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A study of Anderson Localization with application to the Quantum Kicked Rotor

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Abstract

In this work is presented an extended study of Hamiltonians with random potential of the kind involved in the Anderson and Lloyd lattice models. Here are presented the most important results regarding the spectra of this Hamiltonians and their localization. These considerations are also accompanied by an overview of the possible measure of localization for the states and an analysis of the effects induced by the introduction of boundary conditions. An application of the results obtained is provided for the model of a Hamiltonian with a time-periodic delta potential (the Quantum kicked rotor), which will be reduced to the tight binding dynamic proper of Anderson models.

Contents

1 Introduction

The intention of this work is to introduce the reader to the instruments that have been utilized to solve what was initially the Hamiltonian problem describing impurities on a discrete lattice: the behaviour of electrons in a disordered material is described with a Hamiltonian that couples the state of the electron in a site, $u(i)$, with the state in other positions:

$$(Hu)(i) = k_i u(i) + \sum_n V_{ni} u(i-n) \tag{1.0.1}$$

where the sum runs over all the other sites of the lattice. This Hamiltonian models the in-site potential with the magnitude of k_i and the coupling with other sites through the potential V_{ni}. In particular, the Hamiltonian that models the impurity systems initially analyzed is the one with random variables for each site and constant coupling factors. With the introduction of various models, it became clear that such systems required the study of a particular kind of matrices, which in the case of an in-site random potential and near-sites coupling the matrices are of the form

$$H := \begin{pmatrix} k(1) & V_{12} & 0 & \cdots & \cdots & \cdots \\ V_{21} & k(2) & V_{23} & 0 & \cdots & \cdots \\ 0 & V_{32} & k(3) & V_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \tag{1.0.2}$$

with random diagonal entries: in this regard is important the study of random matrices. Moreover, since the Schrödinger equation arising from this Hamiltonian is described in terms of a product of random matrices, some of the most important achievements in this field will be introduced: we will present the famous results of Furstenberg, Osceledesc and others regarding the behaviour of the eigenstates, eigenvalues and of the spectrum of such products, and more in general some of the peculiarities of random processes and product of random matrices on probability spaces. From these analyses it will emerge how electron states are localized on the lattice, and how the randomness is responsible for the localization: in this regard, the Lyapunov exponents characterizing the exponentially localized states will play an important role and will be responsible for the quantification of localization. We will focus on the models introduced for the modelization of these systems by Anderson ([1]) and mainly by Lloyd ([16]), whose model admits an analytic solution for finding the most important properties of the eigenstates. Particular attention will be directed toward the measures of localization: much emphasis has been put on this characteristic since it is also experimentally observed in systems like extrinsic semiconductors. After discussing some of the quantities that describe this characteristic, a method introduced by Hatano and Nelson about the influence on localization will be presented: in their work they proposed an introduction of boundary conditions on a finite lattice to study the localization of eigenstates through the reaction of their energy to the perturbation, in the context of the depinning of

flux lines in superconductors; it turns out that this method is quite general and can be utilized to probe the influence of eventual lattice boundary couplings in an Anderson Model.

The methods provided by these theories will turn out to be important also in a model which apparently doesn't have any relation to the systems mentioned: the delta kicked rotor, first studied by Chirinkov, Izrailev, Ford and Casati ([3]). A rotor is basically an object rotating freely: if we add to this dynamic periodic delta impulses (the kicks, as if we would kick periodically the rotor in a fixed direction) the system shows classically a chaotic behaviour for certain values of the parameters involved, such as the moment of inertia and the intensity of the kicks. The study of the quantum mechanical version shows the presence of a particular behaviour, such that for a period of the kicks which is not commensurable with the intrinsic period of the rotor a localization in angular momentum space emerges, similarly to the localization of the lattice hopping model: we will in fact, through the so-called Maryland construction proposed by Grempel, Fishman and Prange [9], reach a form of the Hamiltonian that resembles that of 1.0.1 of a tight binding model. Before this analysis, an approach to the study of time-dependent Hamiltonians will be also presented, with the introduction of the Floquet states and the quasi-energy theory. Here we mention an experimental setting that shows the behaviour of a kicked rotor, performed by Bayfield and Koch [2]: a Hydrogen atom with high energy quantum numbers is put in a microwave cavity and ionization energies are measured; it has been shown that classically ionization occurs for values of the frequency involved related to the beginning of chaotic diffusion in action space; the quantum mechanical analysis instead predicts a higher ionization energy than the classical one due to the localization of electrons wave packets, which are peculiar to random potential as the ones discussed.

2 Anderson's Hopping Model

2.1 Hamiltonian in a lattice and random potentials

Let's consider a discrete d-dimensional lattice, in which we indicate the location of a point with an integer i; let $|n| := max|n_j|,\ |n|_+ := \sum_{j=1}^{d} |n_j|,\ n \,\epsilon\, \mathbf{Z}^d$; the analog of the laplacian operator on functions of this lattice is defined as

$$(\Delta_d(u))(i) := \sum_{j;|j-i|_+=1} (u(j) - u(i)); \qquad u(i) : \mathbf{Z}^d \to \mathbf{C} \tag{2.1.1}$$

It involes a summation running over the indices j that are a unit distance from i in the d-lattice, mimicking the differential operation represented by the correspondent continuous operator. It can be seen that Δ_d is bounded on $\tilde{l}^2(\mathbf{Z}^d)$ with an absolute continuum spectrum $\sigma_{ac}(\Delta_d) = [-4d, 0]$. Introducing a potential $\tilde{V}(i) : \mathbf{Z}^d \to \mathbf{R}$, the general form for the Schrödinger equation for the eigenvalue ε is

$$\tilde{H}u(i) = -(\Delta_d(u))(i) + \tilde{V}(i)u(i) = \varepsilon u(i) \tag{2.1.2}$$

The equation can be brought to a more convenient form by the unitary operator $(-1)^N$, defined through $[(-1)^N u](i) = (-1)^{|i|_+} u(i)$, which transforms the Hamiltonian to $[(-1)^N]\tilde{H}[(-1)^N]^{-1} = 4d + \Delta_d + V$. If we furthermore define $H_0(u)(i) = \sum_{j;|j-i|_+=1}(u(j))$ and $V = \tilde{V} + I$, we get the Hamiltonian in the form

$$Hu(i) := H_0(u(i)) + V(i)u(i) \tag{2.1.3}$$

which up to a constant and a unitary transformation corresponds to 2.1.2. We have thus separated the "hopping" terms of H_0 from the potential V in each site of the lattice.

We can now consider the case in which the potential V assumes random values at every site: to do so we first introduce a probability space represented by the tern (Ω, F, M), where F is a σ-algebra on the set Ω, and M is a probability measure on (Ω, F): in the probability interpretation this measure represents the probability density of the distribution chosen for the random set. For the case considered we choose

$$\Omega = I^{\mathbf{Z}^d}$$

where I is a subset of $\mathbf{R}$, so that the probability set corresponds to a set of real intervals for every site on the lattice. The corresponding F algebra is generated by the sets $\{r_i | r_{i_1} \in I_1, ..., r_{i_n} \in I_n\}$, $i_j \in \mathbf{Z}^d$, I_i subsets of $\mathbf{R}$.
Of importance for the study of the probability space and for the introduction of some properties of measures are the shift operators S_i on Ω, defined by

$$S_i r(j) = r(j - i), \qquad i, j \in \mathbf{Z}^d$$

These operators shift the lattice of random values by i in one direction.

A probability measure on Ω is said to be stationary if $P(S_i^{-1}R) = P(R)$, $\forall R \in F$. A stationary measure is ergodic if $\forall\, I$ such that $S_i^{-1}(I) = I\ \forall\, i \in \mathbf{Z}^d$ $P(I) = 0$ or $P(I) = 1$.

The Anderson model refers to the case of random independent identically distributed variables (i.i.d.): this distribution is characterized by a probability measure of the kind dP_0^D, $D \subset \mathbf{Z}^d$, where P_0 is the distribution of the random variables $r(i)$ on one site, so that for $I \subset \mathbf{R}$, $P_0(I) = P_0(r(i) \in I)$, $\forall I \subset \mathbf{R}$, $i \in \mathbf{Z}^d$. If we consider this distribution for every point of the lattice we get to the lattice measure dP_0^D. We can then make the identification $V_r(i) = r_i$ such that a random potential is realized by random values on $\mathbf{Z}^d$, and if the variables are i.i.d. we have the Anderson Hamiltonian

$$H_A(u(i)) := H_0(u(i)) + V_r(i)u(i) \tag{2.1.4}$$

For fixed r, this is a normal equation which in principle could be solved in a deterministic way. However, it is interesting to study the shared properties of the spectrum of these Hamiltonians for r varying on the probability space.

2.2 Localization of eigenstates in random potentials: Lyapunov exponents

In the one dimensional case, $d = 1$, the Schrodinger equation is

$$u(i+1) - u(i-1) + (V_r(i) - E)u(i) = 0 \tag{2.2.1}$$

By introducing the vector $\bar{u}(i) := (u(i+1), u(i))^T$ and the matrix

$$L_i(E, r) := \begin{pmatrix} E - V_r(i) & -1 \\ 1 & 0 \end{pmatrix} \tag{2.2.2}$$

then the solutions to equation 2.2.1 satisfy also

$$\bar{u}(i) = L_i(E)\bar{u}(i-1), \tag{2.2.3}$$

and

$$\begin{aligned} \bar{u}(i) &= \Phi_i(E)\bar{u}(0) \\ \bar{u}(-i) &= \Phi_{-i}(E)\bar{u}(0) \end{aligned} \tag{2.2.4}$$

where $\Phi_i(E) = L_i(E)L_{i-1}(E)...L_0(E)$, $\Phi_{-i}(E) = L_{-i+1}(E)^{-1}...L_0(E)^{-1}$ and $\bar{u}(0)$ is the initial conditions vector $\bar{u}(0) = (u(1), u(0))^T$. The above equations permit to calculate the behaviour of the solutions for big values of $|i|$ by studying the behaviour of the product of the matrices $L_i(E, r)$. In this regard, some useful theorems give information on the asymptotic behaviour of $\Phi_i(E)$ and $\bar{u}(i)$ for increasing i. Let's start by defining the Lyapunov exponents, which we will se characterize the asymptotic trend of $\bar{u}(i)$:

$$\begin{aligned} \bar{\gamma}^{\pm}(E, r) &:= \limsup_{N \to \pm\infty} \frac{1}{|N|} \log ||\Phi_N(E, r)|| \\ \underline{\gamma}^{\pm}(E, r) &:= \liminf_{N \to \pm\infty} \frac{1}{|N|} \log ||\Phi_N(E, r)|| \end{aligned} \tag{2.2.5}$$

These quantities are related to the behaviour of the norm of $\Phi_i(E, r)$ for fixed r and E.
We will now introduce important results regarding these quantities and their relation to the spectrum of the Hamiltonian of the one dimensional lattice. In the one dimensional case, the shift operators are such that $S_i = (S_1)^i$, so that a finite translation is a series of one-site translations. A sequence of random variables $\{L_i\}_{i \epsilon \mathbf{N}}$ is a subadditive process if, for a measuring preserving transformation S, $L_{i+j}(r) \leq L_i(r) + L_j(S^i r)$: this condition guarantees in some sense when applied to translation operator S that the process doesn't grow excessively with i, and recalls for the definition of subadditivity given for sequences of numbers. For these processes we have the important theorem found in [14]

Theorem 1.1 (Kingman, without proof)**.** *If $\{L_i\}_{i \epsilon \mathbf{N}}$ is a subadditive process, for which the expected value over $\Omega < |L_i| > < +\infty \ \forall i$ and $\Gamma(L) := inf \frac{< L_i >}{i} > -\infty$, then $L_i(r)/i$ converges for almost all r in Ω. If moreover the traslation operator S is ergodic then for almost all r we have that $lim_{i \to +\infty} \frac{1}{i} L_i(r) = \Gamma(L)$.*

What the theorem says is that if a process satisfies the subadditivity limitation and S is ergodic then we have a convergence for $\frac{L_i(r)}{i}$ which doesn't depend on r, namely on the instance of the randomness. This result is used to prove one important theorem about 2.2.5

Theorem 1.2 (Furstenberg, Kesten ([7])). *For fixed E and almost all r in* Ω

$$\gamma^{\pm}(E) := \lim_{N\to\pm\infty} \frac{1}{|N|} \log ||\Phi_N(E,r)|| \tag{2.2.6}$$

exists indipendently of r and $\gamma^+(E) = \gamma^-(E)$.

Proof, only sketched. Defyning the process as $L_N = ||\Phi_N(E,r)||$, it can be shown that L is subadditive, $< |L_N| > < \infty$ and $inf(< L_N > /N) > -\infty$, so for theorem 1.1 we have that

$$\lim_{N\to+\infty} \frac{1}{|N|} \log ||\Phi_N(E,r)|| = \inf_{N>0} \frac{1}{|N|} < \log ||\Phi_N(E,r)|| > \quad for\ a.e.\ r \tag{2.2.7}$$

$$\lim_{N\to-\infty} \frac{1}{|N|} \log ||\Phi_N(E,r)|| = \inf_{N<0} \frac{1}{|N|} < \log ||\Phi_N(E,r)|| > \quad for\ a.e.\ r \tag{2.2.8}$$

Moreover for the stationarity we have that $< \log ||\Phi_N^{-1}|| > = < \log ||\Phi_{-N+1}|| >$ and for

J := $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

we have both $(J\Phi_N J^{-1})^t = \Phi_N^{-1}$ and $||J\bar{u}|| = ||J^{-1}\bar{u}|| = ||u||$, so that $\gamma^+ = \gamma^-$.

□

The next result due to Osceledets ([17]) gives information about the asymptotic behaviour of the solution of the Schrodinger equation with the potential.

Theorem 1.3 (Osceledets). *Given a sequence of* 2×2 *matrices* $\{L_i\}_{i\in\mathbf{N}}$ *such that* $\lim_{n\to+\infty}(1/n)\log||L_n|| = 0$ *and* $\det L_n = 1$, *then if* $\gamma := \lim_{n\to+\infty}(1/n)\log||L_n.$ $..L_1|| > 0$ *there exist a one dimensional vector subspace* $V \subset \mathbf{R}^2$ *such that*

$$\lim_{n\to+\infty}(1/n)\log||L_n...L_1\bar{u}|| = -\gamma \quad for\ \bar{u}\ \in\ V,\ \bar{u}\neq\ 0 \tag{2.2.9}$$

and

$$\lim_{n\to+\infty}(1/n)\log||L_n...L_1\bar{u}|| = \gamma \quad for\ \bar{u}\ \notin\ V,\ \bar{u}\neq\ 0 \tag{2.2.10}$$

This important theorem tells us that for a process satisfying the condition stated the corresponding Hamiltonian admits exponentially decaying and growing solutions: since the theorem admits similar results for the behaviour at $n \to -\infty$, we can say that the solutions in which we are interested are the exponentially decaying ones with solutions for $n \to -\infty$ and $n \to \infty$ coincident. Another aspect to note is that theorem 1.3, along with theorem 1.2,

guarantees that at fixed energy E for almost all r in the probability space Ω the solutions has the same exponential behaviour at large n, with a Lyapunov exponent $\gamma(E) := \lim_{N\to\pm\infty} \frac{1}{|N|} \log ||\Phi_N(E,r)||$; we can't conclude immediately from this that for every E such condition is met since at varying E, the sets of r for which the condition is not true could add to a set of measure non-zero, thus invalidating the conclusions we reached from theorem 1.3 (the existence of only exponentially decreasing and increasing solutions). However, the following theorem proved initially by Ishii ([13]) will characterize at least the continuous spectrum of the Hamiltonian in terms of the Lyapunov exponents.

Before we need to introduce some definitions: given a Lebesque measure μ on $\mathbf{R}$ and its absolutety continuous part μ_{ac}, a set A is an essential support of μ_{ac} if there is a set B with $\mu(B) = 0$ such that $\mu(\mathbf{R}\backslash(A \cup B)) = 0$ and for C such that $\mu(C) = 0$ then $\mu(A \cap C) = 0$. The essential closure is defined as $\bar{A}^{ess} := \{\lambda | \mu(A \cup (\lambda - \epsilon, \lambda + \epsilon)) > 0 \ \forall \ \epsilon\}$: we can see the similarity with the definition of the closure of a set. We can now state without proof:

Theorem 1.4 (Ishii, Pastur, Kotani)**.** *If $\{V_i\}$ is a bounded ergodic process, then*

$$\sigma_{ac}(H) = \bar{A}^{ass}$$

where $\sigma_{ac}(H)$ is the Hamiltonian with potential associated to the process and $A = \{E|\gamma(E) = 0\}$.

Thus we have at least partially obtained information about the measure of the spectrum of H. Another theorem, stated without proof and initially proposed by Ruelle ([19]) and then expanded by Andrein, Georgescu and Enss (thus the name RAGE), strongly characterizes the continuous spectrum of an Hamiltonian in terms of the localization of continuous eigenstates.

Theorem 1.5 (RAGE)**.** *Consider the self-adjoint operator H and the bounded operator $\chi(|i| \leq R)$, namely the characteristic function of the set $\{i \in \mathbf{Z}^d | |i|_+ \leq R\}$. If ψ_c belongs to the continuous spectrum of H then*

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t dt' |\chi(|i| \leq R) e^{-\frac{i}{\hbar} H t'} \psi_c|^2 = 0 \quad \forall R \tag{2.2.11}$$

What the theorem says is that for every distance R the eigenstate will eventually leave the region included in that distance, as it should be for a non-localized state, thus relating the continuum spectrum with the non-localizability of states. The last theorem of the section finally guarantees the presence of a point spectrum for Hamiltonians of the kind 2.1.3 with random potentials $V_r(i)$ of our interest:

Theorem 1.6 (Kunz, Souillard ([15]))**.** *Suppose that the V_r in a d-dimensional lattice Hamiltonian are random distributed variables with a common distribution $\rho(x)dx$: if $\rho \in L^\infty$ and has compact support, then the corresponding Hamiltonian as a pure point spectrum for almost every r in Ω, and the eigenfunctions are exponentially localized for almost every r.*

This important theorem thus characterizes completely the eigenstates of the Hamiltonian we have introduced, with distribution in the space of l^∞ function.

With the help of the theorem discussed we have been able to characterize strongly the spectrum of our random Hamiltonian: the spectrum is characterized by localized eigenstates for almost all the instances of the randomness.

2.3 Thouless Formula

Until now we have studied the property of the spectrum of a Hamiltonian with a general random potential V related to a random process on a probability space: the mentioned theorems give information on the behaviour of the eigenstates and their localization, described by the Lyapunov exponent; these results don't apply only to a single instance of the randomness but refer to almost all of the probability space. Nonetheless, even if we have ascertained the existence of these exponents, we haven't yet provided a method to calculate them. In this regard, we will introduce a very important formula, the Thouless formula (introduced by Thouless in [21]), which connects the density of states to the Lyapunov exponents. This formula descends from some property of the function $\gamma(E)$ encountered in the previous theorems. The route to this formula is a bit long so we will not go through it: however, it is interesting to state at least that it derives from properties of subharmonicity of $\gamma(E)$. However, we can present a qualitative demonstration on why the formula holds in the one-dimensional finite case: consider the Green function relative to the Hamiltonian of the finite Lloyd model:

$$G_{nm} := \frac{1}{N}\left(\frac{1}{E-H}\right)_{nm} = \frac{1}{N}(-1)^{m-n}\frac{\det_{nm}(E-H)}{\det(E-H)} \tag{2.3.1}$$

and its spectral representation:

$$G(E) = \frac{1}{N}\sum_{\lambda}\frac{\left|u^\lambda\right\rangle\left\langle u^\lambda\right|}{E-E_\lambda} \tag{2.3.2}$$

From the tridiagonality of the matrix Hamiltonian results that

$$G_{1N}(E) = \frac{1}{N}\frac{1}{\prod_{\nu=1}^{N}(E-E_\nu)} \tag{2.3.3}$$

Where E_ν are the eigenvalues of the corresponding Hamiltonian. On the other hand, from 2.3.2 we have that

$$G_{1N} = \frac{1}{n}\sum_{\nu}\frac{u_\nu(1)u_\nu(N)}{E-E_\nu} \tag{2.3.4}$$

Whit u_ν eigenstate relative to the eigenvalue E_ν. Confronting the two precedent expressions we get by evaluation of the residual pole at $E = E_\nu$:

$$u_\nu(1)u_\nu(N) = \frac{1}{\prod_{\mu\neq\nu}(E_\nu - E_\mu)} \tag{2.3.5}$$

If the state is exponentially localized, then we have that $u(1)u(N) = Ae^{-\gamma_\nu N}$ with A normalization factor and γ_ν corresponding Lyapunov exponent. Comparing this expression with 2.3.4 we obtain

$$\gamma_\nu = \frac{1}{N} \sum_{\mu \neq \nu} \log |E_\nu - E_\mu| \tag{2.3.6}$$

In switching to the limit $N \to +\infty$ we have to substitute the sum with an integral and consider also the density of eigenstates. In the general case, we have thus

Theorem 1.7 (Thouless formula)**.** *The Lyapunov exponent is such that*

$$\boxed{\gamma(E) = \int \log |E - E^{'}| \rho(E^{'}) dE^{'}} \tag{2.3.7}$$

where $\rho(E^{'})$ is the density of state of the Hamiltonian (2.1.4)*.*

The interesting aspect of this result is that it connects the density of state to their localization: we see that the presence of other eigenstates influences the exponential behaviour of the localization, and from 1.7 we see that biggest contribute comes from the region of highest density of states.
This formula gives the possibility of calculating the Lyapunov exponent of a random system: it is however quite difficult to compute analytically 2.3.7 in most of the cases, and one of the few solvable models is Lloyd model.

Since we are interested in the Lloyd model we don't execute the calculations for determining $\gamma(E)$ and $\rho(E)$ for the Anderson model, anyway it may be insightful to see some practical results: in figures 1 and 2 are shown the outputs of a simulation which models an Anderson random potential in a finite lattice (with a finite number of sites), with random variables distributed in a symmetric interval on the real line centered around 0. It's clearly visible, expecially in the logarithmic scale, the exponential decay of the eigenstates. The simulation is held for 700 lattice sites and an interval $[-6, 6]$.

In Figure 1 the linear trend of the eigenstate is clearly evident. In Figure 2 the function $\gamma(E)$ from (1.20) is plotted: even if it is broadened by the high number of states, a parabolic-like trend is visible.

3 Lloyd's Model

3.1 Adding a Lorentzian random potential

As stated in the previous section, one of the cases in which the expression for the Lyapunov exponent can be calculated analytically through 2.3.7 is Lloyd's model proposed initially by Lloyd ([16]). In addition to that, this model is essential for the description of the eigenstates of the kicked pendulum. In this section, we will introduce the model and calculate both the Lyapunov exponent and the density of states.

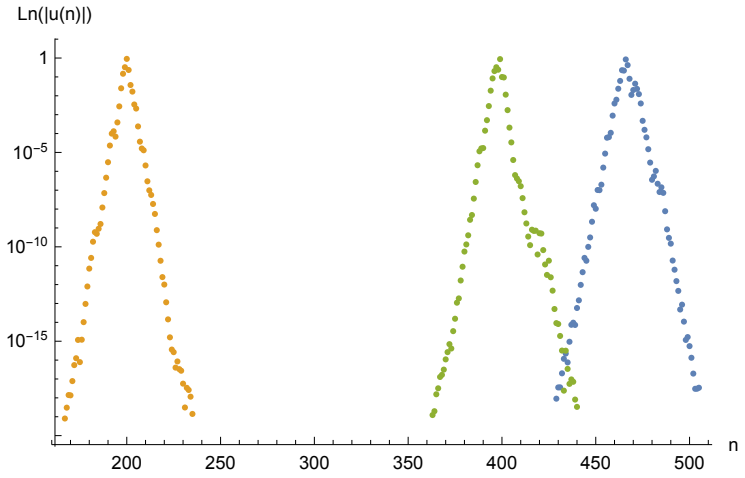


Figure 1: Plot of three eigenstates of an Anderson model with uniform distributed random potential: natural logarithm of the amplitude modulus of states versus site position.

Consider a Schrödinger equation for a one dimensional lattice of the type:

$$\frac{k}{2}(u(i-1)+u(i+1))+V(i)u(i)=Eu(i) \tag{3.1.1}$$

Where we have introduced the factor $\frac{k}{2}$ to quantify the hopping probability to neighboring sites. The Hamiltonian has the matrix form:

$$H:=\begin{pmatrix} V(1) & \frac{k}{2} & 0 & \cdots & \cdots & \cdots \\ \frac{k}{2} & V(2) & \frac{k}{2} & 0 & \cdots & \cdots \\ 0 & \frac{k}{2} & V(3) & \frac{k}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \tag{3.1.2}$$

It's a tridiagonal matrix with random entries on the diagonal. In the case of the Lloyd's model, the random potential follows the Cauchy distribution centered in zero and caracterized by the width δ:

$$\rho(V(i))=\frac{\delta}{\pi(\delta^2+V(i)^2)} \tag{3.1.3}$$

We start by recalling the Green function of the finite Hamiltonian with N sites:

$$G_{nm}:=\frac{1}{N}\left(\frac{1}{E-H}\right)_{nm}=\frac{1}{N}(-1)^{m-n}\frac{\det_{nm}(E-H)}{\det(E-H)} \tag{3.1.4}$$

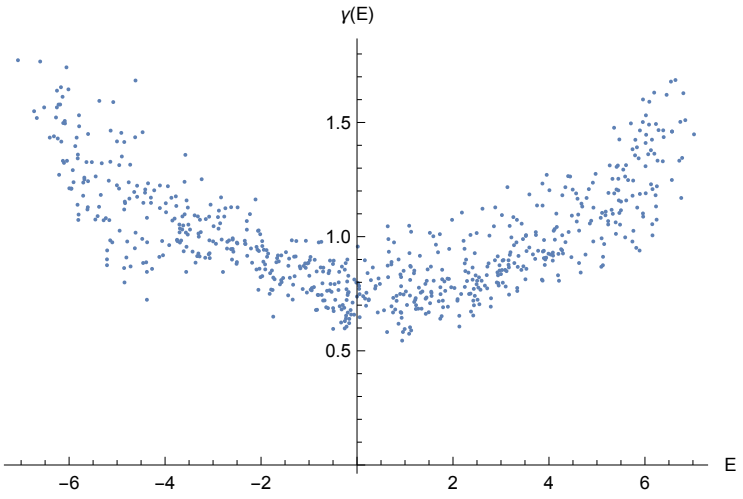


Figure 2: Trend of Lyapunov exponent as a function of E for the Anderson model of i.i.d. variables

Where $\det_{nm}(E-H)$ is the nm minor of $(E-H)$.
Deriving 2.3.7 with respect to the energy E one obtains the useful formula

$$\gamma^{'}(E) = \mathcal{P}\int dE^{'}\,\frac{\rho(E^{'})}{E-E'} \tag{3.1.5}$$

where the principal part of the integral appears.

From the representation of the Green function in the finite case in term of eigenstates u^{λ} with eigenvalues E_{λ}:

$$G(E) = \frac{1}{N}\sum_{\lambda}\frac{\left|u^{\lambda}\right\rangle\left\langle u^{\lambda}\right|}{E-E_{\lambda}}$$

we get that

$$\mathrm{Tr}(G(E)) = \frac{1}{N}\sum_{\lambda}\frac{1}{E-E_{\lambda}} \tag{3.1.6}$$

which in the continuous case, $N\to+\infty$, turns to

$$\mathrm{Tr}(G(E)) = \int dE^{'}\,\frac{\rho(E^{'})}{E-E'} \tag{3.1.7}$$

The denominator of 3.1.7 is the cause of a cut on the complex plane along the real axis, which in turn generates a jump of the function through this axis; for this reason above and below the real axis we have that

$$\mathrm{Tr}(G(E\pm i0)) = \mathcal{P}\int dE^{'}\,\frac{\rho(E^{'})}{E-E'} \mp i\pi\rho(E) \tag{3.1.8}$$

Where $i0$ represents an arbitrary small imaginary value. The real part of this expression gives the derivative of the Lyapunov exponent and is thus used to calculate the asymptotic behaviour of eigenstates; the imaginary part is directly connected to the density of states $\rho(E)$ and will be used to calculate this quantity.

3.2 Lyapunov Exponents and Density of States

It is to be noted that all the precedent discussion holds true for a realization of the randomness: to obtain information common to all the probability state we need to take the averages of the quantities involved. In the case of a Lorentzian distribution of independent variables, the average of a quantity Q is calculated with

$$< Q >= \left(\prod_i \int_{-\infty}^{+\infty} dV(i) \frac{\delta}{\pi(\delta^2 + V(i)^2)} \right) Q(V(i)) \tag{3.2.1}$$

In the case of $\overline{\gamma(E)}$, the average is to be carried out on $\bar{G}(E)$ in the trace expression. The way in which the integral 3.2.1 is calculated is via the replica trick, which is usually employed for calculation in spin glass theories ([6]).
In order to calculate the integral involved, we can employ the theory of Grassman variables and superymmetric integrals. We introduce n anticommuting variables θ_i, $\theta_i\theta_j + \theta_j\theta_i = 0$, and their independent products:

$$(1 + \theta_1)(1 + \theta_2)...(1 + \theta_n) = \Theta_1 + \Theta_2 + ...\Theta_{2n}$$

The linear combinations $F = f_0 + \sum_i f_i\Theta_i$ span a vector space of dimension $2n$ which is also an algebra.
The properties of the integrals of these variables are such that:

1) $\int d\theta_k F = \sum f_i \int d\theta_k \Theta_i \quad linearity$

2) $\int d\theta_k \Theta_i = 0 \quad if\ \Theta_i\ doesn't\ contain\ \theta_k$

3) $\int d\theta_k (\theta_r...)\theta_k(\theta_m..) = (-1)^l(\theta_r...)(\theta_m..)\ where\ l\ is\ the\ number\ of\ variables\ before\ \theta_k$

It can be seen that the integral operation behaves as a derivation on Grassman variables. We can also introduce a conjugation operation, such that for variables Ψ_i we have $\overline{\overline{\Psi}_k} = \Psi_k$ and $\overline{\Psi_k \Psi m} = \bar{\Psi}_k \bar{\Psi}_m$. It can be shown that, given a $n \times n$ matrix M, we have the useful formula

$$\int d\theta_1 d\bar{\theta}_1 ... d\theta_n d\bar{\theta}_n e^{-\bar{\theta} M \theta} = \det M$$

Where θ represents the vector with components the Grassman variables θ_i. This formula can be used to calculate the trace of the Green function through 3.1.4

and the corresponding formula for the inverse of the determinant:

$$\frac{\det(E'-H)}{\det(E-H-i0^+)} = \int \prod_{k=1}^{n} d\theta_k d\bar{\theta}_k \frac{d^2\phi_k}{\pi} e^{-i\bar{\theta}(E'-H)\theta - i\phi^\dagger(E-H-i0^+)\phi} \tag{3.2.2}$$

The average of the diagonal terms is then evaluated as in 3.2.1:

$$\prod_i \int_{-\infty}^{+\infty} dV(i) \frac{\delta}{\pi(\delta^2+V(i)^2)} \exp\left[iV(i)(\bar{\theta}_i\theta_i + |\phi_i|^2)\right] = e^{-\delta\sum(\bar{\theta}_i\theta_i + |\phi_i|^2)}$$

So that we get a final expression for the average of the determinant fraction:

$$\begin{aligned}\left\langle \frac{\det(E'-H)}{\det(E-H-i0^+)} \right\rangle &= \int \prod_{k=1}^{n} d\theta_k d\bar{\theta}_k \frac{d^2\phi_k}{\pi} e^{-i\bar{\theta}(E-H_0-i\delta)\theta - i\phi^\dagger(E-H_0-i\delta)\phi} \\ &= \frac{\det(E'-H_0-i\delta)}{\det(E-H_0-i\delta)}\end{aligned} \tag{3.2.3}$$

Where H_0 is the adiacent matrix of the lattice.
We can obtain the same result with the expression deduced from the Gaussian integral: start by expressing the finite lattice Green function 3.1.4 as a multiple gaussian integral:

$$\begin{aligned}\left(\prod_{i=1}^{N}\prod_{\alpha=1}^{n}\int_{-\infty}^{+\infty} dS_i^\alpha\right) S_p^1 S_q^1 \exp\left[-i\sum_{ij\alpha}(E-H-i0^+)_{ij} S_i^\alpha S_j^\alpha\right] = \\ \frac{N}{2} G_{pq}(E-i0^+)\left[\frac{\pi}{det(E-H-i0^+)}\right]^n\end{aligned} \tag{3.2.4}$$

The trick consists in letting the parameter n vary in a continuous way and getting the limit of the integral for $n \to 0$; we get thus

$$\begin{aligned}NG(E-i0^+)_{pq} = 2\lim_{n\to 0}\left(\prod_{i=1}^{N}\prod_{\alpha=1}^{n}\int_{-\infty}^{+\infty} dS_i^\alpha\right) \\ S_p^1 S_q^1 \exp\left[-i\sum_{kj\alpha}(E-H-i0^+)_{ij} S_k^\alpha S_j^\alpha\right]\end{aligned} \tag{3.2.5}$$

Then in the average for $G(E+i0^+)_{pq}$ the Hamiltonian is the one at 3.1.2: the random diagonal terms get an average of the form:

$$\left\langle \exp\left[-i\sum_j V(j) S_j^\alpha S_j^\alpha\right]\right\rangle = \prod_{j=1}^{N}\int dV(j)\frac{\delta\exp\left[-iV(j)(S_j^\alpha)^2\right]}{\pi(\delta^2+V(j)^2)} = \exp\left[-i\delta\sum_j (S_j^\alpha)^2\right]$$

Thus getting an average for the Green function:

$$N\left\langle G(E-i0^+)_{pq}\right\rangle = 2\lim_{n\to 0}\left(\prod_{i=1}^{N}\prod_{\alpha=1}^{n}\int_{-\infty}^{+\infty} dS_i^\alpha\right) S_p^1 S_q^1$$
$$\exp\left[-i\sum_{ij\alpha}(E-\tilde{H}-i0^+)_{ij}S_i^\alpha S_j^\alpha\right] \tag{3.2.6}$$

Where $\tilde{H}_{pq} = i\delta\delta_{pq} + \frac{k}{2}(\delta_{p,q+1}+\delta_{p,q-1})$. Now the new integral involves an Hamiltonian with a diagonal term i instead of a random potential. This means that formaly as G_{pq} is related to the original Hamiltonian so $< G_{pq} >$ is related to $\tilde{H}$. Notice how the method involving the supersymmetric integral is more direct and elegant.

We can write then

$$(N < G >)^{-1}_{pq} = (E-i\delta)\delta_{pq} - \frac{k}{2}(\delta_{p,q+1}+\delta_{p,q-1})$$

To calculate $Tr(N < G >)$, which is the final objective of all these calculations, we have to invert $\tilde{H}$. Exploy for the average the form of the Green function 3.1.4 and write the trace as

$$\mathrm{Tr}(N < G >) = \sum_m \frac{\det_{mm}(E-\tilde{H})}{\det\Big(E-\tilde{H}\Big)} = \frac{\partial}{\partial E}\log\Big(\det\Big(E-\tilde{H}\Big)\Big) \tag{3.2.7}$$

The evaluation of the determinant in 3.2.7 procedes by induction: denoting by det_N the determinant in the finite case of a $N\times N$ matrix, we have that

$$\begin{aligned}
\det_1 &= E - i\delta \\
\det_2 &= (E-i\delta)^2 - \frac{k^2}{4} \\
\det_n &= (E-1)\det_{n-1} - \frac{k^2}{4}\det_{n-2}
\end{aligned} \tag{3.2.8}$$

Given the form of the ricurrence equations, and given the fact that the coefficients in 3.2.8 do not depend on n, one can guess a solution of the form $\det_n = x^n$, getting the equation $x^2-(E-i\delta)x-\frac{k^2}{4}$; the solutions to the equation are then

$$x_\pm = \frac{1}{2}\left[E-i\delta \pm \sqrt{(E-i\delta)^2-k^2}\right] \tag{3.2.9}$$

Using the initial conditions $det_0 = 1$ and $det_1 = E-i$ we have that $det_1 = (x_+ + x_-)$ and so the determinant is

$$\det_N = \det\Big(E - \tilde{H}\Big) = \frac{x_+^{N+1} - x_-^{N+1}}{x_+ - x_-}$$

We get finally an expression for the average of the Lyapunov exponent

$$< \gamma^{'}(E) >= \frac{\partial}{\partial E} \operatorname{Re}\left[\frac{1}{N}\log\left(\frac{x_+^{N+1} - x_-^{N+1}}{x_+ - x_-}\right)\right]$$

From 3.2.9 we can see that $|x_+| > |x_-|$ thus in the limit of an infinite lattice, $N \to +\infty$, we get

$$< \gamma^{'}(E) >= \frac{\partial}{\partial E}\log|x_+|$$

So that

$$< \gamma(E) >= \log\left|\frac{E - i\delta}{k} + \sqrt{\left(\frac{E - i\delta}{k}\right)^2 - 1}\right| \tag{3.2.10}$$

Which can be rewritten as

$$\cosh(< \gamma(E) >) = \frac{1}{2k}(\sqrt{(E-k)^2 + \delta^2} + \sqrt{(E+k)^2 + \delta^2}) \tag{3.2.11}$$

For the spectral density we have seen in 3.1.8 that

$$\rho(E) = \frac{1}{\pi}\operatorname{Im}\{Tr(G(E - i0))\}$$

which also need to be averaged over the disorder. We have seen that this average brings the result

$$< \rho(E) >= \frac{1}{\pi}\operatorname{Im}\{\operatorname{Tr}\Big(\tilde{G}(E - i0)\Big)\} \tag{3.2.12}$$

Where $\tilde{G}$ is the Green function associated to the Hamiltonian $\tilde{H}$ found in 3.2.6: we have thus reduced the density of states to the one relative to the Hamiltonian found in 3.2.6. If we indicate with $\tilde{E}_k$ the eigenvalues of H_0, the adjacency Hamiltonian with only the hopping components of 2.1.3, the spectral density of H averaged on the chaos is given by a sum of Lorentzian distributions (as one find from explicit calculations of 3.2), so for the finite case

$$< \rho(E) >= \frac{\delta}{N}\sum_{k=1\dots N}\frac{1}{\pi}\frac{1}{(E - \tilde{E}_k)^2 + \delta^2} \tag{3.2.13}$$

For H_0 the density of eigenvalues is given by a sum on the possible periodicity on the lattice in the finite case:

$$\rho_0(E) = \frac{1}{L}\sum_{1\leq l\leq L}\delta(E - 2\cos\left(\frac{2\pi l}{L}\right)) = \int_{\mathbf{R}}\frac{dl}{2\pi}\delta(E - 2\cos l)$$
$$= \int_0^{+\infty}\frac{ds}{\pi}J_0(2s)\cos(Es) \tag{3.2.14}$$

where L is the lattice distance, J_0 the Bessel function and we have gone to the continuous case. The spectral density for the original Hamiltonian can then be evaluated as

$$< \rho(E) >= \int_{\mathbf{R}} dE^{'}\frac{\delta\rho_0(E^{'})}{\pi[(E-E^{'})^2+\delta^2]} = \int_0^{+\infty}\frac{ds}{\pi}J_0(2s)\cos(Es)e^{-\delta s}$$

The integral gives the complicated formula

$$< \rho(E) >= \frac{1}{\pi\sqrt{2}}\frac{\sqrt{4+\delta^2-E^2+\sqrt{(4+\delta^2-E^2)^2+4E^2\delta^2}}}{\sqrt{(4+\delta^2-E^2)^2+4E^2\delta^2}} \tag{3.2.15}$$

We have thus obtained two important facts about the spectrum of the Lloyd's Hamiltonian: we have the exponential behaviour of eigenstates thanks to 3.2.10 and the density of eigenstates thanks to 3.2.15.

Similarly to Anderson's model case, a simulation with a finite lattice and a certain degree of Lorentzian disorder gives some interesting results: in particular eigenstates shows a similar behaviour as in the Anderson Hamiltonian, with an exponential decay which is directly computable from the graphs; moreover the plot of the Lyapunov exponent resembles the logarithmic trend obtained in the formula. Figures 3 and 4 refer to a simulation of an 700×700 Hamiltonian with Cauchy disorder of 2.5.

4 Measures of localization of states

Until now we have obtained insights on the characteristics of tridiagonal Hamiltonian with random diagonal elements: we have seen how the eigenstates of such Hamiltonian behave at distant sites from their localization center, and have focused on the quantitative properties of the localization obtaining even analytical results about the so-called Lyapunov exponents. What we have is thus a spectrum of exponentially decaying states that we have defined as localized. However, it is worth noticing that there are other means to qualify the localization of states, without limiting oneself to the value of the decadence exponents. In this section other quantities that characterize localized states will be presented and confronted; moreover, it will be shown how an introduction of a certain kind of "boundary conditions" can change the eigenstates so to highlight their localization in the un-conditioned case.

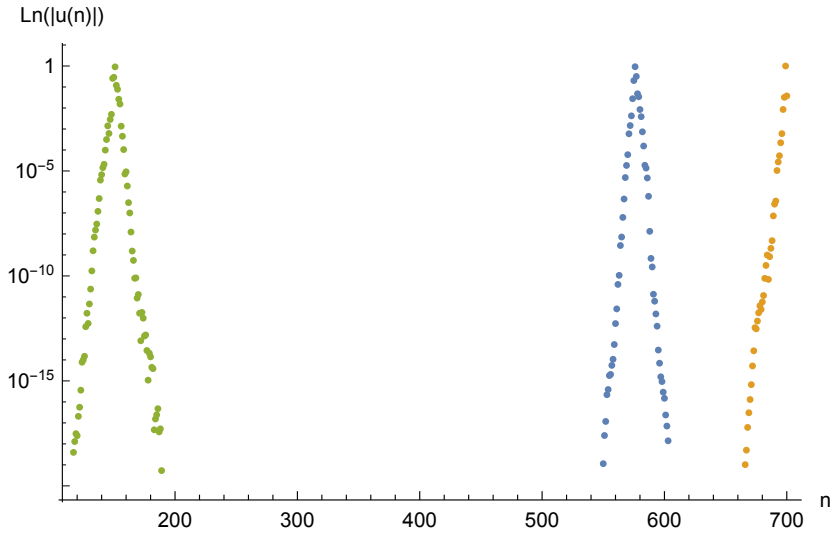


Figure 3: Plot of the natural logarithm of eigenstates site amplute for the Lloyd model. The exponential trend is clearly visible as in the Anderson case.

4.1 Quantities related to localization

For all this section the vectors labeled $\bar{u} \in l^2(\mathbf{R})$ will represent states of the random one dimensional Hamiltonian and $u(i)$ will be the eigenstate's amplitude at site $i \in \mathbf{Z}$. The state will be taken normalized, so that $\sum_i |u(i)|^2 = 1$.

The first quantity to be introduced is the classical root mean square, which gives information about the width of the probability distribution of states. It is defined as

$$\Delta^2 \bar{u} = \sum_i [(i - < p >)^2 |u(i)|^2] =< p^2 > - < p >^2 \tag{4.1.1}$$

Where $< p >$ is the mean position of the eigenstates, $< p >= \sum_i [i|u(i)|^2]$, and $< p^2 >$ is the mean value of i^2, $< p^2 >= \sum_i [i^2|u(i)|^2]$. Obviously, the more this value is large the more the states is unlocalized as the distribution gets wider and thus is less localized. It is interesting how a pertubation of the Hamiltonian can give information about the terms in 4.1.1: introducing a linear perturbative term of the kind $H^{\epsilon}_{ij} = \epsilon \delta_{ij} j$ the perturbation theory affirms that the first order change in the eigenvalue for a state $\bar{u}$ is

$$\Delta E = \epsilon \sum_i i|u(i)|^2 = \epsilon < p > \tag{4.1.2}$$

Essentially, introducing a "V-shaped discrete" potential will affect the furthest eigenstates, changing their energy proportionally to their barycenter. Repeating the same argument with a "parabolic discrete" perturbative potential $H^{\epsilon}_{ij} = \epsilon \delta_{ij} j^2$ one obtains that

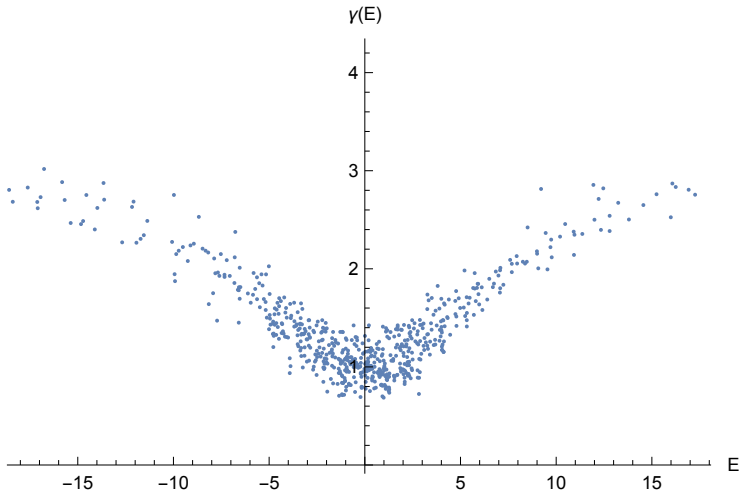


Figure 4: Plot of the Lloyd model Lyapunov exponents as function of the energy. From the distribution of points in the graph one can also infer the density of state, which is bigger near the origin.

$$\Delta E = \epsilon \sum_i i^2 |u(i)|^2 = \epsilon < p^2 > \tag{4.1.3}$$

Thus obtaining the two quantities in 4.1.1: this is a first example of how a slight modification of the potential can alter the eigenstates. The root mean square, as in the classical case, gives information about the width of the distribution of the eigenstates and is thus related directly to its localization.

The next quantity for localization measure is the participation ratio: its inverse is defined as

$$P_r(\bar{u})^{-1} = \sum_i |u(i)|^4 \tag{4.1.4}$$

Its meaning is related to the probability of a state to return in the inizial position under an Hamiltonian time evolution: given the Hamiltonian 2.1.3 and its time propagator $U(t) = e^{\frac{-i}{\hbar}Ht}$, the probability, being the state initially at a lattice position i, to be after a time t at a lattice position j is given by:

$$P_{i\to j} = \frac{1}{t}\int_0^t dt^{'} |\langle j| U(t^{'}) |i\rangle|^2 \tag{4.1.5}$$

Taking the limit for $t \to +\infty$ and inserting two identity relations in 4.1.5 involving eigenstates of the Hamiltonian one obtains

$$\sum_{\lambda\omega} \langle j|E_\lambda\rangle \langle E_\lambda|i\rangle \langle i|E_\omega\rangle \langle E_\omega|j\rangle \lim_{t\to+\infty} \frac{1}{t}\int_0^t dt' \exp\left[\frac{-it'(E_\lambda - E_\omega)}{\hbar}\right]$$
$$= \sum_\lambda |\langle j|E_\lambda\rangle|^2 |\langle i|E_\lambda\rangle|^2$$

Calculating the probability of return to the initial position i one obtains

$$P_{i\to i} = \sum_\lambda |\langle i|E_\lambda\rangle|^4 = \sum_\lambda |u_\lambda(i)|^4 \tag{4.1.6}$$

The sum of 4.1.6 over all sites gives the "average" of the participation numbers 4.1.4 over all eigenstates. This average reflects the probability for the system of returning to a site: if the eigenstates are non localized, then for everyone of them the participation number will be infinite, as this probability is zero: the unlocalized state, in fact, will leave the initial position and will probably not return to it.

Another function that measures the level of localization of a states is it's entropy, defined as in statistical physics:

$$S(\bar{u}) = -\sum_i |u(i)|^2 \log\big(|u(i)|^2\big) \tag{4.1.7}$$

With associated entropy length

$$l_S(\bar{u}) = \exp(S(\bar{u})) \tag{4.1.8}$$

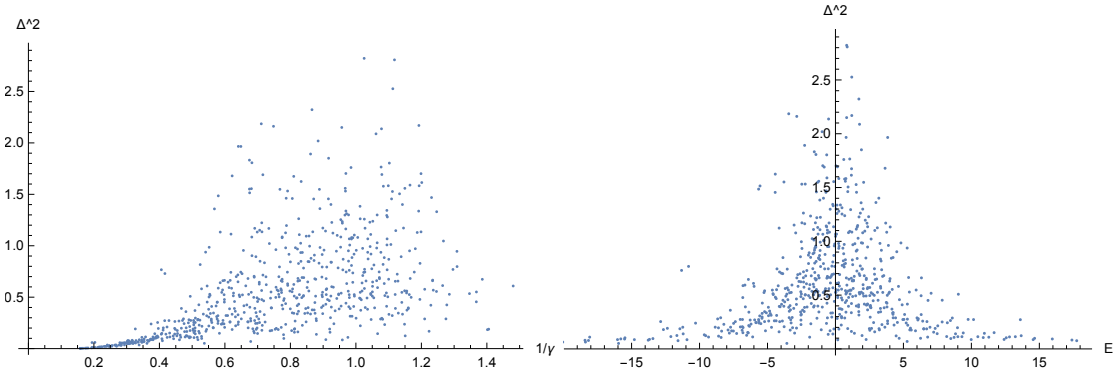
Its interpretation comes from the thermodynamic theory: as the entropy of the system describes its disorder, the entropy of an eigenstates 4.1.7 retains information about the localization, seen as a grade of disorder; it is in fact known that if for example we have a site-localized state, for which $|u(i')|^2 = 1$ for a certain i' and $|u(i)|^2 = 0$ for $i \neq i'$, then the quantity 4.1.7 is zero, so that the case of maximal localization has null entropy or disorder; instead for a state of maximal disorder, which has the same probability amplitude $|u(i)|^2$ for every position i, the function $S(\bar{u})$ has a maximum, to which thus correspond the state of highest disorder. In conclusion, to states of high localization corresponds low disorder, and states with high delocalization have high disorder quantified by the entropy.

In the case of exponentially decaying states, we can suppose that the eigenstates, once normalized, have the form

$$u(i) = \tanh(\gamma)e^{-\gamma|i|} \quad if\ centered\ in\ zero \tag{4.1.9}$$

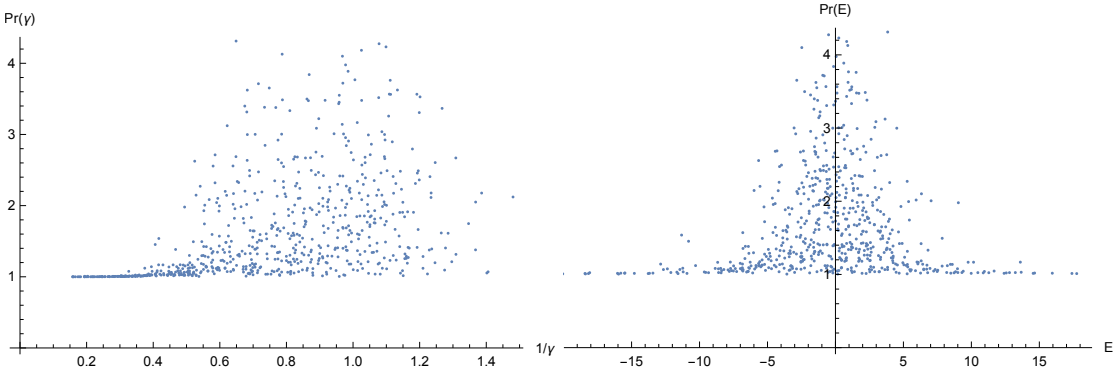
For these eigenstates, explicit calculations of the quantities introduced before give the values

$$\Delta^2(\bar{u}) = \frac{1}{\sqrt{2\sinh(\gamma)}}$$
$$P_r(\bar{u})^{-1} = \frac{\tanh(2\gamma)}{\tanh^2(\gamma)}$$
$$l_S(\bar{u}) = \frac{\exp\left(\frac{2\gamma}{\sinh(\gamma)}\right)}{\tanh(\gamma)} \tag{4.1.10}$$



(a) Mean square value on Lyapunov exponent (b) Mean square value on Energy

Figure 5: Graphs representing the behaviour of the localization of eigenstates for a Lloyd model: simulation of a 700 sites lattice with disorder 2.5.



(a) Participation number on Lyapunov exponent (b) Participation Number on Energy

Figure 6: Here are plotted the graphs of participation numbers for the LLoyd model; the graphs are taken from the same simulation as before

It's visible from figures 5 and 6 how Lyapunov localized eigenstates are also localized in terms of the distribution width and their participation number.

4.2 Influence of boundary conditions on localization

The last method introduced involves a little modification of the initial Hamiltonian: in particular, it will be shown how an introduction of a factor in the hopping coefficients and the modification of boundary conditions in 2.1.3 changes the localization of the eigenstates. This method was initially introduced by Hatano and Nelson ([11]) for the study of the depinning of flux lines in superconductors in presence of a magnetic field. Here is presented the resolution due to Goldsheid (for details refer to [8]).

Consider the Schrödinger equation

$$\begin{aligned} &- e^{a_{i-1}}u(i-1) - e^{b_i}u(i+1) + q_i u(i) = zu(i) \quad 1 \leq i \leq n \\ &u(0) = u(n), \quad u(1) = u(n+1) \end{aligned} \tag{4.2.1}$$

with $\{a_i, b_i, q_i\}$ randomly distributed. To this equation corresponds a matrix Hamiltonian:

$$H := \begin{pmatrix} q_1 & -e^{b_1} & 0 & \cdots & \cdots & -e^{a_n} \\ -e^{a_1} & q_2 & -e^{b_2} & 0 & \cdots & \cdots \\ 0 & -e^{a_2} & q_3 & -e^{b_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ -e^{b_n} & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{4.2.2}$$

We can make some transformation to express this Hamiltonian in a more familiar and symmetric way: let's put $u(i) = w_i v(i)$, with the coefficients w_i given by

$$w_0 = 1, \quad w_i = e^{\frac{1}{2}\sum_{k=0}^{i-1}(a_k - b_k)} \; if \; i \geq 1$$

If moreover we put $c_i = e^{[(a_i+b_i)/2]}$ then 4.2.1 becomes

$$\begin{aligned} &- c_{i-1}v(i-1) - c_i v(i+1) + q_i v(i) = zv(i) \\ &v(n+1) = w_{n+1}^{-1} w_1 v(1), \; v(n) = w_n^{-1} v(0) \end{aligned} \tag{4.2.3}$$

which is a form that resembles more closely an Anderson-like model. Along with this equation, we will refer also to the same problem with the boundary conditions

$$v(n+1) = v(0) = 0 \tag{4.2.4}$$

These conditions in a sense cancel the presence of the boundary terms in 4.2.1 and produce the unperturbed hopping system.

The values a_i and b_i if fixed to 0 produce the unperturbed Hamiltonian 2.1.3 with unitary boundary conditions. The objective is to retrieve the distribution $d\rho_n(x,y)$ of the eigenvalues of 4.2.1, that, since the Hamiltonian is no more a self-adjoint operator, could be complex numbers

$$d\rho_n(z) = d\rho_n(x,y) = \frac{1}{n}\sum_{j=1}^{n}\delta(x-x_j)\delta(y-y_j)$$

It can be shown that the limit distribution for $n \to +\infty$ is found by calculating the limit of a "potential"

$$F_n(z) = \int_C log|z-z'|d\rho_n(z') = \frac{1}{n}log|det(H-zI)| \tag{4.2.5}$$

Which can be see as generated by the "charge distribution" $d\rho_n(z')$ on the complex plane. The existence of the limit of 4.2.5 implies also the existence of the limit of $d\rho_n$ through Poisson equation $d\rho(x,y) = \frac{1}{2\pi}\Delta F(x,y)$. The results is that the limit potential assumes the form

$$F(z) = \begin{cases} a & if\ \Phi(z) < a \\ \Phi(z) & if\ \Phi(z) > a \end{cases}$$

In which $a = \max(< a_0 >, < b_0 >)$ and $\Phi(z)$ is the limit potential of

$$\Phi_n(z) = \frac{1}{n}\log|det(H_0 - zI)| = \int_{-\infty}^{+\infty}\log|z-\lambda|d\tilde{\rho}(\lambda) \tag{4.2.6}$$

Here H_0 is the Hamiltonian associated to 4.2.3 with boundary conditions given by 4.2.4, namely the same Hamiltonian without any effective boundary conditions; $d\tilde{\rho}$ is the corresponding distribution of eigenvalues.

The limit function $\Phi(z) = \lim_{n\to+\infty}\Phi_n(z)$ is harmonic on the whole complex plane except on the support of $d\tilde{\rho}$; for this reason the complex part of the limit spectrum is determined by the equation $\Phi(z) = a$: this equation defines a curve $\mathcal{C}$ on which the density of the eigenvalues respect to the measure of the arc-length ds is related to the jump of the normal derivative of the potential $F(z)$ across the same curve

$$\frac{dv}{ds} = \frac{1}{2\pi}\left|\int_{-\infty}^{+\infty}\frac{d\rho_0(\lambda)}{\lambda - z}\right|, \quad z \in \mathcal{C}$$

From equation 4.2.6 one sees that $\Phi(z)$ must be equal, up to an additive constant, to the Lyapunov exponent thanks to the Thouless formula. It can be proven that $\Phi(z) = \gamma(z) + \frac{1}{2}(< a_0 > + < b_0 >)$ ([8]).

The equation that defines the curve $\mathcal{C}$ is equivalent to the implicit equation

$$\gamma(z) = \frac{1}{2}|< a_0 > - < b_0 >| \tag{4.2.7}$$

This form highlights the fact that if $< a_0 >=< b_0 >$ then the spectrum doesn't have a complex part.

There is a way to determine the form of such curve: consider the following disequation for the average over disorder of the real Lyapunov exponent for the initial Hamiltonian:

$$\bar{\gamma}(E) \leq \frac{1}{2}| < a_0 > - < b_0 > | \tag{4.2.8}$$

Since $\bar{\gamma}(E)$ is a continuous function, the solution is an union of disjoint intervals $\cup_j [x_j, x'_j]$. For every x in one of these intervals, there's a solution to the equation in 4.2.7 for a $y(x)$ since $\gamma(x+iy)$ is monotonous continuous in y and $\lim_{y\to+\infty} = +\infty$; thus the curve $\mathcal{C}$ is an union of disconnected curves $\mathcal{C} = \cup_j \mathcal{C}_j$, for every one of which there are two symmetric arcs for $y(x)$ and $-y(x)$, for x belonging to one of the previous intervals. Moreover, from the property of upper-semi continuity of $\bar{\gamma}(E)$ follows that for every $\epsilon > 0$ the spectrum of 4.2.1 for n large enough lies outside a region of the complex plane surrounded by

$$\mathcal{B}_{j,\epsilon} = \{z \in \mathbf{C} : dist(|z|, \mathcal{L}_j) \leq \epsilon\}$$

This means that the spectrum is wiped away from the interior of the curve $\mathcal{C}$.

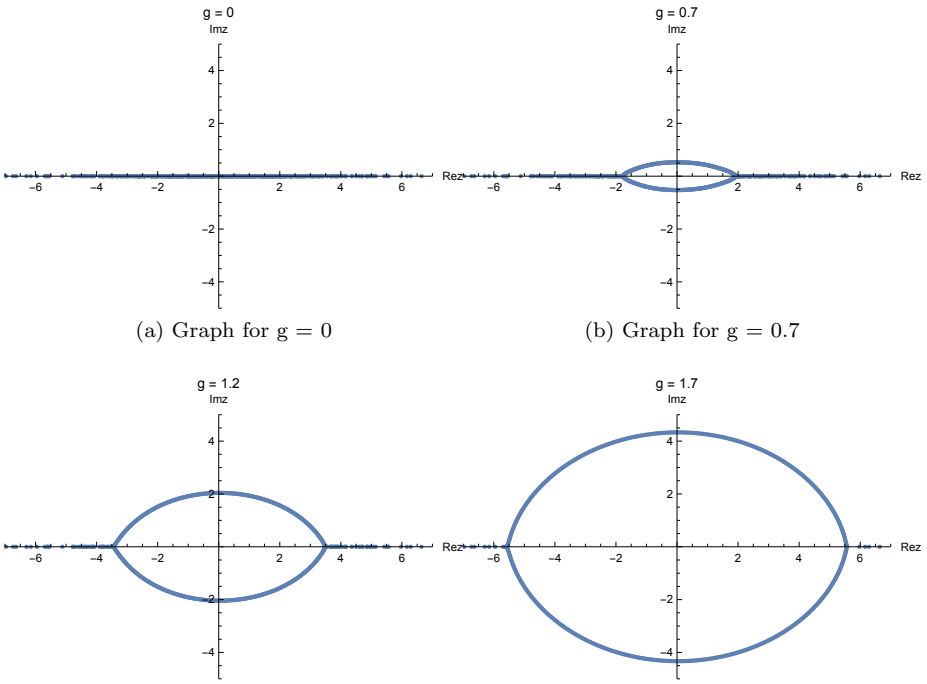
The case of the Lloyd model is covered by choosing the diagonal elements of 4.2.1 within a Cauchy distribution. If one chose the other parameters so that for every i $a_i = -b_i$, for example choosing $a_i = g$ and consequentely $b_i = -g$, then the transormed Hamiltonian 4.2.3 assumes the form of the Lloyd model with $c_i = 1$ $\forall i$. One obtains for the equation of the curve

$$y(x) = \pm \left[\sqrt{\frac{(K^2-4)(K^2-x^2)}{K^2}} - \delta \right], \quad -x_b \leq x \leq x_b$$

With $K = 2\cosh(< b_0 >)$ and x_b determined by the condition $y(x_b) = 0$.

It's interesting to note how the behaviour of the eigenvalues depends on the disorder: if $\delta = 0$ then the two arcs form an ellipse; as the disorder increases, these two arcs come closer and the complex spectrum shrinks, until $\delta = \sqrt{K^2-4}$, when the arcs disappear and the spectrum is completely real: the critical value of g for which this happens is such that $K_{cr} = \sqrt{4+\delta^2}$, and when $K < K_{cr}$ only a real spectrum is present. From the construction starting at 4.2.8 it is also possible to infer that the eigenvalues for which the divergence of the branches occurs are the one corresponding to an energy near zero, so that the curve starts to form from these values: from the relation between the Lyapunov exponent and the energy of the eigenstates 3.2.11 one sees that these are the lowest values for the exponents, thus belonging to the less localized states; these particular states are the ones that feel the most the boundary conditions introduced as are more expanded in the lattice.

We have seen how other means of localizations agree with the definition of localized states through Lyapunov exponents and how these states can, in virtue of their localization, feel a "perturbation" at the boundaries of the lattice. The considerations introduced help to further quantify the principal property of



(a) Graph for g = 0

(b) Graph for g = 0.7

(c) Graph for g = 1.2

(d) Graph for g = 1.7

Figure 7: Real and Imaginary part of eigenvalues of an Hamiltonian of the kind 4.2.2 with $a_i = -b_i = g \ \forall i$ and diagonal values distributed according to a Lorentzian, for a single instance of the randomness. The initial figure (a) corresponds to the unperturbed case with boundary counditions equal to 1. As the perturbation increases, the curve described in section 4.2 starts to appear and delocalization of the eigenstates occurs. It is interesting to note that the eigenstates that first delocalize are the one with minor energies, and thus are less localized (3.2.11).

Lloyd eigenstates, namely the one of localizability, through other properties of the distribution of the eigenstate on the lattice.

5 Delta Kicked Rotor

The previous discussion about localization in lattice Hamiltonians and in particular about Lloyd model served as a preliminary preparation for the following argument: the study of the so-called delta kicked rotor. In classical terms, the kicked rotor is a system driven by the Hamiltonian ([4])

$$H = \frac{J^2}{2I} + G\cos\theta \sum_{n=-\infty}^{+\infty} \delta(t - nT) \tag{5.0.1}$$

Where J is the angular momentum, I the moment of inertia, K the intensity of the "kicks" and T period of the kicks. This system can be interpreted as a rotor (a spinning object) subject to periodic impulses imposed in one direction on it (thus the presence of $\cos(\theta)$). The Hamiltonian can also be written in the form

$$H = \frac{J^2}{2I} + \frac{G}{T} \sum_{n=-\infty}^{+\infty} \cos\left(\theta - \frac{2\pi nT}{T}\right) \tag{5.0.2}$$

So that the system can be seen as subject to cosine wave potentials travelling at the same speed.

From a classical viewpoint the system is chaotic, and without giving a rigorous characterizations it can be seen why: the influence of the periodic kicks on the rotor depends widely on the initial phase space point, since different points receive at different instants of the motion the kicks and thus propagate in a quite various way, producing positive maximum Lyapunov exponent for the phase-space trajectories. Just for the sake of curiosity, the classical Hamiltonian written in terms of the momentum and the associated position (angle)

$$H = \frac{p^2}{2m} + G\cos(x) \sum_{n=-\infty}^{n=+\infty} \delta(nT - t) \tag{5.0.3}$$

produces a relation between phase space points before and after the $n+1$ kick:

$$\begin{cases} p_{n+1} = p_n + \frac{G}{T}\sin(x_n) \\ x_{n+1} = x_n + \frac{T}{I}p_n \end{cases} \tag{5.0.4}$$

The quantum version of the problem, with the operators substituted for the variables in equations 5.0.1 and 5.0.2, produce eigenstates with a peculiar property of localization that is directly connected to the type of localization discussed in previous sections.

The discussion will start from the theory of time-dependent Hamiltonians so that we can reformulate the problem presented by the quantum version.

5.1 Time-Dependent Hamiltonians and Floquet Operators

Consider a time-dependent Hamiltonian $\hat{H}(t)$ with Schrödinger equation for a state $\Psi(t)$

$$\hat{H}(t)\Psi(t) = i\hbar\frac{\partial\Psi(t)}{\partial t} \tag{5.1.1}$$

The state evolves via the Unitary propagator $\hat{U}(t,s)$ such that

$$\Psi(t) = \hat{U}(t,s)\Psi(s)$$

The form of the propagator can be derived iteratively through the infinitesimal time evolution given by 5.1.1

$$\hat{U}(t,s) = I - \frac{i}{\hbar}\int_s^t d\tau \hat{H}(\tau)\hat{U}(\tau,s)$$

The solution of which is given by the Dyson expansion with time ordering due to the non-commutativity of Hamiltonians at different times

$$\hat{U}(t,s) = T\exp\left[-\frac{i}{\hbar}\int_s^t \hat{H}(\tau)d\tau\right] \tag{5.1.2}$$

It is possible to resolve the time dependence in another way, derived from the classical formulation of time-dependent Hamiltonians.

Consider a classical time-dependent Hamiltonian $H(q,p,t)$: it is possible to reduce the problem to a time independent Hamiltonian introducing a fictitious parameter that accounts for the time evolution. This introduction permits to consider both the energy of the system and the motion time as phase space variables and treat them as such. In this way, we introduce the new parameter η and the new phase space coordinates t and E: the new Hamiltonian is

$$K(p,q;E,t) = H(q,p,t) + E \tag{5.1.3}$$

With corresponding equations of motion

$$\begin{aligned}\frac{dt}{d\eta} &= \frac{\partial K}{\partial E} = 1\\ \frac{\mathrm{d}E}{\mathrm{d}\eta} &= -\frac{\partial K}{\partial t} = -\frac{\partial H}{\partial t}\end{aligned} \tag{5.1.4}$$

From these equations emerges that E acts like Lagrangian multiplier and forces the parameter η to "flow" at the same rate as the time t, thus providing the same equation of motion in η for q and p. In addition to that, the second equation expresses how the energy missing from the Hamiltonian "enters" into the energy parameter E, as if it would be the energy exchanged by the system with an external universe or field.

From this construction one proceeds to implement the quantum mechanical case ([12]): consider the Hilbert space $\mathcal{H}$ and its norm $||-||_{\mathcal{H}}$, and define an extended Hilbert space $\mathcal{L}^2(\mathbf{R},\mathcal{H})$ of time-dependent functions $\Psi(t)$ with the normalizability condition in time

$$\int_{-\infty}^{+\infty} dt||\Psi(t)||^2_{\mathcal{H}} \leq +\infty$$

The transition to quantum mechanics consists in considering the variables E and t as operators acting on the introduced Hilbert space: these operators act respectively as a derivative operator and a multiplication operator, in correspondence to the behaviour of the quantum variables p and q:

$$(\hat{T}\Psi)(t) = t\Psi(t)$$
$$(\hat{E}\Psi)(t) = -i\hbar\frac{\partial\Psi(t)}{\partial t}$$
$$[\hat{T}, \hat{E}] = i\hbar$$

The corresponding Hamiltonian is

$$\hat{K} = \hat{H}(t) - i\hbar\frac{\partial}{\partial t} \tag{5.1.5}$$

with Schrödinger equation with respect to the parameter η

$$\hat{K}\Psi = i\hbar\frac{\partial\Psi}{\partial\eta} \tag{5.1.6}$$

The solution to this equation can be obtained, as in the time-independent case, through the action of a one-parameter unitary group on the initial condition ϕ_0 in the Hilbert space

$$\Psi(\eta) = e^{-i\eta\hat{K}/\hbar}\Psi_0$$

This can be the case if $\phi(\eta)$ is an eigenstate of the operator $\hat{K}$ as seen from 5.1.6. By substituing the function $\hat{U}(t, t-\eta)\phi_0(t-\eta)$, which is a time evolution to the time t starting from time $t-\eta$ of the state ϕ_0, into 5.1.6 one can see that this same function is also a solution of the Schrödinger equation with same initial conditions Ψ_0, so that we can relate the Dyson integral time evolution with the "parametric" time evolution scheme with

$$(e^{-i\eta\hat{K}/\hbar}\Psi)(t) = \hat{U}(t, t-\eta)\Psi(t-\eta) \tag{5.1.7}$$

for any function Ψ in the Hilbert space. This time evolution connects the function at time $t-\eta$ to the function at time t. One can also verify the commutation relation $[\hat{T}, \hat{K}] = i\hbar$, thus the von Neumann theorem asserts the existence of a unitary operator $\hat{Y}$ which performs the transformations

$$\hat{K} = \hat{Y}^\dagger\hat{E}\hat{Y}$$
$$\hat{T} = \hat{Y}^\dagger\hat{T}\hat{Y}$$

The corresponding relation of the groups operators generated by $\hat{K}$ and $\hat{E}$ is then

$$e^{-i\eta\hat{K}/\hbar} = \hat{Y}^\dagger e^{-i\eta\hat{E}/\hbar}\hat{Y}$$
$$(e^{-i\eta\hat{K}/\hbar}\phi)(t) = \hat{Y}^\dagger(t)\hat{V}(t-\eta)\phi(t-\eta)$$

From which can be deduced the relation

$$\hat{U}(t,s) = \hat{Y}^\dagger(t)\hat{Y}(s)$$

Let's now consider the periodic case: a time periodic Hamiltonian has the property $\hat{H}(t+T) = \hat{H}(t)$ for a $T \in \mathbf{R}$ (the period) and the corresponding group property

$$\hat{U}(t+T, s+T) = \hat{U}(t,s) \tag{5.1.8}$$

that permits to limit the study of the time evolution to a set $\hat{U}(s+\tau, s)$ for $\tau \leq T$. The Floquet operator is defined as the time propagator over one period, $\hat{F}_t = \hat{U}(t+T, t)$. Notice that the spectral properties of this operator don't depend on the time t, since holds the unitary relation $\hat{F}_\tau = \hat{U}(\tau, t)\hat{F}_t\hat{U}(\tau, t)^\dagger$. This equivalence of spectra permits to focus only on the operator at initial time $\hat{F}_0 = \hat{U}(T, 0)$. From this also follows that $\hat{U}(nT, 0) = \hat{F}^n$.

Another interesting aspect of 5.1.5 is to be analyzed: since the Hamiltonian is periodic in time and the kinetic term is time-invariant, one has that

$$[\hat{K}, \hat{P}] = 0, \quad \hat{P} = \exp\left(\frac{i}{\hbar}T\hat{E}\right) \tag{5.1.9}$$

Namely, the operator $\hat{K}$ is invariant for time-translations of one period. This property allows one to choose the eigenfunctions of the operator among the periodic functions on the interval $[0, T]$, or $\mathcal{L}^2([0,T], \mathcal{H})$ with boundary conditions $\Psi(T) = \Psi(0)$. The restriction of the operator to this subset of the original Hilbert space, indicated with $\hat{K}_T$, is called quasi-energy operator, and for periodic systems this has the same role as the Hamiltonian in the time-independent case, that is governs the time evolution.

The eigenvectors of the period-translation operator have the form

$$\Psi_\lambda(t) = e^{i\lambda t}\phi_\lambda(t), \quad \phi_\lambda(t+T) = \phi_\lambda(t) \tag{5.1.10}$$

A function $\Psi(t)$ in the original Hilbert space $\mathcal{L}^2(\mathbf{R}, \mathcal{H})$ can then be written as a superposition of these eigenstates

$$\Psi(t) = \int d\lambda c(\lambda)\Psi_\lambda(t)$$

where $c(\lambda)$ stands for the coefficients of the linear combination. This is a decomposition of the original Hilbert space in terms of periodic-translations invariant function subspaces. On one of these subspaces the eigenvalue equation of $\hat{K}$ is

$$\begin{aligned} &(\hat{H} - i\hbar\frac{\partial}{\partial t})e^{i\lambda t}\phi_\lambda(t) = \epsilon e^{i\lambda t}\phi_\lambda(t) \\ &=> \hat{K}\phi_\lambda = (\epsilon - \hbar\lambda)\phi_\lambda \end{aligned} \tag{5.1.11}$$

Restricted to the space of periodic functions on $[0, T]$, the quasi-energy equation is

$$\hat{K}_T \phi = \epsilon \phi \tag{5.1.12}$$

The solutions to 5.1.12 have an interesting property: from the periodicity of the functions involved, one notices that if $\phi(t)$ is a solution to 5.1.12 then also $e^{-i2\pi nt/\hbar T}\phi(t)$ is a solution with eigenvalue $\epsilon + 2\pi nT$: as in the case of electron in crystals, the spectrum is constituted of bands of energy produced by replicas of the continuum spectrum.

Another possible approach is to solve the original time-dependent Hamiltonian equation 5.1.1 supposing a solution of the form $\Psi(t) = e^{-iEt/\hbar}\phi(t)$, with $\phi(t) = \phi(t + T)$. Moreover, from the time evolution relation 5.1.7 one sees that for periodic functions the Floquet operator of period-time traslations is unitary equivalent to the operator $e^{-iT\hat{K}_T/\hbar}$: the eigenvalues can then be found, up to a multiple of $2\pi/T$, through the equation

$$\hat{F}\phi(t) = e^{-iET/\hbar}\phi(t) \tag{5.1.13}$$

Since Floquet eigenstates are eigenvectors of a Hermitian operator, $\hat{K}_T$, then we can write eigenstates of equation 5.1.1 as a linear combination of them

$$\Psi(t) = \sum_\lambda A_\lambda e^{-itE_\lambda/\hbar}\phi_\lambda(t) \tag{5.1.14}$$

Where the sum is now discrete and E_λ is the eigenvalue relative to ϕ_λ through 5.1.12.

The coefficients A_λ can be determined by the initial conditions of the states, $\Psi(0)$:

$$A_\lambda = \langle \phi_\lambda(0) | \Psi(0) \rangle$$

since Floquet eigenstates are orthonormal. It results in

$$\Psi(t) = \sum_\lambda \langle \phi_\lambda(0) | \Psi(0) \rangle \, e^{-itE_\lambda} \phi_\lambda(t)$$

From this, the state after one period is simply

$$\Psi(T) = \sum_\lambda \langle \phi_\lambda(0) | \Psi(0) \rangle \, e^{-iTE_\lambda} \phi_\lambda(0) \tag{5.1.15}$$

from the T periodicity of $\phi_\lambda(t)$. We are thus able to obtain the state after a period knowing only the initial conditions on the state.

If we now consider the time-dependent Hamiltonian $\hat{H}(t)$ as a sum of a time-independent Hamiltonian and a time-periodic contribute

$$\hat{H} = \hat{H}_0 + \epsilon \hat{V}(t), \quad V(t) = V(t + T)$$

we can write 5.1.1 in terms of eigenstates of $\hat{H}_0$, $\hat{H}_0 |n\rangle = E_n |n\rangle$:

$$i\hbar\frac{\partial}{\partial t}\langle n|\Psi(t)\rangle = \sum_m \langle n|\,\hat{H}(t)\,|m\rangle\,\langle m|\Psi(t)\rangle \tag{5.1.16}$$

The expression of 5.1.15 is in this way

$$\langle n|\Psi(T)\rangle = \sum_m \hat{U}_{nm}(T)\,\langle m|\Psi(T)\rangle$$

where has been introduced the Floquet Matrix

$$\hat{U}_{nm}(T) = \sum_\lambda \langle n|\phi_\lambda(0)\rangle\,\langle\phi_\lambda(0)|m\rangle\,e^{-iTE} \tag{5.1.17}$$

This is an unitary matrix with eigenvalues $e^{-iE_\lambda T/\hbar}$ and eigenvectors the Floquet states $\phi_\lambda(0)$. Once this matrix is known, the state at time NT is simply given by

$$\langle n|\Psi(NT)\rangle = \sum_m \hat{U}^N_{nm}(T)\,\langle m|\Psi(0)\rangle$$

5.2 Delta Kicked Rotor Hamiltonian and Dynamics

We now apply what has been previously introduced to the case of the quantum kicked rotor. Take the corresponding Hamiltonian 5.0.1: the Schrödinger equation in the one-dimensional Schrödinger representation is

$$i\hbar\frac{\partial\Psi(\theta,t)}{\partial t} = \frac{-\hbar^2}{2I}\frac{\partial^2\Psi(\theta,t)}{\partial\theta^2} + G\cos(\theta)\delta_T(t)\Psi(\theta,t) \tag{5.2.1}$$

Where θ is the variable associated to the angle position of the rotor and $\Psi(\theta,t)$ is the wavefunction of the position of the rotor at time t. In this particular case, the time independent Hamiltonian $\hat{H}_0$ is given by the angular dynamics of the system, namely

$$\hat{H}_0 = \frac{J^2}{2I} = \frac{-\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}$$

which has as eigenstates in one dimension the angular eigenfunctions $e^{-in\theta}$ with eigenvalues $\hbar^2n^2/2I$. Indicating these eigenstates with $|n\rangle$, we can write any state $\Psi(\theta,t)$ as

$$\Psi(\theta,t) = \sum_n \langle n|\Psi(\theta,t)\rangle\,e^{-in\theta} \tag{5.2.2}$$

For brevity from now we will refer to $\langle n|\Psi(\theta,t)\rangle$ with $\Psi_n(t)$. In this decomposition equation 5.2.1 can be written as

$$i\hbar\frac{\partial\Psi_n(t)}{\partial t} = \frac{\hbar^2n^2}{2I}\Psi_n(t) + \frac{G}{2}\delta_T(t)(\Psi_{n+1}(t) + \Psi_{n-1}(t)) \tag{5.2.3}$$

Rewriting the equation from the form of 5.0.2 one obtains

$$i\hbar\frac{\partial\Psi_n(t)}{\partial t} = \frac{\hbar^2 n^2}{2I}\Psi_n(t) + \frac{G}{2T}\sum_{k=-\infty}^{k=+\infty}(e^{-ik\omega t}\Psi_{n-1}(t) + e^{ik\omega t}\Psi_{n+1}(t)) \quad (5.2.4)$$

Where $\omega = \frac{2\pi}{T}$ is the angular velocity.

We now proceed to study the dynamics of the system: the periodicity of the kicks is such that at every time mT, m integer, the system receive an impulse that modifies its "motion"; between these kicks the Hamiltonian acts like the Hamiltonian of a free rotor: right after the initial kick at $t = 0$, the system is in a state indicated by the wave function $\Psi(\theta, 0^+)$. Decomposing this state on the basis of angular momentum eigenfunctions, the free-rotor time evolution is given by

$$\Psi(\theta, t) = \sum_n \Psi_n(0^+)e^{in\theta}e^{-\frac{i\hbar n^2 t}{2I}}, \quad for\ 0^+ \leq t < T \quad (5.2.5)$$

After the time $t = T$ the system receives the kick: from 5.2.1 one sees that because of the presence of the delta function, $\Psi(t)$ is a discontinuous function of time, with a jump at every time nT. Such discontinuity can be analyzed by integrating the Schrödinger equation across a time T at which the kick occurs

$$i\hbar\int_{T-\epsilon}^{T+\epsilon} dt\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2I}\int_{T-\epsilon}^{T+\epsilon} dt\frac{\partial^2\Psi}{\partial\theta^2} - G\int_{T-\epsilon}^{T+\epsilon} dt\cos(\theta)\delta_T(t)\Psi = 0$$

Supposing the regularity of Ψ in θ, when ϵ tends to zero the term with the double derivative in the angular variable gives no contribution, and the jump in the function is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = G\cos(\theta)\delta_T(t)\Psi \quad T - \epsilon < t < T + \epsilon \quad (5.2.6)$$

This equation has the solution

$$\Psi(\theta, T^+) = e^{-i\frac{G}{\hbar}\cos(\theta)}\Psi(\theta, T^-)$$

where T^+ and T^- stand for moments of time right after and before the time T. If we then take in consideration the free-rotor evolution 5.2.5 and the kick the states has the function

$$\Psi(\theta, T^+) = e^{-i\frac{G}{\hbar}\cos(\theta)}\sum_n \Psi_n(0^+)e^{in\theta}e^{-\frac{i\hbar n^2 T}{2I}} \quad (5.2.7)$$

From this equation one can see that the system has a kind of periodicity determined by the system parameter I that emerges from the free evolution exponential: this periodicity is such that the motion does not change from a time T to a time $T+\frac{4\pi I}{\hbar}$, so that we can consider a period such that $0 < T \leq \frac{4\pi I}{\hbar}$. The discontinuity equation 5.2.6 can also be written in the operator form considering the states relative to the functions,

$$|\Psi(t+T)\rangle = e^{\frac{-i}{\hbar}\hat{V}} e^{\frac{-i}{\hbar}\hat{H}_0 T} |\Psi(t)\rangle \tag{5.2.8}$$

Where the operators $\hat{V}$ and $\hat{H}_0$ are the potential and kicketic terms, with $\langle\theta|\hat{V}|\theta'\rangle = G\cos(\theta)\delta(\theta-\theta')$ and $\langle n|\hat{H}_0|n'\rangle = \frac{n^2\hbar^2}{2I}\delta_{nn'}$.

Considering the identity involving Bessel functions $J_n(z)$

$$e^{-iz\cos(\theta)} = \sum_n (-i)^n J_n(z) e^{in\theta}$$

we can project 5.2.7 on eigenstates of the angular momentum operator and obtain the useful equation

$$\Psi_m(\theta, T^+) = \sum_{n=-\infty}^{+\infty} (-i)^{m-n} J_{m-n}\left(\frac{G}{\hbar}\right) e^{-\frac{i\hbar n^2 T}{2I}} \Psi_n(0^+) =$$

$$\sum_{n=-\infty}^{+\infty} \hat{U}_{mn}(T)\Psi_n(0^+) \tag{5.2.9}$$

We have thus obtained the Floquet matrix for the propagation over one time period, and this operator mixes at each kick different momentum eigenstates: thinking about the classical model for one moment only for the sake of analogy, this behaviour can be interpreted as if some angular eingestates are "pushed" while others are "damped" by the kicks, in dependence in some sense of the verse of rotation of the states at the moment of the kick. We will see in the next section how this same alternation of angular momentum eigenstate is connected to the Lloyd model.

Lastly, something can be said about the qualitative property of the energy spectrum of the kicked delta rotor, as shown in [3]: if $T = 4\pi\frac{I}{\hbar}\alpha$, with α a rational positive number less than one and different from $\frac{1}{2}$, then the spectrum has a continuum part as well as some discrete components, with energy growing quadratically after some time in which it undergoes a diffusive behaviour; if $\alpha = 1$ then the energy has only a continuum part and grows quadratically with time; for $\alpha = \frac{1}{2}$ the energy oscillates with time. When α is irrational, then the energy grows linearly in a diffusive way for a period of time and then stops, due to the discreteness of the spectrum. We will specify this phenomenon later.

5.3 Tight Binding Model for the Kicked Rotor

We proceed to write the Hamiltonian in a different manner: let $t = \tau T$, so that time is written as a multiple of the period, and let $\xi = \frac{\hbar T}{I}$ and $\kappa = \frac{G}{\hbar}$ be two dimensionless parameters; the Schrödinger 5.2.1 equation now reads

$$i\hbar\frac{\partial\Psi(\theta,\tau)}{\partial\tau} = -\frac{\xi^2}{2}\frac{\partial^2\Psi(\theta,\tau)}{\partial\theta^2} + \kappa\cos(\theta)\delta_T(\tau)\Psi(\theta,\tau) \tag{5.3.1}$$

or in an operator form

$$i\hbar\frac{\partial\left|\Psi(\tau)\right\rangle}{\partial\tau}=(\hat{H}_0+\hat{V}\delta_T(\tau))\left|\Psi(\tau)\right\rangle \tag{5.3.2}$$

with $\langle n|\,\hat{H}_0\,|n'\rangle=\xi^2/2n^2\delta_{nn'}$ and $\langle\theta|\,\hat{V}\,|\theta'\rangle=\kappa\cos(\theta)\delta(\theta-\theta')$.

As found in 5.2.8 the time evolution in an interval right before the Nth kick and right before the (N+1)th kick, characterized by the impulse and the free rotor dynamics, is now

$$\left|\Psi(N+1)\right\rangle^{-}=e^{-i\hat{H}_0}e^{-i\hat{V}}\left|\Psi(N)\right\rangle^{-} \tag{5.3.3}$$

with $\left|\Psi(N)\right\rangle^{-}$ indicating the state before the Nth kick.

Let's now introduce Floquet theory: suppose the solution to 5.3.3 is of the form

$$\left|\Psi(N)\right\rangle^{-}=e^{-i\omega_\lambda N}\left|\phi_\lambda(N)\right\rangle$$

where $\omega_\lambda=\frac{E_\lambda T}{\hbar}$ and E_λ is the eigenvalue for $|\phi_\lambda\rangle$ in the Floquet equation 5.1.12. For the Floquet states, the time evolution equation takes the form

$$\left|\phi_\lambda(N)\right\rangle^{-}=e^{-i(\hat{H}_0-\omega_\lambda)}e^{-i\hat{V}}\left|\phi_\lambda(N)\right\rangle^{-}$$

since Floquet states are periodic, so that $|\phi_\lambda(N+1)\rangle=|\phi_\lambda(N)\rangle$.

We now introduce two operators, $\hat{Z}_\lambda$ and $\hat{W}$, defined by the following

$$\begin{aligned}e^{-i(\hat{H}_0-\omega_\lambda)}&=\frac{1-i\hat{Z}_\lambda}{1+i\hat{Z}_\lambda}\\ e^{-i\hat{V}}&=\frac{1-i\hat{W}}{1+i\hat{W}}\end{aligned} \tag{5.3.4}$$

With inverse relations

$$\begin{aligned}\hat{Z}_\lambda&=\tan\left(\frac{1}{2}(\hat{H}_0-\omega_\lambda)\right)\\ \hat{W}&=\tan\left(\frac{1}{2}\hat{V}\right)\end{aligned} \tag{5.3.5}$$

This is the Maryland construction, proposed by Grempel, Fishman and Prange in [9], that permits to map the quantum delta kicked rotor problem on a tight binding Hamiltonian which resembles the Lloyd model. If we now set the state $\left|\nu_\lambda(N)\right\rangle^{-}:=(1+i\hat{W})^{-1}\left|\phi_\lambda(N)\right\rangle^{-}$ the equation for this state has the simple form

$$(\hat{Z}_\lambda+\hat{W})\left|\nu_\lambda(N)\right\rangle^{-}=0$$

Indicating $\langle n|\nu_\lambda(N)\rangle$ (projection over n angular momentum eigenstate) with the notation $\nu_n(\lambda)$, the equation becomes

$$Z_n(\lambda)\nu_n(\lambda) + \sum_{m=-\infty, m\neq n}^{+\infty} W_{nm}\nu_m(\lambda) = \epsilon\nu_n(\lambda) \tag{5.3.6}$$

with $Z_n(\lambda) = \langle n|\, \hat{Z}_\lambda\, |n\rangle = \tan\left(\frac{1}{4}\xi n^2 - 2\omega_\lambda\right)$, $W_{nm} = \langle n|\, \hat{W}\, |m\rangle$, $\epsilon = -\langle n|\, \hat{W}\, |n\rangle$.

We have just written the equation for the delta Kicked rotor as an equation consisting of a discrete kinetic term $\hat{Z}$ and a "hopping" potential term $\hat{W}$ as in the lattice Hamiltonian: in this case, the one dimensional lattice consists in the angular momentum space and the points are represented by angular momentum eigenstates. It is also interesting to note that the "random" term is due to the free rotor dynamic of the system, while the hopping term is due to the periodic kicks. Taking into account the translation invariance in angular momentum functions space for the operator W_{nm}, namely $W_{n+r.m+r} = W_{nm}$ and expanding in Fourier series $\hat{W}$, since it is a periodic operator

$$\begin{aligned} &\hat{W} = \sum_q W_q e^{iq\theta} \\ &\implies W_{nm} = \sum_q W_q \langle n|\, e^{iq\theta}\, |m\rangle = \sum_q W_q \delta_{n-q,m} \end{aligned}$$

5.3.6 can be rewritten as

$$Z_n(\lambda)\nu_n(\lambda) + \sum_{q\neq 0} W_q \nu_{q-n}(\lambda) = \epsilon\nu_n(\lambda) \tag{5.3.7}$$

Which emphasize the interaction between sites.

Let's now consider the operator $\hat{Z}_\lambda$, which plays the role of the on-site potential: in the case $\omega_\lambda = 0$ we have $Z_n(0) = \tan\left(\frac{1}{4}\xi n^2\right) = \tan\left(\pi n^2 \beta\right)$, with $\beta = \xi/4\pi$: it is known that if β is irrational, as in the analyzed case, then $x = \beta n^2 (\mod 1)$ is uniformly distributed in the interval $[0, 1]$ as n spans the relative number space; its probability distribution is

$$P(x) = \begin{cases} 1 & if\ 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

This probability distribution, in turn, produces a distribution of the operator $\hat{Z}(0)$ of the kind

$$P(Z) = \int_0^1 dx \delta(Z - \tan(\pi x)) = \frac{1}{\pi}\frac{1}{1+Z^2} \tag{5.3.8}$$

or alternatively

$$P(Z(x)) = P(x)\frac{\mathrm{d}x}{\mathrm{d}Z} = \frac{1}{\pi}\frac{1}{1+Z^2} \tag{5.3.9}$$

The values of the operator projected on eigenstates of the angular momentum follow a Cauchy distribution: as it runs on the angular momentum eigenstates

the distribution is the same as in the case of the Lloyd model for the diagonal part of the operator. It nevertheless can't be considered a total random operator, since the distribution is not random but fixed: the Cauchy distribution is followed by the diagonal elements.

In respect to the values of W_{nm}, for $\kappa < \pi$ it can be shown analytically that

$$\begin{aligned} &W_{nm} = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\theta(n-m)} \tan\left(\frac{\kappa}{2}\cos(\theta)\right) = \\ &\frac{2}{\kappa}\left[\frac{\pi}{\kappa}(1-S(\kappa))\right]^{|n-m|-1}\left(\frac{2\pi^2 S(\kappa)+\kappa^2-2\pi^2}{\pi^2 S(\kappa)+\kappa^2-\pi^2}\right) \quad for\ |m-n|\ odd \\ &W_{nm} = 0 \quad for\ |n-m|\ even \end{aligned} \tag{5.3.10}$$

with $S(\kappa) = \sqrt{1-\frac{\kappa^2}{\pi^2}}$. For $\kappa > \pi$ the elements W_{nm} are singular due to the transformation introduced in 5.3.4: this singularity is however mathematical and doesn't affect the localization of the Floquet states. This operator is the source of the difference from the Lloyd model introduced in previous sections: while that model has only a coupling between neighbouring sites, and thus a tridiagonal matrix, the kicked rotor tight binding model presents a correlation of sites with even distant angular momentum states; nonetheless a plot of 5.3.10 (Fig 9) shows that this correlation fall of exponentially for increasing site distance $|n-m|$ and thus influences only slightly the interaction between momentum states.

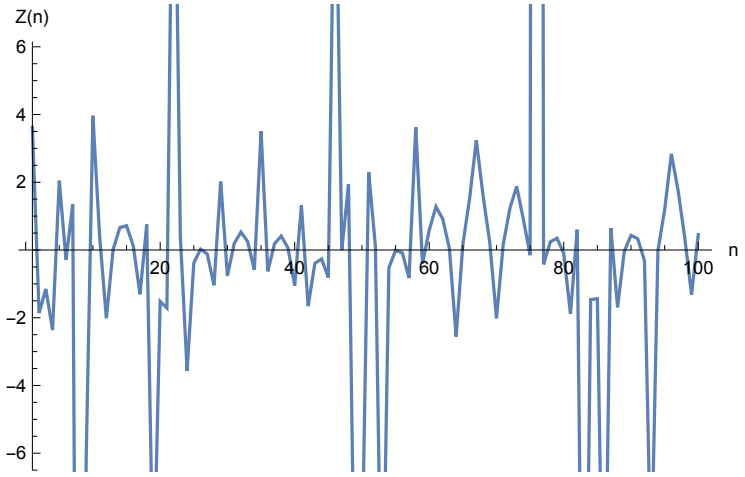


Figure 8: Graph of the in-site operator $Z(\lambda)$ as n increases: it is possible to se a random behaviour with n. In this case the plot is made for $\tan\left(\pi\sqrt{2}n^2\right)$.

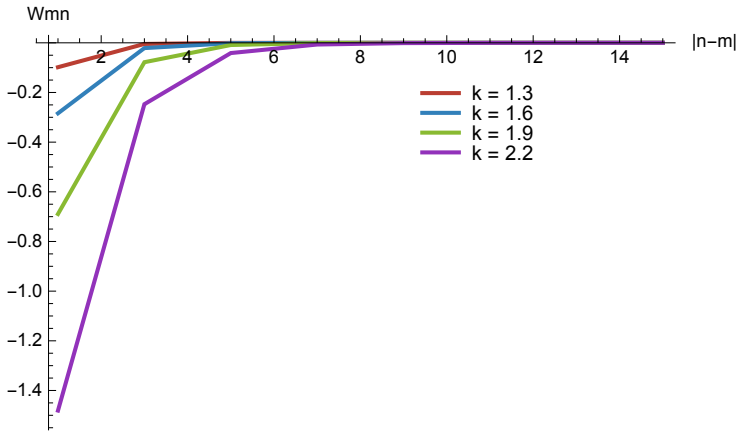


Figure 9: Plot of the behaviour of the hopping potential W_{mn} as $|n-m|$ increases: as seen from the graphic, even for small distance sites (of the order of 3) the influence is small.

5.4 Comparison with Lloyd model

Given the form of the tight binding equation for the delta kicked rotor 5.3.6 we notice immediately similarities with the Lloyd one dimensional lattice equation for states 3.1.1: both have a lattice-like form, with operators connecting different sites, and an in-site potential which is randomly distributed through the lattice with a Cauchy distribution; the differences lay in the more extended interaction of the delta kicked rotor, in which not only neighbouring sites are connected but all sites interact with each other through W_{nm}; moreover the interaction potential is not constant but varies from site to site according to 5.3.10, which is nonetheless symmetric with respect to the position considered due to the presence of $|n-m|$. We can despite everything make some assumptions: the matrix form of the operator involved in 5.3.6 is

$$H_\delta(\lambda) := \begin{pmatrix} Z_1(\lambda) & W_{12} & W_{13} & W_{14} & \cdots & \cdots \\ W_{21} & Z_2(\lambda) & W_{23} & W_{24} & \cdots & \cdots \\ W_{31} & W_{32} & Z_3(\lambda) & W_{34} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{5.4.1}$$

considering only the positive n angular momentum states. Having seen the trend of the quantities W_{nm} (fig. 9) we can consider this matrix as a sum of a tridiagonal matrix and a perturbation matrix: the triagonal matrix has diagonal elements Cauchy distributed and the off-diagonal matrix with elements $W_{n,n-1}$ and $W_{n,n+1}$ which are symmetric with respect to the diagonal and are of the same magnitude; the off-tridiagonal matrix can be considered as a perturbation. What we have obtained is thus a Lloyd-like matrix with a perturbation: we

can then suppose that the eigenstates are a perturbed version of the discrete localized states of the Lloyd model; since the perturbation is weak as we go far off from the centre of the states and the unperturbed state themselves are localized, we can also suppose that these states have an exponential-like decay for large $|n|$. A numerical simulation of the finite case (restricted to a limited number of angular momentum eigenstates) shows indeed this kind of behaviour (fig. 10).

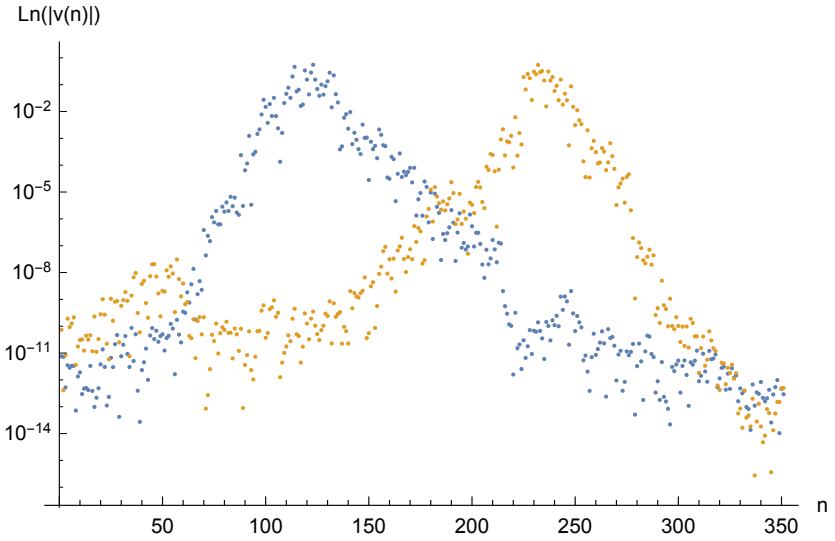


Figure 10: Plot of two states ν_λ: there are plotted in logarithmic scale the moduli of the projection onto angular eigenstates of $\nu_n(\lambda)$, for $\lambda = 0$; similarly that in the Lloyd case, the states present a localization and the trend is exponential for large $|n|$ away from the center of localization. Here $\xi = \sqrt{2}$ and $\kappa = 2.2$.

5.5 Diffusion and localization lenght

As already mentioned, the evolution of the energy when α in $T = 4\pi\frac{I}{\hbar}\alpha$ is an irrational number shows a temporary diffusive behaviour: it grows linearly in time and then stops at a certain moment.

Let's investigate the classical case before: the equations of motion from one period to the next are given by 5.0.4. If the kicks are particularly strong, then the angle variable x_n may cycle many times 2π in one period, so that we can suppose that ideally x_n fills uniformly the interval $[0, 2\pi]$ during the motion. As a consequence, the term $\sin(x_n)$ in the momentum equation shows no correlation between kicks and brings p to undergo a random motion in phase space. The equation for the momentum can then be approximated by

$$p_n = p_0 + \sum_{k=0}^{n} G \sin(x_k)$$

Then, the average carried out on initial conditions of the momentum after n periods is given by

$$\bar{p}_n = \bar{p}_0 + G \sum_{k=0}^{n} \overline{\sin(x_k)} = \bar{p}_0 \tag{5.5.1}$$

since the average of the periodic function $\sin(x)$ over one period is zero. The average for the square of the momentum is instead

$$\begin{aligned} \bar{p}_n^2 &= \bar{p}_0^2 + \sum_{k,m}^{n} G^2 \overline{\sin(x_k)\sin(x_m)} \\ &= \bar{p}_0^2 + \left(\frac{G^2}{2\pi} \int_0^{2\pi} dx \sin(x)^2 \right) nT \end{aligned} \tag{5.5.2}$$

where the last part is due to the uncorrelatedness of the kicks: $\overline{\sin(x_k)\sin(x_m)} = \delta_{km} \overline{sin(x)^2}$. Thus the momentum shows a diffusive behaviour with diffusion constant

$$D = \frac{G^2}{2\pi} \int_0^{2\pi} dx \sin(x)^2 = \frac{G^2}{2}$$

The quantum mechanical system shows a different trend, and the diffusion stops at a moment instead of continuing in time as suggested by 5.5.2. This aspect can be explained in a qualitative manner taking into account the discreteness of the spectrum ([20]). Lets indicate with t_s the time at which the growth of the momentum stops: we can assume that at time t_s the mean value of the energy of a state is $< E >= \frac{p^2}{2I} \approx D t_s$. Let's also suppose that initially the rotor is in the $0th$ angular momentum state $|0\rangle$, so that $p_0 = 0$: at time t_s the rotor will be in an angular momentum state $|n_s\rangle$ and will have thus an energy

$$< E > \approx \frac{\hbar^2 n_s^2}{2I} \approx D t_s \tag{5.5.3}$$

In turn, the angular momentum cutoff is determined by the localization length of the states γ and thus we have $n_s \approx \gamma$: this is due to the fact that the Floquet eigenstates which are near the initial state $|0\rangle$ and can significantly influence it are in a number proportional to their localization length, so that these states are in number $N \approx n_s \approx \gamma$. The mean spacing between two Floquet States is thus $\Delta l = \gamma^{-1}$; since the evolution of the initial state $|0\rangle$ continues until this state can feel the presence of the others, we have that the time at which diffusion stops is $t_s \approx \Delta l^{-1} \approx \gamma$. Combining these relations, we reach

$$D = a\gamma \tag{5.5.4}$$

This relation permits to obtain prevision for the value of the localization length of state without carrying out an analytic calculation on the Hamiltonian. Numerical calculations for a carried out by Shepelyanksy have provided a value for a of $\frac{1}{2}$.

6 Conclusion

What has been showed in this work are the most important results in the field of Anderson model and the study of lattice Hamiltonian with random in site potential: the peculiar characteristic of these systems is the existence of exponentially localized states; to arrive to this result, various important theorem have been exploited: first of all the Kingman, Furstenberg and Osceledesc theorem are essential to determine the existence of such localized state and introduce the definition of the Lyapunov exponent, then the Ishii and Kunz-Souillard theorems guarantee the discreteness of the spectrum. Through Thouless formula we were able to obtain a general expression for the Lyapunov exponent that is directly connected to the density of eigenvalues: it is interesting to notice how the presence of eigenstates influence the localization of other states. An analytical result for the Lyapunov exponent and the density of state has been obtained for the Lloyd model with a Lorentzian disorder for the in-site potentials. Given the importance of the localization of such states, a particular attention has been given to the concept of localization, and various quantities that define this behaviour have been introduced and confronted. What stands out is the method of Nelson-Hatano: we have seen how the presence of boundary condition on the finite chain of lattice points influenced the localization of the states and even delocalized some of them: the curve in the complex plane traced out by the eigenvalues (some of which become complex) well describes this behaviour of delocalization, which turns out to affect the state with the smaller Lyapunov exponent and thus the less localized ones.

The delta kicked rotor is a peculiar system with a classical chaotic behaviour. The quantum mechanical study of the delta kicked rotor needed a propaedeutic analysis of time-dependent Hamiltonian: the restriction to time-periodic potentials then introduced the concepts of Floquet states, Floquet matrix and quasi-energy. With these instruments and with the help of the particular Maryland construction we were able to map the delta kicked rotor Hamiltonian to a tight-binding Anderson model on the space of angular momentum eigenstates, with hopping and in-site potentials: the results previously obtained permitted us to characterized the rotor states as localized (not in space, but in angular momentum space) for certain values of the intensities of the kicks. As in the Anderson model, the important result of this analysis is the localization of state, which we can say characterizes the research of the entire work.

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