B Problem set II

This is the second problem set of the course, to be completed and sent to the course instructors by **Thursday 1st of February 2018** at the latest. The solutions to the problems in this set will be discussed in the final tutorial session of Friday 2nd of February. Problems marked with **(*)** are optional.

Problem II.1: Furry's theorem. Furry's theorem states the following relation for time-ordered expectation values of gauge photon fields and gauge currents:

$$\langle T\left(A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\right)\rangle = 0\,, \tag{B.1}$$

$$\langle T\left(j_{\mu_1}(x_1)\ldots j_{\mu_n}(x_n)\right)\rangle = 0\,, \tag{B.2}$$

with n being an odd integer and j_μ is the Noether's current in either scalar QED or in standard QED. This theorem follows directly from the requirement of charge-conjugation symmetry. The goal is this problem is to derive a proof of this theorem.

To begin with, consider scalar QED. In this theory, the charge conjugation transformation of the complex scalar field ψ amounts to

$$C^{-1}\psi(x)C = \psi(x)^*\,. \tag{B.3}$$

Based on this property, derive the transformation law of the photon field A_μ to ensure that the scalar QED Lagrangian is invariant under charge conjugation. Using this result, demonstrate Furry's theorem in scalar QED without using Feynman diagrams, that is, without assuming a weak coupling situation where the usual perturbative techniques can be applied. Discuss whether or not Furry's theorem holds only when the photons are on-shell or if it also holds for off-shell (virtual) photons.

Then, building on the same strategy, prove Furry's theorem in standard QED, exploiting now the fact that under charge conjugation the Dirac fermion field Ψ transforms as

$$C^{-1}\overline{\Psi}\gamma^\mu\Psi C = -\overline{\Psi}\gamma^\mu\Psi\,. \tag{B.4}$$

Discuss similarities and differences between the derivations of Furry's theorem in scalar and standard QED.

Problem II.2: The Proca Lagrangian. In this problem we consider the so-called Proca Lagrangian, which describes the theory of a massive free photon, and is defined by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\,, \tag{B.5}$$

with m being the photon mass. This theory is of course non invariant under gauge transformations, since as we have discussed the gauge symmetry of Maxwell's theory forbids the presence of a photon mass.

First of all, derive the equations of motion of the theory and show that they imply that

$$\begin{aligned} \left(-\partial^2 + m^2\right)A^\mu &= 0\,, \\ \partial_\mu A^\mu &= 0\,, \end{aligned} \tag{B.6}$$

and determine the solutions for these equations of motion. Show that, unlike the case of massless QED, the Proca Lagrangian leads to three different polarization states for the photon rather than two. Discuss why this outcome was expected. Then, derive in detail the commutation relations that follow from the canonical quantization of the Proca Lagrangian. Discuss the differences and similarities with respect to the analogous process in standard massless QED.

Finally, evaluate the energy-momentum tensor $T_{\mu\nu}$ for the Proca theory, and show that the energy density is positive definite. To achieve this, you help yourselves with the equations of motion of the theory.

Problem III.3(*): The photon propagator in the Coulomb gauge. The aim of this problem is to demonstrate that in Quantum Electrodynamics the following relation for the time-ordered product of photon fields holds

$$\left\langle T\left(A^i(x)A^j(y)\right)\right\rangle = \frac{1}{i}\Delta^{ij}(x-y)\,, \tag{B.7}$$

where the photon propagator in position space is given by

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-i\epsilon}\left(\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}\right)\,. \tag{B.8}$$

In particular, you need to explain the physical origin of the factor $i\epsilon$ that appears in the propagator. To carry out this derivation, you can use the following representation of the spatial components of the photon field:

$$\vec{A}(x) = \sum_{\lambda=\pm}\int\frac{d^3k}{(2\pi)^3 2\omega}\left(\left(\vec{\epsilon}_\lambda(k)\right)^* a_\lambda(k)e^{ikx} + \vec{\epsilon}_\lambda(k)a^\dagger_\lambda(k)e^{-ikx}\right)\,, \tag{B.9}$$

together with the commutation relations for the creation and annihilation operators $a^\dagger_\lambda(k)$ and $a_\lambda(k)$ as well as the completion relations for the polarization vectors $\vec{\epsilon}_\lambda(k)$. Explain in detail the various steps that you have used in your derivation.