NOTES ON RANDOM MATRICES

LUCA GUIDO MOLINARI

LESSON 3

1. The Laplace-Beltrami operator

In a (pseudo)-Riemannian space, with metric g_{ij} and Christoffel connection Γ^k_{ij}, the Laplace operator of a scalar field is the scalar field (indices are summed)

$$\nabla^2\varphi = \nabla_j g^{jk}\nabla_k\varphi = \nabla_j g^{jk}\partial_k\varphi = \nabla_j g^{jk}\partial_k\varphi + \Gamma^j_{jl}g^{lk}\partial_k\varphi$$

Now we use the property of Christoffel symbols with two indices summed: $\Gamma^j_{jl} = \partial_l \log\sqrt{g}$, where $g = \det[g_{ij}]$. The result is the Laplace-Beltrami expression:

$$\boxed{\nabla^2\varphi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k} g^{kj}\sqrt{g}\frac{\partial}{\partial x^j}\varphi} \tag{1}$$

2. The Laplacian in the space of Hermitian matrices

A Hermitian matrix $n\times n$ is specified by the n^2 real coordinates H_{ii}, $\mathrm{Re}H_{ij}$ and $\mathrm{Im}H_{ij}$, $i<j$. The squared distance of two matrices $d(H,H')^2 = \mathrm{tr}(H-H')^2$ is invariant for unitary transformations $H\to UHU^\dagger$. For $H' = H + dH$:

$$ds^2 = \mathrm{tr}(dHdH) = \sum_i (dH_{ii})^2 + 2\sum_{i<j} d(\mathrm{Re}H_{ij})^2 + d(\mathrm{Im}H_{ij})^2$$

Thus the metric tensor is diagonal, with $g_{ii,ii} = 1$ for the n coordinates H_{ii}, and $g_{ij,ij} = 2$ for the other $n^2 - n$ coordinates. The determinant is $g = 2^{n(n-1)}$.
The invariant Laplacian (1) contains the inverse of the metric matrix. This gives

$$\nabla^2_H = \sum_{i=1..n}\frac{\partial^2}{\partial H_{ii}^2} + \frac{1}{2}\sum_{i<j}\frac{\partial^2}{\partial(\mathrm{Re}H_{ij})^2} + \frac{\partial^2}{\partial(\mathrm{Im}H_{ij})^2} \tag{2}$$

The change of coordinates $H = UXU^\dagger$ represents a matrix by its n eigenvalues and $n(n-1)$ parameters ξ_a for the unitary matrix $U\in U(n)/U(1)^n$ (for example, the Euler angles). The invariant distance was obtained in terms of the Hermitian matrix $dT = iU^\dagger dU$:

$$ds^2 = \sum_{i=1..n}(dx^i)^2 + \sum_{ij}(x_i - x_j)^2 dT_{ij}dT_{ji}$$

If we specify the parameters: $ds^2 = g_{ij}dx^i dx^j + g_{ab}d\xi^a\, d\xi^b$ with

$$g_{ab}(x,\xi) = \sum_{i,j}(x_i - x_j)^2(\partial_a T_{ij})(\partial_b T^*_{ij}) = 2\sum_{i<j}(x_i-x_j)^2\mathrm{Re}(\partial_a T_{ij})(\partial_b T^*_{ij}). \tag{3}$$

Date: 18 april 2019.

The new metric tensor is block-diagonal, with unit matrix in the eigenvalue sector and matrix g_{ab} in the unitary sector. The latter is a matrix product: $g_{ab} = (VDV^\dagger)_{ab}$ with $V_{a,ij} = \partial_a T_{ij}$ and D the diagonal matrix with diagonal elements $D_{ij} = (x_i - x_j)^2$. Then: $\sqrt{g} = \Delta^2 |\det V|$, where $\Delta = \prod_{i>j}(x_i - x_j)$ is the Vandermonde determinant of the eigenvalues.
The inverse of the metric tensor is block-diagonal, with a unit block in the eigenvalue sector, and $(VDV^\dagger)^{-1}$ in the unitary sector. The Laplace-Beltrami operator is:

$$\nabla_H^2 = \nabla_X^2 + \sum_{i<j} \frac{1}{(x_i - x_j)^2} L_{ij}(\xi, \partial_\xi) \tag{4}$$

$$L_{ij} = \frac{1}{|\det V|} \sum_a \frac{\partial}{\partial \xi_a} (V^{-1})^*_{a,ij} |\det V| \sum_b (V^{-1})_{ij,b} \frac{\partial}{\partial \xi_b} \tag{5}$$

where ∇_X^2 is the Laplace-Beltrami operator in the eigenvalue sector:

$$\nabla_X^2 \varphi(X) = \frac{1}{\Delta^2} \sum_{k=1..n} \frac{\partial}{\partial x^k} \Delta^2 \frac{\partial}{\partial x^k} \varphi \tag{6}$$

$$= \frac{1}{\Delta} \sum_{k=1..n} \left(\frac{\partial}{\partial x^k} \right)^2 (\Delta \varphi)(X) \tag{7}$$

The last equality follows from the special property $\nabla_X^2 \Delta = 0$.

Example 2.1. Hermitian matrices 2×2 have 4 real parameters: two eigenvalues x_1, x_2 and two parameters for $U(2)$ matrices whose elements of the first row are real positive:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad 0 \le \theta \le \tfrac{\pi}{2}, \quad 0 \le \alpha < 2\pi.$$

One evaluates: $(dT)_{12} = i(U^\dagger dU)_{12} = \sin\theta\cos\theta d\alpha + i d\theta$. Then: $\partial_\alpha T_{12} = \sin\theta\cos\theta$, $\partial_\theta T_{12} = i$. The off diagonal terms of the metric tensor in the unitary tensor $VDV^\dagger$ vanish, and it is:

$$ds^2 = dx_1^2 + dx_2^2 + 2(x_1 - x_2)^2 [\sin^2\theta\cos^2\theta (d\alpha)^2 + (d\theta)^2].$$

Then: $\sqrt{g} = \Delta^2 \sin(2\theta)$, $\Delta = x_2 - x_1$, and $d^4 H = \Delta^2 dx_1 dx_2 \ \sin(2\theta) d\alpha d\theta$. The volume of $U(2)/U(1)^2$ is $\int_0^{\pi/2} d\theta \sin(2\theta) \int_0^{2\pi} d\alpha = 2\pi$.
The Laplace-Beltrami operator is:

$$\nabla^2 = \nabla_X^2 + \frac{1}{\Delta^2} \left[\frac{1}{2\sin^2\theta\cos^2\theta} \frac{\partial^2}{\partial\alpha^2} + \frac{1}{2\sin(2\theta)} \frac{\partial}{\partial\theta} \sin(2\theta) \frac{\partial}{\partial\theta} \right]$$

3. Harish-Chandra formula for the unitary group (1957)

The Harish-Chandra[1] formula is a useful integral on the Haar measure of the unitary group [10]. It was rediscovered by Itzykson and Zuber [9] for the 2-matrix model.

[1]Harish-Chandra (India 1923, Princeton 1983) after the Master degree in physics and research with Bhabha, moved to Cambridge as research student of Dirac. He became interested in representation theory, harmonic analysis, and received several awards.

Theorem 3.1. *Let A, B be Hermitian matrices with eigenvalues x_i and y_i, then:*

$$\boxed{\int_{\mathsf{U}(n)} dU \exp[\frac{1}{t}\mathrm{tr}(AUBU^\dagger)] = t^{\frac{1}{2}n(n-1)} \frac{\det[\exp\frac{1}{t}(x_i y_j)]}{\Delta(x)\Delta(y)} \prod_{j=0}^{n-1} j!} \tag{8}$$

In the integral the matrices A and B can be taken real diagonal.

Proof. This simple proof was later provided by Edouard Brézin [2]. Consider the Laplacian operator ∇_A^2 in the space of Hermitian matrices. Its eigenfunctions are plane waves:

$$\nabla_A^2 \exp[i\mathrm{tr}(BA)] = -\mathrm{tr}(B^2)\exp[i\mathrm{tr}(BA)]$$

The eigenvalue is unchanged if the Hermitian matrix B is replaced by $U^\dagger BU$, or in the continuous superposition:

$$\Psi_B(X) = \int dU \exp[i\mathrm{tr}(BUAU^\dagger)]$$

$\Psi_B(A)$ only depends on the eigenvalues x_i of A, and is a symmetric function of them. It solves the eigenvalue equation with ∇_A^2 being replaced by ∇_X^2:

$$\frac{1}{\Delta(X)} \sum_{k=1..n} \frac{\partial^2}{\partial x_k^2} \Delta(X)\Psi_B(X) = -\mathrm{tr}(B^2)\Psi_B(X)$$

The function $\Delta(X)\Psi_B(X)$ is totally antisymmetric in the eigenvalues x_i and totally symmetric in the eigenvalues y_j of B. It can be obtained as the Slater determinant of the elementary eigenfunctions $\psi_j''(x) = -y_j^2\psi_j(x)$, where $\sum_j y_j^2 = \mathrm{tr}(B^2)$, i.e. $\Delta(X)\Delta(Y)\Psi_B(X) = C\det[\exp(iy_jx_k)]$, where the factor $\Delta(Y)$ has been included for symmetry in the exchange of Y with X, and C is a constant. □

A weaker statement, that is useful for the solution of the 2-matrix model, is the following one. The proof is interesting:

Proposition 3.2 (Mehta, [13])**.**
For symmetric functions $\xi_0(Y)$ of the eigenvalues, such that integrals exist, it is:

$$0 = \int dY \xi_0(Y) \left[\int dU \exp[\tfrac{1}{t}\mathrm{tr}(XU^\dagger YU)] - (2\pi t)^{\frac{1}{2}n(n-1)} \frac{\exp[\frac{1}{t}\sum_i x_i y_i]}{\Delta(x)\Delta(y)}\right] \tag{9}$$

Proof. The Heat Equation with diffusion constant D,

$$\left(\frac{\partial}{\partial t} - D\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}\right) u(x,t) = 0$$

and initial condition $u(x,0) = u_0(x)$, is solved for $t > 0$ with the aid of the Heat kernel $K_t(x)$:

$$u(x,t) = (K_t \star u_0)(x) = \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} dy\, e^{-\frac{1}{4Dt}\sum_k (x_k-y_k)^2} u_0(y)$$

The Heat kernel is a special solution that, for $t \to 0$ is a delta function.
Now, consider the Heat equation in the space of Hermitian matrices $\mathbb{R}^{n^2}$, for a scalar function $\xi(A,t)$, where A denotes the set of n^2 variables:

$$(\frac{\partial}{\partial t} - \frac{1}{2}\nabla_A^2)\xi(A,t) = 0 \tag{10}$$

The solution, with initial condition $\xi(A,0)=\xi_0(A)$ is provided by the Heat kernel:

$$\xi(A,t)=\int dB\,\frac{1}{(2\pi t)^{n^2/2}}\exp[-\tfrac{1}{2t}\mathrm{tr}(A-B)^2]\xi_0(B) \tag{11}$$

Suppose that the initial condition only depends on the eigenvalues Y of B. We change variables $B=WYW^\dagger$, $dB=dWdY\Delta^2(Y)$:

$$\xi(A,t)=\frac{1}{(2\pi t)^{n^2/2}}\int dY\Delta(Y)^2\,\xi_0(Y)\int dW\exp[-\tfrac{1}{2t}\mathrm{tr}(V^\dagger XV-W^\dagger YW)^2]$$

Now, $\mathrm{tr}(V^\dagger XV-W^\dagger YW)^2=\mathrm{tr}[X-(WV^\dagger)^\dagger Y(WV^\dagger)]^2$. We put $WV^\dagger=U$ and use the property of the Haar measure $dW=dU$. Then the solution only depends on eigenvalues:

$$\xi(X,t)=\frac{e^{-\frac{1}{2t}\mathrm{tr}X^2}}{(2\pi t)^{n^2/2}}\int dY\Delta(Y)^2\,\xi_0(Y)e^{-\frac{1}{2t}\mathrm{tr}Y^2}\int dU\exp[\tfrac{1}{t}\mathrm{tr}(XU^\dagger YU)] \tag{12}$$

The function $\xi(X,t)$ is also solution of the Heat equation $(\partial_t-\frac{1}{2}\nabla_X^2)\xi(X,t)$. Then $\Delta(X)\xi(X,t)$ solves the equation with the Laplace-Beltrami operator (6):

$$\frac{\partial\xi}{\partial t}-\frac{1}{2\Delta}\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}\Delta\xi(X,t)=0 \tag{13}$$

with initial condition $\Delta\xi_0$. The solution is provided by the Heat kernel in $\mathbb{R}^n$:

$$\Delta(X)\xi(X,t)=\frac{1}{(2\pi t)^{n/2}}\int dY\Delta(Y)\xi_0(Y)\exp[-\tfrac{1}{2t}\sum_{i=1}^n(x_i-y_i)^2]$$

By equating this expression for $\xi(X,t)$ with the expression (12), and absorbing the factors $\Delta(Y)^2\exp[-\frac{1}{2t}\mathrm{tr}Y^2]$ in the arbitrary function $\xi_0(Y)$, we obtain Mehta's result. □

4. The two-matrix model (Hermitian)

After the solution of 1-matrix models with orthogonal polynomials, the 2-matrix model was attacked by Claude Iztykson and Jean-Bernard Zuber (1980). They reduced the integral of two non-commuting matrices in $2n^2$ variables to an integral in $2n$ eigenvalues by their rediscovery of the Harish-Chandra integral [9]. The saddle point approximation was then used. Madan Lal Mehta formally solved the model by introducing the bi-orthogonal polynomials, and gave an expression for the planar free energy [13]. The model is:

$$\mathbb{Z}_n(c,g)=\int dAdB\;e^{-n\mathrm{tr}(A^2+B^2-2cAB+4gA^4+4gB^4)} \tag{14}$$

A,B are Hermitian $n\times n$ matrices, $0<c<1$.
In 1986 Kazakov showed that it corresponds to the Ising model on random planar graphs with magnetic field $H=0$ [11]. Soon after, with Boulatov, they generalized and solved the model with couplings ge^H and ge^{-H} for A and B to describe the Ising model with constant H [5].

For all n, the integral (14) is amenable to the eigenvalues x_i and y_i of A and B by means of the integral (9). If $A = UXU^\dagger$ and $B = VYV^\dagger$, it is:

$$\mathbb{Z}_n = \int dXdY\Delta^2(X)\Delta^2(Y)e^{-n\sum_i(x_i^2+y_i^2+4gx_i^4+4gy_i^4)}\int dWe^{2nc\,\mathrm{tr}(WXW^\dagger Y)}$$
$$\approx \int dXdY\Delta^2(X)\Delta^2(Y)e^{-n\sum_i(x_i^2+y_i^2+4gx_i^4+4gy_i^4)}\frac{e^{2nc\sum_i x_iy_i}}{\Delta(X)\Delta(Y)}$$

Omitting irrelevant factors that cancel with normalization ($g = 0$), and introducing the potential $v(x,y) = x^2 + y^2 - 2c\,xy + 4gx^4 + 4gy^4$, we arrive at:

$$\mathbb{Z}_n = \int dXdY\Delta(X)\Delta(Y)e^{-n\sum_i v(x_i,y_i)} \tag{15}$$

Bi-orthogonal polynomials. In the integral, the Vandermonde determinant $\Delta(X)$ can be rewritten as $\det[P_m(x_k)]_{k=1..n}^{m=0..n-1}$, where $P_m(x)$ are arbitrary monic polynomials of degree $m = 0...n-1$. The same is done for $\Delta(Y)$, with polynomials $Q_m(y_k)$. Then:

$$\mathbb{Z}_n = \int dXdY\det[P_r(x_k)]\det[Q_s(y_k)]e^{-n\sum_i v(x_i,y_i)}$$
$$= \epsilon_{r_1,\dots,r_n}\epsilon_{s_1,\dots,s_n}\prod_{k=1\dots n}\int dx_kdy_ke^{-nv(x_k,y_k)}P_{r_k}(x_k)Q_{s_k}(y_k).$$

The partition functionis formally evaluated by choosing bi-orthogonal polynomials:

$$\int dxdy\,w(x,y)\,P_k(x)\,Q_j(y) = h_k\delta_{kj}, \qquad w(x,y) = e^{-nv(x,y)}$$
$$\mathbb{Z}_n = n!h_0h_1...h_{n-1} \tag{16}$$

The polynomials are fully determined by the conditions of being monic and bi-orthogonal. Since $v(-x,y) = v(x,-y)$ they have definite parity, and since $v(x,y) = v(y,x)$, the polynomials P_k and Q_k are the same.

Bi-orthogonal polynomials do not have the simple recursive properties of orthogonal ones. In this case with polynomial potential, they satisfy:

Proposition 4.1.

$$xP_k(x) = P_{k+1}(x) + R_kP_{k-1}(x) + S_kP_{k-3}(x) \tag{17}$$

Proof. Suppose that the expansion of $xP_k(x)$ contains a term $T_kP_{k-5}(x)$. Multiply (17) by $P_{k-5}(y)$ and integrate with the measure. It is $\int dxdy\,w(x,y)\,xP_k(x)P_{k-5}(y) = T_kh_{k-5}$. The first integral is dealt with the identity:

$$\frac{1}{2n}\frac{\partial w}{\partial x} + (x + 8gx^3 - cy)w(x,y) = 0 \tag{18}$$

Then $cT_kh_{k-5} = \int dxdy\,w(x,y)(y + 8gy^3)P_k(x)P_{k-5}(y) = 0$. □

The following proposition gives conditions for the zeros of bi-orthogonal polynomials to be real and simple (see Mehta, Random Matrices, 3rd ed.):

Proposition 4.2. *If $w(x,y) > 0$, all moments $\langle x^jy^k\rangle$ are finite, $\det\langle x^jy^k\rangle \neq 0$, $i,j = 0,...,n$ for all n, $\det w(x_i,y_j) > 0$ for $x_1 < ... < x_n$, $y_1 < ... < y_n$, then the bi-orthogonal polynomials $\int dxdy\,w(x,y)\,p_j(x)q_k(x) = h_k\delta_{jk}$ have real and simple zeros in the respective supports of $w(x,y)$.*

With $f_k = h_k/h_{k-1}$ the partition function (16) becomes $\mathbb{Z}_n = n! h_0^n f_1^{n-1} \cdots f_{n-1}$. The normalized free energy is

$$F_n(c,g) = -\frac{1}{n^2}\log\frac{\mathbb{Z}_n(c,g)}{\mathbb{Z}_n(c,0)} = -\frac{1}{n^2}\log\frac{h_0(c,g)}{h_0(c,0)} - \frac{1}{n}\sum_{k=0}^{n-1}(1-\frac{k}{n})\log\frac{f_k(c,g)}{f_k(c,0)} \tag{19}$$

The coefficients f_k, R_k and S_k solve recursive relations that depend on w, with initial conditions that must be evaluated:

Proposition 4.3.

$$cS_k = 8g f_k f_{k-1} f_{k-2} \tag{20}$$

$$cR_k = [1 + 8g(R_{k+1} + R_k + R_{k-1})] f_k \tag{21}$$

$$\frac{k}{2n} = -c\, f_k + R_k + 8g[R_k(R_{k+1} + R_k + R_{k-1}) + S_{k+2} + S_{k+1} + S_k] \tag{22}$$

Proof. 20): multiply (17) by $c\, P_{k-3}(y)$ and integrate with the weight, then use (18)

$$cS_k h_{k-3} = 8g \int dxdy w(x,y) y^3 P_{k-3} P_k(x) = 8g h_k$$

21): multiply (17) by $c\, P_{k-1}(y)$ and integrate with the weight, and use (18):

$$cR_k h_{k-1} = \int dxdy w(x,y)(y + 8gy^3) P_{k-1}(y) P_k(x) = h_k[1 + 8g(R_{k+1} + R_k + R_{k-1})]$$

22): multiply (17) by $c\, P_{k+1}(y)$ and integrate with the weight, and use (18):

$$\begin{aligned} ch_{k+1} &= \int dxdy w(x,y)(y + 8gy^3) P_{k+1}(y) P_k(x) - \tfrac{k+1}{2n} h_k \\ &= 8g h_k [R_{k+1}(R_{k+2} + R_k + R_{k-2}) + S_{k+3} + S_{k+2} + S_{k+1}] + R_{k+1} h_k - \tfrac{k+1}{2n} h_k \end{aligned}$$

The initial conditions are found in the construction of the first polynomials: $S_0=0$, $R_0 = 0$, $h_1 = \langle xy\rangle$, $S_1 = 0$, $R_1 = \langle x^2\rangle/h_0$, $h_2 = \langle x^2y^2\rangle - \langle x^2\rangle^2/h_0$, ... □

The large n limit. The large n limit is obtained by interpolating the coefficients f_k, R_k and S_k with continuous functions. The interpolation depends on the initial conditions. In the simplest case, single interpolating functions suffice e.g. $f_k = f(k/N) = f(x)$, $0 \le x \le 1$. For the double-well potential, two interpolating functions are needed for each coefficient [14].

The equations (20)–(22) become algebraic. For single functions:

$$\begin{aligned} &cS(x) = 8g f^3(x) \\ &cR(x) = [1 + 24gR(x)] f(x) \\ &cx + 2c^2 f(x) - 24(4g)^2 f^3(x) = 2c\, R(x)[1 + 24gR(x)] \end{aligned}$$

They give, for $g = 0$: $f^0(x) = \frac{1}{2}cx/(1-c^2)$ and, for non-zero g:

$$\frac{x}{2} = -cf(x) + \frac{12(4g)^2}{c} f^3(x) + c\frac{f(x)}{[c - 24gf(x)]^2}$$

The planar free energy becomes the integral:

$$F_{\rm pl}(c,g) = -\int_0^1 dx(1-x)\log\frac{f(x)}{f^0(x)} \tag{23}$$

The expansion in g counts the connected vacuum planar diagrams:

$$F_{\rm pl}(c,g) = \sum_{V=1..\infty} g^V F_{{\rm pl},V}(c) = g\frac{4}{(1-c^2)^2} - g^2\frac{4c^4+32c^2+36}{(1-c^2)^4} + \dots$$

The diagrams contributing to $V=1,2$ can be identified in fig.1. They are 4 and 72 (in the quartic 1-matrix model they are 2 and 18. Graphs with two vertices A or B total 4 and 36. The other 36 come from replacing a vertex A with a vertex B in all ways). The planar series has a finite radius of convergence, that allows to determine the large V behaviour of $F_{{\rm pl},V}(c)$ (thermodynamic limit). The radius is obtained from the critical values g_{cr} of $F_{\rm pl}(g,c)$ nearest to the origin, at given value c. As c changes, the critical points may collide and exghange, and this manifests as a phase transition. This analysis of the two-matrix solution was done by Kazakov, and the phase transition is the magnetic transition of the Ising model.

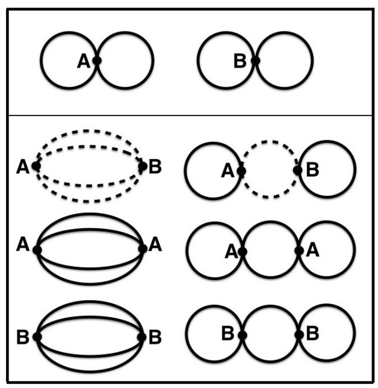

FIGURE 1. The first and second order diagrams. Dashed (AB) or full (AA, BB) lines correspond to factors $c/(1-c^2)$ or $1/(1-c^2)$. Vertices A, B have weight g. Multiplicities are given by combinatorics of planar edging of vertices.

4.1. **The phase transition.** With $z(x) = (24g/c)f(x)$ the fifth order equation becomes:

$$4gx = -\tfrac{1}{3}c^2 z + \tfrac{1}{9}c^2 z^3 + \tfrac{1}{3}\frac{z}{(1-z)^2} \equiv w(z) \tag{24}$$

The function $w(z)$ is plotted in fig.2.
An integration by parts of the integral for the planar free energy (23) gives:

$$\begin{aligned}
F_{\rm pl}(c,g) &= -\frac{1}{2}\log\frac{f(1)}{f^0(1)} + \int_0^1 dx\frac{f'(x)}{f(x)}(x-\tfrac{1}{2}x^2) - \frac{3}{4}\\
&= -\frac{1}{2}\log\frac{\zeta(1-c^2)}{12g} + \frac{1}{4g}\int_0^\zeta \frac{dz}{z}w(z) - \frac{1}{32g^2}\int_0^\zeta \frac{dz}{z}w^2(z) - \frac{3}{4}\\
&= -\frac{1}{2}\log\frac{\zeta(1-c^2)}{12g} + \frac{c^2}{108g}\zeta(\zeta^2-9) - \frac{1}{12g}\frac{\zeta}{\zeta-1} - \frac{3}{4}\\
&\quad - \frac{c^4}{32\cdot 54g^2}\zeta^2(3-\zeta^2+\frac{\zeta^4}{9}) - \frac{c^2}{27\cdot 32g^2}\frac{\zeta^2(\zeta+3)}{\zeta-1} - \frac{1}{54\cdot 32g^2}\frac{\zeta^2(\zeta-3)}{(\zeta-1)^3}.
\end{aligned}$$

The result is a free energy that depends on $\zeta = z(1)$, solution of the fifth order equation $4g = w(\zeta, c)$. In the change from $0 \le x \le 1$ to $0 \le z \le \zeta$ the function $z(x)$ is one-to-one. This breaks down at the critical points $w'(z) = 0$, listed below:

$w'(z) = 0$	$w(z)$	$c < 1/4$	$c > 1/4$
$z_c = -1$	$\frac{2}{9}c^2 - \frac{1}{12}$	min	max
$z_- = 1 - 1/\sqrt{c}$	$\frac{2}{9}(3c - c^2 - 2\sqrt{c})$	max	min
$z_+ = 1 + 1/\sqrt{c}$	$\frac{2}{9}(3c - c^2 + 2\sqrt{c})$	min	min
$1 \pm i/\sqrt{c}$	complex		

The critical points. For $c = 1/4$, $z_c = z_-$ is a triple zero of $w(z)$ and $w(z_c) = -5/72$.

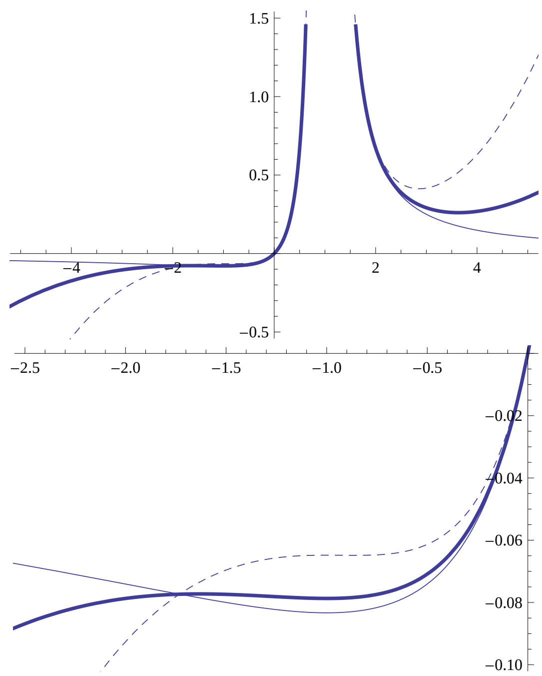

FIGURE 2. The function $w(z)$ for $c = 0.01$ (thin), $c = 1/4$ (thick) and $c = 0.5$ (dashed). For $z \to \pm\infty$ $w(z) \to \pm\infty$. The point $z_+ = 1 + 1/\sqrt{c}$ is a local minimum. The other extrema $z = -1$ and $z_- = 1 - 1/\sqrt{c}$ have a dual behaviour (see Table).

As $4gx$ varies from 0 to $4g$, there is a real intersection $4gx = w(z)$ moving continuously from $z(0) = 0$ to some value ζ solving $4g = w(\zeta)$. For $g > 0$ it is $4g \le w(z_+)$. For $g < 0$ there are two phases:

$c < 1/4$: $|4g| < |w(z_c)|$ (the local minimum). $z(x)$ moves from 0 to $\zeta = -1$.

$c > 1/4$: $|4g| < |w(z_-)|$ (the local minimum). $z(x)$ moves from 0 to $\zeta = z_-$.

The planar free energy and its first two derivatives are continuous functions of g, while $\partial_g^3 F_{\rm pl}(c, g)$ is divergent where $w'(z) = 0$ (note that $\partial_\zeta F = 0$, when $4g = w(\zeta)$ is used).

The asymptotic behaviours of the counting numbers of diagrams with V vertices (up to factors V^p) are:

$$F_{\mathrm{pl},V}(c) \approx \left(\frac{(1-c^2)^2}{c|g_{\mathrm{cr}}(c)|}\right)^V , \quad 4g_{\mathrm{cr}}(c) = \begin{cases} \frac{2}{9}c^2 - \frac{1}{12} & c < 1/4 \\ \frac{2}{9}(3c - c^2 - 2\sqrt{c}) & c > 1/4 \end{cases} \tag{25}$$

The discussion of the Ising model, and the modifications needed to allow for the magnetic field, are the subject of another lesson.

5. The 1-Hermitian matrix model in d=1, large n

The solution of the 1-matrix in $d = 1$ model appeared in the same paper of the saddle-point solution of the model in $d = 0$ (Brézin et al. [3]). Both solutions were rediscussed for the double-well case (Cicuta et al.[7]).

The partition function of the 1-matrix model in one dimension (time)

$$Z = \int dH(t) \exp[-\int_0^\beta \mathrm{tr}(\tfrac{1}{2}\dot{H}(t)^2 + \tfrac{1}{2}m^2 H(t)^2 + \frac{g}{n}H^4)] = e^{-n^2\beta E}$$

describes the thermal equilibrium for n^2 particles with positions $H_{ii}(t)$, $\mathrm{Re}H_{ij}(t)$ and $\mathrm{Im}H_{ij}(t)$ $(i < j)$, with Hamiltonian $\mathcal{H} = -\frac{1}{2}\nabla_H^2 + \frac{1}{2}m^2\mathrm{tr}H^2 + g/n\mathrm{tr}H^4$, where ∇_H^2 is the Laplacian in matrix space (2). The Hamiltonian is invariant for the change of coordinates $H' = UHU^\dagger$.

The ground state is searched in the singlet sector of symmetric functions $\phi(x_1, \ldots, x_n)$, where the eigenvalue equation of $\mathcal{H}$ now involves n bosons

$$-\tfrac{1}{2}\nabla_X^2\phi + \sum\nolimits_{k=1}^n (\tfrac{1}{2}m^2 x_k^2 + \frac{g}{n}x_k^4)\phi = E_0\phi \tag{26}$$

With $\phi = \psi/\Delta$ the Laplacian takes the simpler form (7) and the Schrödinger equation becomes separable, $\psi = \prod \psi_i(x_i)$ where ψ_i are the eigenfunctions of a 1-particle problem:

$$\left[-\tfrac{1}{2}\frac{d^2}{dx^2} + \tfrac{1}{2}m^2x^2 + \frac{g}{n}x^4\right]\psi_i(x) = \epsilon_i\psi_i(x)$$

Since ϕ is totally symmetric, then ψ is antisymmetric, and is the Slater determinant $\psi(x_1 \ldots x_n) = \det[\psi_i(x_j)]$, with eigenvalue $E_0 = \sum_{i=1..n} \epsilon_i$. For a large number n of particles, the sum is evaluated as an integral involving the density of states:

$$E_0 = \int d\epsilon\, \rho(\epsilon)\, \epsilon\, \theta(\epsilon_F - \epsilon), \quad n = \int d\epsilon\, \rho(\epsilon)\, \theta(\epsilon_F - \epsilon)$$

where ϵ_F is the Fermi energy. The density is related to $\mathcal{N}(\epsilon)$, the number of states with energy below ϵ, by $\mathcal{N}'(\epsilon) = \rho(\epsilon)$. For large energy, it is approximated by the semiclassical formula[2]:

$$\mathcal{N}(\epsilon) = \int \frac{dpdx}{2\pi}\theta(\epsilon - \tfrac{1}{2}p^2 - \tfrac{1}{2}m^2x^2 - \frac{g}{n}x^4) = 2\int_a^b \frac{dx}{2\pi}\sqrt{2\epsilon - m^2x^2 - 2(g/n)x^4}$$

where (a, b) is the range for classical motion (positive kinetic energy); for $m^2 > 0$, $a = 0$. The Fermi energy ϵ_F is determined by $n = \mathcal{N}(\epsilon_F)$ and the ground state energy is:

$$E_0 = \int_{\epsilon_0}^{\epsilon_F} d\epsilon\, \epsilon\mathcal{N}'(\epsilon) = n\epsilon_F - \int_{\epsilon_0}^{\epsilon_F} d\epsilon\, \mathcal{N}(\epsilon) = n\epsilon_F - \tfrac{2}{3}\int_a^b \frac{dx}{2\pi}(2\epsilon_F - m^2x^2 - 2\frac{g}{n}x^4)^{\frac{3}{2}}$$

[2] the volume in phase space enclosed by the constant energy surface, in Planck units.

The integrals are elliptic. The rescaling $\epsilon_F = ne_F$, $x = \sqrt{n}s$ gives the expected behaviour E_0 proportional to n^2. This ground state energy is the planar free energy of the matrix model. Its value with $n = 1$ is confronted with the 'exact' energy E of the anharmonic oscillator in $d = 1$ computed by Bender and Wu.

g	E	$E_{\rm pl}$
0.01	0.507	0.505
0.1	0.559	0.547
1	0.804	0.740
50	2.500	2.217
1000	6.694	5.915

Table 5. The ground state energy of the $d = 1$ anharmonic oscillator ($n = 1$) [1] and the planar energy $E_{\rm pl}$, for various values of g, $m = 1$ (from [3], [8]).

Marchesini and Onofri studied the excited states of the matrix Hamiltonian in the singlet and adjoint sectors. For singlets they obtained equally spaced eigenvalues [12]. The approximate planar propagator in $d = 1$ was studied by Canali et al. [6], and its poles were evaluated

References

[1] C. M. Bender and T. T. Wu, *Anharmonic oscillator*, Phys. Rev. **184** n.5 (1969) 1231–1260.

[2] E. Brézin, in: *Two Dimensional Quantum Gravity and Random Surfaces.* Gross, D.J., Piran, T., Weinberg, S. (ed.), Singapore: World Scientific (1992), p. 37.

[3] E. Brézin, C. Itzykson, G. Parisi and J. P. Zuber, *Planar diagrams*, Comm. Math. Phys. **59** (1978) 35–51.

[4] E. Brézin and S. Hikami, *An Extension of the HarishChandra-Itzykson-Zuber Integral*, Comm. Math. Phys. **235** (2003) 125–137.

[5] D. V. Boulatov and V. A. Kazakov, *The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents*, Phys. Lett. B **186** n.3-4 (1987) 379–384.

[6] C. M. Canali, G. M. Cicuta, L. Molinari and E. Montaldi, *The quantum mechanical planar propagator: from vector models to planar graphs*, Nucl. Phys. B **265** [FS15] (1986) 485–505.

[7] G. M. Cicuta, L. Molinari and E. Montaldi, *Large-N phase transitions in low dimensions*, Mod. Phys. Lett. A **1** n.2 (1986) 125–129.

[8] C. Itzykson and J.-M. Drouffe, Théorie statistique des champs, vol.2, Editions du CNRS (1989).

[9] C. Itzykson and J. B. Zuber, *The planar approximation (II)*, J. Math. Phys. **21** (1980) 411–421.

[10] Harish-Chandra, *Differential operators on a semi-simple Lie algebra*, Amer. J. Math. **79** (1957) 87–120.

[11] V. A. Kazakov, *Ising model on a dynamical planar random lattice: exact solution*, Phys. Lett. A **119** n.3 (1986) 140–144.

[12] G. Marchesini and E. Onofri, *Planar limit for SU(N) symmetric quantum dynamical systems*, J. Math. Phys. **21** (1980) 1103–1110.

[13] Madan Lal Mehta, *A method of integration over matrix variables*, Comm. Math. Phys. **79** (1981) 327–340.

[14] L. Molinari, *Phase structure of matrix models through orthogonal polynomials*, J. Phys. A: Math. Gen. **21** (1988) 1–6.

[15] P. J. Zinn-Justin and J. P. Zuber, *On some integrals over the U(n) unitary group and their large N limit*, J. Phys. A: Math. Gen. **36** n.12 (2003) 3173–3193.