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Hydrodynamics in twisted spacetimes

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Abstract

Since twisted spacetimes were defined by B-Y Chen in 1979, they have been deeply studied by geometers. A Lorentzian manifold is twisted if a frame of reference exists where the metric tensor takes the form

$$ds^2 = -dt^2 + f^2(t, \vec{x}) g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu$$

where $g^*_{\mu\nu}$ is the metric of a Riemannian submanifold. This great set of spacetimes contains two important subsets: the Generalised Robertson-Walker spacetimes, where the scale function f depends only on time, and the Robertson-Walker spacetimes, that are spatially homogeneous an isotropic. The latter are well known since they are the basis of the standard model of cosmology.

C.A. Mantica and L.G. Molinari showed that twisted spacetimes can be characterised by the existence of a unique vector field u^i with $u^i u_i = -1$, called torse forming, whose covariant derivative is $\nabla_j u_i = h_{ij}\varphi$.

The aim of this thesis is to analyse the properties of a fluid living in such manifolds. With this purpose in the first chapter a recap of the relativistic hydrodynamical and thermodynamical framework is done and in particular of the duality between the Eckart's frame comoving with the particles' four-current and the other used by Landau, respect to which the energy current is null. In both cases, when the system is near to equilibrium, it is possible to make a first order approximation of the entropy four-current as a function of the thermodynamical quantities of the system, that gives a relation between the components of the energy momentum tensor and the fluid velocity's derivatives.

In the second part of the thesis, the main features of the steps of the hierarchy are presented, including a covariant form of the Ricci tensor. Then the hydrodynamic framework treated in the first chapter is applied to them. We recover some results that were obtained by Coley and MacManus.

If one considers a fluid whose rest-mass four-current is comoving with the torse forming vector, the resulting motion has null acceleration and shear tensor, while the stress tensor is found to be proportional to the Weyl tensor contracted with u^i. At the same time the Einstein equation gives, in a 4-dimensional spacetime, a generalisation of the famous Friedmann equations, obtained in Robertson-Walker spacetimes, where energy and pressure are related to the scale function f and the curvature scalar of the spatial submanifold. In the general case of a generic twisted spacetime new terms containing spatial derivatives of the scale function appear, which are null in GRW and RW.

We showed that only one possible form of the fluid's state equation exists

in a dishomogeneous GRW. Next, we found the timelike eigenvector of the Ricci tensor, which corresponds to Landau's fluid velocity, when the heat flux vector is an eigenvector of the stress tensor.
In the thermodynamical approximation cited above, the anisotropic stress tensor is null for any twisted spacetime. This observation pushed us to consider the particular case where the spatial submanifold is flat: in such situation we have been able to evaluate one of the spatial eigenvalues of the energy momentum tensor and the exact form of the stress tensor in Landau's frame, which has the same role of heat conduction in Eckart's formalism.

During this thesis we have used the signature (-,+,+,...,+) for the lorentzian metrics and we have taken both the gravitational constant G and the speed of light c equal to 1. Moreover we have used latin indices for all the spacetime's dimensions ($i = 0, 1, ..., n$) and greek letters for spatial dimensions ($\mu = 1, 2, ..., n$).

Contents

Chapter 1

Relativistic Hydrodynamics

The problem of understanding the motion of a great number of particles in a relativistic fluid can be treated in two radically different ways: the first option consists in developing a kinetic theory from the rules governing the behaviour of a single particle in a gravitational field, as made by Cercignani and Kremer in [7]. This approach has the advantage to be more satisfying from a formal point of view, but is difficult to apply in non equilibrium situation or when the interactions between particles are not limited to binary collisions. Moreover, given a fluid, is not always possible to know the exact form of the potential between its components.
Another approach consists in studying the evolution of macroscopic quantities, such as the local average particles' speed u^i, the energy-momentum (or stress-energy) tensor T_{ij}, the rest-mass density 4-current J_i or the entropy 4-current S^i, which gives a more intuitive representation and provides results compatible with the kinetic theory in the majority of cases, under the condition of small mean free path respect to the fluid dimensions, or equivalently relaxation time smaller than any macroscopical characteristic time. In this dissertation we will follow the latter method, in the same way as [6] and [3].
The evolution of an infinitesimal volume of the fluid is usually described by a set of scalar and tensorial characteristics which derive from a decomposition of the fluid 4-velocity derivatives

$$\boxed{\nabla_j u_i = \omega_{ij} + \sigma_{ij} + \frac{1}{3}\theta h_{ij} - a_i u_j} \tag{1.0.1}$$

where $h_{ij} := u_i u_j + g_{ij}$ is the projection operator on the subspace orthogonal to u^i and a_i is the fluid 4-acceleration, given by the expression $a_i := u^j \nabla_j u_i$. The scalar $\theta := h^{ij}\nabla_i u_j = \nabla_i u^i$, called *expansion scalar*, is the 4-divergence of velocity and represents the volume variation of the infinitesimal fluid element. The *shear tensor* $\sigma_{ij} := \nabla_{<i}u_{j>} = \nabla_{(i}u_{j)} + a_{(i}u_{j)} - \frac{1}{3}\theta h_{ij}$ is the trace less symmetric part of the covariant derivative of the fluid velocity

projected on the space orthogonal to u_i. It is usually linked to the distortion of the shape of the element, in fact his eigenvalues represent the change rates of axial lengths of an ideal infinitesimal fluid ellipsoid. The trace free feature of the shear tensor reflects the property that deformations represented by this tensor hold the volume of the ellipsoid constant. Finally the *vorticity tensor* $\omega_{ij} := \nabla_{[j}u_{i]} + a_{[i}u_{j]}$ is the antisymmetric part of the derivative of u_i, computed on its orthogonal space. It describes rotations respect to the principal axes of the ellipsoid in the local inertial rest frame.
Some evolution laws for these quantities can be obtained from the Riemann tensor and its action on the 4-velocity through

$$\nabla_j \nabla_i u_k - \nabla_i \nabla_j u_k = R^l{}_{kij} u_l \tag{1.0.2}$$

that, with a multiplication by u^j, becomes

$$u^j \nabla_j \nabla_i u_k = R^l{}_{kij} u_l u^j + u^j \nabla_i \nabla_j u_k \tag{1.0.3}$$

and with Leibniz rule

$$u^j \nabla_j \nabla_i u_k = R^l{}_{kij} u_l u^j + \nabla_i (u^j \nabla_j u_k) - (\nabla_i u^j)(\nabla_j u_k) \tag{1.0.4}$$

The symmetrization, antisymmetrization and trace respect to the indices i and k of the last equation, after some substitutions with the decomposition of $\nabla_j u_i$ (1.0.1), give three evolution laws for the above defined quantities and in particular

$$u^j \nabla_j \theta = -\frac{1}{3}\theta^2 - \sigma^{ij}\sigma_{ij} + \omega^{ij}\omega_{ij} - R_{ij}u^i u^j + \nabla_i a^i \tag{1.0.5}$$

As told before, the other important elements for a complete thermodynamical description of a fluid motion are the stress-energy tensor T_{ij}, which represents the flux of 4-momentum, the mass current or rest-mass flux $J_i = \rho u^i$, where ρ is the rest-mass density, and the entropy current S^i. Such relevant roles for T_{ij} and J_i are an effect of momentum and particles conservation laws, since the request, in absence of external forces, of null flux of particles and momentum through any close surface is equivalent, thanks to Gauss theorem, to request

$$\boxed{\begin{aligned} \nabla_j T^{ij} &= 0 \qquad (1.0.6) \\ \nabla_i J^i &= 0 \qquad (1.0.7) \end{aligned}}$$

The Einstein equation establishes a relation between the stress energy tensor and the curvature of the spacetime where the fluid lives:

$$\boxed{R_{ij} - \frac{1}{2}R g_{ij} = 8\pi T_{ij}} \tag{1.0.8}$$

from this point of view, the energy conservation law is equivalent to the second Bianchi identity,

$$R^i{}_{j[kl;m]} = 0 \tag{1.0.9}$$

which is respected by any Riemann tensor and implies the following relation on the Ricci tensor R_{ij} and the curvature scalar R:

$$\nabla_i R^{ij} - g^{ij}\partial_i \frac{1}{2} R = 0 \tag{1.0.10}$$

that is obviously the same of (1.0.6).

The components of the stress-energy tensor and his contracted with 4-vectors forms have a physical interpretation too. Given a timelike vector ξ^i, the energy density in the frame comoving with ξ^i is $e = T_{ij}\xi^i\xi^j$ and $-T^{ij}\xi_i$ represent the mass-energy density 4-current, while spatial components are the relativistic generalisation of classical stress tensor. Finally there is a statistical definition for the stress-energy tensor and the mass current: they are proportional to the second and the first moment of the distribution function $f(x^i, p^i)$ of the system, which describes how particles occupy the 8-dimensional phase space.

$$J^i := m \int p^j f \frac{d^3p}{p^0} \tag{1.0.11}$$

$$T^{ij} := \int p^i p^j f \frac{d^3p}{p^0} \tag{1.0.12}$$

In particular, the second one imposes symmetry to the stress-energy tensor. It remains to discuss the form of the entropy current S^i, that will be related to the quantities defined above. We will assume that the entropy current is a strictly local function, so independent of gradient of T^{ij} and J^i, as expected from the kinetic theory's definition as a function determined purely by the local form of the distribution function (see [7]). It is possible to operate a decomposition in the projection along u^i and a part orthogonal to it

$$S^i = s\rho u^i + R^i \tag{1.0.13}$$

where s is the entropy density and R^i, with $u^i R_i = 0$, can be considered as a dissipative term, since it is the only vector with not null spatial components respect to the comoving frame.

If we define the entropy S by integrating the current over a spacelike surface Σ

$$S(\Sigma) := \int_\Sigma S^i n_i d^3x \tag{1.0.14}$$

the second principle of thermodynamics becomes $\nabla_i S^i \geq 0$, then, when the system is in equilibrium, it is $\nabla_i S^i = 0$ and, since in that situation all dissipative effects are absent, the entropy current is just the product of the

local entropy, given by the entropy density s times the rest-mass density ρ, multiplied by its speed, i.e. $S^i = s\rho u^i$.
The first law of thermodynamics is the same as in flat space in the fluid comoving frame because of the equivalence principle

$$dU = TdS - pdV + \mu dN \tag{1.0.15}$$

If the number of particles is conserved, its intensive form respect to the rest mass is

$$d\epsilon = Tds - pd\left(\frac{1}{\rho}\right) \tag{1.0.16}$$

where ϵ is the average internal energy per rest-mass unit and is in relation with the volumic energy density used before through equation $e = \rho(\epsilon + 1)$.
In a relativistic dissertation there is a more handly form for the first principle cited above, that is

$$de = \frac{e+p}{\rho} d\rho + \rho T ds \tag{1.0.17}$$

or equivalently

$$\boxed{Tds = d\left(\frac{e}{\rho}\right) + pd\left(\frac{1}{\rho}\right)} \tag{1.0.18}$$

1.1 Energy conditions

Common matter is usually expected to fulfil some relations, called energy conditions, in order to respect its observed behaviours. Firstly classical matter must have positive energy in any frame of reference, that correspond to the disequality

$$\boxed{T_{ij}\xi^i\xi^j \geq 0} \tag{1.1.1}$$

where ξ^i is a generic future-directed timelike vector. It is called the weak energy condition. On the other side matter can't be faster than light, so the energy-momentum 4-current density of matter $-T^{ij}\xi_i$ must always be a future-directed timelike or, in extreme conditions, lightlike vector field. These requests can be translated in the equation

$$-T^{ij}\xi_i = Au^j + Bk^j \tag{1.1.2}$$

with A and B non-negative scalars and k^j a lightlike vector, named dominant energy condition[6].
Another constraint usually requested, the strong energy condition, guarantees the attractiveness of the gravitational force. If ξ^i is a timelike unitary geodesic vectorial field with proper time τ and everywhere orthogonal to the spatial hypersurfaces in some foliation of the spacetime, it is possible to evaluate its relative expansion scalar and vorticity and shear tensors. In this

case, the Raychaudhuri's equation[2], which is deduced from (1.0.5) , states that the change rate of the expansion scalar θ along ξ^i behaves as follows

$$\xi^j \nabla_j \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{ij}\sigma_{ij} + \omega^{ij}\omega_{ij} - R_{ij}\xi^i\xi^j \tag{1.1.3}$$

If the vector ξ^j has an orthogonal foliation, the vorticity tensor is null, while the shear tensor is always only spatial, because $\sigma_{ij}\xi^j = 0$, and then $\sigma_{ij}\sigma^{ij} \geq 0$.
Under the condition $R_{ij}\xi^i\xi^j \geq 0$ and a new relation appears:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \;\Rightarrow\; \frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3} \tag{1.1.4}$$

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau \tag{1.1.5}$$

where θ_0 is the initial value of θ. The latter one implies that, if θ_0 is negative, θ^{-1} must pass through zero, thus $\theta \to -\infty$ and the volume of the fluid will become 0. So the strong energy condition needed to have an attractive gravity is

$$\boxed{R_{ij}\xi^i\xi^j \geq 0} \tag{1.1.6}$$

for any timelike unitary vector ξ^i.
The last law is called null energy condition and states that the weak one and the strong hold for any null vector ξ^i, namely $\xi^i\xi_i$.
The definitions given above are clearly difficult to verify for any possible timelike, or lightlike, vector, consequently a formulation through the eigenvalues of T^{ij} is usually preferred. The stress energy tensor is a real symmetric second order tensor on a n-dimensional real space, thus it must have n real eigenvalues, moreover it must have n-1 spatial eigenvectors ξ^i_μ and one timelike ξ^i_0, which can all be chosen orthogonal respect to each other. that means it is possible to write

$$T^{ij} = e\xi^i_0\xi^j_0 + \sum_{\mu=1}^{n-1} p_\mu \xi^i_\mu \xi^j_\mu \tag{1.1.7}$$

A generic vector is a linear combination of the eigenvectors, so the relations defined above can be expressed in terms of the eigenvalues e and p_μ:
the weak energy condition in this way becomes

$$e \geq 0 \;\; and \;\; e + p_\mu \geq 0 \; \forall \mu = 1, 2, ..., n-1 \tag{1.1.8}$$

while the strong one is

$$e + \sum_{\mu=1}^{n-1} p_\mu \geq 0 \;\; and \;\; e + p_\mu \geq 0 \; \forall \mu = 1, 2, ..., n-1 \tag{1.1.9}$$

and the dominant condition takes the form

$$e \geq 0 \;\; and \;\; -e \leq p_\mu \leq e \; \forall \mu = 1, 2, ..., n-1 \tag{1.1.10}$$

Finally the null energy condition is

$$e + p_\mu \geq 0 \; \forall \mu = 1, 2, ..., n-1 \tag{1.1.11}$$

It is important to emphasise the linkages between previous relations: it can be demonstrated that dominant condition implicates the weak one, while the strong is independent from these two. The null energy condition is automatically valid if any one of the others is respected.

1.2 Perfect fluids

A fluid is called perfect if it has no viscous effects and heat flux and it has a diagonal stress tensor. Under isotropic condition and called e the total energy density $T_{ij}u^i u^j$ respect to the fluid velocity u^i, the stress-energy tensor in the comoving frame looks like

$$T^{ij} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

or, in a covariant way,

$$\boxed{T_{ij} = (e+p)u_i u_j + p g_{ij}} \tag{1.2.1}$$

with the letter p standing for the fluid pressure. As told above, the rest-mass density current appears as

$$J^i = \rho u^i \tag{1.2.2}$$

where ρ represents the rest-mass density. Conservation laws (1.0.6) and (1.0.7) than become:

$$\nabla_i J^i = \nabla_i(\rho u^i) = u^i \nabla_i \rho + \rho \nabla_i u^i = u^i \nabla_i \rho + \rho\theta = 0 \tag{1.2.3}$$

$$\nabla_i T^{ij} = \nabla_i[(e+p)u^i u^j + g^{ij} p] = 0 \tag{1.2.4}$$

The second one can be separated in a temporal equation projected along u^i

$$u_i \nabla_j T^{ij} = u^i \nabla_i e + (p+e)\theta = 0 \tag{1.2.5}$$

and three momentum conservation laws projected on its perpendicular hyperplane

$$h^i{}_j \nabla_k T^{kj} = u^i \nabla_i u_j + \frac{1}{e+p} h^i{}_j \nabla_i p = 0 \tag{1.2.6}$$

From the conservation of particles' number we have(1.2.3)

$$\theta = -\frac{1}{\rho} u^i \nabla_i \rho \tag{1.2.7}$$

and by a substitution in (1.2.5)

$$u^i \nabla_i e - \frac{(p+e)}{\rho} u^i \nabla_i \rho = 0 \tag{1.2.8}$$

so, if we recall the form (1.0.17) of the first principle of thermodynamics and take the covariant derivatives along u^i in spite of differentials, the energy conservation law implies that

$$u^i \nabla_i s = 0 \tag{1.2.9}$$

Thus the specific entropy $s = S/\rho$ is conserved along the motion of the fluid. In perfect fluid irreversible processes are absent, than the second principle of thermodynamics takes the form $\nabla_i S^i = 0$ and the entropy current is given by the simple expression $S^i = \rho s u^i$, exactly like happens at equilibrium. It is important to observe that the conservation of entropy is directly implied by the conservation of particles' current and eq (1.2.9), since

$$\nabla_i(\rho s u^i) = s \nabla_i(\rho u^i) + \rho u^i \nabla_i s \tag{1.2.10}$$

1.3 Imperfect fluids

In the treatment of non-perfect fluids with a relativistic framework the definition of the fluid 4-velocity has a primary role. In perfect fluids without a heat flux, the energy density 4-current and the rest-mass density current are the same vector, because energy is given by the particles mass and their motions. In a dissipative condition the heat flux affects the energy density current, thus it differs from the particles' one.

Lemma. *Let S_{ij} be a symmetric tensor and ξ a timelike unit vector, then S_{ij} can be written as*

$$\boxed{S_{ij} = (A+B)\xi_i \xi_j + B g_{ij} - S_i \xi_j - S_j \xi_i + \sigma_{ij}} \tag{1.3.1}$$

where A and B are scalar fields, S^i is a spacelike field such that $S_i \xi^i = 0$ and σ_{ij} is a symmetric tensor such that $\sigma_{ij}\xi^j = 0$ and $\sigma^k{}_k = 0$. If ξ^i is also an eigenvector of S_{ij}, then $S_i = 0$.

Proof. thanks to the projector h_{ij} it is possible to write

$$\begin{aligned} S_{ij} &= S^{mn}[h_{mi} - u_m u_i][h_{nj} - u_n u_j] \\ &= (S^{mn} u_m u_n) u_i u_j - (S^{mn} u_n h_{mi}) u_j - (S^{mn} u_m h_{nj}) u_i + S^{mn} h_{mi} h_{nj} \\ &= A u_i u_j - S_i u_j - S_j u_i + B h_{ij} + [S^{mn} - B h^{mn}] h_{mi} h_{nj} \end{aligned}$$

The scalar field B is chosen such that the last tensor σ_{ij} is traceless, i.e. $S^k_{\ k} = -A + B(n-1)$. With this choice the result is obtained.
Note that $S^i u_i = 0$, and $S^i S_i = S^{mn} S^{ab} h_{ma} u_n u_b = (S^2)_{mn} u^m u^n + A^2$. The orthogonality $S^i u_i = 0$ implies that $S^0 = 0$ in the rest frame, i.e. $S^i S_i > 0$ (spacelike vector). □

Eckart in 1940 proposed to take as fluid velocity the unitary timelike vector u^i parallel to the rest-mass density current J^i, so we have $J^i = \rho u^i$, as usually happens in perfect fluids. In the frame of reference comoving with u^i, $T^{ij} u_i u_j = T^{ij}_{PF} u_i u_j$, where T^{ij}_{PF} is the stress-energy tensor of a perfect fluid, since the energy density in such situation is locally the same as in a perfect fluid. Under these definitions, the decomposition (1.3.1) applied to T_{ij} takes the form

$$T_{ij} = (e + p' + \Pi) u_i u_j + (p' + \Pi) g_{ij} + q_i u_j + q_j u_i + \pi_{ij} \tag{1.3.2}$$

where Π is the viscous pressure, q^i is the heat flux vector and π_{ij} represents the anisotropic stress tensor. Finally p' is the pressure of the fluid without the viscous effect, sometimes, for brevity, we will take $p = p' + \Pi$.
There exists an alternative approach, formulated by Landau[10], which defines the fluid 4-velocity with the statement that, in the fluid proper rest frame, there is not net energy flux. Under this convention Landau's velocity w^i is an eigenvector of T_{ij} and

$$T^{ij} = (\hat{e} + \hat{p}) w^i w^j + g^{ij} \hat{p} + \tau^{ij} \tag{1.3.3}$$

On the other hand $J^i = \rho w^i + \hat{J}^i$. The trace of a second order tensor is an invariant scalar respect to a change of system of reference, so we have

$$T^k_{\ k} = e + (n-1)p = \hat{e} + (n-1)\hat{p} \tag{1.3.4}$$

An important question is how to pass from a decomposition to another, depending on the preferred timelike vector. This theme is treated in a more complete way by [11], anyway considering a first order approximation of w^i as $u^i + \delta^i$ in a condition near to equilibrium, the two expressions for the energy momentum tensor must coincide in a first order approach.

$$T^{ij} = (\hat{e} + \hat{p})(u^i + \delta^i)(u^j + \delta^j) + g^{ij} \hat{p} + \tau^{ij} \tag{1.3.5}$$

$$T^{ij} = (\hat{e} + \hat{p}) u^i u^j + g^{ij} \hat{p} + (\hat{e} + \hat{p}) u^i \delta^j + (\hat{e} + \hat{p}) u^j \delta^i + \tau_{ij} \tag{1.3.6}$$

By a comparison with (1.3.2), appears

$$\hat{e} = e + O(q^2), \quad \hat{p} = p + O(q^2) \tag{1.3.7}$$

$$\pi_{ij} = \tau_{ij} + O(q^2) \tag{1.3.8}$$

$$w^i = u^i + \tfrac{q^i}{e+p} + O(q^2) \tag{1.3.9}$$

However in such situation the viscous pressure Π is of the same order of q^i, so the right formula for w^i is

$$\boxed{w^i = u^i + \frac{q^i}{e+p'} + O(q^2)} \tag{1.3.10}$$

where only the thermodynamical pressure appears. In both cases the conservation laws $\nabla_i T^{ij} = 0$ and $\nabla_i J^i = 0$ must be respected in the fluid motion. In the particles' velocity model (the Eckart's one) the first equation becomes

$$\begin{aligned} \nabla_i T^{ij} &= u^j u^i \nabla_i (e + p' + \Pi) + (e + p' + \Pi)(u^j \theta + a^j) \\ &+ g^{ij} \nabla_i (p' + \Pi) + q^j \theta + u^i \nabla_i q^j + q^i \nabla_i u^j + u^j \nabla_i q^i + \nabla_i \pi^{ij} = 0 \end{aligned} \tag{1.3.11}$$

while the second one has the same form as in perfect fluids.
The stress-energy tensor conservation law is usually translated in an energy equation and a momentum equation: the definition of spatial covariant derivative[3], namely the derivative on the orthogonal plane respect to u^i,

$$D_i f := h_i^{\ j} \tfrac{\partial f}{\partial x^j} \tag{1.3.12}$$

$$D_i q_j := h_i^{\ l} h_j^{\ m} \nabla_l q_m \tag{1.3.13}$$

$$D_i \sigma_{lk} := h_i^{\ j} h_l^{\ m} h_k^{\ n} \nabla_j \sigma_{mn} \tag{1.3.14}$$

permits to operate this separation:

$$\begin{aligned} (e + p' + \Pi) a_i + D_i (p' + \Pi) + D_j \pi^j_{\ i} + a^j \pi_{ij} + h^j_{\ i} u^k \nabla_k q_j \\ + (\omega_{ij} + \sigma_{ij} + \tfrac{4}{3} \theta h_{ij}) q^j = 0 \end{aligned} \tag{1.3.15}$$

$$u^i \nabla_i e + (e + p' + \Pi)\theta + 2 q_i a^i + D_i q^i + \pi^{ij} \sigma_{ij} = 0 \tag{1.3.16}$$

in order momentum and energy equations.
If one follows the energy's velocity formalism, the conservation laws (1.0.7), (1.3.15) and (1.3.16) take the forms

$$\nabla_i \hat{J}^i + w^i \nabla_i \rho + \rho \hat{\theta} = 0 \tag{1.3.17}$$

$$(\hat{e} + \hat{p}) \hat{a}^k + D^k \hat{p} + D_i \tau^{ik} + \hat{a}_i \tau^{ik} = 0 \tag{1.3.18}$$

$$w^i \nabla_i (\hat{e}) + (\hat{e} + \hat{p}) \theta + \tau^{ik} \hat{\sigma}_{ik} = 0 \tag{1.3.19}$$

where quantities $\hat{\sigma}_{ij}$, $\hat{\theta}$ and $\hat{a}_i$ are evaluated respect to vector w^i.
When irreversible processes are involved,the second principle of thermodynamics $\nabla_i S^i \geq 0$ must be considered, where the entropy current has been

called S^i and will assume the value $S^i = s\rho u^i + R^i$, i.e. its perfect form plus a dissipative term R^i. The different order of expansion considered in evaluating the dissipative term separates the first order theories, also called CIT (*Classical Irreversible Thermodynamics*), from more advanced causal and hyperbolic models.

1.3.1 Classical Irreversible Thermodynamics (CIT)

CIT predicts a linear dependence of R^i on the dissipative terms of the energy-momentum tensor and can be formulated in both the conventions explained above.
In Eckart theory the entropy density 4-current's dissipative term will look like

$$R^i = f(\rho, e)\Pi u^i + g(\rho, e)q^i \tag{1.3.20}$$

By the second principle of thermodynamics in equilibrium condition the entropy density $-S^i u_i$ must have a stationary point respect to variations in the flux given by the viscous pressure Π:

$$\frac{\partial}{\partial \Pi}(s\rho + f(\rho, e)\Pi)\Big|_{eq} = 0 \tag{1.3.21}$$

The only way to satisfy this condition is $f = 0$. On the other hand in the fluid velocity frame $\frac{q^i}{T} = (0, \vec{q}/T)$ which corresponds to the entropy current caused by heat flow, so we obtain

$$S^i = s\rho u^i + \frac{q^i}{T} \tag{1.3.22}$$

The four-divergence of entropy current is

$$\nabla^i S_i = s\nabla_i(\rho u^i) + \rho u^i \nabla_i s + \frac{1}{T}\nabla_i q^i - \frac{q^i}{T^2}\nabla_i T \tag{1.3.23}$$

By particles' conservation law (1.2.3)

$$T\nabla^i S_i = T\rho u^i \nabla_i s + \nabla_i q^i - q^i \nabla_i \ln T \tag{1.3.24}$$

So if we consider that relation with the Gibbs equation (1.0.18), the entropy generation rate is

$$T\nabla^i S_i = \rho u^i \nabla_i \left(\frac{e}{\rho}\right) + \rho p' u^i \nabla_i \left(\frac{1}{\rho}\right) + \nabla_i q^i - q^i \nabla_i \ln T \tag{1.3.25}$$

Now, thanks to (1.2.7) and the imperfect fluid energy conservation (1.3.16), it becomes

$$\begin{aligned} T\nabla^i S_i &= u^i \nabla_i e + (e + p')\theta + \nabla_i q^i - q^i \nabla_i \ln T = \\ &= -(\Pi\theta + 2q_i a^i + D_i q^i + \pi^{ij}\sigma_{ij}) + \nabla_i q^i - q^i \nabla_i \ln T \end{aligned} \tag{1.3.26}$$

Finally it is possible to decompose the covariant derivative of a vector with the relation $\nabla_i q^i = D_i q^i + a_i q^i$, consequently we have

$$T\nabla^i S_i = -\Pi\theta - (D_i \ln T + a_i)q^i - \pi_{ij}\sigma^{ij} \tag{1.3.27}$$

If the entropy 4-divergence has to be non-negative as result of the second principle of thermodynamics, the simpler way to obtain it is establishing the so called *constitutive equations* of CIT

$$\boxed{\begin{aligned} \Pi &= -\zeta\theta \\ q^i &= -\kappa(D^i T + T a^i) \\ \pi_{ij} &= -2\eta\sigma_{ij} \end{aligned}} \qquad \begin{aligned}(1.3.28)\\(1.3.29)\\(1.3.30)\end{aligned}$$

An interesting aspect of these relations is that in general relativity it is possible to have a heat flow without a difference of temperature, since an acceleration of particles will provoke an energy flux with a not null component orthogonal to u^i.
In the non-relativistic limit the constitutive equations take the form

$$\Pi = -\zeta\vec{\nabla}\cdot\vec{v} \tag{1.3.31}$$

$$\vec{q} = -\kappa(\vec{\nabla}T) \tag{1.3.32}$$

$$\pi_{ij} = -2\eta\sigma_{ij} \tag{1.3.33}$$

similar to parts of Navier-Stokes, Fourier and viscosity Newton's equations, thus the non-negative coefficients are identified in the following way:
• ζ is the bulk viscosity
• κ is the thermal conductivity
• η is the shear viscosity
If one prefers to use Landau's hydrodynamic model the results of CIT are similar, as expected from the fist order equalities which have been shown before. From energy conservation (1.3.19)

$$\nabla_i[w^i(\hat{e}+\hat{p}')] - w^i\nabla_i\hat{p}' + \hat{\Pi}\nabla_i w^i + \tau^{ij}\nabla_i w_j = 0 \tag{1.3.34}$$

The substitution $(\hat{e}+\hat{p}')w^i = \frac{\hat{e}+\hat{p}'}{\rho}\rho w^i$ and continuity equation (1.3.17) transform the last relation in

$$\rho w^i \nabla_i\left(\frac{\hat{e}+\hat{p}'}{\rho}\right) - \frac{\hat{e}+\hat{p}'}{\rho}\nabla_i\hat{J}^i - w^i\nabla_i\hat{p}' + \hat{\Pi}\nabla_i w^i + \tau^{ij}\nabla_i w_j = 0 \tag{1.3.35}$$

Now, introducing the relativistic chemical potential $\mu = \frac{\hat{e}+\hat{p}'}{\rho} - T\hat{s}$ and through (1.0.18), a new thermodynamical relation comes to light:

$$d\mu = \frac{1}{\rho}d\hat{p}' - \hat{s}dT \tag{1.3.36}$$

So we obtain

$$\nabla_i \left(\rho\hat{s}w^i - \frac{\mu}{T}\hat{J}^i\right) = -\hat{J}^i\nabla_i\frac{\mu}{T} - \frac{\hat{\Pi}}{T}\nabla_i w^i - \frac{\tau^{ij}}{T}\nabla_i w_j \tag{1.3.37}$$

The left side is the entropy 4-current in Landau's formalism,

$$S^i = \hat{s}\rho w^i - \frac{\mu}{T}\hat{J}^i \tag{1.3.38}$$

then the second principle of thermodynamics requests all therms are positive in the right side. This condition, together with orthogonality of both $\hat{J}^i$ and τ^{ij} respect to w^i and symmetry and null trace of τ^{ij}, uniquely determines a new set of constitutive equations:

$$\begin{aligned}\tau_{ij} = -\eta(\nabla_j w_i + \nabla_i w_i + w_j w^l \nabla_l w_i + w_i w^l \nabla_l w_j) +\\ +\tfrac{2}{3}\eta\nabla_l w^l (g_{ij} + w_i w_j) = -2\eta\hat{\sigma}_{ij}\end{aligned} \tag{1.3.39}$$

$$\hat{\Pi} = -\zeta\hat{\theta} \tag{1.3.40}$$

$$\begin{aligned}\hat{J}^i &= -\kappa\left(\tfrac{\rho T}{\hat{e}+\hat{p}'}\right)^2\left[\nabla^i\left(\tfrac{\mu}{T}\right) + w^i w^j \nabla_j\left(\tfrac{\mu}{T}\right)\right]\\ &= -\kappa\left(\tfrac{\rho T}{\hat{e}+\hat{p}'}\right)^2 D^i\left(\tfrac{\mu}{T}\right)\end{aligned} \tag{1.3.41}$$

where coefficients ζ, η and κ have the same physical meanings as in the particles' formulation. A rewrite of eq (1.3.36) gives

$$d\frac{\mu}{T} = -\frac{\hat{e}+\hat{p}'}{\rho T^2}dT + \frac{1}{\rho T}d\hat{p}' \tag{1.3.42}$$

and the component of particles' current orthogonal to w^i assumes a new form

$$\hat{J}^i = \kappa\left(\frac{\rho T}{\hat{e}+\hat{p}'}\right)^2\left(\frac{\hat{e}+\hat{p}'}{\rho T^2}D^i T - \frac{1}{\rho T}D^i\hat{p}'\right) \tag{1.3.43}$$

By momentum conservation (1.3.18), when $\tau_{ij} = 0$, the third constitutive equation is

$$\hat{J}^i = \kappa\left(\frac{\rho T}{\hat{e}+\hat{p}'}\right)^2\left(\frac{\hat{e}+\hat{p}'}{\rho T^2}D^i T + \frac{1}{\rho T}\hat{a}^i\right) \tag{1.3.44}$$

and is very similar to the corresponding equation of Eckart's convention, if one consider (1.3.10) and consequently

$$J^i = \rho u^i = \rho(w^i - \frac{q^i}{e+p'}) = \rho w^i + \hat{J}^i \Rightarrow \hat{J}^i = -\frac{q^i\rho}{e+p'} \tag{1.3.45}$$

Anyway is necessary to remember that a^i and $\hat{a}^i$ do not correspond to the same vector in general.

1.3.2 Beyond CIT

The classical irreversible thermodynamics is an easy to manage formalism with a lot of analogies with non-relativistic hydrodynamic theories. That is a good feature because makes of CIT a useful and relatively simple model that gives good results in a lot of circumstances. however it has some problems too: the algebraic nature of constitutive equations permits an instantaneous response of thermodynamic fluxes to the action of thermodynamics forces, that is admitted in a Newtonian theory, but is incompatible with the principle of finite speed of light. The result is that first-order theories, in both Landau's and Eckart formulations, do not admit relaxation times for dissipative processes. Another critical aspect of CIT is represented by the existence of some non-selfgravitating special-relativistic and realistic systems which assume unstable behaviours if CIT is applied them, with small perturbations causing an exponential departure from equilibrium. To response to this problems Israel and Stewart [11] proposed in 1976 a second order approach to the evaluation of the dissipative part of entropy four-current, called Extended Irreversible Thermodynamics (EIT). This model introduces more parameters than the three present in CIT and permits to have a causal theory with hyperbolic constitutive equations in almost all situations. Anyway exist some situations where EIT fails in this effort, so an alternative way is represented by the divergence-type theories, which have a strict connection with kinetic theory of gases and are built in such a way that they satisfy the requests of covariance and hyperbolicity and they respect the second principle of thermodynamics.

Chapter 2

The twisted spacetimes hierarchy

A spacetime M is told twisted if it can be seen as a twisted product between the temporal one-dimensional interval I with a metric $g_I = -1$ and a spatial Riemannian manifold M^*. A twisted product $B \times_f C$ of two manifolds B and C is a $dim(B) + dim(C)$ dimensional space, whose metric tensor has the form $g_B + f^2 g_C$ and f is a scalar function on $B \times C$[1].
This definition gives

$$\boxed{ds^2 = -dt^2 + f^2(t, \vec{x}) g^*_{\mu\nu} dx^\mu dx^\nu} \tag{2.0.1}$$

A manifold (M, g) where the canonical foliations L_B and L_C intersect perpendicularly everywhere is twisted if and only if each leaf $B \times c$ ($c \in C$) in L_B is totally geodesic, i.e. each geodesic in the submanifold respect to the induced metric is geodesic for (M, g), and each leaf $C \times b$ ($b \in B$) in L_C is totally umbilical, which means the second fundamental form is proportional to the metric in each point.
The scale function f appears in (2.0.1) only with his second power, consequently we can arbitrary chose a positive sign for f.
As can be simply seen, the less complicated elements of this big family are the RW spacetimes, which appear under the condition of taking M^* as a constant curvature manifold and f as a function depending only from t. These metrics respect the hypothesis of homogeneity and isotropy and so they have a great role in many cosmological models of universe[2]. The Robertson-Walker spacetimes have a null Weyl tensor C_{klmn} and correspond to perfect fluids.
If one removes the constraint of constant curvature for the spatial part and takes in spite of it a generic Riemannian manifold, obtains another subgroup of spacetimes, called generalised Robertson-Walker, shorter GRW. We will discuss later the properties of these spaces.
As shown by G. A. Mantica and L. G. Molinari[4], the twisted spacetimes

hierarchy can be characterised by the existence of a time-like unique unitary vector field u^i, called torse forming, which has the relevant property $\nabla_j u_i = h_{ij}\varphi$. Here φ is a scalar field and, in the frame of reference where is valid (2.0.1), it is equal to the mean curvature scalar $H = \frac{\dot{f}}{f}$, with the dot representing the total derivative respect to the proper time $\dot{f} = \frac{df}{d\tau} = \frac{dx^i}{d\tau}\nabla_i f = u^i \nabla_i f$. φ has a covariant expression too, which is $\varphi = u^k \nabla_k \log f$. As consequence the torse forming is a geodetic and his acceleration is null, indeed $u^j h_{ij}\varphi = u^j h_{ji}\varphi = 0$.
In the frame of reference where the metric appears as (2.0.1), the covariant derivative of the timelike vector u^i with component $(1, 0, 0, 0)$ is $\nabla_j u_i = \partial_j u_i - \Gamma^k{}_{ji} u_k$, while the Christoffel symbols can be evaluated with the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{lj} - \partial_l g_{ij}) \tag{2.0.2}$$

Then, since $g^{0\mu} = 0$, it results that $\nabla_j u_i$ is null if i or j is 0, while $\nabla_\mu u_\nu$ is equal to $\frac{\dot{f}}{f} g_{\mu\nu}$, thus this vector fulfils the torse forming condition. That means the metric (2.0.1) represents the frame of reference comoving with the torse forming vector.
Once introduced the decomposition $\nabla_j \varphi = v_j - u_j u^k \nabla_k \varphi$ and the definition $v_j := h_j{}^k \nabla_k \varphi$, the Ricci tensor appears in the form

$$R_{lk} = -(n u_k u_l + g_{lk})(u^r \nabla_r \varphi + \varphi^2) + \frac{R}{n-1} h_{kl} + (n-2)(u_k v_l + u_l v_k - u^r u^s C_{rkls}) \tag{2.0.3}$$

By means of R_{ij} and the Einstein equation, the resulting stress-energy tensor looks like

$$T_{ij} = (p + e) u_i u_j + p g_{ij} + q_i u_j + u_i q_j + \pi_{ij} \tag{2.0.4}$$

$$e = -\frac{n-1}{8\pi}(u^r \nabla_r \varphi + \varphi^2) + \frac{R}{16\pi} \tag{2.0.5}$$

$$p = -\frac{1}{8\pi}(u^r \nabla_r \varphi + \varphi^2) - \frac{R}{8\pi}\frac{n-3}{2(n-1)} \tag{2.0.6}$$

$$q_j = \frac{n-2}{8\pi} v_j \tag{2.0.7}$$

$$\pi_{ij} = -\frac{n-2}{8\pi} u^r u^s C_{rkls} \tag{2.0.8}$$

which matches perfectly the stress energy tensor of an imperfect fluid with the particles velocity field equal to the torse forming vector u^i (1.3.2) and an anisotropic stress tensor strictly bound with the Weyl tensor. The symmetry of the torse forming covariant derivative and its null acceleration force the shear and vorticity tensors to zero, thus twisted spacetimes belong to a category of spaces which has been widely studied by Coley and McManus in [12].
In the rest frame is possible to evaluate the local form of Riemann tensor

and other geometrical quantities (appendix A), in particular the curvature scalar is

$$R = R^{*} + (n-1)\left[(n-2)\frac{\dot{f}^2}{f^2} + 2\frac{\ddot{f}}{f}\right] - (n-2)(n-5)\frac{f^{\sigma}f_{\sigma}}{f^4} + \\ -2(n-2)\frac{\nabla^{*}_{\sigma}f^{\sigma}}{f^3} \qquad (2.0.9)$$

where f^{σ} is the partial derivative of f along σ component and the covariant derivative ∇^{*} represent a derivation on the submanifold M^{*}.
The Weyl tensor itself has some relevant features in a twisted spacetime. If we separate this tensor in an orthogonal and parallel part with respect to an observer with velocity u^i, we can define an electric $(C_{+})_{rkls}$ and magnetic part $(C_{-})_{rkls}$, similarly to how usually happens with the Maxwell tensor[15].

$$(C_{+})^{rk}{}_{ls} = h^{re}h^{kf}h^{g}{}_{l}h^{h}{}_{s}C_{efgh} + 4u^{[r}u_{[l}C^{k]e}{}_{s]f}u_{e}u^{f} \qquad (2.0.10)$$

$$(C_{-})^{rk}{}_{ls} = 2h^{re}h^{kf}C_{efg[h}u_{d]}u^{h} + 2u_{h}u^{[r}C^{k]hef}h_{le}h_{sf} \qquad (2.0.11)$$

A symmetric second order tensor tensor A_{ij} is Weyl compatible[14] if the following relation is valid:

$$A_{im}C_{jkl}{}^{m} + A_{jm}C_{kil}{}^{m} + A_{km}C_{ijl}{}^{m} = 0 \qquad (2.0.12)$$

In a twisted spacetime the symmetric tensor $u^i u^j$, with u^i torse forming, is Weyl compatible, then

$$u_{i}u_{m}C_{jkl}{}^{m} + u_{j}u_{m}C_{kil}{}^{m} + u_{k}u_{m}C_{ijl}{}^{m} = 0 \qquad (2.0.13)$$

That implies some relations:

$$u^{m}C_{jklm} = u_{k}(u^{i}u^{m}C_{ijlm}) - u_{j}(u^{i}u^{m}C_{iklm}) \qquad (2.0.14)$$

$$u^{m}C_{jklm} = 0 \Longleftrightarrow u^{i}u^{m}C_{iklm} = 0 \qquad (2.0.15)$$

Finally the Weyl tensor is purely electric, as shown in [14], that means $C_{rkls} = (C_{+})_{rkls}$ and $(C_{-})_{rkls} = 0$, in accordance with how expected from the metric form (2.0.1) in [15].

2.1 Generalised Robertson-Walker spacetimes

The metric tensor of a GRW spacetime is

$$\boxed{ds^2 = -dt^2 + f^2(t)g^{*}_{\mu\nu}dx^{\mu}dx^{\nu}} \qquad (2.1.1)$$

thus GRW manifolds belong to the warped product category, which is a subclass of twisted product with a scale factor f depending only on B. A direct

consequence of this fact is that GRW's spatial submanifolds have constant mean curvature, since $\frac{\dot{f}}{f}$ has the same value in each point with equal t.
The subclass of GRW comes to light another time due to the torse forming characteristics, considering that a twisted space is GRW if and only if its torse forming vector is eigenvector of R_{ij}[5]: the derivative of φ in the hyperplane orthogonal to u^i is null and so $v_i = 0$, in this case the form of the Ricci tensor is

$$R_{kl} = -(n-2)u^r u^s C_{rkls} - (nu_l u_k + g_{lk})(u^r \nabla_r \varphi + \varphi^2) + \frac{R}{n-1} h_{kl} \quad (2.1.2)$$

Hence the only dissipative part of the stress energy tensor is the anisotropic stress tensor π_{ij}, for which is valid the relation $\pi_{ij} u^j = 0$. The direct consequence is that the torse forming vector is an eigenvector of T_{ij}, as expected. Despite we are in an imperfect fluid domain, the energy frame of reference and the particles' one coincide perfectly in GRW, because the heat flux is absent.
Moreover, as proved by Mantica and Molinari [8], in GRW[1] exist the following important implication:

$$C_{jklm} u^m = 0 \Longleftrightarrow \nabla^m C_{jklm} = 0 \quad (2.1.3)$$

This correlation was originally proved for the Chen vector X^m [9], whose torse forming is a renormalization. All preceding relations permit to affirm that a GRW spacetime represent a perfect fluid if and only if $\nabla^m C_{jklm} = 0$. However, in the special 4-dimensional case, such condition on the covariant derivative of the Weyl tensor implies $C_{rkls} = 0$, then the fluid described has a perfect behaviour if and only if it lives in a RW spacetime.

2.2 The Robertson-Walker spacetimes

In case of RW spacetimes, which are a subset of GRW, $v_j = 0$ and $C_{rlks} = 0$, thus T_{ij} is

$$T_{ij} = (p+e)u_i u_j + p g_{ij} \quad (2.2.1)$$

and represents a perfect fluid, as told before, while the Ricci tensor has the form

$$R_{kl} = -(nu_l u_k + g_{lk})(u^r \nabla_r \varphi + \varphi^2) + \frac{R}{n-1} h_{kl} \quad (2.2.2)$$

The motion laws for a fluid which generates a RW spacetime are equations (1.2.5) and (1.2.3). RW metrics are often classified by the properties of

[1]has been recently shown by Mantica and Molinari that it is true for any twisted spacetime, howevwer such resunt hasn't been published yet

the spatial submanifold M^*. It has been shown in [2] and [13] that a n-1 dimensional constant curvature manifold has a Riemann tensor of the form

$$R^*_{\mu\nu\rho\sigma} = k(g^*_{\mu\rho}g^*_{\nu\sigma} - g^*_{\mu\sigma}g^*_{\nu\rho}) \tag{2.2.3}$$

where $k = \frac{R^*}{(n-1)(n-2)}$ must be constant. The resulting metric, in a set of coordinates ψ,θ,ϕ,t, is

$$\boxed{ds^2 = -dt^2 + f^2(t)\left\{\frac{d\psi^2}{1-k\psi^2} + \psi^2 d\theta^2 + \psi^2 \sin^2\theta\ d\phi^2\right\}} \tag{2.2.4}$$

With a suitable choice of units for ψ and a rescaling of $f(t)$, the scalar k can always assume a value +1, 0 or -1. At this point can be interesting to study which shape is assumed by the manifold with the three possible k. If $k = 1$, with the substitution $\psi = \sin r$, the spatial metric can be written as

$$f^2(t)\{dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta\ d\phi^2)\} \tag{2.2.5}$$

and represents the surface of a 3-sphere with radius f^2. If $k = 0$, the resulting metric is

$$f^2(t)\left\{d\psi^2 + \psi^2 d\theta^2 + \psi^2 \sin^2\theta\ d\phi^2\right\} \tag{2.2.6}$$

and is the parametrisation with spherical coordinates of a 3 dimensional flat euclidean space, where ψ is used in spite of more common r. The third option, $k = -1$, returns an hyperbolic spatial hypersurface after the variable exchange $\psi = \sinh r$, since the metric has the form

$$f^2(t)\{dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta\ d\phi^2)\} \tag{2.2.7}$$

Then, if the cosmological hypothesis of a homogeneous and isotropic universe is respected, if $k = 1$ we live in a closed manifold, otherwise in an open one.

Chapter 3

Hydrodynamics in twisted spacetimes

Now we will try to analyse some features of hydrodynamics in a generic twisted spacetime.
At the moment we do not know the eigenvalues of the energy momentum tensor of a twisted spacetime, thus we can only observe what happens when the three energy conditions (1.1.1)(1.1.6)(1.1.2) are applied to the torse forming vector. The weak condition with eq. (2.0.5) gives

$$-(n-1)y + \frac{R}{2} \geq 0, \tag{3.0.1}$$

where the expression $u^r \nabla_r \varphi + \varphi^2$ has been renamed y.
The strong condition, through Einstein equation, is

$$-(n-1)y + \frac{R}{2} - \frac{R}{2} = -(n-1)y \geq 0 \tag{3.0.2}$$

while the dominant requests $-T^{ij}u_i = eu^j - q^j$ to be future directed and not spacelike.
u^j and q^j are orthogonal, so the energy-momentum 4-current density of matter is a timelike vector if and only if $q^j q_j \leq e^2$ and is future directed if the product $-T^{ij}u_i u_j$ is negative, thus only if weak energy condition is fulfilled.
Relations (2.0.5) and (2.0.6), after a recombination and a substitution with (2.0.9), which is valid in the rest frame, take the form

$$\tfrac{(n-1)(n-2)}{2}\tfrac{\dot{f}^2}{f^2} = 8\pi e - \tfrac{R^*}{2f^2} + \tfrac{(n-2)(n-5)}{2}\tfrac{f^\sigma f_\sigma}{f^4} + (n-2)\tfrac{\nabla^*_\sigma f^\sigma}{f^3} \tag{3.0.3}$$

$$(n+2)\tfrac{\ddot{f}}{f} = -8\pi(e+3p) + (n-4)R \tag{3.0.4}$$

In a four-dimensional spacetime they are

$$\boxed{\begin{aligned} 3\frac{\dot{f}^2}{f^2} &= 8\pi e - \frac{R^*}{2f^2} - \frac{f^\sigma f_\sigma}{f^4} + 2\frac{\nabla^*_\sigma f^\sigma}{f^3} \\ 3\frac{\ddot{f}}{f} &= -4\pi(e+3p) \end{aligned}} \qquad \begin{aligned}(3.0.5)\\(3.0.6)\end{aligned}$$

which clearly are a generalised form of Friedmann equations in RW spacetimes[2]:

$$3\frac{\dot{f}^2}{f^2} = 8\pi e - \frac{R^*}{2f^2} \qquad (3.0.7)$$

$$3\frac{\ddot{f}}{f} = -4\pi(e+3p) \qquad (3.0.8)$$

They coincide when the spatial derivatives of the scale function $f(t, \vec{x})$ are null. Such request is fulfilled in GRW spacetimes, where $f(t)$ is a function of t only.
In a twisted spacetime the conservation laws can be derived from relations (1.3.15) and (1.3.16) with the conditions of null shear and vorticity tensors and null acceleration.

$$h_i{}^j \nabla_j \left(-ny + \frac{R}{n-1}\right) - h^{km} h_i{}^n \nabla_k((n-2)C_{rmns}u^r u^s) + \\ + h^j{}_i u^k \nabla_k((n-2)v_j) + \tfrac{4}{3}(n-2)v_i \nabla_k u^k = 0 \qquad (3.0.9)$$

$$u^i \nabla_i \left(-(n-1)y + \frac{R}{2}\right) + \\ + \left(-ny + \frac{R}{n-1}\right)\nabla_k u^k + h_i{}^j \nabla_j((n-2)v^i) = 0 \qquad (3.0.10)$$

These relations can be written only in terms of f and his derivatives and R^* with eq (2.0.9). However they can't give any constraint on the form of the scale function f, since the conservation of energy-momentum is equivalent to the second Bianchi identity (1.0.9), which is valid for any possible Riemann tensor.
So it is more interesting to write the equations in terms of thermodynamical quantities

$$D_i p + D_j \pi^j{}_i + h^j{}_i \dot{q}_j + \frac{4}{3}\theta q_i = 0 \qquad (3.0.11)$$

$$\dot{e} + \theta(e+p) + D_i q^i = 0 \qquad (3.0.12)$$

3.1 Dynamics of GRW

In a GRW $v^i = 0$, consequently the heat flux q^i is null too, and conservation of energy-momentum is equivalent to

$$D_i p + D_j \pi^j{}_i = 0 \qquad (3.1.1)$$

$$\dot{e} + \theta(e+p) = 0 \qquad (3.1.2)$$

and Friedmann equations (3.0.7) (3.0.8) are found.
The expansion scalar θ is correlated with the mean curvature, which is usually called Hubble's constant in cosmology, $H = \frac{\dot{f}}{f}$, indeed

$$\theta = \nabla_i u^i = H h^i{}_i = (n-1)H \tag{3.1.3}$$

thus, in a 4-dimensional spacetime, the energy conservation law becomes

$$\dot{e} + 3(e+p)\frac{\dot{f}}{f} = 0 \tag{3.1.4}$$

At this point some relevant observations need to be stressed: in GRW e is an eigenvalue of the stress energy tensor, that means the weak energy condition (3.0.1) respect to u^i is equivalent to the first request of weak itself and dominant conditions, written in terms of eigenvalues of T_{ij}. Anyway the stress tensor has a fundamental role in the definition of spatial eigenvectors and eigenvalues, then it is not possible to complete this type of formulation of the energy conditions in GRW spacetimes without choosing a particular spatial manifold.
If the weak energy condition is respected, the disequality

$$R^* \geq -3\dot{f}^2 \tag{3.1.5}$$

regulates the relation between the scale function and the curvature scalar. Given a spatial manifold M^*, the minimum value of R^* establishes the minimum of the modulus of $\dot{f}$: if a point has negative curvature, the evolution of $f(t)$ will be monotonic, if discontinuities are excluded in $\dot{f}$. On the other side, a stationary universe can't have any points with negative curvature scalar respect to M^*.
Another important aspect of GRW spaces is that spatial independence of f and its temporal derivatives impose that the energy density e spatial distribution at an instant t is totally determined by scalar curvature R^* through eq. (3.0.7)

$$8\pi e = \frac{R^*}{2f^2} - c'(t) \tag{3.1.6}$$

with $c'(t) = 3\frac{\dot{f}^2}{f^2}$. For the same reason, through (3.0.8), $e + 3p$ is a constant scalar in M^*. A state equation for GRW spacetimes will appear as

$$\boxed{p = c(t) - \frac{e}{3}} \tag{3.1.7}$$

where $c(t) = -\frac{1}{4\pi}\frac{\ddot{f}}{f}$. As consequence an equation of state of the form $p = de + c$, with c, d spatially constant and $d \neq -\frac{1}{3}$, is compatible with a GRW only in case of constant curvature: the second Friedmann equation requests $e(1+3d) + c = \textit{constant}$, thus the spatial distribution of energy

must be homogeneous. On the other hand by the first Friedmann equation $8\pi e - \frac{R^*}{2f^2}$ must be constant too, then R^* is constant an the space is RW.
It is possible to insert the generic equation of state for a GRW in the momentum conservation equation and to obtain an interesting relation between the anisotropic stress tensor and the spatial distribution of energy density in the particles' rest frame:

$$3D_j\pi^j{}_i = D_i e = \frac{1}{16\pi f^2} D_i R^* \tag{3.1.8}$$

Such equation is in accordance with the statement that a RW spacetime, where $\pi_{kl} \propto C_{rkls}u^r u^s = 0$, needs to have an homogeneous energy distribution.
In a similar way the energy conservation equation becomes

$$\dot{e} = -2e\frac{\dot{f}}{f} + \frac{\ddot{f}\dot{f}}{4\pi f^2} \tag{3.1.9}$$

The last equation is also the time derivative of the first Friedmann equation.

3.2 Dynamics of RW

A RW has R^* constant in M^* and the first Friedmann equation becomes

$$3\frac{\dot{f}^2}{f^2} = 8\pi e - \frac{3k}{f^2} \tag{3.2.1}$$

Moreover conservation of momentum is

$$D_i p = 0 \tag{3.2.2}$$

that means pressure is an homogeneous quantity. In RW spacetimes we knows all the eigenvalues of T_{ij}, since these spacetimes are spatially isotropic, then any spatial eigenvector has p as eigenvalue. The second formulation of energy conditions can be simply applied in this case. In particular the strong energy condition, in a four dimensional spacetime, states that $e + 3p \geq 0$, thus the scale factor of the universe f can't be stationary, since $\dot{f}$ is zero only for one value of f, but in this case $\ddot{f}$ is not null, because of (3.0.8).
As told above, RW spaces admit state equations with the form $p = de + c$. When c and d have static values, the conservation of energy implies

$$\dot{e} + 3[(d+1)e + c]\frac{\dot{f}}{f} = 0 \Rightarrow (e + \frac{c}{d+1})f^{3d+3} = a = constant \tag{3.2.3}$$

Hence the first Friedmann equation is

$$3\dot{f}^2 = \frac{8\pi a}{f^{3d+1}} - \frac{8\pi c}{d+1}f^2 - 3k \tag{3.2.4}$$

RW spacetimes are homogeneous and isotropic, then they have been used as base for many cosmological models.
From measurements of the cosmological redshift we know that universe is in an expansion phase, while $\ddot{f} < 0$ thanks to (3.0.8). If we go backward in time, the universe becomes smaller and smaller with a faster rate and then, if the homogeneous and isotropic model is true, there must have been a moment when it was $f = 0$, and the universe was pointlike. The instant when the expansion of universe began is usually called big bang.
In this scope, two important state equations are those describing dust and radiation. Galaxies, which represent the greatest part of visible energy density of present universe, are considered comoving with the fluid velocity u^i of the perfect cosmological fluid. Then their velocities in this frame of reference are low, and the pressure negligible, thus is usually taken $p = 0$ in the description of matter, also called dust, on very large scale.
If $p = 0$, (3.2.3) and (3.2.4) are

$$ef^3 = a \quad and \quad 3\dot{f}^2 = \frac{8\pi a}{f} - 3k \tag{3.2.5}$$

The quantity a/f decreases with expansion, hence, if $k = 1$, exist a maximum value of f over that the right part of left equation would become negative, in disagreement with the positive left side. So, in a spherical spacetime,

$$f \leq \frac{8\pi a}{3k} \tag{3.2.6}$$

Once reached its maximum size, the negative expansion acceleration from (3.0.8) will prevail an bring the universe to collapse in a big crunch. If $k \leq 0$ all value of f are possible, because c_1 is positive if is respected the weak energy condition. In case $K = 0$, $\dot{f}$ will tend to 0 and a universe of dust will grow always slower. If $K = -1$, the universe will go on in its expansion forever, with an asymptotically constant velocity $\dot{f} = \sqrt{-k} = 1$.

Radiation's state equation is $p = \frac{e}{3}$, then

$$ef^4 = a \quad and \quad 3\dot{f}^2 = \frac{8\pi a}{f^2} - 3k \tag{3.2.7}$$

A universe of radiation has a phenomenological behaviour similar to dust, with the difference it reaches its asymptomatic condition faster, since there is a $\frac{1}{f^2}$ term.
The exact solutions in these cases can be written in a parametric form

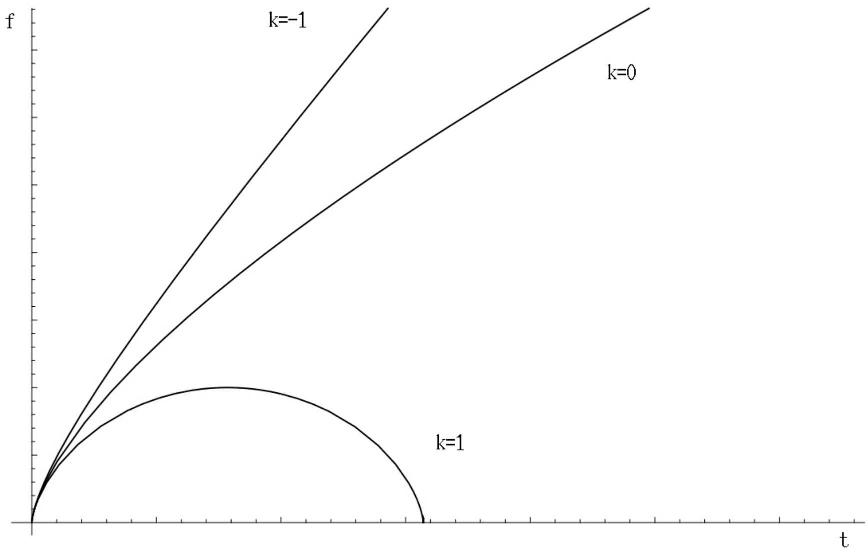

Figure 3.1: evolution of a dust universe with different values of k

respect to the conformal time η:

	$p = 0$	$p = \frac{1}{3}$
$k = +1$	$f = \frac{1}{2}a(1 - \cos\eta)$ $t = \frac{1}{2}a(\eta - \sin\eta)$	$f = \sqrt{a}[1 - (1 - t/\sqrt{a})^2]^{1/2}$
$k = 0$	$f = (9a/4)^{1/3}t^{2/3}$	$f = (4a)^{1/4}t^{1/2}$
$k = -1$	$f = \frac{1}{2}a(\cosh\eta - 1)$ $t = \frac{1}{2}a(\sinh\eta - \eta)$	$f = \sqrt{a}[(1 + t/\sqrt{a})^2 - 1]^{1/2}$

A surprising result consists in the similarity between the equation of time in case $k = 1$ and the Kepler equation describing the motion of a body in a classical central gravitational field, with T corresponding the revolution period and η the angle respect to the principal axis,

$$t = \frac{T}{2\pi}(\eta - \epsilon\sin\eta) \tag{3.2.8}$$

in the particular case with eccentricity $\epsilon = 1$, namely a parabolic orbit.
The measurement of cosmological redshift brought out an acceleration in

universe expansion in the present phase, with a behaviour incompatible with all models presented before. This discrepancy is explained in many theories with the presence of dark energy, that should represent a great part of energy in the universe and cause its accelerated expansion. Dark energy effect is usually implemented with a vacuum energy term in Einstein equation, also called cosmological constant:

$$8\pi T_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} \tag{3.2.9}$$

A discussion of GRW spacetimes with cosmological constant can be found in [16].

3.3 Eigenvector for Ricci tensor

In a twisted space the fluid velocity field u^i is not generally an eigenvector of the energy-momentum tensor T^{ij}. It can be useful finding the exact form of the vector field w^i which is time like, unitary and eigenvector of R^{ij}, and consequently of T^{ij}, corresponding to the Landau energy rest frame fluid velocity.
If the condition $u^r u^s C_{rkls} v^l = \gamma v_k$ is respected, w^i is a linear combination of u^i and v^i. To fulfil these requests one can consider firstly the action of the Ricci tensor on the fluid velocity and the heat current vector:

$$R_{kl}u^l = (nu_k - u_k)y + (n-2)v_k \tag{3.3.1}$$

$$R_{kl}v^l = -v_k y + \tfrac{R}{n-1}v_k + (n-2)v^2 u_k - (n-2)u^r u^s C_{rkls}v^l \tag{3.3.2}$$

To obtain an eigenvector of Ricci $w^l = \alpha u^l + \beta v^l$ with eigenvalue W must be:

$$\begin{aligned} R_{kl}w^l = y((n-1)u_k\alpha - \beta v_k) + \\ +(n-2)(v^2 u_k \beta - v_k\alpha - \gamma v_k \beta) + \tfrac{R}{n-1}\beta v_k = W(\alpha u_k + \beta v_k) \end{aligned} \tag{3.3.3}$$

The vector fields u^l and v^l are orthogonal, so eq. (3.3.3) gives two different equations, which, combined with the normalisation condition, return a system of three equations

$$\begin{cases} W\alpha = (n-1)\alpha y + (n-2)v^2\beta & (3.3.4) \\ W\beta = -y\beta - (n-2)(\alpha + \gamma\beta) + \beta\dfrac{R}{n-1} & (3.3.5) \\ w^l w_l = -1 & (3.3.6) \end{cases}$$

If $W \neq (n-1)y$, i.e. the fluid isn't perfect and $v^l \neq 0$, and if $W \neq \frac{R}{n-1} - y - (n-2)\gamma$, that means the torse-forming vector field is not null, the equations can be rewritten as

$$\begin{cases} \alpha = \dfrac{(n-2)v^2\beta}{W-(n-1)y} & (3.3.7) \\ \beta = \dfrac{-(n-2)\alpha}{W+(n-2)\gamma - \frac{R}{n-1} + y} & (3.3.8) \\ w^l w_l = -1 \end{cases}$$

From eq. (3.3.7) and (3.3.8) can be easily obtained the quadratic equation for W,

$$W^2+W((n-2)\gamma-\frac{R}{n-1}-(n-2)y)-(n-1)y((n-2)\gamma-\frac{R}{n-1}+y)+(n-2)^2v^2 = 0 \tag{3.3.9}$$

that gives

$$W = \frac{(n-2)y + \frac{R}{n-1} - \gamma(n-2) \pm \sqrt{[ny - \frac{R}{n-1} + (n-2)\gamma]^2 - (n-2)^2 4v^2}}{2} \tag{3.3.10}$$

When the expression for W is replaced in eq. (3.3.7) and (3.3.8) and is operated the substitution $\frac{ny-\frac{R}{n-1}+(n-2)\gamma}{2} = b$, the system of equations becomes

$$\begin{cases} \alpha = \dfrac{(n-2)v^2\beta}{-b \pm \sqrt{b^2 - v^2(n-2)^2}} & (3.3.11) \\ \beta = \dfrac{-(n-2)\alpha}{b \pm \sqrt{b^2 - v^2(n-2)^2}} & (3.3.12) \\ w^l w_l = -1 \end{cases}$$

where eq. (3.3.6) gives a normalisation for scalar fields α and β, because it can be write as $-\alpha^2 + \beta^2 v^2 = -1$.
Considering together eq. (3.3.11) and (3.3.6) returns

$$\alpha = \pm \frac{b \pm \sqrt{b^2 - v^2(n-2)^2}}{\sqrt{[b \pm \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.13}$$

and consequently, thanks to eq. (3.3.12)

$$\beta = \mp \frac{n-2}{\sqrt{[b \pm \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.14}$$

At this point we have obtained four solutions for the problem, however two of them, corresponding to the $\pm$ at the beginning of the expressions for α and β, aren't meaningful because they are equivalent to take $-w^l$ in place

of w^l.
We can observe that the argument of the square roots which appear in the denominators of the right sides in both equations is > 0, as required by the existence condition in a real space, only for one of the remaining solutions. Indeed, expanding the square term, the argument becomes $2b^2 - 2v^2(n-2)^2 \pm 2b\sqrt{b^2 - v^2(n-2)^2}$ and is positive if and only if $\pm b$ is positive. By such existence constraint the formulae for α and β are:

$$\alpha = \frac{|b| + \sqrt{b^2 - v^2(n-2)^2}}{\sqrt{[|b| + \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.15}$$

$$\beta = -\frac{n-2}{\sqrt{[|b| + \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.16}$$

The value of γ can be obtained with the first order expansion for $|v| \to 0$, that gives $\alpha \to 1 + O(v^2)$, $\beta \to -\frac{n-2}{2b} + O(|v|)$ and then

$$w^l \to u^l - \frac{(n-2)}{yn - \frac{R}{(n-1)} + \gamma(n-2)} v^l \tag{3.3.17}$$

From (1.3.10) w^l should be $u^l + \frac{q^l}{p'+e} + O(q^2)$ in order to have a convergence between the energy (1.3.3) and particles (1.3.2) rests of frame stress-energy tensors, where, as seen above, $q^l = v^l \frac{n-2}{8\pi}$, $e = -\frac{n-1}{8\pi} y + \frac{R}{16\pi}$ and $p' \approx p = -\frac{1}{8\pi} y - \frac{R(n-3)}{16\pi(n-1)}$ since we are in a situation near to equilibrium. So we have

$$u^l + \frac{n-2}{-ny + \frac{R}{n-1}} v^l = w^l \tag{3.3.18}$$

$$\Rightarrow -(yn - \tfrac{R}{(n-1)} + \gamma(n-2)) = -ny + \tfrac{R}{n-1} \tag{3.3.19}$$

$$\Rightarrow \gamma = O(v) \tag{3.3.20}$$

and

$$2b = -8\pi(p+e) = ny - \frac{R}{n-1} \tag{3.3.21}$$

The eigenvalue γ must be at least of the same order of q, that is in accordance with the expectation of $\pi^{ij} = 0$ with perfect fluids. Disequalities (3.0.1) and (3.0.2) impose that, if $n \geq 3$, $-ny + \frac{R}{n-1} \geq 0$ and consequently $b \leq 0$. Thanks to this statement all the signs in previous results about w^l can now be fixed in the following way:

$$\alpha = \frac{b - \sqrt{b^2 - v^2(n-2)^2}}{\sqrt{[b - \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.22}$$

$$\beta = -\frac{n-2}{\sqrt{[b - \sqrt{b^2 - v^2(n-2)^2}]^2 - v^2(n-2)^2}} \tag{3.3.23}$$

$$W = \frac{(n-2)y + \frac{R}{n-1} - \gamma(n-2)}{2} - \sqrt{b^2 - (n-2)^2 v^2} \tag{3.3.24}$$

Once established the value of α, β and W as function of y, R, v_i and n, which depend all, except n, on the scalar function f and its derivatives, on the torse forming vector and on the curvature scalar R^* of the spatial submanifold M^*[4], the energy density in the energy rest frame can be deduced in a straightforward way and the three energy conditions for ordinary matter can be readapted for this situation.
From Einstein equation

$$\hat{e} = T_{ij}w^i w^j = \frac{-W + \frac{1}{2}R}{8\pi} \tag{3.3.25}$$

thus energy conditions respect to w^i are

$$-W + \frac{1}{2}R \geq 0 \qquad \textit{weak} \tag{3.3.26}$$

$$-W \geq 0 \qquad \textit{strong} \tag{3.3.27}$$

$$-W + \frac{1}{2}R \geq 0 \qquad \textit{dominant} \tag{3.3.28}$$

Clearly weak and dominant conditions bring to the same relation, because w^l is timelike by hypothesis and, if it is future-directed, the future-direction condition for the energy-momentum 4-current density of matter is equivalent to the request of $\hat{e} \geq 0$.
Now we know an eigenvalue of T_{ij}, and more in particular the eigenvalue with a timelike eigenvector, thus eq(3.3.26) represent also the first request in the eigenvalue's formulation of energy conditions (1.1.8) and(1.1.10).

3.3.1 CIT approximation

Hervik, Ortaggio and Wylleman showed in [15] the electric part of Weyl tensor, in a 4-dimensional spacetime, is proportional to Weyl two time contracted with u^i. Since a twisted spacetime is purely electric,

$$C^{ik}{}_{lj} = (C_+)^{ik}{}_{lj} = 4(h^{[i}{}_{[l} + u^{[i}u_{[l]})C_{|r|}{}^{k]}{}_{j]s}u^r u^s \tag{3.3.29}$$

When one tries to apply CIT approximation to a twisted spacetime, from constitutive equations,

$$C_{rkls}u^r u^s = \pi_{kl} = -2\eta\sigma_{kl} = 0 \tag{3.3.30}$$

That means in classical irreversible thermodynamics, from preceding equations, $C_{iklj} = 0$, thus they are equivalent to conformally flat spaces. A conformally flat manifold is a manifold where each point has a neighbourhood that can be mapped to a flat space by a conformal transformation, namely a transformation which preserves angles. In a simpler way, a spacetime is conformally flat if exists a system or reference where its metric is

$$g_{ij} = f(t, \vec{x})\eta_{ij} \tag{3.3.31}$$

with η_{ij} representing the Minkowski flat metric. But what does such condition mean in the family of spacetimes we are studying?
By a simple reparametrization with the conformal time

$$\eta = \int_0^t \frac{dt'}{f} \tag{3.3.32}$$

the twisted metric (2.0.1) looks like

$$ds^2 = f^2(t, \vec{x})(-d\eta^2 + g^*_{\mu\nu}dx^\mu dx^\nu) \tag{3.3.33}$$

A similar space is conformally flat if and only if exists a spatial set of coordinates where $g^*_{\mu\nu}$ is flat.
With this request a generic twisted spacetime corresponds to a fluid where all irreversible effects are caused by heat flow and q^i is proportional to the temperature gradient from the second constitutive equation. Moreover GRW spaces are equivalent to RW and represent perfect fluids. That is coherent with CIT's $R^i = \frac{q^i}{T}$: for a separation between GRW an RW spaces, it is necessary to consider stress tensor's effects in the evaluation of entropy current (as happens, for example, in Israel-Stewart theory), since $\pi_{ij} \neq 0$ is the only difference between these two categories.
The equations of motion of a fluid with zero anisotropic stress tensor are

$$D_i p + h^j{}_i \dot{q}_j + \frac{4}{3}\theta q_i = 0 \tag{3.3.34}$$

$$\dot{e} + \theta(e + p) + D_i q^i = 0 \tag{3.3.35}$$

From the point of view of energy velocity w^i, CIT approximation at the first order respect to q^i or equivalently $\hat{J}^i$ requests that

$$\tau_{ij} = -2\eta\hat{\sigma}_{ij} = -2\eta\left(\sigma_{ij} + \nabla_{<i}\frac{1}{e + p'}q_{j>}\right) \tag{3.3.36}$$

because covariant derivatives are linear operators. In this expression $<>$ represents the symmetric traceless part projected on the orthogonal plane. As anticipated, $\sigma_{ij} = 0$, so the stress tensor respect to Landau's velocity is composed only by the symmetric and traceless part of the covariant derivative of $\frac{q^i}{e+p'}$ projected on the plane orthogonal to the heat flux vector itself.

3.3.2 Zero anisotropic stress tensor

We have just shown that in a first order approximation some particles in a twisted spacetime behave like a fluid with zero anisotropic stress tensor, then it is interesting to look what happens when $C_{rkls}u^r u^s = 0$.
In the particles' frame dissipative effects are provoked by q^i, however in the energy frame there is not heat flux and the same irreversible phenomena will

be represented by a not null stress tensor τ_{ij}. Now we will try to write the exact form of this tensor.
First of all it is possible to introduce some new symbols to simplify successive calculations. We have already stressed that in order to have $w^i w_i = -1$ must be respected the equation $\alpha^2 - \beta^2 v^2 = 1$. It is possible to define a new vector z^i orthogonal to w^i with $z^i z_i = 1$:

$$\boxed{z^i = u^i v\beta + \frac{v^i}{v}\alpha} \tag{3.3.37}$$

Obviously the inverse formulae exist too and they are

$$u^i = w^i\alpha - z^i v\beta \tag{3.3.38}$$

$$\frac{v^i}{v} = -w^i v\beta + z^i\alpha \tag{3.3.39}$$

In the first chapter we have defined the component of particle's current not parallel to w^i with $\hat{J}^i = \rho(u^i - w^i)$. We can affirm that the just defined vector z^i is proportional to the projection on the plane orthogonal to w^i of $\hat{J}^i$, since $\hat{J}^i = \rho[w^i(\alpha - 1) - z^i v\beta]$ and thus

$$\hat{J}^i(g_{ij} + w_i w_j) = -z^i v\beta \tag{3.3.40}$$

The Ricci tensor in this subgroup of twisted spacetimes is

$$R_{ij} = \left(\frac{R}{n-1} - ny\right) u_i u_j + \left(\frac{R}{n-1} - y\right) g_{ij} + (n-2)(u_i v_j + u_j v_i) \tag{3.3.41}$$

but it can be written respect to the two new vectors w^i and z^i. If one requests that mixed terms with both z^i and w^i are null, R_{ij} becomes

$$\begin{aligned} R_{ij} = & \left[\left(\tfrac{R}{n-1} - ny\right)\alpha^2 - 2v^2(n-2)\alpha\beta\right] w^i w^j + \\ & + \left(\tfrac{R}{n-1} - y\right) g_{ij} + \left[\left(\tfrac{R}{n-1} - ny\right) v^2\beta^2 - 2v^2(n-2)\alpha\beta\right] z^i z^j \end{aligned} \tag{3.3.42}$$

From this form of R_{ij} it is simple to see that w^i is eigenvector of Ricci with eigenvalue

$$W = (n-1)y - \left(\frac{R}{n-1} - y\right) v^2\beta^2 + 2v^2(n-2)\beta\alpha \tag{3.3.43}$$

By using equality

$$v^2(n-2)^2 = -[b \pm \sqrt{b^2 - v^2(n-2)^2}]^2 + 2b[b \pm \sqrt{b^2 - v^2(n-2)^2}] \tag{3.3.44}$$

and making some computation, the latter expression for W results exactly equal to (3.3.24) with $\gamma = 0$. On the other side, the decomposition of a generic symmetric second order tensor (1.3.1) with eigenvector w^i is:

$$R_{ij} = \frac{R - Wn}{n-1} w_i w_j + \frac{R - W}{n-1} g_{ij} + \tau_{ij} \tag{3.3.45}$$

and then

$$\tau_{ij} = R_{ij} - \frac{R - Wn}{n-1} w_i w_j - \frac{R - W}{n-1} g_{ij} \tag{3.3.46}$$

A substitution with the preceding representation of R_{ij} gives

$$\tau_{ij} = \left(\frac{W}{n-1} - y \right) (g_{ij} + w_i w_j) - [W - y(n-1)] z_i z_j \tag{3.3.47}$$

After a first order evaluation with CIT in the preceding section, now we have been able to find out the exact form of the stress tensor in the energy frame when the tensor $C_{rkls} u^r u^s$ is null. τ_{ij} is traceless and $\tau_{ij} w^i = 0$, while z^i is an eigenvector of this tensor, since

$$\tau_{ij} z^i = -(n-2) \left(\frac{W}{n-1} - y \right) z_j \tag{3.3.48}$$

That implies z^i is an eigenvector of Ricci tensor too, this time with relative eigenvalue

$$\boxed{Z = \frac{R}{n-1} - W + (n-2) y} \tag{3.3.49}$$

With these calculations we have one of the spatial eigenvector of T_{ij}, thus some disequalities of the formulation through eigenvalues of the energy conditions can be written for this subset of twisted spacetime: weak strong and null conditions request

$$\hat{e} + Z - \frac{R}{2} \geq 0 \tag{3.3.50}$$

while the dominant can be fulfilled only if

$$-\hat{e} \leq Z - \frac{R}{2} \leq \hat{e} \tag{3.3.51}$$

Appendix A

Twisted spacetimes in torse forming's frame

Christoffel symbols: $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{lj} - \partial_l g_{ij})$

$$\Gamma^0{}_{i0} = \Gamma^k{}_{00} = 0, \quad \Gamma^\rho{}_{\mu 0} = (\dot f/f)\delta^\rho{}_\mu, \quad \Gamma^0{}_{\mu\nu} = f\dot f g^*_{\mu\nu} \tag{A.0.1}$$

$$\Gamma^\rho{}_{\mu\nu} = \Gamma^{*\rho}{}_{\mu\nu} + (f_\nu/f)\delta^\rho{}_\mu + (f_\mu/f)\delta^\rho{}_\nu - (f^\rho/f)g^*_{\mu\nu} \tag{A.0.2}$$

where $\dot f = \partial_t f$, $f_\mu = \partial_\mu f$ and $f^\mu = g^{*\mu\nu} f_\nu$.
Riemann tensor: $R_{jkl}{}^m = -\partial_j \Gamma^m_{kl} + \partial_k \Gamma^m_{jl} + \Gamma^p{}_{jl}\Gamma^m{}_{kp} + \Gamma^p{}_{kl}\Gamma^m{}_{jp}$

$$R_{\mu 0\rho}{}^0 = (f\ddot f) g^*_{\mu\rho} \tag{A.0.3}$$

$$R_{\mu\nu\rho}{}^0 = g^*_{\mu\rho}(f\partial_\nu \dot f - \dot f f_\nu) - g^*_{\nu\rho}(f\partial_\mu \dot f - \dot f f_\mu) \tag{A.0.4}$$

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma = R* \;\; \sigma_{\mu\nu\rho} + (\dot f^2 - \tfrac{f^\lambda f_\lambda}{f^2})(g^*_{\mu\rho}\delta^\sigma{}_\nu - g^*_{\nu\rho}\delta^\sigma{}_\mu) + \\ + \tfrac{2}{f^2}(f^\sigma f_\nu g^*_{\mu\rho} - f^\sigma f_\mu g^*_{\nu\rho} + f_\mu f_\rho \delta^\sigma{}_\nu - f_\nu f_\rho \delta^\sigma{}_\mu + \\ + \tfrac{1}{f}[\nabla^*_\mu(f^\sigma g^*_{\nu\rho} - f_\rho\delta^\sigma{}_\nu) - \nabla^*_\nu(f^\sigma g^*_{\mu\rho} - f_\rho \delta^\sigma{}_\mu)] \end{aligned} \tag{A.0.5}$$

Ricci tensor: $R_{jl} = R_{jkl}{}^k$

$$R_{00} = -(n-1)(\ddot f/f) \tag{A.0.6}$$

$$R_{\mu 0} = -(n-2)\partial_\mu(\dot f/f) \tag{A.0.7}$$

$$\begin{aligned} R_{\mu\nu} = R^*_{\mu\nu} + g^*_{\mu\nu}[(n-2)\dot f^2 + f\ddot f] + 2(n-3)\tfrac{f_\mu f_\nu}{f^2} + \\ -(n-4)\tfrac{f^\sigma f_\sigma}{f^2} g^*_{\mu\nu} - (n-3)\tfrac{1}{f}\nabla^*_\mu f_\nu - \tfrac{1}{f} g^*_{\mu\nu}\nabla^*_\sigma f^\sigma \end{aligned} \tag{A.0.8}$$

Curvature scalar: $R = R^k{}_k$

$$\begin{aligned} R = R^* + (n-1)\left[(n-2)\tfrac{\dot f^2}{f^2} + 2\tfrac{\ddot f}{f}\right] - (n-2)(n-5)\tfrac{f^\sigma f_\sigma}{f^4} + \\ -2(n-2)\tfrac{\nabla^*_\sigma f^\sigma}{f^3} \end{aligned} \tag{A.0.9}$$

Weyl tensor: $C_{jklm} = R_{jklm} - \frac{2}{n-2}(g_{j[l}R_{m]k} - g_{k[l}R_{m]j}) + \frac{2}{(n-1)(n-2)}Rg_{j[l}g_{m]k}$

$$-(n-2)C_{0\mu\nu 0} = R^*_{\mu\nu} - R^* \frac{g^*_{\mu\nu}}{n-1} + 2(n-3)\left[\frac{f_\mu f_\nu}{f^2} - \frac{g^*_{\mu\nu}}{n-1}\frac{f_\sigma f^\sigma}{f^2}\right] + \\ -(n-3)\left[\frac{\nabla^*_\mu f_\nu}{f} - \frac{g^*_{\mu\nu}}{n-1}\frac{\nabla^*_\sigma f^\sigma}{f}\right] \qquad \text{(A.0.10)}$$

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