UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

CORSO DI LAUREA IN FISICA

TESI DI LAUREA TRIENNALE

ADM formalism: a Hamiltonian approach to General Relativity

Autore:
Alessandro Danieli

Matricola:
916330

Relatore:
Prof. Luca Guido Molinari

Codice P.A.C.S.:
04.20.-q

Anno accademico 2019-2020

Contents

Abstract

The advent of the 3+1 formalism of General Relativity in the 1920's is a milestone in the history of the theory. Indeed, this approach gave an impetus to the inquiry of the initial value formulation of Einstein's equations, which led in 1952 to the local uniqueness theorem by Y. Choquet-Bruhat. It is on this fertile ground that in the late 1950's R. Arnowitt, S. Deser and C. W. Misner (ADM) proposed a novel Hamiltonian formulation of General Relativity. Their seminal work is a forerunner of Wheeler's geometrodynamics and has far-reaching consequences in the quantization program of the theory. The 3+1 decomposition of spacetime is achieved through a foliation by spacelike hypersurfaces, on which the three-dimensional counterparts of the intrinsic curvature and stress-energy tensors can be defined. Once the Einstein-Hilbert Lagrangian is recast into the ADM Hamiltonian, the variational principle gives rise to a constrained set of Hamilton's equations. Furthermore, the fundamental Poisson brackets between the canonical field variables can be computed.

In this thesis, after a preliminary presentation of the variational principle of General Relativity (chapter 1), we introduce the mathematical apparatus required for the realization of the 3+1 decomposition (chapter 2). The pivotal role of the extrinsic curvature tensor will be elucidated in two phases, starting from the Gauss-Codazzi relations between four and three-dimensional intrinsic curvature tensors, and eventually in the identification of corrective boundary terms to the Einstein-Hilbert action. Chapter 3 is devoted to the three-dimensional conversion of the spacetime metric $g_{\mu\nu}$ by means of the lapse and shift functions, which ultimately leads to the projections of the field equations. The crux of the discussion lies in chapter 4, which thoroughly details the derivation of the ADM Hamiltonian and the path to Hamilton's equations. Accordingly, the Poisson brackets between the conjugate variables are recovered, paving the way for a brief examination of the Wheeler - DeWitt equation, a major step in the quest for a quantum theory of gravity (chapter 5). Finally, we focus on the notions of total energy and momentum of the system, which naturally stem from the evaluation of the Hamiltonian at spatial infinity. In particular, we apply these definitions to the case of a Schwarzschild spacetime, proving the reasonableness of the result.

Chapter 1

Introduction

General Relativity has proved to be one of the most elegant and successful physical theories since its first appearance in the paper *The Field Equations of Gravitation* on November 25, 1907. Albert Einstein's theory relies on the Equivalence Principle, which states that gravity affects all bodies in the same way, making it impossible to disentangle the effects of a gravitational field from those of a uniform accelerating frame, and on the independence of physical laws from the reference frame. These assumptions, combined with the hypothesis that spacetime is a curved manifold structure described by a metric tensor $g_{\mu\nu}$, lead to ascribe the distribution of matter to the geometry of spacetime itself. In particular, this relation is specified by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci Tensor, R is the scalar curvature, Λ is the cosmological constant, G is the gravitational constant, c is the speed of light and $T_{\mu\nu}$ is the stress-energy tensor.
From now on we adopt the geometrized unit system, with $G = c = 1$, and we neglect the contribution of the cosmological constant, setting $\Lambda = 0$.

1.1 Einstein-Hilbert action

The Lagrangian formulation of a field theory allows to deduce the field equations given a region $\mathcal{V}$ of the spacetime manifold and a scalar function $\mathcal{L}(\psi, \partial_\alpha\psi)$, called Lagrangian density, which depends on the field variables ψ and their first derivatives $\partial_\alpha\psi$. Although the fields ψ could be of any type, we will consider only generic tensors of type (r, s) (omitting the indices for brevity). In analogy with the Lagrangian formulation of Newtonian mechanics, the action functional $\mathcal{S}[\psi]$ is defined as the integral

$$\mathcal{S}[\psi] = \int_{\mathcal{V}} \mathcal{L}(\psi, \partial_\alpha\psi)\sqrt{-g}\,\mathrm{d}^4x \tag{1.2}$$

where g is the (negative) determinant of the metric $g_{\mu\nu}$ and $\sqrt{-g}\,\mathrm{d}^4x$ is the proper volume element. The field equations are then recovered by requiring that $\mathcal{S}[\psi]$ is stationary under an arbitrary variation $\delta\psi$ about the actual fields

ψ_0. If one is given a smooth one-parameter family of field configurations ψ_λ, a natural definition of variation comes from the derivative

$$\delta\psi = \left.\frac{\mathrm{d}\psi_\lambda}{\mathrm{d}\lambda}\right|_{\lambda=0} \tag{1.3}$$

which we demand to vanish on the boundary $\partial\mathcal{V}$ of our spacetime region

$$\delta\psi|_{\partial\mathcal{V}} = 0 \tag{1.4}$$

We now assume that there exists a smooth tensor field χ of type (s, r) (thus dual to ψ) such that the action functional is

$$\mathcal{S} = \int_{\mathcal{V}} \mathrm{d}^4x\,\chi\,\psi \tag{1.5}$$

where the contraction between the indices of χ ans ψ is implied. Taking the derivative of $\mathcal{S}$ with respect to the parameter λ leads to the relation

$$\delta\mathcal{S} \doteq \left.\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\lambda}\right|_{\lambda=0} = \int_{\mathcal{V}} \mathrm{d}^4x\,\chi\,\delta\psi \tag{1.6}$$

Therefore, the variation of $\mathcal{S}$ with respect to ψ about ψ_0 is defined as the functional derivative

$$\chi = \left.\frac{\delta\mathcal{S}}{\delta\psi}\right|_{\psi_0} \tag{1.7}$$

which must vanish identically by virtue of the stationarity of the action:

$$\chi = 0 \tag{1.8}$$

These relations ensure that ψ_0 is a solution of the field equations, enclosed in the identity 1.8.
The variational approach to general relativity was first considered by Hilbert and Einstein in 1915, who proposed the simple gravitational action:

$$\mathcal{S}_H = \frac{1}{16\pi}\int_{\mathcal{V}} R\sqrt{-g}\,\mathrm{d}^4x \tag{1.9}$$

We will refer to $\mathcal{S}_H$ as the Hilbert term. This is indeed the simplest gravitational action that can be conceived, since the only nontrivial scalar function that can be constructed from the metric and its derivatives up to the second order is the scalar curvature R. The choice

$$\mathcal{L}_H \doteq \frac{1}{16\pi}R\sqrt{-g} \tag{1.10}$$

not only proves to be particularly compelling, due to the complexity of other possible alternatives, but also establishes a straightforward correspondence between weak field limit and Newtonian theory of gravitation. In addition to $\mathcal{S}_H$, we shall include the contributions from the matter fields, denoted by ϕ, in the term

$$\mathcal{S}_M = \int_{\mathcal{V}} \mathcal{L}_M(\phi, \partial_\alpha\phi;\, g_{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x \tag{1.11}$$

called the matter action. For simplicity, we assume that $\mathcal{L}_M$ depends only on the metric coefficients $g_{\mu\nu}$, together with the field ϕ and its first derivatives. The total action functional is given by the sum of the Hilbert and matter terms

$$\mathcal{S} = \mathcal{S}_H + \mathcal{S}_M \tag{1.12}$$

We are thus in the position to show that the Einstein field equations 1.1 stem from the stationarity of $\mathcal{S}$ under arbitrary variations of $g_{\mu\nu}$.

1.2 Variation of the action

Firstly, let us consider the Hilbert term alone. It will prove more convenient to use the variation of the inverse metric $\delta g^{\mu\nu}$ instead of $\delta g_{\mu\nu}$. This does by no means affect the results, since the two variations are related by

$$g^{\alpha\lambda} g_{\lambda\beta} = \delta^{\alpha}{}_{\beta} \qquad \Longrightarrow \qquad \delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} \tag{1.13}$$

We can perform the variation of $\mathcal{S}_H$ (following the definition 1.6) by focusing on the integrand, namely the Hilbert Lagrangian density $\mathcal{L}_H$, as the variation can be brought under the integral sign:

$$\begin{aligned} (16\pi)\, \delta\mathcal{L}_H &= \delta\left(g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\right) \\ &= -\frac{\delta g}{2\sqrt{-g}}\, g^{\mu\nu} R_{\mu\nu} + \left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}\right)\sqrt{-g} \end{aligned} \tag{1.14}$$

The variation of the metric determinant δg is given by Jacobi's formula:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{1.15}$$

By exploiting the second form of this identity and recalling that $g < 0$, we can replace δg in 1.14:

$$(16\pi)\, \delta\mathcal{L}_H = \left[\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}\right] \sqrt{-g} \tag{1.16}$$

It is now clear that the purely gravitational component of the field equations is recovered if $\delta R_{\mu\nu}$ vanishes. However, this assumption need not hold in the general case, as the first derivatives of $\delta g^{\mu\nu}$ enter the variation $\delta R_{\mu\nu}$, giving rise to extra boundary terms. Indeed, if we resort to the Palatini identity (proved in section 6.2.3 of the Appendix), we find that

$$\delta R_{\mu\nu} = \nabla_\rho \left(\delta \Gamma^{\rho}{}_{\mu\nu}\right) - \nabla_\mu \left(\delta \Gamma^{\rho}{}_{\rho\nu}\right) \tag{1.17}$$

By introducing the contravariant vector $V^\rho \doteq g^{\mu\nu} \delta \Gamma^{\rho}{}_{\mu\nu} - g^{\rho\nu} \delta \Gamma^{\mu}{}_{\mu\nu}$ (whose explicit expressions are discussed in Appendix, section 6.2.2) and using the property $\nabla_\rho g_{\mu\nu} = 0$ of Levi-Civita connections, the last term of equation 1.16 can be recast into a divergence:

$$\begin{aligned} \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} &= \sqrt{-g}\, g^{\mu\nu} \left[\nabla_\rho \delta \Gamma^{\rho}{}_{\mu\nu} - \nabla_\mu \Gamma^{\rho}{}_{\rho\nu}\right] \\ &= \sqrt{-g}\, \nabla_\rho \left[g^{\mu\nu} \delta \Gamma^{\rho}{}_{\mu\nu} - g^{\rho\nu} \delta \Gamma^{\mu}{}_{\mu\nu}\right] \doteq \partial_\rho \left(\sqrt{-g}\, V^\rho\right) \end{aligned} \tag{1.18}$$

Once these results are replaced in the action integral 1.9 and the multiplicative constant (16π) reintroduced, by means of Stokes' theorem the variation $\delta\mathcal{S}_H$ splits into the volume and boundary parts

$$\delta\mathcal{S}_H = \frac{1}{16\pi}\int_{\mathcal{V}}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right)\sqrt{-g}\,\delta g^{\mu\nu}\,\mathrm{d}^4x + \frac{1}{16\pi}\oint_{\partial\mathcal{V}} V^{\mu}\mathrm{d}\sigma_{\mu}\mathrm{d}^3x \tag{1.19}$$

where $\mathrm{d}\sigma_{\mu}$ is the oriented volume element of the hypersurface $\partial\mathcal{V}$. For the moment, we ignore the second integral and proceed as if the surface terms can be safely discarded. Nevertheless, after the introduction of a necessary mathematical apparatus, in section 2.4 we will deal properly with this term.
We now consider the variation of the matter action 1.11 (whose dependence on the matter fields ϕ and on $g_{\mu\nu}$ is utterly generic):

$$\begin{aligned}\delta\mathcal{S}_M &= \int_{\mathcal{V}}\left[\frac{\delta\,\mathcal{L}_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\sqrt{-g} + \mathcal{L}_M\,\delta\sqrt{-g}\right]\mathrm{d}^4x \\ &= \int_{\mathcal{V}}\left[\frac{\delta\,\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}\,\mathcal{L}_M\,g_{\mu\nu}\right]\sqrt{-g}\,\delta g^{\mu\nu}\,\mathrm{d}^4x \end{aligned} \tag{1.20}$$

If we define the stress-energy tensor $T_{\mu\nu}$ by

$$T_{\mu\nu} \doteq -2\frac{\delta\,\mathcal{L}_M}{\delta g^{\mu\nu}} + \mathcal{L}_M\,g_{\mu\nu} \tag{1.21}$$

we see that the variation of the total action $\mathcal{S}$ becomes:

$$\begin{aligned}\delta\mathcal{S} &= \int_{\mathcal{V}}\left[\frac{1}{16\pi}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) + \frac{\delta\,\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}\,\mathcal{L}_M\,g_{\mu\nu}\right]\sqrt{-g}\,\delta g^{\mu\nu}\mathrm{d}^4x \\ &= \frac{1}{16\pi}\int_{\mathcal{V}}\left[R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R - 8\pi T_{\mu\nu}\right]\sqrt{-g}\,\delta g^{\mu\nu}\mathrm{d}^4x \end{aligned} \tag{1.22}$$

Due to the arbitrariness of $\delta g^{\mu\nu}$, the stationarity of $\mathcal{S}$ requires that the integrand be identically zero, leading eventually to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{1.23}$$

which can be rewritten in an equivalent form using the Einstein tensor $G_{\mu\nu}$, corresponding to the left-hand side of 1.23:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.24}$$

The desired conservation of the stress-energy tensor $T_{\mu\nu}$, expressed by the four-divergence

$$\nabla_{\mu}T^{\mu\nu} = 0 \tag{1.25}$$

is ensured by the Bianchi identities $\nabla_{\mu}G^{\mu\nu} = 0$, which follow from the symmetries of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$. This result can also be proved by considering the invariance of the action under an infinitesimal transformation of coordinates (see for instance Ref. [17], section 4.1.8).

Chapter 2

Mathematical prelude to the ADM formalism

In this chapter we introduce some fundamental mathematical notions of differential geometry required for the development of the ADM formalism. From now on we consider a spacetime $(\mathcal{M}, \boldsymbol{g})$, where $\mathcal{M}$ is a real smooth 4-dimensional manifold and $\boldsymbol{g}$ a Lorentzian metric of signature $(-,+,+,+)$ on $\mathcal{M}$.

2.1 Hypersurfaces and embeddings

Definition 1. Given a three-dimensional manifold $\tilde{\Sigma} \subset \mathcal{M}$ and an embedding (i.e. a homeomorphism) $\Phi : \tilde{\Sigma} \to \mathcal{M}$, Σ is said to be a hypersurface of $\mathcal{M}$ if it is the image of $\tilde{\Sigma}$ through Φ:

$$\Sigma = \Phi(\tilde{\Sigma}) \tag{2.1}$$

This embedding defines in a natural way two mappings between tangent and cotangent spaces, respectively called the push-forward and the pull-back.

Definition 2. Let f be a smooth real-valued function $f : \mathcal{M} \to \mathbb{R}$ and $X \in T_p(\tilde{\Sigma})$ a tangent vector of $\tilde{\Sigma}$. The push-forward of Φ is a map between tangent spaces $\Phi_* : T_p(\tilde{\Sigma}) \to T_p(\mathcal{M})$ such that

$$\Phi_* X = X(f \circ \Phi) \tag{2.2}$$

Let $\omega \in T^*_p(\mathcal{M})$ a one-form on $\mathcal{M}$. Then the pull-back is defined as the mapping $\Phi^* : T^*_p(\mathcal{M}) \to T^*_p(\tilde{\Sigma})$ given by

$$\left(\Phi^* \omega\right)(X) = \omega\left(\Phi_* X\right) \tag{2.3}$$

The induced metric $\boldsymbol{\gamma}$ on $\tilde{\Sigma}$ is obtained by pulling back the metric $\boldsymbol{g}$ of $\mathcal{M}$:

$$\boldsymbol{\gamma} \doteq \Phi^* \boldsymbol{g} \tag{2.4}$$

Let $\hat{t} : \mathcal{M} \to \mathbb{R}$ be a regular scalar field on $\mathcal{M}$. Since the embedding Φ ensures that its image Σ is not self-intersecting, each hypersurface Σ can be locally defined as a level surface $\hat{t} = t$ (with t constant) and thus labeled with Σ_t:

$$\Sigma_t = \left\{ p \in \mathcal{M} \,\middle|\, \hat{t}(p) = t \right\} \tag{2.5}$$

If $\tilde{\Sigma}$ is endowed with a coordinate system $\left(x^i\right)$, $i \in \{1,2,3\}$, then the mapping Φ takes the explicit form:

$$\Phi : (x^1, x^2, x^3) \in \tilde{\Sigma} \longrightarrow (t, x^1, x^2, x^3) \in \Sigma_t \subset \mathcal{M} \tag{2.6}$$

Let us consider the isomorphism $\psi : T_p(\Sigma) \longrightarrow T_p(\tilde{\Sigma})$ which associates tangent vectors of Σ and $\tilde{\Sigma}$. Choosing the coordinates of equation 2.6 allows to identify the basis tangent vectors of the two spaces in a straightforward way. Besides, the pull-back ψ^* provides a simple relation between covariant tensors defined on Σ and on $\tilde{\Sigma}$. For instance, the $(0,2)$ tensor $T_{ij} \in T_p^*(\tilde{\Sigma})^2$ gets mapped to

$$T_{\mu\nu} = \left(\psi^* T\right)_{\mu\nu} = \left(\begin{array}{c|c} 0 & 0_j \\ \hline 0_i & T_{ij} \end{array}\right)_{\mu\nu} \in T_p^*(\Sigma)^2 \tag{2.7}$$

From now forth, unless otherwise stated, we will implicitly make use of the embedding map Φ and the isomorphism ψ to identify tensors defined on Σ (whose components are labeled with lowercase Greek indices, running from 0 to 4) and on $\tilde{\Sigma}$ (with lowercase Latin indices restricted to $\{1,2,3\}$).
In this context, the signature of the metric $\boldsymbol{\gamma}$ falls into three categories and provides a useful classification of the hypersurfaces Σ.

Definition 3. A hypersurface Σ is said to be:

- spacelike if $\boldsymbol{\gamma}$ is positive-definite (signature $(+,+,+)$), with timelike normal vector;
- timelike if $\boldsymbol{\gamma}$ is Lorentzian (signature $(-,+,+)$), with spacelike normal vector;
- null if $\boldsymbol{\gamma}$ is degenerate (signature $(0,+,+)$).

If Σ is a non-null hypersurface, the normal $\boldsymbol{n}$ is uniquely defined at every point as the unit vector collinear to $\partial\hat{t}$, the metric dual of the gradient one-form $d\hat{t}$. In particular, we denote with ε the norm:

$$\varepsilon \doteq n_\mu n^\mu = \pm 1 \tag{2.8}$$

If Σ is a spacelike hypersurface, then $\boldsymbol{n}$ is a timelike unit vector, with $\varepsilon = -1$. In the following chapters we will adopt the notation $g^{tt} = g^{00}$, which means that $\hat{t}$ is chosen as the time coordinate, corresponding to the index 0. Consequently, if we write n_μ as $n_\mu = \Omega\, \partial_\mu \hat{t}$, where $\Omega = \Omega(x^\alpha)$ takes care of the normalization, the condition 2.8 fixes the absolute value of Ω:

$$-1 = n_\mu n_\nu g^{\mu\nu} = \Omega^2 g^{tt} \implies \Omega = \pm \frac{1}{\sqrt{-g^{00}}} \tag{2.9}$$

Henceforth we focus on spacelike hypersurfaces Σ and its induced positive-definite metric $\boldsymbol{\gamma}$. We shall set $\Omega < 0$, which ensures that n^μ is a future-directed timelike vector (namely, it points toward the direction of increasing $\hat{t}$). Hence we can express n_μ and n^μ in the natural bases of $T_p(\mathcal{M})$ and $T_p^*(\mathcal{M})$ as

$$n_\mu = -\frac{\delta_\mu{}^0}{\sqrt{-g^{00}}} \tag{2.10}$$

$$n^\mu = -\frac{g^{0\mu}}{\sqrt{-g^{00}}} \tag{2.11}$$

2.2 Extrinsic and intrinsic curvature

In our brief introduction to the Hilbert-Einstein variational principle we considered the Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$ and its contracted forms associated to the Levi-Civita connection ∇ on $(\mathcal{M},\, \boldsymbol{g})$, which express the intrinsic curvature of the spacetime. In a similar way, we can construct a unique Levi-Civita connection D on the manifold $(\Sigma,\, \boldsymbol{\gamma})$ and its corresponding curvature tensors. In order to distinguish between three-dimensional and four-dimensional tensors, we shall adopt the following convention:

- the quantities marked with the superscript "4" relate to $(\mathcal{M},\, \boldsymbol{g})$: the spacetime Riemann tensor becomes ${}^4R^{\rho}{}_{\sigma\mu\nu}$;
- conversely, the quantities marked with "3" or without any superscript refer to the manifold $(\Sigma,\, \boldsymbol{\gamma})$: for example, ${}^3R_{\mu\nu} = R_{\mu\nu}$.

Besides the intrinsic curvature, when dealing with embedded manifolds one may define an extrinsic curvature tensor, which measures the variation of the normal $\boldsymbol{n}$ along a tangent vector. To this end, let us introduce the Weingarten map χ (or shape operator), which acts on tangent vectors of $T_p(\Sigma)$ (seen as a subspace of $T_p(\mathcal{M})$):

$$\begin{aligned} \chi :\ T_p(\Sigma) &\longrightarrow T_p(\Sigma) \\ \boldsymbol{v} &\longmapsto \nabla_{\boldsymbol{v}}\boldsymbol{n} \end{aligned} \tag{2.12}$$

χ is truly an endomorphism of $T_p(\Sigma)$, since $\chi(\boldsymbol{v})$ is orthogonal to $\boldsymbol{n}$:

$$n_\mu[\chi(\boldsymbol{v})]^\mu = n_\mu \nabla_{\boldsymbol{v}} n^\mu = \frac{1}{2}\nabla_{\boldsymbol{v}}(n_\mu n^\mu) = 0 \tag{2.13}$$

This implies that we can use the Latin indices i, j instead of μ, ν to denote the components of χ:

$$\chi_i{}^k = \nabla_i n^k = -\left[\partial_i\left(\frac{g^{0k}}{\sqrt{-g^{00}}}\right) + \frac{1}{\sqrt{-g^{00}}}\,{}^4\Gamma^k{}_i{}^0\right] \tag{2.14}$$

Furthermore, the Weingarten map is self-adjoint with respect to the metric $\boldsymbol{\gamma}$, so its eigenvalues $\kappa_1, \kappa_2, \kappa_3$ are real numbers. These are called principal curvatures of the surface Σ, while the corresponding eigenvectors identify the principal directions. The mean extrinsic curvature is then defined as the arithmetic mean of the principal curvatures:

$$H = \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3) \tag{2.15}$$

Another consequence of the self-adjointness of χ is the existence of a bilinear form on $T_p(\Sigma)$, called second fundamental form, such that:

$$\begin{aligned} \boldsymbol{K} :\ T_p(\Sigma) \times T_p(\Sigma) &\longrightarrow \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) &\longmapsto -u_k\,[\chi(\boldsymbol{v})]^k \end{aligned} \tag{2.16}$$

From the definition of $\chi(\boldsymbol{v})$ we obtain an explicit expression of the components of $\boldsymbol{K}$:

$$K_{ij} = -\nabla_j n_i = {}^4\Gamma^\mu{}_{ij} n_\mu = -\frac{1}{\sqrt{-g^{00}}}\,{}^4\Gamma^0{}_{ij} \tag{2.17}$$

The trace of $\boldsymbol{K}$ with respect to the metric γ turns out to be a multiple of the mean curvature (equation 2.15) and of the trace of the Weingarten map:

$$K = \gamma^{ij} K_{ij} = -\mathrm{Tr}\,\chi = -3H \tag{2.18}$$

We now focus on the properties of $\boldsymbol{K}$ and on the relations between the connections ∇ and D. To this end, we define the orthogonal projector on $T_p(\Sigma)$ as the map

$$\begin{aligned} \boldsymbol{\pi} :\ T_p(\mathcal{M}) &\longrightarrow \quad T_p(\Sigma) \\ v^\alpha \quad &\longmapsto \quad v^\alpha + (n_\mu v^\mu)\, n^\alpha \end{aligned} \tag{2.19}$$

whose components can be explicitly written in terms of n^μ and in matrix form using equations 2.10, 2.11:

$$\pi^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu = \left(\begin{array}{c|c} 0 & -g^{0j}/g^{00} \\ \hline 0^i & \delta^i{}_j \end{array} \right) \tag{2.20}$$

In fact, since $\pi^\mu{}_\nu n^\nu = n^\nu - n^\nu = 0$, it maps any vector in the direction of n^μ to zero, while it acts as the identity on $T_p(\Sigma)$. The pull-back of the orthogonal projector provides a natural extension of the covariant tensors defined on Σ to the spacetime $\mathcal{M}$. In particular, $\boldsymbol{\pi}^*$ yields the extended metric $\boldsymbol{\gamma}_{\mathcal{M}}$ when applied to $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma}_{\mathcal{M}} \doteq \boldsymbol{\pi}^* \boldsymbol{\gamma} \tag{2.21}$$

By definition, $\boldsymbol{\gamma}_{\mathcal{M}}$ is equivalent to $\boldsymbol{\gamma}$ when its action is restricted to $T_p(\Sigma)$ and vanishes if one of the arguments is normal to Σ. This property can be translated into the completeness relation:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{2.22}$$

We shall apply the extension via $\boldsymbol{\pi}^*$ also to the second fundamental form $\boldsymbol{K}$ defined before:

$$\boldsymbol{K}_{\mathcal{M}} \doteq \boldsymbol{\pi}^* \boldsymbol{K} \tag{2.23}$$

This process will be carried out automatically (unless otherwise stated) for every covariant tensor defined on Σ, whereas for contravariant tensors on Σ the extension to $\mathcal{M}$ is trivial (since each $(0, r)$ tangent tensor in $T_p(\Sigma)^r$ is already vanishing upon contraction with n_μ). To lighten the notation, however, throughout the subsequent discussion we will refrain from introducing new symbols and implicitly refer to the corresponding extended quantities.
We now clarify the relation between the spacetime connection ∇ and the three-dimensional connection D. Given a tensor field $\boldsymbol{T}$ on Σ, its (extended) covariant derivative $\boldsymbol{DT}$ satisfies the relation:

$$\boldsymbol{DT} = \boldsymbol{\pi}^*(\nabla \boldsymbol{T}) \tag{2.24}$$

in components:

$$D_\lambda T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \pi^\rho{}_\lambda \pi^{\mu_1}{}_{\alpha_1} \dots \pi^{\mu_r}{}_{\alpha_r} \pi^{\beta_1}{}_{\nu_1} \dots \pi^{\beta_s}{}_{\nu_s} \nabla_\rho T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} \tag{2.25}$$

This stems from the uniqueness of Levi-Civita connections, which can be proved by direct computation. Note that in the right-hand side of 2.24 we implicitly considered the extension of $\boldsymbol{T}$ to $\mathcal{M}$. Applying equation 2.25 to a tangent vector $\boldsymbol{v} \in T_p(\Sigma)$ leads to a useful specific case that involves the extrinsic curvature tensor:

$$D_{\boldsymbol{u}} v^\alpha = \nabla_{\boldsymbol{u}} v^\alpha + K_{\mu\nu} u^\mu v^\nu n^\alpha \qquad \forall\, \boldsymbol{u} \in T_p(\Sigma) \tag{2.26}$$

We conclude this paragraph showing the relations between $K_{\mu\nu}$ and the normal vector n^μ. Firstly, let us define the 4-acceleration of n^μ as the covariant derivative along itself:

$$a^\mu \doteq \nabla_{\boldsymbol{n}} n^\mu \tag{2.27}$$

From the constancy of the norm of n^μ, we deduce that a^μ is orthogonal to n^μ and thus belongs to $T_p(\Sigma)$:

$$g_{\mu\nu} a^\mu n^\nu = n_\mu \nabla_{\boldsymbol{n}} n^\mu = \frac{1}{2} \nabla_{\boldsymbol{n}} (n_\mu n^\mu) = 0 \qquad \Longrightarrow \qquad a^\mu \in T_p(\Sigma) \tag{2.28}$$

By virtue of these results we can now express the extended tensor $K_{\mu\nu}$ in terms of more elementary quantities. Given two vectors $u^\mu, v^\mu \in T_p(\mathcal{M})$, consider the quantity

$$K_{\mu\nu} u^\mu v^\nu \tag{2.29}$$

Since the employment of $\pi^\mu{}_\nu$ in definition 2.23 makes $K_{\mu\nu}$ orthogonal to n^μ, the only contribution to 2.29 comes from the spatial components of the vectors. Therefore, thanks to our choice of coordinates, we can use $K_{ij} = -\nabla_j n_i$ (equation 2.17) simply by replacing the Latin indices with the Greek ones:

$$\begin{aligned} K_{\mu\nu} u^\mu v^\nu &= -\nabla_\mu n_\nu \left[\pi^\mu{}_\alpha u^\alpha\right] \left[\pi^\nu{}_\beta v^\beta\right] \\ &= -\nabla_\mu n_\nu \left[u^\mu + n^\mu n_\alpha u^\alpha\right] \left[v^\nu + n^\nu n_\beta v^\beta\right] \\ &= -u^\mu v^\nu \nabla_\mu n_\nu - u^\mu v^\nu n_\mu \nabla_{\boldsymbol{n}} n_\nu \\ &= -u^\mu v^\nu \left[\nabla_\mu n_\nu + n_\mu a_\nu\right] \end{aligned} \tag{2.30}$$

Hence we arrive at the following fundamental equations:

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu \tag{2.31}$$

$$K = g^{\mu\nu} K_{\mu\nu} = -\nabla_\mu n^\mu \tag{2.32}$$

The contribution of the acceleration vector a_μ drops out in the contraction with n_μ due to the orthogonality relation 2.28.
Before continuing with our discussion, we clarify a relation that has been ignored until now. In particular, raising or lowering an index of the projector $\pi^\mu{}_\nu$ with $g_{\mu\nu}$ yields the metric $\gamma_{\mu\nu}$ and its inverse:

$$\pi_{\lambda\nu} = g_{\lambda\mu} \pi^\mu{}_\nu = g_{\lambda\mu} (\delta^\mu{}_\nu + n^\mu n_\nu) = g_{\lambda\nu} + n_\lambda n_\nu = \gamma_{\lambda\nu} \tag{2.33a}$$

$$\pi^{\mu\lambda} = g^{\lambda\nu} \pi^\mu{}_\nu = g^{\lambda\nu} (\delta^\mu{}_\nu + n^\mu n_\nu) = g^{\mu\lambda} + n^\lambda n^\nu = \gamma^{\mu\lambda} \tag{2.33b}$$

This is a direct consequence of the completeness relation 2.22 involving $\gamma_{\mu\nu}$ and $g_{\mu\nu}$. Thus from now on we replace the notation $\pi^\mu{}_\nu$ with the equivalent $\gamma^\mu{}_\nu$.

2.3 Gauss-Codazzi relations

This section is devoted to the development of the 3+1 dimensional splitting involving the curvature tensors of Σ and $\mathcal{M}$. In particular, as we shall see,

the three and four-dimensional Riemann tensors are related by the extrinsic curvature tensor $K_{\mu\nu}$.
Let us consider the Ricci identity, which relates $R^{\rho}{}_{\sigma\mu\nu}$ to the commutator of covariant derivatives:

$$R^{\rho}{}_{\sigma\mu\nu}v^{\sigma} = [D_{\mu}, D_{\nu}]v^{\rho} \tag{2.34}$$

This equation holds for every $v^{\mu} \in T_p(\mathcal{M})$ as $R^{\rho}{}_{\sigma\mu\nu}$ is a tangent vector of Σ. Each derivative D can be replaced by ∇ using twice equation 2.24 and the explicit expansion of the projector $\gamma^{\mu}{}_{\nu}$. After some passages, we arrive at the Gauss relation:

$$\gamma^{\rho}{}_{\alpha}\gamma^{\beta}{}_{\sigma}\gamma^{\gamma}{}_{\mu}\gamma^{\delta}{}_{\nu}\,{}^{4}R^{\alpha}{}_{\beta\gamma\delta} = R^{\rho}{}_{\sigma\mu\nu} + K^{\rho}{}_{\mu}K_{\sigma\nu} - K^{\rho}{}_{\nu}K_{\sigma\mu} \tag{2.35}$$

By exploiting the idempotence of the projector $\gamma^{\alpha}{}_{\lambda}\gamma^{\lambda}{}_{\beta} = \gamma^{\alpha}{}_{\beta}$, the contraction on ρ and μ gives the contracted Gauss relation:

$${}^{4}R_{\alpha\beta}\,\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu} + {}^{4}R^{\alpha}{}_{\beta\gamma\delta}\,\gamma_{\alpha\mu}n^{\beta}\gamma^{\gamma}{}_{\nu}n^{\delta} = R_{\mu\nu} + KK_{\mu\nu} - K_{\mu\lambda}K^{\lambda}{}_{\nu} \tag{2.36}$$

Finally, contracting again with the metric $\gamma^{\mu\nu}$ yields a generalization of Gauss' Theorema Egregium:

$${}^{4}R + 2\,{}^{4}R_{\mu\nu}n^{\mu}n^{\nu} = R + K^{2} - K_{\mu\nu}K^{\mu\nu} \tag{2.37}$$

It is worth to observe that these three remarkable equations, apart from the notational particularization, hold true for any kind of embedding (in any dimension). We can push further our analysis by considering their projections on Σ and on n^{α}: in particular, let us focus on the total projection of the four-dimensional Ricci identity

$$[\nabla_{\mu}, \nabla_{\nu}]n^{\rho} = {}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma} \tag{2.38}$$

Using the projector $\gamma^{\alpha}{}_{\beta}$ once for every index and exploiting the expansion 2.31 of $K_{\mu\nu}$ results in:

$$\begin{aligned}
\gamma^{\alpha}{}_{\rho}\gamma^{\mu}{}_{\beta}\gamma^{\nu}{}_{\gamma}\,{}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma} &= \gamma^{\alpha}{}_{\rho}\gamma^{\mu}{}_{\beta}\gamma^{\nu}{}_{\gamma}[\nabla_{\mu}, \nabla_{\nu}]n^{\rho} \\
&= -\gamma^{\alpha}{}_{\rho}\gamma^{\mu}{}_{\beta}\gamma^{\nu}{}_{\gamma}[\nabla_{\mu}(K_{\nu}{}^{\rho} + n_{\nu}a^{\rho}) - \nabla_{\nu}(K_{\mu}{}^{\rho} + n_{\mu}a^{\rho})] \\
&= -[D_{\beta}K_{\gamma}{}^{\alpha} - D_{\gamma}K_{\beta}{}^{\alpha}] - [a^{\alpha}K_{\beta\gamma} - a^{\alpha}K_{\gamma\beta}]
\end{aligned}$$

The symmetry of $K_{\alpha\beta}$ allows to displace the last term in square brackets, giving the Codazzi relation:

$$\gamma^{\alpha}{}_{\rho}\gamma^{\mu}{}_{\beta}\gamma^{\nu}{}_{\gamma}\,{}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma} = -[D_{\beta}K_{\gamma}{}^{\alpha} - D_{\gamma}K_{\beta}{}^{\alpha}] \tag{2.39}$$

As usual, we can perform the contraction of the remaining free indices with the aim to obtain new identities. In particular, applying this to α and β on the left-hand side leads to

$$\begin{aligned}
\gamma^{\mu}{}_{\rho}\gamma^{\nu}{}_{\gamma}\,{}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma} &= (\delta^{\mu}{}_{\rho} + n^{\mu}n_{\rho})\gamma^{\nu}{}_{\gamma}\,{}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma} \\
&= \gamma^{\nu}{}_{\gamma}\,{}^{4}R_{\sigma\nu}n^{\sigma} + \gamma^{\nu}{}_{\gamma}\,{}^{4}R_{\rho\sigma\mu\nu}n^{\rho}n^{\sigma}n^{\mu}
\end{aligned} \tag{2.40}$$

The last term drops out due to the antisymmetry of ${}^{4}R_{\rho\sigma\mu\nu}$ with respect to the first two indices. Since the contraction of the covariant derivatives is trivial, we arrive at

$$\gamma^{\mu}{}_{\lambda}\,{}^{4}R_{\mu\nu}n^{\nu} = D_{\lambda}K - D_{\mu}K_{\lambda}{}^{\mu} \tag{2.41}$$

Unsurprisingly, this is called the contracted Codazzi relation.

2.4 Corrections to the Hilbert term of the action

Having introduced the extrinsic curvature tensor $K_{\mu\nu}$, we are now in the position to deal with the pending issue presented in section 1.2. Indeed, in order to displace the boundary term arising in the computation of $\delta\mathcal{S}$ (equations 1.16 and 1.19), we shall add a corrective term to the action which depends on the trace of the extrinsic curvature. To simplify the notation, in this section we omit the superscript "4" since we consider only four-dimensional quantities.

2.4.1 Boundary term $\mathcal{S}_B$ of the action

We recall that the variation of $\mathcal{S}_H$ performed in section 1.2 yields the boundary term

$$\oint_{\partial\mathcal{V}} \delta V^\alpha \mathrm{d}\sigma_\alpha = \oint_{\partial\mathcal{V}} \left[g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\alpha\nu}\delta\Gamma^\mu{}_{\mu\nu}\right] \mathrm{d}\sigma_\alpha \tag{2.42}$$

which in general cannot be discarded due to the presence of nonvanishing partial derivatives $\delta g_{\mu\nu,\alpha}$. We shall now give a precise characterization of the surface element $\mathrm{d}\sigma_\alpha$ in the case of a non-null hypersurface. To this end, let us introduce the defining equation $\Phi(x^\alpha) = \mathit{const}$ of the hypersurface $\partial\mathcal{V}$. To avoid confusion, we denote with r_μ the generic unit normal (not necessarily timelike) pointing in the direction of increasing Φ. This follows immediately if we fix $r_\mu \doteq \varepsilon\,\Omega\,\partial_\mu\Phi$, where Ω is the positive normalization factor, as the contraction with $\partial_\mu\Phi$ gives:

$$r^\mu\partial_\mu\Phi \;=\; \frac{1}{\varepsilon\,\Omega}\, r^\mu r_\mu \;=\; \frac{1}{\Omega} > 0 \tag{2.43}$$

Finally, by demanding $\mathrm{d}\sigma_\alpha$ to be an invariant volume element of $\partial\mathcal{V}$, proportional to r_α and such that $r^\alpha\mathrm{d}\sigma_\alpha > 0$, we arrive at

$$\mathrm{d}\sigma_\alpha = \varepsilon r_\alpha \sqrt{|h|}\,\mathrm{d}^3x \tag{2.44}$$

where h_{ij} is the induced metric on $\partial\mathcal{V}$, h is its determinant and d^3x here refers to the coordinates of the hypersurface. In analogy with $\gamma_{\mu\nu}$, the four-dimensional extension $h_{\mu\nu}$ satisfies the completeness relation

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon r_\mu r_\nu \tag{2.45}$$

which can be easily proved by taking the contraction with normal and tangent vectors of $\partial\mathcal{V}$. Substituting equation 2.44 in the integral 2.42 gives

$$\int_{\mathcal{V}} g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \oint_{\partial\mathcal{V}} \varepsilon\,\delta V^\alpha r_\alpha \sqrt{|h|}\,\mathrm{d}^3x \tag{2.46}$$

In order to recast δV^α as a function of $\delta g_{\mu\nu,\lambda}$ we must evaluate the variation of the Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$. Since $\delta g_{\mu\nu}|_{\partial\mathcal{V}} = 0$, we have

$$\delta\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\sigma}\delta\Gamma_{\sigma\mu\nu} = \frac{1}{2}\,g^{\lambda\sigma}\left(\delta g_{\sigma\mu,\nu} + \delta g_{\sigma\nu,\mu} - \delta g_{\mu\nu,\sigma}\right) \tag{2.47}$$

For simplicity, we focus momentarily on the argument of the surface integral 2.46, discarding the multiplicative term $\sqrt{|h|}$. Replacing the variation $\delta\Gamma^\lambda{}_{\mu\nu}$ with equation 2.47 and gathering the common terms yield

$$\varepsilon\,\delta V^\alpha r_\alpha = \varepsilon g^{\alpha\beta} g^{\mu\nu}\left(\delta g_{\beta\mu,\nu} - \delta g_{\mu\nu,\beta}\right) r_\alpha$$

$$= \varepsilon \left(h^{\mu\nu} + \varepsilon r^{\mu} r^{\nu}\right) \left(\delta g_{\beta\mu,\nu} - \delta g_{\mu\nu,\beta}\right) r^{\beta}$$
$$= \varepsilon r^{\beta} h^{\mu\nu} \left(\delta g_{\beta\mu,\nu} - \delta g_{\mu\nu,\beta}\right) \tag{2.48}$$

In the last line we have exploited the antisymmetry of $(\delta g_{\beta\mu,\nu} - \delta g_{\mu\nu,\beta})$ with respect to the indices β, ν, dropping out upon contraction with $r^{\beta} r^{\nu}$. In addition, since the variation $\delta g_{\alpha\beta}$ vanishes everywhere on the boundary, its derivative along a tangent vector u^{ρ} of $\partial\mathcal{V}$ must be zero. The arbitrariness of u^{ρ} provides the relation

$$u^{\rho} \delta g_{\mu\nu,\rho} = u_{\sigma} h^{\sigma\rho} \delta g_{\mu\nu,\rho} = 0 \quad \Longrightarrow \quad h^{\sigma\rho} \delta g_{\mu\nu,\rho} = 0 \tag{2.49}$$

thereby allowing to discard one term of the last line of 2.48:

$$\varepsilon\, \delta V^{\alpha} r_{\alpha} = -\varepsilon h^{\mu\nu} \delta g_{\mu\nu,\alpha} r^{\alpha} \tag{2.50}$$

By virtue of these results, our surface integral becomes

$$\int_{\mathcal{V}} g^{\mu\nu} \delta R_{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x = -\oint_{\partial\mathcal{V}} \varepsilon h^{\mu\nu} r^{\alpha} \delta g_{\mu\nu,\alpha} \sqrt{|h|} \mathrm{d}^3 x \tag{2.51}$$

We are now in the position to recast this integral into a much more elegant form. Indeed, from the variation of ${\Gamma^{\lambda}}_{\mu\nu}$ (equation 2.47) and the property 2.49, the right-hand side turns into

$$-\oint_{\partial\mathcal{V}} \varepsilon h^{\mu\nu} r^{\alpha} \delta g_{\mu\nu,\alpha} \sqrt{|h|}\, \mathrm{d}^3 x = -\oint_{\partial\mathcal{V}} \varepsilon h^{\mu\nu} r^{\alpha} \left(-2\delta\Gamma_{\alpha\mu\nu}\right) \sqrt{|h|}\, \mathrm{d}^3 x$$
$$= 2 \oint_{\partial\mathcal{V}} \varepsilon h^{\mu\nu} r_{\alpha} \delta {\Gamma^{\alpha}}_{\mu\nu} \sqrt{|h|}\, \mathrm{d}^3 x \tag{2.52}$$

The product $r_{\alpha}\, \delta {\Gamma^{\alpha}}_{\mu\nu}$ can be traced back to the variation of the covariant derivative

$$r_{\alpha} \delta {\Gamma^{\alpha}}_{\mu\nu} = -\delta \left(\nabla_{\mu} r_{\nu}\right) \tag{2.53}$$

If we recall equation 2.32, that gives the relation between K and the (spacetime) covariant derivative of the unit vector r_{μ}, we see that on $\partial\mathcal{V}$ the following relation holds:

$$h^{\mu\nu} r_{\alpha} \delta {\Gamma^{\alpha}}_{\mu\nu} = -\left(g^{\mu\nu} - \varepsilon r^{\mu} r^{\nu}\right) \delta \left(\nabla_{\mu} r_{\nu}\right)$$
$$= -\delta \left(\nabla_{\mu} r^{\mu}\right) = \delta K \tag{2.54}$$

Notice that we used property $r^{\nu} \nabla_{\mu} r_{\nu} = 0$ to eliminate $\varepsilon r^{\mu} r^{\nu}$. Since $\delta g_{\mu\nu} = \delta h_{\mu\nu} = 0$ on the boundary, the variation of $\sqrt{|h|}$ vanishes, thus enabling us to rewrite the integral 2.42 as

$$\int_{\mathcal{V}} g^{\mu\nu} \delta R_{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x = 2\delta \oint_{\partial\mathcal{V}} \varepsilon K \sqrt{|h|}\, \mathrm{d}^3 x \tag{2.55}$$

By virtue of this result, we can add a new term to the action functional $\mathcal{S}$, called the boundary term $\mathcal{S}_B$, such that the variation of $\mathcal{S}$ provides the correct gravitational component of the Einstein field equations. Indeed, if we define this contribution by the integral

$$\mathcal{S}_B = -\frac{1}{8\pi} \oint_{\partial\mathcal{V}} \varepsilon K \sqrt{|h|}\, \mathrm{d}^3 x \tag{2.56}$$

the extra boundary term which tainted equation 1.19 cancels out with 2.55, leading to the left-hand side of the field equations 1.23.

2.4.2 Nondynamical term $\mathcal{S}_0$ of the action

Although the gravitational part of the action $\mathcal{S}_G = \mathcal{S}_H + \mathcal{S}_B$ is now fully consistent with the Einstein field equations, the integral 2.56 might diverge for flat (or asymptotically flat) spacetimes. In fact, by considering the vacuum solution $R_{\mu\nu} = 0$ on a region of spacetime delimited by two hypersurfaces $t = t_1$, $t = t_2$ (with t_1, t_2 constants) and by a three-cylinder with radius ρ, the action reduces to the boundary term

$$\mathcal{S}_B = -\frac{1}{8\pi} \oint_{\partial\mathcal{V}} \varepsilon K \sqrt{|h|}\, \mathrm{d}^3x = \rho(t_2 - t_1) \tag{2.57}$$

which diverges when $\rho \to \infty$. For this reason we might introduce a nondynamical term $\mathcal{S}_0$, which does by no means affect the field equations, such that the total action is bounded even for non-compact (asymptotically) flat manifolds. One possible choice is given by

$$\mathcal{S}_0 = -\frac{1}{8\pi} \oint_{\partial\mathcal{V}} \varepsilon K_0 \sqrt{|h|}\, \mathrm{d}^3x \tag{2.58}$$

with K_0 corresponding to the scalar extrinsic curvature of the embedding of $\partial\mathcal{V}$ in flat spacetime. Accordingly, reintroducing the superscript "4", the well-defined gravitational action $\mathcal{S}_G$ becomes

$$\begin{aligned} \mathcal{S}_G &= \mathcal{S}_H + \mathcal{S}_B - \mathcal{S}_0 \\ &= \frac{1}{16\pi} \int_{\mathcal{V}} {}^4R \sqrt{-g}\, \mathrm{d}^4x - \frac{1}{8\pi} \oint_{\partial\mathcal{V}} \varepsilon \left(K - K_0\right) \sqrt{|h|}\, \mathrm{d}^3x \end{aligned} \tag{2.59}$$

Chapter 3

3+1 decomposition of spacetime

In this chapter we carry on the dimensional splitting of spacetime $(\mathcal{M}, \boldsymbol{g})$ into a purely spatial part and time. We start by fixing the three-geometry of spacelike hypersurfaces Σ, endowing each of them with a coordinate system and with the induced metric $\gamma_{\mu\nu}$. However, this does not fully determine the four-geometry of spacetime: one must in addition set the geometry between two neighbouring hypersurfaces. To this end, we shall define four new functions, which supplement the information required for a complete description of $\mathcal{M}$. Once this has been done, we will be able to rewrite the gravitational action 2.59 in terms of the extrinsic curvature tensor and of three-dimensional quantities inherent to the hypersurfaces.
The feasibility of this process restricts the analysis to a specific class of spacetimes, called globally hyperbolic spacetimes.

Definition 4. A spacetime $\mathcal{M}$ is said to be globally hyperbolic if it admits a spacelike hypersurface Σ (called Cauchy surface) such that every timelike or null curve without endpoints intersects Σ once and only once.

Any globally hyperbolic spacetime admits a foliation by a family of spacelike hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$, which means that each Σ_t is a level surface of a regular scalar field $\hat{t}$ on $\mathcal{M}$. We thus focus on this class of spacetimes and proceed with the introduction of the lapse and shift functions, following the notation adopted by Arnowitt, Deser and Misner in their 1962 article (Ref. [4]).

3.1 The lapse function

In section 2.1 we considered the normal covariant vector collinear to the gradient one-form $d\hat{t}$. In particular, we defined the function Ω as the negative factor which ensures that $n_\alpha = \Omega \left(d\hat{t}\right)_\alpha$ is normalized. Let us introduce a closely related positive quantity N and call it the lapse function:

$$N \doteq \frac{1}{\sqrt{-g^{00}}} = -\Omega \tag{3.1}$$

This name is justified by the physical significance that N acquires: indeed, it determines the lapse of proper time between two neighbouring hypersurfaces $\Sigma_{t+\delta t}$ and Σ_t. To clarify this fact, let us define the normal evolution vector:

$$m^\alpha \doteq N n^\alpha \tag{3.2}$$

We see that it satisfies the relation

$$(d\hat{t})_\alpha m^\alpha = -N^2 g^{\alpha\beta} (d\hat{t})_\alpha (d\hat{t})_\beta = -N^2 g^{00} = 1 \tag{3.3}$$

where we have used equation 2.11. Next we consider $p \in \Sigma_t$ and generate an infinitesimally close point $q \in \mathcal{M}$, such that $x^\alpha(q) = x^\alpha(p) + m^\alpha \delta t$. Substituting into the scalar field $\hat{t}$ and expanding to first order we find that:

$$\hat{t}(q) = \hat{t}(p) + (d\hat{t})_\alpha m^\alpha \delta t = t + \delta t \tag{3.4}$$

Therefore we have proved that $q \in \Sigma_{t+\delta t}$, which means that the displacement $\delta x^\alpha = m^\alpha \delta t$ connects Σ_t to $\Sigma_{t+\delta t}$. Besides, if we consider an observer moving with four-velocity n^α, the elapsed proper time $\delta\tau$ measured between the events p and q is given by

$$\delta\tau \doteq \sqrt{-\left(m^\alpha \delta t\right) g_{\alpha\beta} \left(m^\beta \delta t\right)} = N \delta t \tag{3.5}$$

This implies that the lapse function N associates an infinitesimal interval of coordinate time t to the proper time measured by an observer whose world lines are orthogonal to Σ_t. In order to simplify the notation, henceforth we implicitly identify $\hat{t}$ with the coordinate t.

3.2 The shift functions

Let $(x^i) = (x^1, x^2, x^3)$ be the coordinate system on each hypersurface of the foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ and let (t, x^1, x^2, x^3) be the natural smooth extension to $\mathcal{M}$. The tangent vectors $(\partial_\mu) = (\partial_t, \partial_1, \partial_2, \partial_3)$ constitute the natural basis of $T_p(\mathcal{M})$, while (dx^μ) denotes the corresponding dual basis of $T_p^*(\mathcal{M})$. In particular, ∂_t is the tangent vector to the curve $x^i = const$, whereas $\partial_i \in T_p(\Sigma_t)$. Albeit ∂_t connects two neighbouring hypersurfaces similarly to m^α, they in general differ, as ∂_t is not necessarily orthogonal to Σ_t. In fact, asking the coordinate systems (x^i) to vary smoothly between neighbouring hypersurfaces does by no means fix the direction of ∂_t. In order to fully determine the geometry of spacetime we have to specify the displacement of ∂_t from m^α in any point of $\mathcal{M}$: let us consider the vector difference

$$\beta^\alpha \doteq (\partial_t)^\alpha - m^\alpha \tag{3.6}$$

β^α is tangent to Σ_t, since the projection on n^α is zero:

$$n_\alpha \beta^\alpha = n_\alpha (\partial_t)^\alpha - n_\alpha m^\alpha = n_0 + N = 0$$

In addition, applying the basis one-forms $dx^i \in T_p^*(\Sigma)$ to β^α gives:

$$dx^i(\boldsymbol{\beta}) = \beta^i = \delta^i{}_0 - N n^i = N^2 g^{i0} \tag{3.7}$$

We thus define the three shift functions N^i to be the spatial components of β^α:

$$N^i \doteq \beta^i = N^2 g^{0i} \tag{3.8}$$

Once the lapse function and the shift functions are specified, together with the three-dimensional metric γ_{ij}, the spacetime geometry is completely determined. Let us express $g_{\mu\nu}$ in terms of γ_{ij} and of these new functions N, N^i. At present, we have identified only few components of the metric and its inverse:

$$g_{\mu\nu} = \left(\begin{array}{c|c} A & B_j \\ \hline B_i & \gamma_{ij} \end{array}\right) \qquad g^{\mu\nu} = \frac{1}{N^2}\left(\begin{array}{c|c} -1 & N^j \\ \hline N^i & C^{ij} \end{array}\right) \tag{3.9}$$

where A, B_i and C^{ij} are unknown. These entries can be determined by exploiting the identity $g_{\mu\rho}g^{\rho\nu} = \delta_\mu{}^\nu$:

$$g_{i\rho}g^{\rho 0} = \frac{1}{N^2}(-B_i + \gamma_{ij}N^j) = 0 \qquad \Longrightarrow \qquad B_i = \gamma_{ik}N^k \tag{3.10}$$

$$g_{0\rho}g^{\rho 0} = \frac{1}{N^2}(-A + \gamma_{ik}N^k N^i) = 1 \qquad \Longrightarrow \qquad A = \gamma_{ik}N^k N^i - N^2 \tag{3.11}$$

$$g_{i\rho}g^{\rho j} = \frac{1}{N^2}\gamma_{ik}(N^k N^j + C^{kj}) = \delta_i{}^j \quad \Longrightarrow \quad C^{ij} = N^2\gamma^{ij} - N^i N^j \tag{3.12}$$

Let us adopt the notation $N_i \doteq \gamma_{ik}N^k$. Replacing A, B_i, C^{ij} in equation 3.9 leads to the result we sought:

$$g_{\mu\nu} = \left(\begin{array}{c|c} N_i N^i - N^2 & N_j \\ \hline N_i & \gamma_{ij} \end{array}\right) \quad g^{\mu\nu} = \left(\begin{array}{c|c} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \hline \dfrac{N^i}{N^2} & \gamma^{ij} - \dfrac{N^i N^j}{N^2} \end{array}\right) \tag{3.13}$$

It should be emphasized that $g_{ij} = \gamma_{ij}$, whereas in general $g^{ij} \neq \gamma^{ij}$. Indeed, this is true only if $N^i \equiv 0$, which means that ∂_t and m^α coincide (in this case, the coordinates (x^μ) are said to be Gaussian normal coordinates).
The determinants g and γ respectively of $g_{\mu\nu}$ and γ_{ij} are related by the equation:

$$g = -N^2\gamma \longrightarrow \sqrt{-g} = N\sqrt{\gamma} \tag{3.14}$$

which gives the density $\sqrt{-g}$ of spacetime in terms of the density $\sqrt{\gamma}$ and the lapse function. This follows from the definition of the inverse metric $g^{\mu\nu}$ by means of the adjugate matrix:

$$g^{00} = \frac{\det \boldsymbol{\gamma}}{\det \boldsymbol{g}} = \frac{\gamma}{g} = -\frac{1}{N^2} \tag{3.15}$$

Ultimately, combining equations 2.10, 2.11 and 3.8 provides an explicit form of n_μ and n^μ in terms of the functions N and N^i:

$$n_\mu = (-N, 0, 0, 0)_\mu \tag{3.16}$$

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right)^\mu \tag{3.17}$$

3.3 Final decomposition of the Riemann tensor

Let us return to the contracted Gauss relation (equation 2.36). We now aim at replacing the Riemann tensor ${}^4R^\rho{}_{\sigma\mu\nu}$ with the three-dimensional intrinsic and extrinsic curvature tensors and with the lapse and shift functions. To begin with, we rewrite the acceleration covector a_μ in terms of the lapse function using the definition of n_μ:

$$a_\mu = \nabla_{\boldsymbol{n}} n_\mu = -n^\sigma \nabla_\sigma (N \nabla_\mu t) = -n^\sigma (\nabla_\sigma N)(\nabla_\mu t) - n^\sigma N \nabla_\sigma \nabla_\mu t$$

Since ∇ is torsion free, the covariant derivatives commute if applied to a scalar field, namely $[\nabla_\alpha, \nabla_\beta] t = 0$. Thus:

$$\begin{aligned} a_\mu &= \frac{1}{N} n_\mu n^\sigma \nabla_\sigma N + n^\sigma N \nabla_\mu \left(\frac{n_\sigma}{N} \right) \\ &= n_\mu n^\sigma \nabla_\sigma \ln N - n_\sigma n^\sigma \nabla_\mu \ln N + n^\sigma \nabla_\mu n_\sigma \\ &= (n^\sigma n_\mu + \delta^\sigma{}_\mu) \nabla_\sigma \ln N = \gamma^\sigma{}_\mu \nabla_\sigma \ln N = D_\mu \ln N \end{aligned} \tag{3.18}$$

Plugging a_μ into equation 2.31 gives:

$$\nabla_\mu n_\nu = -K_{\mu\nu} - n_\mu D_\nu \ln N \tag{3.19}$$

In a similar way, the covariant derivative of m^α can be easily computed:

$$\nabla_\mu m^\nu = \nabla_\mu (N n^\nu) = -N K_\mu{}^\nu - n_\mu D^\nu N + n^\nu \nabla_\mu N \tag{3.20}$$

Another mathematical tool we shall consider is the Lie derivative (whose characterization is given in section 6.1.4 of the Appendix). Let us consider the tangent tensors on Σ, namely the tensors invariant by projection:

$$T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} = \gamma^{\alpha_1}{}_{\mu_1} \ldots \gamma^{\alpha_r}{}_{\mu_r} \gamma^{\nu_1}{}_{\beta_1} \ldots \gamma^{\nu_s}{}_{\beta_s} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \tag{3.21}$$

The Lie derivative acts as an endomorphism of the space of tangent tensors on Σ, as it can be proved that $\mathcal{L}_{\boldsymbol{m}} \gamma^\mu{}_\nu = 0$. In fact, combining this with the product rule on 3.21 confirms our assertion:

$$(\mathcal{L}_{\boldsymbol{m}} \boldsymbol{T})^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} = \gamma^{\alpha_1}{}_{\mu_1} \ldots \gamma^{\alpha_r}{}_{\mu_r} \gamma^{\nu_1}{}_{\beta_1} \ldots \gamma^{\nu_s}{}_{\beta_s} (\mathcal{L}_{\boldsymbol{m}} \boldsymbol{T})^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \tag{3.22}$$

The Lie derivative $\mathcal{L}_{\boldsymbol{m}}$ provides some useful and concise relations between the tensorial quantities previously introduced. In particular, since ∇ is torsion free, we can apply the property 6.19 from the Appendix and replace the partial derivatives with their covariant counterparts. This being said, thanks to equation 3.20 the Lie derivative of $\gamma_{\mu\nu}$ results in

$$\mathcal{L}_{\boldsymbol{m}} \gamma_{\mu\nu} = m^\alpha \nabla_\alpha \gamma_{\mu\nu} + \gamma_{\alpha\nu} \nabla_\mu m^\alpha + \gamma_{\mu\alpha} \nabla_\nu m^\alpha = -2N K_{\mu\nu} \tag{3.23}$$

which means that the evolution of $\gamma_{\mu\nu}$ along m^α is related to the lapse function and to $K_{\mu\nu}$. Using $m^\alpha = N n^\alpha$ and the orthogonality $\gamma_{\mu\nu} n^\nu = 0$, equation 3.23 can be rewritten in a significant form:

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_{\boldsymbol{n}} \gamma_{\mu\nu} \tag{3.24}$$

Although this relation is sometimes used as a definition of the extrinsic curvature, it should be noted that it is meaningful only if Σ_t belongs to a foliation, as the derivative of $\gamma_{\mu\nu}$ along n^α may not be defined. Let us now turn to the evaluation of $\mathcal{L}_{\boldsymbol{m}} K_{\mu\nu}$ using the covariant version of the expansion:

$$\mathcal{L}_{\boldsymbol{m}} K_{\mu\nu} = N \left(\nabla_{\boldsymbol{n}} K_{\alpha\beta} + K_{\alpha\rho} \nabla_\beta n^\rho + K_{\rho\beta} \nabla_\alpha n^\rho \right) \tag{3.25}$$

Since $K_{\mu\nu}$ is a tangent tensor of Σ, we can resort to the property 3.22 and apply the projector $\gamma^{\mu}{}_{\nu}$ without altering the result:

$$\begin{aligned}\mathcal{L}_{\boldsymbol{m}} K_{\mu\nu} &= \gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\,\mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} \\ &= N\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\nabla_{\boldsymbol{n}} K_{\alpha\beta} - 2NK_{\mu\rho}K^{\rho}{}_{\nu}\end{aligned} \tag{3.26}$$

If we replace the Greek indices with Latin indices running from 1 to 3, equations 3.23 and 3.26 continue to hold, inasmuch as $K_{\mu\nu}$ and $\gamma_{\mu\nu}$ are three-dimensional tangent tensors defined on Σ:

$$\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = -2NK_{ij} \tag{3.27}$$

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = N\gamma^{k}{}_{i}\gamma^{l}{}_{j}\nabla_{\boldsymbol{n}} K_{kl} - 2NK_{ik}K^{k}{}_{j} \tag{3.28}$$

Exploiting the identity $\mathcal{L}_{\boldsymbol{m}}(\gamma^{ik}\gamma_{kj}) = 0$ and the product rule, we are able to recover the Lie derivative of the inverse metric γ^{ij} and of the trace K, whose significance will be apparent later in this section:

$$\mathcal{L}_{\boldsymbol{m}}\gamma^{ij} \;=\; -\gamma^{ik}\gamma^{jl}\mathcal{L}_{\boldsymbol{m}}\gamma_{kl} \;=\; 2NK^{ij} \tag{3.29}$$

$$\mathcal{L}_{\boldsymbol{m}} K \;=\; 2NK_{ij}K^{ij} + \gamma^{ij}\mathcal{L}_{\boldsymbol{m}}(K_{ij}) \;=\; N\gamma^{ij}\,\nabla_{\boldsymbol{n}} K_{ij} \tag{3.30}$$

Let us now project twice on Σ and once along n^{μ} the four-dimensional Ricci identity 2.38 applied to n^{μ}:

$$\gamma_{\rho\alpha}\gamma^{\mu}{}_{\beta}n^{\nu}\left({}^{4}R^{\rho}{}_{\sigma\mu\nu}n^{\sigma}\right) = \gamma_{\rho\alpha}\gamma^{\mu}{}_{\beta}n^{\nu}\,[\nabla_{\mu},\nabla_{\nu}]n^{\rho} \tag{3.31}$$

Expanding the commutator and using formula 3.19, after some laborious passages we manage to eliminate the derivatives $\nabla_{\mu}n^{\rho}$ in favour of the extrinsic curvature and arrive at

$$\gamma_{\rho\alpha}\gamma^{\mu}{}_{\beta}\,{}^{4}R^{\rho}{}_{\sigma\mu\nu}\,n^{\sigma}n^{\nu} = -K_{\alpha\lambda}K^{\lambda}{}_{\beta} + \gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\nabla_{\boldsymbol{n}} K_{\mu\nu} + \frac{1}{N}D_{\alpha}D_{\beta}N \tag{3.32}$$

This relation will enable to replace the spacetime Riemann tensor ${}^{4}R^{\rho}{}_{\sigma\mu\nu}$ with three-dimensional quantities. In fact, if we compare it with the contracted Gauss relation 2.36, namely

$$\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\,{}^{4}R_{\alpha\beta} + \gamma_{\mu\rho}\gamma^{\alpha}{}_{\nu}\,{}^{4}R^{\rho}{}_{\sigma\alpha\beta}\,n^{\sigma}n^{\beta} = R_{\mu\nu} + KK_{\mu\nu} - K_{\mu\lambda}K^{\lambda}{}_{\nu}$$

we notice the two common terms (up to a notational change) containing ${}^{4}R^{\rho}{}_{\sigma\mu\nu}$ and $K_{\alpha\lambda}K^{\lambda}{}_{\beta}$. Hence after the combination of these equations we are left with the simpler expression

$$\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\,{}^{4}R_{\alpha\beta} = R_{\mu\nu} + KK_{\mu\nu} - \gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\nabla_{\boldsymbol{n}} K_{\alpha\beta} - \frac{1}{N}D_{\mu}D_{\nu}N \tag{3.33}$$

Finally, let us perform the last contraction with $\gamma^{\mu\nu}$ and replace in the right-hand side the Greek indices with Latin ones, exploiting the aforementioned purely spatial character of K and γ:

$$\gamma^{\mu\nu}\,{}^{4}R_{\mu\nu} = R + K^{2} - \gamma^{ij}\nabla_{\boldsymbol{n}} K_{ij} - \frac{1}{N}\,\gamma^{ij}D_{i}D_{j}N \tag{3.34}$$

Equation 3.30 suggests that we should substitute $\gamma^{ij}\nabla_{\boldsymbol{n}} K_{ij}$ with the Lie derivative of K. However, given scalar field f and a vector X, the relation $\mathcal{L}_{X}f = \nabla_{X}f$

always holds, implying that in this particular case the contraction with γ^{ij} commutes with ∇, even though $\nabla_\lambda \gamma^{ij} \neq 0$. Therefore we get

$$\begin{aligned}\gamma^{\mu\nu}\,{}^4R_{\mu\nu} &= R + K^2 - \frac{1}{N}\nabla_{\boldsymbol{m}} K + \frac{1}{N} D^i D_i N \\ &= R + K^2 - \nabla_{\boldsymbol{n}} K + \frac{1}{N} D^i D_i N \end{aligned} \tag{3.35}$$

Recalling that $\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$, we now split the left-hand side into

$${}^4R + {}^4R_{\mu\nu} n^\mu n^\nu = R + K^2 - \nabla_{\boldsymbol{n}} K - \frac{1}{N} D^i D_i N \tag{3.36}$$

Eventually, by virtue of Gauss' Theorema Egregium (equation 2.37), the replacement of ${}^4R_{\mu\nu} n^\mu n^\nu$ leads to the relation we sought:

$${}^4R = R + K^2 + K^{ij} K_{ij} - 2\,\nabla_{\boldsymbol{n}} K - \frac{2}{N} D^i D_i N \tag{3.37}$$

The 3+1 decomposition of the spacetime scalar curvature ${}^4R$ is thus complete.

3.4 Projection of the Einstein field equations

We end this chapter with the projection of Einstein's field equations, resorting to the relations between three and four-dimensional tensors hitherto considered. Let us focus on the case $\Lambda = 0$:

$${}^4R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\,{}^4R = 8\pi T_{\mu\nu} \tag{3.38}$$

We can recast these equations in an equivalent form by displacing the scalar curvature ${}^4R$ with the trace T of the stress-energy tensor. Indeed, contracting 3.38 with $g^{\mu\nu}$ gives

$$g^{\mu\nu}\left[{}^4R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\,{}^4R\right] = 8\pi g^{\mu\nu} T_{\mu\nu} \implies {}^4R = -8\pi T \tag{3.39}$$

After the substitution, we arrive at

$${}^4R_{\mu\nu} = 8\pi\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right] \tag{3.40}$$

It is worth to introduce some tensorial quantities describing the 3+1 splitting of $T_{\mu\nu}$. Since the normal vector n^α is timelike, it may be identified with the four-velocity of some observer, which moves perpendicularly to the hypersurfaces Σ_t. The energy density E measured by this observer is given by the formula:

$$E \doteq T_{\mu\nu} n^\mu n^\nu \tag{3.41}$$

This is analogous to the definition adopted in special relativity. In a similar fashion, we introduce the momentum density as the one-form

$$p_\alpha \doteq -T_{\mu\nu} n^\mu \gamma^\nu{}_\alpha \tag{3.42}$$

The projector $\gamma^\nu{}_\alpha$ ensures that p_α is tangent to the hypersurface Σ_t. Lastly, the total projection of $T_{\mu\nu}$ onto Σ_t produces the stress tensor $S_{\alpha\beta}$:

$$S_{\alpha\beta} \doteq T_{\mu\nu} \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \; = \; T_{\alpha\beta} + E n_\alpha n_\beta + n^\rho (T_{\alpha\rho} n_\beta + T_{\rho\beta} n_\alpha) \tag{3.43}$$

and the corresponding trace with respect to $\gamma_{\mu\nu}$:

$$S = S_{\alpha\beta}\gamma^{\alpha\beta} = S_{ij}\gamma^{ij} \tag{3.44}$$

Combining equations 3.43 and 3.44, we can write T in terms of the trace S and the energy E:

$$T = T_{\mu\nu}g^{\mu\nu} = T_{\mu\nu}(\gamma^{\mu\nu} - n^\mu n^\nu) = S - E \tag{3.45}$$

These definitions describe all the quantities arising from the projections of the stress-energy tensor. Since $T_{\mu\nu}$ is a symmetric rank-2 tensor, we are now in the position to analyze individually each of the three possible combinations.

3.4.1 Total projection onto Σ_t

Let us apply twice the projector $\gamma^\mu{}_\nu$ to 3.40. On the left-hand side we get the projection of the Ricci tensor, which has already been computed in the previous section (formula 3.33). However, we can reshape it into a slightly different form with the introduction of the Lie derivative $\mathcal{L}_{\boldsymbol{m}}K_{\mu\nu}$ through equation 3.26:

$$\gamma^\mu{}_\alpha\gamma^\nu{}_\beta{}^4R_{\mu\nu} = R_{\alpha\beta} - 2K_{\alpha\lambda}K^\lambda{}_\beta + KK_{\alpha\beta} - \frac{1}{N}\left[\mathcal{L}_{\boldsymbol{m}}K_{\alpha\beta} + D_\alpha D_\beta N\right]$$

On the right-hand side, instead, we can replace $T_{\mu\nu}$ and T with the stress tensor $S_{\mu\nu}$, its trace S and the energy density E thanks to 3.43 and 3.45:

$$8\pi\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\Big[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big] = 8\pi\Big[S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S-E)\Big] \tag{3.46}$$

Therefore the total projection onto Σ_t of the Einstein equations reads:

$$\begin{aligned} &R_{\alpha\beta} - 2K_{\alpha\lambda}K^\lambda{}_\beta + KK_{\alpha\beta} - \frac{1}{N}\left[\mathcal{L}_{\boldsymbol{m}}K_{\alpha\beta} + D_\alpha D_\beta N\right] \\ &= 8\pi\Big[S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S-E)\Big] \end{aligned} \tag{3.47}$$

It is worth to note that all the tensors involved in this equation are tangent to Σ_t, thus their components are completely described by the Latin indices ij. Isolating the evolution of the extrinsic curvature $\mathcal{L}_{\boldsymbol{m}}K_{ij}$ leads to the equivalent form:

$$\begin{aligned} \mathcal{L}_{\boldsymbol{m}}K_{ij} = -D_iD_jN &+ N\left[R_{ij} - 2K_{il}K^l{}_j + KK_{ij}\right] \\ &+ 4\pi N\left[\gamma_{ij}(S-E) - 2S_{ij}\right] \end{aligned} \tag{3.48}$$

3.4.2 Total projection along n^μ

Again, let us project the Einstein equations, this time twice along n^μ. This means that we shall contract equation 3.38 with $n^\mu n^\nu$. Using the definition 3.41 of the energy E we find:

$$n^\mu n^\nu{}^4R_{\mu\nu} + \frac{1}{2}{}^4R = 8\pi E \tag{3.49}$$

Comparing the left-hand side of this equation with the generalized Theorema Egregium (formula 2.37) gives the relation:

$$R - K_{ij}K^{ij} + K^2 = 16\pi E \tag{3.50}$$

This is called the Hamiltonian constraint, as it will appear in the Hamiltonian approach to general relativity, described in chapter 4.

3.4.3 Mixed projection onto Σ_t and along n^μ

The last projection that can be performed is the mixed projection along n^μ and onto Σ_t. Since the metric $g_{\mu\nu}$ vanishes upon contraction with n^μ and $\gamma^\nu{}_\alpha$, the field equations reduce to

$$ {}^4R_{\mu\nu} n^\mu \gamma^\nu{}_\alpha = -8\pi p_\alpha \tag{3.51} $$

where p_α is the momentum density previously defined. The left-hand side can be transformed into three-dimensional covariant derivatives by means of the contracted Codazzi relation 2.41:

$$ D_\beta K_\alpha{}^\beta - D_\alpha K = 8\pi p_\alpha \tag{3.52} $$

or equivalently, by restricting the indices to the spatial components:

$$ D_j K_i{}^j - D_i K = 8\pi p_i \tag{3.53} $$

This is called the momentum constraint. In fact, its left-hand side will be rewritten in terms of the conjugate momenta arising in the Hamiltonian description of general relativity.

3.5 Summary of the results

Having completed the projection of the field equations, it is worth to summarize and analyze the results before proceeding with the introduction of the Hamiltonian formalism. In particular, we shall first recall the equation 3.27, which relates the evolution of $\gamma_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$:

$$ \mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = -2NK_{ij} \tag{3.54} $$

Let us split $\boldsymbol{m}$ into $\partial_t - N^k \partial_k$ and rewrite the Lie derivative as

$$ \mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = \partial_t \gamma_{ij} - \left[\gamma_{kj} D_i N^k + \gamma_{ik} D_j N^k + N^k D_k \gamma_{ij}\right] \tag{3.55} $$

From now forth we denote the time derivatives with a dot (Newton notation), so $\partial_t \gamma_{ij} = \dot{\gamma}_{ij}$. Since the metric γ_{ij} is covariantly constant with respect to the connection D, the latter term vanishes and we get

$$ \mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = \dot{\gamma}_{ij} - D_i N_j - D_j N_i \tag{3.56} $$

Combining this equation with 3.54 gives a useful result, which will be repeatedly used in the next chapter:

$$ K_{ij} = \frac{1}{2N}\left[D_i N_j + D_j N_i - \dot{\gamma}_{ij}\right] \tag{3.57} $$

Then we have the projections of the field equations:

$$ \mathcal{L}_{\boldsymbol{m}} K_{ij} = -D_i D_j N + N\Big[R_{ij} - 2K_{il}K^l{}_j + KK_{ij} + 4\pi\big(\gamma_{ij}(S-E) - 2S_{ij}\big)\Big] $$
$$ R - K_{ij}K^{ij} + K^2 = 16\pi E $$

$$D_j K_i{}^j - D_i K = 8\pi p_i$$

These four relations constitute a set of 16 second-order nonlinear partial differential equations in the unknowns γ_{ij}, K_{ij}, N and N^i (assuming that the matter contribution from E, p_i and S_{ij} is given). However, since the number of independent four-dimensional Einstein equations is 10, this must be true also for the system shown above. The 3+1 formalism allowed to treat the field equations as a Cauchy problem, leading to many significant results on the existence and uniqueness (up to isometry) of local and global solutions which are a "development" of the initial data set. In particular, once the four constraint equations are specified on a spacelike hypersurface Σ, they prove to be both necessary and sufficient conditions for the possibility to embed Σ in a spacetime $\mathcal{M}$ which satisfies Einstein equations. A thorough discussion of the subject can be found in Ref. [8], [10] and [11] by Choquet-Bruhat and Geroch.

Chapter 4

ADM Hamiltonian formulation of General Relativity

In this chapter we discuss the Hamiltonian formulation of general relativity proposed by Arnowitt, Deser and Misner which stems from the gravitational action functional 2.59. The significance of the canonical formulation lies in two primary hallmarks:

- time holds a privileged position among the coordinates (x^μ). In particular, the original four-dimensional description is replaced by the evolution of tensor fields on a spacelike three-dimensional hypersurface Σ;
- the time evolution of the system is defined by Hamilton's equations, which are first-order differential equations in the time derivatives.

The canonical form of general relativity also sheds light on the issues originated from the redundancy of the variables $g_{\mu\nu}$. Indeed, although this ensures the general covariance of the theory, it encumbers the identification of the minimal set of data needed to provide a consistent initial value formulation. This reduction to the independent dynamical modes of the gravitational field is highly desirable, as it is a necessary prerequisite for the quantization program of general relativity. In fact, the correspondence between Poisson brackets of the Hamiltonian theory and commutators in quantum mechanics can be considered only when the unconstrained canonical variables are singled out from the corresponding overall set.
In the following sections we present a detailed derivation of Hamilton's equations and arrive at the fundamental Poisson brackets (section 4.5) between the constrained variables. For a formal discussion on the isolation of the independent variables of the system, the reader can refer to the 1962 article by ADM (Ref. [4]).

4.1 Einstein-Hilbert action in 3+1 formalism

Let us return to the Hilbert Lagrangian density $\mathcal{L}_H = {}^4R\sqrt{-g}$, temporarily omitting the irrelevant multiplicative factor $(16\pi)^{-1}$. We shall replace the four-dimensional quantities ${}^4R$ and $\sqrt{-g}$ with their three-dimensional counterparts, using respectively equations 3.37 and 3.14:

$$\mathcal{L}_H = \left[R + K^2 + K^{ij}K_{ij} - 2\nabla_{\boldsymbol{n}}K - \frac{2}{N}D^iD_iN\right]N\sqrt{\gamma} \tag{4.1}$$

In chapter 1 we introduced the action functional $\mathcal{S}_H$ as the integral of $\mathcal{L}_H$ over a region $\mathcal{V}$ of the spacetime manifold. The 3+1 dimensional decomposition allows to carry on the integration on $\mathcal{V}$ by subdividing this region into a family of hypersurfaces Σ_t, labeled by the time t:

$$\mathcal{S}_H = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[R + K^2 + K^{ij}K_{ij} - 2\nabla_{\boldsymbol{n}}K - \frac{2}{N}D^iD_iN\right]N\sqrt{\gamma}\,\mathrm{d}^3x \tag{4.2}$$

where t_1 and t_2 are generic lower and upper time limits. Before proceeding with our analysis, we shall unveil the divergences hidden in the last two terms of $\mathcal{L}_H$ by rewriting them in the following form:

$$\sqrt{\gamma}\,D^iD_iN = \sqrt{\gamma}\,D_i\left(\partial^iN\right) = \partial_i\left(\sqrt{\gamma}\partial^iN\right) \tag{4.3}$$

$$N\sqrt{\gamma}\nabla_{\boldsymbol{n}}K \;=\; N\sqrt{\gamma}n^\alpha\nabla_\alpha K \;=\; \partial_\alpha\left(\sqrt{\gamma}NKn^\alpha\right) + \sqrt{\gamma}NK^2 \tag{4.4}$$

Substituting in the Lagrangian density $\mathcal{L}_H$, the term K^2 changes sign and we arrive at

$$\begin{aligned}\mathcal{L}_H = &\left(R - K^2 + K^{ij}K_{ij}\right)N\sqrt{\gamma} \\ &- 2\left[\partial_i\left(\sqrt{\gamma}\,\partial^iN\right) + \partial_\alpha\left(\sqrt{\gamma}NKn^\alpha\right)\right]\sqrt{\gamma}\end{aligned} \tag{4.5}$$

It will prove more informative to recast temporarily the two divergences in four-dimensional notation. To this end, we consider the generalized Theorema Egregium 2.37

$${}^4R = R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\,{}^4R_{\mu\nu}n^\mu n^\nu$$

and replace the last term with a commutator of spacetime connections, exploiting the contracted Ricci identity:

$${}^4R = R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2n^\mu[\nabla_\alpha, \nabla_\mu]n^\alpha$$

Now we use equations 2.31, 2.32 and the orthogonality relation 2.28 to rewrite the commutator in the form

$$\begin{aligned}n^\mu[\nabla_\alpha, \nabla_\mu]n^\alpha &= n^\mu\left(\nabla_\alpha\nabla_\mu - \nabla_\mu\nabla_\alpha\right)n^\alpha \\ &= \nabla_\alpha\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right) - \left(\nabla_\alpha n^\mu\right)\nabla_\mu n^\alpha + \left(\nabla_\alpha n^\alpha\right)^2 \\ &= \nabla_\alpha\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right) - K^{\alpha\mu}K_{\alpha\mu} + K^2\end{aligned}$$

Substituting this result in the Lagrangian density $\mathcal{L}_H$ and adopting the Latin indices for the contraction $K_{\mu\nu}K^{\mu\nu} = K_{ij}K^{ij}$ give

$$\mathcal{L}_H = \left(R - K^2 + K_{ij}K^{ij}\right)N\sqrt{\gamma} - 2\sqrt{-g}\,\nabla_\alpha\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right) \tag{4.6}$$

Comparing the two expressions 4.5 and 4.6 shows that the "divergence-free" components are the same, thereby implying that the latter terms must be equivalent. In particular, the analysis of the four-dimensional divergence part will be the subject of the following section.

4.1.1 Boundary terms in the 3+1 Lagrangian density

We shall now consider the contribution of $\mathcal{S}_H$ which must be added to $\mathcal{S}_B$, the boundary term of the gravitational action (equation 2.56). Using Stokes' theorem and reintroducing the multiplicative constant $(16\pi)^{-1}$, we obtain

$$-\frac{1}{8\pi}\int_{\mathcal{V}}\sqrt{-g}\,\nabla_\alpha\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right)\mathrm{d}^4x = \\ = -\frac{1}{8\pi}\oint_{\partial\mathcal{V}}\varepsilon\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right)\sqrt{|h|}\,r_\alpha\mathrm{d}^3x$$

where r_α denotes the unit normal to $\partial\mathcal{V}$ and $\mathrm{d}\sigma_\alpha = \varepsilon r_\alpha\sqrt{|h|}\,\mathrm{d}^3x$ is the oriented volume element on $\partial\mathcal{V}$. Further progress can be made if we assume that $\partial\mathcal{V}$ is the union of two spacelike hypersurfaces Σ_{t_1} and Σ_{t_2} (with $t_2 > t_1$) connected by a timelike hypersurface $\mathcal{T}$. Since on Σ_{t_2} the unit normal corresponds to n_α and $\varepsilon = n_\alpha n^\alpha = -1$, the contribution of Σ_{t_2} to the surface integral is

$$-\frac{1}{8\pi}\int_{\Sigma_{t_2}}\varepsilon\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right)\sqrt{|h|}\,r_\alpha\mathrm{d}^3x = \frac{1}{8\pi}\int_{\Sigma_{t_2}}K\sqrt{|h|}\,\mathrm{d}^3x \tag{4.7}$$

with $h > 0$ being the determinant of the induced metric on $\partial\mathcal{V}$ and $K = -\nabla_\alpha n^\alpha$. Similarly, the contribution of Σ_{t_1} is

$$-\frac{1}{8\pi}\int_{\Sigma_{t_1}}\varepsilon\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right)\sqrt{|h|}\,r_\alpha\mathrm{d}^3x = -\frac{1}{8\pi}\int_{\Sigma_{t_1}}K\sqrt{|h|}\,\mathrm{d}^3x \tag{4.8}$$

where the minus sign accounts for the negative orientation of Σ_{t_1} with respect to the future-directed normal, namely $r_\alpha = -n_\alpha$. We see that 4.7 and 4.8 cancel out the corresponding integrals over Σ_{t_1} and Σ_{t_2} contained in $\mathcal{S}_B$ (equation 2.56). The contribution coming from $\mathcal{T}$, though, does not neutralize the remaining term of $\mathcal{S}_B$. In fact, it gives

$$-\frac{1}{8\pi}\int_{\mathcal{T}}\varepsilon\left(n^\mu\nabla_\mu n^\alpha - n^\alpha\nabla_\mu n^\mu\right)\sqrt{|h|}\,r_\alpha\mathrm{d}^3x \tag{4.9}$$

$$= -\frac{1}{8\pi}\int_{\mathcal{T}}\left(n^\mu\nabla_\mu n^\alpha\right)r_\alpha\sqrt{|h|}\,\mathrm{d}^3x = \frac{1}{8\pi}\int_{\mathcal{T}}n^\mu n^\alpha\left(\nabla_\mu r_\alpha\right)\sqrt{|h|}\,\mathrm{d}^3x$$

In the second line we have used the orthogonality relation $n^\alpha r_\alpha = 0$, due to the spacelike character of r_α. Since $\varepsilon = 1$ on $\mathcal{T}$, by merging the integral 4.9 with the remaining term of $\mathcal{S}_B$ we arrive at

$$\frac{1}{8\pi}\int_{\mathcal{T}}n^\mu n^\alpha\left(\nabla_\mu r_\alpha\right)\sqrt{|h|}\,\mathrm{d}^3x - \frac{1}{8\pi}\int_{\mathcal{T}}K\sqrt{|h|}\,\mathrm{d}^3x$$

$$= \frac{1}{8\pi}\int_{\mathcal{T}}\left(n^\mu n^\nu + g^{\mu\nu}\right)\nabla_\mu r_\nu\sqrt{|h|}\,\mathrm{d}^3x \tag{4.10}$$

To simplify the last integral of 4.9 we can introduce a foliation of $\mathcal{T}$ by the two-surfaces S_t, each corresponding to the boundary of Σ_t:

$$S_t = \partial\Sigma_t$$

By considering S_t as a two-hypersurface embedded in the three-dimensional space Σ_t, we can define the (extended) extrinsic curvature tensor of S_t as

$$\kappa_{ij} \doteq -D_i r_j + r_i D_{\boldsymbol{r}} r_j \tag{4.11}$$

where i, j refer to the coordinates of Σ_t and r^i is the normal to S_t. This is analogous to the defining relation of $K_{\mu\nu}$ (equation 2.31) except for a sign, due to $\varepsilon = 1$. Contracting κ_{ij} with the induced metric h^{ij} gives the scalar curvature:

$$\kappa \doteq \kappa_{ij} h^{ij} = -h^{ij} D_i r_j = -D_i r^i \tag{4.12}$$

In addition, we can form the four-dimensional tensor $\kappa_{\mu\nu}$ by extending κ_{ij} as we did with $K_{\mu\nu}$. We therefore proceed using the relations presented in section 2.2 and substitute the connection $\boldsymbol{D}$ with ∇. In particular, applying equation 2.25 results in

$$\begin{aligned} \kappa = -D_\mu r^\mu &= -g^{\mu\nu}(\delta^\alpha{}_\mu + n^\alpha n_\mu)(\delta^\beta{}_\nu + n^\beta n_\nu)\nabla_\alpha r_\beta \\ &= -\nabla_\alpha r^\alpha - n^\alpha n^\beta \nabla_\alpha r_\beta = -\left(g^{\alpha\beta} + n^\alpha n^\beta\right)\nabla_\alpha r_\beta \end{aligned} \tag{4.13}$$

We see that the integral 4.10 contains precisely κ with the opposite sign. Also, in analogy with the relation 3.14 between the determinants γ and g, h can be rewritten as the product of the lapse function N and the determinant σ of the induced metric on S_t:

$$\sqrt{|h|} = N\sqrt{\sigma} \tag{4.14}$$

By virtue of these results, we are now in the position to rewrite the total boundary term of $\mathcal{S}$ as a surface integral on S_t:

$$\mathcal{S}_B = -\frac{1}{8\pi}\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \kappa N\sqrt{\sigma}\, \mathrm{d}^2x \tag{4.15}$$

The nondynamical term $\mathcal{S}_0$ can be equally rewritten in terms of κ_0, the extrinsic curvature of S_t embedded in flat space. Therefore, the gravitational action in 3+1 formalism becomes

$$\begin{aligned} \mathcal{S}_G = \frac{1}{16\pi}\int_{t_1}^{t_2}\mathrm{d}t \Bigg[&\int_{\Sigma_t} \left(R - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\,\mathrm{d}^3x \\ &- 2\oint_{S_t} \left(\kappa - \kappa_0\right) N\sqrt{\sigma}\,\mathrm{d}^2x \Bigg] \end{aligned} \tag{4.16}$$

4.2 The Hamiltonian formalism

Having set most of the mathematical background of the theory, the time is now ripe to undertake the dissertation of the Hamiltonian formalism. In particular, we shall focus on the vacuum case, thus ignoring the matter contribution $\mathcal{S}_M$ and considering the action $\mathcal{S} = \mathcal{S}_G$. In order to follow the ADM notation, from now forth we shall suppress the immaterial multiplicative constant $(16\pi)^{-1}$ contained in the action 4.16.

The first fundamental observation is that $\mathcal{S}$ depends on γ_{ij}, $\dot{\gamma}_{ij}$, the lapse and shift functions N, N^i and their spatial derivatives. Since the time derivatives of N, N^i do not appear in the action integral, the lapse and shift functions, despite being four configuration variables, do not belong to the set of dynamical variables. Indeed, we will prove that N and N^i act as four Lagrange multipliers, each giving rise to a constraint equation. Let us denote with L the gravitational Lagrangian:

$$L = \int_{\Sigma_t} \left(R - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\, \mathrm{d}^3x - 2\oint_{S_t} (\kappa - \kappa_0)\, N\sqrt{\sigma}\, \mathrm{d}^2x \tag{4.17}$$

The first integral is the volume part of L, which we label as L_0. The corresponding Lagrangian density is:

$$\mathcal{L}_0 = \left(R - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma} \tag{4.18}$$

In Hamiltonian mechanics, each configuration variable q is associated with a canonically conjugate momentum p, given by the partial derivative of the Lagrangian with respect to $\dot{q}$. Similarly, the canonical momentum density π is defined as

$$\pi \doteq \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{4.19}$$

The Hamiltonian density $\mathcal{H}$ is then recovered by performing the Legendre transformation of $\mathcal{L}$, with π as the dual variables:

$$\mathcal{H} \doteq \sum_q \pi\dot{q} - \mathcal{L}$$

Due to the aforementioned absence of $\dot{N}$ and $\dot{N}^i$ in 4.17, the corresponding momenta π_N and π_{N^i} vanish:

$$\pi_N \doteq \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \qquad \pi_{N^i} \doteq \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0 \tag{4.20}$$

Therefore, we are left with the six independent momenta π^{ij} conjugate to the components of γ_{ij}:

$$\pi^{ij} \doteq \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} \tag{4.21}$$

In order to find the explicit expression of π^{ij}, we first notice that the boundary term of the Lagrangian 4.17 is independent of the time derivative $\dot{\gamma}_{ij}$. Thus we only need to evaluate the following partial derivatives

$$\frac{\partial R}{\partial \dot{\gamma}_{ij}} = 0 \qquad \frac{\partial K_{rs}}{\partial \dot{\gamma}_{ij}} = -\frac{1}{2N}\delta^i{}_r\delta^j{}_s \tag{4.22}$$

which follow from the absence of $\dot{\gamma}_{ij}$ in the three-dimensional scalar curvature R and from the explicit form of K_{ij}, given by equation 3.57. Combining these results, we obtain:

$$\begin{aligned}
\pi^{ij} &= -\frac{\sqrt{\gamma}}{2}(\gamma^{rk}\gamma^{sl} - \gamma^{rs}\gamma^{kl})(\delta^i{}_r\delta^j{}_s K_{kl} + \delta^i{}_k\delta^j{}_l K_{rs}) \\
&= -\frac{\sqrt{\gamma}}{2}(2K^{ij} - 2\gamma^{ij}K) = \sqrt{\gamma}(K\gamma^{ij} - K^{ij})
\end{aligned} \tag{4.23}$$

Notice that π^{ij} is a contravariant tensor density of weight 1, inasmuch $\sqrt{\gamma}^W$ enters the expression with $W = 1$. The covariant version π_{ij} is recovered by lowering the indices of π^{ij} with the metric γ_{ij}. In the 1962 article by Arnowitt, Deser and Misner (Ref. [4]), the momenta π^{ij} are presented in another equivalent form, which in our case stems from equation 2.17 and from the relation $\sqrt{-g} = N\sqrt{\gamma}$:

$$\pi^{ij} = \sqrt{-g}\left({}^4\Gamma^0{}_{pq} - {}^4\Gamma^0{}_{kl}\gamma^{kl}\gamma_{pq}\right)\gamma^{ip}\gamma^{jq} \tag{4.24}$$

We may also dispense with K^{ij} and K by resorting to equation 3.57 and raising the indices:

$$\pi^{ij} = \frac{\sqrt{\gamma}}{2N}\left[2\gamma^{ij}D_kN^k - D^iN^j - D^jN^i + (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})\dot{\gamma}_{kl}\right] \tag{4.25}$$

Conversely, since the Hamiltonian is a functional of the configuration variables and their conjugate momenta, we shall rewrite the extrinsic curvature tensor and $\dot{\gamma}_{ij}$ as functions of γ_{ij} and π_{ij}. To this end, let us compute the trace of π^{ij}:

$$\pi \doteq \gamma_{ij}\pi^{ij} = 2\sqrt{\gamma}K \tag{4.26}$$

We then combine 4.23 and 4.26 to obtain the desired inversion:

$$K^{ij} = \frac{1}{2\sqrt{\gamma}}(\pi\gamma^{ij} - 2\pi^{ij}) \tag{4.27}$$

$$K = \frac{\pi}{2\sqrt{\gamma}} \tag{4.28}$$

$$\dot{\gamma}_{ij} = D_iN_j + D_jN_i - \frac{N}{\sqrt{\gamma}}(\pi\gamma_{ij} - 2\pi_{ij}) \tag{4.29}$$

This allows to rewrite the volume part $\mathcal{L}_0$ of the Lagrangian density as a function of the canonical variables:

$$\mathcal{L}_0 = N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) \tag{4.30}$$

We denote by $\mathcal{H}_0$ the Hamiltonian density corresponding to $\mathcal{L}_0$, namely $\mathcal{H}_0 \doteq \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_0$. By means of equations 4.29 and 4.30 we can replace $\dot{\gamma}_{ij}$ and $\mathcal{L}_0$, thus arriving at

$$\begin{aligned} \mathcal{H}_0 &= 2\pi^{ij}D_iN_j - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) \\ &= 2D_i\left(\pi^{ij}N_j\right) - 2N_jD_i\pi^{ij} - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) \end{aligned} \tag{4.31}$$

where $D_i\pi^{ij}$ is the covariant derivative of a tensor density, whose definition is given in Appendix (section 6.1.2). The total Hamiltonian H is recovered by combining the integral of $\mathcal{H}_0$ over Σ_t with the contribution of $\kappa - \kappa_0$ computed in section 4.1.1:

$$H = \int_{\Sigma_t} \mathcal{H}_0 \, \mathrm{d}^3x + 2\oint_{S_t} (\kappa - \kappa_0)\, N\sqrt{\sigma}\, \mathrm{d}^2x \tag{4.32}$$

Let H_Σ and H_S denote respectively the volume and boundary parts of H, such that $H \doteq H_\Sigma + H_S$. Since the divergence $2D_i\left(\pi^{ij}N_j\right)$ contained in $\mathcal{H}_0$ gives rise to a surface integral, it must be added to H_S, leaving only the true volume terms in H_Σ:

$$H_\Sigma = \int_{\Sigma_t} \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma}R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2} \right) \right] \mathrm{d}^3x \tag{4.33}$$

$$H_S = 2 \oint_{S_t} \left[N(\kappa - \kappa_0) + N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j \right] \sqrt{\sigma}\,\mathrm{d}^2x \tag{4.34}$$

We shall rewrite H_Σ following the ADM notation, with the aim to emphasize the role of N and N^i. If we define the quantities

$$R^0 = -\sqrt{\gamma}R - \frac{1}{\sqrt{\gamma}} \left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij} \right) \tag{4.35}$$

$$R^i = -2D_j \pi^{ij} \tag{4.36}$$

we immediately see that the volume term takes on the simple form:

$$H_\Sigma = \int_{\Sigma_t} \left[NR^0 + N_i R^i \right] \mathrm{d}^3x \tag{4.37}$$

or equivalently

$$H_\Sigma = \int_{\Sigma_t} N_\mu R^\mu \mathrm{d}^3x \tag{4.38}$$

where we adopted the notation $N = N_0$. The peculiar expression of H_Σ suggests that the lapse and shift functions behave as Lagrangian multipliers. In the following sections we will prove that this is the case, thus showing that H_Σ vanishes identically. Indeed, demanding $\mathcal{S}_G$ to be stationary originates four constraint equations, which force R^0 and R^i to be zero.

4.3 Parametric form of the canonical equations

In order to proceed with our analysis, we shall introduce the notion of parametric form of the canonical equations (following Ref. [15]). Let us consider for simplicity the action of a system with a finite number M of degrees of freedom:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t\, L(q, \dot{q}, t) = \int_{t_1}^{t_2} \mathrm{d}t \left(\sum_{k=1}^{M} p_k \dot{q}_k - H(p, q, t) \right) \tag{4.39}$$

The time t is singled out from the configuration variables of the system since it is the only coordinate lacking the definition of a conjugate momentum. However, this asymmetry can be circumvented by the introduction of a new arbitrary parameter τ, which allows the promotion of t to the set of dynamical variables together with its conjugate momentum p_t. Indeed, if we make the notational change $t = q_{M+1}$ and let the configuration variables $\{q_k\}_{k=1}^{M+1}$ become functions of τ, by direct substitution into 4.39 we obtain the so called action in parameterized form:

$$\tilde{\mathcal{S}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \tilde{L}\left(q_1, \ldots, q_{M+1}; q'_1, \ldots, q'_{M+1} \right) \tag{4.40}$$

where the derivative with respect to τ is denoted by a prime. The modified Lagrangian $\tilde{L}$ is related to L through the equation

$$\tilde{L}\left(q_1,\ldots,q_{M+1};\, q'_1,\ldots,q'_{M+1}\right) = L\left(q_1,\ldots,q_{M+1};\, \frac{q'_1}{q'_{M+1}},\ldots,\frac{q'_M}{q'_{M+1}}\right) q'_{M+1}$$

Consequently, the momentum $p_t = p_{M+1}$ associated to the time $t = q_{M+1}$ can be defined with the standard procedure, and it turns out to be just negative the Hamiltonian H:

$$\begin{aligned} p_{M+1} \doteq \frac{\partial \tilde{L}}{\partial q'_{M+1}} &= L - \left(\sum_{k=1}^{M} \frac{\partial L}{\partial \dot{q}_k} \frac{q'_k}{(q'_{M+1})^2}\right) q'_{M+1} \\ &= L - \sum_{k=1}^{M} p_k \dot{q}_k = -H \end{aligned} \tag{4.41}$$

Therefore q_{M+1} and p_{M+1} belong to the new $2M+2$-dimensional phase space. Let us now focus on a remarkable property of the Lagrangian $\tilde{L}$, namely that of being a homogeneous function of the first order in the variables $q'_1,\ldots,q'_{M+1}$. If we compute the partial derivatives of $\tilde{L}$ with respect to q'_k, namely

$$\frac{\partial \tilde{L}}{\partial q'_k} = \frac{\partial L}{\partial \dot{q}_k} \qquad\qquad \frac{\partial \tilde{L}}{\partial q'_{\mathrm{M+1}}} = \tilde{L} - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}$$

then we see that the following relation holds:

$$\sum_{k=1}^{M+1} \frac{\partial \tilde{L}}{\partial q'_k} q'_k = \tilde{L} \tag{4.42}$$

This in turn allows to prove our claim by applying Euler's theorem on homogeneous functions. Once the partial derivatives in equation 4.42 are replaced with the $M+1$ momenta p_k, we are in the position to show that the action in parameterized form becomes

$$\tilde{\mathcal{S}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sum_{k=1}^{M+1} p_k q'_k\right) \tag{4.43}$$

while the Hamiltonian $\tilde{H}$ of the extended system vanishes identically:

$$\tilde{H} \doteq \sum_{k=1}^{M+1} p_k q'_k - \tilde{L} = 0 \tag{4.44}$$

This striking feature motivates our interlude on the parameterized form, since the volume term H_Σ (equation 4.38) falls into this category. In particular, we can recast the parameterized action 4.43 in an enlightening form by resorting to the Lagrangian multiplier method. Since the equation $p_{M+1} = -H$ (4.41) acts as a constraint on p_{M+1}, there must exist a relation between the $M+1$ conjugate momenta which impairs the independence of the canonical variables. This constraint can be explicitly stated in the action integral by means of an auxiliary function

$$C(q_1,\ldots,q_{M+1};\, p_1,\ldots,p_{M+1}) = p_{M+1} + H \tag{4.45}$$

and a Lagrangian multiplier $\lambda = \lambda(\tau)$, which remains unspecified due to the arbitrariness of τ. Hence the parameterized action becomes

$$\tilde{S} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sum_{k=1}^{M+1} p_k q_k' - \lambda C \right) \tag{4.46}$$

Independent variations of λ and q_k give respectively the constraint equation $C = 0$ (equivalent to the identity 4.41) and the $M+1$ canonical equations of motion, showing that the action 4.46 retains the full informative content of the original system. Moreover, since $\tilde{L}$ and C do not depend directly on τ, the extended system is conservative, regardless of the nature of L.
The above process can be generalized to the case of a field theory with M degrees of freedom by introducing four new external parameters τ^μ and just as many configuration variables $q^{M+1+\mu} = x^\mu(\tau^\alpha)$, together with their respective momenta $p_{M+1+\mu}$. The additional four constraint equations $C^\mu = 0$ and Lagrangian multipliers $\lambda^\mu(\tau^\alpha)$ are required to relate $p_{M+1}, \ldots, p_{M+4}$ with the Hamiltonian and momentum densities of the field. The relevance of the parameter formalism lies in the possibility to reverse this process via the "reduction" of the parameterized action $\tilde{S}$ to the canonical form. This consists in the specification of coordinate conditions (which fix the arbitrary parameters τ^μ) followed by the insertion of the constraint equations into $\tilde{S}$. The reduced action will then reveal the intrinsic degrees of freedom of the system. Indeed, the volume term L_Σ of the Lagrangian (which stems from the Hamiltonian H_Σ, equation 4.38) appears in the guise of a parameterized form Lagrangian, with N_μ and R^μ respectively in the role of Lagrange multipliers and constraints:

$$L_\Sigma \doteq \pi^{ij}\dot{\gamma}_{ij} - H_\Sigma = \pi^{ij}\dot{\gamma}_{ij} - N_\mu R^\mu \tag{4.47}$$

By implication, our aim is to prove that N_μ truly behave as Lagrange multipliers when the variation of the action is considered.

4.3.1 Variation of the lapse function

Let us return to the gravitational action 4.16 (ignoring the immaterial constant factor $(16\pi)^{-1}$), here reproduced for convenience:

$$\mathcal{S}_G = \int_{t_1}^{t_2} \mathrm{d}t \left[\int_{\Sigma_t} \left(R - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\, \mathrm{d}^3x - 2 \oint_{S_t} (\kappa - \kappa_0)\, N\sqrt{\sigma}\, \mathrm{d}^2x \right]$$

Recalling the definitions given in section 1.1, we demand the variation δN to vanish on the boundary. This implies that we can safely ignore the surface integral over S_t, due to the absence of derivatives of N. By resorting to relation 3.57, the variation of the volume term is straightforward:

$$\begin{aligned} \frac{\delta\mathcal{S}}{\delta N} &= \sqrt{\gamma}\left(R - K^2 + K^{ij}K_{ij}\right) + N\sqrt{\gamma}\left(-\frac{2}{N}\right)\left(-K^2 + K^{ij}K_{ij}\right) \\ &= \sqrt{\gamma}\left(R + K^2 - K^{ij}K_{ij}\right) \end{aligned} \tag{4.48}$$

The action is then extremized by setting equation 4.48 to zero. This gives the Hamiltonian constraint 3.50 in the vacuum case, namely when $E = 0$. The replacement of K_{ij} with the conjugate momenta (using 4.26 and 4.27) leads to the relation

$$R^0 = 0 \tag{4.49}$$

which shows that N is truly a Lagrangian multiplier, as R^0 is not affected by the variation of N.

4.3.2 Variation of the shift functions

The subsequent variation with respect to the shift functions involves some subtleties, due to the presence of the covariant derivatives $D_i N_j$ in the extrinsic curvature tensor. Therefore, we begin this proof by explicitly considering a smooth one-parameter family $(N_i)_\lambda$ of shift functions. By evaluating the derivative with respect to λ of the volume term of $\mathcal{S}_G$ we obtain:

$$\begin{aligned}\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_{\lambda=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[R - K^2 + K^{ij}K_{ij}\right] N\sqrt{\gamma}\, \mathrm{d}^3x \\ &= 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} N\sqrt{\gamma}\left(-K\gamma^{ij} + K^{ij}\right) \left.\frac{\mathrm{d}K_{ij}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{d}^3x \\ &= -2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \pi^{ij} D_i \left(\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) \mathrm{d}^3x\end{aligned}$$

In the passage from the second to the third line we recognized the expression 4.23 of the conjugate momenta π^{ij} and substituted it with the identity 3.57, exploiting the symmetry of K_{ij}. Recalling the definition of covariant derivative of a tensor density, we rewrite the integrand as

$$\pi^{ij} D_i \left(\left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) = D_i \left(\pi^{ij} \left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}\right) - \left(D_i \pi^{ij}\right) \left.\frac{\mathrm{d}N_j}{\mathrm{d}\lambda}\right|_{\lambda=0}$$

Upon substitution in the integral (resorting to the definition of δN_i), the divergence can be reduced to a boundary term:

$$\begin{aligned}\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_{\lambda=0} = 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(D_i \pi^{ij}\right) \delta N_j \, \mathrm{d}^3x \\ - 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \frac{\pi^{ij}}{\sqrt{\gamma}}\, r_i \, \delta N_j \sqrt{\sigma}\, \mathrm{d}^2x\end{aligned}$$

The surface integral vanishes as $\delta N_i = 0$ on the boundary. Hence we can discard this term and demand $\mathcal{S}_G$ to be stationary, using the notion of functional derivative 1.7

$$\frac{\delta \mathcal{S}}{\delta N_j} = 2D_i \pi^{ij} = 0$$

thereby leading to the three constraint equations:

$$R^i = 0 \tag{4.50}$$

These correspond to the momentum constraints 3.53 in the vacuum, with $p_i = 0$. Together with 4.49, they constitute the aforementioned four constraint equations of the system. This result concludes the proof and implies that the volume term H_Σ (4.38) vanishes identically when the constraints are imposed:

$$H_\Sigma = 0 \tag{4.51}$$

It is worth to emphasize that these conditions do by no means imply that H_S must vanish. In fact, in section 5.1 we will discuss the relation between the value of H_S in an asymptotically flat spacetime and a notion of energy of the system, which is in general different from zero.

4.4 Hamilton's equations

We are now in the position to determine the twelve Hamilton equations which describe the time evolution of the canonical variables γ_{ij} and π^{ij}:

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \tag{4.52}$$

$$\dot{\pi}^{ij} = -\frac{\delta H}{\delta \gamma_{ij}} \tag{4.53}$$

To this end, we rewrite the total gravitational action $\mathcal{S}_G$ (equation 4.16) in terms of the canonical variables, preserving only the term $\pi^{ij}\dot{\gamma}_{ij}$:

$$\begin{aligned}\mathcal{S}_G &= \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(\pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}\right) \mathrm{d}^3x \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[\pi^{ij}\dot{\gamma}_{ij} + 2N_j D_i \pi^{ij} + N\sqrt{\gamma}\,R - \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \mathrm{d}^3x \\ &\quad - 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \left[N\left(\kappa - \kappa_0\right) + N_i\,\frac{\pi^{ij}}{\sqrt{\gamma}}\,r_j\right] \sqrt{\sigma}\,\mathrm{d}^2x \end{aligned} \tag{4.54}$$

We require that the variation of the configuration variables vanishes on the boundary $S_t = \partial\Sigma_t$, namely:

$$\delta N|_{S_t} = \delta N_i|_{S_t} = \delta\gamma_{ij}|_{S_t} = 0 \tag{4.55}$$

However, we shall by no means impose restrictions on the conjugate momenta, which are treated as independent variables. In agreement with this, the variation of H with respect to N and N_i is equivalent (up to an immaterial overall sign) to the variation of the Lagrangian performed in sections 4.3.1 and 4.3.2, which led to the four constraint equations 4.49 and 4.50. Instead, the variations of γ_{ij} and π^{ij} require a more laborious analysis, that we carry on separately in the following paragraphs.

4.4.1 Variation of the conjugate momenta

We start from the second set of equations 4.53, which are recovered by setting to zero the variation with respect to π^{ij} of $\mathcal{S}_G$. In particular, we first consider the second term from the volume integral of 4.54:

$$\begin{aligned}\mathcal{P} &\doteq \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left(2N_j D_i \pi^{ij}\right) \mathrm{d}^3x \\ &= 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \left[D_i\left(N_j\pi^{ij}\right) - \pi^{ij} D_i N_j\right] \mathrm{d}^3x \end{aligned} \tag{4.56}$$

We now transform the total covariant derivative in a divergence and then apply Stokes' theorem:

$$\mathcal{P} = 2\int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} N_i \,\frac{\pi^{ij}}{\sqrt{\gamma}}\, r_j \sqrt{\sigma}\, \mathrm{d}^2x - 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \pi^{ij} D_i N_j \,\mathrm{d}^3x$$

The first integral of $\mathcal{P}$ cancels out the last part of the boundary term of $\mathcal{S}_G$, thereby leaving only a surface integral which is independent of π^{ij}. Accordingly, the variation of $\mathcal{S}_G$ reduces to:

$$\delta_\pi \mathcal{S}_G = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \delta\pi^{ij} \left[\dot{\gamma}_{ij} - 2D_i N_j - \frac{N}{\sqrt{\gamma}}\,(2\pi_{ij} - \pi\gamma_{ij})\right] \mathrm{d}^3x$$

The stationarity of $\mathcal{S}_G$ and the arbitrariness of $\delta\pi^{ij}$ force the argument of the first integral to vanish. In order to perform the functional derivative and suppress $\delta\pi^{ij}$, we shall replace $2D_iN_j$ with its symmetrization $D_iN_j + D_jN_i$. Hence we obtain the relation

$$\frac{\delta \mathcal{S}_G}{\delta \pi^{ij}} = \dot{\gamma}_{ij} - D_i N_j - D_j N_i - \frac{N}{\sqrt{\gamma}}\,(2\pi_{ij} - \pi\gamma_{ij}) = 0 \tag{4.57}$$

or equivalently

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} = D_i N_j + D_j N_i - \frac{N}{\sqrt{\gamma}}\,(2\pi_{ij} - \pi\gamma_{ij}) \tag{4.58}$$

By replacing π^{ij} and its trace with the extrinsic curvature K_{ij}, equation 4.58 becomes

$$\dot{\gamma}_{ij} = D_i N_j + D_j N_i - 2NK_{ij}$$

We see that the variation of π^{ij} produces the relation 3.57, which fixes the time evolution of the three-dimensional metric by means of the lapse and shift functions.

4.4.2 Variation of the metric

This variation is more involved than the previous one, thus we shall proceed gradually. Firstly, we consider the variation of the volume term of the Hamiltonian density $\mathcal{H}_\Sigma$:

$$\delta_\gamma \mathcal{H}_\Sigma = \delta_\gamma \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma} R + \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \tag{4.59}$$

We recall that the variation of γ follows from Jacobi's formula

$$\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} \tag{4.60}$$

while the computation of the last term in parentheses is straightforward:

$$\delta_\gamma\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) = \left(2\pi^a{}_i\pi^{ib} - \pi\pi^{ab}\right)\delta\gamma_{ab} \tag{4.61}$$

For the variation of $N\sqrt{\gamma}R$ we need the relations presented in sections 6.2.2 and 6.2.3 of the Appendix, which we already encountered in chapter 1. If we denote the three-dimensional contravariant Einstein tensor by G^{ab}, we obtain

$$\delta\big[-N\sqrt{\gamma}R\big] = N\sqrt{\gamma}\left(R^{ab} - \frac{1}{2}\gamma^{ab}R\right)\delta\gamma_{ab} - N\sqrt{\gamma}D_a\delta V^a$$

$$= N\sqrt{\gamma}\, G^{ab}\delta\gamma_{ab} + \sqrt{\gamma}\,\delta V^a D_a N - \sqrt{\gamma}\, D_a\left(N\delta V^a\right) \tag{4.62}$$

In order to lighten the computations, we introduce the two quantities

$$\delta\mathcal{B}_1 \doteq -2\delta_\gamma D_i\left(\pi^{ij}N_j\right)$$
$$\delta\mathcal{B}_2 \doteq -D_a\left(2\pi^{ab}N^c\delta\gamma_{bc} - \pi^{bc}N^a\delta\gamma_{bc}\right)$$

Thanks to this, we now express the variation of the first term of 4.59 as

$$\delta_\gamma\left(-2N_jD_i\pi^{ij}\right) = 2\delta_\gamma\left(\pi^{ij}D_iN_j\right) + \delta\mathcal{B}_1 \tag{4.63}$$

Using the relation 6.21 from the Appendix, we can rewrite $2\delta_\gamma\left(\pi^{ij}D_iN_j\right)$ in terms of $\delta\gamma_{ij}$ and $\delta\mathcal{B}_2$:

$$\begin{aligned}\delta_\gamma\left(2\pi^{ij}D_iN_j\right) &= -2\pi^{ij}N_a\delta\Gamma^a{}_{ij} = -2\pi^{ij}N_a\left(\gamma^{ab}D_i\delta\gamma_{jb} - \frac{1}{2}\gamma^{ab}D_b\delta\gamma_{ij}\right)\\ &= D_a\left(2\pi^{ab}N^c - \pi^{bc}N^a\right)\delta\gamma_{bc} + \delta\mathcal{B}_2\end{aligned} \tag{4.64}$$

By virtue of the three constraints 4.50, namely $D_i\pi^{ij} = 0$, the previous equation turns into

$$\delta_\gamma\left(2\pi^{ij}D_iN_j\right) = \left(2\pi^{ab}D_aN^c - \pi^{bc}D_aN^a\right)\delta\gamma_{bc} + \delta\mathcal{B}_2 \tag{4.65}$$

Combining equations 4.60 to 4.65, the variation $\delta_\gamma\,\mathcal{H}_\Sigma$ becomes

$$\begin{aligned}\delta_\gamma\mathcal{H}_\Sigma = &\left(2\pi^{ab}D_aN^c - \pi^{bc}D_aN^a\right)\delta\gamma_{bc} + N\sqrt{\gamma}\,G^{ab}\delta\gamma_{ab}\\ &+ \frac{N}{\sqrt{\gamma}}\left[-\frac{1}{2}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ab} + 2\pi^a{}_c\pi^{bc} - \pi\pi^{ab}\right]\delta\gamma_{ab}\\ &+ \delta\mathcal{B}_1 + \delta\mathcal{B}_2 + \sqrt{\gamma}\,\delta V^aD_aN\end{aligned} \tag{4.66}$$

The variation of H_Σ is recovered by integrating $\delta_\gamma\mathcal{H}_\Sigma$ over the hypersurface Σ_t. In particular, the three terms from the last line of equation 4.66 give rise to a surface integral, which we denote by δB. However, since the integral of $\delta\mathcal{B}_1 + \delta\mathcal{B}_2$ does not include any derivative of $\delta\gamma_{ab}$ and $\delta\gamma_{ab}|_{S_t} = 0$, the only non-vanishing boundary contribution is given by

$$\delta B \doteq \int_{\Sigma_t}\left(\delta\mathcal{B}_1 + \delta\mathcal{B}_2 + \sqrt{\gamma}\,\delta V^aD_aN\right)\mathrm{d}^3x = -\oint_{S_t} N\delta V^a r_a\sqrt{\sigma}\,\mathrm{d}^2x \tag{4.67}$$

By virtue of the argument used in section 2.4.1 and of the relation 6.25 applied to the three-dimensional case, we can show succinctly that the contraction $\delta V^a r_a$ in 4.67 reduces to

$$\delta V^a r_a = -\sigma^{bc}\delta\gamma_{bc,a}r^a = -2\sigma^{bc}D_br_c = 2\kappa \tag{4.68}$$

where $\sigma^{ab} = r^ar^b + \gamma^{ab}$ is the induced metric on S_t extended to Σ_t. Comparing the variation of H_S (equation 4.34) with 4.67, we see that $\delta B + \delta_\gamma H_S$ vanishes:

$$\delta_\gamma H_S = 2\oint_{S_t} N\delta\kappa\,\sqrt{\sigma}\,\mathrm{d}^2x = -\,\delta B \tag{4.69}$$

This implies that the variation of H comes down to the remaining terms of H_Σ and we can safely ignore H_S. We now recast the product δV^aD_aN contained in the first line of equation 4.66 in a convenient form (using the relation 6.26 from the Appendix):

$$\begin{aligned}\delta V^a D_a N &= \gamma^{ab}\gamma^{cd}\left(D_a\delta\gamma_{bc} - D_c\delta\gamma_{ab}\right)D_d N\\ &= D_a\left[\left(\gamma^{ab}D^c N - \gamma^{bc}D^a N\right)\delta\gamma_{bc}\right] - \left(D^a D^b N - \gamma^{ab}D_c D^c N\right)\delta\gamma_{ab}\end{aligned}$$

Another integration by parts allows to displace the divergence due to the restriction $\delta\gamma_{ab}|_{S_t} = 0$. Thus we get

$$\delta V^a D_a N = -\left(D^a D^b N - \gamma^{ab}D_c D^c N\right)\delta\gamma_{ab} \tag{4.70}$$

Finally, after the symmetrization of the indices a and b in $2\pi^{bc}N^a$, the combination of equations 4.66 to 4.70 lead to

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta\gamma_{ab}} &= D_c\left(\pi^{ac}N^b + \pi^{bc}N^a - \pi^{ab}N^c\right) + N\sqrt{\gamma}\left(R^{ab} - \frac{1}{2}\gamma^{ab}R\right)\\ &\quad - \sqrt{\gamma}\left(D^a D^b N - \gamma^{ab}D_c D^c N\right) - \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right)\gamma^{ab}\\ &\quad + \frac{2N}{\sqrt{\gamma}}\left(\pi^a{}_c\pi^{bc} - \frac{1}{2}\pi\pi^{ab}\right)\end{aligned} \tag{4.71}$$

By definition, the equations of motion are recovered by demanding the action to be stationary. Integrating by parts the product $\pi^{ij}\dot{\gamma}_{ij}$ with respect to the time coordinate results in

$$\begin{aligned}\delta\mathcal{S}_G &= \delta_\gamma\int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t}\left(\pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}\right)\mathrm{d}^3x\\ &= -\int_{t_1}^{t_2}\mathrm{d}t\left[\int_{\Sigma_t}\delta\gamma_{ij}\left(\dot{\pi}^{ij} + \frac{\delta\mathcal{H}}{\delta\gamma_{ij}}\right)\mathrm{d}^3x\right] = 0\end{aligned} \tag{4.72}$$

Thanks to the arbitrariness of $\delta\gamma_{ij}$, this variation provides the second set of Hamilton equations:

$$\dot{\pi}^{ij} = -\frac{\delta\mathcal{H}}{\delta\gamma_{ij}} \tag{4.73}$$

It is worth to summarize the main results of the previous chapters, namely the four constraints 4.49, 4.50 and Hamilton's equations 4.58 for $\dot{\gamma}_{ij}$, together with the explicit form of 4.73:

$$R^0 = -\sqrt{\gamma}R - \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) = 0 \tag{4.74}$$

$$R^i = -2D_j\pi^{ij} = 0 \tag{4.75}$$

$$\dot{\gamma}_{ij} = D_i N_j + D_j N_i - \frac{N}{\sqrt{\gamma}}\left(2\pi_{ij} - \pi\gamma_{ij}\right) \tag{4.76}$$

$$\begin{aligned}\dot{\pi}^{ij} &= -N\sqrt{\gamma}\left(R^{ij} - \frac{1}{2}\gamma^{ij}R\right) + \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ij}\\ &\quad - \frac{2N}{\sqrt{\gamma}}\left(\pi^{ic}\pi_c{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{\gamma}\left(D^i D^j N - \gamma^{ij}D_c D^c N\right)\\ &\quad + D_c\left(\pi^{ij}N^c\right) - \pi^{ic}D_c N^j - \pi^{jc}D_c N^i\end{aligned} \tag{4.77}$$

4.5 Poisson brackets

In classical Hamiltonian mechanics, given two differentiable functions $f(q_k, p_k, t)$ and $g(q_k, p_k, t)$ of the canonical variables q_k, p_k, with $k \in \{1, \ldots, M\}$, the Poisson bracket of f and g is defined as the function

$$\{f, g\} = \sum_{k=1}^{M} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) \tag{4.78}$$

Therefore the Poisson bracket can be considered as a bilinear, anticommutative binary operation acting on the space of functions which depend on the phase space and time. Also, for any three functions f, g, h of this kind, it satisfies the equations

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \tag{4.79}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{4.80}$$

called respectively the Leibniz's rule and the Jacobi identity. By virtue of this definition, Hamilton's equations of motion can be rewritten as

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = \{q_k, H\} \tag{4.81}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \{p_k, H\} \tag{4.82}$$

and, in general, the time evolution of any function $f(q_k, p_k, t)$ is determined by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{4.83}$$

The Poisson bracket for a field theory can be defined by analogy with equation 4.83. In fact, by considering the total time derivative of a differentiable function $f = f(\gamma_{ij}, \pi^{ij}, t)$ we obtain

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\delta f}{\delta \gamma_{ij}} \dot{\gamma}_{ij} + \frac{\delta f}{\delta \pi^{ij}} \dot{\pi}^{ij} + \frac{\partial f}{\partial t} \\ &= \frac{\delta f}{\delta \gamma_{ij}} \frac{\delta \mathcal{H}}{\delta \pi^{ij}} - \frac{\delta f}{\delta \pi^{ij}} \frac{\delta \mathcal{H}}{\delta \gamma_{ij}} + \frac{\partial f}{\partial t} \end{aligned} \tag{4.84}$$

which can be recast in the familiar form 4.83 if we introduce the Poisson bracket

$$\{f, g\} \doteq \frac{\delta f}{\delta \gamma_{ij}} \frac{\delta g}{\delta \pi^{ij}} - \frac{\delta f}{\delta \pi^{ij}} \frac{\delta g}{\delta \gamma_{ij}} \tag{4.85}$$

By virtue of this definition we can compute the fundamental Poisson brackets among the canonical variables of the system:

$$\begin{aligned} \{\gamma_{ij}, \gamma_{kl}\} &= 0 \\ \{\pi^{ij}, \pi^{kl}\} &= 0 \\ \{\gamma_{ij}, \pi^{kl}\} &= \delta_i{}^k \delta_j{}^l \end{aligned} \tag{4.86}$$

These are closely related to the ones which arise in classical mechanics. However, since the canonical coordinates γ_{ij} and π^{ij} are subject to the constraints

$R^\mu = 0$, the relations 4.86 do not represent a minimal set of Poisson brackets between unconstrained variables. As we mentioned at the beginning of the chapter, a full treatment of this issue is given in Ref. [4]. In section 4 of this article, in fact, the generating functions arising from the parameterized-form Lagrangian 4.47 and the linearized theory are considered. This analysis leads to the choice of four coordinate conditions, which together with the four constraint equations allow to eliminate the extra variables of the system. It should be emphasized that the canonical form is not unique, as in addition to the usual canonical transformations there exists a general class of coordinate conditions which produce different canonical variables. This arbitrariness can be exploited to recast the dynamical equations in a suitable form, according to the particular aspect of the theory to be investigated.

4.6 Reintroduction of the cosmological constant

In this section we consider the effects on the main results of our discussion due to the reintroduction of the cosmological constant Λ. We recall that if we restore momentarily the constants G, c, the Einstein field equations with $\Lambda \neq 0$ are given by

$$ {}^4R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} {}^4R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} $$

In order to account for the term $\Lambda g_{\mu\nu}$, we shall introduce a new contribution $\mathcal{S}_\Lambda$ to the gravitational action, depending only on the metric density $\sqrt{-g}$:

$$ \mathcal{S}_\Lambda \doteq \int_\mathcal{V} (-2\Lambda) \sqrt{-g}\, \mathrm{d}^4x \tag{4.87} $$

It is a trivial task to show that the variation of 4.87 with respect to the inverse metric $g^{\mu\nu}$ gives

$$ \frac{\delta \mathcal{S}_\Lambda}{\delta g^{\mu\nu}} = -2\Lambda \left(-\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \right) = \Lambda g_{\mu\nu} \sqrt{-g} \tag{4.88} $$

Eventually, adding this term to the overall variation of $\mathcal{S}$ (equation 1.22) and gathering $\sqrt{-g}$ leads to the correct field equations. Therefore, the gravitational part of the action in four-dimensional formalism takes the form

$$ \begin{aligned} \mathcal{S}_G &= \mathcal{S}_H + \mathcal{S}_\Lambda + \mathcal{S}_B - \mathcal{S}_0 \\ &= \frac{c^4}{16\pi G} \left[\int_\mathcal{V} \left({}^4R - 2\Lambda\right) \sqrt{-g}\, \mathrm{d}^4x - 2 \oint_{\partial\mathcal{V}} \varepsilon \left(K - K_0\right) \sqrt{|h|}\, \mathrm{d}^3x \right] \end{aligned} \tag{4.89} $$

while the total action is recovered by adding $\mathcal{S}_G$ to the matter term (equation 1.11), namely

$$ \mathcal{S} = \mathcal{S}_G + \int_\mathcal{V} \mathcal{L}_M \sqrt{-g}\, \mathrm{d}^4x \tag{4.90} $$

Hereafter we will discard the matter contribution and return to the units $G = c = 1$, adopted throughout the preceding discussion. In addition, we drop the multiplicative constant $(16\pi)^{-1}$, in agreement with the ADM notation. Since the 3+1 decomposition of the spacetime manifold can be implemented in $\mathcal{S}_\Lambda$ via the straightforward substitution $\sqrt{-g} = N\sqrt{\gamma}$ (equation 3.14), the gravitational Lagrangian 4.17 becomes

$$L = \int_{\Sigma_t} \left(R - 2\Lambda - K^2 + K^{ij}K_{ij}\right) N\sqrt{\gamma}\,\mathrm{d}^3x - 2\oint_{S_t} (\kappa - \kappa_0)\, N\sqrt{\sigma}\,\mathrm{d}^2x \quad (4.91)$$

We now move on to the Hamiltonian formalism. The introduction of $\mathcal{S}_\Lambda$ leaves the conjugate momenta unchanged, as it is independent of the time derivatives $\dot{\gamma}_{ij}$. Hence the volume and boundary terms of the Hamiltonian can be computed without effort, leading to

$$H_\Sigma = \int_{\Sigma_t} \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma}\,(R - 2\Lambda) + \frac{N}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right] \mathrm{d}^3x \quad (4.92)$$

$$H_S = 2\oint_{S_t} \left[N\,(\kappa - \kappa_0) + N_i\,\frac{\pi^{ij}}{\sqrt{\gamma}}\,r_j\right] \sqrt{\sigma}\,\mathrm{d}^2x \quad (4.93)$$

We notice that Λ always reveals itself as the "notational change" $R \longrightarrow (R - 2\Lambda)$ involving only the scalar curvature. Indeed, due to the simplicity of $\mathcal{S}_\Lambda$, this holds true also for the derivation of the constraint equations

$$R^0 = -\sqrt{\gamma}\,(R - 2\Lambda) - \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) = 0 \quad (4.94)$$

$$R^i = -2D_j\pi^{ij} = 0 \quad (4.95)$$

as well as for Hamilton's equations, among which the first set (referring to $\dot{\gamma}_{ij}$) is unaltered:

$$\dot{\gamma}_{ij} = D_iN_j + D_jN_i - \frac{N}{\sqrt{\gamma}}\,(2\pi_{ij} - \pi\gamma_{ij}) \quad (4.96)$$

$$\begin{aligned}\dot{\pi}^{ij} = &-N\sqrt{\gamma}\left[R^{ij} - \frac{1}{2}\gamma^{ij}\,(R - 2\Lambda)\right] + \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ij} \\ &- \frac{2N}{\sqrt{\gamma}}\left(\pi^{ic}\pi_c{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{\gamma}\left(D^iD^jN - \gamma^{ij}D_cD^cN\right) \\ &+ D_c\left(\pi^{ij}N^c\right) - \pi^{ic}D_cN^j - \pi^{jc}D_cN^i \end{aligned} \quad (4.97)$$

In conclusion, the reintroduction of Λ has no effect on the definition of π^{ij} and thus only slightly alters the main results of the Hamiltonian formulation. In particular, since $\mathcal{S}_\Lambda$ does not give rise to divergences, the boundary terms of both the Lagrangian and the Hamiltonian are unaffected by this change.

Chapter 5

Conclusion

The ADM Hamiltonian formalism here discussed provides an essential contribution to the path towards the quantization of general relativity. Indeed, when the unconstrained canonical variables γ_{ij}, π^{ij} of the system are identified, we can promote them to the corresponding quantum operators $\hat{\gamma}_{ij}$, $\hat{\pi}^{ij}$. Accordingly, once the fundamental Poisson brackets are defined, the canonical commutation relations are readily recovered:

$$\begin{aligned} \{\gamma_{ij}, \gamma_{kl}\} &= 0 & \longrightarrow \quad \left[\hat{\gamma}_{ij}, \hat{\gamma}_{kl}\right] &= 0 \\ \{\pi^{ij}, \pi^{kl}\} &= 0 & \longrightarrow \quad \left[\hat{\pi}^{ij}, \hat{\pi}^{kl}\right] &= 0 \\ \{\gamma_{ij}, \pi^{kl}\} &= \delta_i^{\ k}\delta_j^{\ l} & \longrightarrow \quad \left[\hat{\gamma}_{ij}, \hat{\pi}^{kl}\right] &= i\hbar\, \delta_i^{\ k}\delta_j^{\ l} \end{aligned} \tag{5.1}$$

Therefore, the Poisson bracket of any two functions A and B turns into the commutator between $\hat{A}$ and $\hat{B}$

$$\{A, B\} \quad \longrightarrow \quad \frac{1}{i\hbar}[\hat{A}, \hat{B}] \tag{5.2}$$

The next step requires the introduction of a wave functional $\Psi[\gamma_{ab}]$ defined on the space of field configurations γ_{ab}. If we adopt the Dirac notation, we can characterize the action of $\hat{\gamma}_{ij}$ and $\hat{\pi}^{ij}$ on the state $|\Psi\rangle$:

$$\hat{\gamma}_{ij}\,|\Psi\rangle \doteq \gamma_{ij}\,\Psi[\gamma_{ab}] \tag{5.3}$$

$$\hat{\pi}^{ij}\,|\Psi\rangle \doteq -i\hbar\,\frac{\delta}{\delta\gamma_{ij}}\,\Psi[\gamma_{ab}] \tag{5.4}$$

Consequently, the constraints 4.94 and 4.95 (with $\Lambda \neq 0$) shall be rewritten as a set of operator equations:

$$\hat{R}^0\,|\Psi\rangle \doteq -\left[\sqrt{\gamma}\,(R - 2\Lambda) + \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right)\right]\Psi[\gamma_{kl}] = 0 \tag{5.5}$$

$$\hat{R}^i\,|\Psi\rangle \doteq -2D_j\pi^{ij}\,\Psi[\gamma_{kl}] = 0 \tag{5.6}$$

Using the representation 5.4 of the momentum operators $\hat{\pi}^{ij}$, we arrive at

$$\left[\sqrt{\gamma}\,(R - 2\Lambda) - \frac{\hbar^2}{\sqrt{\gamma}}\left(\frac{1}{2}\gamma_{ab}\gamma_{cd} - \gamma_{ac}\gamma_{bd}\right)\frac{\delta}{\delta\gamma_{ab}}\frac{\delta}{\delta\gamma_{cd}}\right]\Psi[\gamma_{kl}] = 0 \tag{5.7}$$

$$2D_j \frac{\delta}{\delta \gamma_{ij}} \Psi[\gamma_{kl}] = 0 \tag{5.8}$$

The operator equation 5.7 is called the Wheeler-DeWitt equation (WDW). This remarkable result was first obtained by Bruce DeWitt and John Wheeler in 1967 (Ref. [13]) shortly after Asher Peres published its Hamilton-Jacobi equation of general relativity. Albeit being tainted by a not negligible issue of ill-definedness, the WDW equation constitutes a major step in general relativity and provided new blood in the quest for a theory of quantum gravity.
These results, however, do not exhaust the relevance of the 3+1 formalism. Indeed, the ADM approach is just one among the many related 3+1 formulations which initiated the field of numerical relativity, devoted to the search for approximate solutions to the Einstein equations. In particular, we mention the BSSN scheme (or "conformal ADM"), originated from the ADM one by the addition of extra variables, whose main quality lies in an enhanced stability of the simulations over time.
We conclude our discussion with a brief analysis of the notions of total energy and momentum, restricted to a specific class of spacetimes. These naturally stem from the evaluation of H at spatial infinity after an appropriate choice of lapse and shift.

5.1 ADM mass and momentum

Let us return to the full Hamiltonian 4.32, with the multiplicative constant $(16\pi)^{-1}$ restored, given by:

$$\begin{aligned} H = &\frac{1}{16\pi} \int_{\Sigma_t} \left[-2N_j D_i \pi^{ij} - N\sqrt{\gamma} R + \frac{N}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2} \right) \right] \mathrm{d}^3 x \\ &+ \frac{1}{8\pi} \oint_{S_t} \left[N(\kappa - \kappa_0) + N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j \right] \sqrt{\sigma}\, \mathrm{d}^2 x \end{aligned} \tag{5.9}$$

If we restrict our analysis to the fields γ_{ij} and K_{ij} satisfying the vacuum field equations, the volume term H_Σ (corresponding to the first line of 5.9) vanishes and the only contribution to H is given by the boundary term:

$$H = \frac{1}{8\pi} \oint_{S_t} \left[N(\kappa - \kappa_0) + N_i \frac{\pi^{ij}}{\sqrt{\gamma}} r_j \right] \sqrt{\sigma}\, \mathrm{d}^2 x \tag{5.10}$$

We expect that with a suitable choice of N, N^i this equation shall provide a definition of total energy. However, in order to ensure that the value of H is finite, we must impose further restrictions on the nature of the spacetime under analysis.

Definition 5. Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic spacetime admitting a foliation by the family $\{\Sigma_t\}_{t\in\mathbb{R}}$ of spacelike hypersurfaces. This spacetime is said to be asymptotically flat if and only if there exists on each Σ_t a background metric f_{ij} such that:

1. f_{ij} is flat, i.e. the Riemann tensor $R^i{}_{jkl}$ associated to f_{ij} is identically zero, except on a compact domain $\mathcal{C} \subset \Sigma_t$;

2. given a coordinate system (y^i) on Σ_t, $f_{ij} = \mathrm{diag}(1,1,1)$ outside $\mathcal{C}$ and $r \doteq \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ is unbounded;
3. as $r \longrightarrow \infty$, the metric γ_{ij} and its spatial derivatives exhibit the following asymptotic behaviour:

$$\gamma_{ij} = f_{ij} + \mathcal{O}(r^{-1}) \tag{5.11}$$

$$\frac{\partial \gamma_{ij}}{\partial y^a} = \mathcal{O}(r^{-2}) \tag{5.12}$$

4. as $r \longrightarrow \infty$, the extrinsic curvature K_{ij} and its spatial derivatives satisfy:

$$K_{ij} = \mathcal{O}(r^{-2}) \tag{5.13}$$

$$\frac{\partial K_{ij}}{\partial y^a} = \mathcal{O}(r^{-3}) \tag{5.14}$$

Let us consider a Lorentz reference frame (ξ^μ) in the asymptotic region of Σ_t. We demand that this portion of Σ_t is described by the condition $\xi^0 = const$. Therefore, by denoting with (y^i) the coordinates on Σ_t, we are able to introduce the asymptotic relation $y^i = y^i(\xi^j)$ between spatial coordinates and $x^\mu = x^\mu(\xi^\alpha)$. An observer at rest in the Lorentz frame moves with four-velocity

$$u^\mu = \frac{\partial x^\mu}{\partial \xi^0} \tag{5.15}$$

as ξ^0 corresponds to its proper time. Since $u_\mu u^\mu = -1$ and u^μ is orthogonal to the surfaces $\xi^0 = const$ (or equivalently $t = const$), the following relation must hold at spatial infinity:

$$n^\mu = u^\mu \tag{5.16}$$

where n^μ is the normal to the hypersurface Σ_t. Recalling the definition 3.6, we can write:

$$(\partial_t)^\mu = N n^\mu + N^i (\partial_i)^\mu = N u^\mu + N^i \frac{\partial x^\mu}{\partial y^i} \tag{5.17}$$

We see that setting $N = 1$ and $N^i = 0$ allows to identify the vector ∂_t tangent to the curves $y^i = const$ with the four-velocity of the observer. This choice provides a reasonable definition of energy of the system, called ADM mass, given by the evaluation of the boundary term 5.10 at spatial infinity:

$$M_{ADM} \doteq \lim_{S_t \to \infty} \frac{1}{8\pi} \oint_{S_t} (\kappa - \kappa_0) \sqrt{\sigma}\, \mathrm{d}^2 x \tag{5.18}$$

M_{ADM} is a conserved quantity, since it accounts for the total mass present on Σ_t, even in the case of radiating systems.
Similarly, the choice

$$N = 0, \quad N^i = \frac{\partial y^i}{\partial \xi^a} \tag{5.19}$$

(with a fixed) produces a correspondence between ∂_t and the spatial translations along the coordinate curve of ξ^a. Consequently, using 5.19, the evaluation of 5.10 at spatial infinity provides the definition of ADM momentum:

$$P_a^{ADM} \doteq \lim_{S_t \to \infty} \frac{1}{8\pi} \oint_{S_t} N^i r^j \left(K\gamma_{ij} - K_{ij}\right) \sqrt{\sigma}\, \mathrm{d}^2 x \tag{5.20}$$

It can be proved that the set $\mathcal{P}_\mu = (M_{ADM}, P_1, P_2, P_3)_\mu$ behaves as a four-dimensional one-form under the general coordinate transformations $x^\mu = x^\mu(x'^\alpha)$ which preserve the asymptotic flatness conditions (equations 5.11 - 5.14). At this point, it seems that a definition of angular momentum naturally arises by choosing the shift functions N^i so as to identify ∂_t with the spatial rotations. However, it turns out that this procedure may impair the four-vector properties of the ADM four-momentum $\mathcal{P}_\mu$. This issue can be fixed by adding stronger conditions on γ_{ij} and K_{ij} to the asymptotic flatness conditions previously defined (see Ref. [23] for a detailed analysis).
We shall say that although the definition of M_{ADM} provides a reasonable notion of energy, it is not unique: there exist other relevant definitions of energy which in general disagree with 5.18. For instance, in place of a spacelike boundary S_t, we might take the limit to a null infinity by demanding the following conditions to hold:

$$u \doteq t - r = const \tag{5.21}$$

$$v \doteq t + r \longrightarrow \infty \tag{5.22}$$

This approach gives rise to the Bondi-Sachs mass M_{BS}, whose physical relevance unveils in the analysis of radiating systems. Indeed, since the gravitational radiation propagates along null geodesics, M_{BS} cannot account for the radiation loss (as S_t is now a null hypersurface, parallel to the direction of propagation) and diminishes with increasing u, while M_{ADM} remains constant. It has been shown that the rate of change of M_{BS} with respect to the retarded time u corresponds to negative the outward flux of radiated energy.
An alternative definition of energy was given by Komar (1959) for stationary, asymptotically flat spacetimes as the conserved quantity associated with time translations. In addition, if the spacetime is axisymmetric, a similar procedure provides a definition of angular momentum which is independent of the choice of S_t.
We conclude this section by presenting an example of computation of ADM mass, which endorses the reasonableness of this definition, followed by the evaluation of the ADM momentum.

5.1.1 Example: Schwarzschild spacetime with standard coordinates

Let us consider a Schwarzschild spacetime described by the standard coordinates (t, r, θ, ϕ):

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega \tag{5.23}$$

where $d\Omega = d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric on the two-sphere. Using (r, θ, ϕ) as the coordinates of each hypersurface Σ_t, the three-dimensional metric γ_{ij} becomes

$$\gamma_{ij} = \mathrm{diag}\left[\left(1 - \frac{2m}{r}\right)^{-1}, \, r^2, \, r^2 \sin^2\theta\right] \tag{5.24}$$

Consequently, the six non-vanishing Christoffel symbols ${\Gamma^i}_{jk}$ related to γ_{ij} are

$$
\begin{aligned}
&{\Gamma^r}_{rr} = -\tfrac{m}{r(r-2m)} && {\Gamma^r}_{\theta\theta} = -(r-2m) \\
&{\Gamma^r}_{\phi\phi} = -(r-2m)\sin^2\theta && {\Gamma^\theta}_{r\theta} = \tfrac{1}{r} \\
&{\Gamma^\phi}_{r\phi} = \tfrac{1}{r} && {\Gamma^\phi}_{\theta\phi} = \cot\theta
\end{aligned}
\tag{5.25}
$$

The background metric, on the other hand, is given by the flat metric in standard spherical coordinates (r, θ, ϕ):

$$f_{ij} = \mathrm{diag}\left(1,\, r^2,\, r^2\sin^2\theta\right) \tag{5.26}$$

Let us evaluate the scalar extrinsic curvature κ of S_t, embedded in the three-dimensional hypersurface Σ_t, using equation 4.12, namely $\kappa = -D_i n^i$ (we write n^i instead of r^i to avoid confusion). Since the components of the normal unit vector n^i pointing outside S_t are

$$n_i = \left(\sqrt{\frac{r}{r-2m}},\, 0,\, 0\right)_i \implies n^i = \left(\sqrt{\frac{r-2m}{r}},\, 0,\, 0\right)^i \tag{5.27}$$

we obtain

$$
\begin{aligned}
\kappa &= -\gamma^{ij}\left(\partial_i n_j - {\Gamma^a}_{ij} n_a\right) \\
&= -\frac{r-2m}{r}\,\partial_r\left(\sqrt{\frac{r}{r-2m}}\right) + \sqrt{\frac{r}{r-2m}}\left[-\frac{m}{r^2} - 2\frac{r-2m}{r^2}\right] \\
&= -\frac{2}{r}\sqrt{\frac{r-2m}{r}}
\end{aligned}
\tag{5.28}
$$

The scalar curvature κ_0 referred to the embedding in a flat spacetime can be effortlessly recovered by virtue of the relation $\kappa_0 = \kappa|_{m=0}$ (or equivalently by using the Christoffel symbols associated to the background metric f_{ij}). Hence we have

$$\kappa_0 = -\frac{2}{r} \tag{5.29}$$

Let us identify the boundary S_t with a two-sphere of radius r. This implies that the determinant σ of the induced metric on S_t is simply $\sigma = r^2 \sin\theta$. We can now evaluate M_{ADM} replacing these quantities in the integral 5.18 and taking the limit:

$$
\begin{aligned}
M_{ADM} &= \frac{1}{8\pi}\lim_{S_t\to\infty}\oint_{S_t}(\kappa - \kappa_0)\sqrt{\sigma}\,\mathrm{d}^2x \\
&= -\frac{1}{4\pi}\lim_{r\to\infty}\int_0^\pi d\theta\int_0^{2\pi} d\phi\,\frac{1}{r}\left(\sqrt{\frac{r-2m}{r}} - 1\right) r^2\sin\theta \\
&= -\lim_{r\to\infty} r\left(\sqrt{\frac{r-2m}{r}} - 1\right) = m
\end{aligned}
\tag{5.30}
$$

This is the expected result for a spherically symmetric body of mass m.
Regarding the ADM momentum, the standard coordinates of the Schwarzschild spacetime lead to the trivial result

$$\mathcal{P}_i^{ADM} = 0 \tag{5.31}$$

In order to prove this, we resort to the relations 2.17 and 3.16:

$$K_{ij} = -\nabla_j n_i = {}^4\Gamma^{\mu}{}_{ij} n_\mu = -N\,{}^4\Gamma^{0}{}_{ij} \tag{5.32}$$

This requires the computation of a subset of Christoffel symbols relative to the four-dimensional metric $g_{\mu\nu}$. In particular, the only non-zero term of the kind ${}^4\Gamma^{t}{}_{\mu\nu}$ is

$${}^4\Gamma^{t}{}_{tr} = \frac{1}{2}\partial_r \ln |g_{tt}| = \frac{m}{r\,(r-2m)} \tag{5.33}$$

Since all the Christoffel symbols with lower spatial indices vanish, the extrinsic curvature tensor is nothing but zero:

$$K_{ij} = 0 \tag{5.34}$$

Consequently, replacing K_{ij} in the integral 5.20 gives a null ADM momentum and concludes the proof. The four-momentum of the Schwarzschild spacetime in standard coordinates is thus represented by

$$\mathcal{P}_\mu = (m, 0, 0, 0)_\mu \tag{5.35}$$

Chapter 6

Appendix

6.1 Definitions

6.1.1 Covariant derivative or connection

Let $\mathcal{M}$ be a differentiable manifold. A covariant derivative (or connection) ∇ is a map from the tensor fields of rank (r, s) to the tensor fields $(r, s+1)$ such that:

1. ∇ is linear, namely $\nabla(T+S) = \nabla T + \nabla S$ with T, S tensor fields of the same rank.
2. $\nabla(fT) = \mathrm{d}f \otimes T + f\nabla T$, where f is scalar field and $\mathrm{d}f$ is the $(0,1)$ tensor with components $\partial_\mu f$.
3. given the bases $\{\boldsymbol{e}_\mu\}$ and $\{\boldsymbol{\theta}^\mu\}$ of the tangent and cotangent spaces $T_p(\mathcal{M})$, $T_p^*(\mathcal{M})$, it satisfies
$$\nabla \boldsymbol{e}_\mu = \Gamma^\alpha{}_{\beta\mu}\, \boldsymbol{\theta}^\beta \otimes \boldsymbol{e}_\alpha \tag{6.1}$$
where $\Gamma^\lambda{}_{\mu\nu}$ are the connection coefficients.

In particular, ∇ is said to be a metric connection if given a metric $g_{\mu\nu}$ on $\mathcal{M}$ the following relation holds:
$$\nabla g_{\mu\nu} = 0 \tag{6.2}$$
In this case, the connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are called Christoffel symbols and they are determined by the equation
$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\alpha}\left(g_{\alpha\nu,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}\right) \tag{6.3}$$

6.1.2 Tensor density

The covariant derivative ∇ introduced in the previous section is a map from tensors of rank (r, s) to $(r, s+1)$ tensors. However, in order to simplify the calculations, we can extend its applicability to the class of tensor densities, which are defined by
$$\mathcal{T}^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} = \sqrt{|g|}^{\,W}\, T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} \tag{6.4}$$

where $T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s}$ is a tensor of type (r,s), g is the determinant of the metric $g_{\mu\nu}$ and W is a real number, called the weight of the tensor density. The covariant derivative of $\mathcal{T}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s}$ is then a straightforward generalization of the ordinary derivation:

$$\nabla_\mu \mathcal{T}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} \doteq \sqrt{|g|}^{\,W} \nabla_\mu \left[\frac{\mathcal{T}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s}}{\sqrt{|g|}^{\,W}} \right] = \sqrt{|g|}^{\,W} \nabla_\mu T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} \tag{6.5}$$

6.1.3 Curvature tensors

The curvature of a manifold is completely determined by the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$. We define it following the sign convention of MTW (Ref. [16]):

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho{}_{\sigma\nu,\mu} - \Gamma^\rho{}_{\sigma\mu,\nu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu} \tag{6.6}$$

The Ricci tensor is given by the contraction of the first and third indices in 6.6:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \frac{1}{\sqrt{|g|}} \partial_\lambda \left[\sqrt{|g|}\, \Gamma^\lambda{}_{\mu\nu} \right] - \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\rho\nu} - \partial_\mu\partial_\nu \ln \sqrt{|g|} \tag{6.7}$$

Finally, from the contraction of $R_{\mu\nu}$ with the inverse metric $g^{\mu\nu}$ we obtain the scalar curvature:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{6.8}$$

6.1.4 Lie derivative

Let $\mathcal{M}$ be a differentiable manifold. Given a regular vector field $X = X^\mu \partial_\mu$ on $\mathcal{M}$ and an open subset $I \subset \mathbb{R}$, we define the integral curve of X by

$$\alpha_p : I \longrightarrow \mathcal{M} \tag{6.9}$$

$$s \longmapsto \alpha_p(s) \tag{6.10}$$

such that

$$\alpha_p(0) = p \tag{6.11}$$

$$\forall s_0 \in I \qquad \left.\frac{\mathrm{d}\alpha_p}{\mathrm{d}s}\right|_{s_0} = \dot{\alpha}_p(s_0) = X_{s_0}(\alpha_p) \tag{6.12}$$

Let $U \subset \mathcal{M}$ be an open subset. Each integral curve is associated in a natural way to the map

$$\phi_s^X : U \longrightarrow \mathcal{M} \tag{6.13}$$

$$p \longmapsto \alpha_p(s) \tag{6.14}$$

called the flow along X, such that

$$\forall s_0 \in I \qquad \left.\frac{\mathrm{d}\alpha_p}{\mathrm{d}s}\right|_{s_0} = \dot{\alpha}_p(s_0) = X_{s_0}(\alpha_p) \tag{6.15}$$

The flow ϕ_s^X has the following properties:

1. $\phi_0^X(p) = \alpha_p(0) = p \quad \Longrightarrow \quad \phi_0^X = \mathbb{I}$
2. $\phi_s^X \circ \phi_t^X = \phi_{s+t}^X \quad \forall\, s,t \in \mathbb{R}$

3. ϕ_s^X is a diffeomorphism and $[\phi_s^X]^{-1} = \phi_{-s}^X$

By means of this map, we can define the Lie derivative of a differentiable tensor field T of rank (m,n) along X, evaluated at a point p, by

$$[\mathcal{L}_X(T)]_p = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\left(\phi_{-s}^X\right)_* T_{\phi_s^X(p)}\right] \tag{6.16}$$

which in components reads

$$\begin{aligned}[\mathcal{L}_X(T)]^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} &= X^\lambda \partial_\lambda T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} \\ &- T^{\lambda\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\partial_\lambda X^{\mu_1} - \ldots - T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_n}\partial_\lambda X^{\mu_m} \\ &+ T^{\mu_1\ldots\mu_m}{}_{\lambda\ldots\nu_n}\partial_{\nu_1} X^\lambda + \ldots + T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\lambda}\partial_{\nu_n} X^\lambda\end{aligned} \tag{6.17}$$

If the connection ∇ is torsion-free, namely if the Christoffel symbols are symmetric in the last two indices

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} \tag{6.18}$$

equation 6.17 can be rewritten by replacing the partial derivatives ∂_μ with the covariant counterparts ∇_μ:

$$\begin{aligned}[\mathcal{L}_X(T)]^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} &= X^\lambda \nabla_\lambda T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} \\ &- T^{\lambda\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\nabla_\lambda X^{\mu_1} - \ldots - T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_n}\nabla_\lambda X^{\mu_m} \\ &+ T^{\mu_1\ldots\mu_m}{}_{\lambda\ldots\nu_n}\nabla_{\nu_1} X^\lambda + \ldots + T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\lambda}\nabla_{\nu_n} X^\lambda\end{aligned} \tag{6.19}$$

The main properties of the Lie derivative easily follow from the definition and from the component relation 6.17:

1. the Lie derivative of a tensor field T of rank (m,n) is a tensor field of rank (m,n).

2. $\mathcal{L}_X(T)$ is linear both in X and in T.

3. the Lie derivative satisfies the Leibniz rule

$$\mathcal{L}_X(T\otimes S) = \mathcal{L}_X(T)\otimes S + T\otimes\mathcal{L}_X(S)$$

4. If f is a scalar field, $\mathcal{L}_X(f) = X(f)$.

6.2 Variation with respect to the metric

6.2.1 Christoffel symbols

For the sake of brevity, we denote with $\theta_{\mu\nu} = \delta g_{\mu\nu}$ the variation of the metric $g_{\mu\nu}$ in part of the intermediate passages:

$$\begin{aligned}\delta\Gamma_{\lambda\mu\nu} &= \frac{1}{2}\left(\delta g_{\lambda\nu,\mu} + \delta g_{\mu\lambda,\nu} - \delta g_{\mu\nu,\lambda}\right) \\ &= \frac{1}{2}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) \\ &+ \frac{1}{2}\Big[\Gamma^\sigma{}_{\mu\lambda}\theta_{\sigma\nu} + \Gamma^\sigma{}_{\mu\nu}\theta_{\lambda\sigma} + \Gamma^\sigma{}_{\nu\mu}\theta_{\sigma\lambda} + \Gamma^\sigma{}_{\nu\lambda}\theta_{\mu\sigma} - \Gamma^\sigma{}_{\lambda\mu}\theta_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu}\theta_{\mu\sigma}\Big]\end{aligned}$$

$$= \frac{1}{2}\left(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right) + \Gamma^\sigma{}_{\mu\nu}\delta g_{\sigma\lambda} \tag{6.20}$$

It follows that

$$\begin{aligned}
\delta\Gamma^\rho{}_{\mu\nu} &= \delta g^{\rho\lambda}\Gamma_{\lambda\mu\nu} + g^{\rho\lambda}\delta\Gamma_{\lambda\mu\nu} \\
&= -g^{\rho\lambda}\Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda} + \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) + g^{\rho\lambda}\Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda} \\
&= \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right) \qquad (6.21)
\end{aligned}$$

We shall also consider the contracted version of 6.21:

$$\begin{aligned}
\delta\Gamma^\mu{}_{\mu\nu} &= \delta g^{\mu\lambda}\Gamma_{\lambda\mu\nu} + g^{\mu\lambda}\delta\Gamma_{\lambda\mu\nu} \\
&= -g^{\alpha\mu}g^{\beta\lambda}\theta_{\alpha\beta}\Gamma_{\lambda\mu\nu} + g^{\mu\lambda}\left[\frac{1}{2}\left(\nabla_\mu\theta_{\lambda\nu} + \nabla_\nu\theta_{\mu\lambda} - \nabla_\lambda\theta_{\mu\nu}\right) + \Gamma^\sigma{}_{\mu\nu}\theta_{\sigma\lambda}\right] \\
&= -\Gamma^\beta{}_{\mu\nu}g^{\alpha\mu}\theta_{\alpha\beta} + \frac{1}{2}g^{\lambda\mu}\nabla_\nu\theta_{\lambda\mu} + \Gamma^\beta{}_{\mu\nu}g^{\alpha\mu}\theta_{\alpha\beta} \\
&= \frac{1}{2}g^{\lambda\mu}\nabla_\nu \delta g_{\lambda\mu} \qquad (6.22)
\end{aligned}$$

6.2.2 The vector δV^ρ

Let us consider the vector

$$\delta V^\rho \doteq g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\rho\nu}\delta\Gamma^\mu{}_{\mu\nu} \tag{6.23}$$

which appears in the variation of R. This is equivalent to

$$\delta V^\rho = \left(g^{\mu\nu}\delta g^{\rho\lambda} - g^{\rho\nu}\delta g^{\mu\lambda}\right)\Gamma_{\lambda\mu\nu} + \left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\delta\Gamma_{\lambda\mu\nu} \tag{6.24}$$

We focus on the second product, which contains the variation $\delta\Gamma_{\lambda\mu\nu}$, and we adopt the notation $\theta_{\mu\nu,\lambda} = \delta g_{\mu\nu,\lambda}$:

$$\begin{aligned}
\left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\delta\Gamma_{\lambda\mu\nu} &= \frac{1}{2}\Big[\left(g^{\mu\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\mu\lambda}\right)\left(\delta g_{\lambda\nu,\mu} + \delta g_{\mu\lambda,\nu} - \delta g_{\mu\nu,\lambda}\right)\Big] \\
&= g^{\mu\nu}g^{\rho\lambda}\delta g_{\lambda\mu,\nu} - g^{\mu\nu}g^{\rho\lambda}\delta g_{\mu\nu,\lambda}
\end{aligned}$$

It follows that

$$\delta V^\rho = \left(g^{\mu\nu}\delta g^{\rho\lambda} - g^{\rho\nu}\delta g^{\mu\lambda}\right)\Gamma_{\lambda\mu\nu} + g^{\mu\nu}g^{\rho\lambda}\left(\delta g_{\lambda\mu,\nu} - \delta g_{\mu\nu,\lambda}\right) \tag{6.25}$$

Using equations 6.21 and 6.22, we are able to rewrite δV^ρ by introducing the covariant derivatives of $\delta g_{\mu\nu}$:

$$\begin{aligned}
\delta V^\rho &= \frac{1}{2}g^{\mu\nu}g^{\rho\lambda}\big(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\big) - \frac{1}{2}g^{\rho\nu}g^{\lambda\mu}\nabla_\nu \delta g_{\lambda\mu} \\
&= g^{\mu\nu}g^{\rho\lambda}\big(\nabla_\mu \delta g_{\lambda\nu} - \nabla_\lambda \delta g_{\mu\nu}\big) \qquad (6.26)
\end{aligned}$$

6.2.3 Curvature

Let us consider the variation $\delta\Gamma^\lambda{}_{\mu\nu}$ of the Christoffel symbols induced by a variation of the metric. Since $\delta\Gamma^\lambda{}_{\mu\nu}$ is the difference of two connections, it is a tensor of rank $(1,2)$. By choosing a local inertial frame, we can make the Christoffel symbols to vanish:

$$\Gamma^{\lambda}{}_{\mu\nu} \overset{*}{=} 0 \tag{6.27}$$

The sign $\overset{*}{=}$ emphasizes that the equality is valid in a Lorentz frame. By virtue of equation 6.27 we can now express the variation of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ substituting the partial derivatives ∂_μ with ∇_μ:

$$\delta R^{\rho}{}_{\sigma\mu\nu} \overset{*}{=} \delta\left[\Gamma^{\rho}{}_{\sigma\nu,\mu} - \Gamma^{\rho}{}_{\sigma\mu,\nu}\right] \overset{*}{=} \nabla_\mu \delta\Gamma^{\rho}{}_{\sigma\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\sigma\mu} \tag{6.28}$$

The left-hand side is a tensorial quantity, which implies that the equality must be valid in any reference frame. Hence replacing $\overset{*}{=}$ with $=$ leads to the Palatini identity:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_\mu \delta\Gamma^{\rho}{}_{\sigma\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\sigma\mu} \tag{6.29}$$

and its contracted version, involving the Ricci curvature tensor:

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_\mu \delta\Gamma^{\lambda}{}_{\lambda\nu} \tag{6.30}$$

By virtue of the Palatini identity, we are able to perform a straightforward computation of the variation δR:

$$\begin{aligned}
\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\left[\nabla_\lambda \delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_\mu \delta\Gamma^{\lambda}{}_{\lambda\nu}\right] \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_\lambda \delta V^\lambda
\end{aligned} \tag{6.31}$$

where δV^λ was defined in equation 6.23. Substituting the relation 6.26 we obtain:

$$\begin{aligned}
\nabla_\lambda \delta V^\lambda &= g^{\mu\nu} g^{\rho\lambda} \nabla_\rho \big(\nabla_\mu \delta g_{\lambda\nu} - \nabla_\lambda \delta g_{\mu\nu}\big) \\
&= \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \nabla^\lambda \nabla_\lambda \delta \ln|g|
\end{aligned} \tag{6.32}$$

Therefore we have

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \nabla^\lambda \nabla_\lambda \delta \ln|g| \tag{6.33}$$

Bibliography

[1] Arnowitt R., Deser S. and Misner C. W., *Canonical Variables for General Relativity*, Phys. Rev. 117, 1595 (1960).

[2] Arnowitt R., Deser S. and Misner C. W., *Consistency of the Canonical Reduction of General Relativity*, in *Journal of Mathematical Physics* 1, 434 (1960).

[3] Arnowitt R., Deser S. and Misner C. W., *Dynamical Structure and Definition of Energy in General Relativity*, Phys. Rev. 116, 1322 (1959)

[4] Arnowitt R., Deser S. and Misner C. W., *The Dynamics of General Relativity*, in *Gravitation: an introduction to current research*, L. Witten, ed. Wiley, New York (1962). *arXiv:gr-qc/0405109.*

[5] Bishop R. L., Goldberg S. I., *Tensor Analysis on Manifolds.* Dover Publications (1980).

[6] Brown J. D., Lau S. R. and York J. W., *Action and Energy of the Gravitational Field* (2000). *arXiv:gr-qc/0010024.*

[7] Carroll S. M., *Lecture Notes on General Relativity* (1997). *arXiv:gr-qc/9712019.*

[8] Choquet-Bruhat Y., *Beginnings of the Cauchy problem* (2014). *arXiv:1410.3490 [gr-qc].*

[9] Choquet-Bruhat Y., *General Relativity and the Einstein Equations.* Oxford University Press (2009).

[10] Choquet-Bruhat Y., *Théorème d'existence pour certains systèms d'équations aux dérivées partielles non linéaires.* Acta Mathematica 88, 141 (1952). Available at http://fanfreluche.math.univ-tours.fr/

[11] Choquet-Bruhat Y., Geroch R., *Global Aspects of the Cauchy Problem in General Relativity.* Commun. math. Phys. 14, 329-335 (1969).

[12] Cook G. B., *Initial Data for Numerical Relativity* (2000). *arXiv:gr-qc/0007085.*

[13] DeWitt B. S., *Quantum Theory of Gravity. I. The Canonical Theory*, Phys. Rev. 160, 1113 (1967)

[14] Gourgoulhon E., *3+1 Formalism and Bases of Numerical Relativity*, lecture notes (2007). *arXiv:gr-qc/0703035.*

[15] Lanczos C., *The Variational Principles of Mechanics*, Dover Publications (1986).

[16] Misner C. W., Thorne K. S., Wheeler A., *Gravitation.* W. H. Freeman and Company (1973).

[17] Poisson E., *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge: Cambridge University Press (2004).

[18] Rovelli C., *Notes for a brief history of quantum gravity* (2001). arXiv:gr-qc/0006061v3.

[19] Rovelli C., *The strange equation of quantum gravity* (2015). arXiv:1506.00927 [gr-qc].

[20] Tong F., *A Hamiltonian Formulation of General Relativity* (2006). Available at http://www.math.toronto.edu/mccann/assignments/426/Tong.pdf

[21] Wald R. M., *General Relativity.* University of Chicago Press (1984).

[22] Weinberg S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Wiley, New York (1972).

[23] York J. W., *Kinematics and dynamics of general relativity*, in *Sources of Gravitational Radiation*, edited by L. L. Smarr, Cambridge University Press, Cambridge (1979).