Università degli Studi di Milano

FACOLTÀ DI SCIENZE E TECNOLOGIE
Corso di Laurea in Fisica

Tesi di laurea triennale

Slater decomposition of fractional quantum Hall states

Candidato:
Vittorio Erba
Matricola 837548

Relatore:
Prof. Luca Guido Molinari

Correlatore:
Dr. Pietro Rotondo

Anno Accademico 2015-2016

Abstract

Fractional quantum Hall effect was discovered more than 30 years ago by Tsui, Stormer and Gossard [TSG82], but because of its intrinsically strongly correlated nature, is not yet completely understood.

Among the relevant questions, it is not clear why the celebrated Laughlin's ansatz is such a good approximation for exact ground states of the system. From a more mathematical point of view, another unsolved problem is to represent this correlated wavefunction within the second quantization formalism, i.e. to understand its Slater decomposition. Many authors, for example [Dun93] and [DFGIL94], attacked this problem in the nineties without finding any conclusive answer.

New light on the problem was shed in 2008 by Haldane and Bernevig [BH08], who discovered that bosonic Laughlin's wavefunctions belong to a special class of symmetric functions, *Jack symmetric functions*, widely studied in the mathematical literature.

As a byproduct of this correspondence, they were able to discover a remarkable recurrence relation fulfilled by the coefficients of the monomial decomposition of Laughlin states. This recurrence relation can be non-trivially generalized to the fermionic case.

In this thesis, we systematically review these recent results, whose comprehension requires a deep knowledge of advanced purely mathematical literature on special orthogonal polynomials.

Contents

Chapter 1

Open Problem: Slater decomposition of fractional quantum Hall states

Consider a system of electrons confined in a plane and subject to a constant perpendicular magnetic field B. When a current flows through it, the sample will show the so called Hall effect [Hal79], i.e. will generate a transverse potential drop. The resistivity tensor ρ will be anisotrope; its transverse component ρ_{xy} is called Hall resistance.

Hall resistance is a function of the applied magnetic field; in different regimes, different behaviours arise. In the classical regime, one measures a linear dependence between ρ_{xy} and B. As the magnetic field grows ($B \sim$ T) and the temperature is lowered ($T \sim$ mK), plateaux appear at values h/ne^2, with n integer. This is explained by Landau level formation and impurity induced localization. In extreme regimes, a whole new variety of plateaux arise at values h/fe^2, with f suitable rational non-integer numbers.

It is widely accepted that the fractional plateaux are mainly related to the correlations between electrons induced by Coulomb repulsion. Moreover, it is believed that understanding the physics of the system in a strong field limit, which freezes the dynamics of the system to the lowest energy level (lowest Landau level, LLL), could be enough to understand the whole fractional behaviour.

The high degeneracy of LLL is usually solved by choosing a particular geometry along with a suitable gauge. In disk geometry with symmetric gauge, the LLL Hilbert space is generated by properly symmetrized homogeneous monomials, apart from a non relevant exponential factor and a normalization constant, i.e.

$$|\lambda_1 \dots \lambda_N\rangle = \mathcal{S}/\mathcal{A}[z_1^{\lambda_1} \dots z_N^{\lambda_N}],$$

where $\mathcal{S}/\mathcal{A}$ denotes symmetrization/antisymmetrization.

In 1983, Laughlin proposed [Lau83] a variational ansatz for the ground state of the simplest filling fractions $\nu = 1/q$ observed in the experiments (in particular 1/3). It is based on qualitative assumptions: it's a polynomial (i.e. describes a LLL state), it has definite angular momentum (i.e. it's an homogeneous polynomial), it has poles every time two particles meet and respects Pauli principle (q is an integer, odd for fermions and even for bosons):

$$\phi^L_{GS}(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^q.$$

This family of wavefunctions has been particularly successful in describing fermionic systems at filling fraction $1/q$, with extremely high overlaps with computed ground states; nevertheless, they are not exact groundstates themselves.

It is still not clear why these wavefunctions work so well for the fractional quantum Hall effect problem. Some hints on the problem may be given by a second quantization form of Laughlin's wavefunctions, i.e. by their expansion on the LLL non interacting basis of homogeneous monomials.

To be more concrete, here is an example of such an expansion (manually computed), for 3 particles and for $q = 3$:

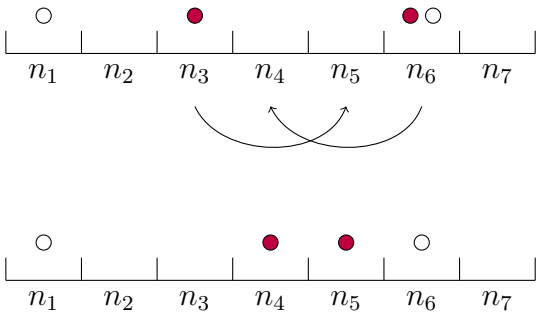
$$\begin{aligned}\phi^L_{GS}(z_1, z_2, z_3) &= [(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)]^3 \\ &= sl_{(6,3,0)} - 3sl_{(6,2,1)} - 3sl_{(5,4,1)} + 6sl_{(5,3,1)} - 15sl_{(4,3,2)}.\end{aligned}$$

Other examples can be found in [Dun93]. It can be proved that the number of slater determinants of the decomposition grows exponentially in the system size N.

A reasonable framework to work with LLL in the symmetric gauge and with Laughlin's wavefunctions is given by symmetric functions theory (restricting for the moment to bosons). Symmetric functions are a generalization of polynomials, in the sense that they can be thought as polynomials in an indeterminate number of variables.

Many features of fractional quantum Hall effect can be recognised in this framework:

- symmetric monomials appear as a natural basis for symmetric functions (see [Sta89] or [Mac95] for example);
- symmetric functions are usually parametrized by partitions, which can be thought as the quantum numbers $\lambda_1 \ldots \lambda_N$ of the LLL basis;
- bosonic Laughin's wavefunctions are symmetric functions.



Example of squeezing

Dots represent electrons, n_i's occupation numbers.
Red electrons are being squeezed by 2 units, i.e. their angular momenta are being modified by two units.
Notice that only *inward* squeezings are considered.

The last point is of particual interest. Dunne [Dun93] proved some results about Laughlin's decomposition on the Schur functions, a particular family of symmetric functions. In [DFGIL94], some sum rules for the coefficients of the Slater expansion were found. Nevertheless, the first big progress on the problem was made by Bernevig and Haldane [BH08], who recognised Laughlin's wavefunctions to be Jack polynomials.

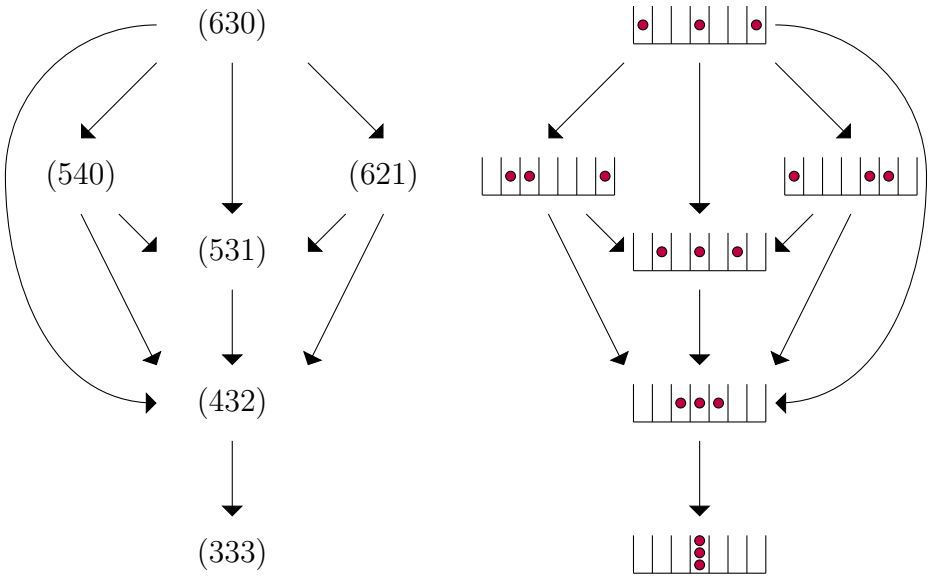
Jack polynomials are a set of symmetric functions already well known for their relevance in integrable systems theory (they are excitations of the Sutherland model, i.e. eigenvalues of Laplace Beltrami differential operator). They are parametrized by a real parameter and a partition. Their intrinsic properties allowed proving a recursion law [TERB11] for the coefficients of the expansion of Laughlin's wavefunctions over the monomial basis.

The idea is based on the concept of squeezing. Consider a particular configuration of the LLL, i.e. n_0 electrons with $L_z = 0$, n_1 with $L_z = \hbar$ etc.... Then, a squeezed configuration consists of the same distribution of electrons, apart from a finite number of pairs of particles. For each of these pairs, the particle with highest angular momentum has it reduced of a number of units, while the other has it augmented of the same amount.

Squeezing allows for the introduction of a partial ordering relation (indicated by the usual ordering symbols) over the set of partitions. Moreover, one has:

- Jack polynomials (of partition λ) expand over the monomial basis with non null coefficients only for partition squeezed from λ:

$$J_\lambda^\alpha = \sum_{\mu \leq \lambda} c_{\mu\lambda} m_\mu;$$



Squeezed partitions from (630)

Given a partition $\lambda = (630)$, i.e. a configuration with 3 electrons with momenta 0, $3\hbar$ and $6\hbar$, one can identify all possible squeezed partition that are generated by λ. Reiterating the process, one finds a chain of partitions.
On the left, partition language is used. For a clearer physical picture, on the right the corresponding particle configurations are shown.

- Laplace Beltrami operator acts on the λ monomial generating a sum over squeezed partitions:

$$H_{LB} m_\lambda = \sum_{\mu \leq \lambda} x_{\mu\lambda} m_\mu .$$

It's the interplay of these two properties that allows proving recursion laws. A concrete example is given by the above computed expansion of Laughin's wavefunction for 3 particles and $q = 3$. In fact, the partition labeling it as a Jack polynomial is (630), and its squeezed partitions are exactly those appearing in the expansion, as in Fig. (**??**).

The aim of this thesis is to review some of the useful results of quantum Hall effect (first chapter), symmetric functions and Jack polynomials (second chapter). In the third chapter, results from [TERB11] will be presented with detailed calculations.

Chapter 2

Quantum Hall Effect

This chapter focuses on quantum Hall effects. First, an historical section is given to sum up qualitative and experimental features of quantum Hall effects. Next, Landau levels theory is explained and some quantitative insights of quantum Hall effects are given.

The main results introduced in the chapter are:

- the basis for the lowest Landau level is given by properly symmetrized monomials, apart from an exponential factor;
- Laughlin's wavefuntions are good approximations for quantum Hall effect groundstates.

2.1 Brief Hystory of Hall Effects

Hall effect is a long known physical phenomenon that has recently revealed surprising features.

Hall effect was first discovered by E. H. Hall in 1879. He noticed that a conductor, under the effect of an electric and a magnetic field perpendicular one to the other, exibits a current flow in the direction orthogonal to both the fields. Conversely, a magnetic field and a current perpendicular to each other generate a voltage in the third direction (see Fig. (2.1.1)).

Carrying out the classical theory, the Hall resistivity is defined as the ρ_{xy} component of the resistivity tensor relating the current density causing the Hall effect and the induced electric field. One can see (see [Jai07] pg.13 for example) that the Hall resitivity equals

$$\rho_H = \rho_{xy} = \frac{B}{\rho q c}, \tag{2.1}$$

Figure 2.1.1: The classical Hall effect

The picture on the left represents the Hall induced current I due to a magnetic field $\vec{B}$ and a voltage difference ΔV. The picture on the right shows the opposite effect, i.e. the Hall induced voltage difference ΔV due to a magnetic field $\vec{B}$ and a current I.

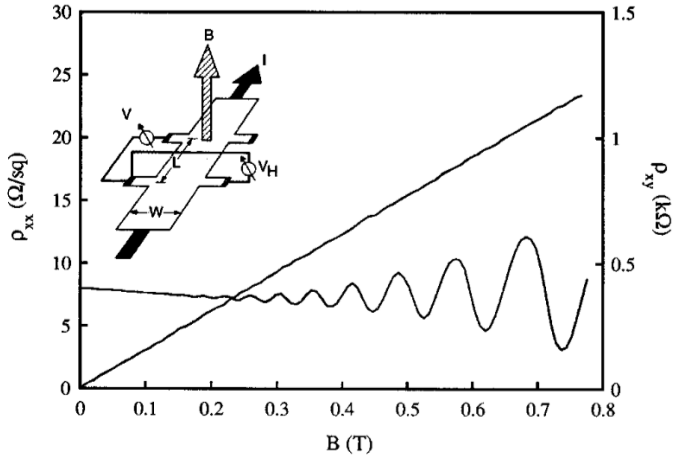


Figure 2.1.2: Classical Hall effect: plot of the resistivity

Plot of the two relevant components of the resistivity tensor as functions of the magnetic field. ρ_{xy} show the expected linear dependence. The plot is taken from [Tsu83].

where ρ is the density of charge carriers, q is their charge and c is the speed of light; ρ_H is linear in B.

In 1980, Von Klitzing and his collaborators observed the first significant deviation from this simple model. They were studying the Hall effect caused by two-dimensional electrons in silicon MOSFET (metal oxide-semiconductor field-effect transistor); as the magnetic field strength was raised, they observed the simple linear dependence of ρ_{xy} with respect to B deform to show plateaux (see Fig. (2.1.3)) at quantized values given by h/ne^2 (independent from the sample).

This phenomenon is called Integer Quantum Hall Effect (IQHE). IQHE can be justified in an independent electron picture: the magnetic field quantizes electrons' energy, creating a sequence of conduction states and localized states (due to impurities). This alternation explains both the plateaux formation and the universal value for the quantum of Hall resistivity.

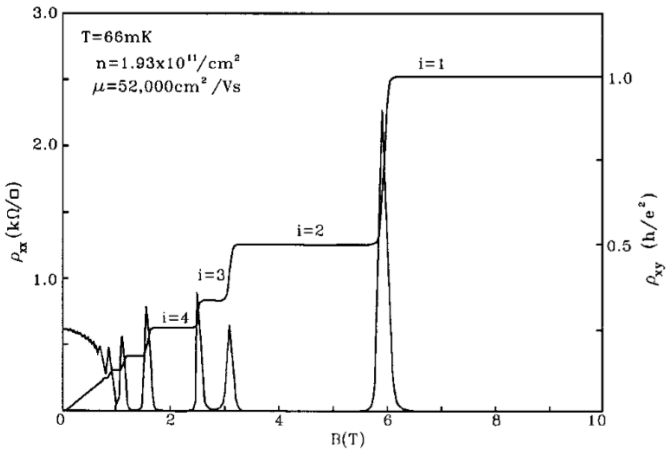


Figure 2.1.3: Integer quantum Hall effect: plot of the resistivity

Plot of the two relevant components of the resistivity tensor as functions of the magnetic field. As the magnetic field is increased, ρ_{xy} shows the characteristic plateaux. The plot is taken from [Tsu83].

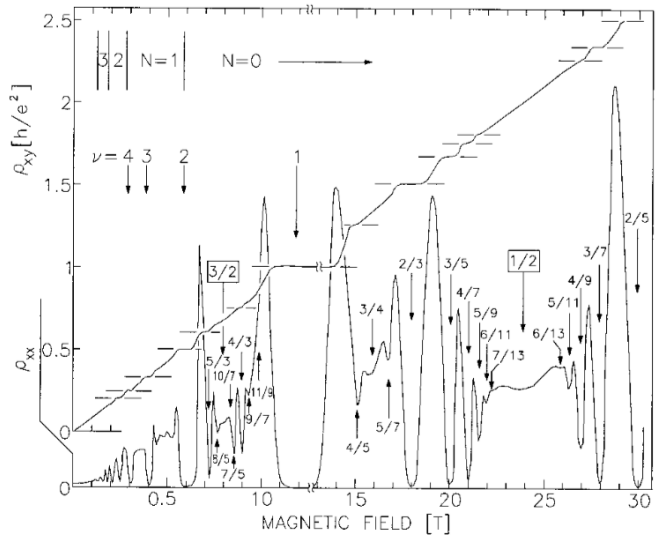


Figure 2.1.4: Fractional quantum Hall effect: plot of the resistivity

Plot of the two relevant components of the resistivity tensor as functions of the magnetic field. In critical conditions for the magnetic field and the temperature, a variety of fractional plateaux appear. The plot is taken from [Tsu83].

Soon, as the experimental setups allowed for stronger magnetic fields and lower temperatures, a variety of new plateaux appeared at *fractional* multiples of h/e^2 (with fractions of the form integer divided by odd integer).

This effect is called Fractional Quantum Hall Effect (FQHE). FQHE cannot be justified without taking into account the strong correlation between electrons due to Coulomb repulsion. This new effect can be seen as the IQHE for a new kind of quasi-particles, *composite fermions*; this theory will not be treated here (see [Jai07]).

A fundamental result in the study of FQHE was given by Laughlin in 1983 (see [Lau83]): he proposed an ansatz for the wavefunction of the ground state of the FQHE system. This ansatz is meaningful, simple to write and overlaps extremely well with computed ground states (for a small number of electrons); nevertheless, it is not the exact ground state. It's unclear why this ansatz should work so well as the ground state of FQHE systems.

2.2 Motion of an electron in a magnetic field: Landau Levels

In this section, a physical system of one electron moving in the xy plane and subject to a $\vec{B} = B\vec{z}$ magnetic field is studied.

2.2.1 Hamiltonian for the free electron

The Hamiltonian for a free electron in a magnetic field is

$$H = \frac{1}{2m_e}\left(\vec{p} - e\vec{A}\right)^2 + g\mu_B \vec{s}\cdot\vec{B}, \tag{2.2}$$

where:

- e is the electron charge ($e < 0$);
- m_e is the electron mass; in the case of an electron in a periodic external potential, as in crystal structures, it is its effective mass;
- $\vec{p}$ is the canonical momentum operator;
- $\vec{A}$ is the vector potential operator;
- g is the Landé factor for the electron;
- μ_B is the Bohr magneton;

- $\vec{s}$ is the elctron spin operator.

The interaction between the spin of the electron and the magnetic field leads to a term proportional to s_z. As it will be discussed, in the strong field limit the electron is confined in the spin down eigenstate; for this reason, the spin dependence will be ignored.

In the Hamiltonian, the canonical momentum operator $\vec{p}$ appears, with the usual commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. In absence of external magnetic field, this operator conicides with the linear momentum $m_e\vec{v}$ and with the generator of translations $\vec{K}$. When an external magnetic field is turned on, this coincidence holds no more.

The linear momentum is given by

$$\vec{\pi} = m_e\vec{v} = m_e\frac{i}{\hbar}[H, \vec{r}] = \vec{p} - e\vec{A}, \tag{2.3}$$

with commutation relations

$$[\pi_x, \pi_y] = i\hbar eB = -i\frac{\hbar^2}{l^2}. \tag{2.4}$$

Above, the magnetic lenght l was introduced ($l = \sqrt{\hbar/|e|B}$).

The generator of translations (called *pseudomomentum*) is given by (see [Yos02], pg. 20)

$$\vec{K} = \vec{p} - e\vec{A} + e\vec{B} \times \vec{r} = \vec{\pi} + e\vec{B} \times \vec{r}, \tag{2.5}$$

with commutation realations

$$[K_x, K_y] = i\frac{\hbar^2}{l^2}. \tag{2.6}$$

An interesting feature of the non commutativity of $\vec{K}$ components is that translation operators doesn't commute.

One finally has:

$$[\pi_i, K_j] = [\pi_i, \pi_j]. \tag{2.7}$$

2.2.2 Energy and Angular Momentum spectra

Using the linear momentum components, one can rewrite the Hamiltonian as

$$H = \frac{1}{2m_e}\left(\pi_x^2 + \pi_y^2\right). \tag{2.8}$$

Since $\vec{\pi}$ components are a pair of canonical observables, one recognises H to be in the form of an 1D harmonic oscillator and tries to construct ladder operators.

First, another pair of canonical observables must be chosen: usually, the center coordinate operators (X, Y) are used. They are defined as

$$\vec{r} = \left(X + \frac{l^2}{\hbar}\pi_y, Y - \frac{l^2}{\hbar}\pi_x \right), \tag{2.9}$$

with commutation relations

$$[X, Y] = il^2. \tag{2.10}$$

They represent the center for the electron's classical cyclotron motion. Notice also that they commute with the linear momentum components and thus with the Hamiltonian.

Having chosen two pairs of canonical operators, mutually commuting, one has a complete set of coordinates to describe the problem.

Ladder operators for the linear momentum components are introduced as

$$\begin{aligned} a &= \frac{l}{\sqrt{2}\hbar}\left(\pi_x - i\pi_y\right), \\ a^\dagger &= \frac{l}{\sqrt{2}\hbar}\left(\pi_x + i\pi_y\right), \end{aligned} \tag{2.11}$$

such that the Hamiltonian can be rewritten, introducing the cyclotron frequency $\omega_c = |e|B/m_e$, as

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right). \tag{2.12}$$

The energy spectrum is $E_n = \hbar\omega_c\left(n + 1/2\right)$ for $n \geq 0$, with an additional degeneracy with respect to the freedom in X and Y, due to the fact that the center coordinates commute with H.

These energy levels are called Landau Levels.

To examine the degeneracy induced by X and Y, the symmetric gauge is choosen, i.e. $\vec{A} = (-By/2, Bx/2, 0)$. In this gauge, the z component of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is given by

$$L_z = -\frac{\hbar}{2l^2}\left(X^2 + Y^2\right) + \frac{l^2}{2\hbar}\left(\pi_x^2 + \pi_y^2\right) = -\frac{\hbar}{2l^2}\left(X^2 + Y^2\right) + \frac{1}{\omega_c}H. \tag{2.13}$$

L_z commutes with H since it is function of H, X and Y. Moreover, the first term is again in the form of a 1D harmonic oscillator. Ladder operators for X and Y are introduced as

$$\begin{aligned} b &= \frac{1}{\sqrt{2}l}\left(X + iY\right), \\ b^\dagger &= \frac{1}{\sqrt{2}l}\left(X - iY\right), \end{aligned} \tag{2.14}$$

such that L_z can be rewritten as

$$L_z = \hbar\left(a^\dagger a - b^\dagger b\right), \tag{2.15}$$

with spectrum $\hbar\left(n - m\right)$ for $n, m \geq 0$.

In the symmetric gauge, the electron state is then characterized by two integer quantum numbers, one identifying the Landau Level, the other the angular momentum.

2.2.3 Schrödinger representation for the symmetric gauge

To give a coordinate representation of the electron states found in an abstract way in the previous sections, one has to express the two pairs of ladder operators as functions of $\vec{r}$ components and their respective derivations.

$$\begin{aligned} a &= \frac{-i}{\sqrt{2}}\left[\frac{1}{2l}\left(x - iy\right) + l\left(\partial_x - i\partial_y\right)\right], \\ b &= \frac{1}{\sqrt{2}}\left[\frac{1}{2l}\left(x + iy\right) + l\left(\partial_x + i\partial_y\right)\right]. \end{aligned} \tag{2.16}$$

Substantial semplifications arise passing to the complex variables $z = (x - iy)/l$ and $z^* = (x + iy)/l$ and introducing exponential terms:

$$\begin{aligned} a &= -i\sqrt{2}e^{-\frac{|z|^2}{4}}\partial_{z^*}e^{+\frac{|z|^2}{4}}, \\ a^\dagger &= \frac{i}{\sqrt{2}}e^{-\frac{|z|^2}{4}}\left(z^* - 2\partial_z\right)e^{+\frac{|z|^2}{4}}, \\ b &= \sqrt{2}e^{-\frac{|z|^2}{4}}\partial_z e^{+\frac{|z|^2}{4}}, \\ b^\dagger &= \frac{1}{\sqrt{2}}e^{-\frac{|z|^2}{4}}\left(z - 2\partial_{z^*}\right)e^{+\frac{|z|^2}{4}}. \end{aligned} \tag{2.17}$$

The ground state wave function $\psi_{0,0}(\vec{r}) = \langle \vec{r}|0,0\rangle$ is given by solving $a\,|0,0\rangle = 0$ and $b\,|0,0\rangle = 0$. These equations state that $\psi_{0,0}$ must be constant function in both z and z^* (i.e. a numerical constant) multiplied by an exponential term $e^{-\frac{|z|^2}{4}}$. Normalization and real choice of phase give

$$\psi_{0,0}(\vec{r}) = \frac{1}{\sqrt{2\pi}l} e^{-\frac{|z|^2}{4}}. \tag{2.18}$$

All the other simultaneous eigenfunctions of H and L_z are built from the ground state as

$$\psi_{n,m}(\vec{r}) = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} \frac{\left(b^\dagger\right)^m}{\sqrt{m!}} \psi_{0,0}(\vec{r}). \tag{2.19}$$

The explicit form for $\psi_{n,m}$ can be found in [Yos02], pg. 24. As it will be shown, in the strong field limit one can restrict the analysis to the Lowest Landau Level (LLL), i.e. to the subspace generated by $\psi_{0,m}$; the explicit form for this particular subset of eigenfunctions is useful:

$$\psi_{0,m}(\vec{r}) = \frac{1}{\sqrt{2^{m+1}\pi m! l}} z^m e^{-\frac{|z|^2}{4}}. \tag{2.20}$$

Notice that the action of $b^\dagger$ restricted to the LLL (generated by functions of the z variable only) is a multiplication by $z/\sqrt{2}$.

Given an explicit coordinate representation for the energy eigenfunction, one can also explicitly count the available states per level, given a particular finite geometry.

The symmetric gauge is particularly suited for the so called disk geometry: the xy plane is reduced to a single disk of radius R. The probability distributions of energy eigenfunctions $|\psi_{n,m}(\vec{r})|^2$ are peaked on a circle of radius $\sqrt{2m}\,l$, so that the maximum angular momentum allowed is given by $m = \lfloor R^2/2l^2 \rfloor$ ($\lfloor\ \rfloor$ is the floor function). The degeneracy per unit area is $(2\pi l^2)^{-1} = eB/hc$.

A particularly useful thing to notice is that all the eigenfunctions $\psi_{n,m}$ have the same exponential factor, which is also applied to the ladder operators a and b as a similarity transformation. This observation leads to consider the exponential factor as part of the measure of the integral defining the scalar product. Such a change of measure allows dropping the exponentials both in the eigenfunctions and in the ladder operators.

In particular, one has

$$\begin{aligned} b &= \sqrt{2}\partial_z, \\ b^\dagger &= \frac{1}{\sqrt{2}}\left(z - 2\partial_{z^*}\right) \overset{LLL}{=} \frac{z}{\sqrt{2}}, \end{aligned} \tag{2.21}$$

$$\psi_{0,m}(\vec{r}) = \frac{z^m}{\sqrt{2^{m+1}\pi m! l}}. \tag{2.22}$$

The modified wave functions are polynomials in z, and the ladder operators appear in a particularly simple form.

2.2.4 Limit of strong field

The limit of strong field is fundamental in the study of the quantum Hall effects. Here, this limit is not given a physical justification, which is going to be discussed later. Instead, its effects on the single electron Hamiltonian are described.

From expression (2.2) and (2.12) one can deduce a linear dependence of H from B, not only for the orbital part, but also for the spin part.

This implies that, in the strong field limit, the electron will be confined in the LLL and in the spin ground state, i.e. the spin down state. It's only freedom is related to the degeneracy of the LLL.

2.3 Many particle problem

In the following, electrons will be substituted with bosons/fermions with the same charge and mass of the electron for the sake of generality.

When treating the many particle problem, one has to take into account the single particle Hamiltonians, which describe the independent behaviours of the particles, and the interaction Hamiltonian. For electrons, the interaction term is the Coulomb interaction $V = \sum_{i<j} \frac{e^2}{|\vec{r_i} - \vec{r_j}|}$.

The new Hamiltonian is then

$$H = \sum_{i=0}^{N} H_i + \sum_{i<j} \frac{e^2}{|\vec{r_i} - \vec{r_j}|}, \tag{2.23}$$

where H_i is the single particle Hamiltonian studied in the preceding section.

A basis of the N LLL particles Hilbert space is given by the tensor product of N single particle bases of the LLL, symmetrized or antisymmetrized according to the statistics obeyed by particles:

$$\begin{aligned}\Psi^{\mathcal{S}/\mathcal{A}}_{m_1,\dots,m_N}(\vec{r_1},\dots,\vec{r_N}) &= \mathcal{N}_{\mathcal{S}/\mathcal{A}}\,\mathcal{S}/\mathcal{A}\left[\psi_{m_1}(\vec{r_1})\dots\psi_{m_N}(\vec{r_N})\right] \\ &= \mathcal{N}_{\mathcal{S}/\mathcal{A}}\,\mathcal{S}/\mathcal{A}\left[\frac{z_1^m}{\sqrt{2^{m_1+1}\pi m_1! l}}\dots\frac{z_N^m}{\sqrt{2^{m_N+1}\pi m_N! l}}\right]\exp\left[\sum_{i=1}^{N}\frac{|z_i|^2}{4}\right],\end{aligned} \tag{2.24}$$

where N is the total number of particles and $\mathcal{S}/\mathcal{A}$ are the symmetrization and antisymmetrization operators. Notice that these wave functions are not normalized; one needs the extra term $\mathcal{N}_{\mathcal{S}/\mathcal{A}}$ to correct for permutations:

$$\mathcal{N}_{\mathcal{S}/\mathcal{A}} = \sqrt{\frac{N!}{n_1!n_2!\dots n_\infty!}}, \tag{2.25}$$

where n_i is the number of particles in the same state m_i.

The same logic described in section 2.2.3 applies here; thus, the exponentials can be dropped:

$$\begin{aligned}\Psi^{\mathcal{S}/\mathcal{A}}_{m_1,\dots,m_N}(\vec{r_1},\dots,\vec{r_N}) &= \mathcal{N}_{\mathcal{S}/\mathcal{A}}\,\mathcal{S}/\mathcal{A}\left[\frac{z_1^m}{\sqrt{2^{m_1+1}\pi m_1! l}}\dots\frac{z_N^m}{\sqrt{2^{m_N+1}\pi m_N! l}}\right] \\ &= \frac{\mathcal{N}_{\mathcal{S}/\mathcal{A}}}{\sqrt{2^{M+N}\pi^N m_1!\dots m_N! l^N}}\mathcal{S}/\mathcal{A}\left[z^{m_1}\dots z^{m_N}\right],\end{aligned} \tag{2.26}$$

where the total momentum $M = \sum_{i=1}^{N} m_i$ was introduced.

These wave functions, eigenstates of the independent particle problem, are of particular interest. In fact, one recognises that $\mathcal{S}/\mathcal{A}\left[z^{m_1}\dots z^{m_N}\right]$ are the **symmetric/antisymmetric monomials of degree M in N variables**.

In general, N particles states will be a linear combinations of these independent particles bases.

2.4 Integer Quantum Hall Effect

Integer quantum Hall effect is caused by the interplay of two mechanisms: formation of Landau Levels and disorder. While disorder is fundamental to explain plateaux formation in the IQHE, it's really the underlying structure of Landau Levels that triggers Hall resistance quantization.

In experimental conditions, electrons are not only affected by the magnetic field and the mutual repulsion. An important contribution is given by the background potential generated by ions and impurities.

Thus, the background to the Hamiltonian (2.23) is not a constant potential; more reasonably, one can model it as a slowly varing (with respect to the magnetic lenght) smooth potential.

One can expect that this background will solve at least partially the high degeneracy of Landau levels, broadening them into a energy spectrum without any gap.

The main result, here just sketched (see [Jai07] for a complete overview), is that only those states relative to unperturbed energy are extended, i.e. not localized in an area much smaller than the sample.

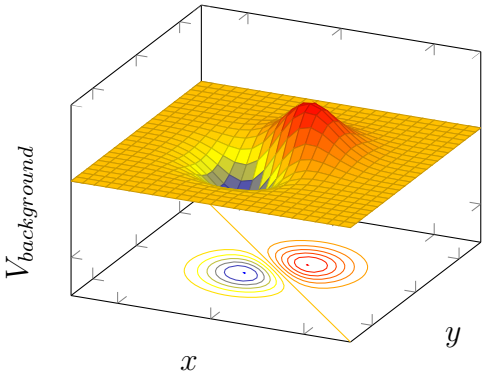


Figure 2.4.1: Example of background potential

The lower contour plot shows equipotential lines, along which electrons move. The straight line represents the behaviour of an electron in the unperturbed Landau level. The closed lines are the semiclassical trajectories of electrons in the gap between Landau levels.

A semiclassical picture useful to justify this claim is the following. The background potential creates a "frame" for the motion of the electron composed by peaks and valleys. If the electron's energy coincides with one of the unperturbed Landau levels, the electron will be unaffected by those background features and will be in an extended state. If the electron's energy is in the gap between unperturbed Landau levels, it will be automatically confined on peaks or hills, moving along closed equipotential lines. Thus, its state will be localized (see Fig. 2.4.1).

To show Hall resistance quantization, it is useful to switch from fixed number of electrons and variable magnetic field to the opposite but equivalent picture.

Fixing B, the system is gradually filled with electrons, and so the Fermi energy will increase. As long as the Fermi energy ϵ_F is in a gap between Landau Levels, the contribution to the conduction mechanism will be constant, being generated by the same amount of extended states. Only when the Fermi energy crosses a Landau level, one would observe a change in the conductance, hence in the Hall resistance.

The number of filled Landau levels is called *filling factor* ν, and equals the total electron density divided by the degeneracy per unit area, $\nu = \frac{\rho hc}{eB}$.

Disorder explains Hall resistance quantization. Still a big question is to be addressed: why R_H has such a universal value, independent from all the sample-dependent features (material, impurities, etc...)?

Moreover: in the classical picture, one finds that $R_H = B/\rho ec$. Following the previous section, one could expect to find the correct R_H value using

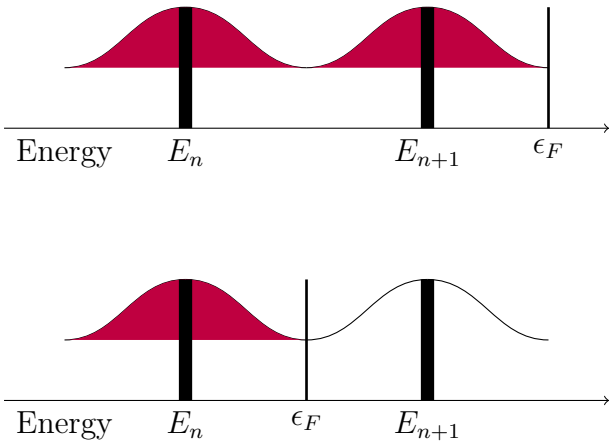


Figure 2.4.2: Progressive filling of broadened Landau levels

The sine like profile indicates the energy level degeneracy as a function of energy, peaked around Landau levels (shown as broad black lines). The shading represents the filling of the system.
Only when the Fermi energy ϵ_F crosses a Landau level the resistance changes.

$\rho = \rho_{extended}$. As the electron density in extended states is very small and sample-dependent, R_H could be very large, and highly dependent on disorder features.

This puzzle was solved by Laughlin, who showed a remarkable fact: R_H is quantized at h/ne^2 as long as $E_n < \epsilon_F < E_{n+1}$.

Intuitively: a disorder-free system with $\nu = n$ has exactly $R_H = B/\rho ec = h/ne^2$, as all the electrons contribute to conduction. The Hall resistance will not change if a sufficently weak disorder is taken into account, such that no mixing occurs between different Landau levels. Adding some electrons, i.e. tuning ν away from n, will neither change R_H, beacuse the procedure will just result in localized charge carriers.

2.5 Fractional Quantum Hall Effect

If Landau levels were the unique feature underlying QHE, no fractional quantization could be observed.

Fractional quantum Hall effect (FQHE) is the expression of correlation in electrons' behaviour, due to Coulomb repulsion.

As above, detailed treatment of the topic can be found in [Jai07] or [Yos02].

From now on, the strong field limit will be used, to restrict the Hilbert space of states to the LLL. In fact, filled Landau levels are believed not to

contribute to FQHE plateaux formation. It seems a good approximation to study electrons' correlations in a single Landau level, namely the lowest one.

The LLL restriction can be carried on in two ways: the simplest one is to restrict the analysis to the space generated by the LLL eignefunctions. This approach is particularly easy: in fact LLL is generated by properly symmetrized or antisymmetrized homogeneus monomials of at most degree m (highest possible L_z eigenvalue) in N variables (where N is the number of electrons in the sample). The other approach consists in projecting all the operators using a similarity transformation. The problem is that LLL projection operator is not simple, and the projected Hamiltonian is not easy to be used (at least in the usual first quantization coordinate representation).

2.5.1 Laughlin's Ansatz

A relevant quantity of theory has been built to explain FQHE. One of the first successful approaches is the so called Laughlin's Ansatz.

Driven by the use of Ritz variational principle, Laughlin proposed a simple yet meaningful ansatz $\phi_{GS}(z_1, \ldots, z_N)$ for the ground state of N interacting electrons.

Here the main ideas are reported:

- the LLL is generated by monomials (see Eq. (2.26)). If no geometric requirement is given, the LLL is an infinite dimensional Hilbert space, and its generic element would be a non-trivial function. Disk geometry, or other geometries with bounded surface, will lower the LLL degeneracy below infinity; in this case, the generic LLL function would be a polynomial in the N complex variables z_i. Keeping in mind disk geometry, $\phi_{GS}(z_1, \ldots, z_N)$ will be a polynomial;

- homogeneus polynomials are eigenfunctions of L_z, with the degree of the polynomial as eigenvalue (in the LLL L_z is represented by $\sum z_i \partial_{z_i}$). As Coulomb interaction commutes with L_z, a simultaneous basis to diagonalize both of them can be chosen. Thus, $\phi_{GS}(z_1, \ldots, z_N)$ will be an homogeneous polynomial of degree M (the total angular momentum);

- electrons are fermions, so Pauli principle implies that $\phi_{GS}(z_1, \ldots, z_N)$ will be an homogeneous antisymmetric polynomial of deegree M;

- Coulomb interaction is strongly repulsive at short distances. One can reasonably assume that $\phi_{GS}(z_1, \ldots, z_N)$ should have nodes every time two coordinates coincide. This implies that $\phi_{GS}(z_1, \ldots, z_N)$ should be divisible by the Vandermonde determinant, i.e. $\prod_{i<j}(z_i - z_j)$.

Combining all the observations and assumptions above, one finds the so called Laughlin's wavefunction

$$\phi^L_{GS}(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^q, \tag{2.27}$$

with q arbitrary odd integer.

At first sight, one could think of q as a variational parameter to minimize the energy functional. It's not the case: q is determined by the filling fraction ν. First of all, it is to be noticed that ϕ^L_{GS} contains $z_i^{q(N-1)}$ as the highest power of each coordinate; this means that each electron can have at most $L_z = q(N-1)$, i.e. his density distribution is peaked on a circle of radius at most $\sqrt{2(q(N-1)+1)}l$ (see Sec. 2.2.3). The enclosed area is $2(q(N-1)+1)\pi l^2$. Assuming a uniform density of electrons on the circle, one has $\nu = 2\pi l^2 N/S \overset{N\to\infty}{\longrightarrow} 1/q$.

Laughin's wavefunctions are thus modeling FQHE ground states for a specific series of fractional plateaux, not for all of them. Other model wavefunctions exists for different filling fractions, which will not be treated here.

Laughlin's wave functions has proven to be an enourmously good ansatz for $1/q$ filling fractions. Section 4.4 of [Yos02] provides some insights. Two points are of fundamental interest:

- Laughlin's wave functions is not the exact groundstate for the FQHE Hamiltonian. Laughlin's wavefunctions are ground states of particular Hamiltonians (called Model Hamiltonians) built using Haldane pseudopotentials (see [Jai07] or [Yos02]);

- Laughlin's wave functions overlaps extremely well ($\sim 99\%$) with the exact ground state of FQHE Hamiltonian, computed numerically for a small number of electrons (~ 10).

Chapter 3

Symmetric Functions and Jack Polynomials

Jack Polynomials belong to a wide class of special functions, the Symmetric Functions. Some preliminaries will be developed to define Jack Polynomials and to show some of their properties.

Two key results will be presented:

- Laughlin's bosonic states are Jack Polynomials;
- it is possible to generalize Jack Polynomials to take care also of fermionic Laughlin's states.

3.1 Partitions

A partition λ is a non increasing, definitively null sequence of integers. The finite number of non null entries is called the *lenght* $l(\lambda)$ of the partition.

Partitions are represented in various ways:

- indicating all the non null (and possibly also part of the null) entries in decreasing order, $(\lambda_1, \dots, \lambda_{l(\lambda)}, 0, \dots, 0)$;
- indicating all the entries in a contracted form and in decreasing order, using exponents to indicate the number of repetitions of a certain component, $(\lambda_1^{e_1}, \dots, \lambda_{l(\lambda)}^{e_{l(\lambda)}})$;
- using the multeplicities $n_i(\lambda)$ (number of appearences of i in λ), $(n_1(\lambda), n_2(\lambda), \dots, n_i(\lambda), \dots)$;
- using a Young tableaux, a diagram composed by vertically stacked, left aligned rows of squares. The i-th row is λ_i squares long.

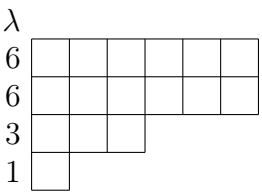


Figure 3.1.1: Young tableau representation of $(6, 6, 3, 1, \dots)$

For example, the partition $(6631\dots)$ can be written also as $(6^2 3^1 1^1)$, $(1010020\dots)$ or as in Fig. (3.1.1).

The sum $\sum_{i=0}^{l(\lambda)} \lambda_i$ is often referred to as $|\lambda|$.

Partitions are pure combinatorial objects. Nevertheless, they are particularly useful. For example, Young tableaux and several bases of polynomial spaces are in one to one correspondence with partitions.

It is important to find a meaningful physical picture for partitions, to gain a better understanding and to avoid using them as a mere mathematical object.

Consider the third representation given for partitions, in the example above $(1010020\dots)$. This is formally equivalent to a N-particles state expressed in an occupation numbers basis. Following the example, one has 1 particle in the first orbital, 1 in the third and 2 in the sixth. In this picture, other representations of partitions assume meaning: $(6631\dots)$ lists in decreasing order orbitals occupied by every single particle, and $(6^2 3^1 1^1)$ does the same in a contracted way.

With this idea in mind, a generic N-particle state can be expressed on an occupation number basis as $|\lambda\rangle$, with $l(\lambda) = N$.

Notice that the fact that partitions are ordered sequences accounts for the permutation symmetry of indistinguishable particles. One could choose not to order the sequences of orbitals occupied by every single particle, but this would destroy the one to one correspondence between partitions and physical states (for example (6631) and (1636) would represent the same state).

To relate to LLL, orbitals are labeled by the angular momentum eigenvalue m. A partition $(1010020\dots)$ will represent a state in which one electron has vanishing angular momentum, one electron has $L_z = 2\hbar$ and two electrons have $L_z = 5\hbar$. (6631) will be the list of electrons' angular momenta, in decreasing order. Notice that $|\lambda|$ is the total angular momentum of the system.

From now on, the decreasing listing representation will be called the *usual* representation (λ_i will indicate the i-th component of λ). The multeplicity representation will be called the *occupation numbers* representation.

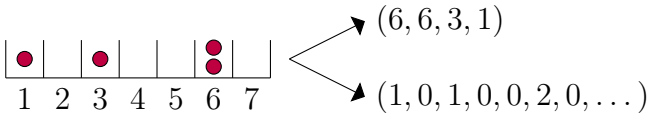
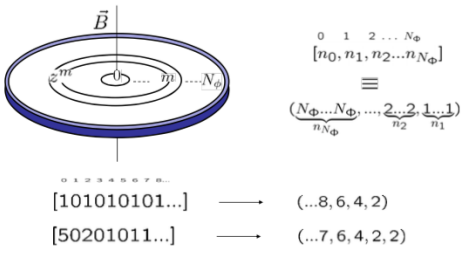


Figure 3.1.2: Example of different notation for partitions



On the left, disk geometry for the LLL is represented. Each orbital is labeled by m.
On the right, the conversion between occupation number representation and the usual representation.
Below, two examples of conversion between the occupation number representation and the usual representation.
Notice the different convention for the numbering of orbitals: mathematical notation starts from 1, physical notation from 0.
The image is taken from [BH08].

Figure 3.1.3: LLL picture for partitions

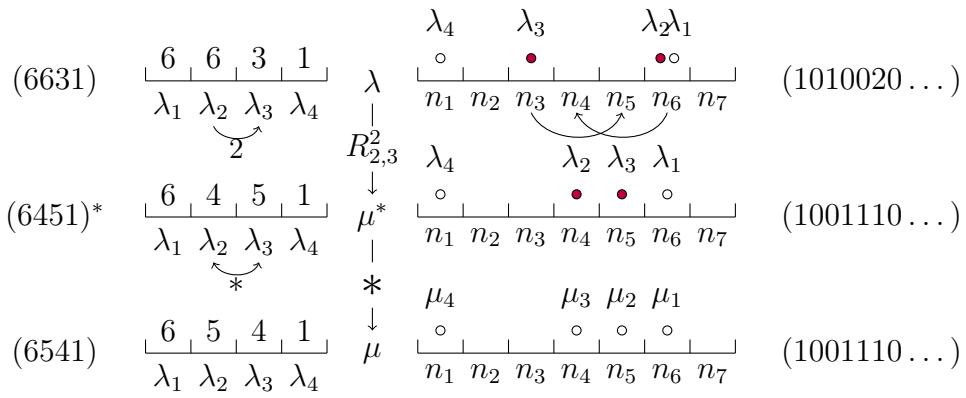


Figure 3.1.4: Example of squeezing

On the left, squeezing in the usual representation is shown. First, the rough squeeze is performed as defined in Eq. (3.1); then, reordering to restore decreasing order is done. On the right, the occupation number counterpart of the same squeezing is shown. Notice that no explicit reordering is needed in this representation.

3.1.1 Squeezing

A fundamental (for physical purposes) operation on partitions is called "squeezing".

Given a partition λ, two integers $0 \leq i \leq j \leq l(\lambda)$ and an integer $0 \leq s \leq \lambda_i - \lambda_j$, the squeezing operator R^s_{ij} acts as

$$R^s_{ij}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots) = (\lambda_1, \dots, \lambda_i - s, \dots, \lambda_j + s, \dots)^*, \tag{3.1}$$

where the $*$ means that a reordering to restore the decreasing order of the final partition may be needed. Notice that squeezings preserve $|\lambda|$.

Fig. (3.1.4) shows an example of squeezing, both in the usual partition representation and in the occupation numbers one.

In the second case of Fig. (3.1.4), it is easy to give a physical interpretation of squeezing operations: two particles, the i-th and j-th, are moved inward by s steps from their orbitals.

Squeezings are of fundamental importance. Suppose to have a two body interaction V which is invariant under rotations (for simplicity, have in mind the LLL disk geometry). Then, when passing in a second quantization picture,

$$V = \frac{1}{2} \sum_{r,s} \sum_{m,n} \langle r, s|V|m, n\rangle \, a^\dagger_r a^\dagger_s a_m a_n, \tag{3.2}$$

where creation and annihilation operators for the angular momentum basis have been introduced. As V commutes with the angular momentum, $\langle r, s|V|m, n\rangle \neq 0 \iff r + s = m + n$.

This implies that $a_r^\dagger a_s^\dagger a_m a_n |\lambda\rangle$ must be multiple of $|\mu\rangle$ with μ squeezed from λ (or unsqueezed, i.e. squeezed with negative s).

Therefore squeezing (and unsqueezing) operators become a sort of basis for rotational invariant interactions.

In the following, chains of partitions squeezed from a fixed partition λ will be useful; an example of such a chain is given in Fig. (**??**).

3.1.2 Orderings

Ordering relations can be established on the set of partitions. The *squeezing induced* ordering, also called natural/dominance partial ordering is defined equivalently as:

- two partitions λ and μ are such that $\lambda > \mu$ if μ can be derived from λ with a finite number of squeezings. This implies that if $|\lambda| \neq |\mu|$, λ and μ are not comparable;
- two partitions λ and μ are such that $\lambda > \mu$ if $|\lambda| = |\mu|$ and $\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i$, $\forall r > 0$.

This ordering is not a total ordering on partitions. For examples, see Fig. (**??**).

In [Sta89], two more orderings are established:

- the *reverse lexicographic* ordering: two partitions λ and μ are such that $\mu \overset{R}{\leq} \lambda$ if the first non-vanishing term $\lambda_i - \mu_i$ is positive. This is a total ordering, compatible with the dominance ordering;
- the *Young tableaux induced* ordering: two partitions λ and μ are such that $\mu \subset \lambda$ if the tableau of μ is fully included in the tableau of λ.

In the following, dominance ordering will be used, as squeezing characterizes some of the interesting properties of Jack Polynomials. However, it shall be remarked that this is not a total ordering. Most of the theorems related to ordering are proven in the reverse lexicographic ordering, and then it is shown that all the total orderings compatible with the dominance one are equivalent for the purposes of the proof.

3.2 The Ring of Symmetric functions

Symmetric functions generalize polynomials by removing any constraint on the number of independent variables. This generalization can be useful to describe wavefunctions of very large many body systems.

Let Λ_n be the ring of symmetric polynomials in n independent variables, with coefficients in a generic commutative ring R (in the following, $R = \mathbb{Z}$).

The first thing to be stressed is that Λ_n is a graded ring: in fact, $\Lambda_n = \bigoplus_{r\geq 0} \Lambda_n^r$, where Λ_n^r is the subring of symmetric homogeneous polynomials of degree r (by convention, 0 is homogeneous of every degree).

A collection of surjective homomorphisms (of graded rings) is constructed: let $\omega : \Lambda_{n+1} \to \Lambda_n$ be defined by $\omega : f(x_1, \ldots, x_n, x_{n+1}) \to f(x_1, \ldots, x_n, 0)$ for every $f \in \Lambda_{n+1}$. The restriction of ω to Λ_n^r is again surjective, and it is a bijection if and only if $r \leq n$.

These homomorphisms allow for the construction of the inverse limit $\Lambda^r = \varprojlim_n \Lambda_n^r$.

By definition, an element $f \in \Lambda^r$ is a sequence $\{f_n\}_{n\geq 0}$ such that:

- $f_n \in \Lambda_n^r$ for each $n \geq 0$;
- $f_n = \omega f_{n+1}$.

Λ^r is again a ring, which is called the *ring of homogeneous symmetric functions of degree r*.

Finally, the *ring of symmetric functions* is defined as $\Lambda = \bigoplus_{r\geq 0} \Lambda^r$.

This is a very strict mathematical construction. The point is that symmetric functions are sequences of regular polynomials, each one with an additional variable. Thus, symmetric functions can be regarded as polynomials in infinitely many independent variables.

In the following, the symbol Λ_R will indicate explicitly which ring of coefficients is taken into account.

3.2.1 Relevant subsets of Λ

Monomial symmetric functions

Given a partition λ, a monomial is defined as $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \ldots$

The sum of all distinct monomials obtainable from λ permuting the variables is called the monomial symmetric function $m_\lambda = \sum x^\lambda$.

For example:

$$\begin{aligned} m_{(1,2)} &= \sum_{i\neq j} x_i x_j^2, \\ m_{(1,1)} &= \sum_{i<j} x_i x_j. \end{aligned} \tag{3.3}$$

m_λ's form a $\mathbb{Z}$-basis of $\Lambda_\mathbb{Z}$. For example, restricting to the 3 variables case $(x_1, x_2, x_3) = (x, y, z)$,

$$x^2y^2z^2 + 4xyz + 5xy + 5xz + 5yz = m_{(2,2,2)} + 4m_{(1,1,1)} + 5m_{(1,1,0)}.$$

In the particular case of N bosons in the LLL, it was shown that a basis for the Hilbert space of states is given by symmetric homogeneous monomials of degree M (total angular momentum).

This allows for the following observations:

- partitions λ label N-particles LLL states $|\lambda\rangle$ whose Schrödinger representation is given by a multiple of m_λ. In particular, λ_i are the angular momenta of the N particles and $|\lambda|$ is the total angular momentum of the system;
- the N-particles LLL Hilbert space is the closure of Λ_N^M.

Elementary symmetric functions

For $\lambda = (1^r)$, m_λ is called the r-th elementary symmetric function e_r. For a generic partition λ, let $e_\lambda = e_{\lambda_1}e_{\lambda_2}\dots$. It can be shown (see [Mac88]) that e_λ's form a $\mathbb{Z}$-basis of $\Lambda_\mathbb{Z}$.

Examples of elementary functions are:

$$\begin{aligned} e_1 &= m_{(1)} = \sum x_i, \\ e_2 &= m_{(1,1)} = \sum_{i<j} x_i x_j, \\ e_{(2,1)} &= e_2 e_1 = \left(\sum_{i<j} x_i x_j\right)\left(\sum x_i\right). \end{aligned} \tag{3.4}$$

Power sum symmetric functions

For $\lambda = (r)$, m_λ is called the r-th power sum symmetric function p_r. For a generic partition λ, let $p_\lambda = p_{\lambda_1}p_{\lambda_2}\dots$. It can be shown (see [Mac88]) that p_λ's form a $\mathbb{Q}$-basis of $\Lambda_\mathbb{Z}$.

Examples of power functions are:

$$\begin{aligned} p_1 &= m_{(1)} = \sum x_i, \\ p_2 &= m_{(2)} = \sum x_i^2, \\ p_{(2,1)} &= p_2 p_1 = \left(\sum x_i^2\right)\left(\sum x_i\right). \end{aligned} \tag{3.5}$$

Schur symmetric functions

Let λ be a partition, and let $D_\lambda = \det\left(x_i^{\lambda_j+n-j}\right)$. By the properties of determinants, D_λ vanishes every time $x_i = x_j$; thus, it is divisible by the Vandermonde determinant D_0.

Schur functions are defined as $s_\lambda = D_\lambda/D_0$.

Since $s_\lambda(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n, 0)$, s_λ's are well defined (recall that the number of zeros in the end of λ doesn't alter the partition).

It is possible to introduce a scalar product $\langle , \rangle$ on Λ such that:

$$\begin{aligned} \langle p_\lambda, p_\mu \rangle &= \delta_{\lambda\mu} z_\lambda \\ z_\lambda &= \prod_{r\geq 1} r^{n_r(\lambda)} n_r(\lambda)! = 1^{n_1(\lambda)} 2^{n_2(\lambda)} \ldots n_1(\lambda)! n_2(\lambda)! \ldots \end{aligned} \tag{3.6}$$

with δ being the usual Kroneker's delta.

It can be shown (see [Mac95]) that Schur symmetric functions are characterized uniquely by the following properties:

- $s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\mu\lambda} m_\mu$;
- $\langle s_\lambda, s_\mu \rangle = 0$ for $\lambda \neq \mu$.

i.e. they are an orthogonal system with a particular "triangular" form when written on the monomial basis.

3.2.2 Relevant constructions from Λ

From symmetric functions to symmetric polynomials

Given a symmetric functions, it is always possible to reduce it to a symmetric polynomial in N independent variables by setting $x_{N+1} = x_{N+2} = \cdots = 0$. As all the properties of symmetric functions are valid for an indefinite number of independent variables, they will still hold in the polynomial case.

From symmetric functions to antisymmetric functions

The same construction of Λ applies for the ring of antisymmetric polynomials.

A useful property of antisymmetric polynomials is that they are divisible by the Vandermonde determinant, and that this ratio defines a symmetric polynomial. Thus, one can build every antisymmetric function by multiplying a symmetric function with the Vandermonde determinant.

A useful basis of antisymmetric functions are the so called *Slater determinants* sl_λ, i.e. completely antisymmetrized monomials.

Notice that for N fermions in the LLL, sl_λ's play the same role as m_λ's for bosons.

Another useful remark is that, as antisymmetrization (i.e. Pauli principle) prevents two indeterminates to have the same exponent in a homogeneous polynomial (i.e. two particles to be in the same state), one can refer partitions to the *partition of minimal degree* (i.e. state of minimal angular momentum), which is $(N-1, N-2, \dots, 2, 1, 0, \dots)$ if N is the number of indeterminates (see [Dun93]). Thus, a partition $(\lambda_1, \lambda_2, \dots, \lambda_N)$ for N fermions could be rewritten as $(\lambda_1 - (N-1), \lambda_2 - (N-2), \dots, \lambda_N)$ without losing any kind of information.

3.3 Jack symmetric functions

Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} \alpha^{l(\lambda)} z_\lambda$ be the scalar product on $\Lambda_{\mathbb{R}(\alpha)}$.

Then Jack symmetric functions (also known as *Jacks*) P_λ^α are uniquely characterized by

- $P_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} a_{\mu\lambda} m_\mu$;
- $\langle P_\lambda^\alpha, P_\mu^\alpha \rangle_\alpha = 0$ for $\lambda \neq \mu$.

Sketch of proof (see [Sta89]): the first condition is basically Grahm-Schimdt orthogonalization for the monomial basis with respect to the given scalar product. The only issue is that the ordering used for the partitions is not a total orderding. One should first prove that every total order compatible with the dominance order gives the same Grahm-Schmidt expansion, and then provide an existence theorem for such a total order (in our case, this is not required because reverse lexicographic ordering is a total order and satisifies the compatibility condition). ■

Jacks generalize a variety of other subsets of symmetric functions (modulo a normalization factor), such as Schur funtions ($\alpha = 1$) and monomial symmetric functions ($\alpha \to \infty$).

Jacks' definition implies that P_λ^α are monic, i.e. the first coefficient of the expansion on the monomial basis is 1. This is usually called the *P normalization*.

Other normalizations are possible. One among the most used is the *J normalization*, defined by $a_{\lambda,(1^{|\lambda|})} = |\lambda|!$.

From now on, P_λ^α will indicate Jacks with P normalization and J_λ^α Jacks with J normalization.

3.3.1 Properties

Expansion on the monomial basis

A fundamental property of Jacks is already required in the definition

$$\begin{aligned} P^\alpha_\lambda &= \sum c_{\mu\lambda} m_\mu, \\ c_{\mu\lambda} \neq 0 &\iff \mu \text{ can be squeezed by } \lambda, \end{aligned} \tag{3.7}$$

i.e. Jacks' expansion on the monomial basis has non null coefficients only for those partitions μ that can be squeezed from λ (c's are functions of α).

This means that, since squeezing implies $|\mu| = |\lambda|$, Jacks are homogeneous symmetric functions of degree $|\lambda|$, and therefore are eigenstates of $\sum x_i\partial_i$ with eigenvalue $|\lambda|$.

An example of such an expansion is given for $\lambda = (1001001)$, whose squeezed partitions are listed in Ch. 1:

$$\begin{aligned} P^\alpha_{(1001001)} = {} & c^{(1001001)}_{(1001001)} m_{(1001001)} + c^{(0110001)}_{(1001001)} m_{(0110001)} \\ & + c^{(1000110)}_{(1001001)} m_{(1000110)} + c^{(0101010)}_{(1001001)} m_{(0101010)} \\ & + c^{(0011100)}_{(1001001)} m_{(0011100)} + c^{(0003000)}_{(1001001)} m_{(0003000)}. \end{aligned}$$

Laplace Beltrami operator

Jack symmetric functions can be characterized as the unique polynomial eigenfunctions of certain partial differential operators.

Among the others, Laplace Beltrami operators are of great physical relevance in relation to integrable models (to be precise, Calogero-Sutherland and Calogero-Sutherland-Moser models).

The Laplace Beltrami operator is defined as

$$H^\alpha_{LB} = \sum (x_i\partial_i)^2 + \frac{1}{2\alpha} \sum_{i\neq j} \frac{x_i + x_j}{x_i - x_j}(x_i\partial_i - x_j\partial_j). \tag{3.8}$$

To show that Jacks are eigenvectors of H^α_{LB}, one can notice that $\frac{\alpha}{2} H^\alpha_{LB} = D(\alpha) - \frac{N-1-\alpha}{2} \sum x_i\partial_i$, with $D(\alpha)$ defined in [Sta89], pg. 84. Following the proof given in that reference, and recalling that $\sum x_i\partial_i$ is diagonal on Jacks gives that

$$H^\alpha_{LB} P^\alpha_\lambda = \left[\sum \left(\lambda_i^2 + \frac{\lambda_i}{\alpha}(N + 1 - 2i) \right) \right] P^\alpha_\lambda. \tag{3.9}$$

It is important to notice that this relation can be used as a definition for Jacks, if Eq. (3.7) is also required.

Notice that if $\alpha \to \infty$, $H^\infty_{LB} = \sum (x_i\partial_i)^2$, which is diagonal on the monomial basis, justifying the claim $\lim_{\alpha\to\infty} P^\alpha_\lambda = m_\lambda$.

3.3.2 Negative parameter Jack polynomials

Jacks are defined for positive α. It's of physical interest to study also negative parameter Jacks. In [FJMM01], it is shown that problems may arise only for negative rational values of α; for these values of the parameter, a criterion to select partition whose associated Jack is regular was found. The condition is called (k, r, N)-admissibility, and describes a *generalized Pauli principle* which prevents more than k particles in a N-particles system to occupy r consecutive orbitals. Mathematically, this principle is formulated as $\lambda_i - \lambda_j \geq \lfloor \frac{j-i}{k} \rfloor r$ for each $i < j$ and with $\lfloor \, \rfloor$ being the floor function.

Haldane and Bernevig, in [BH08], first pointed out that Laughlin's wavefunctions (in their bosonic version), as well as other model wavefunctions, are particular Jacks.

Bosonic Laughlin's wavefunctions are defined as usual Laughlin's wavefunctions divided by a Vandermonde determinant, i.e. $\psi^L = \sum_{i<j}^{N} (z_i - z_j)^r$ with r even. It can be shown that these wavefuction satisfy $\psi^L = J^{\alpha_{1,r}}_{\lambda^0(1,r)}$, where

- $\alpha_{k,r} = -\frac{k+1}{r-1}$;
- $\lambda^0(k,r)$ is the (k, r, N)-admissible partition which minimizes $|\lambda^0(k,r)|$, i.e. for $k = 1$ $(10^{r-1}10^{r-1}\dots)$.

The claim that ψ^L is a Jack is motivated in [BH08], constructing H_{LB} as a sum of a constant and an operator which annihilates ψ^L.

This identification is the key observation for the expansion of Laughlin's wavefunctions on the monomial basis, i.e. the non interacting N particle basis.

3.4 Jack antisymmetric functions

Jacks can be useful for the study of bosonic systems, given their symmetry for the exchange of coordinates. An equivalent class of antisymmetric functions is to be defined to properly treat fermionic systems.

The simplest (and with most physical meaning) choice for this definition is (see [TERB11])

$$S^{\alpha}_{\lambda'} = J^{\alpha}_{\lambda} \prod_{i<j} (z_i - z_j), \tag{3.10}$$

i.e., multiplication for a Vandermonde determinant, with $\lambda'_i = \lambda_i + N - i$. This is justified by the fact that Laughin's states and bosonic Laughlin's states

differ only for the multiplication by a Vandermonde determinant. Relative angular momenta λ' are used as sketched in Sec. (3.2.2).

For the following manipulations, it is fundamental to have an operator similar to the Laplace Beltrami also for antisymmetric Jacks.

The basic idea is to use Eq. (3.9):

$$\begin{aligned} E^{\alpha}_{\lambda} S^{\alpha}_{\lambda'} &= E^{\alpha}_{\lambda} \left(\prod_{\substack{k,l \\ k<l}} (z_k - z_l) \right) J^{\alpha}_{\lambda} \\ &= \left(\prod_{\substack{k,l \\ k<l}} (z_k - z_l) \right) E^{\alpha}_{\lambda} J^{\alpha}_{\lambda} = \left(\prod_{\substack{k,l \\ k<l}} (z_k - z_l) \right) H^{\alpha}_{LB}(J^{\alpha}_{\lambda}). \end{aligned} \tag{3.11}$$

Then, by explicit calculation of $(z_i\partial_i)S^{\alpha}_{\lambda'}$ and $(z_i\partial_i)^2 S^{\alpha}_{\lambda'}$ and some manipulations (all the details are reported in App. A), one obtains that the right operator is

$$H^{\alpha}_{LB,\,F} = \sum_i (z_i\partial_i)^2 + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right) \sum_{\substack{i,j \\ i\neq j}} \left[\frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j) - 2\frac{z_i^2 + z_j^2}{(z_i - z_j)^2} \right], \tag{3.12}$$

with eigenvalues

$$E^{\alpha}_{\lambda'} = \sum \lambda'_i \left[\lambda'_i - 2\left(\frac{1}{\alpha} - 1\right) i \right] + \left(\frac{1}{\alpha} - 1\right) \left((N+1)|\lambda'| - N(N-1) \right). \tag{3.13}$$

By definition, $H^{\alpha}_{LB,\,F}$ is diagonal on $S^{\alpha}_{\lambda'}$ with eigenvalue $E^{\alpha}_{\lambda'}$.

Chapter 4

Recursion Laws

In this chapter, recursion laws for the coefficients of the expansion of Jacks on the monomial basis are derived. A generalization to antisymmetric Jacks is then presented.

In the following, an operator H on the symmetric/antisymmetric polynomials will be said to have a triangular action on a particular basis $\{b_\lambda\}$ if

$$Hb_\lambda = C_\lambda^\lambda b_\lambda + \sum_{\mu<\lambda} C_\mu^\lambda b_\mu, \tag{4.1}$$

with $C_\lambda^\lambda \neq 0$.

An operator H will be said to be a two body operator if it's sum of operators acting on all the distinct pairs of variables, i.e. $H = \sum_{i<j} H_{i,j}(x_i, x_j, \partial_i, \partial_j)$. H will always be considered symmetric for exchange of coordinate indexes.

The derivation is organized as follows:

- given an operator diagonal on certain functions f_λ and triangular on a basis $\{b_\lambda\}$, general recurrence relations for the expansion of f_λ on $\{b_\lambda\}$ are derived. To be concrete, think of f_λ as Jacks and of b_λ as the monomial basis;

- a general form for the action of triangular two body operators on the N variable basis $\{b_\lambda\}$ is given as a function of the action of the same operator on the 2 variables basis;

- explicit calculation for Jacks'expansion on symmetric monomials is provided;

- explicit calculation for antisymmetric Jacks'expansion on slater determinants is provided.

4.1 Generic recurrence relations

Let H be an operator triangular on the $\{b_\lambda\}$ basis for the symmetric/antisymmetric polynomials (to be concrete, the symmetric monomial basis or the slater determinants basis). Let f_λ be an eigenvector of H with eigenvalue E_λ for each partition λ, i.e. $E_\lambda f_\lambda = H f_\lambda$.

Suppose that:

$$f_\lambda = X^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda} X^\lambda_\mu b_\mu \quad \text{with} \quad X^\lambda_\lambda \neq 0, \tag{4.2}$$

where X^λ_μ and C^λ_μ are suitable sets of coefficients. This hypothesis is redundant, as triangularity and the eigenvector relation suffice to prove it. Nevertheless, this property is characterizing for Jacks and thus is to be remarked.

Then, plugging Eq. (4.2) into the eigenvector relation one obtains (see explicit calculations at App. B)

$$X^\lambda_\kappa = \frac{1}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa<\mu\leq\lambda}} X^\lambda_\mu C^\mu_\kappa, \tag{4.3}$$

whose validity is guaranteed if E_λ's are distinct for distinct λ's.

This relation is recursive, and in particular needs an initial condition. Usually this is given for monic Jacks P^α_λ as $X^\lambda_\lambda = 1$.

4.2 General form for triangular operators

A physical view point suggests that two body operators' action on N particles states should be understandable by their action on 2 particles states. In polynomial spaces, this means that two body operators' action on N variables base functions should be understandable by their action on 2 variable base functions.

Two equivalent approaches can lead to find an expression for the N variables expression Hb_λ as a function of the same 2 variables expression. Both the derivation rely on the usage of a particular $\{b_\lambda\}$ as the basis: symmetric polynomials require the symmetric monomail basis, antisymmetric monomials require the slater determinants basis.

Let H satisfy

$$\begin{aligned} Hb_{(m,n)} &= \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k b_{(m-k,n+k)} \quad \text{for} \quad m > n, \\ Hb_{(m,n)} &= 0 \quad \text{for} \quad m = n. \end{aligned} \tag{4.4}$$

The first requirement is triangularity in 2 variables; the second requirement allows for a unified treatment of symmetric and antisymmetric cases. This further hypotesis is automatically satisfied in the antisymmetric case; for the symmetric one, in general it is not. If H is a Laplace Beltrami operator and b_λ is the monomial basis, it is satisfied.

The first path relies on explicit computation.

The second one uses a "second quantization" formalism adapted to the polynomial ring; this construction allows for the usage of physical techniques and terminology to study polynomials.

4.2.1 First approach

Explicit calculation requires an explicit form for N variables monomials or slaters, which is given using sums over permutations ($\pm$ is for symmetric/antisymmetric case respectively)

$$b_\lambda = \sum_{\sigma \in S_N} (\pm)^\sigma z_1^{\sigma(\lambda_1)} \dots z_N^{\sigma(\lambda_N)}. \tag{4.5}$$

Thus:

$$\begin{aligned} Hb_\lambda &= \sum_{i<j} H_{i,j} b_\lambda \\ &= \sum_{i<j} \sum_{\sigma \in S_N} H_{i,j} z_1^{\sigma(\lambda_1)} \dots z_N^{\sigma(\lambda_N)} \\ &= \sum_{i<j} \sum_{\sigma \in S_N} (z_1^{\sigma(\lambda_1)} \dots \not{z}_i^{\sigma(\lambda_i)} \dots \not{z}_j^{\sigma(\lambda_j)} \dots z_N^{\sigma(\lambda_N)}) H_{i,j} \left[z_i^{\sigma(\lambda_i)} z_j^{\sigma(\lambda_j)} \right]. \end{aligned} \tag{4.6}$$

The interesting action is:

$$\begin{aligned} H_{i,j} \left[z_i^{\sigma(\lambda_i)} z_j^{\sigma(\lambda_j)} \right] &= \frac{1}{2} H_{i,j} \left[z_i^{\sigma(\lambda_i)} z_j^{\sigma(\lambda_j)} \pm z_i^{\sigma(\lambda_j)} z_j^{\sigma(\lambda_i)} \right] \\ &= \frac{1}{2} H_{i,j} b_{(\sigma(\lambda_i), \sigma(\lambda_j))} \\ &= \frac{1}{2} \sum_{k=0}^{(\sigma(\lambda_i) - \sigma(\lambda_j))/2} F_k^{(\sigma(\lambda_i), \sigma(\lambda_j))} \\ &\qquad \times \left[z_i^{\sigma(\lambda_i) - k} z_j^{\sigma(\lambda_j) + k} \pm z_j^{\sigma(\lambda_i) - k} z_i^{\sigma(\lambda_j) + k} \right] \\ &= \sum_{k=0}^{(\sigma(\lambda_i) - \sigma(\lambda_j))/2} F_k^{(\sigma(\lambda_i), \sigma(\lambda_j))} z_i^{\sigma(\lambda_i) - k} z_j^{\sigma(\lambda_j) + k}, \end{aligned} \tag{4.7}$$

where the second passage was allowed by summing over all the permutation σ seen as $\sigma' \circ s_{i,j}$ (note that the sum is over all the elements of a group).

Recollecting all the pieces one has:

$$\begin{aligned} Hb_\lambda &= \sum_{i<j} \sum_{\sigma \in S_N} (z_1^{\sigma(\lambda_1)} \dots \not{z}_i^{\sigma(\lambda_i)} \dots \not{z}_j^{\sigma(\lambda_j)} \dots z_N^{\sigma(\lambda_N)}) \\ &\quad \times \sum_{k=0}^{(\sigma(\lambda_i)-\sigma(\lambda_j))/2} F_k^{(\sigma(\lambda_i),\sigma(\lambda_j))} z_i^{\sigma(\lambda_i)-k} z_j^{\sigma(\lambda_j)+k} \\ &= \sum_{i<j} \sum_{\sigma \in S_N} \sum_{k=0}^{(\sigma(\lambda_i)-\sigma(\lambda_j))/2} F_k^{(\sigma(\lambda_i),\sigma(\lambda_j))} \\ &\quad \times z_1^{\sigma(\lambda_1)} \dots z_i^{\sigma(\lambda_i)-k} \dots z_j^{\sigma(\lambda_j)+k} \dots z_N^{\sigma(\lambda_N)}. \end{aligned} \tag{4.8}$$

Notice that the second summation implies $\sigma(\lambda_i) \geq \sigma(\lambda_j)$: this is always true modulo a minus sign.

The last line can be recognised as the sum over all the partitions $\mu \leq \lambda$, a part from a sign $(\pm)^{N_{SW}}$ due to restoration of decreasing order of the partition by N_{sw} swaps

$$Hb_\lambda = \sum_{\mu \leq \lambda} (\pm)^{N_{SW}} F_k^{(m,n)} b_\mu, \tag{4.9}$$

with $\mu = (\lambda_1 \dots \lambda_i - k \dots \lambda_i + k \dots \lambda_N)^*$ if $\lambda_i = m$ and $\lambda_j = n$.

4.2.2 Second approach

Notations

Let Λ_1 be the space of one variable polynomials (with complex coefficients and variables). Λ_1 is to be considered as a one particle Hilbert space. Its inner product is defined as $\langle r|s\rangle = \delta_{r,s}$, where $|r\rangle = z^r$ is the monomial basis (orthonormal for construction).

The space of k variable polynomials Λ_k can be seen as the k-particle Hilbert space associated with Λ_1, i.e. the closure of linear combinations of factored states $|r_1, \dots, r_k\rangle = |r_1\rangle \otimes \dots \otimes |r_k\rangle = z_1^{r_1} \dots z_k^{r_k}$ equipped with the inner product $\langle r_1, \dots, r_k | s_1, \dots, s_k\rangle = \langle r_1|s_1\rangle \dots \langle r_k|s_k\rangle = \delta_{r_1,s_1} \dots \delta_{r_k,s_k}$. Factored states have a clear interpretation: they describe the configuration in which the *i-th* particle is in the one particle state $|r_i\rangle$.

Some restrictions occur if one chooses the statistics satisfied by particles. In particular, for bosons $(+)$ or fermions $(-)$, one has to restict Λ_k to the

spaces of, respectively, symmetric polynomials Λ_k^+ or antisymmetric polynomials Λ_k^-. In this picture, factored states $|r_1, \ldots, r_k\rangle$ and $|s_1, \ldots, s_k\rangle$ are different states only if $\{r_i\}$ is not a permutation of $\{s_i\}$. It is thus natural to label factored states by partitions λ. These partition must be such that $l(\lambda) < k$.

Notice that $|\lambda\rangle$ still describe pure factored states.

A basis for $\Lambda_k^\pm$ is given by, respectively, the symmetric monomials $m_\lambda = |\lambda^+\rangle$ or the slater determinants $sl_\lambda = |\lambda^-\rangle$.

Notice that, due to symmetrization and antisymmetrization procedures, the states $|\lambda^\pm\rangle$ are not normalized:

$$\langle\lambda^+|\lambda^+\rangle = \frac{n_1(\lambda)! \ldots n_\infty(\lambda)!}{k!}, \tag{4.10}$$

$$\langle\lambda^-|\lambda^-\rangle = \frac{1}{k!}. \tag{4.11}$$

Creation and annihilation operators $a_i^\dagger$ and a_i are introduced, such that in the occupation number picture:

$$\begin{aligned} a_i^\dagger |n_0, n_1, \ldots, n_i, \ldots\rangle &= |n_0, n_1, \ldots, n_i + 1, \ldots\rangle, \\ a_i |n_0, n_1, \ldots, n_i, \ldots\rangle &= \begin{cases} |n_0, n_1, \ldots, n_i - 1, \ldots\rangle & \text{for} \quad n_i \neq 0 \\ 0 & \text{for} \quad n_i = 0 \end{cases}. \end{aligned} \tag{4.12}$$

Squeezings are written as operators of the form $a_{m-l}^\dagger a_{n+l}^\dagger a_n a_m$, with $m > n$ and $0 < l < \frac{m-n}{2}$ for integer m, n, l. Their action on the occupation numbers basis is (if $n_n \neq 0$ and $n_m \neq 0$, otherwise they annihilate the state):

$$\begin{aligned} &a_{m-l}^\dagger a_{n+l}^\dagger a_n a_m |\ldots, n_n, \ldots, n_{n+l}, \ldots, n_{m-l}, \ldots, n_m, \ldots\rangle \\ &= |\ldots, n_n - 1, \ldots, n_{n+l} + 1, \ldots, n_{m-l} + 1, \ldots, n_m - 1, \ldots\rangle. \end{aligned} \tag{4.13}$$

The same actions in the usual representation is

$$\begin{aligned} a_i^\dagger |\lambda\rangle = a_i^\dagger |\ldots, i, \ldots\rangle &= (\pm)^{N_{SW}} |\ldots, i + 1, \ldots\rangle, \\ a_i |\lambda\rangle = a_i |\ldots, i, \ldots\rangle &= \begin{cases} |\ldots, i - 1, \ldots\rangle & \text{for} \quad i \in \lambda \\ 0 & \text{for} \quad i \notin \lambda \end{cases}, \end{aligned} \tag{4.14}$$

$$a_{m-l}^\dagger a_{n+l}^\dagger a_n a_m |\ldots, m, \ldots, n, \ldots\rangle = (\pm)^{N_{SW}} |\ldots, m - l, \ldots, n + l, \ldots\rangle. \tag{4.15}$$

The $(\pm)^{N_{SW}}$ is a statistics dependent term due to the fact that construction operators create a new particle in front of the partition. To restore the decreasing order, a number of N_{SW} swaps must be performed.

Notice that the operators just described are not the usual creation and annihilation operators for the absence of the typical multiplicative factors. Still, one has the expected bosonic commutation relations

$$\left[a_r^\dagger, a_s^\dagger\right] = 0, \tag{4.16}$$

and fermionic anticommutation relations

$$\left\{a_r^\dagger, a_s^\dagger\right\} = 0. \tag{4.17}$$

Computation

Given a 2 body operator, its second quantization form is

$$H = \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m, \tag{4.18}$$

where $a^\dagger$ and a are creation and annihilation operators defined above.

Notice that this formula is defined on factored states, and not on symmetrized/antisymmetryzed states. To switch to symmetric/antisymmetric polynomials, one has to provide some justification. As H is symmetric, one has:

$$\begin{aligned} H &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m \\ &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| S_\pm^\dagger S_\pm H S_\pm^\dagger S_\pm |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m \\ &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s|^\pm H |m,n\rangle^\pm \, a_r^\dagger a_s^\dagger a_n a_m, \end{aligned} \tag{4.19}$$

where $S_\pm$ are symmetrization/antisymmetrization operators.

From now on the $\pm$ exponent of brakets will be dropped, and all brakets will describe symmetrized/antisymmetrized states.

One has (see detailed computation in App. B):

$$\begin{aligned} H |\lambda\rangle &= \frac{1}{2} \sum_{r,s,m,n} \langle r,s| H |m,n\rangle \, a_r^\dagger a_s^\dagger a_n a_m |\lambda\rangle \\ &= \sum_{\mu \leq \lambda} F_k^{(m,n)} (\pm)^{N_{SW}} |\mu\rangle \, , \end{aligned} \tag{4.20}$$

where:

- in the last line, $\mu = [\lambda_1, \ldots, \lambda_i - k, \ldots, \lambda_j + k, \ldots]$ with $\lambda_i = m$ and $\lambda_j = n$, i.e., μ is a generic partition squeezed from λ. NB.: notice the definition of m and n as the terms of the dominating partition λ modified by the squeezing. This is the opposite as done by [TERB11]; to compare the results, its convention will be used;

- the factor $(\pm)^{N_{SW}}$, where $+$ is for the bosonic case and $-$ for the antisymmetric, is caused by the reordering of μ after it is squeezed from λ.

4.3 Jacks

Laplace Beltrami operator H^{α}_{LB} is defined as:

$$H^{\alpha}_{LB} = K + \frac{1}{\alpha}V = \sum_i (z_i\partial_i)^2 + \frac{1}{\alpha}\sum_{\substack{i,j \\ i<j}} \frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j). \tag{4.21}$$

H^{α}_{LB}'s action on monomials m_λ is calculated as in the following. K is diagonal on m_λ with eigenvalue $\sum_i \lambda_i^2$. V is a two body operator. The explicit calculation of its action on two variables monomials $m_{(n,p)}$, by Eq. (4.9) or Eq. (4.20), gives its action on generic monomials.

For this calculation, suppose $n > p$

$$
\begin{aligned}
Vm_{(n,p)} &= \left[\frac{x+y}{x-y}\left(x\partial x - y\partial y\right)\right]\left(x^n y^p + x^p y^n\right) \\
&= \frac{x+y}{x-y}\left(n-p\right)\left(x^n y^p - x^p y^n\right) \\
&= \frac{x+y}{x-y}\left(n-p\right) x^p y^p \left(x^{n-p} - y^{n-p}\right) \\
&= \left(n-p\right)\left(x+y\right) \sum_{k=0}^{n-p-1} x^{(n-1)-k} y^{k+p} \\
&= \left(n-p\right)\left[\sum_{k=0}^{n-p-1} x^{n-k} y^{k+p} + \sum_{k=0}^{n-p-1} x^{n-(1+k)} y^{(k+1)+p}\right] \qquad (4.22) \\
&= \left(n-p\right)\left[x^n y^p + \sum_{k=1}^{n-p-1} x^{n-k} y^{k+p} + \sum_{k=1}^{n-p-1} x^{n-k} y^{k+p} + x^p y^n\right] \\
&= \left(n-p\right)\left[m_{(n,p)} + 2\sum_{k=1}^{n-p-1} x^{n-k} y^{k+p}\right] \\
&= \left(n-p\right)\left[m_{(n,p)} + 2\sum_{k=1}^{(n-p)/2} m_{(n-k,p+k)}\right].
\end{aligned}
$$

Keeping the notation of the precedent sections, one has $C_\kappa^\mu = F_k^{(n,p)} = \frac{2}{\alpha}(n-p)$ for $k \neq 0$ and $F_0^{(n,p)} = \frac{1}{\alpha}(n-p)$.

4.3.1 Jacks' recursion laws

Using Eq.(4.3) with $C_\kappa^\mu = F_k^{(n,p)} = \frac{2}{\alpha}(n-p)$ one obtains

$$
X_\kappa^\lambda = \frac{\frac{2}{\alpha}}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \leq \lambda}} X_\mu^\lambda((\kappa_i + k) - (\kappa_j - k)), \qquad (4.23)
$$

where $\kappa = [\kappa_1, \ldots, \kappa_i, \ldots, \kappa_j, \ldots, \kappa_N]$, $\mu = [\kappa_1, \ldots, \kappa_i + k, \ldots, \kappa_j - k, \ldots, \kappa_N]$.

4.4 Antisymmetric Jacks

Antisymmetric Laplace-Beltrami operator H_F^α is defined as:

$$\begin{aligned} H_F^\alpha &= K + \left(\frac{1}{\alpha} - 1\right) V \\ &= \sum_i (z_i \partial_i)^2 + \left(\frac{1}{\alpha} - 1\right) \sum_{\substack{i,j \\ i<j}} \left[\frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j) - 2 \frac{z_i^2 + z_j^2}{(z_i - z_j)^2} \right]. \end{aligned} \tag{4.24}$$

H_F^α's action on slaters sl_λ is calculated as in the following. K is diagonal on sl_λ with eigenvalue $\sum_i \lambda_i^2$. V is a two body operator. The explicit calculation of its action on two-particles slaters $sl_{(n,p)}$, by Eq. (4.9) or Eq. (4.20), gives its action on generic slaters.

The full calculation is similar to Eq. (4.22), and is given in App. B.

For this calculation, suppose $n > p$ and let $k = n - p$:

$$V sl_{(n,p)} = (n - p - 2) sl_{(n,p)} + 2 \sum_{l=1}^{(n-p)/2} (n - p - 2l) sl_{(n-l,p+l)}. \tag{4.25}$$

Thus one has $C_\mu^\lambda = F_k^{(n,p)} = \left(\frac{1}{\alpha} - 1\right) 2(n - p - 2k)$ for $k \neq 0$ and $F_0^{(n,p)} = \left(\frac{1}{\alpha} - 1\right)(n - p - 2)$.

4.4.1 Antisymmetric Jacks' recursion laws

Using Eq.(4.3) with $C_\kappa^\mu = F_k^{(n,p)} = \left(\frac{1}{\alpha} - 1\right) 2(n - p - 2k)$ one obtains

$$X_\kappa^\lambda = \frac{2\left(\frac{1}{\alpha} - 1\right)}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa < \mu \leq \lambda}} X_\mu^\lambda (\kappa_i - \kappa_j), \tag{4.26}$$

where $\kappa = [\kappa_1, \ldots, \kappa_i, \ldots, \kappa_j, \ldots, \kappa_N]$, $\mu = [\kappa_1, \ldots, \kappa_i + k, \ldots, \kappa_j - k, \ldots, \kappa_N]$

4.5 Validity of recursion laws

As stated in Sec. (4.1), recursion laws are valid if different Jacks have different eigenvalues. Otherwise, the denominator may vanish.

In [Sta89], pg. 85, a lemma is given that solves the problem for positive α. It states that if two partitions generate the same eigenvalue, the two

partitions are incomparable in the dominance order, thus they never belong to the same chain of squeezings.

For negative α, problems arise for the same negative rational values discussed in Sec. (3.3.2). In [TERB11] it is shown that under a limit prescription, every vanishing denominator is coupled with a vanishin numerator, such that the coefficients are still finite numbers.

4.6 Normalization issues

The recursion laws derived in this chapter describe the decomposition of Jacks on the pure monomial basis. But LLL is generated by multiples of monomials. This is not a great issue, nevertheless it is useful to know the decomposition on normalized N particles wavefunctions.

$$J_\lambda^\alpha = \sum_{\mu\leq\lambda} c_{\mu,\lambda} m_\mu = \sum_{\mu\leq\lambda} c_{\mu,\lambda} \frac{1}{\mathcal{N}_\mu} \mathcal{N}_\mu m_\mu = \sum_{\mu\leq\lambda} \overline{c}_{\mu,\lambda} \overline{m}_\mu, \tag{4.27}$$

where $\overline{c}_{\mu,\lambda} = c_{\mu,\lambda}/\mathcal{N}_\mu$ are the coefficients of Jacks' expansion over a differently normalized monomial basis. See Sec. (2.3) for the right LLL normalization.

Appendices

Appendix A

Explicit calculations for Chapter 3

In this Appendix explicit calculations for Sec. (3.4) will be performed. In the following, let $V = \left(\prod_{\substack{k,l \\ k<l}} (z_k - z_l)\right)$ be the Vandermonde determinant.

First, $(z_i\partial_i)S^{\alpha}_{\lambda'}$ is computed:

$$(z_i\partial_i)(S^{\alpha}_{\lambda'}) = (z_i\partial_i)\left[VJ^{\alpha}_{\lambda}\right] = V(z_i\partial_i)(J^{\alpha}_{\lambda}) + J^{\alpha}_{\lambda}(z_i\partial_i)V. \tag{A.1}$$

The first term is already in the required form, the second needs some work:

$$\begin{aligned}
&(z_i\partial_i)\left(\prod_{\substack{k,l \\ k<l}}(z_k - z_l)\right) = \\
&= z_i \prod_{\substack{k,l \\ k<l;\, k,l\neq i}} (z_k - z_l)\partial_i\left[\prod_{\substack{m \\ m<i}}(z_m - z_i)\prod_{\substack{m \\ i<m}}(z_i - z_m)\right] \\
&= z_i \prod_{\substack{k,l \\ k<l;\, k,l\neq i}} (z_k - z_l)\left[\sum_{\substack{m \\ m\neq i}} \frac{\prod_{n<i}(z_n - z_i)\prod_{i<n}(z_i - z_n)}{z_i - z_m}\right] \qquad \text{(A.2)} \\
&= \left[\sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right]\left(\prod_{\substack{k,l \\ k<l}}(z_k - z_l)\right) \\
&= \left[\sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right] V.
\end{aligned}$$

Collecting and rearranging, one recovers $V(z_i\partial_i)J^\alpha_\lambda$ as a function of $S^\alpha_{\lambda'}$:

$$V(z_i\partial_i)(J^\alpha_\lambda) = \left[(z_i\partial_i) - \sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right] S^\alpha_{\lambda'}. \tag{A.3}$$

Another useful relation is prooven form Eq. (A.2):

$$\begin{aligned}
\sum_{\substack{i,j \\ i\neq j}} (z_i\partial_i)(S^\alpha_{\lambda'}) &= \sum_i \sum_{\substack{j \\ j\neq i}} \left(V(z_i\partial_i)J^\alpha_\lambda + \left[\sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right] S^\alpha_{\lambda'} \right) \\
&= \sum_i (N-1) \left(V(z_i\partial_i)J^\alpha_\lambda + \left[\sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right] S^\alpha_{\lambda'} \right) \\
&= (N-1)\sum_i (\lambda^B_i) S^\alpha_{\lambda'} + (N-1)\sum_{\substack{i,m \\ m\neq i}} \frac{z_i}{z_i - z_m} S^\alpha_{\lambda'} \\
&= \left((N-1)|\lambda^B| + \frac{1}{2}N(N-1)^2 \right) S^\alpha_{\lambda'}.
\end{aligned} \tag{A.4}$$

Next, $(z_i\partial_i)^2 S^\alpha_{\lambda'}$ is computed, keeping in mind that the point is finding $V(z_i\partial_i)^2 J^\alpha_\lambda$ as a function of $S^\alpha_{\lambda'}$.

$$\begin{aligned}
(z_i\partial_i)^2(S^\alpha_{\lambda'}) &= (z_i\partial_i)((z_i\partial_i)S^\alpha_{\lambda'}) \\
&= (z_i\partial_i)\left[V(z_i\partial_i)J^\alpha_\lambda + \sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m} S^\alpha_{\lambda'} \right] \\
&= (z_i\partial_i)[V](z_i\partial_i)[J^\alpha_\lambda] + V(z_i\partial_i)^2[J^\alpha_\lambda] \\
&\quad + S^\alpha_{\lambda'}(z_i\partial_i)\left[\sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}\right] + \sum_{\substack{m \\ m\neq i}} \frac{z_i}{z_i - z_m}(z_i\partial_i)[S^\alpha_{\lambda'}].
\end{aligned} \tag{A.5}$$

Using also Eq. (A.3) one finds:

$$V(z_i\partial_i)^2(J^\alpha_\lambda) = \Bigg[(z_i\partial_i)^2 - 2\sum_{\substack{m\\ m\neq i}} \frac{z_i}{z_i - z_m}(z_i\partial_i) + 2\sum_{\substack{m\\ i\neq m}} \frac{z_i^2}{(z_i - z_m)^2} - \sum_{\substack{m\\ m\neq i}} \frac{z_i}{z_i - z_m} + \sum_{\substack{m,n\\ i\neq m\neq n}} \frac{z_i^2}{(z_i - z_m)(z_i - z_m)}\Bigg] S^\alpha_{\lambda'}. \tag{A.6}$$

Then, using the relations just computed:

$$\begin{aligned} E^\alpha_\lambda S^\alpha_{\lambda'} &= E^\alpha_\lambda V J^\alpha_\lambda = V E^\alpha_\lambda J^\alpha_\lambda = V H^\alpha_{LB}(J^\alpha_\lambda) \\ &= V\left[\sum_i (z_i\partial_i)^2 + \frac{1}{\alpha}\sum_{\substack{i,j\\ i<j}} \frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j)\right] J^\alpha_\lambda \\ &= \Bigg[\sum_i (z_i\partial_i)^2 - 2\sum_{\substack{i,m\\ m\neq i}} \frac{z_i}{z_i - z_m}(z_i\partial_i) + 2\sum_{\substack{i,m\\ i\neq m}} \frac{z_i^2}{(z_i - z_m)^2} \\ &\quad - \sum_{\substack{i,m\\ m\neq i}} \frac{z_i}{z_i - z_m} + \sum_{\substack{i,m,n\\ i\neq m\neq n}} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)}\Bigg] S^\alpha_{\lambda'} \\ &\quad + \frac{1}{\alpha}\Bigg[\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{z_i + z_j}{z_i - z_j}(z_i\partial_i - z_j\partial_j) \\ &\quad - \sum_{\substack{i,j\\ i<j}} \frac{z_i + z_j}{z_i - z_j}\left(\sum_{\substack{m\\ m\neq i}} \frac{z_i}{z_i - z_m} - \sum_{\substack{m\\ m\neq j}} \frac{z_j}{z_j - z_m}\right)\Bigg] S^\alpha_{\lambda'}. \end{aligned} \tag{A.7}$$

This is not the simplest form for the fermionic Laplace Beltrami operator. Using the manipulations described in the following pages, Eq. (A.9) to Eq. (A.13), and in particular hiding all the constants in the eigenvalue, one

finds the wanted expressions:

$$H^{\alpha}_{LB,\,F} = \sum_i (z_i\partial_i)^2 + \frac{1}{2}\left(\frac{1}{\alpha}-1\right)\sum_{\substack{i,j\\ i\neq j}}\left[\frac{z_i+z_j}{z_i-z_j}(z_i\partial_i - z_j\partial_j) - 2\frac{z_i^2+z_j^2}{(z_i-z_j)^2}\right],$$
$$E^{\alpha}_{\lambda'} = \sum \lambda'_i\left[\lambda'_i - 2\left(\frac{1}{\alpha}-1\right)i\right] + \left(\frac{1}{\alpha}-1\right)\left((N+1)|\lambda'| - N(N-1)\right). \tag{A.8}$$

Third factor

$$2\sum_{\substack{i,m\\ i\neq m}}\frac{z_i^2}{(z_i-z_m)^2} = 2\frac{1}{2}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}\right] = \left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}\right] \tag{A.9}$$

Fourth factor

$$\begin{aligned} -\left[\sum_{\substack{i,m\\ m\neq i}}\frac{z_i}{z_i-z_m}\right] &= -\frac{1}{2}\left[\sum_{\substack{i,m\\ m\neq i}}\frac{z_i}{z_i-z_m} + \sum_{\substack{i,m\\ m\neq i}}\frac{z_m}{z_m-z_i}\right] \\ &= -\frac{1}{2}\left[\sum_{\substack{i,m\\ m\neq i}}(1)\right] = -\left[\frac{1}{2}N(N-1)\right] \end{aligned} \tag{A.10}$$

Fifth factor

$$\begin{aligned} &\left[\sum_{\substack{i,m,n\\ i\neq m\neq n}}\frac{z_i^2}{(z_i-z_m)(z_i-z_n)}\right] = \\ &= \frac{1}{3}\left[\sum_{\substack{i,m,n\\ i\neq m\neq n}}\frac{z_i^2(z_m-z_n)+z_n^2(z_i-z_m)-z_m^2(z_i-z_n)}{(z_i-z_m)(z_i-z_n)(z_m-z_n)}\right] \\ &= \frac{1}{3}\left[\sum_{\substack{i,m,n\\ i\neq m\neq n}}(1)\right] = \left[\frac{1}{3}N(N-1)(N-2)\right] \end{aligned} \tag{A.11}$$

Seventh factor

$$
\begin{aligned}
&-\frac{1}{\alpha}\left[\sum_{\substack{i,j\\ i<j}}\frac{z_i+z_j}{z_i-z_j}\left(\sum_{\substack{m\\ m\neq i}}\frac{z_i}{z_i-z_m}-\sum_{\substack{m\\ m\neq j}}\frac{z_j}{z_j-z_m}\right)\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i+z_j}{z_i-z_j}\left(\frac{z_i+z_j}{z_i-z_j}+\sum_{\substack{m\\ m\neq i,j}}\frac{z_m(z_j-z_i)}{(z_i-z_m)(z_j-z_m)}\right)\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{(z_i+z_j)^2}{(z_i-z_j)^2}-\sum_{\substack{i,j,m\\ m\neq i\neq j}}\frac{z_i+z_j}{z_i-z_j}\frac{z_m(z_i-z_j)}{(z_i-z_m)(z_j-z_m)}\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}+\sum_{\substack{i,j\\ i\neq j}}\frac{2z_iz_j}{(z_i-z_j)^2}-\sum_{\substack{i,j,m\\ m\neq i\neq j}}\frac{z_m(z_i+z_j)}{(z_i-z_m)(z_j-z_m)}\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}+\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}\right.\\
&\qquad\left.-2\sum_{\substack{i,j\\ i\neq j}}\frac{z_i}{(z_i-z_j)}-\sum_{\substack{i,j,m\\ m\neq i\neq j}}\frac{z_m(z_i+z_j)}{(z_i-z_m)(z_j-z_m)}\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[2\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}-N(N-1)-\frac{1}{3}\sum_{\substack{i,j,m\\ m\neq i\neq j}}(-1)\right]S^{\alpha}_{\lambda'}\\
&=-\frac{1}{2\alpha}\left[2\sum_{\substack{i,j\\ i\neq j}}\frac{z_i^2+z_j^2}{(z_i-z_j)^2}-N(N-1)+\frac{1}{3}N(N-1)(N-2)\right]S^{\alpha}_{\lambda'}
\end{aligned}
\tag{A.12}
$$

Second factor

$$\left[-2\sum_{\substack{i,j\\ j\neq i}}\frac{z_i}{z_i-z_j}(z_i\partial_i)\right]S^\alpha_{\lambda'} = -\left[\sum_{\substack{i,j\\ j\neq i}}\frac{z_i}{z_i-z_j}z_i\partial_i - \sum_{\substack{i,j\\ j\neq i}}\frac{z_j}{z_i-z_j}z_j\partial_j\right]S^\alpha_{\lambda'}$$

$$= \left[-\sum_{\substack{i,j\\ j\neq i}}\frac{z_i}{z_i-z_j}z_i\partial_i + \sum_{\substack{i,j\\ j\neq i}}\frac{z_j}{z_i-z_j}z_j\partial_j \right.$$

$$\left. +\frac{1}{2}\sum_{\substack{i,j\\ j\neq i}}\frac{z_iz_j(\partial_i-\partial_j)}{z_i-z_j} - \frac{1}{2}\sum_{\substack{i,j\\ j\neq i}}\frac{z_iz_j(\partial_i-\partial_j)}{z_i-z_j}\right]S^\alpha_{\lambda'}$$

$$= -\frac{1}{2}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i+z_j}{z_i-z_j}(z_i\partial_i - z_j\partial_j)\right]S^\alpha_{\lambda'}$$

$$+\frac{1}{2}\left[-\sum_{\substack{i,j\\ j\neq i}}\frac{z_i}{z_i-z_j}z_i\partial_i + \sum_{\substack{i,j\\ j\neq i}}\frac{z_j}{z_i-z_j}z_j\partial_j + \sum_{\substack{i,j\\ j\neq i}}\frac{z_iz_j(\partial_i-\partial_j)}{z_i-z_j}\right]S^\alpha_{\lambda'}$$

$$= -\frac{1}{2}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i+z_j}{z_i-z_j}(z_i\partial_i - z_j\partial_j)\right]S^\alpha_{\lambda'} - \frac{1}{2}\left[\sum_{\substack{i,j\\ j\neq i}}(z_i\partial_i + z_j\partial_j)\right]S^\alpha_{\lambda'}$$

$$= -\frac{1}{2}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i+z_j}{z_i-z_j}(z_i\partial_i - z_j\partial_j)\right]S^\alpha_{\lambda'} - \sum_{\substack{i,j\\ j\neq i}}(z_i\partial_i)S^\alpha_{\lambda'}$$

$$= -\frac{1}{2}\left[\sum_{\substack{i,j\\ i\neq j}}\frac{z_i+z_j}{z_i-z_j}(z_i\partial_i - z_j\partial_j)\right]S^\alpha_{\lambda'} - \left[(N-1)|\lambda^B| + \frac{1}{2}N(N-1)^2\right]S^\alpha_{\lambda'} \tag{A.13}$$

Appendix B

Explicit calculations for Chapter 4

B.1 Generic recurrence relations

Follows from Sec. (4.1):

$$
\begin{aligned}
Hf_\lambda &= H\left[X^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda} X^\lambda_\mu b_\mu\right] = X^\lambda_\lambda H b_\lambda + \sum_{\mu<\lambda} X^\lambda_\mu H b_\mu \\
&= X^\lambda_\lambda\left[C^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda} C^\lambda_\mu b_\mu\right] + \sum_{\mu<\lambda} X^\lambda_\mu\left[C^\mu_\mu b_\mu + \sum_{\kappa<\mu} C^\mu_\kappa b_\kappa\right] \\
&= X^\lambda_\lambda C^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda} X^\lambda_\lambda C^\lambda_\mu b_\mu + \sum_{\mu<\lambda} X^\lambda_\mu C^\mu_\mu b_\mu + \sum_{\substack{\mu \\ \mu<\lambda}}\sum_{\substack{\kappa \\ \kappa<\mu}} X^\lambda_\mu C^\mu_\kappa b_\kappa \\
&= X^\lambda_\lambda C^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda}\left(X^\lambda_\lambda C^\lambda_\mu + X^\lambda_\mu C^\mu_\mu\right) b_\mu + \sum_{\substack{\kappa \\ \kappa<\lambda}}\sum_{\substack{\mu \\ \kappa<\mu<\lambda}} X^\lambda_\mu C^\mu_\kappa b_\kappa \\
&= X^\lambda_\lambda C^\lambda_\lambda b_\lambda + \sum_{\kappa<\lambda}\left(X^\lambda_\lambda C^\lambda_\kappa + X^\lambda_\kappa C^\kappa_\kappa\right) b_\kappa + \sum_{\substack{\kappa \\ \kappa<\lambda}}\sum_{\substack{\mu \\ \kappa<\mu<\lambda}} X^\lambda_\mu C^\mu_\kappa b_\kappa \qquad \text{(B.1)} \\
&= X^\lambda_\lambda C^\lambda_\lambda b_\lambda + \sum_{\kappa<\lambda}\left[X^\lambda_\lambda C^\lambda_\kappa + X^\lambda_\kappa C^\kappa_\kappa + \sum_{\substack{\mu \\ \kappa<\mu<\lambda}} X^\lambda_\mu C^\mu_\kappa\right] b_\kappa \\
&= X^\lambda_\lambda C^\lambda_\lambda b_\lambda + \sum_{\kappa<\lambda}\left[X^\lambda_\kappa C^\kappa_\kappa + \sum_{\substack{\mu \\ \kappa<\mu\leq\lambda}} X^\lambda_\mu C^\mu_\kappa\right] b_\kappa, \\
E_\lambda f_\lambda &= E_\lambda\left[X^\lambda_\lambda b_\lambda + \sum_{\mu<\lambda} X^\lambda_\mu b_\mu\right] = E_\lambda X^\lambda_\lambda b_\lambda + \sum_{\kappa<\lambda} E_\lambda X^\lambda_\kappa b_\kappa.
\end{aligned}
$$

Matching term by term as b_λ is a basis, one obtains the following recurrence relations for the X coefficients:

$$\begin{aligned} E_\lambda X^\lambda_\lambda &= X^\lambda_\lambda C^\lambda_\lambda, \\ E_\lambda X^\lambda_\kappa &= X^\lambda_\kappa C^\kappa_\kappa + \sum_{\substack{\mu \\ \kappa<\mu\leq\lambda}} X^\lambda_\mu C^\mu_\kappa. \end{aligned} \tag{B.2}$$

Semplifications lead to the proper recurrence relation

$$X^\lambda_\kappa = \frac{1}{E_\lambda - E_\kappa} \sum_{\substack{\mu \\ \kappa<\mu\leq\lambda}} X^\lambda_\mu C^\mu_\kappa. \tag{B.3}$$

B.2 General form for triangular operators: Second approach

Follows from Sec. (4.2.2). First H matrix elements are computed. For $r > s$ one has:

$$\begin{aligned} \langle r,s|\, H\, |m,n\rangle &= \langle r,s| \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \,|m-k,n+k\rangle \\ &= \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \,\langle r,s|m-k,n+k\rangle \\ &= \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \frac{1}{2}\delta_{r,m-k}\delta_{s,n+k}, \end{aligned} \tag{B.4}$$

and for $r = s$ (only for the symmetric case, otherwise it's 0):

$$\begin{aligned} \langle r,r|\, H\, |m,n\rangle &= \langle r,r| \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \,|m-k,n+k\rangle = \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \,\langle r,r|m-k,n+k\rangle \\ &= \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \frac{2}{2}\delta_{r,m-k}\delta_{s,n+k} = \sum_{k=0}^{(m-n)/2} F^{(m,n)}_k \delta_{r,m-k}\delta_{s,n+k}. \end{aligned} \tag{B.5}$$

Now, Eq. (4.19) is computed:

$$
\begin{aligned}
H\left|\lambda\right\rangle &= \frac{1}{2}\sum_{r,s,m,n}\left\langle r,s\right|H\left|m,n\right\rangle a_r^\dagger a_s^\dagger a_n a_m\left|\lambda\right\rangle \\
&= \frac{1}{2}\sum_{m}\sum_{n<m}\sum_{r,s}\left\langle r,s\right|H\left|m,n\right\rangle a_r^\dagger a_s^\dagger a_n a_m\left|\lambda\right\rangle \\
&\quad + \frac{1}{2}\sum_{m}\sum_{n>m}\sum_{r,s}\left\langle r,s\right|H\left|m,n\right\rangle a_r^\dagger a_s^\dagger a_n a_m\left|\lambda\right\rangle \\
&\quad + \frac{1}{2}\sum_{m}\sum_{r,s}\left\langle r,s\right|H\left|m,m\right\rangle a_r^\dagger a_s^\dagger a_m a_m\left|\lambda\right\rangle \\
&= \frac{1}{2}\sum_{m}\sum_{n<m}\sum_{r,s}\left\langle r,s\right|H\left|m,n\right\rangle a_r^\dagger a_s^\dagger a_n a_m\left|\lambda\right\rangle \\
&\quad + \frac{1}{2}\sum_{n}\sum_{m<n}\sum_{r,s}\left\langle r,s\right|H\left|n,m\right\rangle a_r^\dagger a_s^\dagger a_m a_n\left|\lambda\right\rangle \\
&= \sum_{m}\sum_{n<m}\sum_{r,s}\left\langle r,s\right|H\left|m,n\right\rangle a_r^\dagger a_s^\dagger a_n a_m\left|\lambda\right\rangle \\
&= \sum_{m}\sum_{n<m}\left\{\frac{1}{2}\sum_{r>s}\sum_{k=0}^{(m-n)/2}F_k^{(m,n)}\delta_{r,m-k}\delta_{s,n+k}a_r^\dagger a_s^\dagger a_n a_m\right. \\
&\quad + \frac{1}{2}\sum_{s>r}\sum_{k=0}^{(m-n)/2}F_k^{(m,n)}\delta_{s,m-k}\delta_{r,n+k}a_s^\dagger a_r^\dagger a_n a_m \\
&\quad \left.\overset{B}{+}\sum_{k=0}^{(m-n)/2}F_k^{(m,n)}\delta_{r,m-k}\delta_{r,n+k}a_r^\dagger a_r^\dagger a_n a_m\right\}\left|\lambda\right\rangle \\
&= \sum_{m}\sum_{n<m}\left\{\sum_{r>s}\sum_{k=0}^{(m-n)/2}F_k^{(m,n)}\delta_{r,m-k}\delta_{s,n+k}a_r^\dagger a_s^\dagger a_n a_m \overset{B}{+} F_{(m-n)/2}^{(m,n)}a_r^\dagger a_r^\dagger a_n a_m\right\}\left|\lambda\right\rangle \\
&= \sum_{m}\sum_{n<m}\left\{\sum_{\substack{k=0\\k\neq(m-n)/2}}^{(m-n)/2}F_k^{(m,n)}a_{m-k}^\dagger a_{n+k}^\dagger a_n a_m \overset{B}{+} F_{(m-n)/2}^{(m,n)}a_r^\dagger a_r^\dagger a_n a_m\right\}\left|\lambda\right\rangle \\
&= \sum_{\mu\leq\lambda}F_k^{(m,n)}(\pm)^{N_{SW}}\left|\mu\right\rangle,
\end{aligned}
\tag{B.6}
$$

where:

- $\overset{B}{+}$ means that the factor is non zero only for bosons;

- in the second last line, for fermions one has no particular issue: in fact, $k = (m-n)/2$ identifies an "improper" squeezing in which 2 particles are created in the same state, and the additional term is zero. For bosons, if $(m-n)/2$ isn't integer the summation condition is automatically satisfied and the additonal term is zero; if $(m-n)/2$ is integer, the additional term replaces exactly the forbidden one in the summation;

- the last passage is made by recognising that a operators form squeezings apart from a permutation of N_{SW} exchanges.

B.3 Antisymmetric Laplace-Beltrami on 2 variables slaters

Follows from Sec. (4.4). For this calculation, suppose $n > p$ and let $k = n-p$:

$$
\begin{aligned}
Vsl_{(n,p)} &= \left[\frac{x+y}{x-y}\left(x\partial x - y\partial y\right) - 2\frac{x^2+y^2}{(x-y)^2}\right]\left(x^n y^p - x^p y^n\right) \\
&= \frac{x^p y^p}{x-y}\left[\sum_{l=1}^{k}(x+y)(x^k+y^k) - 2(x^2+y^2)\sum_{l=1}^{k}x^{k-l}y^{l-1}\right] \\
&= \frac{x^p y^p}{x-y}\left[\sum_{l=1}^{k}(x+y)(x^k+y^k) - (x^2+y^2)\sum_{l=1}^{k}\left(x^{k-l}y^{l-1} + x^{l-1}y^{k-l}\right)\right] \\
&= \frac{x^p y^p}{x-y}\sum_{l=1}^{k}\left[(x^{l-1}-y^{l-1})(x^{k-l+2}-y^{k-l+2}) + (x^l - y^l)(x^{k-l}y - xy^{k-l})\right] \\
&= x^p y^p \sum_{l=1}^{k}\left[(x^{k-l+2}-y^{k-l+2})\sum_{t=1}^{l-1}\frac{x^{l-1-t}y^{t-1} + x^{t-1}y^{l-1-t}}{2}\right] \\
&\quad + x^p y^p \sum_{l=1}^{k}\left[(x^{k-l}y - xy^{k-l})\sum_{t=1}^{l}\frac{x^{l-t}y^{t-1} + x^{t-1}y^{l-t}}{2}\right] \\
&= \frac{1}{2}\sum_{l=1}^{k}\sum_{t=1}^{l-1}\left[sl_{(n-t+1,p+t-1)} + sl_{(n-l+t+1,p+l-t-1)}\right] \\
&\quad + \frac{1}{2}\sum_{l=1}^{k}\sum_{t=1}^{l}\left[sl_{(n-t,p+t)} + sl_{(n-l+t-1,p+l-t+1)}\right] \\
&= \sum_{l=0}^{n-p-2}(n-p-1-l)sl_{(n-l,p+l)} + \sum_{l=1}^{n-p}(n-p-1-l)sl_{(n-l,p+l)} \\
&= (n-p-2)sl_{(n,p)} + 2\sum_{l=1}^{n-p-1}(n-p-l)sl_{(n-l,p+l)} \\
&= (n-p-2)sl_{(n,p)} + 2\sum_{l=1}^{(n-p)/2}(n-p-2l)sl_{(n-l,p+l)}.
\end{aligned}
\tag{B.7}
$$

Bibliography

[BH08] B. Andrei Bernevig and F. D. M. Haldane. Model fractional quantum hall states and jack polynomials. *Phys. Rev. Lett.*, 100:246802, Jun 2008.

[DFGIL94] P. Di Francesco, M. Gaudin, C. Itzykson, and F. Lesage. Laughlin's wave functions, coulomb gases and expansion of the discriminant. *International Journal of Modern Physics A*, 09(24):4257–4351, 1994.

[Dun93] G. Dunne. Slater Decomposition of Laughlin States. *Int. J. Mod. Phys.*, B7:4783–4813, 1993.

[FJMM01] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. A differential ideal of symmetric polynomials spanned by jack polynomials at beta = - (r-1)/(k+1). ArXiv.org, 2001.

[Hal79] E. H. Hall. On a new action of the magnet on electric currents. *American Journal of Mathematics*, 2(3):287–292, 1879.

[Jai07] Jainendra K. Jain. *Composite Fermions*. Cambridge University Press, 2007.

[Lau83] R. B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395–1398, May 1983.

[Mac88] I. G. Macdonald. A new class of symmetric functions. In *Publ. I.R.M.A. Strasbourg, 372/S20, Actes 20 S'eminaire Lotharingien*, pages 131–171, 1988.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 1995.

[Sta89] Richard P Stanley. Some combinatorial properties of jack symmetric functions. *Advances in Mathematics*, 77(1):76 – 115, 1989.

[TERB11] Ronny Thomale, Benoit Estienne, Nicolas Regnault, and B. Andrei Bernevig. Decomposition of fractional quantum hall model states: Product rule symmetries and approximations. *Phys. Rev. B*, 84:045127, Jul 2011.

[TSG82] D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.*, 48:1559–1562, May 1982.

[Tsu83] D. C. Tsui. Nobel lecture: Interplay of disorder and interaction in two-dimensional electron gas in intense magnetic fields. *Rev. Mod. Phys.*, 71:891–895, 1983.

[Yos02] Daijiro Yoshioka. *The Quantum Hall Effect*. Springer, 2002.