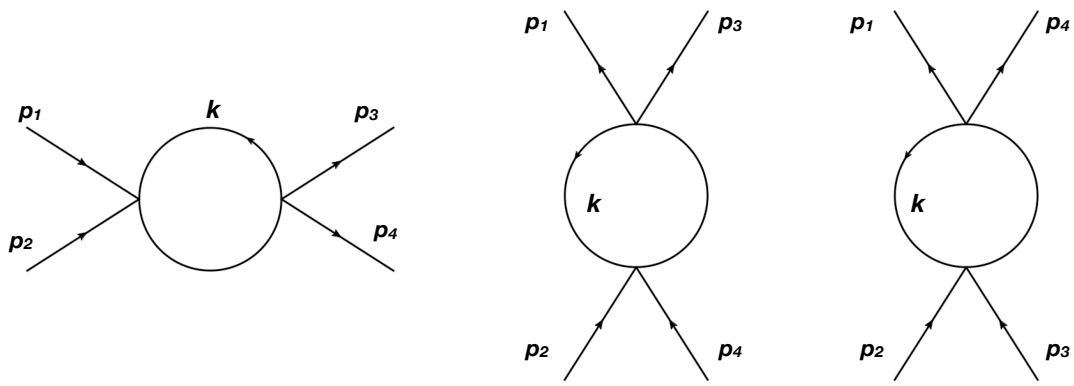


Figure 2.3. The Feynman diagrams relevant the calculation of the $\mathcal{O}(\lambda^2)$ contribution to the $2 \to 2$ scattering amplitude in $\lambda\phi^4$ theory. From left to right the topologies correspond to s-channel, t-channel, and u-channel respectively.

2.2 One-loop calculations in $\lambda\phi^4$ theory

As mention above, the main features of the renormalization procedure of a QFT can be already discussed in the relatively simple $\lambda\phi^4$ theory. This theory contains a massless scalar field ϕ with a quartic self-interaction, and is described by the Lagrangian Eq. (2.1). For the $2 \to 2$ scattering amplitude, it is easy to verify that the tree level diagram gives

$$i\mathcal{M}_0 = -i\lambda\,. \tag{2.24}$$

The goal of this section is to compute $\mathcal{M}_1$, the next term in the perturbative expansion of the amplitude $\mathcal{M}$, which is the term of the order $\mathcal{O}(\lambda^2)$:

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 + \ldots\,. \tag{2.25}$$

We will then encounter a number of infinities in the calculation, which will be dealt with by means of a regularization procedure in the same spirit as that of the Casimir effect.

There are three diagrams that contribute to the one-loop amplitude $\mathcal{M}_1$, corresponding to the s-, t- and u-channels diagrams respectively. These Feynman diagrams are represented in Fig. 2.3. Let us focus for the time being in the contribution from the s-channel diagram, the leftmost one in Fig. 2.3. If we define $p \equiv (p_1 + p_2) = (p_3 + p_4)$ (from momentum conservation), then the application of the Feynman rules of the $\lambda\phi^4$ theory leads to the following amplitude:

$$\mathcal{M}_{1,s} = \frac{(i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \frac{1}{k^2 - i\epsilon} \frac{1}{i} \frac{1}{(p-k)^2 - i\epsilon}\,, \tag{2.26}$$

where $1/2$ is a symmetry factor and we integrate over the four-momentum k^μ going through the loop. See App. A for a reminder of the Feynman rules of $\lambda\phi^4$ theory. Recall that for the time being the scalar field is massless, $m = 0$.

To evaluate loop integrals that arise in quantum field theory, there are a number of mathematical identities that can be used, and go over the name of *Feynman integral formulae*. The one that we need here is the

following:

$$\frac{1}{AB} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{(xA+yB)^2} = \int_0^1 dx\frac{1}{(xA+(1-x)B)^2}\,, \tag{2.27}$$

where one expresses the product of denominators $1/AB$ in terms of an integral over a variable x called a *Feynman parameter*. This integral is particularly useful for those cases where $1/A$ and $1/B$ represent the propagator of loop particles. For reference, the complete Feynman formula to combine denominators is

$$\frac{1}{A_1\ldots A_n} = \int dF_n \left(\sum_{k=1}^n x_k A_k\right)^{-n}\,, \tag{2.28}$$

with the integration measure over the Feynman parameters x_i is given by

$$\int dF_n = (n-1)!\int_0^1 \left(\prod_{k=1}^n dx_k\right)\delta\left(x_1+\ldots+x_n-1\right)\,. \tag{2.29}$$

Let us exploit the mathematical relationship of Eq. (2.27) to compute our loop integral Eq. (2.26). We observe that if we assign $A=(p-k)^2-i\epsilon$ and $B=k^2-i\epsilon$, then the amplitude can be rewritten using Eq. (2.27) and some additional algebra as

$$i\mathcal{M}_{1,s} = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1\frac{dx}{\left((p^2-2p\cdot k)(1-x)+k^2-i\epsilon\right)^2}\,. \tag{2.30}$$

Next, we do the following change of variables within the integral over k:

$$\widetilde{k}\equiv k+p(1-x)\,,\quad d\widetilde{k}=dk\,, \tag{2.31}$$

which does not affect the integration ranges since it is just a constant offset and the integral in Eq. (2.30) is unbounded. One can verify that this change of variables implies

$$p^2(1-x)+k^2-2p\cdot k(1-x) = p^2(1-x)+\widetilde{k}^2-p^2(1-x)^2 = \widetilde{k}^2+p^2x(1-x)\,, \tag{2.32}$$

and therefore (going back to denoting k the loop integration variable) we see that Eq. (2.30) can be expressed as

$$i\mathcal{M}_{1,s} = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{\left(k^2+p^2x(1-x)-i\epsilon\right)^2}\,, \tag{2.33}$$

which as we will see now is rather easier to handle with than the original loop integral.

To show this, one has to change the integration order to have

$$i\mathcal{M}_{1,s} = \frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2+p^2x(1-x)-i\epsilon\right)^2}\,, \tag{2.34}$$

and attempt to compute the integral over the loop four-momentum k^μ. We see that this integral has the following form

$$I_{1,s}\equiv\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2+\Delta-i\epsilon\right)^2}\,, \tag{2.35}$$

with $\Delta\equiv p^2x(1-x)$ being a constant (k-independent) term. It is easy to convince yourselves that an integral

of the form Eq. (2.35) has a logarithmic divergence in the ultraviolet regime, since in this case

$$I_{1,s} \simeq \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \simeq \int \frac{dk}{k} \,, \tag{2.36}$$

and therefore it requires to be regularized (in the same way as we regularized the large energy modes in the Casimir) effect. Physically, this means that our theory Eq. (2.1) is an *effective theory* only valid below a certain energy cutoff, and we cannot expect it to be applicable to arbitrarily large energies (small distances).

A very effective regularization strategy is known as the *dimensional regularization* method, introduced by 't Hooft and Veltman in 1972. This method is based on the realization that if instead of working in $d = 4$ space-time dimensions we assume a generic d-dimensional space-time, then the integral

$$I_{1,s}^{(d)} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + \Delta - i\epsilon)^2} \,, \tag{2.37}$$

is actually convergent for $d < 4$. So the idea of the dimensional regularization method is to evaluate $I_{1,s}^{(d)}$ for those values of d for which it converges, and then analytically continue the result to the physical limit $d \to 4$.[1]

To achieve this goal, the first step is to perform a *Wick rotation* and go to Euclidean signature. The Wick rotation is defined by the transformation $k^0 = ik_E^d$ and $k^i = k_E^i$, such that $k^2 = k_E^2$ where

$$k_E^2 = (k_E^1)^2 + \ldots + (k_E^d)^2 \,, \tag{2.38}$$

and where it can be shown that in general,

$$\int d^dk f\left(k^2 - i\epsilon\right) = i \int d^dk_E f\left(k_E^2\right) \,, \tag{2.39}$$

provided $f(k_E^2) \to 0$ faster than k_E^{-d} as $k_E \to \infty$. After applying the Wick rotation the integral Eq. (2.37) can be expressed as follows using spherical coordinates,

$$I_{1,s}^{(d)} = i \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)^2} = \frac{i}{(2\pi)^d} \int d\Omega_d \int_0^\infty dk_E k_E^{d-1} \frac{1}{(k_E^2 + \Delta)^2} \,, \tag{2.40}$$

where Ω_d is the differential solid angle of the d-dimensional sphere, which is given by

$$\Omega_d \equiv \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,, \tag{2.41}$$

and $\Gamma(x)$ is Euler's Gamma function, see App. B. The computation of the radial integral gives the following result:

$$\int_0^\infty dk_E k_E^{d-1} \frac{1}{(k_E^2 + \Delta)^2} = \Delta^{d/2-2} \Gamma(d/2) \Gamma\left(\frac{4-d}{2}\right) \frac{1}{2\Gamma(2)} \,. \tag{2.42}$$

It is easy to convince yourselves that this integral is only convergent for $d < 4$, and is divergent otherwise.

[1] This integral could also be regularized using the Λ cutoff strategy adopted for the study of the Casimir effect. Actually a wide range of alternative regularization methods exist, all leading to equivalent predictions for physical observables.

The integral Eq. (2.42) is a particular case of the general result

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{\left(k_E^2\right)^a}{\left(k_E^2+\Delta\right)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)} \Delta^{-(b-a-d/2)} \,, \tag{2.43}$$

in the case $a = 0$ and $b = 2$. This result is very useful for general loop calculations in QFT.

Combining the results from angular and radial integrals, we find that the integral that we wanted to compute gives

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \Delta^{d/2-2} \,, \tag{2.44}$$

where we have used that $2\Gamma(2) = 2$. The next step, in preparation for taking the limit $d \to 4$, we define $d \equiv 4 - \epsilon$ and expand around the physical $\epsilon \to 0$ limit, where terms that go to zero in this limit can be neglected, and we find

$$\begin{aligned} \int_0^\infty k_E^{d-1} dk_E \frac{1}{(k_E^2+\Delta)^2} &= \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right)\left(1 - \frac{\epsilon}{2}\ln\Delta + \mathcal{O}(\epsilon)\right) \\ &= \frac{i}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma_E - \ln\Delta + \mathcal{O}(\epsilon)\right) \,, \end{aligned} \tag{2.45}$$

where we have used a number of mathematical identities which are collected in the appendix, and we have Wick-rotated back to Minkowski metric.

Now that we have computed Eq. (2.37), the next step is to perform the integral over the Feynman parameter x, Eq. (2.35). Using Eq. (2.45), we find the following result for the one-loop s-channel matrix element,

$$i\mathcal{M}_{1,s} = \frac{\lambda^2}{2}\left(\frac{-i}{16\pi^2}\right)\int_0^1 dx \left[\ln(p^2 x(1-x)) + \gamma_E - \frac{2}{\epsilon}\right] = -\frac{i\lambda^2}{32\pi^2}\ln\frac{p^2}{\Lambda^2} \,, \tag{2.46}$$

where

$$\ln\Lambda = \frac{1}{\epsilon} + \text{constant} \,, \tag{2.47}$$

and where we have used an definite integral computed in App. B. In the above expression, $\mathcal{O}(\epsilon)$ terms have been neglected. Recall that p^2 is the square of the center-of-mass energy of the scattering particles. This is the final result for the s-channel contribution to the one-loop scattering amplitude in $\lambda\phi^4$ theory, see Fig. 2.3.

At this point we can add together the tree-level and the one-loop contribution to construct the partial $\mathcal{O}(\lambda^2)$ (next-to-leading order, or NLO for short) matrix element for $2 \to 2$ scattering in $\lambda\phi^4$ theory. Note that this is not the full result, since the t- and u-channels are still missing, but as we will show it captures the essential physical features. The final result reads

$$\mathcal{M}_{\rm NLO}(s) = \mathcal{M}_0 + \mathcal{M}_{1,s} = -\lambda - \frac{\lambda^2}{32\pi^2}\ln\left(-\frac{s}{\Lambda^2}\right) \,, \tag{2.48}$$

with $s = -p^2$ being the center-of-mass energy of the $2 \to 2$ scattering. At a first glance, this result does not seem to make much sense, since it is infinite in the $\Lambda \to \infty$ ($\epsilon \to 0$) physical limit. Even the Born approximation becomes meaningless, since the next term in the perturbative expansion appears to be infinitely larger. How can we understand this behaviour?

The origin of the infinity in the NLO scattering amplitude Eq. (2.48) is not too different from that of the Casimir effect discussed in Sect. 2.1, namely extrapolating the theory beyond its range of validity, in this

case the deep ultraviolet regime. We will next show how, despite these infinities, it is possible to make sense of the theory and provide finite well-defined predictions for the physical observables of the theory. The key observation to achieve this is that the coupling constant λ in Eq. (2.48), that is, the Lagrangian parameter, is *not* a physical observable.

2.3 The renormalized coupling constant

While the (partial) NLO amplitude $\mathcal{M}_{\rm NLO}(s)$ Eq. (2.48) seems to be infinite, it is easy to check that the *difference* between two center-of-mass energies s_1 and s_2 is actually finite, since

$$\mathcal{M}_{\rm NLO}(s_1) - \mathcal{M}_{\rm NLO}(s_2) = \frac{\lambda^2}{32\pi^2} \ln \frac{s_2}{s_1} \,. \tag{2.49}$$

In other words, given either a *calculation* or a *measurement* of the amplitude $\mathcal{M}(s_1)$, we can use Eq. (2.49) to predict the behaviour of the scattering amplitude for any other value of s_2 without ever encountering infinities. So the question is: should we expect that $\mathcal{M}(s)$ is finite? It seems so, since after all this amplitude is the building block of the physical cross-sections. But maybe the situation is subtler than that?

The solution of this conundrum lies in the interpretation of the coupling constant λ, that determines the strength of the interactions in the theory. By construction, it is a measure of how strong is the self-interaction of the scalar field in the theory. At the Born level (leading order), the value of λ is determined directly from a measurement of the $\sigma(\phi\phi \to \phi\phi)$ scattering cross-section, recall Eq. (2.24). Now, at NLO the relation between the scattering amplitude and the coupling λ is more involved, since we need to account also for the $\mathcal{O}(\lambda^2)$ terms, see Eq. (2.48).

Let us do the following thing now. It will appear at first a bit arbitrary but its rationale will be soon revealed. We *define* a *renormalized coupling constant* λ_R as $-\mathcal{M}(s_0)$, the value of the $2 \to 2$ scattering amplitude $\sigma(\phi\phi \to \phi\phi)$ for a given fixed center of mass energy s_0. Note that by construction $\lambda_R = \lambda$ at the Born level, but in general the two coupling constants will be different once we account for loop (higher order) effects. From this definition we have that at the one-loop level, see Eq. (2.48), the renormalized coupling is given by

$$\lambda_R \equiv -\mathcal{M}(s_0) = \lambda + \frac{\lambda^2}{32\pi^2} \ln\left(-\frac{s_0}{\Lambda^2}\right) + \mathcal{O}(\lambda^3) \,, \tag{2.50}$$

or in other words, the difference between λ and λ_R is given by

$$\lambda_R - \lambda = \frac{\lambda^2}{32\pi^2} \ln\left(-\frac{s_0}{\Lambda^2}\right) + \mathcal{O}(\lambda^3) \,. \tag{2.51}$$

This equation relates λ, the parameter that appears in the Lagrangian of the theory, with λ_R, the measured value of the $\phi\phi \to \phi\phi$ scattering amplitude at a reference center of mass energy s_0. This identity can also be reversed to give

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln\left(-\frac{s_0}{\Lambda^2}\right) + \mathcal{O}(\lambda_R^3) \,, \tag{2.52}$$

where note that we have used that $\lambda = \lambda_R + \mathcal{O}\left(\lambda_R^2\right)$, Eq. (2.51).

So far, we do not seem to have made much progress: after-all, the relation Eq. (2.50) still is ill-defined, since we need to take $\Lambda \to \infty$ in the physical limit. However, the situation changes once we write the scattering matrix element in terms of λ_R. Indeed, of express the NLO matrix element $\mathcal{M}_{\rm NLO}(s)$, Eq. (2.48),

in terms of the renormalized coupling λ_R. What we find in this case is that

$$\mathcal{M}_{\rm NLO}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s}{s_0}\right) + \mathcal{O}(\lambda_R^3)\,, \tag{2.53}$$

which is not only perfectly finite in the physical $\Lambda \to \infty$ limit, but in addition gives a experimentally testable prediction of how the scattering amplitude s depends with the center-of-mass energy of the collision, given a measurement of the same quantity at a reference center-of-mass-energy s_0.

So where have all the infinities gone? The key point here is that the parameter λ in the theory Lagrangian, Eq. (2.1), is *not an observable quantity* once we account for genuinely quantum (loop) effects. If we express our predictions in terms only of measurable quantities, such as λ_R, the value of the scattering amplitude at s_0, then the results turn out to be well-behaved. Eq. (2.53) illustrates the renormalization paradigm of quantum field theories, where UV infinities can be reabsorbed into a redefinition of the *bare* parameters of the Lagrangian. If you feel that this is some kind of dirty trick you are not alone: some of the founding fathers of quantum theory shared the same feeling when presented with the renormalization program. In the following, we study in some more details how the process works in more generality.

2.4 Absorbing infinities: counterterms

The starting point of this whole discussion was the Lagrangian of $\lambda\phi^4$ theory, Eq. (2.1). We found that if we compute amplitudes in terms of the coupling constant λ that appears in the Lagrangian, beyond the Born level we get unphysical (infinite) results, Eq. (2.48). On the other hand, we found that if the same amplitude if expressed in terms of the renormalized coupling constant λ_R, which is defined in terms of a physical cross-section (the value of the $\phi\phi \to \phi\phi$ scattering amplitude at $s = s_0$) then the amplitude is perfectly well defined, Eq. (2.53).

This observation suggest that in general, in order to tame the infinities that arise in QFT loop calculations, the following approach seems advisable:

(a) write down the Lagrangian of the theory in terms of the physical couplings and other observable parameters, rather than the bare ones; and then

(b) remove the UV divergences by adding additional terms to the Lagrangian, known as the *counter-terms*.

Coupling constant renormalization. In order to illustrate this strategy, let us consider the following redefinitions of the scalar field and the coupling constant in the Lagrangian of the $\lambda\phi^4$ theory:

$$\phi \to Z_\phi^{1/2}\phi\,, \tag{2.54}$$

$$\lambda \equiv \lambda_R Z_4\,, \tag{2.55}$$

so that in terms in terms of the renormalized field and coupling the Lagrangian of the theory now reads

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial_\mu\phi\partial^\mu\phi - \frac{\lambda_R}{4!} Z_\lambda \phi^4\,, \tag{2.56}$$

with $Z_\lambda \equiv Z_\phi^2 Z_4$. In Eq. (2.56) the terms Z_ϕ and Z_λ are general functions of the renormalized coupling λ_R, the energy scale s, and the cutoff of the theory Λ (or what is the same, of the regulator ϵ). That is, in

general we have that

$$Z = Z(\lambda_R, s, \Lambda)\,. \tag{2.57}$$

These terms are called technically *counterterms*, and their role is to make sure that the theory gives finite predictions for the amplitude and other observables. To ensure a smooth matching with the tree-level (Born) results, we should have that

$$Z_\phi = 1 + \mathcal{O}(\lambda_R)\,, \tag{2.58}$$

$$Z_\lambda = 1 + \xi_\lambda \lambda_R + \mathcal{O}(\lambda_R^2)\,, \tag{2.59}$$

where the coefficients of these λ_R expansions can be fixed by explicit loop calculations (as well as by more general methods).

It is easy to check that if one repeats the one-loop calculation presented in Sect. 2.2, but using the Lagrangian Eq. (2.56), one gets the following result,

$$\mathcal{M}_{\rm NLO} = -\lambda_R Z_\lambda - \frac{\lambda_R^2 Z_\lambda}{32\pi^2} \ln\left(-\frac{s^2}{\Lambda^2}\right)\,, \tag{2.60}$$

by comparing with Eq. (2.48). Note that by using Eq. (2.59) we can express the above equation as

$$\mathcal{M} = -\lambda_R\left(1 + \xi_\lambda \lambda_R\right) - \frac{\lambda_R^2 (1 + \xi_\lambda \lambda_R)}{32\pi^2} \ln\left(-\frac{s^2}{\Lambda^2}\right)\,. \tag{2.61}$$

Next, since we have *defined* the renormalized coupling as $\mathcal{M}(s_0) = -\lambda_R$, the above equation can be expressed as

$$\mathcal{M}(s_0) = -\lambda_R\left(1 + \xi_\lambda \lambda_R\right) - \frac{\lambda_R^2}{32\pi^2} \ln\left(-\frac{s_0^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^3\right) = -\lambda_R\,, \tag{2.62}$$

which requiring the cancellation of the coefficient of the $\mathcal{O}\left(\lambda_R^2\right)$ term gives

$$Z_\lambda = 1 - \frac{\lambda_R}{32\pi^2} \ln\left(-\frac{s_0^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^2\right)\,. \tag{2.63}$$

Therefore, we have determined that that renormalizing the bare fields and coupling constants of the $\lambda\phi^4$ theory as indicated in Eq. (2.56), one automatically obtains finite results at the one-loop order provided that the coupling constant counterterm is taken to be Eq. (2.63).

Note that the counterterm is also infinite in the physical limit $\Lambda \to \infty$, but this has no impact on observable quantities since these turn out to be independent of Λ. We also note here that that to cure the ultraviolet divergence that appears in $\phi\phi \to \phi\phi$ scattering we do not need the scalar field renormalization counter-term Z_ϕ, at least at the one-loop level, but this will be eventually necessary for other processes.

While here we have only sketched the surface of a rather complex procedure, it is enough to understand that renormalization is a program that systematically makes observables finite which gives testable predictions about the IR physics despite the UV divergences. The core idea is that in renormalizable QFT physical observables are well-defined once a finite number of counterterms are introduced, whose values in turn are fixed by a finite number of measurements. Also note that loops can produce behaviour which are completely different from anything that can be generated at tree level. In particular, non-analytic behaviour such as $\ln(s/s_0)$ is characteristic of loops, while tree-level amplitudes are always rational polynomials.

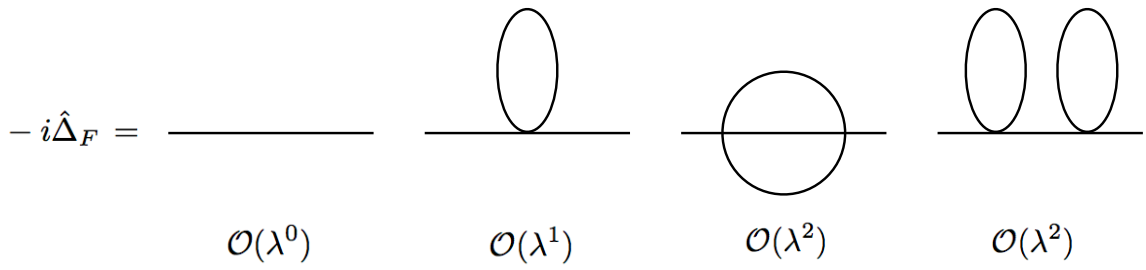


Figure 2.4. Some of the the Feynman diagrams that contribute to the calculation of the propagator $\mathbf{\Delta}_F$ in the interacting $\lambda\phi^4$ theory up to $\mathcal{O}\left(\lambda^2\right)$.

Mass renormalization and the renormalized propagator. Another important aspect of the renormalization program is the renormalization of the free-particle propagator. Similarly to the case of the scattering amplitudes, this will lead to the realization that the bare mass m of the Lagrangian is not a physical observable, and we will introduce a better defined *renormalized mass*.

Our starting point to study the renormalized propagator is the Lagrangian of the $\lambda\phi^4$ theory before adding the counterterms, this time accounting for the mass of the scalar field:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,, \tag{2.64}$$

and now try to compute the two-point function $\phi \to \phi$, in other words, the scalar field propagator. Recall the definition of the Feynman propagator:

$$\Delta_F(x_1 - x_2) = i\left\langle T\left(\phi(x_1)\phi(x_2)\right)\right\rangle_{(0)} \,, \tag{2.65}$$

where the subindex (0) indicates that the expectation value should be computed using the free-theory states, rather than those of the interacting theory, and where T indicates time ordering. Once we account for the self-interactions of the scalar field, the two point function will be modified, leading to an *interacting propagator* $\mathbf{\Delta}_F$ given by

$$\mathbf{\Delta}_F(x_1 - x_2) \equiv i\left\langle T\left(\phi(x_1)\phi(x_2)\right)\right\rangle \,. \tag{2.66}$$

Our goal here is to evaluate the interacting propagator, but before that it is useful to understand its structure in perturbation theory. The propagator in the interacting theory will be determined by computing those diagrams in Fig. 2.4, where we show some of the diagrams that contribute to the result up to $\mathcal{O}\left(\lambda^2\right)$. As illustrated by these diagrams, the self-coupling of the scalar field ϕ affects how this same field propagates in space-time.

An alternative way to represent the perturbative expansion of the propagator in Fig. 2.4 is achieved by reorganizing it in terms of *one-particle-irreducible* (1PI) diagrams, which are defined by those diagrams that cannot be reduced (by cutting) in terms of diagrams with two or more propagators.[2] As shown in Fig. 2.5, the propagator is the sum of the free theory terms, plus all 1PI diagrams, plus all diagrams that contain 2 1PI contributions, and so on. Note that the 1PI diagrams cannot be associated a unique power of the coupling λ, since in general they receive contributions to all orders.

Furthermore, from the Feynman rules of the theory, one observes that the dependence on the momenta

[2] In other words, a diagram is 1PI if it is still connected after any one line is cut.

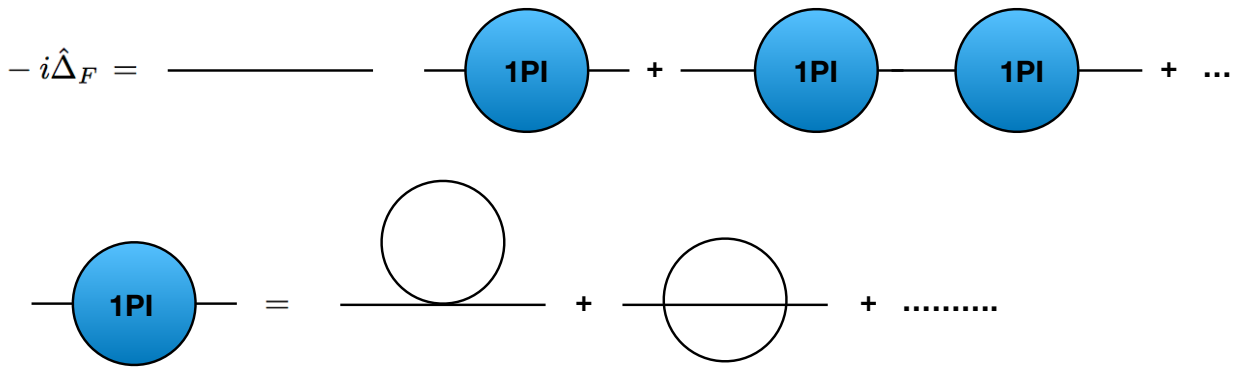


Figure 2.5. The perturbative expansion of the interacting propagator $\mathbf{\Delta}_F$ in Fig. 2.4 can also be organized in terms of diagrams with different number of 1PI diagrams.

k of the various pieces shown in Fig. 2.5 will be the following:

- free theory: $\frac{1}{i}\Delta_F(k^2)$,
- 1PI diagrams: $\frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2)$,
- diagrams with 2 1PI contributions: $\frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2)$,

and so on, where $i\Sigma(k^2)$ is known as the *self-energy* contribution and contains the effects of all 1PI diagrams to all powers of λ without the propagators from the two external lines.

The reason to organize the perturbative expansion of the propagator of $\lambda\phi^4$ theory as illustrated in Fig. 2.5 is that the infinite series

$$\frac{1}{i}\mathbf{\Delta}_F(k^2) = \frac{1}{i}\Delta_F(k^2) + \frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2) + \frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2)i\Sigma(k^2)\frac{1}{i}\Delta_F(k^2) + \ldots\,, \tag{2.67}$$

has the form of a geometric series and therefore can be summed to all orders, assuming of course that the perturbative expansion is convergent. By doing this, we end up with the following expression of the interacting propagator expressed in terms of the self-energy $\Sigma(k^2)$ of the scalar field:

$$\mathbf{\Delta}_F(k^2) = \frac{1}{\left(\Delta_F(k^2)/i\right)^{-1} - i\Sigma(k^2)} = \frac{1}{k^2 + m^2 - i\epsilon - \Sigma(k^2)}\,, \tag{2.68}$$

where we have used the expression for the free-theory propagator

$$\Delta_F(k^2) = \frac{i}{k^2 + m^2 - i\epsilon}\,, \tag{2.69}$$

which has a pole at $k^2 = -m^2$, with m being the bare mass. From Eq. (2.68), we observe that the net effect of the loop corrections (to all orders) in the propagator of $\lambda\phi^4$ theory is a redefinition of the effective mass of the propagating particles, by an amount which is determined by the self-energy $\Sigma(k^2)$ which in turn is computed from the 1PI diagrams.[3]

At this point we see how the classical and QFT definitions of a particle mass start to diverge. At the classical level, the *physical mass* m_p of a particle is simply $m_p = m$, the parameter of the Lagrangian.

[3]Since the self-energy is in general complex, loop corrections will modify not only the physical mass but also the *width* of the propagating particles.

However from Eq. (2.68) we see that a more robust definition of m_p is the solution $k^2 = -m_p^2$ of

$$k^2 + m^2 - \Sigma(k^2) = 0\,, \tag{2.70}$$

namely the pole of the all-orders propagator. In other words, m_p becomes dependent on the perturbative order at which we are working (just as in the case of the renormalized coupling). In addition, this implies that at the quantum level the physical mass $m_p(k^2)$ in general depends on the value of k^2, rather than being fixed as in the classical theory, just as the renormalized coupling depends on s. To be a bit more precise, $k^2 + m^2 - \mathrm{Re}(\Sigma(k^2)) = 0$ determines the physical mass of the particle m_p, while $\mathrm{Im}(\Sigma(k^2))$ determines the its width Γ_p, which fixes the decay rate of the particle. For stable particles, $\Gamma_p = 0$.

In order to determine the quantum corrections to the mass in $\lambda\phi^4$ theory (analogously as that we have done for the quantum corrections to λ from the scattering amplitude), we need to evaluate the one-loop correction to the self-energy $\Sigma(k^2)$. From the Feynman diagram in Fig. 2.5, we see that the one-loop correction to the self-energy is given by

$$i\Sigma(k^2) = -\frac{i\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{i}\frac{1}{p^2+m^2-i\epsilon} + \mathcal{O}(\lambda^2)\,, \tag{2.71}$$

where we have used the fact that the coupling constant λ does not depend on the energy k at this perturbative order, or in other words, that $\lambda - \lambda_R = \mathcal{O}\left(\lambda^2\right)$. This integral is quadratically divergent: to see this, note that in the UV limit one has that

$$i\Sigma(k^2) \simeq -\frac{i\lambda}{2}\int \frac{p^3 dp}{p^2} \simeq \lambda \int p\, dp \to \infty\,. \tag{2.72}$$

However, we are now more confident that we can make sense of this apparently unphysical result by using a similar renormalization strategy as in the case of the coupling constant.

Therefore let us try to compute the integral Eq. (2.71) using dimensional regularization. Computing the integral in d Euclidean dimensions after a Wick rotation, we find that

$$\int \frac{d^d p_E}{(2\pi)^d}\frac{1}{p_E^2+m^2} = \Gamma\left(1-\frac{d}{2}\right)(m^2)^{d/2-1}\frac{1}{(4\pi)^{d/2}} \tag{2.73}$$

$$= \frac{1}{(4\pi)^{d/2}}\left(-\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right)\left(m^2 - \epsilon\frac{m^2}{2}\ln m^2 + O(\epsilon^2)\right)\,, \tag{2.74}$$

where in the second step as usual we have used that $d = 4-\epsilon$, together with some of the properties of the Gamma function collected in App. B. Defining the UV cutoff as $\ln\Lambda \equiv 2/\epsilon + \gamma_E$, we have that the one-loop expression for the scalar field self-energy in $\lambda\phi^4$ theory is

$$\Sigma(k^2) = \lambda\frac{m^2}{32\pi^2}\ln\left(\frac{m^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda^2\right)\,. \tag{2.75}$$

The fact that the self-energy is divergent in the physical $\Lambda \to \infty$ limit is now something that does not worry us, since we know that the bare mass m is not an observable quantity. The relevant physical requirement is

that the physical mass (the pole of the all-orders propagator) is finite, and therefore we need to impose that

$$k^2 + m^2 - \mathrm{Re}\left(\Sigma(k^2)\right) = 0\,, \tag{2.76}$$

when $k^2 = -m_p^2$, is finite as well. In this case we find that the Lagrangian mass m can be expressed in terms of the physical mass m_p and the renormalized coupling λ_R as follows

$$m^2 = m_p^2 \left(1 + \frac{\lambda_R}{32\pi^2} \ln\left(\frac{m_p}{\Lambda^2}\right) + \mathcal{O}(\lambda_R^2)\right)\,, \tag{2.77}$$

where we have used the mass-shell condition $k^2 = -m^2$. Note that as in the case of the renormalized coupling, one has $m - m_p = \mathcal{O}\left(\lambda_R\right)$. In general, when we compute the self-energy $\Sigma(k^2)$ in perturbation theory, its structure will have the form

$$\Sigma(k^2) = Ak^2 + Bm^2\,, \tag{2.78}$$

where A and B are functions of the parameters of the theory as well as of the relevant scales such as λ_R, Λ, m_p^2 and k.

The renormalized Lagrangian. The formal way to regulate the divergences that arise in the two point function is the same as in the case of the coupling constant, namely one needs to add suitable counterterms to the Lagrangian of the theory

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} Z_m m_p^2 \phi^2 - \frac{\lambda_R}{4!} Z_\lambda \phi^4\,, \tag{2.79}$$

which is the same as Eq. (2.56) with the additional mass counterterm Z_m. This Lagrangian can also be written as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\rm bare} + \mathcal{L}_{\rm ct}\,, \\ \mathcal{L}_{\rm bare} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_p^2\phi^2 - \frac{\lambda_R}{4!}\phi^4\,, \\ \mathcal{L}_{\rm ct} &= -\frac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(Z_m - 1)m_p^2\phi^2 - \frac{\lambda_R}{4!}(Z_\lambda - 1)\phi^4\,, \end{aligned} \tag{2.80}$$

explicitly separating the bare terms from the counterterms, which vanish at the Born level where $Z = 1$.

From the renormalized Lagrangian Eq. (2.80) one can derive a new set of Feynman rules from which finite observables will be obtained by construction. These Feynman rules are collected in Fig. (2.6), where now the scalar field propagator is expressed in terms of the physical mass

$$\frac{-i}{k^2 + m_p^2 - i\epsilon}\,, \tag{2.81}$$

and there are three interaction terms: two four-point interactions, with couplings $-\lambda_R$ and $-\lambda_R\left(Z_\lambda - 1\right)$ respectively, and a two-point interaction vertex with an associated Feynman rule

$$-\,i\left(Z_\phi - 1\right)k^2 - i\left(Z_m - 1\right)m_p^2\,. \tag{2.82}$$

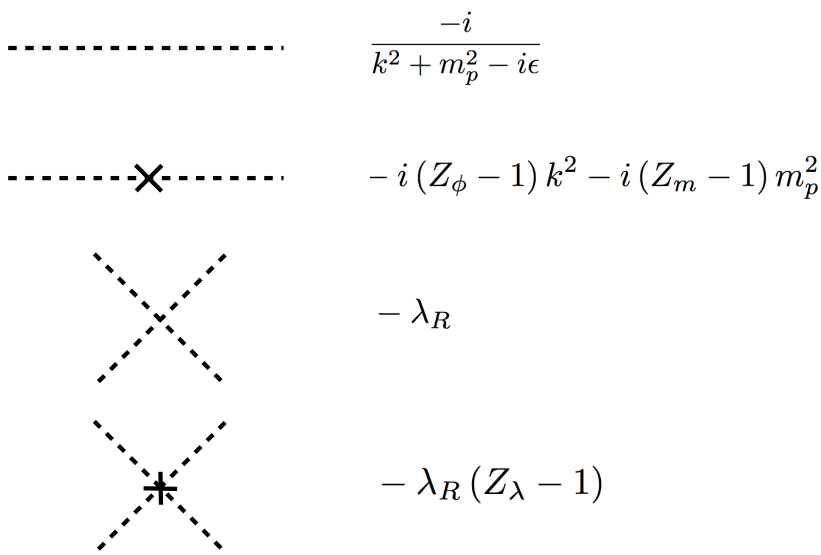


Figure 2.6. The modified Feynman rules of $\lambda\phi^4$ theory once the bare Lagrangian is supplemented by the counterterms required to regulate the UV divergences, see Eq. (2.80). The second term should be understood as an interaction vertex term.

The main advantage of this formalism is that once the expressions for the counterterms Z_i is known at any given order in perturbation theory, any physical quantity computed from Eq. (2.80) will automatically lead to well-defined, finite results.

To summarize, arbitrary physical predictions in $\lambda\phi^4$ will be finite and well-defined once we determine the three counterterms Z_ϕ, Z_m and Z_λ at a given order in perturbation theory. These counterterms must be fixed by requiring the following conditions:

- Scattering amplitudes should be finite, *e. g.* $\mathcal{M}(s_0) = -\lambda_R$
- The interacting propagator $\mathbf{\Delta}_F$ has a pole at the physical mass m_p with residue one, which in turn implies

$$\begin{aligned} \Sigma(k^2=-m_p^2) &= 0\,, \\ \frac{\partial}{\partial k^2}\Sigma(k^2=-m_p^2) &= 1\,. \end{aligned} \tag{2.83}$$

Imposing these conditions, one finds that the $\mathcal{O}(\lambda_R)$ expressions for the counterterms are

$$\begin{aligned} Z_\phi &= 1+\mathcal{O}(\lambda_R^2)\,, \\ Z_m &= 1+\frac{\lambda_R}{32\pi^2}\ln\left(\frac{m_p^2}{\Lambda^2}\right)+\mathcal{O}(\lambda_R^2)\,, \\ Z_\lambda &= 1-3\frac{\lambda_R}{32\pi^2}\ln\left(\frac{-s}{\Lambda^2}\right)+\mathcal{O}(\lambda_R^2)\,, \end{aligned} \tag{2.84}$$

where the factor 3 in the last equation arises from other one-loop diagrams beyond the s-channel one, as will be shown below.

2.5 Complete renormalization of the $2 \to 2$ scattering amplitude.

We can now complete the program that we have initiated in the previous sections, and derive the full renormalization of the $\lambda\phi^4$ at one-loop. Specifically, this requires evaluating the remaining diagrams (beyond the s-channel one) that contribute to the $\phi\phi \to \phi\phi$ scattering at $\mathcal{O}(\lambda_R^2)$. To this purpose, we go back to the massless $m = 0$ limit. To compute the full $\mathcal{M}_1$ scattering amplitude, we need to evaluate all the Feynman diagrams shown in Fig. 2.3. It is easy to check that one should find:

$$\begin{aligned} i\mathcal{M}_1 &= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^2} \left[\frac{1}{i} \frac{1}{k^2 - i\epsilon} \frac{1}{i} \frac{1}{(p_1 + p_2 - k)^2 - i\epsilon} \right. \\ &+ \frac{1}{i} \frac{1}{k^2 - i\epsilon} \frac{1}{i} \frac{1}{(p_1 + p_3 - k)^2 - i\epsilon} \\ &+ \left. \frac{1}{i} \frac{1}{k^2 - i\epsilon} \frac{1}{i} \frac{1}{(p_1 + p_4 - k)^2 - i\epsilon} \right] . \end{aligned} \tag{2.85}$$

Now, following the same strategy as the one used for the case of the s-channel diagram, we can use Feynman parameters and dimensional regularization to compute this integral, and then one finds

$$i\mathcal{M}_1 = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left[\frac{1}{(k^2 - sx(1-x) - i\epsilon)} + \frac{1}{(k^2 - tx(1-x) - i\epsilon)} + \frac{1}{(k^2 - ux(1-x) - i\epsilon)} \right] , \tag{2.86}$$

which upon perform the integrations over x and k leads to

$$i\mathcal{M}_1 = -\frac{i\lambda^2}{32\pi^2} \left[\ln\left(-\frac{s}{\Lambda^2}\right) + \ln\left(-\frac{t}{\Lambda^2}\right) + \ln\left(-\frac{u}{\Lambda^2}\right) \right] , \tag{2.87}$$

where we have used the definition of the Mandelstam variables[4] $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_3)^2$ and $u = -(p_1 - p_4)^2$.

The amplitude Eq. (2.87) is divergent, as we know, and thus needs to be renormalized using the same strategy as for $\mathcal{M}_{1,s}$. The main difference is that now there is additional additional kinematic freedom, and thus we need to specify reference values for two energy scales, in particular for s_0 and t_0, from where u_0 follows from four-momentum conservation ($s + t + u = 4m^2$).[5] Taking into account these considerations, we can then *define* the renormalized coupling by the condition $\mathcal{M}(s_0, t_0, u_0) \equiv -\lambda_R$, which then leads to

$$\lambda = \lambda_R - 3\frac{\lambda_R^2}{32\pi^2} \ln\left(-\frac{\mu^2}{\Lambda^2}\right) , \tag{2.88}$$

where μ is a reference energy scale which assures the consistent definition of λ_R for a given scattering center of mass energy and angle. The factor 3 in Eq. (2.88) reflect the fact that we have regulated three separate divergences, one for each channel. Hence the coupling constant counterterm Z_λ will be given by

$$Z_\lambda = 1 - \frac{3\lambda_R}{32\pi^2} \ln\left(-\frac{\mu^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^2\right) . \tag{2.89}$$

The corresponding calculation in the case of a non-zero mass $m \neq 0$ of the scalar fields proceeds anal-

[4] Recall that we use here the $(-,+,+,+)$ signature of the Minkowski metric.
[5] Equivalently, one can define λ_R in terms of a given energy and scattering angle references.

ogously but is a little more involved. It is an interesting exercise to verify that in this case the one-loop scattering amplitude Eq. (2.87) read

$$\mathcal{M}_1 = -\frac{\lambda^2}{32\pi^2}\left[\ln\left(\frac{m^2}{\Lambda^2}\right) - \sqrt{1-\frac{4m^2}{s}}\ln\left(\frac{\sqrt{1-4m^2/s}+1}{\sqrt{1-4m^2/s}-1}\right)\right] . \tag{2.90}$$

Note that the scattering amplitude has only sense for $s \geq s_0 = 4m^2$, the two-particle production threshold. So for $s \geq s_0$, the amplitude $\mathcal{M}_1$ is real, while for smaller center-of-mass energies it develops an imaginary piece. Remarkably, it can be shown that this non-trivial analytic structure is fixed by the unitarity and causality requirements.

2.6 The renormalization group equations

One of the most important consequences of the renormalization procedure is that quantities such as the mass or the coupling constants are not fixed parameters anymore, but rather functions of the energy of the process we are considering. For instance, in the case of the scattering amplitude $\mathcal{M}_{\rm NLO}(s)$, we have seen that once it is measured at $s = s_0$, then its dependence for any other value of s is determined by perturbation theory. Here we formalize a bit more these ideas by presenting the *renormalization group equations*, a central construction in QFTs which allows to relate physical quantities at different scales.

Let us start with the renormalized NLO scattering amplitude in $\lambda\phi^4$ theory, Eq. (2.53),

$$\mathcal{M}_{\rm NLO}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{s_0} + \mathcal{O}\left(\lambda_R^3\right) , \tag{2.91}$$

where the renormalized coupling had been defined by the condition $\mathcal{M}(s_0) = -\lambda_R$. Since the specific choice of s_0 is ultimately arbitrary, physical observables must be independent of this value of s_0. Therefore, from general considerations one must require that the one-loop scattering amplitude is independent of s_0,

$$\frac{d\mathcal{M}_{\rm NLO}(s)}{ds_0} = 0 , \tag{2.92}$$

a condition which leads to the so-called renormalization group equations (RGE), for reasons that will become apparent soon.

Let us derive what are the consequences of the invariance under the choice of s_0 imposed by Eq. (2.92). For this, we note that $\mathcal{M}_{\rm NLO}(s)$ in Eq. (2.91) has both explicit dependence on s_0 as well as an implicit dependence via the renormalized coupling, since recall that from Eq. (2.50) we had that

$$\lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}\ln\left(-\frac{s_0}{\Lambda^2}\right) + \mathcal{O}(\lambda^3) , \tag{2.93}$$

thus also with a dependence on s_0. Taking into account both the explicit and the implicit dependences of Eq. (2.91) on s_0, we find that the invariance requirement Eq. (2.92) implies that that

$$\frac{d}{ds_0}\mathcal{M}_{\rm NLO}(s) = \frac{d}{ds_0}\left(\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{s_0}\right) = 0 , \tag{2.94}$$

and using the chain rule to account the implicit dependence of λ_R in s_0 we find that

$$\frac{d\lambda_R}{ds_0} \cdot \left(1 + \frac{\lambda_R}{16\pi^2} \ln \frac{s}{s_0}\right) - \frac{\lambda_R^2}{32\pi^2} \frac{1}{s_0} = 0 \,. \tag{2.95}$$

Then rearranging terms, we obtain that the invariance of $\mathcal{M}_{\rm NLO}$ with respect to the arbitrary choice for s_0 leads to the condition:

$$\frac{d\lambda_R}{ds_0} = \frac{\lambda_R^2}{32\pi^2} \frac{1}{s_0} \left(1 + \frac{\lambda_R}{16\pi^2} \ln \frac{s}{s_0}\right)^{-1} = \frac{\lambda_R^2}{32\pi^2} \frac{1}{s_0} + \mathcal{O}\left(\lambda_R^3\right) \,, \tag{2.96}$$

where as usual we can neglect terms which are beyond the perturbative order that we are computing. Eq. (2.96) determines how λ_R must depend on s_0 in order that physical quantities such as the $\mathcal{M}_{\rm NLO}$ amplitude do not depend on this arbitrary choice. In other words, Eq. (2.96) fixes the *scale dependence* of the renormalized coupling constant.

Therefore, the differential equation Eq. (2.96) can be solved to determine the dependence on s_0 of the renormalized coupling constant. Since s_0 is here just a dummy variable, let us call it s and solve

$$\frac{d\lambda_R(s)}{ds} = \frac{\lambda_R^2(s)}{32\pi^2 s} \,, \tag{2.97}$$

subject to the boundary condition that $\lambda_R(s = s_0)$ coincides with the value of $-\mathcal{M}(s = s_0)$. The solution of this differential equation gives

$$\frac{1}{\lambda_R(s_0)} - \frac{1}{\lambda_R(s)} = \frac{1}{32\pi^2} \ln \frac{s}{s_0} \,, \tag{2.98}$$

which can be rearranged to give

$$\lambda_R(s) = \frac{\lambda_R(s_0)}{1 - \frac{\lambda_R(s_0)}{32\pi^2} \ln \frac{s}{s_0}} \,, \tag{2.99}$$

which is the sought-for expression of the scale dependence of $\lambda_R(s)$ as a function of $\lambda_R(s_0)$, which recall it is an observable quantity, since it is identified with a measurement of the $2 \to 2$ scattering amplitude. Since this dependence is determined entirely by the outcome of the renormalization procedure, Eq. (2.97) is called a *renormalization group equation*.

Remarkably, the solution of the RGE equation for $\lambda_R(s)$, Eq. (2.99), can also be expanded in Taylor series to give

$$\lambda_R(s) = \lambda_R(s_0) \sum_{n=0}^{\infty} \left(\frac{\lambda_R(s_0)}{32\pi} \ln \frac{s}{s_0}\right)^n = \lambda_R(s_0) + \frac{\lambda_R^2(s_0)}{32\pi} \ln \frac{s}{s_0} + \ldots \,, \tag{2.100}$$

and therefore we see that the renormalization group equations resum *to all orders* in the perturbative expansion of $\lambda_R(s)$ a specific class of terms, specifically those terms that are logarithmically enhanced in s. To understand why this is quite powerful, not that with a one-loop calculation we have determined the complete tower of *leading logarithmic terms* to all orders in perturbation theory. On the other hand, the full expansion of $\lambda_R(s)$ will also contain terms of the form $\lambda_R^k (\ln s/s_0)^{k-2}$, and to determine these we would need to perform a two-loop calculation, and so on.

The derivative of the renormalized coupling with respect to the energy scale,

$$\frac{d\lambda_R(s)}{ds} \equiv \beta(\lambda_R) \,, \tag{2.101}$$

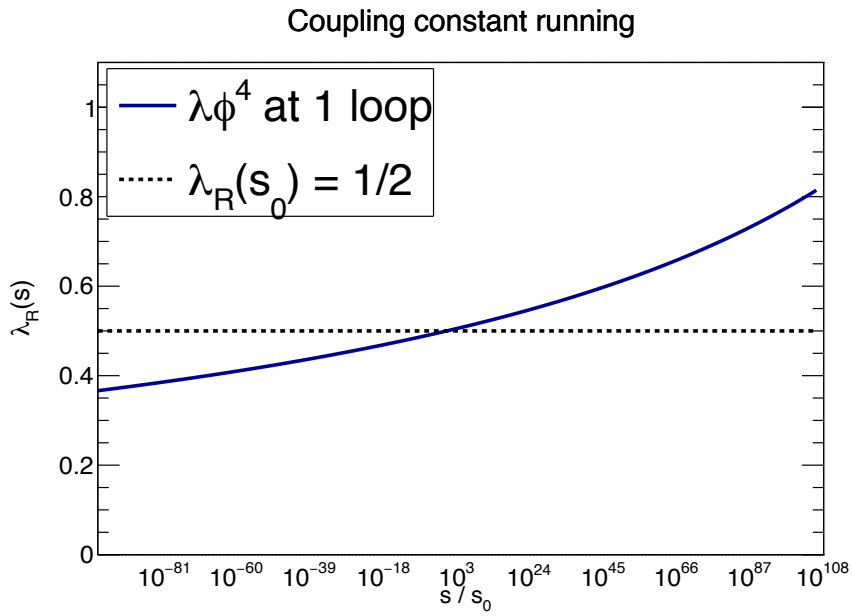


Figure 2.7. The one-loop running of the renormalized coupling constant $\lambda_R(s)$ in $\lambda\phi^4$ theory, Eq. (2.99), as a function of the scale ratio s/s_0. The reference value of the coupling constant has been taken to be $\lambda_R(s_0) = 1/2$.

which determines the running of λ_R with the energy s is called the *beta function* of the theory. The beta function $\beta(\lambda_R)$ can be computed in perturbation theory, with each term providing a full tower of logarithmic terms to all orders. In the specific case of $\lambda\phi^4$ theory, the solution of the RGE for its coupling constant Eq. (2.99) indicates that $\lambda_R(s)$ increases with the energy, in other words, perturbation theory will work better at low scales than at high scales. In this respect there are two interesting limiting behaviors of Eq. (2.99):

(a) For very small energies as compared to the reference value, $s \ll s_0$, then $\lambda_R(s) \to 0$ and the theory becomes effectively free (non-interacting). So in this limit all scattering cross-sections vanish.

(b) For very high energies as compared to the reference value, $s \gg s_0$, the coupling constant increases so perturbation theory will converge more slowly. If we keep increasing the energy, at some finite value given by

$$s = s_0 \times \exp\left(\frac{32\pi^2}{\lambda_R(s_0)}\right) , \tag{2.102}$$

the renormalized coupling becomes *infinite*. This is a sign that we are reaching the non-perturbative regime, where the perturbative description of the theory is not valid anymore. The value of Eq. (2.102) is known as the *Landau pole* of the theory.

In Fig. 2.7 we show the one-loop running of the renormalized coupling constant $\lambda_R(s)$ in $\lambda\phi^4$ theory, Eq. (2.99), as a function of the scale ratio s/s_0. The reference value of the coupling constant has been taken to be $\lambda_R(s_0) = 1/2$. We observe how for $s \gg s_0$ the coupling constant increases, reaching eventually the Landau pole where $\lambda_R(s) \to \infty$, and for $s \ll s_0$ the coupling constant becomes actually smaller. This is one of the most striking features of quantum field theories: the strength of a given interaction is not just a fixed number, but inherently a function of the typical energy scale at which we are applying the theory. As we have discussed, this is a generic consequence of the renormalization procedure of the theory.

From Fig. 2.7 we also see that the coupling in $\lambda\phi^4$ theory runs with the energy but in an extremely slow way: to increase the value of $\lambda_R(s)$ from its reference value of 0.5 up to 0.8 one needs to increase the energy by more than 100 orders of magnitude. This very slow running of the coupling constant of $\lambda\phi^4$ is also visible if we compute the value of the Landau pole of the theory, Eq. (2.102): for $\lambda_R(s_0) = 0.5$, we find that $s/s_0 \simeq 10^{274}$ - so we can be confident that this theory can always be treated with perturbation theory. Note that this is not a generic behaviour of all QFTs: some run much faster, and others, such as QCD, exhibit the inverse behaviour where the coupling becomes stronger as we decrease the energy scale s.

2.7 Formal derivation of the renormalization group equations

In Sect. 2.6 we found that imposing that physical observables, such as the $2 \to 2$ scattering amplitudes, should be independent of arbitrary choices such as the reference value of the energy s_0, one can derive renormalization group equations (RGEs) that *determine the energy dependence* of the renormalized coupling constants.

In this section we develop a more general framework to write down RGEs, based on the dimensional regularization method. The basic observation here is that the logarithms resummed by the RG are associated with ultraviolet divergences. The renormalization of these divergences introduces an *arbitrary scale* μ into physical predictions, which should cancel in the all-order calculation. The RGE equations will then arise from requiring the independent of physical observables on the arbitrary scale μ, which in general is referred to as the *renormalization scale*.

As in the rest of this part of the course, we work with the $\lambda\phi^4$ theory defined by the Lagrangian Eq. (2.64). As we have demonstrated, the bare parameters in the Lagrangian as formally UV divergent in $d = 4$, recall Eq. (2.52) for the coupling constant and the fact that λ_R is a physical observable (and thus finite). We have shown that these infinities can be tamed using dimensional regularization, that is, by working in $d = 4 - \epsilon$ space-time dimensions. Recall that the mass dimensions of the fields and parameters that appear in the Lagrangian are in d space-time dimensions

$$[\phi] = \frac{d-2}{2}\,, \qquad [m] = 1\,, \qquad [\lambda] = 4 - d\,, \tag{2.103}$$

given that $[\mathcal{L}] = d$ and $[\partial_\mu] = 1$, since in natural units the action $S \sim \int d^4x\mathcal{L}$ is dimensionless. In particular, the coupling constant λ is not dimensionless anymore in $d \neq 4$ dimensions.

In the renormalized theory, once we have introduce counterterms as explained in Sect. 2.4, the Lagrangian is expressed in terms of physical parameters (related directly to measurable quantities) and of renormalized fields. In order to formally derive general RGEs, we require two conditions:

- to be able to expand any quantities in the renormalized theory in terms of a *dimensionless parameter*, and
- keep the fields with canonical dimension, irrespective of the value of d.

To achieve this two requirements, a convenient rescaling is given by:

$$\phi \to \sqrt{Z_\phi}\phi\,, \qquad m_p = \frac{1}{\sqrt{Z_1}}m\,, \qquad \lambda_R = \frac{1}{Z_4}\mu^{d-4}\lambda\,. \tag{2.104}$$

In Eq. (2.104), the scale μ has been introduced in order to keep the rescaled coupling λ_p dimensionless in any number of dimensions, and where the counterterms themselves, Z_ϕ, Z_1, and Z_4 are dimensionless. If we now express the Lagrangian of $\lambda\phi^4$ theory, Eq. (2.64), in terms of the renormalized fields defined in Eq. (2.104), we find that in d space-time dimensions

$$\mathcal{L} = -\frac{1}{2}Z_\phi \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_p^2 Z_m \phi^2 - \frac{\lambda_R}{4!}Z_\lambda \mu^{4-d}\phi^4 \,, \tag{2.105}$$

where we have defined the following counterterm combinations

$$Z_m \equiv Z_1 Z_\phi \,, \qquad Z_\lambda = Z_4 Z_\phi^2 \,. \tag{2.106}$$

An important feature of the rescaled Lagrangian Eq. (2.105) is that both the physical coupling λ_R and the counterterms Z_i are *dimensionless* irrespective of the specific value of d.

Now, the crucial observation here is that scale μ in Eq. (2.105) is completely *arbitrary*: indeed, the original Lagrangian of the theory Eq. (2.64) did not exhibited any dependence on μ. Therefore, the *bare* parameters of the Lagrangian, such as the bare coupling constant λ and the mass m, *cannot depend on the choice* of the renormalization scale μ. This requirement leads to the following differential equation:

$$\mu\frac{d}{d\mu}\lambda = 0 \,, \tag{2.107}$$

which combined with the fact that the bare and physical couplings are related by $\lambda = Z_4\mu^{4-d}\lambda_R$, Eq. (2.104), we have that

$$0 = \mu\frac{d}{d\mu}\lambda = \mu\frac{d}{d\mu}\left(Z_4\mu^{4-d}\lambda_R\right) = \mu^\epsilon Z_4\lambda_R\left[\epsilon + \frac{\mu}{\lambda_R}\frac{d\lambda_R}{d\mu} + \frac{\mu}{Z_4}\frac{dZ_4}{d\mu}\right] \,, \tag{2.108}$$

where we have used the chain rule and accounted for the fact that both Z_4 and λ_R depend on μ. In the last step we have used that $\epsilon = 4-d$, in order to facilitate the connection with the physical limit $\epsilon \to 0$.

This differential equation, Eq. (2.108), determines how the renormalized coupling of the theory *runs with the energy scale*, and can be solved order by order in perturbation theory. For instance, at leading order we have that

$$Z_4 = Z_\lambda Z_\phi^{-2} = 1 + \mathcal{O}\left(\lambda_R\right) \quad \to \quad \left(\frac{dZ_4}{d\mu}\right)_0 = 0 \,, \tag{2.109}$$

and then from Eq. (2.108) we find that

$$\mu\left(\frac{d\lambda_R}{d\mu}\right)_0 = -\epsilon\lambda_R \,. \tag{2.110}$$

which determines the running of the coupling with the scale μ at leading order (Born level) in $d = 4-\epsilon$ space-time dimensions. Consistently, we find that if $\epsilon = 0$ then λ_R is scale invariant at tree level, as we know from the explicit computation of scattering diagrams. It is remarkable to note that in $\lambda\phi^4$ theory the coupling constant runs even at leading order for $d \neq 4$.

At the one-loop level (NLO), we have derived above the explicit $\mathcal{O}(\lambda_R)$ expressions for the three counterterms of the theory, from explicit computations of two- and four-point functions using dimensional regu-

larization. The results were the following:

$$\begin{aligned} Z_\phi(\lambda_R,\epsilon) &= 1+\mathcal{O}\left(\lambda_R^3\right) , \\ Z_\lambda(\lambda_R,\epsilon) &= 1+\frac{3\lambda_R}{16\pi^2}\left(\frac{1}{\epsilon}+\text{constant}\right)+\mathcal{O}\left(\lambda_R^2\right) , \\ Z_4(\lambda_R,\epsilon) &= 1+\frac{3\lambda_R}{16\pi^2}\frac{1}{\epsilon}+\mathcal{O}\left(\lambda_R^2\right) , \end{aligned} \tag{2.111}$$

Therefore, at NLO the renormalization group equation for the counterterm Z_4 reads:

$$\mu\left(\frac{dZ_4}{d\mu}\right)_{\text{NLO}} = \frac{3}{16\pi^2}\left(\mu\frac{d\lambda_R}{d\mu}\right)\frac{1}{\epsilon}+\mathcal{O}\left(\lambda_R^2\right) = -\frac{3}{16\pi^2}\lambda_R+\mathcal{O}\left(\lambda_R^2\right) , \tag{2.112}$$

where we have used Eq. (2.110).

Next, we can substitute this result into Eq. (2.108) and doing some algebra leads to the RGE equation for the renormalized coupling constant: at NLO

$$\mu\frac{d\lambda_R}{d\mu} = \frac{3}{16\pi^2}\lambda_R^2 \equiv \beta(\lambda_R) , \tag{2.113}$$

where now we have been able to take the $\epsilon \to 0$ limit safely. Therefore, we find that we can recover the previous result in terms of the beta function of the $\lambda\phi^4$ theory at the first non-trivial order, recall Eq. (2.98). Note that this derivation has not required any explicit calculation, provided the *perturbative expansion of the counterterms* is known at any fixed order. This result highlight that this new method is rather more general and powerful than the one used in the previous section.

Given the structure of the counterterm $Z_4 = Z_\lambda Z_\phi^{-2}$, in principle the beta function of the theory receives contributions from many different diagrams, in particular from all those that are required to evaluate Z_λ and Z_ϕ. For this reason, building the RGEs from the bare Lagrangian is a more systematic way than looking at explicit observables. On the other hand, the calculation of the beta function in terms of physical observables can be more intuitive, so in this respect the two methods are *complementary*.

Using a similar strategy, we can derive RGEs for other parameters of the Lagrangian and determine their dependence with the scale. Specifically, we can determine the RGEs corresponding to the physical mass m_p and to the renormalized scalar field ϕ. From Eq. (2.104) we have that

$$\begin{aligned} m &= \sqrt{Z_1}m_p = \sqrt{Z_m Z_\phi^{-1}}m_p , \\ \phi_0 &= \sqrt{Z_\phi}\phi , \end{aligned} \tag{2.114}$$

where we denote by ϕ_0 is the bare field and ϕ the renormalized one. To derive the RGE equations, we impose the physical requirement that the bare mass m and scalar field ϕ_0 cannot depend on the arbitrary renormalization scale μ, since it was absent in the original Lagrangian of the theory. Therefore we have that:

$$\mu\frac{d}{d\mu}m = 0 , \qquad \mu\frac{d}{d\mu}\phi_0 = 0 , \tag{2.115}$$

which lead to two differential equations that relate the Z_i counterterms to the renormalized mass m_p and

field ϕ. At this point, it becomes convenient to define the following quantity:

$$\gamma_m \equiv \frac{\mu}{m_p}\frac{dm_p}{d\mu}\,, \tag{2.116}$$

which is called the *anomalous dimension* of the mass, and

$$\gamma_\phi \equiv \frac{1}{2}\frac{\mu}{Z_\phi}\frac{dZ_\phi}{d\mu}\,, \tag{2.117}$$

which is known as the *anomalous dimension* of the scalar field. The reason of these names will become apparent in short. In the case of the physical mass m_p, its anomalous dimension γ_m determines its running with the energy, analogously to the beta function in Eq. (2.113) which fixes the running of the renormalized coupling λ_R.

Let us here work out explicitly the renormalization group equation for the physical mass m_p. Taking into account that the mass counterterm Z_m is given, see Eq. (2.84), by

$$Z_m = 1 + \frac{\lambda_R}{32\pi^2}\ln\left(\frac{m_p^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^2\right)\,, \tag{2.118}$$

which combined with the result for the other counterterms, Eq. (2.111), leads to the following relation between the physical and renormalized masses:

$$m = \sqrt{Z_m Z_\phi^{-1}}\, m_p \simeq \left(1 + \frac{\lambda_R}{64\pi^2}\ln\left(\frac{m_p^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^2\right)\right)\times m_p\,, \tag{2.119}$$

and therefore by imposing the invariance condition Eq. (2.115) using Eq. (2.119) we can compute the anomalous dimension γ_m of the physical mass, which determines its running with the energy. This is the analogous relation of Eq. (2.113) for the renormalized coupling.

To be explicit, we will have

$$\mu = \frac{dm}{d\mu} = 0 = \mu\frac{d}{d\mu}\left[\left(1 + \frac{\lambda_R}{64\pi^2}\ln\left(\frac{m_p^2}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^2\right)\right)\times m_p\right]\,, \tag{2.120}$$

which upon some rearrangement leads to

$$\left(\mu\frac{d\lambda_R}{d\mu}\right)\ln\left(\frac{m_p^2}{\Lambda^2}\right) + \lambda_R\left(\frac{\mu}{m_p}\frac{dm_p}{d\mu}\right)\left[2 + \ln\left(\frac{m_p^2}{\Lambda^2}\right)\right] = 0\,. \tag{2.121}$$

This equation relates the beta function of the renormalized coupling with the anomalous mass dimension γ_m determining the scale dependence of the physical mass, Eq. (2.116). Putting everything together, we find that

$$\gamma_m(\lambda_R) = \frac{\mu}{m_p}\frac{dm_p}{d\mu} = -\frac{\frac{3}{16\pi^2}\lambda_R\ln\left(\frac{m_p^2}{\Lambda^2}\right)}{2 + \ln\left(\frac{m_p^2}{\Lambda^2}\right)} \to \frac{3}{16\pi^2}\lambda_R\,, \tag{2.122}$$

where in the last step we have taken the physical limit $\epsilon \to 0$, which is equivalent to $\Lambda \to \infty$. Eq. (2.122) is the sought-for result of the mass anomalous dimension in $\lambda\phi^4$ theory.

To summarize, in this section we have shown how the renormalization group equations are powerful tools, basic on basic invariant requirements, which allows one to resum contributions to all orders in perturbation theory and to infer the running of the masses and the couplings with the scale for any given theory. As such they are ubiquitous in many applications of QFT from particle physics to condensed matter systems.

2.8 General renormalizability conditions

In the discussion above, we have found that if we express the $\phi\phi \to \phi\phi$ scattering amplitude in $\lambda\phi^4$ theory in terms of the physical coupling λ_R, the dependence on the cutoff Λ disappears (or equivalently on the regulator ϵ), rendering the scattering amplitude finite. This is a generic feature of the so-called *renormalizable theories*, those where all physical observables can be rendered well-defined by introducing a *finite number of counterterms*. Other theories don't satisfy this requirement, and go under the name of non-renormalizable QFT. Note that non-renormalizable QFT they can still provide sensible predictions as an effective low energy theory, provided we don't extrapolate to large energies above its regime of validity.

It is possible to determine from first principles whether of not a given QFT is renormalizable, or this can only be found by trial an error? In the following, we make use of *dimensional analysis* and *power counting* to provide some arguments that will provide information about when a quantum field theory is renormalizable or not. The idea is to understand how many counter-terms are required to render physical observables finite. For instance, in massless $\lambda\phi^4$ theory, defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial_\mu \phi \partial^\mu \phi - \frac{\lambda_R}{4!} Z_\lambda \, , \phi^4 \tag{2.123}$$

is enough the presence of Z_ϕ and Z_λ to render all possible physical observables finite? Or would we need additional counter-terms in the Lagrangian? As we will show, it is possible to (mostly) answer this question without actually computing a single diagram.

Dimensional analysis. The interactions between quantum fields can be classified into three different types, depending on the *mass dimensions* of the relevant coupling constant:

- *Relevant interactions*, when the coupling constant has positive mass dimensions $[g] > 0$.
- *Marginal interactions*, when the coupling constant is dimensionless $[g] = 0$.

  This is the case for example of the quartic interaction in $\lambda\phi^4$ theory in $d = 4$ dimensions.
- *Irrelevant interactions*, when the coupling constant has negative mass dimensions $[g] < 0$.

  An example of this is Fermi's theory of beta decay.

Here the symbol [ ] indicates the mass dimensions of given quantity, for example $[E] = 1$ and $[x] = -1$. Within this classification, we can provide simple heuristic argument regarding the (non-)renormalizabiloty of a given QFT. Consider for example a theory with four-point fermion interaction, such as Fermi theory of the weak interaction in nuclear beta decay:

$$\mathcal{L}_F = G_F \bar{\psi}\psi\bar{\psi}\psi \, . \tag{2.124}$$

The coupling constant of this theory[6] has mass units $[G] = -2$, since $[\mathcal{L}] = 4$ and $[\psi]$=3/2. Since this theory predicts the interaction between fermion fields, we can use it to compute in principle the scattering between two electrons, $e^-e^- \to e^-e^-$ via the four-point interaction vertex. We know that in this case the scattering matrix element $\mathcal{M}$ will be given by the following perturbative expansion:

$$\mathcal{M} = Gc_0 + G^2c_1 + \dots , \tag{2.125}$$

The main difference as compared to the corresponding expansion in $\lambda\phi^4$ theory is that now G is not a dimensionless parameter. This is a problem, since a sensible perturbative expansion must be given by the powers of a dimensionless parameter (ensuring that each order is smaller than the previous one). In order to have an expansion with dimensionless coefficients, we need to modify the expansion including the cut-off scale Λ,

$$\mathcal{M} = G\widetilde{c}_0 + G^2\Lambda^2\widetilde{c}_1 + \dots , \tag{2.126}$$

which is a problem, since now not only the matrix element is infinite, but now also differences between scales of the form $\mathcal{M}(s_1) - \mathcal{M}(s_2)$ will still depend on the cutoff Λ.

Fermi theory is an example of an effective field theory that is non-renormalizable. While it is suitable to describe the tree-level interactions between electrons and positrons, it has a pathological UV behaviour. This indicates that this theory requires a UV completion to make sense. The solution is to include the W, Z fields in the full electroweak theory, which regularise the UV physics and make the theory renormalizable. Indeed, to describe the correct UV physics of the weak interactions one needs to account for the fact that the masses of the W, Z photons is large but finite.

This heuristic argument suggests that the renormalizability or not of a theory depends on the types of operators that it includes, namely:

- A *renormalizable theory* includes only *relevant* or *marginal* operators. This implies that a finite set of counterterms if enough to remove all the UV divergences.
- A *non-renormalizable theory* contains at least *one irrelevant operator*, implying that to remove all infinities it would be necessary to include an infinite number counterterms.

Let us now give a bit more weight to these statements.

Power counting. We have seen in our study of loop corrections to the scattering processes in $\lambda\phi^4$ theory that at least one divergent diagram can be effectively regulated. But what about other diagrams that might also exhibit UV divergences, for instance higher-order diagrams? Do we need to include additional modifications to the Lagrangian beyond those already added. As a related question, how can we determine if a Feynman diagram is divergent and how bad this divergence is?

Let us consider a given Feynman diagram. Such diagram will be characterized by a number of features, being in particular the most important ones:

- V: the number of interaction vertices.
- E: the number of external lines.

[6] Recall that the action S is dimensionless, so $S = \int d^4x\mathcal{L}$ fixes the mass dimensions of the Lagrangian density.

- I: the number of internal lines (propagators).
- L: the number of undetermined momenta, which is equivalent to the number of loops.

For simplicity, we assume that this diagram contains only scalar fields. The internal momenta satisfy $(V-1)$ relations among themselves, from the V vertices where momentum conservation is imposed and where where the -1 terms arises from the global conservation of four momenta. Therefore one has the following relation

$$L = I - (V - 1) \ , \tag{2.127}$$

which enables us to compute the naive counting of powers of k^μ in a given diagram.

Let us give two examples showing how this relation holds:

- In the one-loop Feynman diagrams of Fig. 2.3, one has $L = 1$ (one loop), $V = 2$ (two interaction vertices), and $I = 2$ (two internal lines), so indeed Eq. (2.127) holds.
- In the first diagram for the two-loop correction of the propagator, shown in Fig. 2.4. we have $I = 33$ internal lines (propagators), $V = 2$ interaction vertices, $E = 2$ external lines. So applying Eq. (2.127) leads to $L = 3 - (2 - 1) = 2$, corresponding to the number of loops in this diagram.

Next, one defines D as the *superficial (apparent) degree of divergence* of a given Feynman diagram. To do this, we note that:

- Each loop involves an integral over momenta in d-dimensions, $\sim \int d^d k$
- Each internal line involves a propagator with inverse powers of the four-momenta, $\sim k^{-2}$.

Therefore, you can easily convince yourselves that the superficial degree of divergence D of a diagram is

$$D = d \times L - 2 \times I \, , \tag{2.128}$$

that is, the number of loops times the mass dimension of $d^d k$, minus twice the number of propagators (internal lines). This means that a Feynman diagram with superficial degree of divergence D will scale as $\sim k^D$. Note that in this derivation we have exploited the fact that in $\lambda\phi^4$ theory the Feynman rule for the vertex does not depend on momenta. In theories where this is not the case, the definition of Eq. (2.128) would be different.

Using again the same Feynman diagrams as above as an example, we can see that

- The one-loop diagram in Fig. 2.3 for the scattering amplitude has $d = 4$, $L = 1$, and $I = 2$, which corresponds to an apparent degree of divergence of $D = 0$. This is consistent with the *logarithmic divergence*, which is found from the explicit computation of the Feynman diagram.
- The two-loop diagram to the propagator in Fig. 2.4 has $d = 4$, $L = 2$, and $I = 3$, which corresponds to $D = 2$. And indeed an explicit calculation of this Feynman diagram shows that it has a *quadratic divergence*, similarly to the corresponding one-loop diagram.

In addition to the above results, we need one more relation in order to write the superficial degree of divergence D in terms of V and E, in order to assess how the interactions affect the degree divergences of

a given diagram. For this purpose, suppose that V_N stands for the number of vertices with N legs. In a diagram with V_N such vertices, we have NV_N lines which are either internal or external, so one has

$$NV_N = E + 2I\,. \tag{2.129}$$

Note that an internal line counts twice since it originates and ends at a vertex. Combining all the relations that we have derived up to here we find that

$$\begin{aligned} L &= I - V_N + 1\,, \\ D &= dL - 2I\,, \\ NV_N &= E + 2I\,, \end{aligned} \tag{2.130}$$

and therefore the expression for D in terms of N, V_N and E reads:

$$D = d - \frac{1}{2}(d-2)E + V_N\left(\frac{N-2}{2}d - N\right) \tag{2.131}$$

which in $d = 4$ space-time dimensions corresponds to

$$D = 4 - E + (N-4)V_N\,. \tag{2.132}$$

Let's verify that this relation holds for the same two diagrams discussed above:

- The one-loop diagram in Fig. 2.3 for the scattering amplitude has $E = 4$, $N = 4$ and $V_4 = 2$, which corresponds to an apparent degree of divergence of $D = 4 - 4 + (4-4) \times 2 = 0$, consistent with the previous result.
- The two-loop diagram to the propagator in Fig. 2.4 has $E = 2$, $N = 4$, and $V_4 = 2$, which corresponds to $D = 4 - 2 + (4-4) \times 2 = 2$, again consistent with the above result, which implies a quadratic divergence of the diagram.

Putting everything together, one can suggest the following *naive classification* about whether or not a given Feynman diagram is UV divergent:

- $D > 0$: diagram is UV *divergent*.
- $D = 0$: diagram is likely to be *logarithmically divergent*.
- $D < 0$: diagram is likely to be UV *finite*.

Consider specifically the case of the $\lambda\phi^4$ theory. In this case $N = 4$, since the only interaction possible in this theory involves four legs. Then we have that $D = 4 - E$. The interesting result is that the degree of divergence only depends on the number of external legs, but it is actually *independent of the number of vertices* that we have in the diagram. Therefore we have only two classes of *problematic* diagrams in this theory, namely:

- Diagrams with 2 external legs, for which $D = 2$, corresponding to loop corrections to the propagator.

- Diagrams with 4 external legs, for which $D = 0$, corresponding to loop corrections to the $2 \to 2$ scattering amplitude.

This result implies that *all* the counterterms that we need to renormalize the theory are those required to renormalize both the two-point function (the propagator) and the four-point function (the $\phi\phi \to \phi\phi$ scattering amplitude). Therefore, we can conclude that with the counterterms that have already been added above to the Lagrangian of the $\lambda\phi^4$ theory, Eq. (2.123), all possible diagrams that one can compute *will be finite*. In other words, we only need a finite number of terms to make finite the physical predictions of the theory, and therefore we can conclude that it is *renormalizable*. And as we compute more and more loops, the counterterms Z_λ, Z_ϕ and Z_m will get modified with higher order corrections, but no more counterterms will be required.

Why we call D a *naive* degree of divergence? The reason is that a diagram with $D = 0$ will *not* necessarily have a logarithmic divergence. Such divergence is possible and compatible with the diagram's features, but is not necessary. For example, *tree-level diagrams* have $D = 0$ in $\lambda\phi^4$ theory (trivially from Eq. (2.128) since $L = I = 0$), yet they are obviously finite. Also, $D < 0$ does not necessarily imply that a given diagram is finite. Understanding the actual degree of divergence of a diagram is more delicate that this naive analysis. What this analysis can indicate is whether or not a given theory exhibits *UV divergences*, but it does not allow us to prove that it is renormalizable. For example in $d = 4$ we had

$$D = 4 - E + (N - 4)V_N \,. \tag{2.133}$$

For a theory with interactions of the form ϕ^5 or even with a higher power, there are infinitely many diagrams that are divergent with arbitrary number of external lines for which we will potentially need a different counterterm to cancel it.

3 Quantization of abelian gauge theories

In this second part of the course we move to a rather different topic than in the first part: the *quantization of abelian gauge theories*, with emphasis of course on Quantum Electrodynamics (QED), the QFT that describes the electromagnetic interaction, as well as on its simpler scalar version (sQED). The topics that will be covered here build upon what you have seen in the *Quantum Field Theory* course, in particular the quantization of the scalar and of the Dirac fields.

In this part of the course, first of all we will review the covariant formulation of Maxwell's electromagnetism, with emphasis on the classical symmetries and related properties. This will include a description of the coupling between matter and gauge fields. We will then move to describe the canonical quantization of Maxwell's equations and derive the Feynman rules for the interaction between electrons and photons. We will use these rules to compute some simple processes in scalar QED, and highlight the deep connection between the theories of the symmetry and the corresponding physical results.

3.1 A review of Maxwell's electromagnetism

We begin this second part of the course with a succinct review of Maxwell's theory of electromagnetism, specifically of its relativistic (covariant) formulation. As you know, the equations of motion of this theory are Maxwell's equations, which when expressed in terms of the electric and magnetic field read

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \rho\,, \\ \boldsymbol{\nabla}\times\mathbf{B}-\frac{\partial}{\partial\mathbf{t}}\mathbf{E} &= \mathbf{J}\,, \\ \boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial}{\partial\mathbf{t}}\mathbf{B} &= 0\,, \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\,, \end{aligned} \tag{3.1}$$

where quantifies in boldface refer to (three-dimensional) vectors. In these equations, ρ and $\mathbf{J}$ are the charge and current densities respectively, that in general depend on the space time coordinates $(t,\mathbf{x})$, and that we will take as external inputs for the time being.

The last two equations of Eq. (3.1), which do not have external sources, can be solved by introducing a vector potential $\mathbf{A}$ and a scalar potential ϕ so that

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\phi-\frac{\partial}{\partial t}\mathbf{A} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\,. \end{aligned} \tag{3.2}$$

You can verify that the two homogeneous Maxwell's equations are always satisfied if $\mathbf{E}$ and $\mathbf{B}$ are defined as in Eq. (3.2), using the fact that the divergence of the curl of any vector field $\mathbf{X}$ vanishes, as does the curt of the gradient of any scalar field ϕ,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{X})=0\,,\qquad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)=0\,. \tag{3.3}$$

The fact that we can replace the electric and magnetic fields $(\mathbf{E},\mathbf{B})$ by the scalar and vector potentials $(\phi,\mathbf{A})$ reveals something rather deep about the structure of the theory, since the former has *6 independent*

components while the latter only has only *4 components*. Therefore, Maxwell's electromagnetism expressed in terms of $(\mathbf{E}, \mathbf{B})$ has an inherent redundancy, with important consequences as we will now discuss.

Maxwell's theory, described by Eq. (3.1) admits a very elegant description in terms of the covariant formalism of special relativity, which will also be useful in order to later promote it to the quantum field theory level. With this motivation, we define a four-vector potential

$$A^\mu \equiv (\phi, \mathbf{A}) \,, \tag{3.4}$$

which is known as the *gauge field*, and the corresponding tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \,, \tag{3.5}$$

which goes under the name of the *field strength* tensor. Note that by construction this tensor is antisymmetric, $F^{\mu\nu} = -F^{\nu\mu}$. By comparing with Eq. (3.1), we can identity $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk} B_k$ and write the first two equations of motion (the inhomogeneous Maxwell's equations) as

$$\partial_\nu F^{\mu\nu} = J^\mu \,, \tag{3.6}$$

where the four-current is naturally defined as $J^\mu \equiv (\rho, \mathbf{J})$. The last two of Maxwell's equations (the homogeneous ones) can be also written in terms of the field strength tensor as

$$\epsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0 \,, \tag{3.7}$$

which is known as the *Bianchi identity*, and $\epsilon_{\mu\nu\rho\sigma}$ is the maximally antisymmetric tensor. In the absence of external currents, $J = (0, \mathbf{0})$, the Lagrangian of Maxwell's electromagnetism is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right) \,, \tag{3.8}$$

from which Eq. (3.1) can be derived using the Euler-Lagrange equations, as it is easy to check explicitly using This is a direct consequence of the Euler-Lagrange equation for A_0,

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu A_\mu)} \right) = 0 \,. \tag{3.9}$$

The vector field A_μ has of four components, so one could conclude that the gauge field represented by it has 4 degrees of freedom. However, we know that physical (massless) photons have only two degrees of freedom, namely its two possible polarization states. To understand this apparent discrepancy, let us write the Lagrangian as a function of the gauge field A_μ. In this case one finds

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial^\mu A^\nu \partial_\nu A_\mu \tag{3.10}$$

If we inspect this equation, we observe that the Lagrangian does not contain time derivatives of A_0, since these cancel out due to its symmetry structure. Therefore, we know from classical mechanics that is the Lagrangian contains no $\partial A_0 / \partial t$ terms, then this means that the degree of freedom A_0 is not dynamical (in

other words, $\partial_t A_0 = 0$).

The same result, namely that the number of physical components of A_μ is smaller than 4, can also be derived from inspecting the equations of motion, where we have that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \quad \rightarrow \quad \boldsymbol{\nabla}^2 A_0 + \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{A}}{\partial t} = 0 \,, \tag{3.11}$$

an equation which can be solved for $A_0(\mathbf{x})$ to give

$$A_0(\mathbf{x}) = \int d^3x' \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) (\mathbf{x}') \,, \tag{3.12}$$

which illustrates that given some boundary conditions for the spatial components of the gauge field $\mathbf{A}$, then the non-dynamical time component $A_0(\mathbf{x})$ is fully specified. Therefore A_0 is not an independent degrees of freedom, and we are left with *3 independent degrees of freedom*. Which is better than before, but still one too many than the physical degrees of freedom of the photon.

In order to eliminate the other unphysical degree of freedom, we note another symmetry that we have not discussed up to now. Indeed, the Lagrangian of our theory, Eq. (3.8), has a large additional symmetry group, since it is invariant under transformations of the form

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x) \,, \tag{3.13}$$

with $\lambda(x)$ an arbitrary scalar function. Note that not only the Lagrangian, but also the field strength $F_{\mu\nu}$ itself is invariant under this transformation. We call this kind of invariance a *gauge symmetry*, for historical reasons that are not really relevant here. Crucially, $\lambda(x)$ in general depends on the space-time position x.

The existence of this additional *local* symmetry implies that A_μ provides a redundant description of the electromagnetic field $F_{\mu\nu}$. One possible way to show that gauge theories are *redundancies of the theory*, and here are on a quite different conceptual footing as other types invariance such a *global* symmetries, is obtained by looking again to Maxwell's equations in the covariant formulation, and noting that

$$\partial_\mu F^{\mu\nu} = \left(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) A^\nu = 0 \,. \tag{3.14}$$

Crucially, the operator that appears in Eq. (3.14), $\left(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right)$ is *non-invertible*. This means that Eq. (3.14) is satisfied by any function of the form $\partial_\mu \lambda(x)$, which is a total differential. In turn, this property implies that we cannot really distinguish between $A_\mu(x)$ and $A_\mu(x) + \partial_\mu \lambda(x)$ as solutions of the Maxwell's equations: they are physically the same. However, if we are happy to make the formal equivalence

$$A_\mu \approx A_\mu + \partial_\mu \lambda \,, \tag{3.15}$$

then the whole problem disappears, and Maxwell's equations become invertible.

Therefore, in classical electrodynamics all states that are related by a gauge transformation of the form

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x) \,, \tag{3.16}$$

are considered to be *physically equivalent*. By making a *gauge choice*, which is a specific choice of Eq. (3.16),

we can select a representative of each family of states related by gauge transformation. Differences choices of gauge are more suitable than others for specific problems, but at the end of the day results should be the same irrespective of the specific gauge choice. In other words, we say that physical observables should be *gauge independent*.

There are a number of widely used gauge choices, each one defined by a *gauge condition*. Some of the most frequent choices are:

- The *Lorenz gauge*.

  The Lorenz choice of gauge is defined by the following condition:[7]

$$\partial_\mu A^\mu = 0\,, \tag{3.17}$$

  which in terms of the vector $\mathbf{A}$ and scalar potential ϕ reads

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{\partial\phi}{\partial t} = 0\,. \tag{3.18}$$

  This choice does not remove all the redundancies in the theory, since any gauge transformation that satisfies $\partial_\mu\partial^\mu\lambda(x) = 0$ will still preserve the Lorentz gauge condition. The main advantage of Eq. (3.17) is that it is Lorentz invariant (valid in any reference frame), but the drawback is that since there is still some redundancy, it is sometimes difficult to keep track of the physical degrees of freedom.

- The *Coulomb gauge*.

  Starting from the Lorentz gauge, Eq. (3.17), we can make use of the residual freedom in the gauge transformation, which allows to use any function that $\partial_\mu\partial^\mu\lambda(x) = 0$, to set

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0\,. \tag{3.19}$$

  In this case, we find by comparing with Eq. (3.12) that the time component of the gauge field vanishes, $A_0 = 0$. While the Coulomb gauge Eq. (3.19) is not Lorentz invariant, it allows exhibit the physical degrees of theory in an explicit way.

In order to illustrate the usefulness of the Coulomb gauge, let us write Maxwell's equations explicitly in this gauge. Using Eq. (3.19) and $A_0 = 0$, we can write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial^\mu A^\nu\partial_\nu A_\mu = -\frac{1}{2}\partial_t A_i\partial_t A^i - \frac{1}{2}\partial_j A_i\partial^j A^i\,. \tag{3.20}$$

Next, if we now if we compute the Euler-Lagrange equation for the spatial (non-zero) components of the gauge field A_i,

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_i)}\right) - \frac{\partial\mathcal{L}}{\partial A_i} = 0\,, \quad i = 1,2,3\,, \tag{3.21}$$

we find that the classical equation of motion for the dynamical components of the gauge field A_i are

$$\Box A_i = \left(-\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2\right)A_i = 0\,, \tag{3.22}$$

[7] Note that the Lorentz gauge is named after the Danish physicist Ludwig *Lorenz*, not after the Dutch physicist Hendrik *Lorentz* of the eponymous transformations.

where $\Box$ denotes the D'Alambertian operator. Eq. (3.22) is of course nothing but the *Klein-Gordon equation* for the three scalar fields A_1, A_2, and A_3, see also App. (A). Using the known solutions of the Klein-Gordon equation, we find that the general solution for $\mathbf{A}$ can be written as

$$\mathbf{A}(\mathbf{x},t) = \sum_{\lambda} \int \frac{d^3k}{2\omega(2\pi)^3} \left[(\vec{\epsilon}_\lambda)^*(k) a_\lambda(k) e^{i\mathbf{k}\cdot\mathbf{x}} + \vec{\epsilon}_\lambda(k) a^\dagger_\lambda(k) e^{-i\mathbf{k}\cdot\mathbf{x}} \right] , \tag{3.23}$$

where $\vec{\epsilon}_\lambda$ are the gauge field polarization vectors and the sum runs over $\lambda = 1, 2$ the two physical polarizations of the photon. As can be seen from the combination of Eqns. (3.22) and (3.23), the solutions of the massless Klein-Gordon equation satisfy the mass-shell condition $k^2 = 0$, reflecting the fact that classically photons are massless. At the classical level, $a_k(k)$ are just functions, and they will be later promoted to operators once we quantize the photon field.

Let us now determine the form of the gauge field polarization vectors $\vec{\epsilon}_\lambda$ that appear in Eq. (3.23). From the Coulomb gauge condition, Eq. (3.19), it follows that

$$\vec{\epsilon}_\lambda(k) \cdot \mathbf{k} = 0 , \quad \lambda = 1, 2 , \tag{3.24}$$

that is, the polarization vectors $\vec{\epsilon}_\lambda(k)$ are perpendicular to the direction of motion $\mathbf{k}$. To see this, note that

$$0 = \boldsymbol{\nabla} \cdot \vec{A} = i \sum_{\lambda} \int \frac{d^3k}{2\omega(2\pi)^3} \left[\left((\vec{\epsilon}_\lambda)^*(k) \cdot \vec{k} \right) a_\lambda(k) e^{i\mathbf{k}\cdot\mathbf{x}} - \left(\vec{\epsilon}_\lambda(k) \cdot \vec{k} \right) a^*_\lambda(k) e^{-i\mathbf{k}\cdot\mathbf{x}} \right] , \tag{3.25}$$

which requires the photon polarization to be perpendicular to the direction of propagation. This requirement, Eq. (3.24), has two independent solutions, which we label for convenience as $\lambda = \pm$. Assuming that the photon propagates in the z direction, $\mathbf{k} = (0, 0, |k|)$, a suitable choice of polarization vectors corresponds to

$$\begin{aligned} \vec{\epsilon}_+(k) &= \frac{1}{\sqrt{2}} (1, -i, 0) , \\ \vec{\epsilon}_-(k) &= \frac{1}{\sqrt{2}} (1, +i, 0) , \end{aligned} \tag{3.26}$$

which physically corresponds to *circularly polarized* photons, and where the overall prefactor follows from the normalization condition. This explicitly shows how the two physical polarization states of the photon are orthogonal to each other, namely

$$\vec{\epsilon}_+(k) \cdot \vec{\epsilon}^*_-(k) = 0 . \tag{3.27}$$

For a generic direction of the propagation of the photon, these *polarization vectors* form an orthonormal and complete set, and satisfy the following conditions

$$\begin{aligned} \epsilon_\lambda(k) \cdot \epsilon^*_{\lambda'}(k) &= \delta_{\lambda\lambda'} , \\ \sum_{\lambda=\pm} \epsilon^*_{i,\lambda}(k) \epsilon_{j,\lambda}(k) &= \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} , \end{aligned} \tag{3.28}$$

where you can easily check that the orthogonality requirement Eq. (3.24) holds now for a generic vector $\mathbf{k}$.

3.2 Coupling gauge and matter fields

The discussion so far has been restricted to the case of electromagnetism *without external charges or currents*, in other words, without accounting for the effects of the coupling to matter. Let us know study how the quantum field theory of electromagnetism is modified in the presence of a non-zero four-current $j^\mu \neq 0$. In this case the equations of motion of the theory read

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \,, \\ \epsilon_{\mu\nu\lambda\rho}\partial^\nu F^{\lambda\rho} &= 0 \,. \end{aligned} \tag{3.29}$$

The antisymmetry property of the field strength tensor $F_{\mu\nu}$ together with the first of Eq. (3.29) implies that

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0 \,, \tag{3.30}$$

and therefore only conserved currents ($\partial_\nu j^\nu = 0$) are consistent with Maxwell's electromagnetism.

This result can be understood as a direct consequence of gauge invariance. Indeed, adding a current to the equations of motion of the theory can be achieved by means of a covariant coupling of the current to the gauge field in the Lagrangian, leading to an action

$$S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu \right) \,. \tag{3.31}$$

Now let us see how the action is affected if we perform a gauge transformation of the form Eq. (3.16). While $F_{\mu\nu}$ is gauge invariant, the matter-gauge coupling term is modified:

$$S \to S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu \left(A_\mu + \partial_\mu \lambda \right) \right) \,, \tag{3.32}$$

This means that the action transforms under a gauge transformation as

$$S \to S + \int d^4x \partial_\mu \left(j^\mu \lambda(x) \right) - \int d^4x \left(\partial_\mu j^\mu \right) \lambda(x) \,, \tag{3.33}$$

where we have integrated by parts. Since the four-current goes to zero at infinity for all sensible boundary conditions, the second term in Eq. (3.33) vanishes, being a total differential. Therefore we find that the action under a gauge transformation varies as

$$S \to S - \int d^4x \left(\partial_\mu j^\mu \right) \lambda(x) \,, \tag{3.34}$$

and therefore S is only invariant under gauge transformations if the four-current is conserved, $\partial_\mu j^\mu = 0$.

Having realized that only conserved currents are allowed, let us see which further information we can derive on the form of the coupling between matter and gauge fields. In this respect, gauge invariance is extremely useful as a guiding principle. The reason is that if one ensures gauge invariance in the action, we know the the resulting equations of motion will be unchanged and that the number of degrees of freedom will not be modified.

Scalar QED. To illustrate how we can couple photons with matter fields, assume we want to add the following term to the Maxwell Lagrangian

$$\mathcal{L} \to \mathcal{L} + A_\mu \phi \partial^\mu \phi \,, \tag{3.35}$$

which would describe the coupling between the photon and a real scalar field ϕ. It is easy to check that this term is not gauge invariant and thus not allowed. In general, you can convince yourselves that any coupling between the photon and a real scalar field will never satisfy gauge invariance. The reason is that gauge invariance will only be satisfied if the new field transforms in a way that exactly cancels the gauge transformation of the photon.

So while coupling a photon to a real scalar field seems difficult to reconcile with gauge invariance, the situation is more interesting with a complex scalar field, which we will denote[8] by ψ

$$\psi = \phi_1 + i\phi_2 \,. \tag{3.36}$$

Then we will assume that under a gauge transformation this complex field changes as

$$\psi(x) \to e^{-i\alpha(x)}\psi(x) \,, \tag{3.37}$$

with $\alpha(x)$ being an arbitrary real function. Being a scalar field, the kinetic term describing ψ is the Klein-Gordon Lagrangian

$$\mathcal{L}_{\psi,\text{kin}} = -\partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi \,, \tag{3.38}$$

which under a gauge transformation of the form Eq. (3.37) transforms as

$$\mathcal{L}_{\psi,\text{kin}} \to \mathcal{L}_{\psi,\text{kin}} + \left(\partial_\mu \alpha \partial^\mu \alpha\right) \psi^* \psi \,, \tag{3.39}$$

which is not gauge invariant. This Lagrangian Eq. (3.38) can be made invariant if we promote its partial derivative into an object known as the *covariant derivative*, defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu \,, \tag{3.40}$$

with e some unspecified constant. It is now possible to verify that under a gauge transformation of the form $A_\mu \to A_\mu + (\partial_\mu \alpha)/e$ the covariant derivative transforms as

$$D_\mu \psi \to e^{-i\alpha(x)} D_\mu \psi \,, \tag{3.41}$$

that is, exactly in the same way as the field ψ itself, see Eq. (3.37). Therefore, a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(D_\mu \psi\right)^* \left(D_\mu \psi\right) - m^2 \psi^* \psi \,, \tag{3.42}$$

is gauge invariant, and therefore represents a physically sensible interaction between a photon and a complex scalar field. This theory is know as *scalar QED* (sQED), since it has the same structure as the usual quantum

[8] Not to mix with a Dirac fermion field.

electrodynamics but with a scalar field replacing the Dirac fermion. If instead of coupling the photon to a complex scalar we couple it to a fermion field governed by Dirac's equation, then we obtain the QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}i\gamma^\mu D_\mu\psi - m\bar{\psi}\psi\,. \tag{3.43}$$

Note that the gauge invariance requirement fixes completely the coupling between matter and gauge fields, irrespective of the specific type of the former.

Gauge symmetries and conserved currents. In general, the symmetries of a theory can be classified into two main types:

- *Local symmetries.*

  These symmetries are parametrized by a continous function $\lambda(x)$ which depends on the space-time position x. Gauge symmetry is the most representative example of this family.

- *Global symmetries.*

  These are instead given in terms of constant (independent of the space-time position) parameters (numbers, vectors, matrices). Examples of global symmetries include translations, rotations, and Lorentz boosts.

Note that any local symmetry will have also associated a global symmetry, corresponding to the particular case where the function λ is constant.

From *Noether's theorem* we know that for each global (continous) symmetry has associated a conserved charge. Does this also apply to local symmetries such as gauge transformations? In other words, has Maxwell's Lagrangian (without external currents) a conserved charge associated to gauge invariance? If we apply Noether's theorem (see App. A), one finds that in the case of a gauge transformation the conserved four-current would be

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta A_\nu = -F^{\mu\nu}\partial_\nu\lambda(x) = -\partial_\nu\left(F^{\mu\nu}\lambda(x)\right)\,, \tag{3.44}$$

where in the last step we have used the chain rule for differentiation and the fact that the equations of motion of the theory require $\partial_\mu F^{\mu\nu} = 0$. Therefore the *conserved charge* associated to a gauge transformation is:

$$Q = \int d^3x\, j^0 = \int d^3x\, \partial_i\left(F^{0i}\lambda(x)\right) = 0\,, \tag{3.45}$$

since it is a total derivative, and the photon fields vanish at infinity. Therefore, there are no conserved charges associated to local symmetries such as gauge invariance.

Despite this result, when we coupled A_μ to either a complex scalar or a fermion field, we obtained a non-trivial global symmetry, corresponding to the case of a constant gauge transformation. The application of Noether's theorem to the scalar field ψ in sQED Eq. (3.43), where $\delta\psi = -i\alpha\psi$, see Eq. (3.37), where now α is a constant, gives the following result

$$\begin{aligned} j_s^\mu &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^*}\delta\psi^* \\ &= i\alpha\left[\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi + i2eA^\mu\psi^*\psi\right]\,, \end{aligned} \tag{3.46}$$

while in the case of fermions the conserved current is instead

$$j_f^\mu = \alpha \bar{\psi} \gamma^\mu \psi \,. \tag{3.47}$$

We observe that in both cases Noether's current j^μ is equal to the term the multiplies the photon field A_μ in the Lagrangian. This conserved current turns out to correspond the *electric charge* carried by the matter fields, as we will discuss in more detail below.

3.3 Canonical quantization of the photon field

At this point we are ready to discuss the quantization of the gauge field A_μ using the canonical formalism. We will work in the Coulomb gauge, discussed in Sect. 3.1, and defined by Eq. (3.19). The Lagrangian of Maxwell's electromagnetism in the Coulomb gauge is given by

$$\mathcal{L} = \frac{1}{2} \dot{A}_i \dot{A}_i - \frac{1}{2} \partial_j A_i \partial_j A_i \,, \tag{3.48}$$

where we have used that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ and thus $A_0 = 0$ in this gauge. The canonical momenta of the theory are given by

$$\begin{aligned} \pi^0 &= \frac{\delta \mathcal{L}}{\delta \dot{A}_0} = 0 \,, \\ \pi^i &= \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = \dot{A}_i \,, \end{aligned} \tag{3.49}$$

where we note that the fact that $\boldsymbol{\nabla} \cdot \mathbf{A} = \mathbf{0}$ also implies that $\boldsymbol{\nabla} \cdot \pi = \mathbf{0}$.

Now we can write the corresponding Hamiltonian density, finding

$$\mathcal{H} = \pi_i \dot{A}_i - \mathcal{L} = \frac{1}{2} \pi_i \pi_i + \frac{1}{2} \nabla_j A_i \nabla_j A_i \,, \tag{3.50}$$

in terms of the spatial components of the photon field A_i and their corresponding conjugate momenta π_i. In order to quantize the photon field we need to impose suitable *canonical commutation relations* between the field and its conjugate momentum. A naive condition would be the following:

$$[A_i(\vec{x}, t), \pi_j(\vec{y}, t)] = i \delta_{ij} \delta^{(3)}(\vec{x} - \vec{y}) \,, \tag{3.51}$$

but this option is not consistent with the gauge condition. To see this, one should note that in the Coulomb gauge we still want to have $\boldsymbol{\nabla} \cdot \mathbf{A} = \mathbf{0}$ at the level of operators, but if the take the divergence of Eq. (3.51) we find that

$$[\boldsymbol{\nabla} \cdot \mathbf{A}(\vec{x}, t), \boldsymbol{\nabla} \cdot \vec{\pi}(\vec{y}, t)] = i \nabla^2 \delta^{(3)}(\vec{x} - \vec{y}) \neq 0 \,, \tag{3.52}$$

which is inconsistent with the Coulomb gauge fixing condition.

The problem with the naive commutation relations of Eq. (3.51) is that we have not properly taken into account the constraints from gauge fixing. Without deriving a formal proof, here we just state that the

correct set of commutation relations for the gauge field are given by

$$[A_i(\vec{x},t),\pi_j(\vec{y},t)] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{3.53}$$

and we show that this commutation relation is consistent with the Coulomb gauge constraints. To prove this, we note that the RHS of Eq. (3.53) can be rewritten in momentum space as follows:

$$[A_i(\vec{x},t),\pi_j(\vec{y},t)] = i\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\vec{p}|^2}\right)\exp\left(i\vec{p}\cdot(\vec{x}-\vec{y})\right)\,, \tag{3.54}$$

and if we take a derivative of this commutation relation, we find that

$$\begin{aligned}[\partial_iA_i(\vec{x},t),\pi_j(\vec{y},t)] &= -\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\vec{p}|^2}\right)p_i\exp\left(i\vec{p}\cdot(\vec{x}-\vec{y})\right)\\ &= -\int\frac{d^3p}{(2\pi)^3}\left(p_j-p_j\right)\exp\left(i\vec{p}\cdot(\vec{x}-\vec{y})\right) = 0\,,\end{aligned} \tag{3.55}$$

and therefore the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}=0$ is still satisfied by the commutation relations Eq. (3.53), as we wanted to demonstrate.

Therefore, the complete set commutation relations for the gauge field in Maxwell's theory are given by

$$\begin{aligned}[A_i(\vec{x},t),\pi_j(\vec{y},t)] &= i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^{(3)}(\vec{x}-\vec{y})\,,\\ [A_i(\vec{x},t),A_j(\vec{y},t)] &= 0\,,\\ [\pi_i(\vec{x},t),\pi_j(\vec{y},t)] &= 0\,.\end{aligned} \tag{3.56}$$

At this point we are ready to use the solution for $\vec{A}$ in the Coulomb gauge found using the Klein-Gordon equation, Eq. (3.23), namely

$$\mathbf{A}(\mathbf{x},t) = \sum_{\lambda=\pm}\int\frac{d^3k}{2\omega(2\pi)^3}\left[(\vec{\epsilon}_\lambda)^*(k)a_\lambda(k)e^{ikx} + \vec{\epsilon}_\lambda(k)a^*_\lambda(k)e^{-ikx}\right]\,, \tag{3.57}$$

and combine it with the commutation relations in Eq. (3.56), leading to the following relations for the creation and annihilation operators of the photon field:

$$\begin{aligned}\left[a_\lambda(k),a_{\lambda'}(k')\right] &= 0\,,\\ \left[a^\dagger_\lambda(k),a^\dagger_{\lambda'}(k')\right] &= 0\,,\\ \left[a_\lambda(k),a^\dagger_{\lambda'}(k')\right] &= (2\pi)^3 2\omega\delta^{(3)}(k-k')\delta_{\lambda\lambda'}\,,\end{aligned} \tag{3.58}$$

as can be checked explicitly, using the fact that the canonical momentum is given by

$$\vec{\pi}(\mathbf{x},t) = \frac{\partial\vec{A}}{\partial t} = \sum_{\lambda=\pm}\int\frac{d^3k}{2\omega(2\pi)^3}k_0\left[(\vec{\epsilon}_\lambda)^*(k)a_\lambda(k)e^{ikx} - \vec{\epsilon}_\lambda(k)a^*_\lambda(k)e^{-ikx}\right]\,. \tag{3.59}$$

In a similar way as what is done with the quantization of the fermion field from the Dirac Lagrangian, we can physically identify the operators in Eq. (3.57) as follows:

- $a_\lambda^\dagger(k)$: *creation operator.*

  When applied to a given quantum state, $a_\lambda^\dagger(k)$ creates a mode of the photon field with momentum k and polarization λ.

- $a_\lambda(k)$: *annihilation operator.*

  When applied to a given quantum state, $a_\lambda(k)$ annihilates a mode of the photon field with momentum k and polarization λ.

The two physical polarizations of the photon field can be identified as:

- $\lambda = +$: right-circular polarization.
- $\lambda = -$: left-circular polarization.

Finally, we can write the Hamiltonian Eq. (3.50) in terms of the creation and annihilation operators to find

$$H = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} a_\lambda^\dagger(k) a_\lambda(k) + 2E_0 V \,, \tag{3.60}$$

where the zero point energy is defined as

$$E_0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega \,. \tag{3.61}$$

and V is the integration volume. As we know, the fact that the zero point energy is infinite is not relevant since only differences in energy are physically meaningful.

3.4 Interactions in gauge theories: the scalar QED case

Now that we have learned how to canonically quantize the free photon field, Eq. (3.57), we are at a stage where we can formulate the full theory of quantum electrodynamics, the quantum field theory that governs the interactions between electrons and photons. This will require the derivation of Feynman rules that allow us eventually to compute the amplitudes for simple processes. Since the fermion case is a bit more complicated, to make our lives easier in this part of the course we will focus on the *scalar QED case*: the theory of the interactions between the charged scalar fields by means of photons. Recall that scalar QED at the classical level was introduced above in Eq. (3.42). This process will include several steps, with the ultimate goal of performing some simple tree-level computations of scattering processes using this theory.

LSZ reduction formula for photons. At this point we are ready to derive the LSZ reduction formula for photons. In the Coulomb gauge, the spatial components of the photon field, $\vec{A}$, are given by Eq. (3.57), where the gauge field polarization vectors $\vec{\epsilon}_\lambda$ being perpendicular to the direction of propagation, $\vec{k} \cdot \vec{\epsilon}_\lambda(k) = 0$, and satisfying the orthonormality conditions of Eq. (3.28). In the same way as what is done for scalars and fermions, it is possible to invert the relation between the creation and annihilation operators and the photon

field itself to yield

$$\begin{aligned} a_\lambda(k) &= i\vec{\epsilon}_\lambda(k)\cdot\int d^3x\, e^{-ikx}\overline{\partial_t}\vec{A}(x)\,, \\ a^\dagger_\lambda(k) &= -i\vec{\epsilon}^*_\lambda(k)\cdot\int d^3x\, e^{ikx}\overline{\partial_t}\vec{A}(x)\,, \end{aligned} \tag{3.62}$$

where we have defined the following symbol

$$f\overline{\partial}g \equiv f(\partial g) - (\partial f)g\,. \tag{3.63}$$

To verify these equalities, you can plug Eq. (3.62) in Eq. (3.57) can check that the identity is satisfied. From here, the LSZ framework applies in the same way as in the scalar and fermion cases. The only subtlety here is that Note that for an incoming and an outgoing photon we need to make the corresponding replacements:

$$\begin{aligned} a^\dagger_\lambda(k)_{\rm in} &= i\vec{\epsilon}^*_\lambda(k)\cdot\int d^4x e^{+ikx}(-\partial^2)\vec{A}_\mu(x)\,, \\ a^\dagger_\lambda(k)_{\rm out} &= i\vec{\epsilon}_\lambda(k)\cdot\int d^4x e^{-ikx}(-\partial^2)\vec{A}_\mu(x)\,. \end{aligned} \tag{3.64}$$

Feynman rules for scalar QED. We now have all the ingredients that we need to derive the Feynman rules for scalar QED. The full Lagrangian of the theory is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D^\mu\psi^\dagger D_\mu\psi - m^2\psi^\dagger\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial^\mu\psi^*\partial_\mu\psi - m^2\psi^*\psi + ieA^\mu\left[(\partial_\mu\psi^*)\,\psi - \psi^*\,(\partial_\mu\psi)\right] - e^2A_\mu A^\mu\psi^*\psi\,, \end{aligned} \tag{3.65}$$

where we see that there are two *interaction terms*, meaning two terms that couple the photon with the scalar field, whose structure is fixed by the gauge invariance requirements of the theory. These terms lead to the two interaction vertices shown in Fig. 3.1. The first corresponds to a vertex involving two scalars and one photon,

$$ieA^\mu\left[(\partial_\mu\psi^*)\,\psi - \psi^*\,(\partial_\mu\psi)\right]\,, \tag{3.66}$$

which involves also the derivatives of the scalar field, and is proportional to the electric charge e of the scalar field. The second interaction term is

$$-\,e^2A_\mu A^\mu\psi^*\psi\,, \tag{3.67}$$

and leads to a vertex with two photons and two scalar fields, with no dependence on neither the photon or the scalar field momenta. The diagrams in Eq. (3.1) are the only possible ways that photons and scalars can interact being consistent with the requirements of gauge symmetry: as discussed above, other possible interaction terms would not satisfy gauge invariance.

From the discussion of the symmetry properties of scalar QED, we found that a global gauge transformation had associated a conserved current (via Noether's theorem) of the form of Eq. (3.46), which can also be expressed in terms of the covariant derivative as

$$j^\mu = i\left(\psi^*D^\mu\psi - (D^\mu\psi)^*\psi\right)\,. \tag{3.68}$$

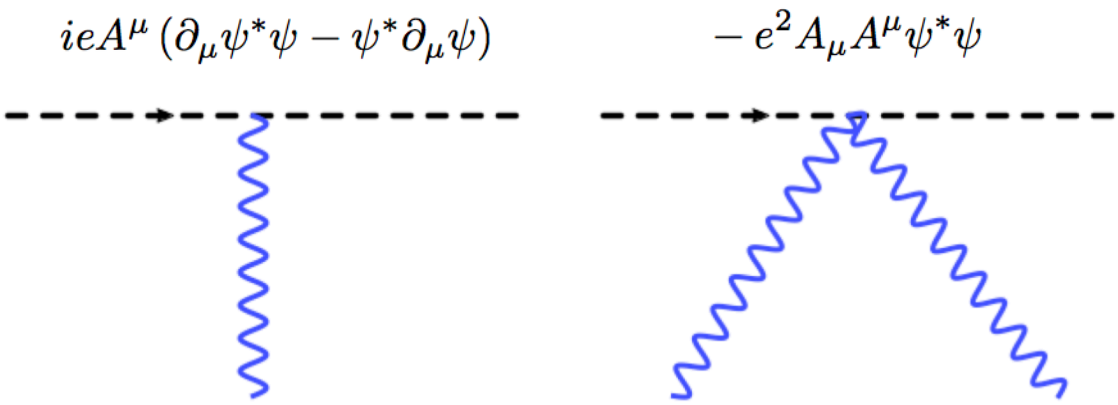


Figure 3.1. The photon-scalar interaction vertices in scalar QED, see Eq. (3.65) and the corresponding discussion in the text more more details.

The opposite sign between the first and second term implies that the charged scalar field ψ contains states with both *positive* and *negative charge*, as opposed to states with only one type of charge. With some abuse of notation we can write the scalar field in terms of creation and annihilation operators as

$$\psi(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3 2\omega}\left(a_s(k)e^{ikx}+b_s^\dagger(k)e^{-ikx}\right)\,, \tag{3.69}$$

so the state created by $a_s^\dagger$ will be called the "positron" e^+ and the state created by $b_s^\dagger$ will be called the "electron" e^-. Needless to say, electron and positron are the antiparticles of each other.

Combining the various ingredients together, is it possible to derive the Feynman rules of scalar QED allowing to construct the corresponding S matrix elements. We do not have the time to carry our their formal derivation, so we just state the results as given:

- For an incoming positron or electron e^+ or e^-, one uses the same rules as in the free scalar theory.
- For an incoming photon: add a factor of $\epsilon^{*\mu}_{\lambda_i}(k_i)$
- For an outgoing photon: add a factor of $\epsilon^{\mu}_{\lambda_i}(k_i)$
- For the interaction vertices, using the sign convention in Fig. 3.2, we add for the $\psi\psi^*A$ vertex a factor

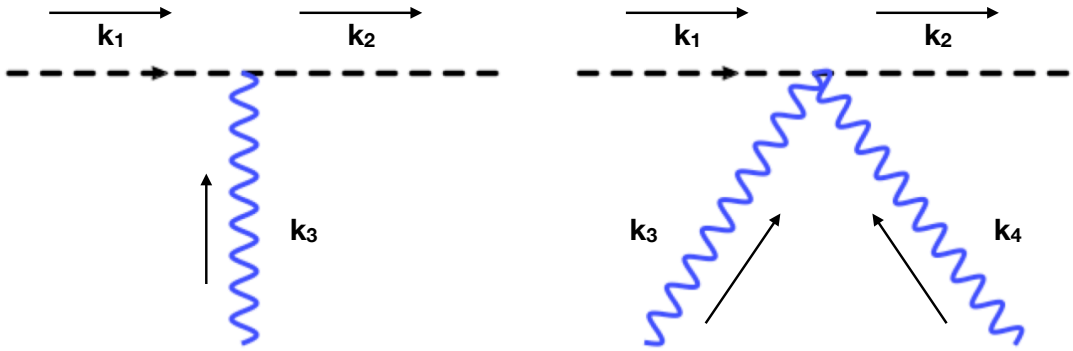
$$ie(k_1+k_2)_\mu(2\pi)^4\delta\left(\sum k_i\right)\,, \tag{3.70}$$

  while for the $\psi\psi^*A^2$ vertex we have instead to include a term

$$-2ie^2\eta_{\mu\nu}(2\pi)^4\delta\left(\sum k_i\right)\,, \tag{3.71}$$

  accounting for all relevant symmetry factors, and where the delta functions ensure momentum propagation at the vertices. The Lorentz indices need to be properly contracted either with a photon propagator, another photon vertex, or with the polarization vectors of an incoming/outgoing photon.

  Note that the second interaction term is formally suppressed by a factor e as compared to the first one.



$$ie(k_1+k_2)_\mu(2\pi)^4\delta\left(\sum k_i\right) \qquad -2ie^2\eta_{\mu\nu}(2\pi)^4\delta\left(\sum k_i\right)$$

Figure 3.2. The Feynman rules for the interaction vertices in scalar QED.

- For the internal lines (propagators) we have

$$\frac{-i}{k^2+i\epsilon}\left(\eta^{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right), \tag{3.72}$$

  for the photon propagator, and

$$\frac{-i}{k^2+m^2-i\epsilon}, \tag{3.73}$$

  for the scalar electron (positron) propagator.

- Integrate over all internal (loop) momenta with a factor $\int d^4k/(2\pi)^4$.

- Sum over all diagrams that contribute to the same final and initial state, including the relevant averages in the initial state case.

With these Feynman rules it is possible to evaluate any process in scalar QED, first at tree level, and then eventually at the loop level using a generalization of the renormalization procedure that has been described in Sect. 2 in the case of $\lambda\phi^4$ theory.

The photon propagator in Eq. (3.72) has been derived using the so-called *generalized Feynman gauge*, also known as the R_ξ gauge. Setting $\xi = 1$ leads to the Feynman gauge, while setting $\xi = 0$ gives the Lorenz gauge (also known as the Landau gauge). Given that the choice for the value of ξ is gauge-dependent, it should drop from all physical quantities, such as scattering amplitudes.

Scattering processes in scalar QED. As a first application of these Feynman rules, let us calculate the tree-level cross-section for *Moller scattering* in scalar QED, namely the scattering between two electrons,

$$e^- + e^- \to e^+ - e^- . \tag{3.74}$$

The two diagrams that contribute to this processes are those shown in Fig. 3.3, which correspond to t-channel and u-channel scattering respectively. Applying the Feynman rules for this theory, we see that the amplitude

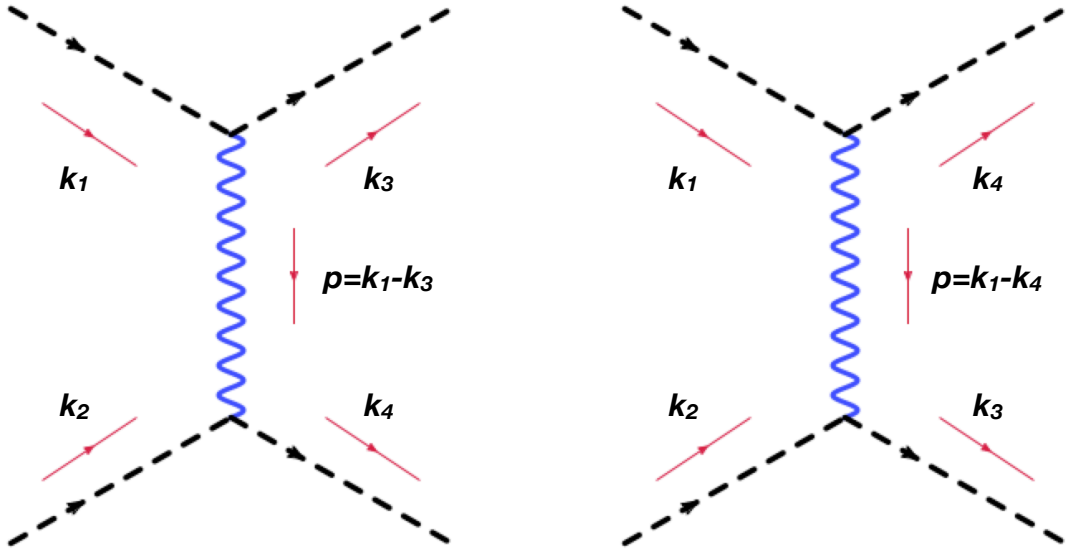


Figure 3.3. Feynman diagrams relevant for the calculation of Moeller scattering, $e^-e^- \to e^-e^-$, at tree level in scalar QED. See text for more details.

for this processes at tree level is obtained by adding the contribution of the s-channel and t-channel diagrams yielding

$$\begin{aligned} \mathcal{M} &= \left[ie(k_1+k_3)_\mu \frac{-i}{p^2}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right) ie(k_2+k_4)_\nu \right. \\ &+ \left. ie(k_1+k_4)_\mu \frac{-i}{\widetilde{p}^2}\left(\eta^{\mu\nu} - (1-\xi)\frac{\widetilde{p}^\mu \widetilde{p}^\nu}{\widetilde{p}^2}\right) ie(k_2+k_3)_\nu \right] (2\pi)^4 \delta(k_1+k_2-k_3-k_4)\,. \end{aligned} \tag{3.75}$$

with $p = k_1 - k_3$ and $\widetilde{p} = k_1 - k_4$ for the $t-$ and $s-$ channel respectively, and the overall delta functions ensure global momentum conservation. Note also that here the $+i\epsilon$ prescription in the photon propagator Eq. (3.72) is not needed since the kinematics of the process ensure that the photon will always be off-shell.

We can further simplify this expression by noting that for the t-channel diagram we have that

$$p^\mu \left(k_1+k_3\right)_\mu = \left(k_1-k_3\right)^\mu \left(k_1+k_3\right)_\mu = k_1^2 - k_3^2 = m^2 - m^2 = 0\,, \tag{3.76}$$

since both electrons have the same mass, and similarly for the u-channel diagram. Taking this property into account, the scattering amplitude simplifies to

$$\mathcal{M}\left(e^-e^- \to e^-e^-\right) = e^2 \frac{\left(k_1+k_3\right)_\mu \left(k_2+k_4\right)^\mu}{t} + e^2 \frac{\left(k_1+k_4\right)_\mu \left(k_2+k_3\right)^\mu}{u}\,, \tag{3.77}$$

where as usual the Mandelstam variables are defined as

$$s = -\left(k_1+k_2\right)^2\,, \quad t = -\left(k_1-k_3\right)^2\,, \quad u = -\left(k_1-k_4\right)^2\,. \tag{3.78}$$

Note that, as expected, the gauge parameter ξ of the photon propagator does not appear in the final result

Eq. (3.77). If it had appeared, it would mean that the result depends on the specific choice of gauge, which is of course unphysical.

From the result Eq. (3.77) we see that the amplitude is the largest when either t or s are small: we say that Moeller scattering in scalar QED has both a *t-channel and a u-channel singularity.*

3.5 Ward identities in QED

In the Coulomb gauge, we found that the photon polarization vectors are perpendicular to the photon propagation direction, $\vec{k} \cdot \vec{\epsilon}_\lambda(k) = 0$, and satisfied the orthonormality conditions in Eq. (3.28). However, upon a closer inspection, one realizes that there might be an inconsistency somewhere in our derivation. To see this, assume that we do a Lorentz transformation, then the four-momentum k^μ is transformed as

$$\Lambda^\mu_\nu k^\nu = k^{'\mu} \,, \tag{3.79}$$

however, in general a Lorentz transformation of the polarization states will mix the physical polarization states with the momentum

$$\Lambda^\mu_\nu \epsilon^\nu_+ = c_1 \epsilon^\nu_+ + c_2 \epsilon^\nu_- + c_3 k^\mu \,, \tag{3.80}$$

where the coefficients c_i depend on the specific choice of transformation matrix Λ.

To see why this is a problem, consider the matrix element that we would get from scattering one photon. Given the Feynman rules of the theory, we know that the matrix element will look like

$$\mathcal{M} = \epsilon_\mu M^\mu \,, \tag{3.81}$$

with ϵ_μ some linear combination of $\epsilon_\pm$ and M_μ transforms as a vector under a Lorentz transformation

$$M_\mu \to M'_\mu = \Lambda^\nu_\mu M_\nu \,. \tag{3.82}$$

Now, a Lorentz transformation of the matrix element $\mathcal{M}$ will lead to

$$\mathcal{M} \to \left(c_1 \epsilon^\mu_+ + c_2 \epsilon^\mu_- + c_3 k^\mu \right) M'_\mu \,, \tag{3.83}$$

which is a problem, since the new polarization vector is *not* part of the physical Hilbert space of the theory, where only linear combinations of $\epsilon^\mu_\pm$ are allowed. In order to ensure that

$$\mathcal{M} = \epsilon_\mu M^\mu \to \epsilon'_\mu M^{'\mu} \,, \tag{3.84}$$

with $\epsilon'_\mu = c_1 \epsilon^\mu_+ + c_2 \epsilon^\mu_-$, we need to impose that

$$\mathcal{M}^\mu k_\mu = 0 \,, \tag{3.85}$$

a property which is known as the *Ward identity.* The formal proof of the Ward identity is somewhat involved and will be covered in the *Advanced Quantum Field Theory* course. Here we will restrict ourselves to verify that the condition Eq. (3.85) holds for explicit cases. The validity of Ward identities is closely related both for the Lorentz and to the gauge invariance of (s)QED. Also, while here we take a look only at tree-level

processes, it can be shown that Eq. (3.85) also holds when loop corrections are included.

In order to highlight the interplay between the Ward and gauge invariance, recall that the propagator of the photon $\Delta_{\mu\mu}(k)$ Eq. (3.72) included a dependence on the unphysical gauge fixing parameter ξ,

$$i\Delta_{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) . \tag{3.86}$$

Since a general matrix element that contains a photon propagator must, due to the Lorentz invariance requirement, have the form

$$\mathcal{M} = \mathcal{M}_{\mu\nu}\Delta^{\mu\nu}(k) , \tag{3.87}$$

then we see that gauge invariance, which demands the independence of $\mathcal{M}$ on the value of ξ, must lead to the following condition:

$$\mathcal{M}_{\mu\nu}k^\mu k^\nu = 0 , \tag{3.88}$$

which is a property closely related to the original Ward identity in Eq. (3.85).

Let us now illustrate the implications of the Ward identify with an explicit example. Let is consider electron-electron scattering into a pair of photons in scalar QED, namely

$$e^- + e^- \to \gamma + \gamma . \tag{3.89}$$

At the tree level, this process receives contributions from both t-channel and u-channel diagrams, as represented in the Feynman diagram of Fig. 3.4. Starting with the t-channel diagram and applying the Feynman rules of the theory, we find that the amplitude reads

$$i\mathcal{M}_t = ie^2\frac{(2k_1 - k_3)_\mu (k_4 - 2k_2)_\nu}{(k_1 - k_3)^2 + m^2}\epsilon_3^{*\mu}\epsilon_3^{*\nu} , \tag{3.90}$$

that after some algebra and imposing the mass-shell condition $k_1^2 = -m^2$, can be written as

$$i\mathcal{M}_t = -ie^2\frac{(k_3\cdot\epsilon_3^* - 2k_1\cdot\epsilon_3^*)(k_4\cdot\epsilon_4^* - 2k_2\cdot\epsilon_4^*)}{k_3^2 - 2k_1\cdot k_3} , \tag{3.91}$$

with the corresponding u-channel diagram giving a closely related result

$$i\mathcal{M}_u = -ie^2\frac{(k_4\cdot\epsilon_4^* - 2k_1\cdot\epsilon_3^*)(k_3\cdot\epsilon_3^* - 2k_2\cdot\epsilon_4^*)}{k_3^2 - 2k_2\cdot k_3} . \tag{3.92}$$

The full matrix element is of course given by the sum of the two diagrams,

$$\mathcal{M}\left(e^- + e^- \to \gamma + \gamma\right) = \mathcal{M}_t\left(e^- + e^- \to \gamma + \gamma\right) + \mathcal{M}_u\left(e^- + e^- \to \gamma + \gamma\right) . \tag{3.93}$$

For this amplitude to be physically consistent, as discussed above, it needs to satisfy the Ward identity, Eq. (3.85). From the practical point of view, this corresponds to replacing the polarization vector ϵ_3^* with the corresponding four-momentum k_3 (or the same with $3 \to 4$). Doing this replacement, some terms cancel out between the numerator and the denominator, and we find

$$\left(\mathcal{M}_t + \mathcal{M}_u\right)_{\epsilon_3\to k_3} = -e^2\left[k_4\cdot\epsilon_4^* - 2k_2\cdot\epsilon_4^* + k_4\cdot\epsilon_4^* - 2k_1\cdot\epsilon_4^*\right] = -2e^2\left[k_4 - k_2 - k_1\right]\times\epsilon_4^* \neq 0 , \tag{3.94}$$

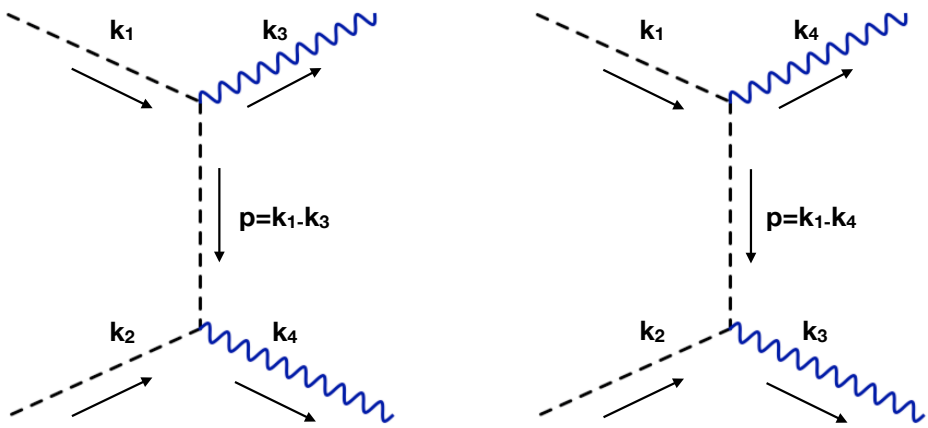


Figure 3.4. Feynman diagrams relevant for the calculation of the tree level scattering process $e^-e^- \to \gamma\gamma$ in scalar QED. See text for more details.

which is different from zero, and thus the Ward identity is not satisfied. Does it mean that the theory is internally inconsistent?

The reason why the Ward identity seems to fail is that the diagrams shown in Fig. 3.4 are not the only diagrams that contribute to the $e^-e^- \to \gamma\gamma$ scattering amplitude at $\mathcal{O}(e^2)$: at this order, also the interaction vertex with two gauge fields and two scalar fields (right diagram in Fig. 3.2) contributes, and therefore needs to be taken into account. Using the convention for the sign of the momenta shown in Fig. 3.5 and the corresponding Feynman rule, the amplitude corresponding to this additional diagram turns out to be

$$i\mathcal{M}_4 = -2ie^2\eta_{\mu\nu}\epsilon_3^{*\mu}\epsilon_4^{*\nu}\,, \tag{3.95}$$

and doing the same replacement as above we get

$$i\mathcal{M}_4\Big|_{\epsilon_3\to k_3} = -2ie^2\eta_{\mu\nu}k_3^\mu\epsilon_4^{*\nu} = -2ie^2 k_3\cdot\epsilon_4^*\,. \tag{3.96}$$

Now if we add the contributions from the three diagrams we get

$$\left(\mathcal{M}_t + \mathcal{M}_u + M_4\right)_{\epsilon_3\to k_3} = -2e^2\left[k_4 + k_3 - k_2 - k_1\right]\times\epsilon_4^* = 0\,, \tag{3.97}$$

due to overall momentum conservation, since $k_1 + k_2 = k_3 + k_4$. So reassuringly enough, the Ward identity is explicitly verified, as expected from the general symmetries of the theory, in particular from the Lorenz and gauge invariance requirements. One can check that Ward identities also hold for similar tree-level process. As mentioned above, a formal proof is beyond the scope of this course.

Crucially, the above *derivation* of the Ward identities did not require at any point to assume that the outgoing photons were *on-shell* and thus *massless*. indeed, at no point we have used that $k_3^2 = 0$ or $k_3^2 = 0$, which would imply that the outgoing photons are on-shell. This has the very important consequence that Ward identities should be valid even if the photon if off-shell, for example, a photon in a *loop* or a photon appearing as an *internal propagator* in a Feynman diagram.

In order to give an explicit example of the other relation introduced above related to the Ward identities,

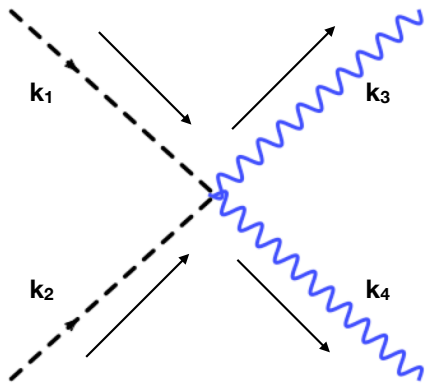


Figure 3.5. The other Feynman diagram that contributes to the scattering amplitude of the $e^-e^- \to \gamma\gamma$ in scalar QED at $\mathcal{O}\left(e^2\right)$, in addition to the two diagrams already shown in Fig. 3.5.

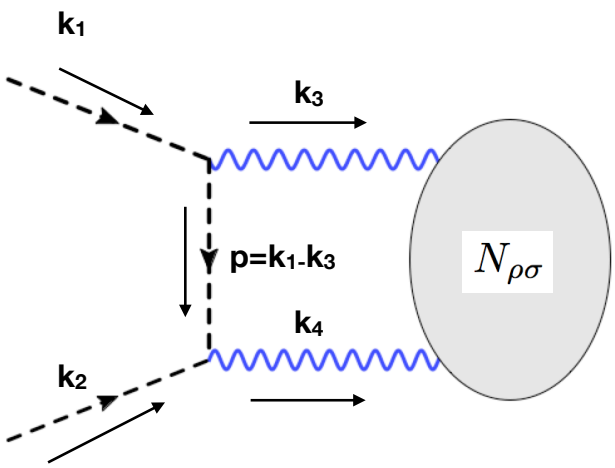


Figure 3.6. The Feynman diagram for a similar scattering amplitude as that of Fig. 3.4 but where now the two photons are internal lines (so off-shell) rather than outgoing particles, and where we denote in full generality the rest of the diagram by $N^{\rho\sigma}$.

Eq. (3.88), which is directly related to the gauge invariance of the theory, we need to consider diagrams with internal photons (since only there the photon propagator, including the gauge fixing parameter ξ, appears). In order to achieve this, let us consider the diagram in Fig. 3.6, which is similar to that of $e^-e^- \to \gamma\gamma$ scattering in Fig. 3.4 but now the two photons are internal lines, and we denote by $N^{\rho\sigma}$ the general reminder of the matrix element (that as we will show we do not need to specify in order to show the gauge independence of the whole scattering amplitude).

Let us now compute the Feynman diagram in Fig. 3.6 by using the Feynman rules of scalar QED. Recall that the two photons in the diagram, with momenta k_3 and k_4 respectively, are internal lines and thus we need to integrate over all their possible values, taking into account overall momentum conservation. Then one finds the following amplitude:

$$\mathcal{M} = e^2 \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \delta^{(4)}(k_1 + k_2 - k_3 - k_4) \frac{\left(k_3^\mu - 2k_1^\mu\right)\left(k_4^\nu - 2k_2^\nu\right)}{k_3^2 - 2k_3 \cdot k_1} \Delta_{\mu\rho}(k_3)\Delta_{\nu\sigma}(k_4) N^{\rho\sigma} \,. \tag{3.98}$$

This result is clearly not gauge invariant, since contracting the momentum dependence from the photon vertices with the propagators does not cancel out the terms proportional to ξ, the gauge fixing terms.

Of course, this is per se not surprising: individual diagrams do not need to be gauge invariant, and only the sum of all diagrams that contribute to a physical scattering amplitude has to be gauge invariant. This is the same property as that illustrated by the previous example: only when adding the two diagrams of Fig. 3.4 with that of Fig. 3.5 the Ward identify was satisfied. In this case, we need to supplement the diagram of Fig. 3.6 with the corresponding versions of the diagrams in Fig. 3.4 (right) and Fig. 3.5 where the photons being internal lines attached to the rest of the diagram, which is denoted generically by $N^{\rho\sigma}$.

Accounting for these two additional diagrams, the result of Eq. (3.98) now reads

$$\begin{aligned}\mathcal{M} &= e^2 \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \delta^{(4)}(k_1+k_2-k_3-k_4) \left[\frac{(k_3^\mu - 2k_1^\mu)(k_4^\nu - 2k_2^\nu)}{k_3^2 - 2k_3\cdot k_1} \right. \\ &+ \left. \frac{(k_3^\mu - 2k_2^\mu)(k_4^\nu - 2k_1^\nu)}{k_3^2 - 2k_3\cdot k_2} + 2\eta^{\mu\nu} \right] \Delta_{\mu\rho}(k_3)\Delta_{\nu\sigma}(k_4) N^{\rho\sigma} \,. \end{aligned} \tag{3.99}$$

Now, the term which is not gauge invariance in the photon propagator $\Delta_{\mu\nu}(k_3)$ is proportional to $k_3^\mu k_3^\nu$. Replacing $\Delta_{\mu\nu}(k_3)$ by $k_3^\mu k_3^\nu$ in Eq. (3.99), in order to isolate the gauge-dependent contribution to the total scattering amplitude, leads to the following result:

$$\begin{aligned}\mathcal{M}_{\rm gauge-dep} &\propto e^2 \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \delta^{(4)}(k_1+k_2-k_3-k_4) \left[\frac{(k_3^\mu - 2k_1^\mu)(k_4^\nu - 2k_2^\nu)}{k_3^2 - 2k_3\cdot k_1} \right. \\ &+ \left. \frac{(k_3^\mu - 2k_2^\mu)(k_4^\nu - 2k_1^\nu)}{k_3^2 - 2k_3\cdot k_2} + 2\eta^{\mu\nu} \right] k_3^\mu k_3^\nu \Delta_{\nu\sigma}(k_4) N^{\rho\sigma} = 0 \,. \end{aligned} \tag{3.100}$$

which vanishes due to the overall four-momentum conservation imposed by the delta function.

This exercise demonstrates that, irrespectively of the specific form that the rest of the diagram $N^{\rho\sigma}$, the part of the diagram with the two internal photon lines will be independent of the gauge fixing parameter ξ. It can be checked using higher order diagrams that this property always holds, as required from the Lorenz and gauge invariance of the theory. An all-order proof can be derived using the path-integral formalism, as will be discussed in the *Advanced Quantum Field Theory* course.

3.6 Lorentz invariance and charge conservation

In the last section of this second part of the course, we present a rather elegant illustration of the direct connection between Lorentz invariance and electric charge conservation, which is independent of the gauge symmetry requirements. This can be achieved by showing that a theory that includes a massless spin-one particle (such as QED or scalar QED) has associated automatically a *conserved charge* (irrespective of whether or not gauge symmetry is present).

To demonstrate this remarkable property, consider a scattering amplitude $\mathcal{M}_0(\{p_i\})$ with n external legs (which can be either incoming or outgoing) in scalar QED, such as that represented in Fig. 3.7. The amplitude will depend in general on the n four-momenta of the external legs, denoted by $(\{p_i\})$. Next, we want to attach an outgoing photon to one of the external legs of the diagram, and consider all possible ways in which this attachment can be done. To begin with, one can attach the photon to an incoming

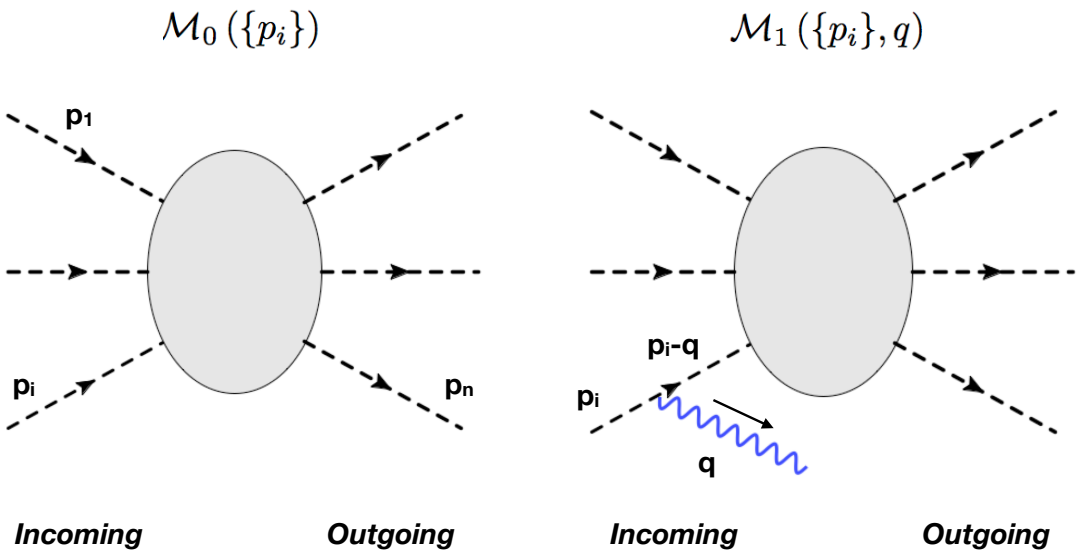


Figure 3.7. Left plot: a generic scattering amplitude in scalar QED, denoted by $\mathcal{M}(\{p_i\})$, consisting of a number of incoming and outgoing (scalar) electrons or positrons. Right plot: the same diagram, now with an outgoing photon with momentum q attached to the incoming scalar leg i. In this case the full scattering amplitude is denoted by $\mathcal{M}(\{p_i\}, q)$, see text for more details.

electron/positron leg, as shown in Fig. 3.7. This leads to the following amplitude, denoted $\mathcal{M}_1(\{p_i\}, q)$, and which now depends in addition on q, the photon momentum,

$$\mathcal{M}_1(\{p_i\}, q) = e\frac{(2p_i^\mu - q^\mu)}{(p_i - q)^2 + m^2}\epsilon_\mu(q)\mathcal{M}_0(p_1, p_2, \ldots, p_i - q, \ldots, p_n)\,, \tag{3.101}$$

where $\mathcal{M}_0(p_1, p_2, \ldots, p_i - q, \ldots, p_n)$ is the original scattering amplitude $\mathcal{M}_0(\{p_i\})$ but with the replacement $p_i \to p_i - q$ due to the effects of attaching the outgoing photon to the i-th incoming scalar leg.

Next, by using the mass-shell relation $p_i^2 = -m^2$ for the (scalar) electrons, the orthogonality between polarization and momentum of the photon, $q \cdot \epsilon = 0$, and the fact that on-shell photons are massless, $q^2 = 0$, we can write the amplitude as

$$\mathcal{M}_{1,\text{in},-} = -e\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_1, p_2, \ldots, p_i - q, \ldots, p_n)\,, \tag{3.102}$$

where the subindex indicates that the photon has been attached to an incoming electron. On the other hand, we could also attach the outgoing photon to a leg j with opposite charge than leg i, and then one would get

$$\mathcal{M}_{1,\text{in},+} = +e\frac{p_j \cdot \epsilon}{p_j \cdot q}\mathcal{M}_0(p_1, p_2, \ldots, p_j - q, \ldots, p_n)\,, \tag{3.103}$$

which is the same as for electrons but with an overall change of sign. Using the same considerations, if the photon is attached to an outgoing scalar leg k one would find for the amplitude,

$$\mathcal{M}_{1,\text{out},\pm} = \pm e\frac{p_k \cdot \epsilon}{p_k \cdot q}\mathcal{M}_0(p_1, p_2, \ldots, p_k + q, \ldots, p_n)\,, \tag{3.104}$$

where note that now the replacement is $p_k \to p_k + q$ since the four-momentum p_k is outgoing. Finally, we

can attach the outgoing photon to some of the internal lines of the diagram, leading to an amplitude denoted by $\mathcal{M}_{1,\text{int}}(\{p_i\}, q)$.

In order to derive the connection between the Lorentz invariance of the theory and the fact that there is a conserved charge (that we can identify with the electric charge) we need to take the so-called *soft limit* for the outgoing photon. This means that all the components of q are much smaller than any of the external momenta $\{p_i\}$. To be precise, we define the soft limit by the following condition:

$$|q \cdot p_i| \ll |p_j \cdot p_k| \,. \tag{3.105}$$

In the soft limit, we can approximate all terms where the photon momentum appears additively as follows

$$\mathcal{M}_0(p_1, p_2, \ldots, p_i \pm q, \ldots, p_n) \simeq \mathcal{M}_0(p_1, p_2, \ldots, p_i, \ldots, p_n) \,, \tag{3.106}$$

and therefore the only relevant contributions in this limit are those where the photon is attached to the external scalar electron/positron legs, where the terms of the form $1/(p_i \cdot q)$ become large. Since for $\mathcal{M}_{1,\text{int}}(\{p_i\}, q)$ there will be no terms of this form, we can write the scattering amplitude with the extra outgoing photon in the soft limit, including all possible ways to attach this photon, as follows

$$\mathcal{M}_{1,\text{soft}} \simeq e\mathcal{M}_0\left(\{p_i\}\right) \left[\sum_{\text{incoming}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} - \sum_{\text{outgoing}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} \,, \right] \tag{3.107}$$

where $Q_i = \pm$ depending on the specific charge carried by the external leg.

Now the crucially observation is that under a Lorentz transformation any scattering amplitude $\mathcal{M}$ should be *Lorentz invariant*, since $\mathcal{M}$ is manifestly a Lorentz scalar. On the other hand, we know that the photon polarization states ϵ_μ themselves are not Lorentz invariant. So assume we have a Lorentz transformation such that

$$\begin{aligned} \epsilon_\mu &\rightarrow \epsilon_\mu + q_\mu \,, \\ q_\mu &\rightarrow q_\mu \,, \end{aligned} \tag{3.108}$$

which defines the so-called *little group* of the Lorentz group. If we apply this Lorentz transformation on the matrix element Eq. (3.107), we find that

$$\mathcal{M}_{1,\text{soft}} \rightarrow \mathcal{M}_{1,\text{soft}} + e\mathcal{M}_0 \left[\sum_{\text{incoming}} Q_i - \sum_{\text{outgoing}} Q_i \right] \tag{3.109}$$

where the original matrix element $\mathcal{M}_0\left(\{p_i\}\right)$ is invariant under this Lorentz transformation since it does not contain any dependence on the photon momentum q.

Therefore, we find that the only way for the scattering amplitude Eq. (3.107) to be Lorentz invariant is to ensure that

$$\sum_{\text{incoming}} Q_i = \sum_{\text{outgoing}} Q_i \,, \tag{3.110}$$

in other words, that electric charge is conserved in all interactions of the scalar QED theory. We emphasize

that this argument does not reply on gauge symmetry, and highlights the close connection between the conservation of electric charge and the Lorentz invariance of (scalar) QED. A similar derivation can be carried out in the case of the spinor electrodynamics.

3.7 Scalar QED at the one loop level

Now we put together a number of important topics that we have learned so far, and try to have a go at computing *one-loop corrections* to the QFT of photons interacting with charged scalar fields, namely scalar QED. Specifically, we will show how to compute some of the counterterms of scalar QED, using the same techniques that were used in the context of $\lambda\phi^4$ theory.

The starting point of the calculation is the Lagrangian of scalar QED at the tree level, namely

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D^\mu\psi^\dagger D_\mu\psi - m^2\psi^\dagger\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^*\psi + ieA^\mu\left[\left(\partial_\mu\psi^\dagger\right)\psi - \psi^\dagger\left(\partial_\mu\psi\right)\right] - e^2A_\mu A^\mu\psi^\dagger\psi\,, \end{aligned} \tag{3.111}$$

where we have expanded the covariant derivative in order to better highlight the different types of interaction operators that appear in the theory. Recall that this Lagrangian already includes all possible terms with positive or zero mass dimension coefficients and that satisfy Lorentz, U(1) gauge symmetry, parity, time reversal, and charge conjugation.

What we need to do now is to include the counterterms to account for the fact that beyond the tree level the bare parameters of the Lagrangian are not physical observables, and we can redefine them in order to renormalize the theory, absorbing this way the ultraviolet infinities that arise in the one-loop scattering amplitudes. Once we account for these counterterms, the full Lagrangian of scalar QED now reads

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1\,, \\ \mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial^\mu\psi^*\partial_\mu\psi - m^2\psi^*\psi\,, \\ \mathcal{L}_1 &= iZ_1e\left[\psi^\dagger\partial^\mu\psi - \left(\partial^\mu\psi^\dagger\right)\psi\right]A_\mu - Z_4e^2\psi^\dagger\psi A^\mu A_\mu - \frac{1}{4}Z_\lambda\lambda\left(\psi^\dagger\psi\right)^2 + \mathcal{L}_{\rm ct}\,, \\ \mathcal{L}_{\rm ct} &= -(Z_2-1)\partial^\mu\phi^\dagger\partial_\mu\phi - (Z_m-1)m^2\phi^\dagger\phi - \frac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu}\,. \end{aligned} \tag{3.112}$$

We have allowed for completeness a self-interaction term of the scalar field, $\sim\lambda\left(\psi^\dagger\psi\right)^2$, but in the following we will not use it and we can confidently set it to $\lambda = 0$.

The structure of the Lagrangian in Eq. (3.112) is the following. First of all, $\mathcal{L}_0$ contains the kinetic terms, namely those corresponding to the free theory. The interaction terms are then contained in $\mathcal{L}_1$, which is divided into the same terms as in the original Lagrangian but with the renormalization factors Z_i included and the $\mathcal{L}_{\rm ct}$ part, which contain the proper counterterms. Note that at tree level one has that $Z_i = 1$, and thus one has that $\mathcal{L}_{\rm ct} = 0$. Although it is beyond the scope of these lectures to give a formal proof, scalar QED is a *renormalizable theory*, which means that once I compute the values of the counterterms $Z_1, Z_4, Z_\lambda, Z_2, Z_m, Z_3$ at any given order in perturbation theory, then any other general physical observable will also be finite at the same perturbative order.

Note that in Eq. (3.112) we have explicitly separated the free theory kinetic terms (contained in $\mathcal{L}_0$) from the interaction terms in $\mathcal{L}_1$. The latter include the pure counterterm operators, $\mathcal{L}_{\rm ct}$, which at this point

are treated as interactions. The advantage of this way of writing the Lagrangian of the theory is that in Feynman diagrams we can always use the free theory propagators of the photon and scalar fields, and the counterterms in $\mathcal{L}_{\rm ct}$ are treated as *new interaction terms* with their associated Feynman rules. In this way the renormalization process is somewhat more transparent, and the only thing we need to do is to determine the explicit values of the Z_i at any given order in perturbation theory.

Specifically, let us illustrate the computation of the photon self-energy $\Pi^{\mu\nu}(k)$ at $\mathcal{O}(e^2)$ in the perturbative expansion. Recall that in the case of $\lambda\phi^4$ theory, the calculation of the one loop corrections to the scalar self-energy lead to the definition of the *physical mass*. A photon is massless and so this is not possible here, and what this calculation will lead us is to the explicit form of the renormalized (interacting) propagator. As in the case of $\lambda\phi^4$, this self-energy arises from the loop corrections to the photon propagator. Indeed, the exact photon propagator in momentum space can be written as

$$\mathbf{\Delta}_{\mu\nu}(k) = \Delta_{\mu\nu}(k) + \Delta_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\Delta_{\sigma\nu}(k) + \ldots \tag{3.113}$$

with $i\Pi^{\mu\nu}(k)$ given by the sum of all 1PI diagrams with two external photon lines (and with the photon propagators removed) and where the free-photon propagator has the standard expression in the R_ξ gauge,

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) . \tag{3.114}$$

This series is a geometric one that can be summed to all orders once the expression for the self-energy $i\Pi^{\mu\nu}(k)$ is known at any given order in perturbation theory, in the same way as in the case of $\lambda\phi^4$ theory.

Recall also that in our discussion of the Ward identities in Sect. 3.5 we reasoned that the photon propagator should be *transverse*, namely the fact that in any amplitude containing a photon propagator $\mathcal{M} = \mathcal{M}_{\mu\nu}\Delta^{\mu\nu}(k)$ should satisfy $\mathcal{M}_{\mu\nu}k^\mu k^\nu = 0$ to ensure that the amplitude is independent of the unphysical gauge fixing parameter ξ. From Eq. (3.114) this has the consequence that the photon self-energy should be transverse as well, that is, it should satisfy the condition

$$\Pi^{\mu\nu}(k)k_\mu k_\nu = 0 . \tag{3.115}$$

As we will show now, this condition is indeed satisfied by the explicit calculation of the self-energy at $\mathcal{O}(e^2)$.

The Feynman diagrams that contribute to the photon self-energy at $\mathcal{O}(e^2)$ are shown in Fig. 3.8. Note that they contain both the one-loop diagrams and the contributions from the counterterms, which as we know are required in order to cancel the ultraviolet divergences of the theory. Specifically, the last diagram in Fig. 3.8 corresponds to the counterterm associated to the photon kinetic term in $\mathcal{L}_{\rm ct}$ in Eq. (3.112). The contribution for these three diagrams can be computed using the Feynman rules of scalar QED, suitably modified in order to account for the counterterm, required once we start doing loop calculations. The result for the sum of the three diagrams is given by

$$\begin{aligned} i\Pi^{\mu\nu}(k) &= -(iZ_e e)^2 \int \frac{d^4l}{(2\pi)^4} \frac{(2l+k)^\mu (2l+k)^\nu}{((l+k)^2+m^2)(l^2+m^2)} \\ &+ (-2iZ_4)e^2\eta^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2+m^2} - i(Z_3-1)\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) , \end{aligned} \tag{3.116}$$

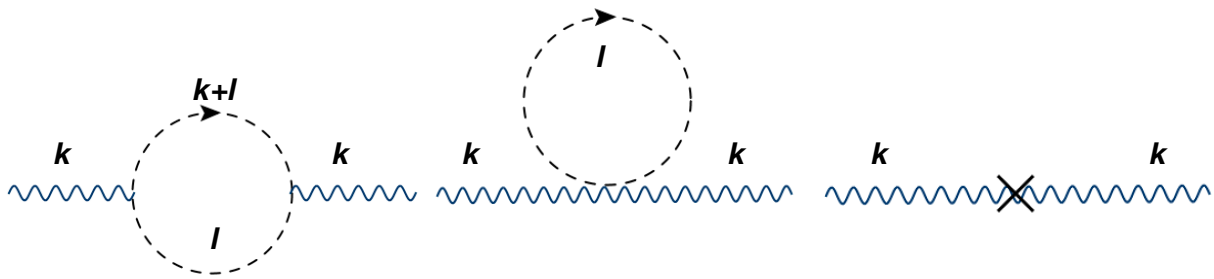


Figure 3.8. The three Feynman diagrams for the $\mathcal{O}\left(e^2\right)$ contributions to the photon self-energy in scalar QED. The last term corresponds to the counterterm associated to the photon propagator being treated as an interaction term.

where note that the last diagram in Fig. 3.8 has associated a Feynman rule proportional to $\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)$.[9] Given that $Z_i = 1 + \mathcal{O}(e^2)$ (since all counterterms are one at the Born level), we can set $Z_1 = Z_4 = 1$ in the expression above since the subsequent terms would be formally higher order. We then see that Z_3 is the only counterterm relevant for the calculation of the photon self-energy at this perturbative order.

Next we can combine the two first terms in Eq. (3.116), those that involve an integral over the loop momentum l, into a single term, to have

$$i\Pi^{\mu\nu}(k) = e^2 \int \frac{d^4 l}{(2\pi)^4} \frac{N^{\mu\nu}}{((l+k)^2+m^2)(l^2+m^2)} - i(Z_3-1)\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) , \tag{3.117}$$

where by doing some simple algebra one can check that the numerator is given by

$$N^{\mu\nu} = (2l+k)^\mu (2l+k)^\nu - 2\left[(l+k)^2 + m^2\right]\eta^{\mu\nu} . \tag{3.118}$$

The integral over l is clearly divergent in the ultraviolet, since at large momenta we have $\sim \int d^4 l \; (l^2/l^4) = \int l^3 dl \; (l^{-2}) \sim \Lambda^2_{\rm uv}$. Therefore in order to preempt the fact that we need to work in $d = 4 - \epsilon$ space-time dimensions in order to regulate the ultraviolet divergent that would arise from this integral we replace $e \to e\mu^{\epsilon/2}$ (so that the electron charge is still dimensionless in d dimensions) and then combine the two denominators using Feynman's formula. The result is given by

$$i\Pi^{\mu\nu}(k) = e^2\mu^\epsilon \int_0^1 dx \frac{d^d q}{(2\pi)^d} \frac{M^{\mu\nu}}{(q^2+D)^2} - i(Z_3-1)\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) , \tag{3.119}$$

where we have defined $q = l + xk$, $D = x(1-x)k^2 + m^2$, and the numerator is given by

$$M^{\mu\nu} = (2q + (1-2x)k)^\mu \left(2q + (1-2x)k\right)^\nu - 2\left[(q+(1-x)k)^2 + m^2\right]\eta^{\mu\nu} . \tag{3.120}$$

At this point we have everything that we need in order to evaluate the photon self-energy $\Pi^{\mu\nu}$ in scalar electrodynamics at $\mathcal{O}\left(e^2\right)$. Doing some algebra with the numerator, including canceling terms linear in q and reshuffling the integration variables one finds that one can make the following replacement:

$$N^{\mu\nu} \to -4y^2(k^2 g^{\mu\nu} - k^\mu k^\nu) , \tag{3.121}$$

[9] See the course textbooks for a derivation of this result. Note that of course at the Born level $Z_3 = 1$ and this term vanishes.

with $x = y + 1/2$. Therefore we find that the self-energy $\Pi^{\mu\nu}(k)$ is transverse, meaning that

$$\Pi^{\mu\nu}(k) k_\mu k_\nu = 0 \,, \tag{3.122}$$

as required from the gauge invariant of the theory, as discussed above in the context of the Ward identities. Is it possible to verify by explicit computation that the same property holds for higher perturbative orders.

Next, we can perform the integral over the rescaled loop momentum q, and focusing only in the divergent part, we can expand the result to get

$$\mu^\epsilon \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{8\pi^2} \frac{1}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \,. \tag{3.123}$$

Following this, we need to compute the integral over y to get

$$\int_{-1/2}^{1/2} dy N^{\mu\nu} = -\frac{1}{3} \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu\right) \,, \tag{3.124}$$

and combining all the results obtained so far, we achieve the following results for the photon-self energy at the first non-trivial order in the perturbative expansion:

$$\Pi^{\mu\nu}(k) = \Pi(k^2) \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu\right) \,, \tag{3.125}$$

where we have defined

$$\Pi(k^2) = -\frac{e^2}{24\pi^2} \frac{1}{\epsilon} + \text{finite} - (Z_3 - 1) \,. \tag{3.126}$$

Therefore, this calculation gives us the answer for the first non-trivial correction to the Z_3 counterterm, namely

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\epsilon} + \ldots \,, \tag{3.127}$$

where the dots indicate possible constant terms.[10] We have thus shown that the quadratic divergence that affects the one-loop diagrams in Fig. 3.8 for the photon self-energy is canceled out thanks to the counterterm Z_3 that redefines the physical photon propagator.

Using a similar approach one can evaluate all the other counterterms of the theory. Specifically, one needs to evaluate the one-loop corrections to the *scalar field propagator* as well as to the *three-point* and *four-point interaction vertices*. This way, the complete set of counterterms of scalar QED will be determined, allowing the computation of general finite physical observables at a given order in the perturbative expansion.

3.8 QED: spinor electrodynamics

We conclude this part of the course with a quick introduction to spinor QED, the Quantum Field Theory describing the interactions between (Dirac) fermions and photons. We will be brief in this discussion, since most of what we have seen for the case of scalar electrodynamics applies also as well for spinor electrodynamics.

In standard QED, the electromagnetic current $j^\mu(x)$ is proportional to the *Noether current* corresponding

[10] Different renormalization choices will lead to different constant terms in the definition of the counterterms of the theory.

to the U(1) global symmetry of the Dirac field. A straightforward computation leads to

$$j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\Psi(x)\,, \tag{3.128}$$

where e is the electron charge in the Heaviside-Lorentz system of units. Taking into account Eq. (3.128), the full Lagrangian of spinor QED will be given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi + e\bar{\Psi}\gamma^\mu\Psi A_\mu\,. \tag{3.129}$$

For the time being we are concerned only with tree-level processes, so we do not need to add counterterms of this Lagrangian yet.

This Lagrangian is invariant under a U(1) gauge symmetry, as we saw in the case of scalar QED, namely:

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\lambda(x)\,, \tag{3.130}$$

$$\Psi(x) \to \exp\left(-ie\lambda(x)\right)\Psi(x)\,. \tag{3.131}$$

Note that the Lagrangian is invariant under this gauge transformation irrespectively of whether of not the fields obey the equations of motion. As we know, this property is more transparent once the Lagrangian is expressed in terms of the covariant derivative

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}D\!\!\!/\Psi - m\bar{\Psi}\Psi\,, \tag{3.132}$$

which has been defined as

$$D_\mu \equiv \partial_\mu - ieA_\mu\,, \tag{3.133}$$

and which transforms in the same way (*covariantly*) as the Dirac field under a gauge transformation, as is easy to check. A useful expression relates the covariant derivative with the field-strength tensor, namely

$$F^{\mu\nu} = \frac{i}{e}\left[D^\mu, D^\nu\right]\,, \tag{3.134}$$

where recall that $F^{\mu\nu}$ is invariant under a gauge transformation.

Using familiar techniques, it is possible to derive the Feynman rules of spinor QED:

- For each interaction vertex: $ie\gamma^\mu$,
- For each internal photon: $-ig^{\mu\nu}/(k^2 - i\epsilon)$,
- For each internal fermion: $-i(-p\!\!\!/ + m)/(p^2 + m^2 - i\epsilon)$,

where we are working in the Feynman gauge $\xi = 1$ and where the rules for the external fermions and photons are the same ones as in the corresponding free theories. When working in spinor QED one has to be rather careful with the spinor order, since spinors do not commute among them: in particular, spinor indices should be contracted starting at one end of a fermion line.

Let us now show how we can use these Feynman rules in order to compute an scattering amplitude (and the corresponding spin-averaged square) in spinor QED. Specifically, we will consider the process of

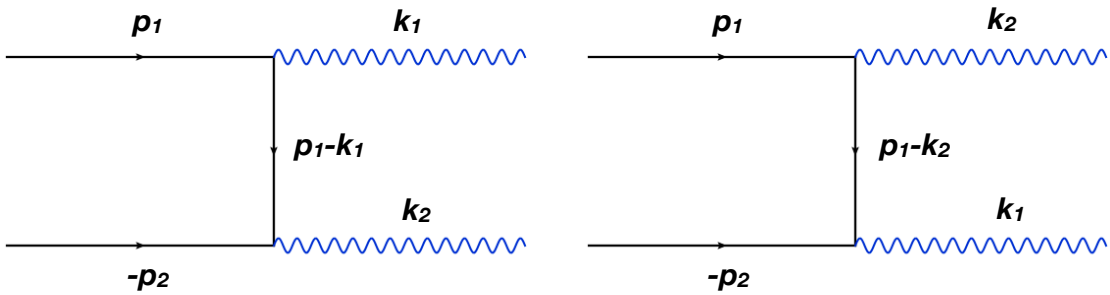


Figure 3.9. The Feynman diagram for the process of electron-positron annihilation into a photon pair in spinor QED, corresponding to t-channel and u-channel scattering respectively.

electron-positron annihilation into a photon pair,

$$e^+ + e^- \to \gamma + \gamma \,, \tag{3.135}$$

whose Feynman diagram is represented in Fig. 3.9. Using the Feynman rules of the theory we find that the scattering amplitude of this process $\mathcal{M}$ will be given by

$$\mathcal{M} = e^2 \epsilon^{\mu}_{1'} \epsilon^{\nu}_{2'} \bar{v}_2 \left[\gamma_\nu \left(\frac{-\not{p}_1 + \not{k}_1 + m)}{-t + m^2} \right) \gamma_\mu + \gamma_\mu \left(\frac{-\not{p}_1 + \not{k}_2 + m)}{-u + m^2} \right) \gamma_\nu \right] u_1 \tag{3.136}$$

where the primes indicate that a particle corresponds to the final state, and where we have defined the usual Mandelstam variables:

$$s = -(p_1 + p_2)^2 \,, \quad t = -(p_1 - k_1)^2 \,, \quad u = -(p_1 - k_2)^2 \,. \tag{3.137}$$

Recall that the three Mandelstam variables are related among them due to four-momentum conservation as $s + t + u = 2m^2$. As is clear from the structure of the amplitude $\mathcal{M}$, these two diagrams will have associated t-channel and u-channel singularities respectively.

The amplitude Eq. (3.136) is not particularly illustrative since it depends on the specific spins of the incoming fermions and on the specific polarization of the outgoing photons. Let us now evaluate the square of the scattering amplitude, as needed in order to compute the cross-section for this process. To do this, we write our scattering amplitude as

$$\mathcal{M} \equiv \epsilon^{\mu}_{1'} \epsilon^{\nu}_{2'} \bar{v}_2 A_{\mu\nu} u_1 \,, \tag{3.138}$$

where $A_{\mu\nu}$ is a matrix in spinor space. Squaring this amplitude leads us, using the properties of the spinor algebra, to

$$|\mathcal{M}|^2 = \epsilon^{\mu}_{1'} \epsilon^{\nu}_{2'} \epsilon^{\rho *}_{1'} \epsilon^{\sigma *}_{2'} \left(\bar{v}_2 A_{\mu\nu} u_1 \right) \left(\bar{u}_1 A_{\sigma\rho} v_2 \right) \,. \tag{3.139}$$

In order to simplify this expression, we will use that fact that we are interested in the squared amplitude averaged over initial state spins of the electron and the positron. The calculation of this average, using the spinor algebra techniques, leads to

$$|\mathcal{M}|^2 = \frac{1}{4} \epsilon^{\mu}_{1'} \epsilon^{\nu}_{2'} \epsilon^{\rho *}_{1'} \epsilon^{\sigma *}_{2'} \mathrm{Tr} \left[A_{\mu\nu} \left(-\not{p}_1 + m \right) A_{\sigma\rho} \left(-\not{p}_2 + m \right) \right] \,. \tag{3.140}$$

In addition to averaging over the spins of the electron and the positron, in general one is not able to measure the polarization of the final-state photons in a hard-scattering reaction, and therefore we wish to sum over the final state photon polarizations.[11] From Eq. (3.140) is clear that we need to evaluate sums over the photon polarization vectors of the form

$$\sum_{\lambda=\pm} \epsilon^{\mu}_{\lambda}\left(\vec{k}\right)\epsilon^{\rho *}_{\lambda}\left(\vec{k}\right) , \tag{3.141}$$

which can be computed *e.g.* in the Coulomb gauge, and where using the fact that photon is on-shell, $k^2 = 0$ and that the amplitudes involving external photons must satisfy the Ward identities, one can derive the following rather compact substitution rule

$$\sum_{\lambda=\pm} \epsilon^{\mu}_{\lambda}\left(\vec{k}\right)\epsilon^{\nu *}_{\lambda}\left(\vec{k}\right) \to g^{\mu\nu} . \tag{3.142}$$

Note that if we multiply the above equation by $k_\mu k_\nu$ we find

$$\sum_{\lambda=\pm} \left(k_\mu \epsilon^{\mu}_{\lambda}\left(\vec{k}\right)\right)\left(k_\nu \epsilon^{\nu *}_{\lambda}\left(\vec{k}\right)\right) = k^2 = 0 , \tag{3.143}$$

which is consistent with the fact that the polarization vector of a photon is orthogonal the corresponding four-momentum. Using this result, we can perform the sum over final state polarizations to get

$$\frac{1}{4}\sum_{\lambda_1',\lambda 2'}\sum_{s_1,s_2}|\mathcal{M}|^2 = e^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2} + \frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2} + \frac{\langle\Phi_{tu}+\Phi_{ut}\rangle}{(m^2-t)(m^2-u)} ,\right] \tag{3.144}$$

where we see that the squared amplitude is composed by three terms: the square of the t-channel amplitude, the square of the u channel amplitude, and the interference term. It is always possible to identify for instance the term that arises from the t channel scattering since it is the one that diverges when $t \to -m^2$.

The various numerators in Eq. (3.144) are given by the traces of matrices in spinor space. For instance, the numerator of the square of the t-channel amplitude is given by

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\mathrm{Tr}\left[\gamma_\nu\left(-\not{p}_1 + \not{k}_1' + m\right)\gamma_\mu\left(-\not{p}_1 + m\right)\gamma^\mu\left(-\not{p}_1 + \not{k}_1' + m\right)\gamma^\nu\left(-\not{p}_2 + m\right)\gamma^\mu\right] , \tag{3.145}$$

which can be computed using properties of the algebra of the Dirac matrices, such as

$$\gamma^\mu \not{A}\gamma_\mu = 2\not{A} , \tag{3.146}$$

and related expressions, as well as the formulae for the traces of matrices in spinor space. After a lengthy but straightforward calculation one gets

$$\langle\Phi_{tt}\rangle = 2\left[tu - m^2(3t+u) - m^4\right] , \tag{3.147}$$

and likewise for the other contributions in Eq. (3.144). It is easy convince yourselves that the structure of the scattering amplitudes in QED simplify significantly in the massless limit, where spin coincides with

[11] Note that one *averages* over initial state polarizations and spins, but one instead *sums* for final state polarization and spin.

helicity, and we can use the so-called *helicity formalism*. For example, for this specific process we have that

$$\frac{\langle \Phi_{tt} \rangle}{(m^2 - t)^2} = \frac{2\left[tu - m^2(3t + u) - m^4\right]}{(m^2 - t)^2} \to \frac{2tu}{t^2} = \frac{2u}{t}\,, \tag{3.148}$$

in the massless limit, and likewise for the other contributions to this process. Using the spinor helicity formalism, the calculations of scattering amplitudes in gauge theories become rather simpler than in the massive case.

The same strategies that we have applied here for the $e^+ + e^- \to \gamma\gamma$ scattering amplitude also apply to all other tree level diagrams in QED, which thus have associated no further conceptual difficulties. Likewise, the computation of loop corrections in spinor electrodynamics follows the same strategy as that outlined in Sect. 3.7 in the case of scalar electrodynamics. This topic will be discussed more extensively in the *Particles and Fields* course.

4 A first look at non-abelian gauge theories

After this discussion of the quantum field theories involving spin-one fields (photons), scalar and spinor QED, we now present a first look at another class of closely related QFTs: *non-abelian gauge theories*, which are constructed in a similar way as QED but not where the gauge symmetry group is non-abelian (that is, non-commutative). For concreteness, we will focus on the non-abelian gauge theory that describes the strong interactions: *Quantum Chromodynamics*, or QCD for short.

As compared to QED, the main difference in the QCD case is the existence of new internal quantum number, *color*, which, in the language of gauge theory, leads to an invariance under a different non-abelian group. QED is an abelian gauge theory, where the relevant gauge group is the abelian $U(1)$. As opposed to it, the gauge group of QCD is $SU(3)$, the group of specially unitary transformations of degree $n = 3$. This group is defined by all $n \times n$ unitary matrices

$$U^T U = \mathbb{1} \tag{4.1}$$

which have determinant equal to 1, det U = 1. The fact that the gauge invariance of QCD is under a non-abelian group has important consequences, as we will see now.

In the *fundamental representation*, a suitable choice of generators for $SU(3)$ are the Gell-Mann matrices, which are hermitian and traceless,

$$t^A \equiv \frac{1}{2}\lambda^A \,, \quad A = 1, \ldots, 8 \,, \tag{4.2}$$

and which obey the commutation relations of the group's Lie algebra

$$\left[t^A, t^B\right] = i f^{ABC} t^C \,, \quad A, B, C = 1, \ldots, 8 \,, \tag{4.3}$$

with f^{ABC} the structure constants of $SU(3)$, namely

$$f_{123} = 1 \,, \tag{4.4}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \,, \tag{4.5}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \,, \tag{4.6}$$

and the corresponding permutations. All other structure constants are zero. The Gell-Mann matrices for the fundamental representation of $SU(3)$ are summarized in Fig. 4.1.

The QCD Lagrangian has formally a rather similar structure as compared to the QED one, though now the invariance is with respect to the non-abelian group $SU(3)$, and this leads to a number of striking differences between the two theories. The QCD Lagrangian is thus the following:

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\psi}_i^{(f)} \left(i\gamma_\mu D^\mu_{ij} - m_f \delta_{ij}\right) \psi_j^{(f)} - \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} \,, \tag{4.7}$$

where a is now a color index that runs from 1 to $N_c - 1 = 8$, i, j are color indices from 1 to $N_c = 3$, since quarks transform in the fundamental representation of $SU(3)$, and we are assuming N_f massive fermions now. Unlike fermions, which transform in the *fundamental* representation of SU(3), gluons transform in the

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Figure 4.1. The Gell-Mann matrices for the fundamental representation of $SU(3)$.

adjoint representation of SU(3). In QCD, the covariant derivative is defined as follows:

$$D^{\mu}_{ij} = \partial^{\mu}\delta_{ij} + ig_s t^a_{ij} A^{\mu}_a \,. \tag{4.8}$$

where now g_s is the strong coupling constant, and the gluon field replaces the photon field that we had in QED. Note that in QCD the covariant derivative has a *matrix structure*: it is a matrix acting on fermions in the fundamental representation, unlike the case of QED. In particular, it mixes the color degrees of freedom. This has important physical consequences. In QCD, the field strength tensor is defined as

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu \,, \tag{4.9}$$

with f_{abc} the structure constants of $SU(3)$ introduced above.

The fermion sector of the QCD Lagrangian must be independently gauge invariant, because there are n_f fermion fields in the theory, each with a different mass. To achieve this, the covariant derivative must transform as the quark field itself

$$D^{\mu}_{ij}\psi_j \to \left(D^{\mu}_{ij}\psi_j\right)' = U_{ik}(x) D^{\mu}_{kj}\psi_j \,. \tag{4.10}$$

With this condition, one can check explicitly that the gauge transformation property of the gluon field is the

$$t^a A^{\mu}_a \to t^a A^{'\mu}_a = U(x) t^a A^{\mu}_a U^{-1}(x) + \frac{1}{g_s}\left(\partial^{\mu} U(x)\right) U^{-1}(x) \,. \tag{4.11}$$

where note that the SU(3) gauge transformation acts on the product $t^a A^{\mu}_a$, rather than on the gluon field itself. One important difference between QED and QCD is that the field-strength tensor itself is not gauge invariant. To show this, a useful relation is provided by the commutator between two covariant derivatives

$$[D_\mu,\, D_\nu] = ig_s\, t^a\, F^a_{\mu\nu} \,, \tag{4.12}$$

which can be derived from the definition of the covariant derivative acting on a fermion field. Now, trans-

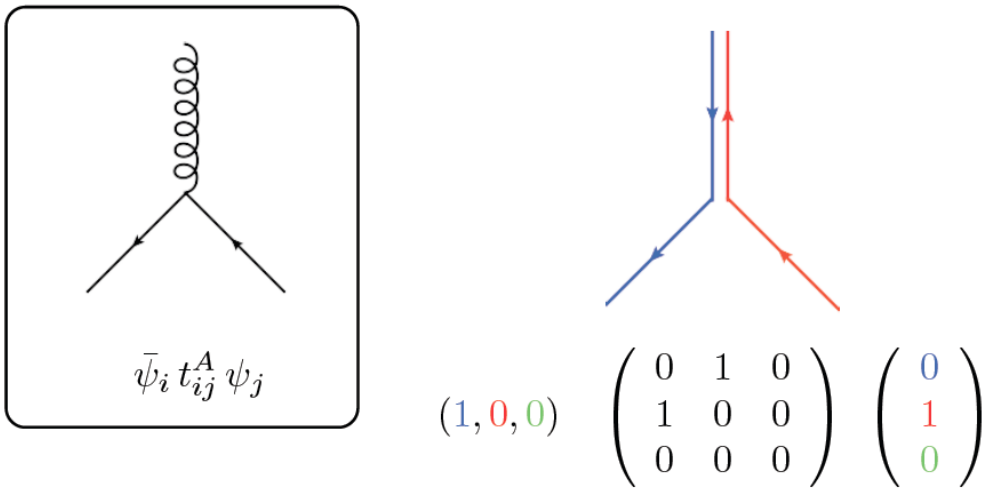


Figure 4.2. The emission of a gluon from a quark interpreted in terms of color flow, with the Gell-Mann matrix performing a rotation in color space.

formation law of the field strength tensor under $SU(3)$ transformations will be given by

$$t^a F^a_{\mu\nu} \to t^a F^{'a}_{\mu\nu} = U(x)\, t^a F^a_{\mu\nu}\, U^{-1}(x)\,, \tag{4.13}$$

to is it in general not invariant under gauge transformations. However, the product $F^a_{\mu\nu} F^{\mu\nu}_a$ *is* invariant under $SU(3)$ transformations, and hence the gluon sector of the Lagrangian is also gauge invariant. We can use the following property of $SU(3)$

$$\text{Tr}\,\left[t^a t^b\right] = \frac{1}{2}\delta^{ab}\,, \tag{4.14}$$

and therefore we can have the following

$$-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a = -\frac{1}{2} F^a_{\mu\nu}\, F^{\mu\nu,b} \text{Tr}\,\left[t^a t^b\right] = -\frac{1}{2} \text{Tr}\,\left[F^a_{\mu\nu}\, t^a\, F^{\mu\nu,b}\, t^b\right]\,, \tag{4.15}$$

which, using the transformation law Eq. (4.13), can be shown to be invariant under $SU(3)$ transformations due to the cyclic properties of the trace. Therefore, the purely gluonic piece of the QCD Lagrangian is gauge invariant, even if the individual field-strength tensor is not.

As in the case of QED, requiring invariance under local SU(3) color transformations together with locality, Lorentz invariance, and the CPT symmetries constraints rather tightly the possible operators that can be present in the QCD Lagrangian. As we will see below, a non-abelian gauge theory leads to the phenomenon of *asymptotic freedom*, which means that QCD looks similar to QED but only at high energies, while at low energies it looks like a very different type of physical theory.

The various contractions of color indices in the QCD Lagrangian can be interpreted as color flows between the different types of fields. In this specific example of Fig. 4.2, an incoming quark with red color emits a gluon and is transformed into an outgoing quark with blue color. Gluons carry both color and anti-color: they change the color charge of quarks and of other gluons. The allowed color flows are restricted by $SU(3)$ symmetry, as can be seen in this example with the Gell-Man matrix.

Following the same techniques as in the case of QED, it is possible to determine the Feynman rules

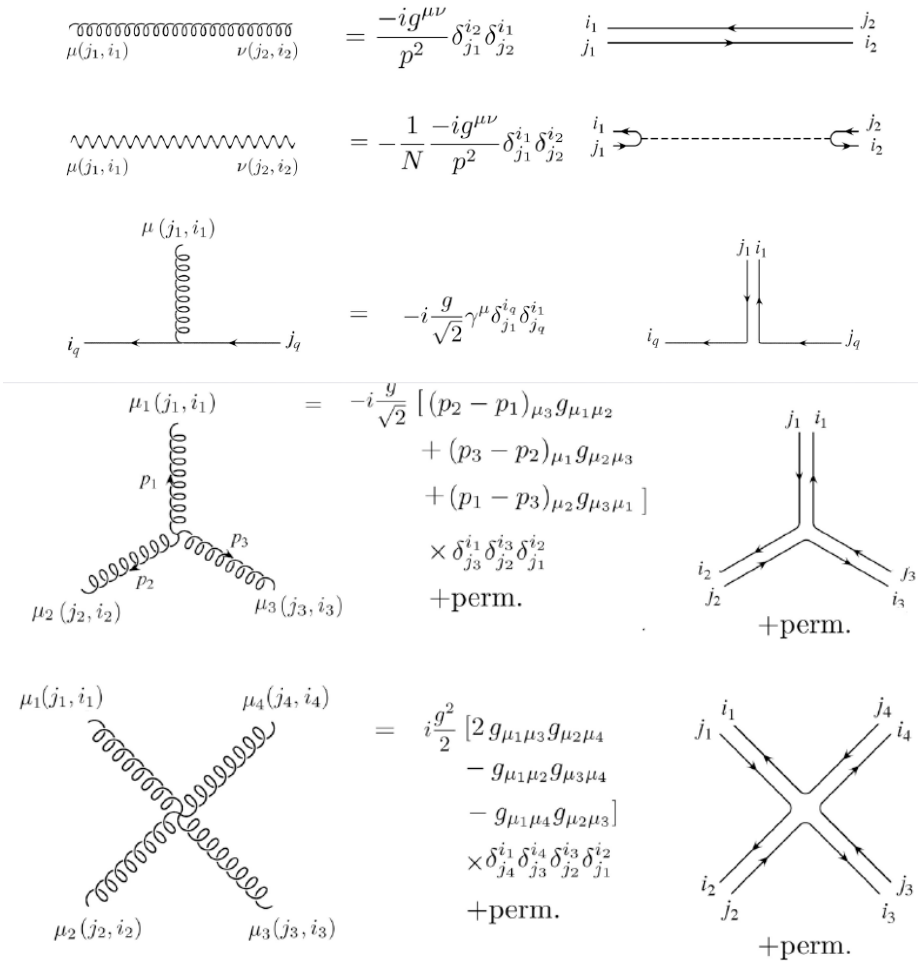


Figure 4.3. The Feynman rules of QCD, for the propagation and interaction of quarks and gluons. In the right part of the plot, we show how the Feynman rules can be also interpreted in terms of color flow between quarks and gluons.

that QCD must obey. These Feynman rules are summarized in Fig. 4.3. In the left part of the plot, we show the various propagators and vertices that are allowed in QCD, with the corresponding Feynman rules. Note the presence of three-gluon and four-gluon vertices, which are absent in QED: this is a genuine new feature of QCD, which arises from the non-abelian nature of $SU(3)$, and that has important consequences, like asymptotic freedom, which we will discuss below. In the three-gluon diagram of Fig. 4.3, p_i are the four momenta of each of the gluons, assuming they are incoming. When computing Feynman diagrams, all possible permutations of the three and four-gluon vertices need to be included.

In order to understand where the three- and four-gluon vertices come from, if we now expand the gluonic term of the QCD Lagrangian, keeping only the terms proportional to the strong coupling constant, we find the following terms

$$\begin{aligned} F^{\mu\nu}_a F^a_{\mu\nu} \quad &\rightarrow \quad \ldots + g_s^2\, f_{abc}\, f_{ade}\, A^{\mu,b}\, A^{\nu,c}\, A^d_\mu\, A^e_\nu \\ &- \quad g_s\, f_{abc}\, A^{\mu,b}\, A^{\nu,c}\left[\partial_\mu\, A^a_\nu - \partial_\nu\, A^a_\mu\right] - g_s\, f_{abc}\, A^b_\mu\, A^c_\nu\left[\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}\right] \end{aligned} \tag{4.16}$$

which lead to the vertices shown in Fig. 4.3. The proportionality to p_i in the three-gluon vertex arises the

derivatives in the second half of the equation above.

In Fig. 4.3, right part, we also show how the Feynman rules can be understood in terms of color flow. In all the diagrams, we can see that color is conserved, a basic property of $SU(3)$ symmetry. Therefore, in QCD only diagrams which conserve color will be physically allowed. Comparing the QCD Feynman rules in Fig. 4.3 with the corresponding QED rules we see that the main difference is the presence of self-interaction terms for gauge bosons: the three-gluon and four-gluon vertices. This is a direct consequence of the non-abelian nature of the gauge group of QCD, SU(3), and is the main reason of why at high energies the strong interaction can be described by the mathematical language of a perturbative quantum field theory.

4.1 Renormalization group equations in QCD

Perhaps the most remarkable property of QCD is that of asymptotic freedom: the strength of the QCD interaction decreases when the momentum transfers involved in the interaction increase. This allows to use the basic tools of perturbative QFT to deal with hard QCD interactions. On the other hand, it also implies that at low scales QCD becomes non-perturbative, since the QCD coupling diverges, and the perturbative interpretation is not valid anymore. While this suggests confinement, since the strength of the interaction between two quarks would increase the more we try to separate them (larger distances corresponds to smaller energy scales), confinement has never been formally derived from the QCD Lagrangian, and only partial numerical evidence from lattice simulations has been obtained.

As we discussed in Sect. 2, the running of the coupling constants in Quantum Field Theories like QED or QCD can be understood neatly from renormalization group arguments. Let us consider a given observable G in a generic QFT (for instance, a cross-section) with (bare) dimensionless coupling constant α, and a number of kinematical invariants $s_1 \ldots s_n$. In quantum field theories we encounter ultraviolet (UV) divergences, which in renormalizable theories can be removed by a suitable redefinition of the couplings and fields. If we denote by M the UV cut-off,[12] after renormalization the observable G will be given as follows

$$G = G(\alpha,\, M,\, s_1, \ldots, s_n)\ . \tag{4.17}$$

Now, the dependence on M can be eliminated by a suitable redefinition of the bare coupling (which does not depend on any scale) in terms of the renormalized coupling, order by order in the perturbative expansion

$$\alpha_{\rm ren} = \alpha + \sum_{l=2} c_l\,(M/\mu)\,\alpha^l\,, \tag{4.18}$$

where the coefficients c_l are dimensionless, and the arbitrary scale μ has been introduced for dimensional reasons.

We can now use a basic property of renormalizable QFTs, namely, that physical quantity can be expressed in terms of the renormalized coupling $\alpha_{\rm ren}$, the scale μ and other kinematical invariants, without the need anymore of the UV cutoff (see also the discussion in Sect. 2.8). Therefore, we can write

$$G\left(\alpha\left(\alpha_{\rm ren}, M/\mu\right),\, M,\, s_1, \ldots, s_n\right) = \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)\ , \tag{4.19}$$

In renormalizable QFTs, a single redefinition of the coupling makes finite all physical observables. The price

[12] Similar arguments can be derived with other renormalization techniques, such as dimensional regularization.

to pay is the dependence of our results on the arbitrary *renormalization scale* μ, which could be eliminated only by computing the theory to all perturbative orders.

The renormalization group equations (RGE) arise from the condition that if we vary $\alpha_{\rm ren}$ and μ, keeping α and M fixed, cross-sections should be invariant (since before renormalization, physical observables depend only on M and α, and we have done here only a change of variables). This leads to the conditions

$$\frac{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)}{\partial \alpha_{\rm ren}} d\alpha_{\rm ren} + \frac{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)}{\partial \mu} d\mu = 0 \tag{4.20}$$

$$\frac{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)}{\partial \alpha_{\rm ren}} d\alpha_{\rm ren} + \frac{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)}{\partial \mu^2} d\mu^2 = 0 \tag{4.21}$$

and from these expressions it is possible to determine that the dependence of the renormalized coupling $\alpha_{\rm ren}$ with the scale μ is given by:

$$\mu^2 \frac{d\alpha_{\rm ren}}{d\mu^2} = \beta(\alpha_{\rm ren})\,. \tag{4.22}$$

This is so because putting together the various equations above we find

$$\mu^2 \frac{d\alpha_{\rm ren}}{d\mu^2} = -\frac{\mu^2 \partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)/\partial \mu^2}{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)/\partial \alpha_{\rm ren}} = -\frac{\mu^2 \partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)/\partial \mu^2}{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)/\partial \alpha_{\rm ren}} \tag{4.23}$$

and therefore the functional dependence of the renormalized coupling with the scale μ is fixed by the following conditions:

- It cannot depend on M, since RHS does not
- It cannot depend on the kinematical invariants, since LHS does not
- It cannot depend on μ, since we dont have other dimensionful variables available and the coupling is dimensionless

Therefore, at the first non-trivial order, the RGE equation Eq. (4.22) must read

$$\frac{d}{d\,\log \mu^2} \alpha_{\rm ren}(\mu^2) = -b_0 \alpha_{\rm ren}^2\,, \tag{4.24}$$

Eq. (4.24) holds for any perturbative renormalizable QFT: the renormalized coupling *runs*, that is, it depends on the typical physical scale that enters each specific scattering amplitude. This equation can be easily solved to give

$$\alpha_{\rm ren}(\mu^2) = \frac{1}{b_0 \log \mu^2/\Lambda^2}\,, \tag{4.25}$$

Now, so far everything holds for a generic theory. In the case of QCD, the coefficient b_0 can be determined by computing the Feynman diagrams for the $\mathcal{O}\left(\alpha_s^2\right)$ corrections to the gluon propagator, summarized in Fig. 4.4. The leading order term for the QCD beta function is given by

$$b_0 = \frac{33 - 2n_f}{12\pi}\,. \tag{4.26}$$

The result that b_0 is positive arises from the gluon self-interactions, that is from the non-abelian nature of QCD (to see this, note that the diagram with quarks in the loop leads to the N_f-proportional piece).

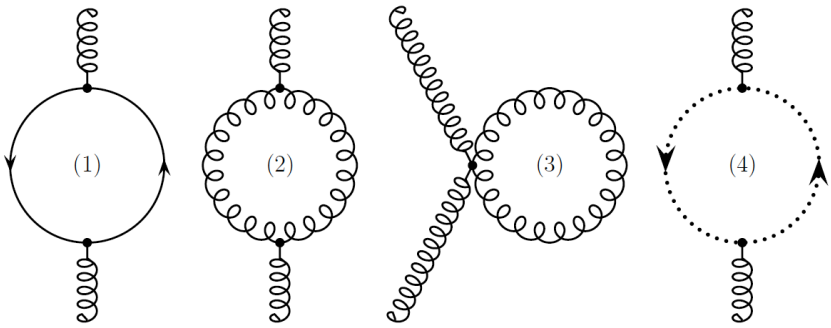


Figure 4.4. Feynman diagrams which contribute to the renormalization of the gluon propagator in QCD, which determine the leading order coefficient of the RGE equations for the running of $\alpha_{\rm ren}(\mu)$ in Eq. (4.24).

In other words, even a pure gluonic version of QCD would exhibit this property. Looking at Eq. (4.25), we see that $b_0 > 0$ implies that the theory becomes free (non-interacting) in the ultra-violet, hence the name *asymptotically free*. In Fig. **??** we summarize recent experimental determinations of the QCD running coupling $\alpha_s(Q)$ for different scales, together with the four-loops QCD prediction. Asymptotic freedom can be seen by the fact that $\alpha_s(Q)$ decreases when Q is increased.

The situation is remarkably different in the case of QED. There, the first term of the beta function is negative, due to the absence of photon self-interactions in the QED Lagrangian:

$$b_0^{\rm QED} = -\frac{4n_f}{12\pi} \tag{4.27}$$

As opposed to QCD, QED becomes strongly interacting (non-perturbative) in the ultraviolet, that is, at very small distances and large scales. The scale where QED stops being weakly coupled is determined by the position of the Landau pole in QED, which is

$$\Lambda = m_e \exp\left(-\frac{1}{2b_0\alpha_{\rm ren}(m_e)}\right) \sim 10^{90}\,{\rm GeV} \gg M_{Pl} \tag{4.28}$$

much larger than any sensible scale, even the Plank scale where quantum gravity effects become important. For this reason non-perturbative effects are not relevant for Quantum Electrodynamics.

In summary, asymptotic freedom is one of the central properties in QCD: it becomes a free theory at very large energies. The renormalization group equations allows to determine the dependence of the running coupling on the scale, in terms of $\alpha_s(q_{\rm ref})$ at some reference scale, which needs to be extracted from experimental data. On the other hand, we still dont understand what are the dynamics that lead to confinement. The QCD coupling is large at low scales, but there perturbation theory breaks down. Confinement is an experimental observation (hadrons are always color singlets) but it is a purely non-perturbative phenomenon, thus non-perturbative methods are required, like lattice QCD.

4.2 QCD radiation in electron-positron annihilation

After this brief introduction to the basic principles QCD, we turn out to explore a representative application to high energy processes, specifically *hadron production in electron-positron annihilation*. The Born-level

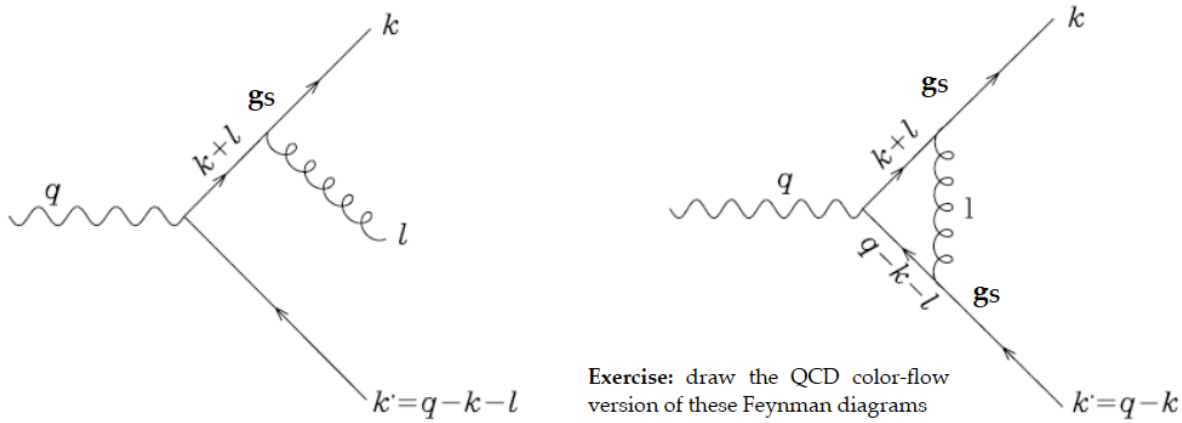


Figure 4.5. QCD corrections to quark-antiquark production in electron-positron annihilation.

matrix element for this process follows straightforwardly from the Feynman rules of spinor electrodynamics:

$$\mathcal{M}_0 = ie\bar{u}(k)\,\epsilon^\mu(q)\,\gamma_\mu\,v(k')\,. \tag{4.29}$$

The real emission and virtual correction to the Born diagram are illustrated in Fig. 4.5, which also sets the notation for the various four-momenta that will be used in the following. For the diagram with real radiation of a gluon out of the outgoing quark, left plot of Fig. 4.5 (there is a similar contribution from the outgoing anti-quark), and neglecting all fermion masses since this is a high-energy processes, we find that the corresponding matrix element is given by

$$\mathcal{M}_1 = -ieg_s\,\bar{u}(k)t^a\epsilon_a^\nu(l)\,\gamma^\nu\,\frac{1}{\not{k}+\not{l}}\,\epsilon^\mu(q)\,\gamma_\mu\,v(k')\,, \tag{4.30}$$

where the appearance of g_s is a sign of the QCD perturbative corrections, and where $\epsilon_a^\nu(l)$ is the gluon polarization vector.[13] The properties of the gluon polarization vectors are very similar to those of the photon from QED, see Sect. 3.1, with the main difference being that now there exist N_c^2-1 different types of gluons.

To isolate the regions where the matrix element is enhanced, we can use a number of well-known properties of the Dirac algebra for massless fermions, namely

$$\frac{1}{\not{k}+\not{l}} = \frac{\not{k}+\not{l}}{(k+l)^2}\,, \qquad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4\,, \qquad \bar{u}(k)\not{k} = 0 \tag{4.31}$$

which allow us to write Eq. (4.30) as follows

$$\mathcal{M}_1 = -ieg_s\,\bar{u}(k)\,\not{\epsilon}(q)\,v(k')\cdot\left[\frac{t^a\epsilon_a(l)\cdot k}{k\cdot l} - \frac{t^a\epsilon_a(l)\cdot k'}{k'\cdot l}\right]\,, \tag{4.32}$$

In the above equation we restrict ourselves in the *soft approximation*, which is $l \ll k, k'$, that is, the gluon momentum is much softer than any of the two quarks. This is useful since this approximation isolates the kinematic region where the matrix element becomes divergent, see also the discussion in Sect. 3.6 in the

[13] Recall that a is an index in the SU(3) adjoint representation, and thus $a = 1,\ldots,8$.

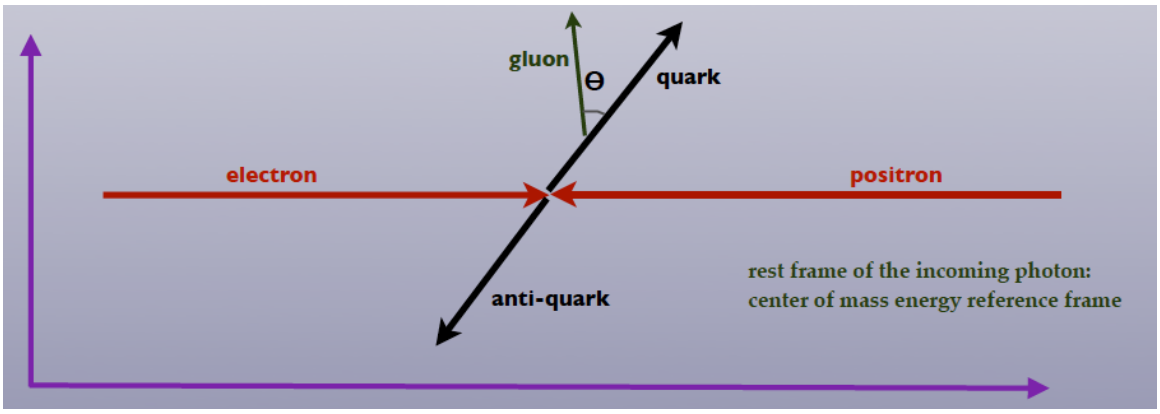


Figure 4.6. Kinematics of real gluon emission in quark-antiquark production via electron-positron annihilation.

context of the QED. In general QCD matrix-elements have a complicated structure, but this can be substantially simplified in the *soft* and *collinear* limits, which is also the limit which dominates many important cross-sections. In Eq. (4.32) we observe that $\mathcal{M}_1$ is singular when $k \cdot l \to 0$ or $k' \cdot l \to 0$.

If now we square the matrix element, we realize than in the soft approximation it can be written as a factorized product of the Born matrix element squared and a term that depends only on the kinematics of the three final state particles,

$$|\mathcal{M}_1|^2 = |\mathcal{M}_0|^2 \, g_s^2 C_F \frac{2k \cdot k'}{(k \cdot l)\,(k' \cdot l)} \,. \tag{4.33}$$

To compute the cross-section for the real emission contribution, we need to add the corresponding phase space factor and integrate over all the relevant phase space (in this case over the gluon momentum l, subject to the mass-shell condition), in order to obtain the following expression:

$$\sigma_1 = C_F \, g_s^2 \, \sigma_{\rm born} \cdot \int \frac{d^3 l}{2 l^0 \, (2\pi)^3} \frac{2k \cdot k'}{(k \cdot l)\,(k' \cdot l)} \,. \tag{4.34}$$

In order to interpret this result, it is useful to express the various Lorentz invariants in the *laboratory frame*, and denoting by θ the angle between the radiated gluon and the quark (see Fig. 4.6), we obtain the following expression

$$\sigma_1 = C_F \, \frac{g_s^2}{4\pi^2} \, \sigma_{\rm born} \int \frac{dl_0}{l_0} \, \frac{d\cos\theta}{(1+\cos\theta)\,(1-\cos\theta)} \,, \tag{4.35}$$

so that the real emission cross-section is the Born times the QCD coupling times an integral over the energy and the angle of the radiated gluon in the laboratory frame. The integral over energies is clearly divergent when the gluon becomes very soft, $l_0 \to 0$. The angular integral is also divergent, as one can see by using the substitution $x \equiv \cos\theta$,

$$\int \frac{x}{1-x^2} = -\frac{1}{2} \log\left(1-x^2\right) \,, \tag{4.36}$$

so the angular integral diverges when the gluon becomes collinear to the quark ($\theta = 0, x = 1$) or to the anti-quark ($\theta = \pi, x = -1$). Therefore, the real emission cross-section is *divergent* due to either *soft* and *collinear* emission of gluons. Indeed, soft and collinear divergences are a generic feature of massless gauge boson emission in QFTs, as we also saw in the case of QED. These divergences are canceled out by virtual corrections in inclusive enough observables, by virtue of the so-called *Kinoshita-Lee-Nauenberg* theorem.

Total cross-section. While the real emission cross-section diverges, it is also possible to compute the virtual diagrams in the same soft approximation. Putting everything together, we find that the QCD corrections to e^+e^- annihilation into hadrons are given by:

$$\begin{aligned} \sigma_0 &= \sigma_{\text{born}} \\ \sigma_{1,\,\text{real}} &= C_F \, \frac{g_s^2}{\pi^2} \, \sigma_{\text{born}} \int \frac{dl_0}{l_0} \, \frac{d\cos\theta}{1-\cos^2\theta} \\ \sigma_{1,\,\text{virt}} &= -C_F \, \frac{g_s^2}{\pi^2} \, \sigma_{\text{born}} \int \frac{dl_0}{l_0} \, \frac{d\cos\theta}{1-\cos^2\theta} \, , \end{aligned} \tag{4.37}$$

where we see that in the soft limit the virtual corrections have the same structure, but different sign, as the real emission corrections. Therefore, if we integrate Eq. (4.37) without any phase space restriction, we find a finite result, with infrared and collinear singularities canceling between real and virtual terms. In order to find the actual result, one needs to perform the full calculation, without resorting to the soft approximation. In this case, it is possible to evaluate the $\mathcal{O}(\alpha_s)$ correction to the R-ratio, and one finds

$$R = R_{\text{born}} \cdot \left(1 + \frac{\alpha_s}{\pi}\right) \, . \tag{4.38}$$

The cancellation between real and virtual soft/collinear singularities allows to make perturbative QCD calculations to any order in the coupling expansion. However, as we discuss now, one needs to suitable define the physical observables to ensure this cancellation to take place.

QCD corrections to inclusive jet distributions. While soft and collider divergences cancel in total inclusive cross-sections, the situation is however different if we try to compute the QCD corrections to less inclusive observables, such as jets. Jets are collimated sprays of hadrons that appear frequently in high-energy collisions. As opposed to total cross-sections, jets are not univocally defined, and what we call a *jet* actually is a definition-dependent statement. The first definition of a QCD jet, by Sterman and Weinberg (SW), stated that a given hard-scattering event contributes to the *jet cross-section* if we can find two cones, with opening angle δ, that contain a fraction $1-\epsilon$ of the total energy in the event.

Note that in e^+e^- collisions the total energy of the event is fixed by the energy of the lepton beams, unlike in an hadronic collision, and therefore the Sterman-Weinberg definition is operative. As opposed to the inclusive calculation above, for the jet cross-section the phase space for gluon emission is now limited (for example, if too much energy falls outside the jet cone, this event will not satisfy the SW definition), and this will have important consequences. In Fig. 4.7 we show the diagrams for the various contributions to the QCD α_s corrections to the calculation for the cross-section for the production of Sterman-Weinberg jets. From left to right we have the Born diagram, the virtual correction, and real emission falling both inside and outside the jet cone.

Let us compute the contributions to the various diagrams in Fig. 4.7 to the Sterman-Weinberg jet cross-section. The Born term is unchanged by the jet definition. To see this, note that if the two partons as back to back all the energy of the event will fall into the two cones. For the real emission contribution, we need to start from Eq. (4.35), obtained in the soft and collinear approximation, but now with the integration restricted to the in-jet or out-jet phase space.

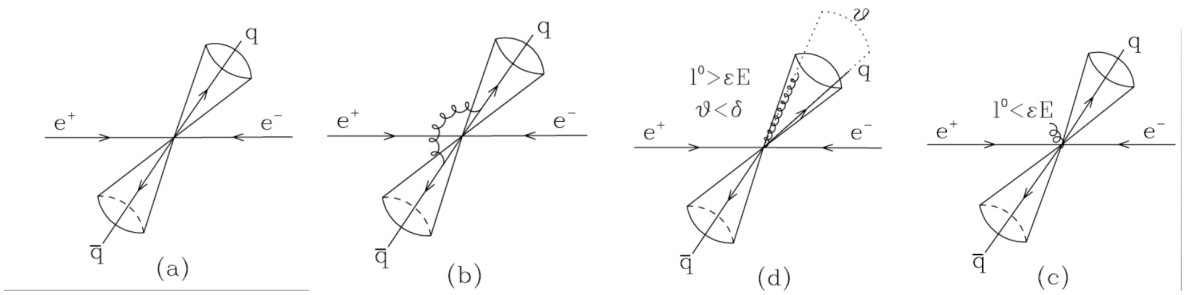


Figure 4.7. Diagrams for the various contributions to the QCD α_s corrections to the calculation for the cross-section for the production of Sterman-Weinberg jets. From left to right we have the Born diagram, the virtual correction, and real emission falling both inside and outside the jet cone.

Hard gluon radiation. In the case of **hard** radiation inside the jet cone, we have

$$\sigma_1^{\text{jet,real,d}} = C_F \frac{g_s^2}{\pi^2} \sigma_{\text{born}} \int_{\epsilon E}^{E} \frac{dl_0}{l_0} \left[\int_0^{\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right] , \tag{4.39}$$

where the lower limit of the integration in the gluon energy arises here because we consider hard radiation only. If hard radiation, with $l_0 \geq \epsilon E$, is emitted outside the cone, the event would not contribute to the jet cross-section.

Soft gluon radiation. In the case of **soft** gluon emission $l_0 \leq \epsilon E$, irrespective of whether or not radiation falls in the jet cone, the event will contribute to the jet cross-section, and its contribution will be:

$$\sigma_1^{\text{jet,real,c}} = C_F \frac{g_s^2}{\pi^2} \sigma_{\text{born}} \int_0^{\epsilon E} \frac{dl_0}{l_0} \left[\int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right] . \tag{4.40}$$

Virtual corrections As in the case of the total inclusive cross-section In order to make the jet cross-section finite, we need to compute the virtual correction. This reads, integrating the energy and the angle of the emitted gluon over the entire phase space:

$$\sigma_1^{\text{jet,virt}} = -C_F \frac{g_s^2}{\pi^2} \sigma_{\text{born}} \int_0^{E} \frac{dl_0}{l_0} \int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} , \tag{4.41}$$

since for virtual corrections there is no restriction on the kinematics of the gluon.

Total jet cross-section. Adding together the Born cross-section, the different real emission terms, and the virtual correction, we find that the jet cross-section is now *finite*. As can be seen, the only term which does not cancel between real and virtual emission is the contribution for hard gluons emitted outside the jet cones:

$$\sigma^{\text{jet}} = \sigma_{\text{born}} \left(1 - \frac{g_s^2}{\pi^2} \int_{\epsilon E}^{E} \frac{dl_0}{l_0} \int_{\delta}^{\pi-\delta} \frac{d\cos\theta}{1-\cos^2\theta} \right) \tag{4.42}$$

$$= \sigma_{\text{born}} \left(1 - \frac{4g_s^2}{4\pi^2} \log\epsilon \, \log\frac{1+\cos\delta}{1-\cos\delta} \right) . \tag{4.43}$$

Interestingly, the cross-section now depends on the jet definition parameters ϵ and δ. The logarithmic dependence of the cross-section with these parameters is a remnant of the cancellation of soft and collinear divergences. The QCD correction becomes unreliable when either ϵ is too small (trying to resolve soft gluons) or the cone is too narrow (trying to resolve collinear gluons). This result illustrates how we can make sense of a Quantum Field Theory that exhibits soft and infrared divergences: they key is defining inclusive enough observables where these divergences will cancel with those of the corresponding virtual contributions.

A Quantum Field Theory recap

In this appendix we provide for completeness some background material that has already been covered in the Quantum Field Theory course.

Lorentz invariance. A Lorentz transformation is a linear, homogeneous change of space-time coordinates $x^\mu \to \bar{x}^\mu$ or the form

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu \,, \tag{A.1}$$

that preserves the distance x^2 between x^μ and the origin, namely

$$x^2 = g_{\mu\nu} x^\mu x^\nu = -t^2 + \vec{x}^2 \,, \tag{A.2}$$

where we use the $(-,+,+,+)$ signature for the metric $g_{\mu\nu}$. This invariance condition implies that the transformation matrix Λ^μ_ν satisfies

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \,. \tag{A.3}$$

Note that the Lorentz transformations also include the ordinary spatial rotations as a subgroup.

The Klein-Gordon equation. The Klein-Gordon equation is the relativistic equivalent of the Schroedinger wave function equation in quantum mechanics. It is given in natural units by

$$\left(\Box + m^2\right) \phi(x) = 0 \tag{A.4}$$

where $\phi(x)$ is the wave function, m is the particle mass and the d'Alembert operator is defined by

$$\Box \equiv -\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla} \,. \tag{A.5}$$

While the Klein-Gordon equation is Lorentz invariant, it has unphysical solutions with negative energy. Therefore it is not the suitable relativistic generalization of Schroedinger equation, which is instead provided by the Dirac equation. For a free particle, the solution of the Klein-Gordon equation is given by plane waves of the form

$$\phi(\mathbf{x}, t) = \exp\left(i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right) \,, \tag{A.6}$$

which have associated the following dispersion relation:

$$E = \sqrt{m^2 + \mathbf{p}^2} \,. \tag{A.7}$$

The general solution of the Klein-Gordon equation, assuming finite boundary conditions, is

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[a(\mathbf{k}) e^{ikx} + a^*(\mathbf{k}) e^{-ikx}\right] \,, \tag{A.8}$$

where the phase-space term $d^3k/((2\pi)^3 2\omega)$ is Lorentz invariant. The quantization of the Klein-Gordon field $\varphi(x)$ can be achieved by promoting the coefficients $a(\mathbf{k})$ to operators satisfying a suitable set of commutation relations.

Feynman parameters. These are useful parametric relations that allow us to compute loop integrals in quantum field theories. One important relation that will be useful in these lectures is:

$$\frac{1}{AB} = \int_0^1 \frac{1}{\left(A + (B-A)x\right)^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{(xA+yB)^2} \tag{A.9}$$

which is a particular case of the more general expression.

$$\frac{1}{A_1 \ldots A_n} = \int dF_n \left(\sum_{k=1}^{n} x_k A_k\right)^{-n} , \tag{A.10}$$

with the integration measure over the Feynman parameters x_i is defined as

$$\int dF_n = (n-1)! \int_0^1 \left(\prod_{k=1}^{n} dx_k\right) \delta\left(x_1 + \ldots + x_n - 1\right) . \tag{A.11}$$

Wick rotation. This mathematical operation is motivated by the fact that the Minkowski metric in natural units

$$ds^2 = -(dt)^2 + dx^2 + dy^2 + dz^2 , \tag{A.12}$$

is equivalent to the Euclidean metric in four spatial dimensions,

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 , \tag{A.13}$$

provided that the time t is allowed to take negative values, $t = i\tau$ with τ a real variable. Wick rotation allows in certain cases to rewrite integration over Minkowski momenta in terms of integration over four-dimensional Euclidean momenta. Specifically, if k is a Minkowskian four-momentum and k_E is the corresponding Wick-rotated Euclidean version, it can be shown that in general:

$$\int d^d k f\left(k^2 - i\epsilon\right) = i \int d^d k_E f\left(k_E^2\right) , \tag{A.14}$$

provided $f(k_E^2) \to 0$ faster than k_E^{-d} as $k_E \to \infty$.

Noether's theorem. Assume a theory which describes some fields $\{\phi_i\}$ in terms of a Lagrangian $\mathcal{L}(\phi_a, \phi_b, \ldots)$. These fields can be either scalars, spinors, or vector fields. Assume we have a *global symmetry* that leaves the *action invariant*. Working at the infinitesimal level, this means that a transformation of the fields of the form

$$\delta\phi_a(x) = X_a(\{\phi_i\}) , \tag{A.15}$$

leaves the Lagrangian invariant up to a total derivative, $\delta\mathcal{L} = \partial_\mu G^\mu$, so that the action itself is invariant under such transformation.

Now, an arbitrary transformation of the fields (not necessarily the one associated to the symmetry that

we are considering here) leads to:

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)\,, \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\,. \end{aligned} \tag{A.16}$$

Now, the terms in brackets vanishes due to the Euler-Lagrange equations of motion. Therefore we find that the total variation of the Lagrangian is given by

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\,, \tag{A.17}$$

and for the transformation of the specific form of Eq. (A.15), this Lagrangian variation should a total differential,

$$\delta\mathcal{L} = \partial_\mu G^\mu\,. \tag{A.18}$$

Therefore, we obtained the important result that the existence of this global symmetry has associated a conserved current

$$\partial_\mu j^\mu = 0\,, \tag{A.19}$$

given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) - G^\mu(\phi)\,, \tag{A.20}$$

which is the result of Noether's theorem. Eq. (A.20) is known as *Noether's current*, and illustrates the deep connection between the symmetries of a theory and the associated conservation laws.

The LSZ reduction formula. The Lehmann-Symanzik-Zimmermann (LSZ) reduction formula allows us to construct appropriate initial and final states for scattering experiments, making possible *e.g.* the calculation of scattering cross-sections. For a real scalar free field, the one-particle states are given by

$$|k\rangle = a^\dagger(\mathbf{k})|0\rangle\,, \tag{A.21}$$

where the creation operator in terms of the field $\varphi(x)$ is given by

$$a^\dagger(\mathbf{k}) = -i\int d^3x e^{ikx}\overline{\partial}_0\varphi(x)\,, \tag{A.22}$$

and it is normalized as follows

$$\langle k||k'\rangle = (2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,. \tag{A.23}$$

In the interacting theory, the creation and annihilation operators will in general depend on the time t. Then a suitable choice for the initial and final states of scattering experiments is

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)|0\rangle\,, \tag{A.24}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)|0\rangle\,, \tag{A.25}$$

and one can show that using appropriately defined normalizations, the matrix element between the initial and final states will be given by

$$\langle i||f\rangle = i^{n+n'} \int d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)\ldots d^4x_{1'} e^{ik'_1x'_1}(-\partial_{1'}^2+m^2)\ldots \times \langle 0|T\left(\varphi(x_1)\ldots\varphi(x'_1)\ldots\right)|0\rangle\,, \quad \text{(A.26)}$$

in the general case of n incoming and n' outgoing particles. This is the LSZ reduction formula, which relates matrix elements between initial and final scattering states and time-ordered matrix elements of the field in the vacuum.

B Mathematical expressions

In this appendix we collect, for ease of reference, a number of useful mathematical expressions which will appear at various stages during the course, as well as in some of the problem sets.

Euler's Gamma function. The Gamma function $\Gamma(x)$ is a generalization of the factorial function to real and complex numbers. For n a positive integer one has

$$\Gamma(n) = (n-1)!\,, \tag{B.1}$$

while for all complex numbers except non-positive integers one has

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz\,. \tag{B.2}$$

Some useful relations involving Gamma functions are

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n!)}{n!2^{(2n)}}\sqrt{\pi}\,, \tag{B.3}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma_E + \sum_{k=1}^{n}\frac{1}{k} + \mathcal{O}(x)\right]\,, \tag{B.4}$$

with $\gamma_E = 0.5772$ the Euler-Mascheroni constant, and

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}\left(\epsilon\right)\,. \tag{B.5}$$

which is useful in the limit $\epsilon \to 0$.

Loop integrals. In the calculation of loop integrals in quantum field theory, a number of definite integrals are useful, such as for example

$$\int_0^1 dx\left[\ln(p^2 x(1-x)) + B\right] = \ln(p^2) + B - 2\,. \tag{B.6}$$

Another integral that appears frequently in loop calculations in QFT is

$$\int_0^\infty dk_E k_E^{d-1}\frac{1}{(k_E^2+\Delta)^2} = \Delta^{d/2-2}\Gamma(d/2)\Gamma\left(\frac{4-d}{2}\right)\frac{1}{2\Gamma(2)}\,, \tag{B.7}$$

with k_E being a d-dimensional Euclidean momentum. This integral is a particular case of the more general expression:

$$\int \frac{d^d k_E}{(2\pi)^d}\frac{\left(k_E^2\right)^a}{\left(k_E^2+\Delta\right)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)}\Delta^{-(b-a-d/2)}\,. \tag{B.8}$$

Note that the two integrals above can be divergent in the physical limit $d = 4$ and thus need to be regulated. Another integral that might arise in some calculations is of the form

$$\int_0^1 dx x(1-x)\left[\ln\left(\frac{-s}{m^2}\right) + \ln\left(\frac{x(1-x)}{1-x(1-x)}\right)\right] = \ln(-s/m^2) + 4 - \pi\sqrt{3}\,, \tag{B.9}$$

which appears for example when computing the self-energy of a massive scalar field.

C Problem set I

This is the first problem set of the course, to be completed and sent to the course instructors by **Thursday 18th of January 2018** at the latest. The solutions to the problems in this set will be discussed in the tutorial session of Friday 19th of January. Problems marked with **(*)** are optional.

Problem I.1: Loop integrals. One-loop calculations in generic quantum field theories often involve the calculation of integrals of the following form:

$$\begin{aligned} I_0 &= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\Delta+2k\cdot p)^n}\,, \\ I_{1\mu} &= \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu}{(k^2+\Delta+2k\cdot p)^n}\,, \\ I_{2,\mu\nu} &= \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2+\Delta+2k\cdot p)^n}\,. \end{aligned} \tag{C.1}$$

In these integrals, both Δ and p_μ are fixed constants independent of the loop four-momentum k_μ, and $n \geq 2$ is an integer number. These integrals are defined in Euclidean space, with d being the number of dimensions. Compute these integrals, describing in detail the various steps of the solution. You might want to use the following result:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \Delta^{(d-4)/2}\,. \tag{C.2}$$

Draw some examples for loop Feynman diagrams where these integrals could arise, for different values of n.

Problem I.2: Beta functions in Yukawa theory. Let us consider Yukawa theory, a quantum field theory defined by the following Lagrangian:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1\,, \\ \mathcal{L}_0 &= iZ_\Psi \bar{\Psi}\gamma^\mu\partial_\mu\Psi - Z_m m\bar{\Psi}\Psi - \frac{1}{2}Z_\phi \partial_\mu\phi\partial^\mu\phi - Z_M\frac{M^2}{2}\phi^2\,, \\ \mathcal{L}_1 &= iZ_g g\phi\bar{\Psi}\gamma^5\Psi - \frac{1}{24}Z_\lambda\lambda\phi^4\,, \end{aligned} \tag{C.3}$$

where Ψ is a Dirac fermion (say an electron) and ϕ is a real scalar field. In this Lagrangian, $\mathcal{L}_0$ contains the (renormalized) kinetic and mass terms for the fermion and scalar fields, while $\mathcal{L}_1$ includes the interaction terms. The masses of the fermion and scalar fields are m and M. There are two types of interactions in Yukawa theory: the fermion-scalar interaction $g\phi\bar{\Psi}\gamma^5\Psi$, and the self-interactions of the scalar field $\lambda\phi^4$, with (massless) coupling constants g and λ respectively.

In the Yukawa theory Lagrangian Eq. (D.23) one has already included the counterterms necessary to regulate that ultraviolet divergences that arise in the loop calculations of the theory. At the first non-trivial

order in the coupling constants g and λ of the theory, one can show that these counterterms are given by:

$$\begin{aligned}
Z_\phi &= 1 - \frac{g^2}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,, \\
Z_M &= 1 + \left(\frac{\lambda}{16\pi^2} - \frac{g^2}{2\pi^2}\frac{m^2}{M^2}\right)\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,, \\
Z_\Psi &= 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,, \\
Z_m &= 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,, \\
Z_g &= 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,, \\
Z_\lambda &= 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\left(\frac{1}{\epsilon} + \text{finite}\right) + \dots\,,
\end{aligned} \tag{C.4}$$

where as usual one has defined $\epsilon = 4 - d$, the regulator used to control the UV divergences in dimensional regularization.

The goal of this problem is to evaluate the beta functions β_g and β_λ for the two coupling constants g and λ, which determine their running with the energy. Discuss what can we learn from this result, and in particular what does this tell us about the low and high energy behaviour of Yukawa theory. How does this behaviour compare with that in $\lambda\phi^4$ theory? Next, evaluate the anomalous mass dimension of m and M as well as the anomalous dimensions for the fermion and scalar fields Ψ and ϕ. Explain in detail the various steps of your calculation and discuss the physical interpretation of the results.

Problem I.3(*): Tadpole diagrams. In this problem we consider the so-called ϕ^3 theory in $d = 4$ space-time dimensions, defined by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3\,, \tag{C.5}$$

with ϕ being a real scalar field with mass m and self-interactions mediated by the coupling constant g. Note that the mass dimensions of the self-coupling are $[g] = 1$. A characteristic feature of this theory is that it exhibits a non-vanishing vacuum expectation value $\langle\phi(x)\rangle \neq 0$ induced by tadpole diagrams.

First of all, explain briefly why one needs $\langle\phi(x)\rangle = 0$ in order to use the Lehmann-Symanzik-Zimmermann reduction formula to evaluate S-matrix elements. Next, show that at leading order in the coupling constant g is it possible to set $\langle\phi(x)\rangle = 0$ by adding to the Lagrangian a counterterm of the form $\mathcal{L}_{\text{ct}} = Yg$, and determine the expression for Y. You should find that

$$Y = \frac{i}{2}g\Delta_F(0) + \mathcal{O}\left(g^3\right)\,. \tag{C.6}$$

Discuss whether or not adding this counterterm to the Lagrangian affects other observables. Eq. (D.32) suggest that Y might be imaginary: explain if we should be worried about this, and if we have any reason to believe that Y should be real.

Then evaluate $\Delta_F(0)$ by introducing an ultraviolet cutoff Λ_{UV} to regulate the ultraviolet divergences.

This can be done by means of the replacement

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{k(x-y)}}{k^2+m^2-i\epsilon} \to \int \frac{d^4k}{(2\pi)^4} \frac{e^{k(x-y)}}{k^2+m^2-i\epsilon} \left(\frac{\Delta_{\rm UV}^2}{k^2+\Lambda_{\rm UV}^2-i\epsilon} \right)^2 \tag{C.7}$$

which makes the integral convergent for $x = y$, and that leaves the infrared physics unchanged. Show that in the case that the cutoff is much higher than the mass, $\Lambda_{\rm UV} \gg m$, then the result for the propagator is

$$\Delta_F(0) = \frac{i}{16\pi^2} \Lambda_{\rm UV}^2 \,. \tag{C.8}$$

Finally, compare the result obtained with the dimensional regularization method used in the lectures with that obtained by introducing an ultraviolet cutoff $\Lambda_{\rm UV}$ and comment on the results.

D Problem set II

This is the second problem set of the course, to be completed and sent to the course instructors by **Thursday 1st of February 2018** at the latest. The solutions to the problems in this set will be discussed in the final tutorial session of Friday 2nd of February. Problems marked with **(*)** are optional.

Problem II.1: Furry's theorem. Furry's theorem states the following relation for time-ordered expectation values of gauge photon fields and gauge currents:

$$\langle T\left(A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\right)\rangle = 0\,, \tag{D.1}$$

$$\langle T\left(j_{\mu_1}(x_1)\ldots j_{\mu_n}(x_n)\right)\rangle = 0\,, \tag{D.2}$$

with n being an odd integer and j_μ is the Noether's current in either scalar QED or in standard QED. This theorem follows directly from the requirement of charge-conjugation symmetry. The goal is this problem is to derive a proof of this theorem.

To begin with, consider scalar QED. In this theory, the charge conjugation transformation of the complex scalar field ψ amounts to

$$C^{-1}\psi(x)C = \psi(x)^*\,. \tag{D.3}$$

Based on this property, derive the transformation law of the photon field A_μ to ensure that the scalar QED Lagrangian is invariant under charge conjugation. Using this result, demonstrate Furry's theorem in scalar QED without using Feynman diagrams, that is, without assuming a weak coupling situation where the usual perturbative techniques can be applied. Discuss whether or not Furry's theorem holds only when the photons are on-shell or if it also holds for off-shell (virtual) photons.

Then, building on the same strategy, prove Furry's theorem in standard QED, exploiting now the fact that under charge conjugation the Dirac fermion field Ψ transforms as

$$C^{-1}\overline{\Psi}\gamma^\mu\Psi C = -\overline{\Psi}\gamma^\mu\Psi\,. \tag{D.4}$$

Discuss similarities and differences between the derivations of Furry's theorem in scalar and standard QED.

Problem II.2: The Proca Lagrangian. In this problem we consider the so-called Proca Lagrangian, which describes the theory of a massive free photon, and is defined by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\,, \tag{D.5}$$

with m being the photon mass. This theory is of course non invariant under gauge transformations, since as we have discussed the gauge symmetry of Maxwell's theory forbids the presence of a photon mass.

First of all, derive the equations of motion of the theory and show that they imply that

$$\begin{aligned}\left(-\partial^2 + m^2\right)A^\mu &= 0\,,\\ \partial_\mu A^\mu &= 0\,,\end{aligned} \tag{D.6}$$

and determine the solutions for these equations of motion. Show that, unlike the case of massless QED, the Proca Lagrangian leads to three different polarization states for the photon rather than two. Discuss why this outcome was expected. Then, derive in detail the commutation relations that follow from the canonical quantization of the Proca Lagrangian. Discuss the differences and similarities with respect to the analogous process in standard massless QED.

Finally, evaluate the energy-momentum tensor $T_{\mu\nu}$ for the Proca theory, and show that the energy density is positive definite. To achieve this, you help yourselves with the equations of motion of the theory.

Problem III.3(*): The photon propagator in the Coulomb gauge. The aim of this problem is to demonstrate that in Quantum Electrodynamics the following relation for the time-ordered product of photon fields holds

$$\left\langle T\left(A^i(x)A^j(y)\right)\right\rangle = \frac{1}{i}\Delta^{ij}(x-y)\,, \tag{D.7}$$

where the photon propagator in position space is given by

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-i\epsilon}\left(\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}\right)\,. \tag{D.8}$$

In particular, you need to explain the physical origin of the factor $i\epsilon$ that appears in the propagator. To carry out this derivation, you can use the following representation of the spatial components of the photon field:

$$\vec{A}(x) = \sum_{\lambda=\pm}\int \frac{d^3k}{(2\pi)^3 2\omega}\left(\left(\vec{\epsilon}_\lambda(k)\right)^* a_\lambda(k)e^{ikx} + \vec{\epsilon}_\lambda(k)a^\dagger_\lambda(k)e^{-ikx}\right)\,, \tag{D.9}$$

together with the commutation relations for the creation and annihilation operators $a^\dagger_\lambda(k)$ and $a_\lambda(k)$ as well as the completion relations for the polarization vectors $\vec{\epsilon}_\lambda(k)$. Explain in detail the various steps that you have used in your derivation.