NOTES ON RANDOM MATRICES

LUCA GUIDO MOLINARI

The large-N limit of the free energy of a multi-matrix model provides the counting of multi-matrix planar graphs, and this counting may correspond to the summation of configurations of a statistical model on random planar graphs. The thermodynamic limit of the statistical model is realised by $g \to g_{\rm cr}$, and may show phase transitions, that are influenced by the fluctuations of the surface that supports it. The critical exponents differ from those of the statistical model on a regular lattice in a way predicted by KPZ [13]. Here are some models:

- Ising model (Kazakov, 1986 [8]);
- Ising model with magnetic field (Boulatov & Kazakov, 1987 [3]);
- q-state Potts model (Kazakov, 1988 [9]);
- O(n) model (Duplantier & Kostov, 1988 [7]);
- Percolation on a fractal (Kazakov, 1989 [10]);
- Three colour problem (Cicuta & al., 1993 [5]);
- 8-vertex model (a case) (Kazakov & P. Zinn-Justin, 1999; [12]);
- Baxter colouring problem (Kostov, 2002 [15])

1. Ising model on random planar graphs

In the Ising model on a connected graph, a spin $\sigma = \pm 1$ is allocated at each vertex, and adjacent spins (i.e. connected by an edge) have interaction energy $J\sigma_i\sigma_j$ with ferromagnetic coupling $J = -1$ (i.e. parallel spins have lower energy). The partition function for the Ising model on a graph[1] in a uniform magnetic field is:

$$Z_{\rm Ising}(G,\beta,H) = \sum_{\sigma_i=\pm 1} \exp\left[-\beta J \sum_{ij} \sigma_i G_{ij}\sigma_j + H\sum_i \sigma_i\right] \tag{1}$$

If V is the number of vertices of the graph, there are 2^V spin configurations.
Given a graph with a spin configuration on it, (G,σ), let E_p and E_a be the numbers of edges connecting parallel and antiparallel spins, and $V_\uparrow$, $V_\downarrow$ be the numbers of vertices with spin $+1$ or -1. The magnetisation is $\sum \sigma_i = V_\uparrow - V_\downarrow$. The statistical weight of (G,σ) is

$$\exp[\beta(E_p - E_a) + H(V_\uparrow - V_\downarrow)] \tag{2}$$

If all vertices have coordination 4 then $4V = 2E$, where $V = V_\uparrow + V_\downarrow$, $E = E_p + E_a$.

The Ising model on the regular square lattice with $H = 0$ was solved in the infinite V limit by Lars Onsager (1944) and for $H \neq 0$ near T_c, by Chen Ning Yang

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[1] A graph with V vertices labelled $1...V$ is described by the $V\times V$ adjacency matrix $G_{ij} = 1$ if vertices ij are connected by an edge, 0 otherwise. $\sum_i G_{ij}$ is the number of edges with extremum j (coordination of vertex j).

(1952) who found spontaneous magnetisation for $T < T_c$. Since the square lattice is self-dual, the critical temperature was obtained, $\beta_c \approx 2.269$.

Amazingly, the analytic solution of the Ising model on a connected planar graph becomes feasible if, besides summing on spin configurations on the graph, one also sums on the planar graphs themselves, with V vertices:

$$Z_{\rm Ising}(V,\beta,H) = \sum\nolimits_{G_{pl}} Z_{\rm Ising}(G_{pl},\beta,H)$$

The Ising model on random planar graphs with coordination 4 and $H=0$ was solved in 1986 by Kazakov [8] by mapping it to a 2-matrix model. Soon after Boulatov and Kazakov [3] modified the 2-matrix model in order to include a magnetic field:

$$\mathbb{Z}_N(c,g,H) = \int dAdB\; e^{\left[-N{\rm tr}(A^2+B^2-2cAB+4ge^H A^4+4ge^{-H}B^4)\right]} \tag{3}$$

A, B are Hermitian $N\times N$ matrices, $0<c<1$.
The power expansion in g corresponds to a sum of Feynman graphs with quartic vertices of type A or B, that correspond to spin orientations $\uparrow$ or $\downarrow$. In a graph the vertices are connected by propagators (edges) of two types:

$$\frac{1}{N}\langle {\rm tr}AA\rangle = \frac{1}{N}\langle {\rm tr}BB\rangle = \frac{1}{1-c^2}, \quad \frac{1}{N}\langle {\rm tr}AB\rangle = \frac{c}{1-c^2}$$

Since $0<c<1$, edges connecting parallel spins are enhanced. A connected graph has weight in the parameters

$$N^\chi (ge^H)^{V_A}(ge^{-H})^{V_B}\langle AA\rangle^{E_p}\langle AB\rangle^{E_a} = N^\chi\left[\frac{gc}{(1-c^2)^2}\right]^V c^{-\frac{1}{2}(E_p-E_a)}e^{H(V_\uparrow - V_\downarrow)}$$

where $\chi = V+F-E$ is the Euler number of the closed surface that hosts the graph. Planar graphs ($\chi=2$) dominate the large-N limit of the model. The generator of connected planar graphs is the planar free energy:

$$F_{\rm pl}(c,g,H) = -\lim_{N\to\infty}\frac{1}{N^2}\log\frac{\mathbb{Z}(c,g,H)}{\mathbb{Z}(c,0,0)} = \sum_{k=1}^{\infty}\left[\frac{gc}{(1-c^2)^2}\right]^V F_V(c,H) \tag{4}$$

The coefficients $F_V(c,H)$ of the power expansion in g take record of all the planar connected Feynman graphs with V vertices. Each one corresponds to a configuration (G,σ) of the Ising model (the Feynman graph is G, with the further information that its vertices are A and B). Comparison among the weight of a graph and of an Ising configuration (2) gives the correspondence:

$$\boxed{F_V(c,H) = Z_{\rm Ising}(V,\beta,H), \qquad c = e^{-2\beta}} \tag{5}$$

The expansion (4) in powers of g of the planar free energy $F_{\rm pl}$ has a finite radius of convergence $g_{\rm cr}(c,H)$. Hadamard's formula gives the leading behaviour of the coefficients, i.e. of the free energy $F_V(c,H)$ for large V:

$$F_V(c,H) \approx \left[\frac{c\,|g_{cr}(c,H)|}{(1-c^2)^2}\right]^{-V} \times \text{sub-leading factors}$$

Accordingly, the free energy per site of the Ising model is evaluated by the formula

$$F_{\rm Ising} = -\frac{1}{V}\log Z_{\rm Ising}(V,\beta,H) \approx \log\left[\frac{c\,|g_{cr}(c,H)|}{(1-c^2)^2}\right]$$

Bi-orthogonal polynomials. For any N, the two-matrix integral (3) is amenable to the eigenvalues x_i and y_i of A and B by means of the HarishChandra-Itzykson-Zuber integral, in the form by Mehta [17] that is here used.
If $A = UXU^\dagger$ and $B = VYV^\dagger$, where U, V are unitary and X, Y are diagonal, it is:

$$\begin{aligned}\mathbb{Z}_N &= \int dXdY\,\Delta^2(x)\Delta^2(y)e^{-N\sum_i(x_i^2+y_i^2+4ge^Hx_i^4+4ge^{-H}y_i^4)}\int dW e^{2Nc\,\mathrm{tr}(WXW^\dagger Y)}\\ &\approx \int dXdY\,\Delta(x)\Delta(y)e^{-N\sum_i v(x_i,y_i)}\end{aligned}$$

with potential $v(x,y) = x^2 + y^2 - 2c\,xy + 4ge^Hx^4 + 4ge^{-H}y^4$.
By writing $\Delta(x) = \det[P_m(x_k)]_{k=1\dots N}^{m=0\dots N-1}$ and $\Delta(y) = \det[Q_m(y_k)]_{k=1\dots N}^{m=0\dots N-1}$, with monic polynomials $P_m(x)$ and $Q_m(y_k)$, and by choosing them bi-orthogonal,

$$\int dxdy\, e^{-Nv(x,y)}\, P_k(x)\, Q_j(y) = h_k\delta_{kj}$$

the partition function is $\mathbb{Z}_N = N!h_0 \dots h_{N-1}$.
The polynomials are fully determined by the condition. Since $v(-x,-y) = v(x,y)$ the polynomials may be chosen with definite parity.

Proposition 1.1.

$$xP_k(x) = P_{k+1}(x) + R_kP_{k-1}(x) + S_kP_{k-3}(x) \tag{6}$$

$$yQ_k(x) = Q_{k+1}(x) + R'_kQ_{k-1}(x) + S'_kQ_{k-3}(x) \tag{7}$$

Proof. Suppose that the expansion of $xP_k(x)$ contains a term $T_kP_{k-5}(x)$. Multiply (6) by $Q_{k-5}(y)$ and integrate with the measure. It is $\int dxdy \exp(-Nv)xP_k(x)Q_{k-5}(y) = T_kh_{k-5}$. The first integral is dealt with the second of the identities:

$$\frac{1}{2N}\frac{\partial}{\partial x}e^{-Nv(x,y)} + e^{-Nv(x,y)}(x + 8ge^Hx^3) = cye^{-Nv(x,y)} \tag{8}$$

$$\frac{1}{2N}\frac{\partial}{\partial y}e^{-Nv(x,y)} + e^{-Nv(x,y)}(y + 8ge^{-H}y^3) = cxe^{-Nv(x,y)} \tag{9}$$

Then $cT_kh_{k-5} = \int dxdye^{-Nv}(y + 8ge^{-H}y^3)P_k(x)Q_{k-5}(y) = 0$.
Similarly, $cT'_kh_{k-5} = \int dxdye^{-Nv}(x + 8ge^Hx^3)P_{k-5}(x)Q_k(y) = 0$. □

Proposition 1.2. *Define* $f_k = h_k/h_{k-1}$*, then:*

$$cS_k = 8ge^{-H}f_kf_{k-1}f_{k-2} \tag{10}$$

$$cS'_k = 8ge^{H}f_kf_{k-1}f_{k-2} \tag{11}$$

$$cR_k = [1 + 8ge^{-H}(R'_{k+1} + R'_k + R'_{k-1})]f_k \tag{12}$$

$$cR'_k = [1 + 8ge^{H}(R_{k+1} + R_k + R_{k-1})]f_k \tag{13}$$

$$\frac{k}{2N} = -c\,f_k + 8ge^{-H}[R'_k(R'_{k+1} + R'_k + R'_{k-1}) + S'_{k+2} + S'_{k+1} + S'_k] + R'_k \tag{14}$$

$$\frac{k}{2N} = -c\,f_k + 8ge^{H}[R_k(R_{k+1} + R_k + R_{k-1}) + S_{k+2} + S_{k+1} + S_k] + R_k \tag{15}$$

Proof. Eq.(10). Multiply (6) by $c\,Q_{k-3}$ and integrate with the weight, then use (9)

$$cS_kh_{k-3} = 8ge^{-H}\int dxdye^{-Nv(x,y)}y^3Q_{k-3}P_k(x) = 8ge^{-H}h_k$$

Eq.(12). Multiply (6) by $c\,Q_{k-1}(y)$ and integrate with the weight, and use (9):

$$\begin{aligned} cR_k h_{k-1} =& \int dxdy e^{-Nv(x,y)}(y+8ge^{-H}y^3)Q_{k-1}(y)P_k(x) \\ =& h_k[1+8ge^{-H}(R'_{k+1}+R'_k+R'_{k-1})] \end{aligned}$$

Eq.(14). Multiply (6) by $c\,Q_{k+1}(y)$ and integrate with the weight, and use (9):

$$\begin{aligned} ch_{k+1} =& \int dxdy e^{-Nv(x,y)}(y+8ge^{-H}y^3)Q_{k+1}(y)P_k(x) - \frac{k+1}{2N}h_k \\ =& 8ge^{-H}h_k[R'_{k+1}(R'_{k+2}+R'_k+R'_{k-2})+S'_{k+3}+S'_{k+2}+S'_{k+1}] \\ &+ R'_{k+1}h_k - \frac{k+1}{2N}h_k \end{aligned}$$

The other equations are similarly obtained. □

The partition function is now expressed in terms of f_k:

$$\log \mathbb{Z}_N(c,g,H) = \log N! + N\log h_0 + \sum_{k=1}^{N-1}(N-k)\log f_k \tag{16}$$

The large N limit selects planar graphs. The coefficients $f_k, R_k, S_k, ...$ are interpolated by functions, and the recursive equations become algebraic. With $c<1$ the boundary conditions $f_0, R_0, ..., f_1, R_1, ...$ allow for interpolation of coefficients by single functions, as $f_k = f(k/N) = f(x)$, $0 \le x \le 1$. One can do more by expanding in $1/N$, $f_{k+1} \approx f(x) + (1/N)f'(x) + ...$ and approach g_{cr} and $N \to \infty$ to account for all topologies (double scaling) [16].
The case $c>1$ and $H=0$, has boundary conditions that require different functions to interpolate even or odd coefficients [18, 4].
The recursive equations become:

$$\begin{aligned} &cS(x) = 8ge^{-H}f^3(x) \\ &cS'(x) = 8ge^{H}f^3(x) \\ &cR(x) = [1+24ge^{-H}R'(x)]f(x) \\ &cR'(x) = [1+24ge^{H}R(x)]f(x) \\ &cx + 2c^2f(x) - 24(4g)^2f^3(x) = 2c\,R'(x)[1+24ge^{-H}R'(x)] \\ &cx + 2c^2f(x) - 24(4g)^2f^3(x) = 2c\,R(x)[1+24ge^{H}R(x)] \end{aligned}$$

The free energy. Since $\lim_{N\to\infty} N^{-2}\log h_0 = 0$, the planar free energy of the 2-matrix model is the integral

$$F_{\rm pl}(c,g,H) = -\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N-1}(1-\frac{k}{N})\log\frac{f_k}{f_k^0} = -\int_0^1 dx(1-x)\log\frac{f(x)}{f^0(x)}$$

The equation for $f(x)$ is obtained from the system:

$$\frac{x}{2} = -cf(x) + \frac{12(4g)^2}{c}f^3(x) + c\frac{f(x)}{[c-24gf(x)]^2} + 48gc^2f^2(x)\frac{(\cosh H - 1)}{[c^2-(24g)^2f^2(x)]^2}$$

and gives for $g, H = 0$: $f_0(x) = \frac{1}{2}cx/(1-c^2)$. The perturbative expansion is

$$F_{\rm pl} = \frac{2ge^H + 2ge^{-H}}{(1-c^2)^2} - \frac{g^2}{(1-c^2)^4}[4c^4 + 32c^2 + 18(e^{2H}+e^{-2H})] + \dots$$

By setting $z(x) = (24g/c) f(x)$:

$$4gx = -\tfrac{1}{3}c^2 z + \tfrac{1}{9}c^2 z^3 + \tfrac{1}{3}\frac{z}{(1-z)^2} + \tfrac{2}{3}\frac{z^2}{(1-z^2)^2}(\cosh H - 1) \equiv w(z) \tag{17}$$

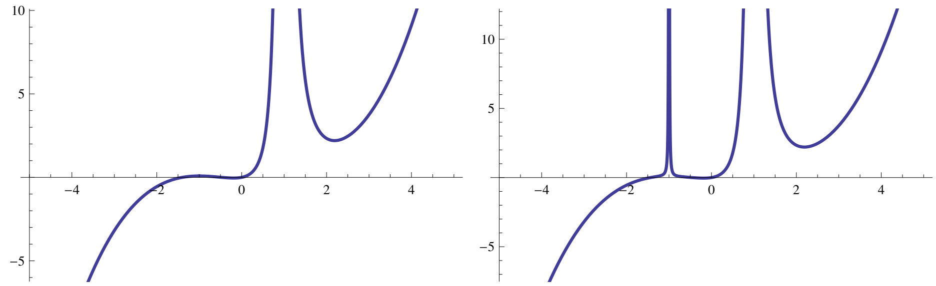

FIGURE 1. The function $w(z, \frac{1}{2}, H)$ versus z, for $H = 0$ (left) and $H = 0.1$ (right).

By assuming that $f(x)$ is one-to-one, integration by parts gives the planar free energy in terms of $\zeta = z(1)$, solution of the equation $4g = w(\zeta, c, H)$:

$$\begin{aligned} F_{\rm pl} &= -\tfrac{1}{2}\log\frac{f(1)}{f^0(1)} + \int_0^1 dx \frac{f'(x)}{f(x)}(x - \tfrac{1}{2}x^2) - \tfrac{3}{4} \\ &= -\tfrac{1}{2}\log\frac{\zeta(1-c^2)}{12g} + \frac{1}{4g}\int_0^\zeta \frac{dz}{z} w(z) - \frac{1}{32g^2}\int_0^\zeta \frac{dz}{z} w^2(z) - \tfrac{3}{4} \end{aligned}$$

The "thermodynamic limit" (when the average number of vertices is divergent) is obtained at the critical values g_{cr}. They result from the equation $w'(z) = 0$:

$$\cosh H - 1 = -\frac{(1+z)^4[1 - c^2(1-z)^4]}{4z(1+z^2)} \tag{18}$$

The solutions $z_{\rm cr}(c, H)$ are entered in $w(z, c, H)$ to give $4g_{cr}(c, H) = w_{\rm cr}(c, H)$.

Case $H = 0$. According to the discussion of the 2-matrix model, the singular behaviour of F is determined by the points: $z_c = -1$ for $c < 1/4$ and $z_- = 1 - \frac{1}{\sqrt{c}}$ for $c > 1/4$ with corresponding values $w(z_c, c, 0) = \frac{2}{9}c^2 - \frac{1}{12}$ and $w(z_-, c, 0) = -\frac{2}{9}c^2 + \frac{2}{3}c - \frac{4}{9}\sqrt{c}$. As the parameter z is varied from 0 to $1 - 1/\sqrt{c}$, the first singularity that is encountered is $z = -1$ for $0 < c < 1/4$ and $z = 1 - 1/\sqrt{c}$ for $1/4 < c < 1$. The value $c = 1/4$ marks a phase transition.

$$4g \le 4g_{cr}(c, 0) = \begin{cases} \frac{2}{9}c^2 - \frac{1}{12} & 0 < c \le \frac{1}{4} \text{ (low T)} \\ -\frac{2}{9}c^2 + \frac{2}{3}c - \frac{4}{9}\sqrt{c} & \frac{1}{4} < c < 1 \text{ (high T)} \end{cases} \tag{19}$$

Case $H \neq 0$. For small H the zeros of $w'(z) = 0$ are :

$$z_1(c, H) = -1 + \frac{\sqrt{2H}}{(1-16c^2)^{1/4}} - \frac{H}{(1-16c^2)^{3/2}} + \dots$$
$$z_2(c, H) = (1 - \frac{1}{\sqrt{c}})\left[1 - \frac{c}{2}\frac{2c - 2\sqrt{c} + 1}{(2\sqrt{c} - 1)^4} H^2 + \dots\right]$$

and correspond to two phases:

$$4g_{cr}(c,H) = \begin{cases} w(z_1) = \frac{2}{9}c^2 - \frac{1}{12} + \frac{\sqrt{1-16c^2}}{12}H + \dots & 0 < c \le \frac{1}{4} \\ w(z_2) = -\frac{2}{9}c^2 + \frac{2}{3}c - \frac{4}{9}\sqrt{c} + \kappa H^2 + \dots . & \frac{1}{4} \le c < 1 \end{cases} \tag{20}$$

1.1. **Magnetization.** The average magnetisation per vertex in the thermodynamic limit is

$$\begin{aligned} M(c,H) &= \lim_{V\to\infty} \frac{1}{V}\frac{\partial}{\partial H} F_V(c,H) \\ &= \frac{\partial}{\partial H}\log g_{cr}(c,H) = \frac{\partial}{\partial H}\log w_{cr}(\zeta(H,c),H) = \frac{1}{w_{cr}}\frac{\partial w_{cr}}{\partial H} \end{aligned}$$

because $w'(\zeta) = 0$. The equations for M and $w' = 0$ provide M and H parametrically in ζ:

$$M = 3\frac{\sqrt{[1-c^2(1-\zeta)^4][1-c^2(1+\zeta)^4]}}{4c^2(1-\zeta^2)^2 + 3 - 8c^2} \tag{21}$$

$$\cosh H = 1 - \frac{(1+\zeta)^4[1-c^2(1-\zeta)^4]}{4\zeta(1+\zeta^2)} \tag{22}$$

- for $\zeta \to 0$ it is $H \to \infty$ and $M \to 1$ i.e. all spins are aligned with H.
- for $\zeta = -1$ it is $H = 0$ and $M = 3\sqrt{1-16c^2}/(3-8c^2)$ (spontaneous magnetiz.).
- for $\zeta = 1 - 1/\sqrt{c}$ it is $H = 0$ and $M = 0$.

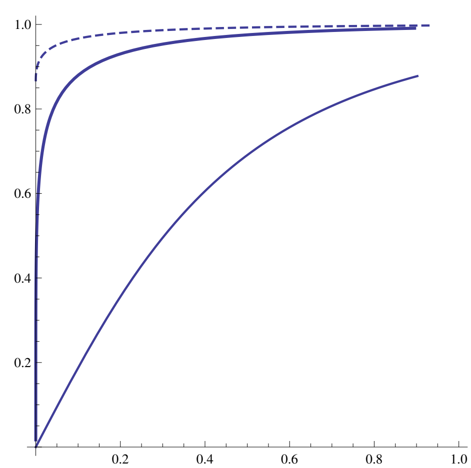

FIGURE 2. The magnetisation M per vertex as a function of H for $c = 1/7$ (dashed, low temperature phase), $c = 1/4$ (thick, critical temperature) and $c = 1/\sqrt{2}$ (line, high temperature phase). Note the spontaneous magnetisation for $c = 1/7$ and the different slopes in the origin. $M = 1$ is the saturation value.

The critical exponents. (See the book by Stanley for definitions [21]) We study the free energy and the magnetisation near the phase transition temperature $c = 1/4$, $H \to 0$.

- Specific heat at constant H, $\alpha = -1$.
  Definition: $C_H \approx \epsilon^{-\alpha}$, where $\epsilon = (T - T_c)/T_c$ is the reduced temperature. The free energy near $c_{\rm cr} = 1/4$ has continuous first and second derivatives in c, and finite discontinuity of the third derivative (derivative of specific heat). This means $\alpha = -1$.
- Spontaneous magnetization, $\beta = \frac{1}{2}$.
  Definition: $M(c,0) = (-\epsilon)^\beta$. The average magnetization for $H \to 0$ is:

$$M(c,0) = \begin{cases} 0 & \text{high T} \\ \frac{3\sqrt{1-16c^2}}{8c^2-3} & \text{low T} \end{cases}$$

  Near $c_{\rm cr} = \frac{1}{4}$, $M(c,0) = -\frac{12\sqrt{2}}{5}\sqrt{c_{cr} - c}$ i.e. $M \approx (T - T_c)^{1/2}$.
- Magnetic susceptibility, $\gamma = 2$.
  Definition: $\chi = \frac{\partial M}{\partial H}\Big|_{H=0} = \frac{1}{5}(2\sqrt{c} - 1)^{-2} \propto (T_c - T)^{-\gamma}$.
- Exponent $\delta = 5$.
  Definition: $|M(c_{cr}, H)| = |H|^{1/\delta}$.
  At $c_{\rm cr} = 1/4$ and small H, the equation $w'(z) = 0$ is solved by $z_1 = -1 + (2H)^{2/5}$. Correspondingly, $M(\frac{1}{4}, H) \propto H^{1/5}$.

The exponents satisfy the scaling identities of critical phenomena:

$$\begin{aligned} &\alpha + 2\beta + \gamma = 2 && \text{(Rushbrooke)} \\ &\delta - 1 = \frac{\gamma}{\beta} && \text{(Widom)} \\ &2 - \alpha = \nu d && \text{(Josephson)} \end{aligned}$$

critical exp	α	β	γ	δ	νd	γ_{str}
regular	0	1/8	7/4	15	2	–
random	-1	1/2	2	5	3	$-1/3$

Table: the critical exponents of the Ising model on regular 2d lattices and on random planar graphs. The latter fit the predictions of the theory by KPZ [13]. To test universality Boulatov and Kazakov also solved the Ising model on cubic graphs and obtained the same exponents.

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FIGURE 3. Vladimir Kazakov (Gorki 1954) obtained the PhD in 1981 at the Landau Inst. Since 1989 he is at l'École Normal Supérieure (Paris-6). His main interests are QFT, string theory, matrix models, statitical mechanics, integrability. Right: Ivan Kostov (Moscow State Univ., then Saclay).

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