NOTES ON RANDOM MATRICES

LESSON 1: BEAUTIFUL THEOREMS AND SPECIAL MATRICES

LUCA GUIDO MOLINARI

---

1. Beautiful theorems: *Cauchy, Bendixson and Hirsch, Perron and Frobenius, Schur, Cayley and Hamilton, Jordan, Polya Littlewood and Hardy, Fan, Gershgorin, Weyl, Bauer and Fike.*
2. Special matrices: *Vandermonde, Cauchy, Hilbert, Pascal, Discrete Fourier, Kac.*
3. Eigenvalues of random matrices: *the semicircle, the circle, and the elliptic laws.*

Determinants appeared much earlier than matrix algebra: Cramer's formula for solving systems of linear equations dates 1750. Laplace's expansion was devised in 1772, and Gaussian elimination appeared in 1801, to solve least squares problems in celestial mechanics. In 1812 Cauchy obtained new results about minors and adjoints in the context of quadratic forms and, with Binet, the multiplication theorem for determinants.

The term 'matrix' was introduced in 1848 by Sylvester, to name an array of numbers, and matrix algebra was formalized by Arthur Cayley in 1855. He showed that compositions of linear transformations provide the rule of matrix multiplication, and obtained the inverse of a matrix. The Cayley-Hamilton theorem was reported in his *Memoir on the Theory of Matrices* (1858), for small matrices.

Camille Jordan introduced the important canonical form of a matrix in his *Treatise on substitutions and algebraic equations* (1870). His student André-Louis Cholesky, in solving least squares problems in geodesy, devised an algorithm to factor a positive definite matrix as $LL^\dagger$, where L is lower triangular (1910). The method was generalized in 1938 by Tadeusz Banachiewicz who introduced LUP factorization of any square matrix, into a lower and an upper triangular matrices, where L has units in the diagonal, and a permutation P [27].
John Francis (1961 [12]) introduced the QR decomposition, where Q is a rotation and R is a square upper triangular matrix with positive diagonal entries.
At the dawn of digital computers, John von Neumann and Herman Goldstine introduced condition numbers to analyze round-off errors (1947). A famous reference text is G. Golub and C. Van Loan, *Matrix Computations* (1966).

1. Beautiful theorems

In the study of quadratic forms, Cauchy proved that the eigenvalues of a real symmetric matrix are real, and the following:

Date: 18 Jan 2018; revised July 2021.

Theorem 1.1 (**Interlacing theorem, Cauchy, 1829**).
Let H be a Hermitian $n\times n$ matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$, and A a principal submatrix of size $n-1$ with eigenvalues $\mu_1 \ge \cdots \ge \mu_{n-1}$, then the eigenvalues of A and H interlace:

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \ldots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n \tag{1}$$

Proof. The exchange of one row and column brings the matrix H to the form:

$$H = \begin{bmatrix} A & b \\ b^\dagger & c \end{bmatrix}$$

Let $H = U\Lambda U^\dagger$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, then, for $t \notin Sp(H)$:

$$[(tI_n - H)^{-1}]_{nn} = \frac{\det[tI_{n-1} - A]}{\det[tI_n - H]} = \sum_{k=1}^{n} \frac{|U_{nk}|^2}{t - \lambda_k}$$

As a function of t it diverges at the points λ_i and it decreases on each interval $(\lambda_{i+1}, \lambda_i)$. Then it has a zero in each interval: $\mu_i \in [\lambda_{i+1}, \lambda_i]$. These zeros are the eigenvalues of A. □

A simple proof by Fisk [11] is based on the following theorem by Hermite: *The roots of polynomials p_n, q_{n-1} interlace if and only if all roots of $p_n + tq_{n-1}$ are real for all real t.* Then, note that:

$$\det \begin{bmatrix} A - zI_{n-1} & b \\ b^\dagger & c - z + t \end{bmatrix} = \det \begin{bmatrix} A - zI_{n-1} & b \\ b^\dagger & c - z \end{bmatrix} + \det \begin{bmatrix} A - zI_{n-1} & b \\ 0 & t \end{bmatrix}$$

The left hand side polynomial always has real roots, and equals $\det[H - zI_n] + t\det[A - zI_{n-1}]$. Then the roots of H and A interlace.

The theorem can be generalised to the eigenvalues $\mu_1 \ge \cdots \ge \mu_m$ of a principal sub-matrix of size $m \times m$: $\lambda_{k+n-m} \le \mu_k \le \lambda_k$ $(k = 1, \ldots, m)$.

The *numerical range* $\rho(X)$ of a matrix X is the set of values $\{u^\dagger X u,\ u \in \mathbb{C}^n,\ \|u\| = 1\}$. The set is convex (Toeplitz-Hausdorff).

Theorem 1.2 (**Bendixson, 1902; Hirsch, 1902**, [7]).
If $x+iy$ is an eigenvalue of a complex matrix X, then $x \in \rho(\mathrm{Re}X)$ and $y \in \rho(\mathrm{Im}X)$.

Proof. If $Xu = (x + iy)u$ and $\|u\| = 1$, then $x + iy = u^\dagger(\mathrm{Re}X + i\mathrm{Im}X)u$. □

The inclusion can be strengthened in various ways.

Theorem 1.3 (**Oskar Perron, 1907**, [19]).
Let a square matrix X have strictly positive matrix elements ($X_{ij} > 0$). Then the eigenvalue r with largest modulus is real, positive and simple and the eigenvector is real with positive components.

The theorem was extended by Ferdinand Frobenius (1912) to $X_{ij} \ge 0$.

Theorem 1.4 (**Issai Schur, 1909**, [15]).
If X is any real (complex) matrix, there are a orthogonal (unitary) matrix and a upper triangular matrix T such that $X = OTO^T$ ($X = UTU^\dagger$).

Proof. By induction, assume that the statement is true for dimension $n-1$. Consider an eigenvector $Xu = zu$, $u \in \mathbb{C}^n$ with unit norm, and a unitary matrix W whose first column is the eigenvector. Then $XW = [zu, \cdots]$ and

$$W^\dagger X W = \begin{bmatrix} z & b^\dagger \\ 0 & X' \end{bmatrix}$$

b is some vector. The sub matrix X' can be written as $VT'V^\dagger$. Then:

$$X = W \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} z & b^\dagger V \\ 0 & T' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V^\dagger \end{bmatrix} W^\dagger$$

The diagonal elements of the triangular matrix are the eigenvalues of X. $\square$

The fundamental Cayley-Hamilton theorem was proven for any size n by Ferdinand Frobenius, in 1878. The proof is simple with Schur's representation.

Theorem 1.5 (**Cayley-Hamilton theorem**). *Every square complex matrix X satisfies its own characteristic equation $p(X) = 0$, $p(z) = \det(z - X)$.*

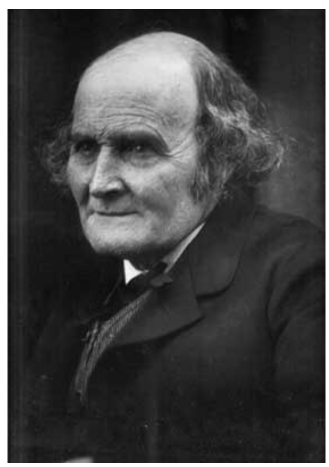

FIGURE 1. James Joseph Sylvester (London 1814 - London 1897) contributed to matrix theory, invariant theory, number theory, partition theory, and combinatorics. He played a leadership role in American mathematics in the later half of the 19th century as professor at Johns Hopkins University and as founder of the American Journal of Mathematics. At his death, he was professor at Oxford.

FIGURE 2. Arthur Cayley (Richmond UK 1821 - Cambridge 1895) was student at Trinity College at age 17. After a period as a lawyer (as De Morgan, while Sylvester was an actuary for some time) in 1863 he earned the Sadleirian chair of mathematics at Cambridge University. In addition to algebra, Cayley made fundamental contributions to algebraic geometry (study of manifolds of zeros of multivariate polynomials), in particular cubic surfaces.

Proof. Let $X = UTU^\dagger$ with T triangular and $T_{ii} = z_i$. Then $p(X) = U\prod_k(T - z_k)U^\dagger$. The triangular matrix $T - z_k$ has a diagonal zero at site k; their product is zero. □

An extension is Phillip's theorem (1919): Given matrices $A_1, ..., A_q$ and $B_1, ..., B_q$ such $[B_i, B_j] = 0$ and $A_1B_1 + ... + A_qB_q = 0$, define the polynomial $p(x_1, ..., x_n) = \det(x_1A_1 + ... + x_nA_n)$. Then $p(B_1, ..., B_n) = 0$.

Theorem 1.6 (Jordan's normal form, 1870). *Any complex square matrix A is similar to the block diagonal matrix*

$$A = P\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}P^{-1}, \quad J_k = \begin{bmatrix} z_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & z_k \end{bmatrix}$$

where J_k is a Jordan block of A, z_k is an eigenvalue of A. The same eigenvalue may appear in more blocks. The decomposition is unique, up to permutations of the blocks.

If J is a Jordan matrix of size n and parameter z then J^k is upper triangular with constant diagonals $\{z^k, \binom{k}{1}z^{k-1}, \binom{k}{2}z^{k-2}, ..., \binom{k}{n-1}z^{k-n+1}\}$ (the diagonals are null for $k - n + 1 < 0$).

The following material is taken from the beautiful book [20].
A real matrix P is *doubly stochastic* if $P_{ij} \geq 0$ and the sum of the elements in every column as well as every row is unity. The last conditions mean that if e is the vector with all $e_i = 1$, then $Pe = e$ and $e^TP = e^T$. Permutation matrices are of this sort. It is simple to see that the product and the convex combination of doubly stochastic matrices is doubly stochastic. The following theorem by Birkhoff (1946) states: any doubly-stochastic matrix is the convex combination of permutation matrices[1].
For every $n \times n$ matrix A with positive entries there exist diagonal matrices D, D' such that DAD' is doubly stochastic (Sinkhorn 1964).

Definition 1.7. For real $x_1 \geq ... \geq x_n$ and $y_1 \geq ... \geq y_n$, then x is majorized by y ($x \prec y$) if: $x_1 \leq y_1$, $x_1 + x_2 \leq y_1 + y_2$, ...,

$$x_1 + \cdots + x_{n-1} \leq y_1 + \cdots + y_{n-1},$$
$$x_1 + \cdots + x_n = y_1 + \cdots + y_n$$

If, in the last line, $x_1 + \cdots + x_n \leq y_1 + \cdots + y_n$, then $x \prec_w y$.

Theorem 1.8 (Polya, Littlewood and Hardy, 1929). *$x \prec y$ iff $x = Py$ where P is a $n \times n$ doubly stochastic matrix.*

A real function ϕ on $\mathscr{D} \subset \mathbb{R}^n$ is Schur-convex on $\mathscr{D}$ if $x \prec y \Rightarrow \phi(x_i) \leq \phi(y_i)$ for all i, $x, y \in \mathscr{D}$. Example: $\phi(x) = \sum_{k=1}^n g(x_i)$ is Schur-convex on I^n if $g : I \to \mathbb{R}$ is convex on the interval I (Schur, Hardy & Littlewood, Polya). A notable case is $g(x) = x\log x$ on $I = [0, 1]$. Since $(1/n, \ldots, 1/n) \prec (p_1 \geq \cdots \geq p_n) \prec (1, 0, \ldots, 0)$, it implies that $0 \leq -\sum_{k=1}^n p_j \log p_j \leq \log n$ (Shannon entropy).

[1] a similar theorem holds for unitary matrices with row and column sums equal to one; see arXiv:1812.08833

Theorem 1.9 (Issai Schur, 1923).
If H is complex Hermitian with diagonal elements $h_1 \geq h_2 \geq \cdots \geq h_n$ and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then $h \prec \lambda$.

Proof. $H = U\Lambda U^\dagger$, then $H_{ii} = \sum_k |U_{ik}|^2\lambda_k$. The matrix $p_{ik} = |U_{ik}|^2$ is doubly stochastic. $\square$

Schur's statement is equivalent to the following theorem by Ky Fan [4]:
If $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of a Hermitian matrix, then

$$\sum_{j=1}^{k} \lambda_j = \max_{\{u_j\}} \sum_{j=1}^{k} (u_j|Xu_j)$$

where max is taken on all othonormal bases $\{u_1 \dots u_k\}$ in $\mathbb{C}^n$. A nice corollary, valid for Hermitian matrices, is $\lambda(A+B) \prec \lambda(A) + \lambda(B)$.
Several majorizations of eigenvalues and singular values of matrices have been obtained by Hermann Weyl, Ky Fan, Lidskii (see the great book by Bhatia, [4]). An example is:

Theorem 1.10 (Ky Fan, 1950, [4]). *For any matrix X:* $\mathrm{Re}\lambda(X) \prec \lambda(\mathrm{Re}X)$

Now comes an amazingly simple and general theorem, that inspired much research on localisation of eigenvalues [31]:

Theorem 1.11 (Gershgorin circles, 1931, [31]).
If z is an eigenvalue of a complex matrix X, then it belongs to a Gershgorin circle $|z - X_{kk}| \leq \sum_j' |X_{kj}|$.

Proof. In the eigenvalue equation $\sum_{m\neq j} X_{jm}u_m = (z - X_{jj})u_j$, $j = 1, ..., n$, let u_k be the component with highest modulus, and choose the phase factor such that $u_k > 0$. Then: $(z - X_{kk})u_k = \sum_{m\neq k} X_{km}u_m$. Divide by u_k and result follows. $\square$

Since $Sp(X)$, the spectrum of X, coincides with $Sp(X^T)$, the result holds also if column, instead of row, elements are summed.
X di strictly (row) *diagonal-dominant* if $|X_{kk}| \geq \sum_m' |X_{km}|$, $\forall k$. In this case no Gershgorin circle contains the origin, and the matrix X is invertible.
The following fixes a unique circle for all eigenvalues $z_j \in Sp(X)$ (see [4] 1.6.16):

$$\left| z_j - \tfrac{1}{n}\mathrm{tr}X \right|^2 \leq \tfrac{n-1}{n} \left[\mathrm{tr}(X^\dagger X) - \tfrac{1}{n}|\mathrm{tr}X|^2 \right]$$

For Hermitian matrices, singular values σ_i and eigenvalues are related by $\sigma_i = |\lambda_i|$. For the general matrix the following holds:

Theorem 1.12 (Hermann Weyl, 1949).
For any $n \times n$ complex matrix with eigenvalues $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and singular values $\sigma_1 \geq \cdots \geq \sigma_n$:

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} \sigma_j \quad (k = 1 \dots n-1); \qquad \prod_{j=1}^{n} |\lambda_j| \leq \prod_{j=1}^{n} \sigma_j \tag{2}$$

If no eigenvalue is zero, it means that $\log|\lambda| \prec \log\sigma$.

Related results are: for any non-singular complex matrix: $|\lambda| \prec_w \sigma$, $|\lambda^2| \prec_w \sigma^2$. In particular (Schur, 1909): $\sum_{k=1}^n |\lambda_k|^2 \le \sum_{k=1}^n \sigma_k^2 = \mathrm{tr} X^\dagger X$.
As a particular case we can obtain the arithmetic-geometric mean inequality ([20]):

$$\frac{1}{n}\sum_{k=1}^n x_k \ge (x_1 \cdots x_n)^{1/n}$$

by applying Schur's inequality to the following matrix, whose eigenvalues solve $\lambda^n = \sqrt{x_1 \dots x_n}$

$$\begin{bmatrix} 0 & \sqrt{x_1} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{x_2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{x_{n-1}} \\ \sqrt{x_n} & 0 & 0 & \dots & 0 \end{bmatrix}$$

Theorem 1.13 (Brauer, 1952, [5]). *If A is a $n \times n$ matrix with eigenvalues $\lambda, \lambda_2, ..., \lambda_n$ with $A\mathbf{u} = \lambda \mathbf{u}$, then the eigenvalues of $A + \mathbf{u}\mathbf{v}^\dagger$ are $\lambda + \mathbf{u}^\dagger \mathbf{v}, \lambda_2, ..., \lambda_n$ (a simple proof in* arXiv:2110.01376*).*

If $A \in \mathbb{C}^{n\times n}$ is invertible, and $\|A\|$ is the sup-norm, the *condition number* of A is:

$$\mathrm{Cond}(A) = \|A\| \cdot \|A^{-1}\| \qquad \text{(1947, von Neumann and Goldstine)}$$

Since $1 = \|A^{-1}A\| \le \|A\|\|A^{-1}\|$, it is $\mathrm{Cond}(A) \le 1$. It is a useful quantity in the study of perturbation of eigenvalues: how much do the eigenvalues of $A+B$ deviate from those of A? A general bound is (see also [2]):

Theorem 1.14 (**Bauer and Fike, 1960**, [3]). *If $A = V\Lambda V^{-1}$, for any eigenvalue μ of $A + B$:*

$$\min_{1\le j\le n} |\mu(A+B) - \lambda_j(A)| \le \mathrm{Cond}(V)\,\|B\| \tag{3}$$

Proof. The proof is simple: if $\mu \in Sp(A+B)$ then $0 = \det(A+B-\mu) = \det(\Lambda + V^{-1}BV - \mu) = \det(\Lambda - \mu)\det[1 + (\Lambda-\mu)^{-1}V^{-1}BV]$. Then -1 is an eigenvalue of $(\Lambda-\mu)^{-1}V^{-1}BV$. It is $1 \le \|(\Lambda-\mu)^{-1}V^{-1}BV\| \le \|(\Lambda-\mu)^{-1}\|\,\|V^{-1}BV\| \le \max_k \frac{1}{|\lambda_k-\mu|}\|V\|\|V^{-1}\|\|B\| \le (\min|\lambda_k-\mu|)^{-1}\|B\|\mathrm{Cond}(V)$. □

2. Some special matrices

2.1. **(Alexandre Théophile) Vandermonde matrix (∼1770).** It is a basic matrix in linear algebra and matrix theory. Given complex numbers $z_1, ..., z_n$ it is:

$$[z_j^{k-1}] = \begin{bmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ 1 & z_2 & & z_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \dots & z_n^{n-1} \end{bmatrix} \tag{4}$$

$$\boxed{\Delta(z_1 \dots z_n) \equiv \det[z_j^{k-1}] = \prod_{j>k}(z_j - z_k)} \tag{5}$$

Example 1: given $X \in \mathbb{C}^{n\times n}$ with distinct eigenvalues, compute the matrix $\exp[tX]$. By Cayley-Hamilton theorem, X solves $0 = \prod_k(X - z_k)$. Then all powers X^p,

$p \geq n$ are linear combinations of powers $n-1, ..., 0$, and $\exp[tX] = \sum_{j=0}^{n-1} c_j(t) X^j$. Since eigenvectors are linearly independent: $\exp[tz_k] = \sum_{j=0}^{n-1} [z_k^j] c_j(t)$. Inversion of the Vandermonde matrix (see Knuth, *The art of computer programming*) gives the coefficients c_j.

Example 2: given $(x_1, y_1), ..., (x_n, y_n)$ find the polynomial $p(x) = a_0 + ... + a_{n-1} x^{n-1}$ such that $p(x_k) = y_k$, i.e. $\sum_{j=0}^{n-1} a_j x_k^j = y_k$. The problem can be solved by inverting the Vandermonde matrix $[x_k^j]$. However, it is more practical to obtain it as a combination of Lagrange interpolating polynomials:

$$p(x) = y_1 L_1(x) + ... + y_n L_n(x), \qquad L_k(x) = \prod_{j \neq k} \frac{x - x_k}{x_j - x_k} \tag{6}$$

2.2. **Cauchy matrices (1841).** Given numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, the $n \times n$ Cauchy matrices are [17]

$$C_{ij} = \frac{1}{x_i + y_j}, \qquad \det C = \frac{\Delta(x_1, \ldots, x_n)\Delta(y_1, \ldots, y_n)}{\prod_{i \leq j}(x_j + y_i)} \tag{7}$$

$$C'_{ij} = \frac{1}{1 + x_i y_j}, \qquad \det C' = \frac{\Delta(x_1, \ldots, x_n)\Delta(y_1, \ldots, y_n)}{\prod_{i \leq j}(1 + x_i y_j)} \tag{8}$$


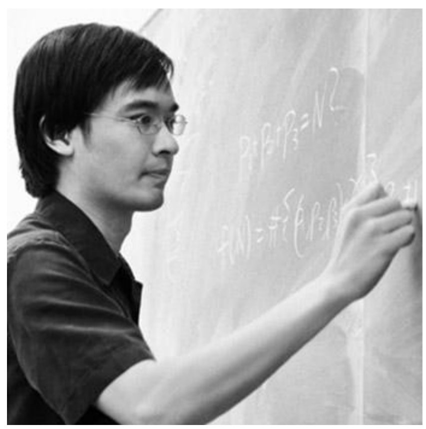

FIGURE 3. Issai Schur (Russia 1895 - Tel-Aviv 1941) graduated in Berlin under the supervision of Frobenius and Fuchs, and taught at Berlin's university. In despite of the racial laws (1933) his fame and the intervention of Erhard Schmidt enabled him to lecture for two more years. Then Polya invited him to Zurich for lecturing on group representation theory. He migrated to Palestine.

FIGURE 4. Terence Tao (Adelaide Australia, 1975) won the gold medal of the Int. Math. Olympiad at age 13. Admitted to Princeton with a letter by Erdös, he got his PhD, supervised by Elias Stein. Professor at UCLA, he won of a Fields medal (2006, for his contributions to partial differential equations, combinatorics, harmonic analysis and additive number theory) and many other awards.

The inverse matrix is $C^{-1}_{ij} = (x_j + y_i)L_j(y_i)L_i(x_j)$, with the Lagrange polynomials evaluated with parameters $\{x_i\}$ and $\{y_i\}$ [26].

2.3. **Hilbert matrix (1894).**

$$H_{ij} = \left(\frac{1}{i+j-1}\right), \qquad \det H_n = \frac{(1!2!\dots(n-1)!)^4}{1!2!\dots(2n-1)!} \tag{9}$$

It is a Hankel matrix (i.e. matrix elements depend on $i+j$). For finite size, the entries of the inverse matrix are integers. H is a positive matrix:

$$u^T H u = \sum_{i,j=1}^{n} \int_0^1 dx x^{i+j-2} u_i u_j = \int_0^1 dx (\sum_{k=1}^{n} u_k x^{k-1})^2 \geq 0$$

The matrix $n=\infty$ is the matrix representation of the operator $(Hf)(z) = \int_0^1 dt f(t)(1-zt)^{-1}$ on holomorphic functions $\mathrm{L}^2(\mathbb{D})$ in the basis z^k. Magnus (1950) proved the continuous spectrum $[0,\pi]$ (see [9]).

2.4. **Discrete Fourier transform.** It is the $n \times n$ symmetric unitary matrix

$$\mathcal{F}_{jk} = \frac{1}{\sqrt{n}} \exp[i\frac{2\pi}{n} jk] \tag{10}$$

Since $\mathcal{F}^4 = 1$ the eigenvalues of $\mathcal{F}$ are ± 1, $\pm i$. The trace is a Gauss sum: $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \exp[i\frac{2\pi}{n}k^2] = \frac{1}{2}(1+i)[1+i^{2n}]$. (See [22])

2.5. **Pascal matrices.** There are various forms of Pascal matrices, and a vast literature [18, 6]. The lower triangular Pascal matrix P_n has size $n+1$, and each row contains the binomial coefficients of the Pascal triangle. For example:

$$P_4 = \begin{bmatrix} \binom{0}{0} & & & & \\ \binom{1}{0} & \binom{1}{1} & & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \exp \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & 3 & 0 & \\ & & & 4 & 0 \end{bmatrix}$$

The inverse of P_n has binomial elements with alternating signs. For example:

$$P_4^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

This property is connected to a nice inversion theorem:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \ \Rightarrow\ b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \tag{11}$$

($\mathbf{a} = P_n \mathbf{b}$, then $\mathbf{b} = P_n^{-1}\mathbf{a}$). The product $Q_n = P_n P_n^T$ is the symmetric Pascal matrix:

$$Q_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

It is a positive definite matrix, with unit determinant and matrix elements $(Q_n)_{ij} = \binom{i+j}{j}$ $i, j = 0, ..., n$. Its positive eigenvalues come in pairs $(q, 1/q)$. If the size is odd (n even), an eigenvalue is 1. The coefficients of the polynomial $\det(z + Q_n)$ are positive and are the sequence A045912 in OEIS (the on-line encyclopedia of integer sequences). The polynomial with $z = 1$ was evaluated by Andrews [1]:

$$\det(1 + Q_{n-1}) = 2\prod_{k=1}^{n-1} \frac{(3k+2)(2k+2)_k}{(3k+1)(k+1)_k} \tag{12}$$

where $a_k = a(a+1)\cdots(a+k-1)$. The polynomials with $z = -1, e^{i\pi/3}, e^{i\pi/6}$ were evaluated by Ciucu et al. [10] to count lozenge coverings of an hexagon.
The **Carlitz matrix** A_n mirrors the matrix P_n. For example:

$$A_4 = \begin{bmatrix} & & & & 1 \\ & & & 1 & 1 \\ & & 1 & 2 & 1 \\ & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}, \qquad A_4^{-1} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -1 & 3 & -3 & 1 & \\ 1 & -2 & 1 & & \\ -1 & 1 & & & \\ 1 & & & & \end{bmatrix}$$

It is $\det(z - A_n) = \prod_{j=0}^{n}(z - \varphi^j \varphi'^{n-j})$, where $\varphi = \frac{1}{2}(1+\sqrt{5})$ and $\varphi' = \frac{1}{2}(1-\sqrt{5})$ [8, 25], and $A_n A_n^T = Q_n$.

2.6. **Mark Kac Matrix.** The following matrix has a long history [30] beginning with Sylvester, who found the eigenvalues. Marc Kac obtained the left and right eigenvectors, in a study of a discrete random walk in an interval, with position-dependent probability for left or right unit displacement [16]. An example of Kac matrix is

$$S_4 = \begin{bmatrix} 0 & 1 & & & \\ 4 & 0 & 2 & & \\ & 3 & 0 & 3 & \\ & & 2 & 0 & 4 \\ & & & 1 & 0 \end{bmatrix}$$

The eigenvalues are integers. For example: $\det(S_4 - z) = z(z^2 - 2^2)(z^2 - 4^2)$, $\det(S_5 - z) = (z^2 - 1^2)(z^2 - 3^2)(z^2 - 5^2)$. Kac showed that if $S_n\mathbf{u} = z\mathbf{u}$ then, with $u_1 = 1$, the generating function for the components is:

$$1 + tu_2 + ... + t^{n-1}u_n + t^n = (1-t)^{(n-z)/2}(1+t)^{(n+z)/2}$$

In the example $S_4 U_4 = U_4 Z_4$, $Z_4 = \text{diag}(4, 2, 0, -2, -4)$ and

$$U_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

3. Distributions of eigenvalues of large random matrices

Random Matrix models are, in general, specified by a set of matrices with a probability measure. Several systems in physics, biology, economy, finance, ... show behaviours that can be described through properties of eigenvalues or eigenvectors of random matrices, or products, or truncations of random matrices, whose randomness and symmetries are suggested by the features of the system being studied.

We review some important distribution laws for the eigenvalues of large random matrices.

3.1. **The Semicircle law.** The semicircle law was obtained by Eugene Wigner in 1955 for symmetric banded matrices with elements $\pm v$ with random sign (bordered matrices). He then extended it to full symmetric matrices with independent random elements with zero mean, equal variance, and finite higher moments [32].

Theorem 3.1 (Wigner, 1955)**.** *Let $X \in \mathbb{C}^{n\times n}$ be Hermitian, with random matrix elements $\{X_{jk}\}_{j<k}$ that are i.i.d. with zero mean and finite variance σ, while diagonal elements X_{kk} are i.i.d. random variables with bounded mean and variance. Then the eigenvalues of $\frac{1}{\sqrt{n}}X_{ij}$ tend in distribution to the semicircle law:*

$$\rho_{sc} = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}$$

For real symmetric matrices with i.i.d. matrix elements with variance σ, it is $\langle\lambda^2\rangle = n\langle X_{ij}^2\rangle$. The second moment of the semicircle of radius R is $R^2/4$, then: $R = 2\sigma\sqrt{n}$. For complex Hermitian matrices with variance σ of i.i.d. real and imaginary parts of matrix elements it is $n\langle\lambda^2\rangle = (2n^2-1)\sigma^2$. Then the radius is $R = 4\sigma\sqrt{2n}$.

The semicircle law holds also for large Hermitian Band Random matrices with same hypothesis on matrix elements H_{ij} in a band $|i-j| < W/2$, $W = cN^\nu$ where $0 < c < 1$ and $0 < \nu < 1$ (Bogachev, Molchanov, Pastur, 1991; Anderson and Zeitouni, 2004).

3.2. **The circle law.** The circle law for complex eigenvalues was obtained by Jean Ginibre in 1965, for real, complex and quaternionic matrices with Gaussian distribution [13]. Vyacheslav Girko (1984) generalised to i.i.d. matrix elements of mean 0 and variance $1/n$.

For a real matrix the number of real eigenvalues is $\sim\sqrt{n}$ (Edelman). For comparison, the number of real roots of a random polynomial with normally distributed real coefficients, scales as $\sim\log n$ (Kac).

Theorem 3.2 (Terence Tao and Van Vu, 2010, [29])**.** *Let $X \in \mathbb{C}^{n\times n}$ have matrix elements X_{jk} that are i.i.d. random variables with zero mean and unit variance.*

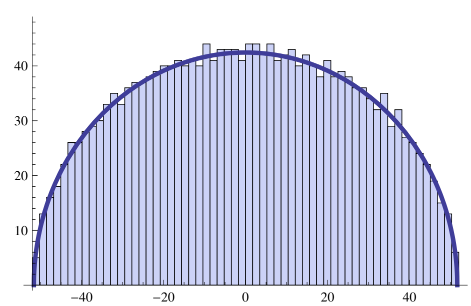

FIGURE 5. Semicircle Law: histogram of the eigenvalues of two random Hermitian matrices $n = 1000$ with Re and Im of matrix elements $i < j$ uniformly distributed in $[-1, 1]$ and real diagonal elements with same distribution.

Then the eigenvalue density converges to

$$\rho_c(z) = \frac{1}{\pi}\theta(1 - |z|)$$

The real (and imaginary) part of the eigenvalues has the semicircle distribution: $\rho(x) = \int dy \frac{1}{\pi}\theta(1 - x^2 - y^2) = \frac{2}{\pi}\sqrt{1 - x^2}$.

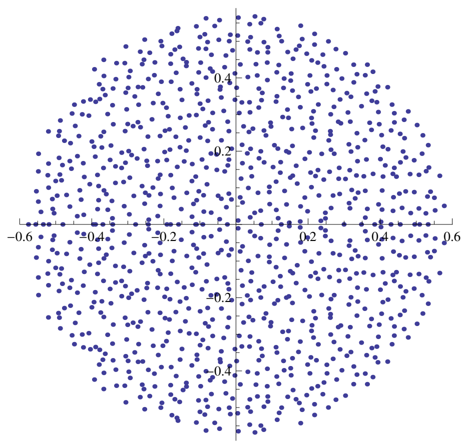

FIGURE 6. The circle distribution of the eigenvalues of 2 random real matrices $n = 1000$ with matrix elements uniformly distributed in $[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}]$.

3.3. The elliptic law.

Theorem 3.3 (Hoi Nguyen and Sean O'Rourke, 2015 [24], see also [28])**.** *Let* $X \in \mathbb{R}^{n \times n}$ *with random pairs* $\{X_{jk}, X_{kj}\}$ *that are i.i.d. in* $\mathbb{R}^2$*, with* X_{12} *and* X_{21} *with zero mean and unit variance,* $\mathbb{E}(X_{12}X_{21}) = \tau$*,* $\tau < 1$*. Then the eigenvalue density converges to uniform on an ellipse:*

$$\rho_{x,y} = \frac{1}{\pi(1 - \tau^2)}, \qquad \frac{x^2}{(1 + \tau)^2} + \frac{y^2}{(1 - \tau)^2} \le 1$$

REFERENCES

[1] G. E. Andrews, *Plane Partitions (III): The Weak Macdonald Conjecture*, lnventiones Math. **53** (1979) 193–225.

[2] J. Banks, A. Kulkarni, S. Mukherjee and N. Srivastava, *Gaussian regularization of the pseudospectrum and Davies' conjecture*, (2019)

[3] F. L. Bauer and C. T. Fike, *Norms and exclusion theorems*, Numer. Math. **2** n.1 (1960) 137–141.

[4] Rajendra Bhatia, *Matrix analysis*, (Springer, 1997).

[5] Alfred Brauer, *Limits for the characteristic roots of a matrix. IV: Applications to stochastic matrices*, Duke Math. J. **19** (1952) 75–91.

[6] R. Brawer and M. Pirovino, *The linear algebra of the Pascal matrix*, LAA **174** (1992). 13–23.

[7] E. T. Browne, *The characteristic roots of a matrix*, Bull. Amer. Math. Soc. **36** n.10 (1930) 705–710.

[8] L. Carlitz, *The characteristic polynomial of a certain matrix of binomial coefficients*, Fibonacci Quarterly **3** 81–89

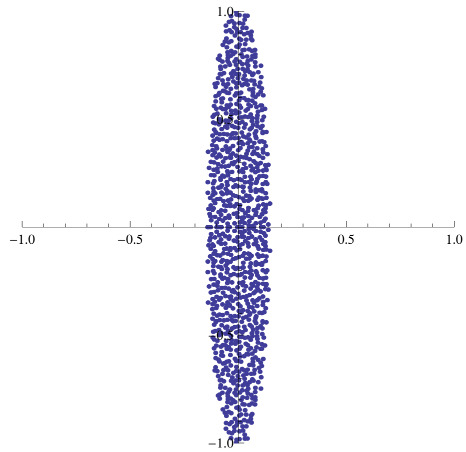

FIGURE 7. The elliptic distribution of the eigenvalues of a random real matrix $n = 1000$ with matrix elements uniformly distributed in $[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}]$ such that $X_{ij}X_{ji} < 0$, $i \neq j$.

[9] Man-Duen Choi, *Trick or treats with the Hilbert matrix*, Am. Math. Monthly **90** n.5 (1983) 301-312.

[10] M. Ciucu, T. Eisenkölbl, C. Krattenthaler, D. Zare, *Enumeration of lozenge tilings of hexagons with a central triangular hole*, J. Combin. Theory Ser. A **95** (2001) 251–334.

[11] S. Fisk, *A very short proof of Cauchy's interlace theorem for eigenvalues of Hermitian matrices*, arXiv:math/0502408 [math.CA].

[12] J. G. F. Francis, *The QR transformation I, II*, The Computer Journal **4** (1961) 265–271, 332–345.

[13] J. Ginibre, *Statistical Ensembles of Complex, Quaternion, and Real Matrices*, J. Math. Phys. **6** n.3 (1965) 440-449.

[14] V. Girko, *The circle law*, Theory of Probability & Its Applications, **29** n.4 (1985) 694–706.

[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. (Cambridge, 2013).

[16] Mark Kac, *Random walk and the theory of Brownian motion*, Amer. Math. Monthly **54** n.7 (1947) 369–390.

[17] Christian Krattenthaler, *Advanced determinant calculus*, 1999 (67pp), arXix:math/9902004 [Math.CO].

[18] W. F. Lunnon, *The Pascal matrix*, The Fibonacci Quaterly **15** (1977) 201–204.
scanned https://www.fq.math.ca/Scanned/15-3/lunnon.pdf

[19] C. R. MacCluer, *The many proofs and applications of Perron's theorem*, SIAM Review **42** n.3 (2000) 487–498.

[20] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: theory of majorization and its applications*, 2nd ed. (Springer, 2009).

[21] Madan Lal Mehta, *Matrix theory, selected topics and useful results*, (Les Editions de Physique, 1977).

[22] M. L. Mehta, *Eigenvalues and eigenvectors of the finite Fourier transform*, J. Math. Phys. **28** n.4 (1987) 781–785.

[23] S. A. Molchanov, L. A. Pastur, A. M. Khorunzhii, *Limiting eigenvalue distribution for band random matrices*, Theor. Math. Phys. **90** n.2 (1992) 108–118.

[24] Hoi Nguyen and Sean O'Rourke, *The Elliptic Law*, International Mathematics Research Notices, 2015 (17) (2015) 7620–7689. arXiv:1208.5883

[25] O. K. Pashaev and M. Ozvatan, *Golden binomials and Carlitz characteristic polynomials*, arXiv:2012.11001[math.QA]

[26] Samuel Schechter, *On the Inversion of Certain Matrices*, Mathematical Tables and Other Aids to Computation **13** n.66 (1959) 73–77.

[27] A. Schwarzenberg-Czerny, *On matrix factorization and efficient least squares solution*, Astronomy and Astrophysics Supplement **110** (1995) 405.
[28] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, *Spectrum of large random asymmetric matrices*, Phys. Rev. Lett. **60** (19) (1988) 1895.
[29] T. Tao and V. Vu, *Random matrices: Universality of ESDs and the circular law*, arXiv:0807.4898.
[30] Olga Taussky and John Todd, *Another look at a matrix of Marc Kac*, LAA **150** (1991) 341–360.
[31] R. S. Varga, *Gershgorin and his circles* (Springer-Verlag, 2004).
[32] E. P. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. Math. **63** n.3 (1955) 548–564; *On the distribution of the roots of certain symmetric matrices*, Ann. Math. **67** n.2 (1958) 325–327.
[33] Fuzhen Zhang, *Matrix theory, basic results and techniques* (Springer, 1999).