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Laplacian ground state eigenvalues
of regular polygons

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1 Introduction

In this paper we will approach the eigenvalue problem of the Laplacian operator in regular polygons. Apart from the mathematical significance that this problem might have it is directly applicable to quantum mechanics as the main equation for the problem is equivalent to the Schrödinger equation for a free particle stripped of all constants. Such physical applications may find fruition in problems relating to particle confinement. By imposing the polygonal boundary conditions, we require that the wave function be null outside the polygonal region, thus obtaining a confined particle. For future reference, we state our problem to be:

Problem: To find the eigenvalues λ_N for the equation:

$$\Delta\psi = -\lambda_N\psi \tag{1.1}$$

restricted to an N sided regular polygon with Dirichlet boundary conditions.

The position and size of the N sided regular polygon is generally irrelevant: a coordinate transformation will suffice to transform a solution into an other. In some cases choosing particular polygons will be preferable, as some choices simplify greatly the solution process. In each case that will be taken into consideration the choice of polygon will be discussed profusely.

Unfortunately the goal we have set ourselves to solve is too ambitious for the moment. In the past this problem has surfaced many times and in different contexts and its solution has proven to be very elusive. Notwithstanding many different and ingenious attempts no solution has yet been found and only approximations of λ_N exist, both in terms of numerical estimates and power series of $\frac{1}{N}$ or other parameters. It is worth noting that an exact solution has been found in three special cases: the triangle (necessarily equilateral since we are considering regular polygons), the square and the circle (which of course is not a polygon but is considered as a limiting case as $N \rightarrow \infty$). These exact solutions will also be discussed in detail in the following pages.

Recent attempts at a solution (starting from the '90s) have confirmed a pattern found by many authors which shows a dependency of the coefficients of the series expansion (in terms of $\frac{1}{N}$) of the eigenvalues on the Riemann Zeta function. What is most surprising is the specific relationship that has been found between the Zeta function and the series expansion: when the transformation is from the unit circle to the polygon with the same area, only powers of $\frac{1}{N}$ that may be expressed as the sum prime numbers greater than two are present and each expansion term is proportional to the product of Zeta functions of those prime numbers that sum up to the order of the expansion term. Given this relationship, some authors offer a solution based on the conjecture of continuing regularity in all terms.

Different solutions have been attempted which have covered several areas of mathematics but most of them approach this problem perturbatively from the case of the circle. Two such solutions will be discussed in this paper: one based on Schwartz-Christoffel Mappings and one on the Calculus of Moving Surfaces. The former, introduced by L. G. Molinari in 1997 [24], involves the construction of a complex mapping function through the Schwartz-Christoffel transform, a mathematical tool used to build mappings from certain compact domains in the complex plane to others (further explanation will be given in due time). By constructing such a mapping L. G. Molinari is then able to express the eigenvalue as a series expansion, relating each expansion term to a summation over Bessel functions. By usage of numerical computations L. G. Molinari is then able to give an approximate result. The second approach to be discussed is that by P. Grinfeld and G. Strang, developed between 2004 and 2012 [9,10] and, afterwards also M. Broady [1] in 2013 which relies on the Calculus of Moving Surfaces. The Calculus of Moving Surfaces is a subbranch of Differential Geometry which deals with time dependent imbedded manifolds. This formalism was developed over a century ago in 1903 by French mathematician Jacques Hadamard [11] in the context of fluid dynamics. Later, in 2012 the theory was improved upon by P. Grinfeld [7] with the definition of a proper *invariant time derivative* $\dot{\nabla}$. His solution is based on the construction of a continuous transformation from the integrable case of the circle to that of a polygon. P. Grinfeld constructs a homotopy between the circle and the polygons, after which, by virtue of the Calculus of Moving Surfaces he is able to relate the variation of the eigenvalue to an expansion of the homotopy in terms of the affine parameter (corresponding to the *time* of the Calculus of Moving Surfaces).

Other solutions which are not to be discussed in this paper are based on numerical methods. Such is the solution given by R. S. Jones in 2017 [13] which instead focuses on the above mentioned conjecture of regularity. Having already numerically calculated the eigenvalues with great precision [14], R. S. Jones offers the coefficients for the expansion terms first through a linear regression, after which he applies the Lenstra Lenstra Lovász algorithm [21] for rational factorization to compute the closest rational coefficients for his expansion.

As will be discussed, it has been proven that in the case of transformation from a circle to a polygon with the same area the first terms (up to $\frac{1}{N^2}$) in the expansion are removed. Including R. S. Jones' work, which adds the seventh and eighth term to the expansion [13], for polygons with area π the known terms of the series for the ground state eigenvalue are:

$$\lambda_N = \lambda_0 \left(1 + \frac{4\zeta(3)}{N^3} + \frac{(12 - 2\lambda_0)\zeta(5)}{N^5} + \frac{(8 + 4\lambda_0)\zeta(3)^2}{N^6} + \frac{(36 - 12\lambda_0 - \frac{1}{2}\lambda_0^2)\zeta(7)}{N^7} + \frac{(48 + 8\lambda_0 + 2\lambda_0^2)\zeta(5)\zeta(3)}{N^8} \right) \quad (1.2)$$

λ_0 being the ground state eigenvalue for a circle.

2 Exact solutions

As anticipated the problem may not in general be solved exactly. There are a few instructive cases however for which the eigenvalue may be found. Among these cases, there are those of the equilateral triangle, the square and the circle. One may try to explain why these cases would be much easier than those not solved. One of the characteristics that these polygons have is that they are the only ones (excluding the circle) that may tessellate completely the euclidean plane. Classically this condition allows for a complete solution, as the particle may be thought of as moving in a straight line on the tessellated plane. It is important to note that the hexagon also offers a tessellation of the plane but by reflecting it through all of its sides one finds two different coverings of the plane. Because of this it is not possible to find a complete solution for the hexagon. As is the case for the classical problem, also in the quantum mechanical problem some solutions of the hexagon may be expressed in terms of the solutions for the equilateral triangle. These solutions however do not constitute the whole spectrum and in particular, they do not include the ground state solution. To prove this result the oscillation theorem may come to mind; this theorem however is not applicable as it requires that the problem be one dimensional. An extension of this theorem to multiple dimensions exists and may be applied to this problem. This extension is known as “Courant’s nodal domain theorem” [3], and is discussed in the next section. Following a brief discussion on the general properties of the Laplacian solutions, we will discuss the exact solutions for the above mentioned polygons, in increasing order of complexity: first the square, then the circle and lastly the triangle. What makes the first two cases considerably less difficult than the last is the fact that symmetry in the boundary conditions is directly exploited by the correct choice of coordinates, namely, the Cartesian coordinates for the square case and the polar coordinates for the circle case. With these choices the Laplacian eigenvalue equation becomes separable and the solution may be found directly. The case of the triangle is a little trickier because, for it to be solved, its symmetry properties must be exploited by other means which are necessarily less evident and require some ingenuity. A couple of solutions for the triangle will be reported in the paper.

2.1 General properties of the Laplacian eigenfunctions and eigenvalues

Two geometrical properties of the Laplacian will be discussed in the following. The first will be the above mentioned Courant nodal domain theorem. This theorem will be used to substantiate the claim that it is not possible to build the ground

state solution of the Laplacian in a regular hexagon from solutions of an equilateral triangle. Although in its modern formulation [2] the theorem is very general (I. Chavel provides a proof for Dirichlet, Von Neumann and mixed boundary eigenvalue problems in Riemannian Manifolds) in this paper we will discuss only the case relevant to the Dirichlet boundary conditions, and we will assume functions to be infinitely smooth. The second geometrical property to be discussed will be the Faber-Krahn inequality, concerning the magnitude of the eigenvalues of the Laplacian on the basis of the shape of the domain.

2.1.1 Courant's nodal domain theorem

Courant's nodal domain theorem is a generalization to higher dimensions of the oscillation theorem for the Laplacian eigenvalue problem. For future reference, we recall the oscillation theorem [19]:

Theorem 1 (Oscillation). *The function $\psi_n(x)$ corresponding to the $(n + 1)$ th eigenvalue E_n (the eigenvalues being arranged by order of magnitude) vanishes n times for finite values of x .*

The nodal theorem was proved by Richard Courant and first published in the famous book authored with David Hilbert in 1924 [3] on Mathematical Physics. While in one dimension it is possible to characterize solutions based on the number of roots the eigenfunction has, in multiple dimensions the attention is shifted towards connected subsets of the domain of the eigenfunction, enclosed by nodal lines. In particular we give the following definition [2]:

Definition 1. Let $f : M \rightarrow \mathbb{R} \in C^0$, M being an orientable n -dimensional Riemannian manifold. Then the *nodal set* of f is the set $f^{-1}[0]$ and a *nodal domain* is a connected component of $\overline{M} \setminus f^{-1}[0]$.

Generalizing this theorem to higher dimensions inevitably is the source of an inconvenient drawback: in more than one dimensions it is not possible to uniquely identify a solution based on the number of nodal domains present. The theorem only offers an upper limit. This however is sufficient for our intent to prove that it is not possible to build the ground state solution for the Laplacian in a regular hexagon by assembling solutions of the equilateral triangle. Specifically the theorem states [2]:

Theorem 2 (Courant's nodal domain theorem). *Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$ be the list of eigenvalues of the Laplacian operator and $\{\phi_1, \phi_2, \phi_3, \dots\}$ be a complete orthonormal basis of $\mathcal{L}^2(M)$ (M being, as before, an orientable n -dimensional Riemannian manifold) with each ϕ_j eigenfunction of λ_j for $j = 1, 2, 3, \dots$. Then the number of nodal domains of ϕ_k is less than or equal to k for all $k = 1, 2, 3, \dots$.*

Since we are looking for the solution with the least possible eigenvalue λ_1 , Courant's theorem states that the eigenfunction will have exactly one nodal domain. Because any solution for the hexagon derived from a solution of the equilateral triangle will have at least six nodal domains, we can conclude that the ground state solution of the hexagon may not be built by assembling solutions of the equilateral triangle.

A proof of the theorem will be offered in the following, however a few results must be recalled before. Most of what follows is taken directly from [2]. The theorem is very general, however we only need to prove it for manifolds that are subsets of \mathbb{R}^2 and infinitely continuous functions. Moreover we will focus on: the problem with Dirichlet boundary conditions and connected manifolds, with compact closures and nonempty piecewise C^∞ boundaries (which we will refer to as *normal domains*). There are three main results needed to be recalled for the theorem: Rayleigh's theorem, the "max-min" method and the unique continuation theorem. Further on, unless otherwise noted, scalar products (and norms) will be those of $\mathcal{L}^2(\mathbb{R}^m)$. We start by defining the space of admissible function $\mathfrak{H}(M)$ as the completion of C^∞ functions compactly supported on M . We will restrict our search for eigenfunctions in this space. We simplify the notation by introducing the Dirichlet integral:

$$D[f, h] := \langle \nabla f, \nabla h \rangle$$

Theorem 3 (Rayleigh's theorem). *We are given a normal domain with the Dirichlet eigenvalue problem having the function space $\mathfrak{H}(M)$ and eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots \tag{2.1}$$

where each eigenvalue is repeated the number of times equal to its multiplicity. Then for any $f \in \mathfrak{H}(M)$, $f \neq 0$:

$$\lambda_1 \leq \frac{D[f, f]}{\|f\|^2}$$

(where $\|\cdot\|$ is the norm in $\mathcal{L}^2(M)$) with the equality if and only if f is an eigenfunction of λ_1 . If $\{\phi_1, \phi_2, \phi_3, \dots\}$ is a complete orthonormal basis of $\mathcal{L}^2(M)$ such that ϕ_j is an eigenfunction of λ_j for each $j = 1, 2, 3, \dots$ then for $f \in \mathfrak{H}(M)$, $f \neq 0$ such that:

$$\langle f, \phi_1 \rangle = \langle f, \phi_2 \rangle = \langle f, \phi_3 \rangle = \dots = \langle f, \phi_{k-1} \rangle = 0 \tag{2.2}$$

we have the inequality:

$$\lambda_k \leq \frac{D[f, f]}{\|f\|^2}$$

with equality if and only if f is an eigenfunction of λ_k

Proof. For $f, h \in \mathfrak{H}(M)$, by Green's first identity we have:

$$\begin{aligned} \int_M (h\Delta f + (\nabla h) \cdot (\nabla f)) dV &= 0 \\ \langle h, \Delta f \rangle + \langle \nabla h, \nabla f \rangle &= 0 \\ D[h, f] &= -\langle h, \Delta f \rangle \end{aligned}$$

Let $f \in \mathfrak{H}(M)$, set:

$$\alpha_j = \langle f, \phi_j \rangle$$

for $k > 1$, (2.2) is equivalent to setting $\alpha_1 = \alpha_2 = \dots = \alpha_{k-1} = 0$. Hence for all $k = 1, 2, \dots$ and $r = k, k + 1, \dots$ we have:

$$\begin{aligned}
 0 &\leq D \left[f - \sum_{j=k}^r \alpha_j \phi_j, f - \sum_{j=k}^r \alpha_j \phi_j \right] \\
 &= D[f, f] - 2 \sum_{j=k}^r \alpha_j D[f, \phi_j] + \sum_{j=k, l}^r \alpha_j \alpha_l D[\phi_j, \phi_l] \\
 &= D[f, f] + 2 \sum_{j=k}^r \alpha_j \langle f, \Delta \phi_j \rangle - \sum_{j=k, l}^r \alpha_j \alpha_l \langle \phi_j, \Delta \phi_l \rangle \\
 &= D[f, f] - \sum_{j=k}^r \lambda_j \alpha_j^2
 \end{aligned}$$

Hence $\sum_{j=k}^r \lambda_j \alpha_j^2$ is finite. Furthermore we have:

$$D[f, f] \geq \sum_{j=k}^r \lambda_j \alpha_j^2 \geq \lambda_k \sum_{j=k}^r \alpha_j^2 = \lambda_k \|f\|^2$$

which proves the theorem. □

When proving the nodal domain theorem, the Rayleigh theorem will be use to provide a lower bound to the eigenvalue of a function. An upper bound is found by the “max-min theorem”. The procedure of identifying a function by finding upper and lower bounds is the “max-min method”.

Theorem 4 (max-min). *Given $v_1, v_2, v_3, \dots, v_{k-1} \in \mathcal{L}^2(M)$ let:*

$$\mu = \inf_{f \in \mathfrak{H}(M)} \frac{D[f, f]}{\|f\|^2}$$

where $f \in \mathfrak{H}(M)$ and f is orthogonal to v_j for $j = 1, 2, 3, \dots, k - 1$ then for eigenvalues given by (2.1),

$$\mu \leq \lambda_k$$

where the equality holds if v_j are the first $k - 1$ orthonormal eigenfunctions.

Proof. Let $f \in \mathfrak{H}(M)$ be:

$$f = \sum_{j=1}^k \alpha_j \phi_j$$

where ϕ_j for $j = 1, 2, 3, \dots, k$ are the first k eigenfunctions. The orthogonality conditions implies that:

$$0 = \sum_{j=0}^k \alpha_j \langle \phi_j, v_l \rangle, \quad l = 1, 2, 3, \dots, k - 1$$

If we think of α_j as the variables and the scalar products as the coefficients, this condition gives rise to $k - 1$ equations of k variables, which implies that there must be nontrivial solutions. Hence:

$$\mu \|f\|^2 \leq D[f, f] = \sum_{j=1}^k \lambda_j \alpha_j^2 \leq \lambda_k \sum_{j=1}^k \alpha_j^2 = \lambda_k \|f\|^2$$

which implies that, $\mu \leq \lambda_k$. □

The unique continuation theorem will not be proven, however it states that [12]:

Theorem 5. *If any solution $f \in \mathfrak{S}(M)$ of the Laplacian eigenvalue equation vanishes on a nonempty open subset of a connected domain, then f is identically zero on the domain.*

We now sum everything up and with a *coup de grâce* give a proof of Courant's theorem.

Proof. We will prove the theorem by a *reductio ad absurdum*. Let ϕ_k be an Laplacian eigenfunction with eigenvalue λ_k . Let $D_1, D_2, \dots, D_k, D_{k+1}, \dots$ be nodal domains of ϕ_k . Define the functions:

$$\psi_j = \begin{cases} \phi_k & x \in D_j \\ 0 & x \notin D_j \end{cases} \quad j = 1, 2, 3, \dots, k$$

Again, we look for a function $f = \sum_{j=1}^k \alpha_j \psi_j$, $f \neq 0$ such that f is orthogonal to the first $k - 1$ eigenfunctions of the Laplacian. This can be done in the same way as we did above, by constructing a system of equations with more variables than conditions. Furthermore each function ψ_j is itself a solution to the eigenfunction problem. Because f is orthogonal to the first $k - 1$ eigenfunctions we may apply Rayleigh's theorem. We then have that:

$$\lambda_k \leq \frac{D[f, f]}{\|f\|^2}$$

However, by the max-min theorem we have that:

$$\inf_{f \in \mathfrak{S}(M)} \frac{D[f, f]}{\|f\|^2} \leq \lambda_k$$

These last two inequalities imply that:

$$\frac{D[f, f]}{\|f\|^2} = \lambda_k$$

Which is equivalent to stating that f is an eigenfunction of the Laplacian operator with eigenvalue λ_k :

$$\begin{aligned} D[f, f] &= \lambda_k \|f\|^2 \\ \langle \Delta f, f \rangle + \langle \lambda_k f, f \rangle &= 0 \\ \langle \Delta f + \lambda_k f, f \rangle &= 0 \end{aligned}$$

Because $f \neq 0$, and the choice of f is general, the first term in the scalar product must be zero. However, by its construction f is identically zero in all the nodal sets D_{k+1}, D_{k+2}, \dots , so by the unique continuation theorem f is identically zero. This contradicts our choice of f . □

2.1.2 Faber-Krahn inequality

Proved independently by G. Faber and E. Krahn in the same years of the publication of R. Courant and D. Hilbert's book, [5, 15], the Faber-Krahn inequality states that, in any number of dimensions, for a fixed volume (or measure), the magnitude of the ground state eigenvalue of the Laplacian operator will be minimized on a ball. Specifically [28]:

Theorem 6 (Faber-Krahn inequality). *Let $D \subset \mathbb{R}^n$ be a bounded domain and let B be the ball centered at the origin with $\text{Vol}(D) = \text{Vol}(B)$. Let $\lambda_1(B)$, $\lambda_1(D)$ be the ground state eigenvalues of the Laplacian in the domains B and D respectively. Then $\lambda_1(B) \leq \lambda_1(D)$ with equality if and only if $D = B$ almost everywhere.*

Modern proofs of the theorem (including [28]) rely on Courant's nodal domain theorem. We will not provide a proof of this theorem, however, for regular polygons, the first term of the eigenvalue expansion 1.2 is positive, thus reflecting the Faber-Krahn inequality. Furthermore, it is known that for any $N \in \mathbb{N}$ there exists a N sided polygon minimizing the eigenvalue however it has only been conjectured [27] that this is the regular one. A few attempts have been made to prove this conjecture but none have been successful yet. Some, including [26] by Carlo Nitsch however, have elucidated the problem by finding previously unknown inequalities that may be used by posterity to prove the full conjecture.

2.2 The square

The square is certainly the easiest of the three exactly solvable cases. The optimum choice of orientation for the square is with the sides parallel to the axes. This choice makes equation (1.1) separable, allowing for a solution on each direction (x and y). The problem becomes one dimensional and its solution is a matter of elementary quantum mechanics (or elementary differential equations). Imposing the boundary conditions $x, y \in [-\frac{L}{2}, \frac{L}{2}]$ the solution is, for $\eta = x, y$:

$$\psi_\eta(\eta) = \sqrt{\frac{2}{L}} \sin\left(\frac{k_\eta \pi}{L} \left(\eta - \frac{L}{2}\right)\right) \quad (2.3)$$

Where k_η is an integer and represents the principal quantum number of the excited state and L is the side length chosen for the square. Then for each k_x, k_y the eigenvalue is:

$$\lambda_{k_x, k_y} = \left(\frac{\pi}{L}\right)^2 (k_x^2 + k_y^2)$$

Notice that an exchange between the values of k_x and k_y maintains the same eigenvalue, so our solutions will be at least doubly degenerate if $k_x \neq k_y$. Also in general the sum of the squares of two different pairs of integers may be the same (e.g. $5^2 + 5^2 = 1^2 + 7^2$) so the degeneracy for some pairs of principal quantum numbers is even greater.

2.3 The circle

The case of a circular boundary requires a little bit more attention. In polar coordinates the Laplacian eigenvalue equation may be expressed as:

$$\Delta\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi = -\lambda\psi \quad (2.4)$$

The symmetry in the boundary conditions suggests a possible solution in terms of a product of eigenfunctions for the angular and radial part, i.e. $\psi(r, \theta) = R(r)\Theta(\theta)$. With this substitution equation (2.4) becomes:

$$\Delta\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) R(r)\Theta(\theta) = -\lambda R(r)\Theta(\theta)$$

Separating the equations by virtue of the introduced constant k we write:

$$\begin{cases} \frac{\partial^2}{\partial \theta^2} \Theta(\theta) + k^2 \Theta(\theta) = 0 \\ \frac{\partial^2}{\partial r^2} R(r) + \frac{1}{r} \frac{\partial}{\partial r} R(r) + \left(\lambda - \frac{k^2}{r^2} \right) R = 0 \end{cases}$$

the first equation is elementary and the second is (up to the variable transformation $r \rightarrow \frac{z}{\rho}$, $\rho^2 = \lambda$) the well known Bessel equation. It is important to note that k must be chosen to be an integer. This condition ensures 2π periodicity of $\psi(r, \theta)$ in θ . These functions were first discovered by Daniel Bernoulli (1700-1782) in the context of hanging chains, but were later generalized and adopted by Friedrich Wilhelm Bessel (1784–1846) in his studies of planetary motion [30]. Their solution is well known and may be derived using Frobenius' method. The solutions are organized in two categories and are:

$$J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} \left(\frac{x}{2} \right)^{2m+k}$$

$$Y_k(x) = \frac{J_k(x) \cos(k\pi) - J_{-k}(x)}{\sin(k\pi)}$$

respectively called Bessel equations of the first and the second kind. Because the latter are not regular in the origin they must be discarded. The infinitely many roots of the Bessel functions may be numerically calculated. Let $\rho^{(n,k)}$ be the n -th root of the k -th Bessel function of the first kind. The general, unnormalized solution for the eigenfunction to the Laplacian in a unit circle is then:

$$\psi_{n,k}(r, \theta) = J_k(\rho^{(n,k)} r) e^{\pm ik\theta}$$

Some useful identities that will be used in a later section are now derived. Having reduced the two-dimensional problem into a one-dimensional one, if we seek the fundamental solution we may apply the oscillation theorem 1 which imposes that we take $k = 0$ as, for $k \neq 0$ the radial part of the wave function is zero in the

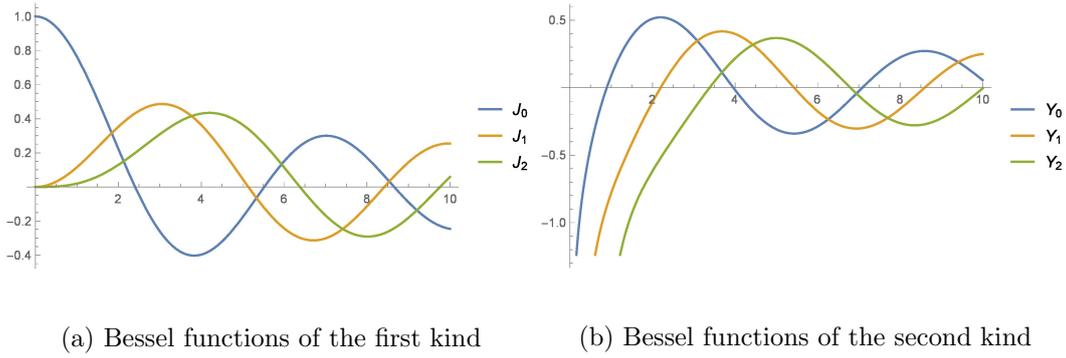


Figure 2.1: First Bessel functions

origin. Obviously then we must also take the first root: $\rho^{(0,0)}$. Hence the ground state eigenvalue is $\lambda_0 = (\rho^{(0,0)})^2$ corresponding to the wave function:

$$\psi_{0,n}(r, \theta) = N J_0(\rho^{(0,0)} r) = N \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+1)} \left(\frac{\rho^{(0,0)} r}{2} \right)^{2m} \quad (2.5)$$

Where N is the normalization factor. It can be shown that the normalized $\psi_{0,n}$ is:

$$\psi_{0,n}(r, \theta) = \frac{J_0(\rho^{(n,0)} r)}{\sqrt{\pi} J_1(\rho^{(n,0)})}$$

Furthermore, using equation (2.5):

$$\begin{aligned} \frac{\partial}{\partial r} \psi_{0,n}(r, \theta) &= -\frac{\rho^{(n,0)} J_1(\rho^{(n,0)} r)}{\sqrt{\pi} J_1(\rho^{(n,0)})} \implies \frac{\partial}{\partial r} \psi_{0,n}(1, \theta) = -\frac{\rho^{(n,0)}}{\sqrt{\pi}} \quad (2.6) \\ \frac{\partial^2}{\partial r^2} \psi_{0,n}(r, \theta) &= -\frac{(\rho^{(n,0)})^2 \left(J_0(\rho^{(n,0)} r) - \frac{J_1(\rho^{(n,0)} r)}{(\rho^{(n,0)} r)} \right)}{\sqrt{\pi} J_1(\rho^{(n,0)})} \implies \frac{\partial^2}{\partial r^2} \psi_{0,n}(1, \theta) = \frac{\rho^{(n,0)}}{\sqrt{\pi}} \end{aligned}$$

These last two results will prove to be useful when we will seek an approximation of the eigenvalue through the Calculus of Moving Surfaces.

2.4 The equilateral triangle

The case of the equilateral triangle is certainly the most interesting among the three exact solutions. This problem first appeared in the contexts of heat diffusion and elasticity, and its first solution is accredited to G. Lamé in 1861 [17, 18]. There are many different modern approaches to the solution of the problem, which exploit different properties of the symmetrical nature of the boundary. The most ingenious of these is certainly that of H. R. Krishnamurthy, H. S. Mani and H. C. Verna published in 1982 [16]. Krishnamurthy's solution is based on a coordinate transformation mapping the problem of a single particle in an equilateral triangle to that of three fermionic point masses restricted to a segment. Given that the latter is easily solvable, a solution for the triangle is obtained. Another elegant solution is given by B. J. McCartin [23], whose solution is an improvement from the original

by G. Lamé. To fully understand the spectrum of the solutions it is instructive to consider N. Stambaugh and M. Semon's work [31], which offers great insight on the eigenfunctions, and in particular on their classification based on their transformations when acted upon by C_{3v} : the group of transformations that transform the equilateral triangle into itself. Although it won't be discussed thoroughly, Li's work [22] is notable as he is able to find a complete list of solutions from the symmetry properties alone. There are many other possible solutions to the problem, such as that by W. Gaddah [6] which is based on gauge transformations of the fundamental equation, however, for brevity's sake these will not be reported.

2.4.1 Three particles solution

In their paper, Krishnamurthy et al. [16], attempt a solution of the problem of a free particle in an equilateral triangle in the context of a different similar problem: that of finding the exact solution for a free particle inside a tetrahedron. In their solution, which follows from a discussion of the problem of N particles on a segment, they are able to relate the problem of a single particle in an equilateral triangle to that of three *hard cores* (as they call them) on a segment. If these are considered to be fermionic (in that their wave function is antisymmetric by pair exchange) and noninteracting, the wave function may be expressed as a Slater determinant of wave functions such as (2.3). However, to simplify the expression, rather than considering trigonometric functions, Krishnamurthy suggests considering periodic boundaries and expressing the wave function as a complex exponential. The segment is also dilated to a length of 2π . The wave function then becomes:

$$\psi_{n_1, n_2, n_3}(\theta_1, \theta_2, \theta_3) = N \begin{vmatrix} \exp(in_1\theta_1) & \exp(in_1\theta_2) & \exp(in_1\theta_3) \\ \exp(in_2\theta_1) & \exp(in_2\theta_2) & \exp(in_2\theta_3) \\ \exp(in_3\theta_1) & \exp(in_3\theta_2) & \exp(in_3\theta_3) \end{vmatrix} \quad (2.7)$$

N being the normalizing factor. Furthermore, without loss of generality we can take:

$$\begin{cases} \theta_1 \geq \theta_2 \\ \theta_2 \geq \theta_3 \\ \theta_3 + 2\pi \geq \theta_1 \end{cases} \quad (2.8)$$

In their paper Krishnamurthy et al. find a transformation which they show maps the problem of three fermionic particles to one particle confined to an equilateral triangle. This transformation may follow from the attempt to transform (2.8) into the boundary conditions for an equilateral triangle. The triangle they choose is that of vertices $(0, 0)$, $(\sqrt{2}\pi, \sqrt{\frac{2}{3}}\pi)$, $(0, 2\pi)$. Let Y_1, Y_2, Y_3 be the transformed coordinates. Let Y_1, Y_2 span the plane containing the equilateral triangle. Then, the boundary conditions become:

$$\begin{cases} Y_1 \geq 0 \\ Y_1 \geq \sqrt{3}Y_2 \\ \frac{Y_1}{\sqrt{2}} \leq 2\pi - \sqrt{\frac{3}{2}}Y_2 \end{cases} \quad (2.9)$$

Suppose we wish to transform in the order given the boundary equations (2.8) into the equations (2.9). We wish to impose conditions on a general matrix $M \in GL(3, \mathbb{R})$ such that the boundary conditions transform as stated. Specifically, let:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & g & h \\ e & f & i \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

The invertibility of the transformation will be checked after the boundary conditions are imposed. It is immediate to see, that, for the correct variables to appear in the transformed equations it must be that $b = g$, $c = h = i$. The first equation does not give any sort of condition on the value of the parameters of M if not that $a - d > 0$. Matching the second and third equations in the boundary conditions we see that:

$$\begin{cases} \frac{f - b}{d - e} = \sqrt{3} \\ e - a = \frac{1}{\sqrt{2}} \\ f - b = -\sqrt{\frac{3}{2}} \end{cases}$$

from which it immediately follows that $d - e = -\frac{1}{\sqrt{2}}$. These conditions do not uniquely identify a solution so we must impose further conditions. Krishnamurthy chooses to pick $c = 3^{-\frac{1}{3}}$, $b = 6^{-\frac{1}{2}}$ and $e = 0$. These choices stem from an intermediate transformation which he defines for the general case of N cores. With these choices it follows that:

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

The wave function (2.7) under such transformation becomes:

$$\begin{aligned} & \psi_{n_1, n_2, n_3}(Y_1, Y_2, Y_3) = \\ & = N \left| \begin{array}{ccc} \exp\left(in_1 \left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_1 \left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_1 \left(-\frac{2Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) \\ \exp\left(in_2 \left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_2 \left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_2 \left(-\frac{2Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) \\ \exp\left(in_3 \left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_3 \left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) & \exp\left(in_3 \left(-\frac{2Y_2}{\sqrt{6}} + \frac{Y_3}{\sqrt{3}}\right)\right) \end{array} \right| \\ & = N e^{i\left((n_1+n_2+n_3)\frac{Y_3}{\sqrt{3}}\right)} \\ & \left| \begin{array}{ccc} 1 & 1 & 1 \\ \exp\left(i(n_2 - n_1) \left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(n_2 - n_1) \left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(n_2 - n_1) \left(-\frac{2Y_2}{\sqrt{6}}\right)\right) \\ \exp\left(i(n_3 - n_1) \left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(n_3 - n_1) \left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(n_3 - n_1) \left(-\frac{2Y_2}{\sqrt{6}}\right)\right) \end{array} \right| \end{aligned}$$

We now suppress the coordinate Y_3 along with its quantum number. This variable represents the position of the center of mass of the points. The wave function then simplifies to:

$$\psi_{ml}(Y_1, Y_2) = \tilde{N} \begin{vmatrix} \exp\left(im\left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(im\left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(im\left(-\frac{2Y_2}{\sqrt{6}}\right)\right) \\ \exp\left(i(l+m)\left(\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(l+m)\left(-\frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{6}}\right)\right) & \exp\left(i(l+m)\left(-\frac{2Y_2}{\sqrt{6}}\right)\right) \end{vmatrix} \quad (2.10)$$

Which is the desired solution to the eigenfunction problem. The eigenvalue is¹:

$$\lambda = \frac{4}{9} (l^2 + m^2 + lm) \quad (2.11)$$

2.4.2 Direct solution

However quick and elegant may the previous solution be, it does not offer much insight into the solutions of the problem. For this, a more direct solution, like the one given by B. J. McCartin [23] is useful. He solves the problem directly and his solution starts from the beginning of G. Lamé's work. It is interesting to compare B. J. McCartin's work to that of N. Stambaugh and M. Semon. [31] whose paper offers much insight into the symmetrical nature of the solutions. More specifically, Stambaugh considers the two dimensional representation of C_{3v} the group associated with the symmetries of the equilateral triangle. Such a group is often called the dihedral group of order 6 and may also be referred to as D_3 or D_6 . As the Hamiltonian (inclusive of the boundary conditions) commutes with the elements of the 2D representation of this group, the action of an element of the group on a solution of the equation (1.1) produces a solution (possibly the same) to the same Hamiltonian. The group elements may be represented by operators on the Hilbert space concerning the problem: if $\Gamma(g)$ is a representation of an element $g \in C_{3v}$ and $\psi(x, y)$ is an \mathcal{L}^2 function in an equilateral triangle then we define the action of the operator \hat{g} in ψ to be $\hat{g}\psi(x, y) = \psi(\Gamma(g^{-1})(x, y))$. Because these operatorial representations commute with the Hamiltonian solutions are mapped to solutions. Stambaugh's focus is on finding relationships between solutions behaving differently under the action of the operatorial representation of C_{3v} .

The two dimensional representation of the group C_{3v} may be considered as generated by three elements: the identity E , a $\frac{2\pi}{3}$ counterclockwise rotation σ and a reflection through one of the axes of the triangle μ . Without loss of generality we may consider an equilateral triangle centered in the origin with a vertex on the positive x axis. We may consider μ to be a reflection about such axis. All of the other transformations may be constructed from these. All products and commutators may be constructed from the following equalities:

$$\mu\sigma = \sigma^{-1}\mu, \quad \sigma^3 = E, \quad \mu^2 = E$$

¹In their paper in the last line of the matrix (2.10) they do not set $l+m$ as a quantum number in the exponent but rather they leave it as l . We chose to put the sum of the quantum number in order to make the eigenvalue (2.11) the same as the one calculated in the next section.

$\hat{\mu} \backslash \hat{\sigma}$	Symmetric	Asymmetric
Symmetric	A_1	E_1
Antisymmetric	A_2	E_2

Table 2.1: Symmetry classes of the Laplacian eigenfunctions in an equilateral triangle

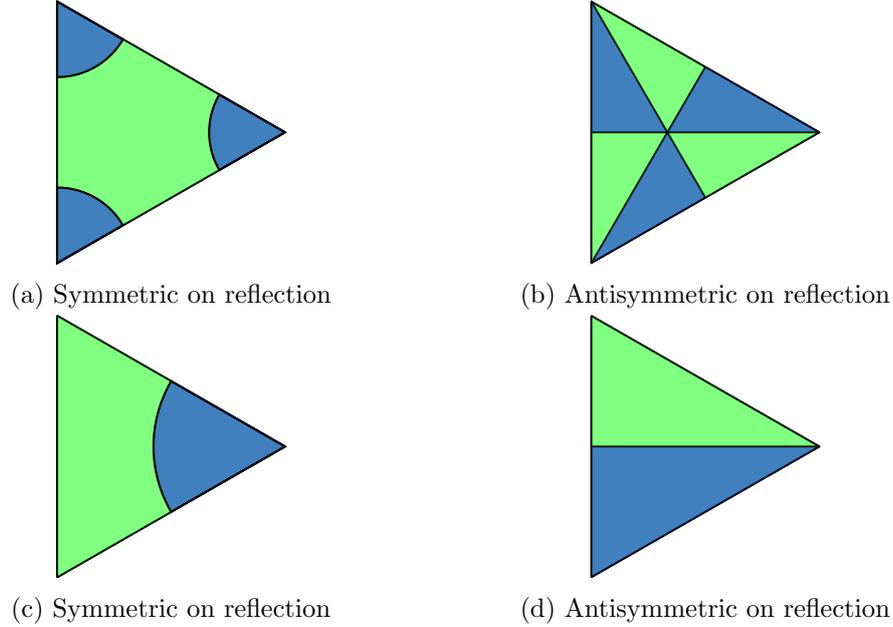


Figure 2.2: Rotationally symmetric and asymmetric patterns on an equilateral triangle.

The 2D representations of the elements μ and σ are then:

$$\Gamma_3(\sigma) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \Gamma_3(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As anticipated Stambaugh focuses on the operatorial representation. Consider for example the rotation $\hat{\sigma}$. There are three possibilities of outcome when this operator is applied to a function ψ . It may be that $\hat{\sigma}\psi = \pm\psi$, in which case we say that ψ is either symmetric or antisymmetric or that it is not an eigenfunction in which case we say that ψ is asymmetric. Furthermore, it is possible to eliminate the antisymmetric case as $\sigma^3 = E$ implies that, given a function f :

$$f = \hat{\sigma}^3 f = -f$$

hence f must be zero everywhere. In the same way, we may divide the set of $\hat{\sigma}$ -symmetric functions by their result when acted upon by $\hat{\mu}$: they can either be symmetric, antisymmetric or asymmetric. In this case it is possible to remove the

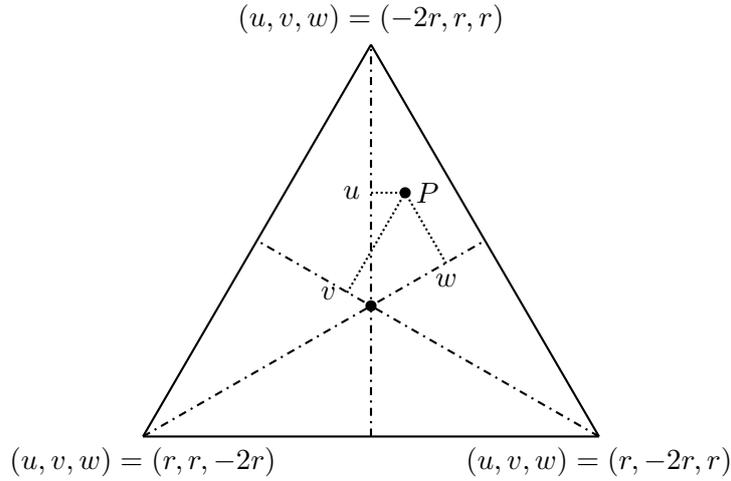


Figure 2.3: Triangular coordinate system

asymmetric case as, if f is $\hat{\mu}$ -asymmetric we may define the two functions:

$$\begin{aligned} f_+ &= f + \hat{\mu}f \\ f_- &= f - \hat{\mu}f \end{aligned}$$

Which are $\hat{\mu}$ -symmetric and $\hat{\mu}$ -antisymmetric respectively, and their sum is $2f$. The same sort of classification may be constructed with $\hat{\sigma}$ asymmetric functions. We then have divided all of the solutions in four categories based on their symmetry properties when acted upon by C_{3v} . We shall refer to these symmetry classes as noted in table 2.1.

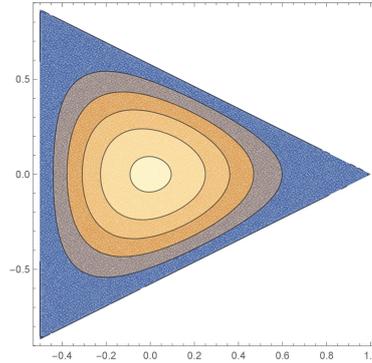
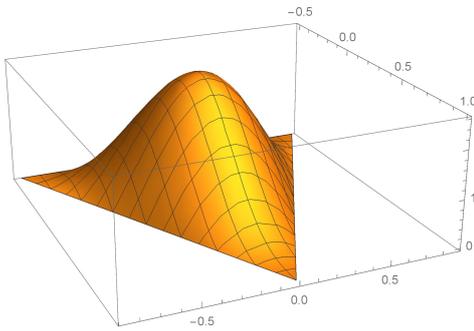
As a visual aid for the symmetry classes, consider figures 2.2. These do not represent functions but patterns that respect the symmetric properties of A_1 , A_2 , E_1 and E_2 . Once a solution is found, others may be generated. The simplest way to do so is by tessellation. Stambaugh shows two ways to tessellate the equilateral triangle, resulting in different outcomes. Stambaugh calls these solutions harmonics.

Having categorized the solutions to the equation (1.1) we may start to look for a solution in algebraic form. Arguing as the original paper by Lamé, McCartin starts with a coordinate transformation, mapping the Cartesian coordinates (x, y) to what he calls triangular coordinate (u, v, w) . The value of these is defined as the distance from the center of the projections on the altitudes. Consider figure 2.3 as a reference. Let the positive direction of each coordinate be assigned as the one directed away from the vertex. This way, if r is the inradius, each vertex is some permutation of $-2r, r, r$. This way the boundary conditions are easily verified. With this coordinate transformation and with an ansatz of variable separation, McCartin finds two general solutions to (1.1), one symmetric and one antisymmetric in the exchange of the pair $v \leftrightarrow w$. He then proceeds to manually impose the boundary conditions, following which he is able to express the eigenvalue. His expression for the eigenvalue agrees with that found by Krishnamurthy. His solution, translated to an equilateral triangle with vertices $(0, 1)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and expressed

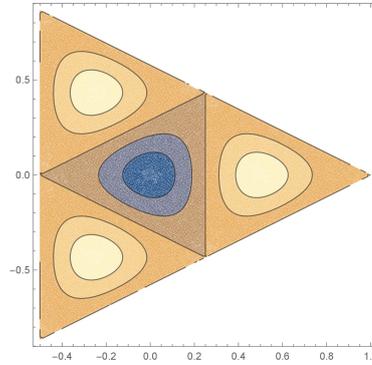
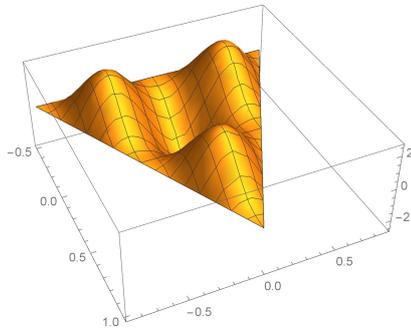
in x, y coordinates reads:

$$\begin{cases} \psi_{n,m}^s(x, y) = \sin\left(\frac{1}{3}\pi m(2-2y)\right) \cos\left(\frac{2\pi x(m+2n)}{3\sqrt{3}}\right) + \sin\left(\frac{1}{3}\pi n(2-2y)\right) \cos\left(\frac{2\pi x(-2m-n)}{3\sqrt{3}}\right) \\ \quad - \cos\left(\frac{2\pi x(m-n)}{3\sqrt{3}}\right) \sin\left(\frac{1}{3}\pi(2-2y)(m+n)\right) \\ \psi_{n,m}^a(x, y) = \sin\left(\frac{1}{3}\pi m(2-2y)\right) \sin\left(\frac{2\pi x(m+2n)}{3\sqrt{3}}\right) + \sin\left(\frac{1}{3}\pi n(2-2y)\right) \sin\left(\frac{2\pi x(-2m-n)}{3\sqrt{3}}\right) \\ \quad - \sin\left(\frac{2\pi x(m-n)}{3\sqrt{3}}\right) \sin\left(\frac{1}{3}\pi(2-2y)(m+n)\right) \end{cases}$$

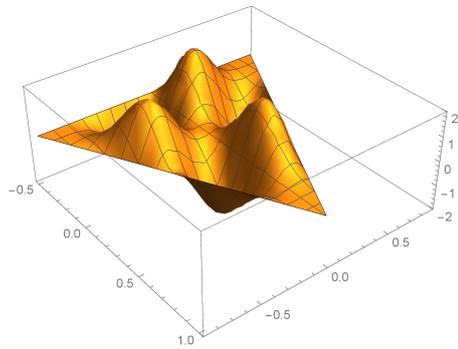
Some of the first few eigenfunctions are represented in figures 2.4.



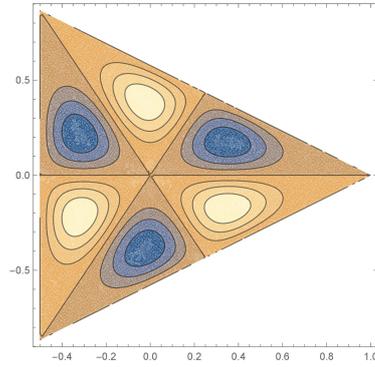
(a) A_1 , $(m, n) = (1, 1)$ ground state solution (b) A_1 , $(m, n) = (1, 1)$ ground state solution, contour plot



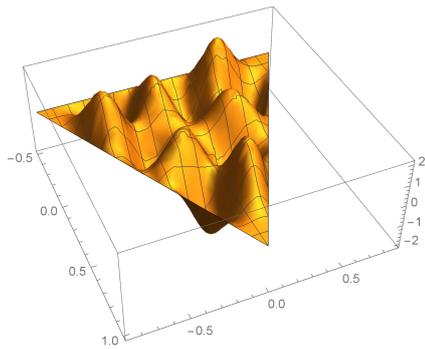
(c) A_1 , $(m, n) = (2, 2)$ first harmonic of ground state (d) A_1 , $(m, n) = (2, 2)$ first harmonic of ground state contour plot



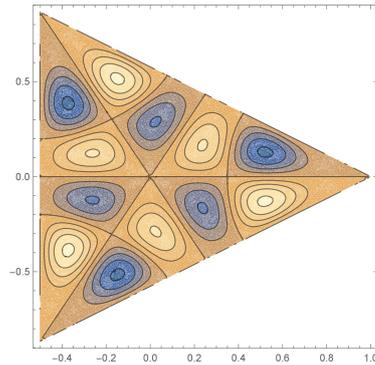
(e) A_2 , $(m, n) = (1, 4)$ lowest A_2 solution



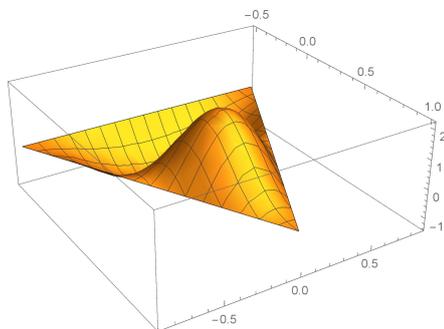
(f) A_2 , $(m, n) = (1, 4)$ lowest A_2 solution contour plot



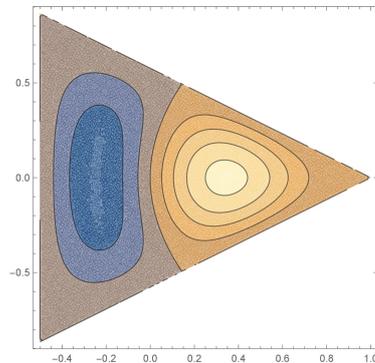
(g) A_2 , $(m, n) = (2, 5)$



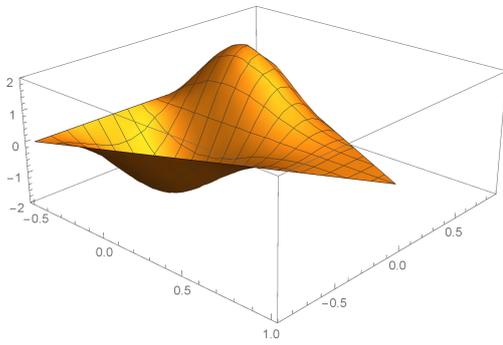
(h) A_2 , $(m, n) = (2, 5)$ contour plot



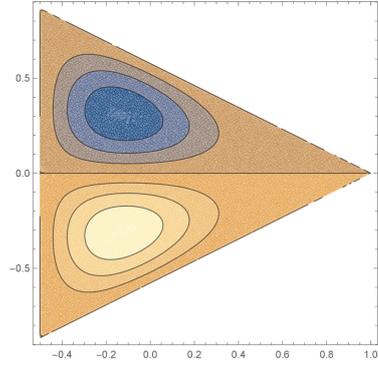
(i) E_1 , $(m, n) = (2, 1)$ lowest E_1 solution



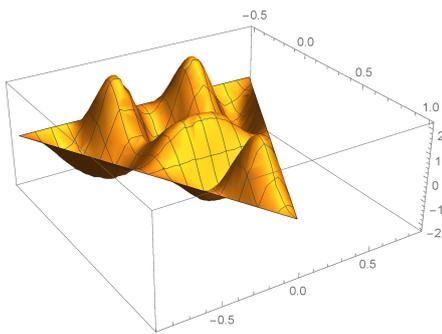
(j) E_1 , $(m, n) = (2, 1)$ lowest E_1 solution contour plot



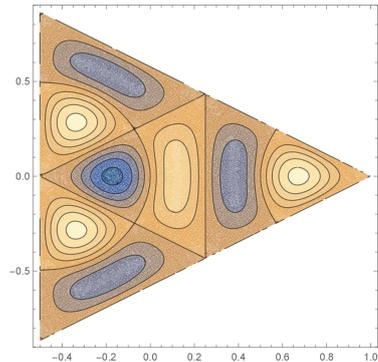
(k) E_2 , $(m, n) = (1, 2)$ lowest E_2 solution



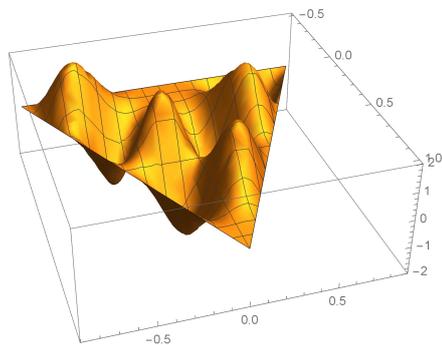
(l) E_2 , $(m, n) = (1, 2)$ lowest E_2 solution contour plot



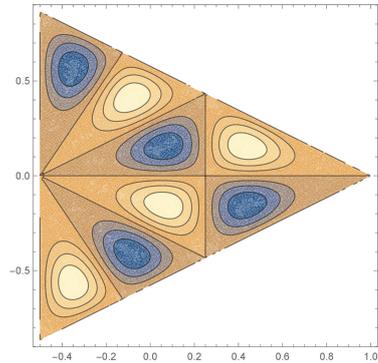
(m) E_1 , $(m, n) = (2, 4)$



(n) E_1 , $(m, n) = (2, 4)$ contour plot



(o) E_2 , $(m, n) = (2, 4)$



(p) E_2 , $(m, n) = (2, 4)$ contour plot

Figure 2.4: Eigenfunctions of (1.1) in an equilateral triangle

3 Perturbative solutions

3.1 Solution through the Calculus of Moving Surfaces

Recent attempts to the solution of the eigenvalue problem [9, 10] of the Laplacian in regular polygons have partially shifted their attention to the Calculus of Moving Surfaces. This mathematical theory offers geometric insight into the problem, which can be exploited in a straightforward fashion to aide to its solution. Through this technique a few illuminating results have been proven. It is possible to prove, and a proof will be offered in this paper, that by rescaling the polygons such that their area is the same as that of the perturbed circle, the first terms to appear in the $\frac{1}{N}$ expansion of the eigenvalue disappear. Moreover it is possible to substantiate analytically the values for the expansion, and to definitively assert their connection to the Riemann Zeta function. The Calculus of Moving Surfaces is thus a powerful tool that may be applied to many other boundary perturbation and optimization problems (M. Boady's Ph.D. thesis [1] offers more examples, as does P. Grinfeld's textbook [8] on differential geometry). In the next few sections a brief account of the theory will be given, after which it will be applied to the problem in question.

3.1.1 Introduction to the Calculus of Moving Surfaces

Not much has been written to date on the Calculus of Moving Surfaces. Most of the following discussion is taken from P. Grinfeld's textbook [8], which offers an introduction into the subject. The main focus of the Calculus of Moving Surfaces are imbedded surfaces thought of as moving and stretching in time. The tool used to study these structures is the invariant time derivative whose definition will be built in the next few pages. It is worth recalling a few basic definitions of differential geometry in order to efficiently approach the Calculus of Moving Surfaces. The next few lines will be dedicated to that purpose.

Topics in imbedded surfaces

In this section we wish now to report the mathematical construction behind the concept of an imbedding. Let N and M be differentiable manifolds and let the dimension of N be equal to that of M a less of one. Let (N, ϕ) , $\phi : N \rightarrow M$ be an imbedding of N onto M . We shall refer to the function ϕ as the imbedding function. Let S^α be a set of coordinates on N , and let the vectors and covectors be defined on N in the standard way. Further on we shall refer to these objects as being surface objects (surface coordinates, surface vectors and surface covectors) and they will be denoted by a Greek lettered superscript. Let Z^i be a coordinate

system on a subset of M containing a subset of the image of the imbedding function ϕ . We shall denote the i -th coordinate of a point $p \in M$ as $Z^i(p)$. Using the Z^i coordinates and the imbedding map one can parameterize $\phi(N)$ in the following way:

$$S^i(\phi(p)) := Z^i(\phi(p)), \quad p \in N$$

These \mathbb{R}^d points are identified by the same letter S (to emphasize that they are to describe the same geometrical object) with a latin index. One must always keep in mind that, as Z^i are coordinates of M they benefit from an extra component. We shall call the Z^i and all of the derived mathematical constructions ambient objects.

By function composition one may express the ambient coordinates in terms of the surface coordinates:

$$Z^i(p) = Z^i(S(p))$$

It is a little bit trickier to define an imbedded tangent space. Intuitively, it is clear that the tangent space of the imbedded manifold will be a subspace of the tangent space of the larger manifold. Before discussing the tangent space as a whole, the attention should be shifted to a single vector. There is a natural connection between functions on surface and ambient coordinates. Let f be a function on N to \mathbb{R} . We can define a “new” function $\tilde{f} : \phi(N) \subset M \rightarrow \mathbb{R}$ defined implicitly by the relation $\tilde{f} \circ \phi = f$. We can view this function as a copy of f on the points which are the “imbedded versions” of those of N in M . Differentiating with respect to the α -th surface coordinate we get:

$$\partial_\alpha f|_p = \partial_\alpha(\tilde{f} \circ \phi)|_p = [d\phi(\partial_\alpha)]\tilde{f}|_{\phi(p)} = Z_\alpha^i \partial_i \tilde{f}|_{\phi(p)} \quad (3.1)$$

We can identify $Z_\alpha^i \partial_i$ with a vector-like object in the tangent space of $\phi(p) \in M$. We shall refer to these vectors as the ambient tangent vectors and the space they span as the ambient tangent space. It is important to notice that this expression is valid only in the subset $\phi(N) \subset M$: no such tangent ambient vectors exist outside the image of ϕ . In this sense we have transported the surface vector ∂_α to an ambient vector in an ambient tangent space in M . As all maps used were regular and linear the mapping of the vector is as well. Also it is important to notice that the object Z_α^i is not really a tensor as the α index has one less component than i . Naively it can be thought of as a rectangular (as opposed to a square) matrix. We call Z_α^i the shift tensor. With the appropriate metrics (the surface metric for the first component and the ambient metric for the second), one may raise or lower the shift tensor’s indices.

Furthermore, as the dimension of the tangent space of N is equal to that of M less of one, and each vector in the ambient tangent vector space is determined by one in the tangent space of N we expect there to be another vector in the tangent space of M orthogonal to all of the ones in the tangent space of the imbedded manifold. When it is normalized we call such vector the normal vector to the surface. There is an ambiguity still which is left, for the direction of the normal vector. For two dimensional closed surfaces imbedded in \mathbb{R}^3 it is common to take the normal vector “pointing outwards.”

Example. Consider a 2-sphere. It is common in topology to parameterize the sphere as a one by one square by identifying all the points on one side with one single point, all the points on the opposite with another and identifying all of the points on each of the remaining sides with the opposite one. This parameterization is useful as it is intrinsically two dimensional: the sphere is not imbedded in any larger manifold. By stretching the domain to a $[0, \pi] \times [0, 2\pi]$ rectangle and naming the first coordinate θ and the second ϕ we get the common imbedding in \mathbb{R}^3 :

$$\begin{cases} x(\theta, \phi) = \sin(\theta) \sin(\phi) \\ y(\theta, \phi) = \sin(\theta) \cos(\phi) \\ z(\theta, \phi) = \cos(\theta) \end{cases}$$

This coordinate transformation constitutes an imbedding map from N , the sphere, to M, \mathbb{R}^3 . In this case θ and ϕ would be the S^α coordinates while x, y and z would be the ambient coordinates Z^i . If f is a function of (θ, ϕ) we then have:

$$f(\theta, \phi) = \tilde{f}(x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$$

As described in equation (3.1) we can obtain the θ tangent vector as:

$$\partial_\theta f(\theta, \phi) = \partial_\theta \tilde{f}(x(\theta, \phi), y(\theta, \phi), z(\theta, \phi)) = Z_\theta^i \partial_i \tilde{f}(x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$$

Where $i = x, y, z$ and for example $Z_\theta^x = \partial_\theta x(\theta, \phi)$. Similarly one can construct the ϕ tangent vector. The normal vector in this case would be:

$$\mathbf{N} = \sin(\theta) \sin(\phi) \partial_x + \sin(\theta) \cos(\phi) \partial_y + \cos(\theta) \partial_z$$

Topics in the Calculus of Moving Surfaces

It is common to define moving surfaces as being imbedded manifolds whose imbedding function ϕ is dependent also on a separate parameter, identified as time. Hence from now on: $\phi : N \times I \rightarrow M$, where $I \subseteq \mathbb{R}$ is a real interval. There is no real reason to identify this new coordinate with time, however in the spirit of the Calculus of Moving Surfaces we will do so. Hence, *time* is not itself a coordinate, it does not define any point or region on any manifold: *time* defines a particular imbedding function. For the purposes of this paper we will not concern ourselves with problems of regularity and will assume everything to be as continuous and smooth as necessary. Specifically, from now on we will consider $\phi(p, t)$, $p \in N$, $t \in \mathbb{R}$ to be smoothly dependent on t .

To clarify the necessity of the Calculus of Moving Surfaces we will now build an example to illustrate one of the problems that this theory aims to solve. Consider a function $T : N \times \mathbb{R} \rightarrow \mathbb{R}$. Because of the additional time dependence, one may interpret such a function to be defined on the *moving surface* in M . Consider two sets of coordinates of N : S^α and S'^α . By composition with the coordinate system, $T(t, p)$ may be expressed as a function of either of the coordinates of N . When evaluated with respect to the primed coordinates we denote such function as $T(t, S')$ whereas in the other case we omit the prime. Let $U(t)$ and $U'(t)$ also be

defined as follows:

$$U(t) := \frac{d}{dt}T(t, S)$$

$$U'(t) := \frac{d}{dt}T(t, S')$$

By function composition we may express one coordinate system in terms of the other. If one wishes to relate points in the image of the imbedding map such an expression will be in general time dependent. In an attempt to reduce redundancy sometimes we will omit the point of which the coordinates we are considering. Hence sometimes we will write $S(t, S')$ rather than $S(t, S'(p))$. We then have:

$$T(t, S') = T(t, S(t, S'))$$

Differentiating with respect to time we get:

$$U'(t) = U(t) + \frac{d}{dS^\alpha}T(t, S(t, S'))\frac{d}{dt}S^\alpha(t, S')$$

Because of the last term, which in general is nonzero, we cannot say that T evolves in the same way in the two coordinate systems, i.e. $U(t) \neq U'(t)$. It is one of the goals of the Calculus of Moving Surfaces to clarify such a divergence from intuition. In what follows, we wish to develop a mathematical formalism which provides tools for our basic intuition. This is done not by changing anything already established by differential geometry but rather by introducing a new concept: a new kind of time derivative. In a sense, just like the covariant derivative was built we wish to build an “invariant time derivative” (*invariant* is meant between changes of coordinate systems). The key to this problem is to construct a purely geometrical derivative definition, one that depends only on the manifold to be studied rather than the coordinate system that is chosen. It will be shown that in order to accomplish this feat we must let the derivative vary as is done for the covariant derivative.

Before introducing the two main results of the Calculus of Moving Surfaces which will be used for our approach to the solution we must first introduce some kind of velocity for our moving manifold. Our goal will be to give a purely coordinate detached definition, in order to ensure it being a property of the object of study rather than one of the coordinate system of our choosing. Simply taking the time derivative of a coordinate function will not suffice. It is however the first step to the solution of this puzzle, hence consider the “coordinate velocity” defined as:

$$V^i(t, p) := \frac{d}{dt}Z^i(t, S(p))$$

where p is a point on the manifold N , $S(p)$ is the set of S coordinates of the point p and Z^i is the i -th coordinate of the point $\phi(p, t)$ in the manifold M .

Proposition 1. *The coordinate velocity defined above is not invariant with respect to coordinate changes and in fact it does not even transform as a tensor.*

Proof. Let Z^i and Z'^i be two ambient coordinate systems in a subset of $\phi(N) \subset M$ and let S^α , S'^α be two surface coordinate systems for N . As defined above we may

write:

$$\begin{aligned} V^i(t, p) &= \frac{d}{dt} Z^i(t, S(p)) \\ V'^i(t, p) &= \frac{d}{dt} Z'^i(t, S'(p)) \end{aligned}$$

As we have already done, we relate S^α and S'^α to one another. By (many) function compositions we may write:

$$V'^i(t, p) = \frac{d}{dt} Z'^i(Z(t, S(t, S')))$$

And expanding the derivation:

$$V'^i(t, p) = \frac{dZ'^i}{dZ^j} \left(\frac{d}{dt} Z^j(t, S(t, S')) + \frac{d}{dS^\alpha} Z^j(t, S(t, S')) \frac{d}{dt} S^\alpha(t, S') \right)$$

The first term, outside of the the parenthesis may be identified with the Jacobian between the Z and Z' coordinate systems. The first term in the parenthesis is exactly $V^i(t, p)$ while the last term is the contraction between the shift tensor and a time derivative of the expression of one surface coordinate system in terms of the other. As the last term is generally nonzero we have proven our claim. \square

The last equation in the proof provides a useful clue to the next step needed to build our definition of a coordinate free velocity. Namely, being proportional to the shift tensor, it is a linear combination of vectors in the ambient tangent vector space to the surface under consideration. To get rid of it then we can just contract it with the normal vector, thus obtaining our final, coordinate free definition:

$$C := V^i N_i$$

The proof of this object being coordinate invariant is the same as that of V^i not being so: the last term simply vanishes. Geometrically C can be thought of as the velocity of a point on the moving surface in the normal direction to the imbedded manifold. Because of this it is often called *interface velocity*.

With this last object we will be able to construct a new type of derivative with which we will be able to operate in a purely coordinate free manner: a derivative not dependent on the coordinate functions we use to describe the manifold, and which preserves the tensor transformation property. One could just state the definition and show that it is well behaved, however due to its intuitive geometrical interpretation before doing so it is worth building in a heuristic way. Consider a coordinate function $\gamma(S^\alpha)$ in the manifold N . Through the imbedding map this curve will be mapped to M and subsequently to \mathbb{R}^d through a set of coordinates on M : $\gamma(S^\alpha) \rightarrow \phi(t, \gamma(S^\alpha)) \rightarrow Z^i(\phi(t, \gamma(S^\alpha)))$. Let A be a point on the image of the curve γ through ϕ and let T be a function on a neighborhood of A . In a small time h the point A as well as the whole curve $\phi(t, \gamma(S^\alpha))$ will move a bit, to a new position on the manifold which we will identify as the point B . Heuristically, one could evaluate T at B in the following way:

$$T(B) = T(A) + h \left. \frac{d}{dt} T \right|_A$$

Now let D be a point on the time translated coordinate function $\phi(\gamma)$ “close” to B . Following the coordinate function along the moving surface one might write

$$T(B) = T(D) + hV^i Z_i^\alpha \nabla_\alpha T(D)$$

Eliminating the point B in the two preceding equations we get:

$$T(D) - T(A) = h \left(\frac{d}{dt} T(A) - V^i Z_i^\alpha \nabla_\alpha T(D) \right)$$

Motivated by the equation above we define:

$$\dot{\nabla} T(t, S) = \frac{d}{dt} T(t, S) - V^i Z_i^\alpha \nabla_\alpha T(t, S) \quad (3.2)$$

In the following lemma we show that this definition of an invariant time derivative is coordinate independent and allows for correct tensor transformations.

Lemma 1. *The time derivative $\dot{\nabla}$ defined in equation (3.2) is invariant and well defined.*

Proof. Let S and S' be coordinate systems on the manifold N . Let $T(t, S)$ and $T(t, S')$ be as defined above. We wish to prove that:

$$\dot{\nabla} T(t, S) = \dot{\nabla} T(t, S')$$

Hence, applying the definition (3.2):

$$\begin{aligned} \dot{\nabla} T(t, S') &= \frac{d}{dt} T(t, S') - V'^i Z'_i{}^\alpha \nabla'_\alpha T(t, S') \\ &= \frac{d}{dt} T(t, S(t, S')) - V'^i Z'_i{}^\alpha \nabla'_\alpha T(t, S(t, S')) \\ &= \frac{\partial}{\partial t} T(t, S(t, S')) + \nabla_\beta T(t, S(t, S')) \frac{dS^\beta}{dt} - \\ &\quad \left[\frac{\partial}{\partial t} Z^i(t, S(t, S')) + \frac{d}{dS^\gamma} Z^i(t, S(t, S')) \frac{dS^\gamma}{dt} \right] \frac{dS'^\alpha}{dZ^i} \nabla_\beta T(t, S(t, S')) \frac{d}{dS'^\alpha} S^\beta(t, S') \\ &= \frac{d}{dt} T(t, S) - V^i Z_i^\beta \nabla_\beta T(t, S) = \dot{\nabla} T(t, S) \end{aligned}$$

□

The reader should keep in mind that the term $V^i Z_i^\alpha$ represents a sort of projection of the coordinate velocity onto the surface expressed however using surface vectors. When this contraction is applied all information regarding velocity in the normal direction is lost, hence it is *not* true that $Z_i^\alpha Z_\alpha^j = \delta_i^j$. Geometrical insight might lead the reader towards the proven result that:

$$N_j N^i + Z_j^\alpha Z_\alpha^i = \delta_j^i \quad (3.3)$$

With this last relation in mind it is possible to simplify the definition of the invariant derivative on tensors defined on the whole host manifold. Specifically,

consider a function $T : M \rightarrow \mathbb{R}$. When evaluated on $\phi(N)$ we may consider the function $T = T(p, t)$ to be defined on the moving surface in M . Furthermore these may be expressed in terms of the surface coordinates. Hence we have $T = T(Z(\phi(p, t))) = T(Z(t, S))$. Applying the definition (3.2) we get:

$$\begin{aligned}\dot{\nabla}T &= \frac{d}{dt}T(Z(t, S)) - V^i Z_i^\alpha \nabla_\alpha T(Z(t, S)) \\ &= (V^j - V^i Z_i^\alpha Z_\alpha^j) \frac{d}{dZ^j}T(Z(t, S)) \\ &= \left(V^j - V^i \left(\delta_i^j - N_i N^j \right) \right) \frac{d}{dZ^j}T(Z(t, S)) \\ &= CN^j \frac{d}{dZ^j}T(Z(t, S))\end{aligned}$$

Where, in the last step, we made use of equation (3.3). Which, again, may be interpreted as a directional derivative of the function, in the direction normal to the surface and proportional to the interface velocity.

We now get to the last two results of the Calculus of Moving Surfaces, one of which will be used in the problem. These are the rules for the derivation of integral relations. One refers to volume integration and is completely analogous to the fundamental theorem of calculus while the second does not have a very intuitive meaning. The rules are:

$$\frac{d}{dt} \int_{\Omega} F dV = \int_{\Omega} \frac{d}{dt} F dV + \int_{\partial\Omega} C F dA \quad (3.4)$$

Where F is any kind of sufficiently smooth and real function on the imbedded manifold, Ω is a subset of the manifold and $\partial\Omega$ is the boarder of Ω . This formula can be intuitively interpreted as adding the contribution of a small change in the volume by integrating on its surface multiplied by its velocity. We will not make use of this second differentiation rule, however, for completeness' sake, the second equation is:

$$\frac{d}{dt} \int_S F dS = \int_S \dot{\nabla} F dS - \int_S C B_\alpha^\alpha F dS$$

where everything is as before and B_α^α is the curvature tensor. This curvature tensor may be interpreted as a direct consequence of the metrilinear property:

$$\nabla_i g_{mn} = 0 \quad (3.5)$$

where g_{mn} is the metric in M . Equation (3.5) implies that, given any two vectors \mathbf{S}^m and \mathbf{S}^n tangent to the m -th and n -th surface coordinate function:

$$\langle \mathbf{S}^m, \nabla_i \mathbf{S}^n \rangle = 0$$

Hence $\nabla_i \mathbf{S}^n$ is proportional to the normal vector. The tensor B_i^j is then defined as:

$$B_i^j = \nabla_i \mathbf{S}^j$$

3.1.2 Application of the Calculus of Moving Surfaces

We now turn our attention from the theory of the Calculus of Moving Surfaces to the problem we originally had: that of the solution of the eigenvalue problem for the Laplacian in regular polygons. The solution reported here is due to P. Grinfeld, and G. Strang and was originally published in [10]. The goal is to find an analytical expression for the first few terms of the expansion of the eigenvalue in terms of the variable $\frac{1}{N}$, N being the number of sides of the regular polygon. Of particular use will be equation (3.4).

General strategy

Before attempting a solution we should first provide an outline of the procedure we will use. First of all we restate the main equation of the problem:

$$\Delta\psi = -\lambda_N\psi \tag{1.1}$$

We wish to study this equation perturbatively from the solution of a circle, which is known and was reported in section 2.3. In particular our objective will be of finding an expansion of λ_N , the ground state eigenvalue, such as:

$$\lambda_N = \lambda_0 \left(1 + \frac{c_1}{N} + \frac{c_2}{N^2} + \frac{c_3}{N^3} + \dots \right)$$

hence our efforts will be targeted towards finding an expression for the coefficients c_i . To accomplish this feat we start by looking for a homotopy $\Phi : I \times [0, 1] \rightarrow \mathbb{R}^2$ (I being an interval in \mathbb{R}) such that: $\Phi(\theta, 0)$ is the parametrization of the circle, and $\Phi(\theta, 1)$ is the parametrization of the polygon. The function $\Phi(\theta, t)$ will be the imbedding function, from a real interval parametrized by θ to the real plane \mathbb{R}^2 . Different values for the second parameter (which is the above mentioned time) will identify different transitional curves between the polygons and the circle. More will be said on the choice of homotopy in a dedicated section.

Having found the homotopy expression for the boundary we can incorporate it in the original problem as a boundary condition dependent on t . Our solution ψ will thus depend on this parameter as well. If our choices are sufficiently well behaved we will also be able to apply calculus to ψ also in this new variable, hence, hopefully we are able to construct a Maclaurin expansion of ψ in terms of the variable t .

We will show that not everything however is so simple, as it is not always possible to express algebraically what we need. In such cases we will Taylor expand over the appropriate variable in order to better arrive at the solution we are seeking.

Hadamard's term

Having established how we intend to approach the problem let us start by differentiating with respect to time equation (1.1). Assuming symmetry of second derivatives we get:

$$\Delta\partial_t\psi = -\partial_t\Delta\psi = \lambda'_N\psi - \lambda_N\partial_t\psi$$

Where λ'_N indicates the time derivative of λ_N . It is a remarkable result of the Calculus of Moving Surfaces that:

$$\lambda'_N = - \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS \quad (3.6)$$

C being the interface velocity of the boundary. Because this relation was found by Hadamard we shall refer to λ'_N as the Hadamard term. As suggested by Strang and Grinfeld to remove this leading term one should keep the area constant. It is intuitive to see why it should be so if only C were inside the integral: the integral acts as a mediator on all of the small displacements of the curve, thus, if the total (signed) area difference is null, the integral should be zero. This intuition may be corroborated by the following argument: suppose we seek to calculate the area at a time t of the subset Ω enclosed by our curve. This may be expressed as an integral as:

$$A(t) = \int_{\Omega} d\Omega$$

Deriving through by t , we apply equation (3.4) and find that, because the integrand is 1:

$$A'(t) = \int_{\partial\Omega} C dS$$

It is a little less obvious to show that this holds also in the case where we include $|\nabla\psi|^2$. Recall that we are seeking the first term in a perturbation series of the eigenvalue solution for the Laplacian in a unit circle. Hence we only need to evaluate (3.6) in that case. It was shown (2.6) that¹:

$$\frac{d}{dr}\psi(1) = \frac{\rho}{\sqrt{\pi}}$$

ρ being such that $\rho^2 = \lambda_0$. Hence applying the polar coordinate transformation we can easily find that:

$$\begin{aligned} \langle \nabla\psi(1), \nabla\psi(1) \rangle &= \left(\frac{d}{dx}\psi(1) \right)^2 + \left(\frac{d}{dy}\psi(1) \right)^2 \\ &= \left(\frac{d}{dr}\psi(1) \right)^2 + \left(\frac{1}{r} \frac{d}{d\theta}\psi(1) \right)^2 = \left(\frac{d}{dr}\psi(1) \right)^2 = \frac{\lambda_0}{\pi} \end{aligned} \quad (3.7)$$

In the last equality the fact that $\psi(r)$ is independent of the angle θ was used. As that value is constant it can be brought out of integration the integration and we get yet again the case where only C is to be integrated. We now intend to prove equation (3.6). Before doing so however it is necessary to demonstrate another simple result:

Lemma 2.

$$\int_{\Omega} \psi \frac{d\psi}{dt} d\Omega = 0$$

¹ ψ and ρ refer to the ground state

Proof. We start with the normalization condition:

$$\int_{\Omega} |\psi|^2 d\Omega = 1$$

We wish to derive this equation now by time. As the boundary is dependent on time as well as the function calculating this derivative is not so simple and requires an application of the Calculus of Moving Surfaces. Specifically it is exactly the case of equation (3.4).

$$\int_{\Omega} \frac{d}{dt} |\psi|^2 d\Omega + \int_{\partial\Omega} C |\psi|^2 dS = 0$$

Under the Dirichlet boundary conditions, the second term on the left hand side is null ($x \in \partial\Omega \implies \psi(x) = 0$). Expanding the derivative we get that ψ and $\frac{d\psi}{dt}$ are orthogonal in $\mathcal{L}^2(\Omega^2)$. \square

We now proceed to prove the validity of the formula for Hadamard's term.

Theorem 7.

$$\lambda'_N = - \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS$$

Proof. We start by expressing λ as a Rayleigh quotient with unit denominator:

$$\lambda = \int_{\Omega} \langle \nabla\psi, \nabla\psi \rangle d\Omega \tag{3.8}$$

This formula may be obtained as follows: multiply the eigenvalue equation by ψ and integrate over Ω . As we can take ψ to be unitarely normed the equation reads:

$$\lambda = - \int_{\Omega} (\Delta\psi)\psi d\Omega$$

By applying Green's first identity we find:

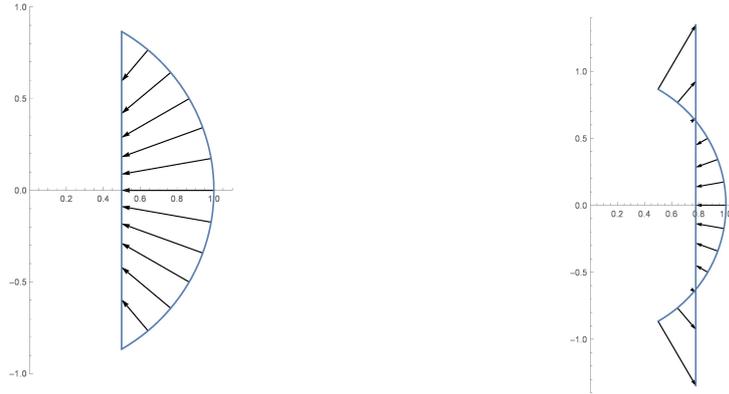
$$\lambda = \int_{\Omega} \langle \nabla\psi, \nabla\psi \rangle d\Omega - \int_{\partial\Omega} \psi \nabla\psi dS$$

Because of the boundary conditions the second integral on the right side of the equation is zero. Hence, we have proven equation (3.8). Deriving through by time we get:

$$\lambda' = \frac{d}{dt} \int_{\Omega} \langle \nabla\psi, \nabla\psi \rangle d\Omega$$

which, by application of equation (3.4), can be expanded as:

$$\begin{aligned} \lambda' &= 2 \int_{\Omega} \left\langle \nabla\psi, \frac{d}{dt} \nabla\psi \right\rangle d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS \\ &= 2 \int_{\Omega} \left[\nabla \left(\psi \nabla \frac{d\psi}{dt} \right) - \psi \Delta \frac{d\psi}{dt} \right] d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS \\ &= 2 \int_{\partial\Omega} \psi \nabla \frac{d\psi}{dt} dS - 2 \int_{\Omega} \psi \Delta \frac{d\psi}{dt} d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS \end{aligned} \tag{3.9}$$



(a) Grinfeld's transformation, non area preserving.

(b) Area preserving transformation.

Figure 3.1: Radial transformations of one side of the equilateral triangle.

The first integral vanishes due to the boundary conditions, and we may express

$$\Delta \frac{d\psi}{dt} = \frac{d}{dt} \Delta \psi = -\frac{d}{dt} (\lambda \psi) = -\lambda' \psi - \lambda \frac{d\psi}{dt}$$

So, plugging this into equation (3.9),

$$\begin{aligned} \lambda' &= 2 \int_{\Omega} \psi \left(\lambda' \psi + \lambda \frac{d\psi}{dt} \right) d\Omega + \int_{\partial\Omega} C \langle \nabla \psi, \nabla \psi \rangle dS \\ &= 2\lambda' + \int_{\partial\Omega} C \langle \nabla \psi, \nabla \psi \rangle dS \\ \lambda' &= - \int_{\partial\Omega} C \langle \nabla \psi, \nabla \psi \rangle dS \end{aligned}$$

Where we used lemma 2 for the first equality. \square

Homotopy construction

The choice of a correct homotopy is a very delicate a subtle process. In general, there are a great number of such transformations which might be used, but due to integration most are prohibitive. In the optimum case, one would like to find a homotopy which is area conserving at all times, in order to eliminate Hadamard's term. It is straightforward to write a homotopy mapping the circle to a polygon with equal area, however it is challenging to write one conserving the area at all times, i.e. such that:

$$A'(t) = \frac{d}{dt} \int_{\Omega} d\Omega = 0 \quad \forall t \in [0, 1]$$

Such a homotopy would have some points moving inward and some points moving outward in such a way that the total signed area *averages out*. At the end of the transformation some of the points of the side of the polygon would be inside the original circle, while the vertices and some other points would be on the outside (figure 3.1 shows Grinfeld's transformation and an area preserving transformation.).

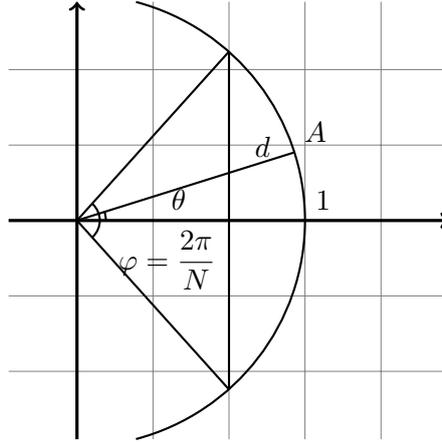


Figure 3.2: Arc of a circle with side of N -gon. The distance traveled by point A is d .

Grinfeld and Strang leave this problem to be solved by posterity. They consider a homotopy such that each point moves radially to the inscribed polygon with constant speed. With reference to figure 3.2 one can write:

$$d(\theta, N) = 1 - \frac{\cos\left(\frac{\pi}{N}\right)}{\cos(\theta)}$$

As each point is moving with constant speed, the speed of each point is proportional to the distance traveled. We set the proportionality constant equal to minus one. Hence, each point moves with speed $V(\theta, N) = -d(\theta, N)$. One can parametrize the homomorphism as:

$$\Phi(\theta, N, t) = (1 + tV(\theta, N)) \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

This way, for $t = 0$ we return to a parametrization of the circle and for $t = 1$ we get that of the polygon. It should be noted that this function is not valid for all θ ; it is only valid for $\theta \in \left[-\frac{\pi}{N}, \frac{\pi}{N}\right]$. To get a complete curve one must extend this result.

Interface velocity calculation

We now seek to calculate $C(\theta, N, t)$ in order to use Hadamard's formula to get the first variation. The normal $\mathbf{N}(\theta, N, t)$ to the curve is obtained from the tangent $\mathbf{T}(\theta, N, t)$ which is easily calculated as a θ derivative of $\Phi(\theta, N, t)$. Furthermore, the coordinate velocity $\mathbf{V}(\theta, N, t)$ vector can be calculated as the time derivative of $\Phi(\theta, N, t)$. We remind the reader that this point-wise velocity is not $C(\theta, N, t)$, as $C(\theta, N, t)$ is the interface velocity. To get the expression for $C(\theta, N, t)$ we must project $\mathbf{V}(\theta, N, t)$ onto $\mathbf{N}(\theta, N, t)$. Such operation yields:

$$C(\theta, N, t) = \frac{\sec^2(\theta) \left(\cos\left(\frac{\pi}{N}\right) - \cos(\theta)\right) \left(\cos(\theta) + t \cos\left(\frac{\pi}{N}\right) - t \cos(\theta)\right)}{\sqrt{\left(\cos(\theta) + t \sec^2(\theta) \cos\left(\frac{\pi}{N}\right) - t \cos(\theta)\right)^2 + (t-1)^2 \sin^2(\theta)}}$$

As expected, this rather complicated expression reduces to $V(\theta, N)$ for $t = 0$. Because our expansion is about $t = 0$ this will be instrumental to our solution.

First expansion term

Having calculated $C(\theta, N, t)$ we now continue in our intent to calculate the first term in the expansion λ' for which we will use Hadamard's formula (3.6). Using (3.7) we get:

$$\begin{aligned}\lambda' &= - \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle dS = - \frac{\lambda_0}{\pi} \int_{\partial\Omega} C(\theta, N, 0) dS \\ &= - \frac{N\lambda_0}{\pi} \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} \left(\frac{\cos\left(\frac{\pi}{N}\right)}{\cos(\theta)} - 1 \right) d\theta = - \frac{\lambda_0}{\pi} \int_{-\pi}^{\pi} \left(\frac{\cos\left(\frac{\pi}{N}\right)}{\cos\left(\frac{\theta}{N}\right)} - 1 \right) d\theta \quad (3.10)\end{aligned}$$

Expressing the above equation in terms of $\xi = \frac{1}{N}$ and Taylor expanding the integrand about $\xi = 0$ we get:

$$\begin{aligned}\lambda' &= - \frac{\lambda_0}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} (\theta^2 - \pi^2) \xi^2 + O(\xi^4) \right) d\theta \\ &= \lambda_0 \left(\frac{2\pi^2}{3} \xi^2 + O(\xi^4) \right)\end{aligned}$$

Direct integration of (3.10) would have shown that the result is true to $O(\xi^6)$ [10]. Recalling that: $\zeta(2) = \frac{\pi^2}{6}$, we get:

$$\lambda' = \lambda_0 (4\zeta(2)\xi^2 + O(\xi^4))$$

This provides the first term in the expansion of λ^2 :

$$\lambda_N = \lambda_0 \left(1 + \frac{4\zeta(2)}{N^2} + \dots \right)$$

To compute further terms it is necessary to also include the successive terms in the time expansion of λ . These calculations are similar to what was done above and are in [10]. A further expansion yields:

$$\lambda_N = \lambda_0 \left(1 + \frac{4\zeta(2)}{N^2} + \frac{4\zeta(3)}{N^3} + \frac{28\zeta(4)}{N^4} + \dots \right)$$

In his Ph.D. thesis Boady [1], working with Grinfeld, is able to derive, applying symbolic computation methods to the Calculus of Moving Surfaces, two other terms in this expansion. His expansion is still referred to the inscribed polygon and is:

$$\begin{aligned}\lambda_N &= \lambda_0 \left(1 + \frac{4\zeta(2)}{N^2} + \frac{4\zeta(3)}{N^3} + \frac{28\zeta(4)}{N^4} + \right. \\ &\quad \left. \frac{12\zeta(5) + 16\zeta(2)\zeta(3) - 2\lambda_0\zeta(5)}{N^5} + \frac{8\zeta(3)^2 + 124\zeta(6) + 4\zeta(3)^2\lambda_0}{N^6} + O\left(\frac{1}{N^7}\right) \right) \quad (3.11)\end{aligned}$$

²Please note that the following expression is not true up to $O\left(\frac{1}{N^6}\right)$. In order to assess the correct expansion order it is necessary to find λ'' and verify its first term in the $\frac{1}{N}$ expansion.

Rescaling

By rescaling it is possible to convert the solution obtained for the inscribed polygon to that of the polygon with area π . The expression for this eigenvalue is much simpler and shows a surprising degree of regularity. In particular, by rescaling, all known terms proportional zeta functions of even numbers disappear. The solution for the eigenvalue of a polygon with area π was reported in equation (1.2). To obtain this result one may compose the function ψ with an appropriate coordinate dilation. By evaluating the Laplacian on this function composition one may then find the correct eigenvalue. Specifically, it is useful to think in terms of polar coordinates as the ratio of the area of a circle and the inscribed N sided polygon is the same. In the case we approached, the radius of the circumscribing circle is 1. In the case where the area of the polygon is chosen to be π it is a matter of simple trigonometry to assess that the radius should be:

$$\nu_N = \sqrt{\xi \csc(\xi) \sec(\xi)}$$

Where $\xi = \frac{\pi}{N}$. The rescaling will then be simply done by a multiplication by ν_N . Let $\tilde{\rho} = \nu\rho$. Equation (1.1) then becomes:

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \psi(\tilde{\rho}) \right) = \frac{\nu^2}{\tilde{\rho}} \frac{d}{d\tilde{\rho}} \left(\tilde{\rho} \frac{d}{d\tilde{\rho}} \psi(\tilde{\rho}) \right) = \nu^2 \Delta \psi = -\lambda_N \psi$$

Hence we get a rescaling factor for the eigenvalue of ν^{-2} . A series expansion to the sixth term of $\frac{1}{N}$ of the eigenvalue rescaling factor offers:

$$\begin{aligned} \nu^{-2} = 1 - \frac{2\pi^2}{3N^2} + \frac{2\pi^4}{15N^4} - \frac{4\pi^6}{315N^6} + O\left(\left(\frac{1}{N}\right)^8\right) = \\ 1 - \frac{4\zeta(2)}{N^2} + \frac{12\zeta(4)}{N^4} - \frac{12\zeta(6)}{N^6} + O\left(\left(\frac{1}{N}\right)^8\right) \end{aligned}$$

Which, when multiplied by the left hand side of equation (3.11) gives the first line in equation (1.2).

3.2 Solution through Schwartz-Christoffel Mapping

Lastly we report the perturbative solution given in terms of Schwartz-Christoffel mappings. This kind of solution has been successful in the elucidation of the problem and the construction of the first few terms in the expansion.

The Schwartz-Christoffel transform is a set of conformal transformations of the complex plane mapping the upper plane ($\Im(z) > 0$) or the centered unit circle ($|z| < 1$) to a large set of possible domains (to be discussed below). The solution will make use of particular Schwartz-Christoffel mappings from the circle to inscribed polygons. In recent times this theory has had a fruitful renaissance as computational methods have become more advanced and readily available. This theory has found applications in fluid dynamics and electrostatics, and many other areas of physics. In general, Schwartz-Christoffel mappings constitute a versatile tool which may simplify some problems which are made complex by their boundary conditions.

3.2.1 The Schwartz-Christoffel transform

Our goal for this section is to provide the intuition behind Schwartz-Christoffel mappings. These transformations represent the *crown jewel* of the study of conformal transformations of the complex plane that took place in Europe in the early to mid 1800s. The roots of the Schwartz-Christoffel transform lay in the studies of C. Gauss and those of B. Riemann. Of particular relevance is the Riemann Mapping Theorem, which was published in Riemann's Ph.D. thesis [29] delivered in 1867, stating [20]:

Theorem 8 (Riemann Mapping Theorem). *Let U be a simply connected open set which is not the whole plane. Then U is analytically isomorphic to the disc of radius 1. More precisely, given $z_0 \in U$, there exists an analytic isomorphism*

$$f : U \rightarrow D(0, 1)$$

of U to the unit disc, such that $f(z_0) = 0$. Such an isomorphism is uniquely determined up to rotation, i. e. multiplication by $e^{i\theta}$ for some real θ , and is therefore uniquely determined by the additional condition:

$$f'(z_0) > 0.$$

This theorem provides the grounds for what we are doing. In particular it states that our search for a map connecting two simply connected regions of the complex plane is not in vain. The proof of this theorem is not constructive, hence further investigation is needed in order to provide a useful transformation. A necessary result, given by Schwartz in order to prove the final formula is the Schwartz reflection principle. This principle is useful for the construction of analytical continuations of functions across their boundary. It has very few restrictions and its result may be applied to most *well behaved* functions. In particular the principle states that [25]:

Theorem 9 (Schwartz Reflection Principle). *Let D_1 and D_2 be two adjacent domains whose common boundary is a smooth arc α . If the analytic functions $f_1(z)$ and $f_2(z)$ are regular in D_1 and D_2 respectively and if the limits for $z \rightarrow \alpha$ of both functions coincide and are continuous on α , then $f_1(z)$ and $f_2(z)$ are analytic continuations of each other.*

It can be shown [25] that this result is equivalent to stating that, if $f(z)$ is an analytic function on the upper half plane, extending to a continuous function on the real axis then $f(z)$ may be analytically continued to the lower half plane by complex conjugation $\Im(z) < 0, \implies f(z) = \overline{f(\bar{z})}$. We now turn to the main result: the Schwartz-Christoffel mapping formula. This formula is often proved in its form with the upper half plane as its domain. This choice, however, is arbitrary as by composition it is possible to construct more general transformations. We will focus on transformations to generalized polygons, for which we take T. Driscoll and L. Trefethen's [4] definition:

Definition 2. We define a generalized polygon to be a collection of vertices $\{w_i\}_{i=0}^N$, $w_i \in \mathbb{C} \cup \{\infty\}$ and real interior angles $\{\alpha_i\}_{i=0}^N$. In addition we require that the polygon make a total turn of 2π and hence the sum of the exterior angles $\{\mu_i\}$ defined to be such that $\mu_i + \alpha_i = 1$ be equal to 2.

For simplicity we treat just the case where all preimages of the vertices (the prevertices, for further reference) are finite. The map we are looking for will transform each point of the upper half plane into a corresponding point in the interior of a polygon D . The vertices of the polygon will be images of points on the real axis. Because of the Riemann Mapping Theorem we are sure that such a transformation exists and is analytical everywhere except at the prevertices. Let $f(z)$ be such a transformation. Let the points $\{a_i\}_{i=0}^N$ be the prevertices. These points divide the real axis into N parts each of which is mapped by $f(z)$ to a side of the polygon. By the Schwartz reflection principle $f(z)$ may be analytically continued across each one of these segments. The image of the continuation will be a mirror image D' of D with respect to one of its sides will thus be the conformal map of the lower half plane. By further reflection of the domain one returns to the original domain, the upper half plane. If the second reflection is done through a different side however the image will be congruent, but shifted and in general also rotated. Hence, if $f(z)$ is *twice reflected* to the function $f_1(z)$ it will be true that:

$$f_1(z) = af(z) + b$$

$$g(z) = \frac{f_1''(z)}{f_1'(z)} = \frac{f''(z)}{f'(z)}$$

furthermore $g(z)$ can be defined by continuation as a single valued analytic function everywhere in the closure of the upper half plane except at the prevertices, where the derivatives might fail to exist. The main idea now is to express $f(z)$ and more precisely its derivatives in terms of products of simpler functions. This intuition stems from the geometric result that the argument of products of complex number is the sum of the argument of the multiplied numbers. Hence by correctly choosing the product functions as step functions on the real axis we are able to guide the image of $f(z)$ along the border of the desired polygon. In particular the optimal choice for the step function is: $f_k(z) = (z - a_k)^{-\mu_k}$, as:

$$f'(z) = \prod_{k=0}^n (z - z_k)^{-\mu_k},$$

$$f''(z) = \sum_{h=0}^n (-\mu_h) (z - z_h)^{-\mu_h-1} \prod_{k=0}^n [(z - z_k)^{-\mu_k}]^{1-\delta_{h,k}},$$

$$\frac{f''(z)}{f'(z)} = \frac{d}{dz} \ln(f'(z)) = \sum_{h=0}^n -\frac{\mu_h}{z - z_h}$$

Where $\delta_{h,k}$ is the Kronecker symbol. Integrating twice one gets the result:

$$f(z) = A + B \int^z \prod_{k=0}^n (\zeta - z_k)^{-\mu} d\zeta$$

When composed with the map from the unit circle to the upper half plane this becomes:

$$f(z) = A + B \int^z \prod_{k=0}^n \left(1 - \frac{\zeta}{z_k}\right)^{-\mu_k} d\zeta \quad (3.12)$$

In most cases the Schwartz-Christoffel mapping formula (3.12) is found to be of difficult application with analytical methods. The main difficulty is in finding the correct prevertices values z_k to fit into the equation. While the angles are simply encoded into the transform through the exponents μ_k the side lengths are given by the distances among the z_k . The search for the prevertices is a parameter problem that may be solved by use of computers and numerical methods. In our case of interest however, due to the regularity of the image of the maps these may be easily found as will be explained further on.

Schwartz-Christoffel mappings to polygons

Mappings from the circle to the inscribed polygons are among the simplest to be found. In these cases, the outer angle $\pi\mu_k$ is $\frac{2\pi}{n}$. This is a widely known result of elementary euclidean geometry. Furthermore, as all the points in the image are evenly spaced, it is natural to pick the prevertices to be evenly spaced across the unit circle. Without initially concerning ourselves with side length and correct overall proportions, we take as prevertices the roots of unity. By Ruffini's theorem, because each and only n -th root of unity is a zero of the polynomial $P(x) = x^n - 1$ then:

$$P(x) = x^n - 1 = \prod_{h=0}^{n-1} (x - \omega^h), \quad \omega = \exp\left(\frac{2\pi i}{n}\right)$$

however:

$$\prod_{h=0}^{n-1} (x - \omega^h) = (-1)^n \prod_{h=0}^{n-1} \left(1 - \frac{x}{\omega^h}\right)$$

Hence we can express the integrand in the Schwartz-Christoffel formula as:

$$\prod_{k=0}^{n-1} \left(1 - \frac{\zeta}{z_k}\right)^{-\mu_k} = \prod_{h=0}^{n-1} \left(1 - \frac{\zeta}{\omega^h}\right)^{-\frac{2}{n}} = (1 - \zeta^n)^{-\frac{2}{n}}$$

As noted by Molinari [24], the integral of this function is the well known Gaussian Hypergeometric function with coefficients:

$$\int^z (1 - \zeta^n)^{-\frac{2}{n}} d\zeta = z {}_2F_1\left(\frac{1}{n}, \frac{2}{n}, 1 + \frac{1}{n}, z^n\right) \quad (3.13)$$

Furthermore, as we wish that the polygon be circumscribed by the unit circle, and as $z = 1$ due to our choice of prevertices, is always a vertex, we multiply by a normalizing term, given by:

$$c_n = {}_2F_1\left(\frac{1}{n}, \frac{2}{n}, 1 + \frac{1}{n}, 1\right)^{-1} = \frac{\Gamma\left(1 - \frac{1}{n}\right)}{\Gamma\left(1 - \frac{2}{n}\right)\Gamma\left(1 + \frac{1}{n}\right)}$$

Thus obtaining a final result of:

$$w_n(z) = c_n z {}_2F_1\left(\frac{1}{n}, \frac{2}{n}, 1 + \frac{1}{n}, z^n\right)$$

3.2.2 Application of the Schwartz-Christoffel transform

The map derived above is used by Molinari [24] to generate the ground state eigenvalues for the free particle in the polygonal domain. More specifically it is the basis for the expansion which is done over a fictitious parameter λ . To accomplish this feat the Laplacian eigenvalue equation (1.1) is expressed by complex coordinates and rewritten as:

$$-4 \frac{\partial^2}{\partial z^* \partial z} \psi(z, z^*) = \varepsilon^2 |w'_n(z)|^2 \psi(z, z^*) \quad (3.14)$$

Molinari then proceeds to introduce in the equation above the expansion parameter λ and, through perturbation theory is then able to derive the first few terms in the expansion.

Introduction of the perturbation parameter and solution procedure

The introduction of the λ parameter is done through a manipulation of the series decomposition of the Hypergeometric function. It is established that:

$${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

With $(a)_k$ being the rising Pochhammer symbol evaluated on a . As noted by Molinari, insertion of the parameters given by equation (3.13) leads to:

$$w(z) = c_n z \sum_{k=0}^{\infty} f_k z^{nk}, \quad f_k = \frac{\left(\frac{2}{n}\right)_k}{k! (nk + 1)}$$

Hence, as $f_0 = 1$ the parameter λ is introduced as:

$$w_{n,\lambda}(z) = c_n z \left(1 + \lambda \sum_{k=1}^{\infty} f_k z^{kn} \right) \quad (3.15)$$

With this choice, when λ is put to equal 0 the map becomes a dilation c_n ; when $\lambda = 1$ the function becomes the desired transformation. The expansion greatly benefits from the use of a polar coordinate system. Insertion of equation (3.15) into (3.14) gives:

$$H_0 \psi(r, \theta) = (\varepsilon c_n)^2 \left| 1 + \lambda \sum_{k=1}^{\infty} f_k (kn + 1) \left(r e^{i\theta} \right)^{kn} \right|^2 \psi(r, \theta)$$

As the solutions in the circle are the Bessel functions, Molinari's perturbation approach is based on the perturbation of an established integral expression for these functions. In particular, it is well known that:

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(nt - z \sin(t))} dt \quad (3.16)$$

Hence Molinari looks for a solution of the form:

$$\psi(z, z^*) = \frac{1}{2\pi} \int_0^{2\pi} h(\alpha) e^{\frac{\varepsilon}{2}(e^{i\alpha}w(z^*) - e^{-i\alpha}w(z))} d\alpha$$

The explicit values of the eigenvalue ε and the weight function $h(\alpha)$ are obtained by imposing the boundary condition on z :

$$0 = \int_0^{2\pi} h(\alpha) e^{\frac{\varepsilon}{2}(e^{i\alpha}w(e^{-i\theta}) - e^{-i\alpha}w(e^{i\theta}))} d\alpha$$

As $h(\alpha)$ is periodic, these last expressions may be further simplified by expressing $h(\alpha)$ in terms of its Fourier transform:

$$h(\alpha) = \sum_{k=-\infty}^{\infty} h_k e^{ik\alpha}$$

Recalling equation (3.16), we can thus get rid of the integral expression:

$$\psi(r, \theta) = \sum_{k=-\infty}^{\infty} h_k e^{ik\theta} J_{-k}(\varepsilon r)$$

After having expressed the Fourier transform of the weight function $h(\alpha)$ in terms of the perturbative parameter λ , by virtue of the boundary condition $\psi(1, \theta) = 0$ Molinari proceeds to explicitly find the first three such expansion, leading to a solution:

$$\varepsilon_n = \frac{1}{C_n} \varepsilon_0 [1 - \lambda^2 \delta_2 - \lambda^3 \delta_3 + \dots]$$

where ε_0 is what before was called λ_0 , the ground state eigenvalue of the free particle restricted to a circle, and:

$$\delta_2 = \frac{\varepsilon_0}{2} \sum_{k=1}^{\infty} f_k^2 \frac{J_{kn+1}(\varepsilon_0)}{J_{kn}(\varepsilon_0)}$$

$$\begin{aligned} \delta_3 = \frac{\varepsilon_0^2}{4} \sum_{k=2}^{\infty} f_k \sum_{s=1}^{k-1} f_{k-s} f_s \frac{J_{sn+1}(\varepsilon_0)}{J_{sn}(\varepsilon_0)} & \left[2 \frac{J_{kn+1}(\varepsilon_0)}{J_{kn}(\varepsilon_0)} + \frac{J_{(k-s)n+1}(\varepsilon_0)}{J_{(k-s)n}(\varepsilon_0)} \right] \\ & - \frac{\varepsilon_0^2}{4} \sum_{k=2}^{\infty} f_k \frac{J_{kn+2}(\varepsilon_0)}{J_{kn}(\varepsilon_0)} \sum_{s=1}^{k-1} f_{k-s} f_s \end{aligned}$$

4 Conclusion

We have covered many aspects of the eigenvalue problem of the Laplacian in polygonal domains. We showed solutions for the integrable cases of the square, the circle and the equilateral triangle. These solutions were reported in detail and some aspects of the eigenfunctions were studied as well. We focused in particular on the equilateral triangle case as it has proved to be the hardest of the three.

A brief account was also given of the Calculus of Moving Surfaces and the invariant time derivative was constructed. We recalled some of the problems and main results of the theory. Following this we were able to apply the Calculus of Moving Surfaces to fruition to the problem of the ground state eigenvalue for the Laplacian in an n -sided polygon. The first term was calculated explicitly and full known result was reported. We also presented a solution method in terms of Schwartz-Christoffel mappings, which were also discussed in detail.

The main nuances of the problem were shown and some solutions, and their mathematical background were explained in detail.

Table 4.1: Numerical results for the eigenvalue in polygons with area π , calculated using Grinfeld, Strang and Broady's and Jones' expansion formulae. These are compared to the numerical results obtained by Jones, correct until the 51st digit.

N	λ_{Exact}	$\lambda_{\text{G. S. B.}}$	$\lambda_{\text{G. S. B.}} - \lambda_{\text{Exact}}$	λ_{Jones}	$\lambda_{\text{Jones}} - \lambda_{\text{Exact}}$
3	$\frac{4\pi}{\sqrt{3}} \approx 7.25519746$	7.18063883	-7.46×10^{-2}	7.22405394	-3.11×10^{-2}
4	$2\pi \approx 6.28318531$	6.28372255	5.37×10^{-4}	6.28360915	4.24×10^{-4}
5	6.02213793	6.02312309	9.85×10^{-4}	6.02235583	2.18×10^{-4}
6	5.91741783	5.91783196	4.14×10^{-4}	5.91747950	6.17×10^{-5}
7	5.86644931	5.86662158	1.72×10^{-4}	5.86646819	1.89×10^{-5}
8	5.83849143	5.83856804	7.66×10^{-5}	5.83849791	6.48×10^{-6}
9	5.82182680	5.82186339	3.66×10^{-5}	5.82182927	2.47×10^{-6}
10	5.81126036	5.81127900	1.86×10^{-5}	5.81126139	1.03×10^{-6}
11	5.80423064	5.80424068	1.00×10^{-5}	5.80423110	4.61×10^{-7}
12	5.79936980	5.79937548	5.68×10^{-6}	5.79937003	2.21×10^{-7}
13	5.79590027	5.79590361	3.35×10^{-6}	5.79590038	1.12×10^{-7}
14	5.79335701	5.79335905	2.04×10^{-6}	5.79335706	5.92×10^{-8}
15	5.79145001	5.79145130	1.29×10^{-6}	5.79145004	3.27×10^{-8}
Circle	$\lambda_0 = 5.78318596$	5.78318596	0	5.78318596	0

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