NOTES ON RANDOM MATRICES

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LESSON 2

Many matrix models have been studied, and for different reasons [1]. Wishart ensembles (matrices $R^T R$ where R is random rectangular) were introduced in multivariate statistics. The Gaussian and Circular ensembles were deeply studied for spectral properties [26, 24], that well compared with those of complex systems in physics and chemistry, as nuclear resonances, molecular spectra, chaotic billiards, disordered wires (and zeros of Riemann's zeta function [8]). The solution of 1-matrix models with invariant non-Gaussian probability density, showed universal properties for the correlators at small scales, and allowed for the counting of Feynman graphs of different genera. The progress was possible for the connection with the theory of orthogonal polynomials [15]. The solution of a 2-matrix model as Ising model on random planar graphs, opened a field of statistical mechanics on random graphs, with universality and critical indices predicted by 2D quantum gravity. Products of random matrices are studied, as models for transfer matrices. The distributions of single (largest, second largest etc) levels shows connections with the theory of Painlevé differential equations. Almost every field in physics has been touched by RMT.

Now, we discuss 1-matrix models, described by the partition function

$$\mathcal{Z}_n = \int dH \exp[-n \operatorname{tr} v(H)]$$

where $v(x)$ is a polynomial independent of n, and the factor n is placed for a proper large n limit of the density of eigenvalues. The ensembles may consist of real symmetric, Hermitian or quaternionic self-dual matrices, and are invariant under the action of the orthogonal $\mathsf{O}(n)$, unitary $\mathsf{U}(n)$ or symplectic $\mathsf{Sp}(2n)$ groups.

The simplest choice $\operatorname{tr} v(H) = \frac{1}{2}\operatorname{tr}(H^2)$ implies that the free matrix elements are i.i.d. normal variables for $i < j$, and i.i.d. normal variables (with different variance) for $i = j$. The three ensembles are the GOE (Gaussian Orthogonal), GUE (Gaussian Unitary) and the GSE (Gaussian Symplectic). For GUE: $\operatorname{tr} v(H) = \frac{1}{2}\sum_k H_{kk}^2 + \sum_{j<k}(\mathrm{Re}H_{jk}^2 + \mathrm{Im}H_{jk}^2)$. Higher powers in the potential introduce correlations among matrix elements. For various reasons the unitary symmetry is simpler to handle, and we restrict to Hermitian models.

1. From matrix elements to eigenvalues

A Hermitian $n \times n$ matrix H is a point in the Euclidean space $\mathbb{R}^{n^2}$ of free parameters, with metric $ds^2 = \operatorname{tr}(dH^2)$, and volume element $dH = \prod_{i=1}^n dH_{ii} \prod_{i<j} d\mathrm{Re}H_{ij}$

Date: 18 apr 2019.

$d\mathrm{Im}H_{ij}$. The factorization $H = UXU^\dagger$ (where X is the diagonal matrix of eigenvalues and $U \in \mathsf{U}(n)/\mathsf{U}(1)^n$ is the unitary matrix of column eigenvectors with real positive first component) is a change of coordinates. We wish to express dH in terms of eigenvalues and eigenvectors.

Proposition 1.1.

$$dH = \Delta(x_1 \dots x_n)^2 \prod_{i=1}^n d\lambda_i \, dU \tag{1}$$

where $\Delta(x_1, \dots, x_n) = \prod_{i>j}(x_i - x_j)$.

Proof. The key point is the evaluation of the metric tensor in the new coordinates. $dH = dUXU^\dagger + U(dX)U^\dagger + UXdU^\dagger$. Since $(dU)U^\dagger + UdU^\dagger = 0$, then: $dH = U([U^\dagger dU, X] + dX)U^\dagger$. The Euclidean distance of neighbouring points (matrices) becomes: $ds^2 = \mathrm{tr}([U^\dagger dU, X] + dX)^2$. Now put $U^\dagger dU = idT$, where T is Hermitian.

$$ds^2 = \mathrm{tr}(i[dT, X] + dX)^2 = \mathrm{tr}(dX^2) - \mathrm{tr}[dT, X]^2 + 2i\mathrm{tr}(X[dT, X])$$

The last term vanishes by the cyclic property of the trace, and

$$\begin{aligned} ds^2 &= \sum_k (dx_k^2) - 2\mathrm{tr}(dTXdTX - (dT)^2 X^2) \\ &= \sum_k (dx_k^2) - 2\sum_{ij} (dT)_{ij}(dT)_{ji}(x_i x_j - x_i^2) \\ &= \sum_k (dx_k^2) + \sum_{ij} |dT_{ij}|^2 (x_i - x_j)^2 \\ &= \sum_k (dx_k^2) + 2\sum_{i<j} [d(\mathrm{Re}T_{ij})^2 + d(\mathrm{Im}T_{ij})^2](x_i - x_j)^2 \end{aligned}$$

The quadratic form defines the metric tensor, with unit elements in the sector of eigenvalues $x_1, \dots, x_n$. Therefore, the determinant factors: $\det g = \prod_{i<j}(x_i - x_j)^4 F^2(\xi)$, where ξ_a are the independent variables that parametrize unitary matrices. The volume element in the new variables has Jacobian equal to $\sqrt{g}$. □

For the three matrix ensembles the joint probability density of eigenvalues is:

$$p_\beta(x_1, \dots, x_n) = Z_{n,\beta}^{-1} \prod_{i>j} |x_i - x_j|^\beta \exp[-n \sum_{j=1}^n v(x_j)]$$

with exponents $\beta = 1, 2, 4$. Matrix ensembles for any value of β have been built in 2002 by Dumitriu and Edelman [17], with tridiagonal matrices with Gaussian distribution of diagonal elements, and χ distributions for off-diagonal elements.
For all n and β the normalization can be evaluated with Selberg's integral:

$$Z_{n,\beta} = (2\pi)^{\frac{n}{2}} \beta^{-\frac{n}{2} - \frac{1}{4}\beta n(n-1)} \Gamma(1 + \tfrac{1}{2}\beta)^{-n} \prod_{j=1}^n \Gamma(1 + \tfrac{1}{2}\beta j)$$

The Vandermonde factor is responsible for the *level repulsion*. The distribution of spacings among next eigenvalues, normalized with the average separation, behaves as s^β. The whole distribution is a complicate function, but it is well described by *Wigner's surmise* (exact for 2×2 matrices):

$$P(s) = C_\beta s^\beta e^{-K_\beta s^2} \qquad (K_1 = \tfrac{\pi}{4},\ K_2 = \tfrac{4}{\pi},\ K_4 = \tfrac{64}{9\pi}) \tag{2}$$

It is realized in the energy spectra of several physical systems, as billiards with or without magnetic field, molecular spectra, nuclear resonances ..., with β fixed by the symmetry of the Hamiltonian for time-reversal [24].
Ex: obtain C_β and K_β in $P(s)$ from the normalisation condition and $\langle s\rangle = 1$.

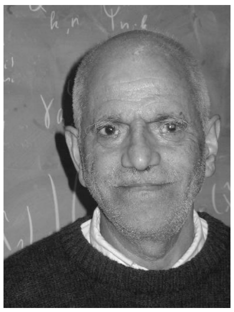

FIGURE 1. Madan Lal Mehta (Rajasthan 1932 - Udaipur 2006) joined in 1958 the Mathematical Physics group in Saclay and received his PhD in 1961 under Claude Bloch. After periods in Delhi, Princeton and Argonne Nat. Lab. he was permanent in Saclay until he retired and returned to India.

FIGURE 2. Éduard Brézin (Paris 1938) studied at École Polytechnique before his PhD and worked at the theory division of the Commissariat à l'énergie atomique in Saclay until 1986, when he became professor at the École Normale Supérieure. He contributed to high energy physics and used QFT to study macroscopic properties of matter, critical phenomena, and RMT with Hikami and Zee. Among various awards, he received the Dirac Medal 2011 of the ICTP with John Cardy and Alexander Zamolodchikov.

FIGURE 3. Anthony Zee (China 1945) graduated at Princeton and obtained his PhD from Harvard in 1970, supervised by Sidney Coleman. In 1970/72, 1977/78, he was at the Institute for Advanced Study. During his first year as assistant professor at Princeton, E.Witten was his teaching assistant and grader. He is now at Kavli Inst. for Theor. Phys. (UCSB). His researches range on particle physics, condensed matter physics, anomalies, RMT, superconductivity, QHE, evolutionary biology. He's author of successful textbooks.

2. **Hermitian 1-matrix models**

The partition function Z_n for the eigenvalues of Hermitian random matrices describes a gas of n equally charged particles in the real line that interact via a log potential and are confined by a polynomial potential $v(x)$ of degree ν:

$$Z_n = \int d^n x \; \exp\{-n^2[\tfrac{1}{n}\sum_k v(x_k) - \tfrac{1}{n^2}\sum_{j\neq i} \log|x_i - x_j|]\} \tag{3}$$

Recall that in 2 dimensions $\nabla^2 \log|z-z_0| = 2\pi\delta_2(z-z_0)$. Therefore, the point charges may be thought of as n parallel infinite, uniformly charged wires in 3D, lying in a plane.
Several methods of solution in the large-n limit have been devised: *Stieltjes method, saddle point, orthogonal polynomials, loop equations, replica trick, superintegrals ...* Here, the first three are reviewed.

3. The Stieltjes method

In the original problem (1885) Stieltjes was interested in the equilibrium configuration of n unit charges free to move in $(-1,1)$ in presence of charges $\frac{1}{2}(\alpha+1)$ and $\frac{1}{2}(\beta+1)$ at the ends. The charges interact via the log potential. He proved that the positions of the unit charges are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ (see [26], [29]).

Here we apply the method to n unit charges with log-interaction, in a polynomial potential well $v(x)$. The equilibrium configuration is the solution of the n equations

$$v'(x_k) = \frac{2}{n}\sum_{j\neq k}\frac{1}{x_k-x_j} \qquad k=1\dots n \tag{4}$$

Let $p(z)=\prod_k(z-x_k)$ be the polynomial whose zeros are the equilibrium positions. If the zeros are distinct, it is $\sum_j' \frac{2}{x_k-x_j} = p''(x_k)/p'(x_k)$. The equation becomes $p''(x_k) - n\,v'(x_k)p'(x_k)=0$. The polynomial $p''(z)-nv'(z)p'(z)$ has degree $n+\nu-2$, and n zeros match those of $p(z)$; therefore:

$$p''(z) - n\,v'(z)p'(z) + n^2\,q(z)p(z) = 0 \tag{5}$$

where $q(z)$ is a polynomial of degree $\nu-2$ whose coefficients ensure equality.
It is useful to introduce the function $F_n(z)$, and its large n limit

$$F_n(z) = \frac{1}{n}\frac{p'(z)}{p(z)} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{z-x_k} \quad \underset{n\to\infty}{\longrightarrow} \quad F(z) = \int_\sigma dx\frac{\rho(x)}{z-x} \tag{6}$$

where $\rho(x)$ is the normalized density of zeros with support σ. For large $|z|$:

$$F(z) = \sum_{k=1}^{\infty}\frac{1}{z^k}\int_\sigma dx\rho(x)\,x^{k-1} = \frac{1}{z}+\frac{\langle x\rangle}{z^2}+\frac{\langle x^2\rangle}{z^3}+\dots \tag{7}$$

Once F is found, the density is

$$\rho(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\operatorname{Im}F(x-i\epsilon) \qquad x\in\sigma \tag{8}$$

The differential equation (5) for $p(z)$ becomes a Riccati equation for $F(z)$:

$$\tfrac{1}{n}F_n'(z) + F_n(z)^2 - v'(z)F_n(z) + q(z) = 0. \tag{9}$$

For large n it is algebraic with solution

$$F(z) = \tfrac{1}{2}v'(z) - \tfrac{1}{2}\sqrt{v'(z)^2 - 4\,q(z)}$$

The requirement $F(z)\to 1/z$ for large $|z|$ fixes $q(z)$ with some residual freedom, and the expansion (7) gives the moments of the eigenvalue density[1].

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[1] Eq.(9) for F_n can be obtained directly by inserting eq.(4) in the algebraic identity $\sum_{i=1}^n \frac{1}{x_i-z}\sum_{j=1}^{n}{}' \frac{1}{x_i-x_j} = \frac{1}{2}\sum_{i=1}^n\frac{1}{(z-x_i)^2} - \frac{1}{2}\left[\sum_{i=1}^n\frac{1}{z-x_i}\right]^2$. The result, compared with (9), gives the expression: $q(z) = \frac{1}{n}\sum_{i=1}^n \frac{v'(x_i)-v'(z)}{x_i-z}$

Remark 3.1. *The support of the eigenvalue density may consist of one or more cuts in the real line: $\sigma = \cup_i [a_i, b_i]$. They result from coalescence of the zeros of $p(x)$, for $n \to \infty$. By setting $\psi(x) = e^{-\frac{n}{2}v(x)} p(x)$, the function solves the differential equation $-\psi''(x) + v_{\rm eff}(x)\psi(x) = 0$, with effective potential*

$$v_{\rm eff}(x) = \frac{n^2}{4}[v'(x)^2 - 4q(x) - \tfrac{2}{n}v''(x)]$$

that may display one or more wells. As the barrier heights of the wells increase with n, at the points $v'(x)^2 - 4q(x) = 0$ the density vanishes. *Thus*

$$\sigma = \{x : v'(x)^2 - 4q(x) < 0\}. \tag{10}$$

The requirement that $F(z)$ is analytic on $\mathbb{C}/\sigma$ imposes that the only branch points of F are the extrema of σ. *Then necessarily $v'(z)^2 - 4q(z) = g(z)^2 \prod_i (z - a_i)(z - b_i)$ where $g(z)$ is a non-vanishing real polynomial.*

Example 3.2. $v(x) = \frac{1}{2}x^2$, then $v_{\rm eff}(x) = x^2 - 4$. The eigenvalue density is Wigner's *semicircle law*

$$\boxed{\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}} \tag{11}$$

It is the density of the zeros of the Hermite polynomial $p_n(x) = H_n(x\sqrt{\frac{n}{2}})$ for $n \to \infty$. The large z expansion of $F(z) = \frac{1}{2}[z - \sqrt{z^2 - 4}]$ generates the moments of the semicircle distribution (Catalan numbers).

Minimization of the action and positivity of the density, fix the support σ. As the parameters of the potential are changed, in the large n limit, various phase transitions are possible: splitting of σ (because the density develops a zero inside it), change of the edge singularity from power $\frac{1}{2}$ to power $\frac{3}{2}$ or higher (because the density evolves a zero at an endpoint).

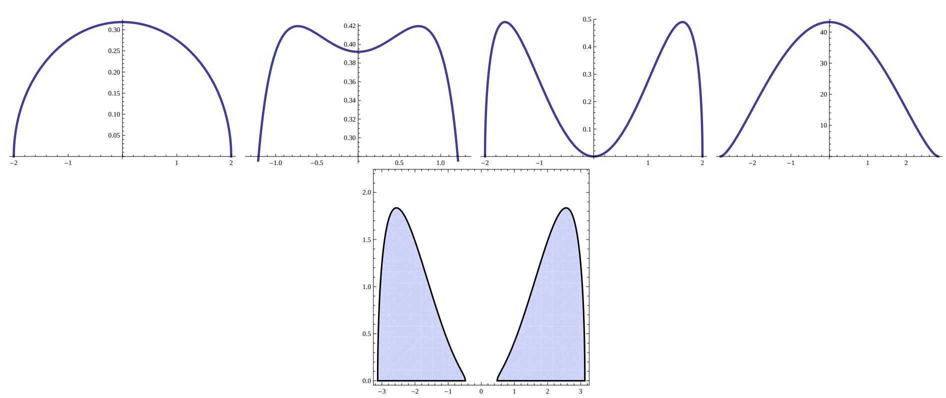

FIGURE 4. One-cut eigenvalue distributions for the quartic model: $g_4 = 0, \mu = 1$ (semicircle), $g_4 = 1/4, \mu = 1$, $g_4 = 1/4, \mu = -2$ (onset of two-cut solution), $g_4 = -1/48, \mu = 1$ (boundary of perturbative series - edge singularity). Two-cut eigenvalue distribution for the quartic model: $g_4 = 1/4, \mu = -2.5$.

Example 3.3. $v(x) = \frac{1}{2}\mu x^2 + g_4 x^4$. It is $q(z) = q_0 + q_1 z + 4g_4 z^2$ to match the highest power in (5). If $F(z) = \frac{1}{z} + \frac{1}{z^2}\langle x\rangle + \frac{1}{z^3}\langle x^2\rangle + \dots$, from eq.(9) and neglecting F' we obtain: $q_1 = 4g_4\langle x\rangle$ and $q_0 = 4g_4\langle x^2\rangle + \mu$, $\mu\langle x\rangle + 4g_4\langle x^3\rangle = 0$ etc.
For a symmetric density with support $\sigma = (-a, a)$ it is $q_1 = 0$ and

$$\begin{aligned}\rho(x) =& \frac{1}{2\pi}\sqrt{4q_0 + (16g_4 - \mu^2)x^2 - 8\mu g_4 x^4 - 16g_4^2 x^6} \\ =& \frac{2g_4}{\pi}(x^2 + b^2)\sqrt{a^2 - x^2}\end{aligned} \tag{12}$$

The condition that the argument of the square root factors as $16g_4^2(x^2+b^2)^2(a^2-x^2)$ comes from the analyticity of F. The factorization implies $12g_4(a/2)^4 + \mu(a/2)^2 - 1 = 0$ i.e. $6g_4a^2 = -\mu + \sqrt{\mu^2 + 48g_4}$ and $b^2 = \frac{1}{4}(\mu/g_4) + \frac{1}{2}a^2$. This is the solution obtained by Brézin et al. [9] with the saddle point method.
The end-point a^2 is singular at $g_4^* = -\frac{\mu^2}{48}$, that determines the radius of convergence of the planar perturbative series. At this critical point, the argument in (12) is a perfect cube and the density changes its edge behaviour:

$$\rho(x) = \frac{\mu^2}{24\pi}(\tfrac{8}{\mu} - x^2)^{3/2}$$

This critical value g_4^* will reappear in the double scaling limit, with orthogonal polynomials.

Another critical point appears when $b^2 = 0$, at $\mu_{cr} = -4\sqrt{g_4}$. A negative value μ corresponds to a double-well potential $v(x)$. For $\mu \le \mu_{cr}$ the density (12) would become negative in $x = 0$ and a two-cut phase $\sigma_2 = (-a, -b) \cup (b, a)$ must be considered [11], with new density

$$\rho(x) = \frac{4g_4}{2\pi}|x|\sqrt{(a^2 - x^2)(x^2 - b^2)}$$

$a^2, b^2 = [-\mu \pm 4\sqrt{g_4}]/4g_4$. At $\mu = -4\sqrt{g_4}$ the gap $(-b, b)$ closes. Asymmetric 1 and 2-cut solutions ($q_1 \neq 0$) are also admissible in certain regions of parameters, and were studied by Shimamune (1982) [31]. The choice of which solution holds for given μ, g_4, is that it minimizes the free energy. On each line of phase transition in parameter space, the free energy is singular in the third derivative.
For high order potentials, the large parameter space may accomodate multi-critical points [12].

4. Saddle point - large n

The well known solution [9] by Brézin et al. was obtained by this method. Suppose that the equilibrium configuration is described, for large n, by a normalized density $\rho(x)$ with support σ. The action is written as a functional of ρ, with a normalisation constraint for ρ,

$$S[\rho, \mu] = \int_\sigma dx\, v(x)\rho(x) - \iint_{\sigma\times\sigma} dxdy\, \rho(x)\rho(y)\log|x - y| - \mu\int_\sigma dx\rho(x)$$

The extremum occurs for $v(x) - 2\int_\sigma dy\, \rho(y)\log|x - y| = \mu$, $x \in \sigma$. A derivative yields the saddle point equation:

$$\boxed{\fint_\sigma dy\frac{\rho(y)}{x - y} = \tfrac{1}{2}v'(x), \qquad x \in \sigma} \tag{13}$$

Brézin et al. obtained a 1-cut solution $\sigma = [-a, a]$ as follows: they introduced the function $F(z) = \int_\sigma dy \rho(y)/(z-y)$ and noted that there is only one function $F(z)$, analytic in $\mathbb{C}/\sigma$, that goes as $1/z$ for large z, such that $F(x \mp i\epsilon) = v'(x) \pm i\pi\rho(x)$, $x \in \sigma$ and real for $|x| > 2a$. The function $F(z)$ is the one discussed above.
Another possibility is to use the **Poincaré-Bertrand formula**, which provides the exchange of principal-part integrals:

$$\fint_\sigma \frac{dx}{x-y} \fint_\sigma \frac{dx'}{x'-x} f(x,x') = -\pi^2 f(y,y) + \fint_\sigma dx' \fint_\sigma \frac{dx}{(x-y)(x'-x)} f(x,x')$$

In particular, for a factored function:

$$\fint_\sigma \frac{dx}{\pi} \frac{g(x)}{x-y} \fint_\sigma \frac{dx'}{\pi} \frac{f(x')}{x'-x} = -f(y)g(y) + \fint_\sigma \frac{dx'}{\pi} f(x') \fint_\sigma \frac{dx}{\pi} \frac{g(x)}{(x-y)(x'-x)}$$

If σ is the union of intervals, the following integrals are useful:

$$\fint_\sigma \frac{dx}{\pi} \frac{1}{\phi(x)} \frac{1}{x-y} = \begin{cases} 0 & y \in \sigma \\ 1/\psi(y) & y \notin \sigma \end{cases} \tag{14}$$

$$\sigma = [a,b], \quad \phi(x) = \sqrt{(b-x)(x-a)}, \quad \psi(x) = \sqrt{|(x-a)(x-b)|} \tag{15}$$

$$\sigma = [a,b] \cup [c,d], \quad \phi(x) = \begin{cases} \sqrt{(x-a)(x-b)(x-c)(d-x)} & x \in [c,d] \\ -\sqrt{(x-a)(b-x)(c-x)(d-x)} & x \in [a,b] \end{cases} \tag{16}$$

$$\psi(x) = \sqrt{|(x-a)(x-b)(x-c)(x-d)|} \times \begin{cases} 1 & x < a \text{ or } x > d \\ -1 & b < x < c \end{cases}$$

4.1. **Solution of saddle-point equation.** For a $k-$cut solution there is an appropriate square-root function $\phi(x)$ that inverts the saddle-point equation (13) by means of the Poincaré-Bertrand formula. Just integrate it as follows:

$$\fint_\sigma dx \frac{1}{\phi(x)} \frac{1}{x-x'} \fint_\sigma dy \frac{\rho(y)}{x-y} = \frac{1}{2} \fint_\sigma dx \frac{1}{\phi(x)} \frac{1}{x-x'} v'(x)$$

By the Poincaré-Bertrand formula the left-hand site becomes:

$$-\fint_\sigma dy \rho(y) \fint_\sigma dx \frac{1}{\phi(x)} \frac{1}{(x-x')(x-y)} + \pi^2 \frac{\rho(x')}{\phi(x')}$$

where the integral splits into two integrals with value zero. Therefore:

$$\boxed{\rho(x') = \frac{1}{2\pi^2} \phi(x') \fint_\sigma dx \frac{1}{\phi(x)} \frac{v'(x)}{x-x'}} \tag{17}$$

The general 1-cut solution is:

$$\rho(x) = \frac{1}{\pi} \sqrt{(b-x)(x-a)} \int_a^b \frac{dy}{2\pi} \frac{v'(y)-v'(x)}{y-x} \frac{1}{\sqrt{(b-x)(x-a)}}$$

(we subtracted a term that yields a null integral).
- For $v'(x) = x$, $\sigma = (-a, a)$, the semicircle distribution $\rho(x) = \frac{1}{2\pi}\sqrt{a^2-x^2}$ is obtained. Normalization of the density fixes $a = 2$.
- For $v'(x) = \mu x + 4g_4 x^3$, $\sigma = (-a, a)$, we reobtain the solution (12)

$$\rho(x) = \frac{1}{\pi}\sqrt{a^2-x^2} \int_{-a}^{a} \frac{dy}{2\pi} \frac{\mu + 4g_4(x^2+xy+y^2)}{\sqrt{a^2-y^2}} = \frac{4g_4}{2\pi}(x^2+b^2)\sqrt{a^2-x^2}$$

where $b^2 = \frac{1}{2}a^2 + \frac{\mu}{4g_4}$, and a is fixed by normalisation. The 2-cut solutions are obtained in a similar manner.

5. Orthogonal Polynomials

The elegant and powerful technique of orthogonal polynomials gives a formally exact solution of Hermitian 1-matrix models, for all n. It was devised by Daniel Bessis [4] and allowed to determine sub-leading terms of the $1/n$ (topological) expansion [5]. It is based on the following property of the Vandermonde determinant: if $p_k(x) = x^k + \dots$ are any monic polynomials, then

$$\det \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} = \det \begin{bmatrix} p_0(x_1) & p_1(x_1) & \dots & p_{n-1}(x_1) \\ p_0(x_2) & p_1(x_2) & \dots & p_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ p_0(x_n) & p_1(x_n) & \dots & p_{n-1}(x_n) \end{bmatrix}$$

and $\Delta^2(\mathbf{x}) = \det[p_r(x_j)]\det[p_s(x_k)] = \epsilon_{r_1\dots r_n}\epsilon_{s_1\dots s_n}\prod_k p_{r_k}(x_k)p_{s_k}(x_k)$.
The polynomials are chosen orthogonal for the weight $\exp[-nv(x)]$:

$$\int_{-\infty}^{+\infty} dx\, e^{-nv(x)} p_k(x) p_m(x) = h_k \delta_{km} \tag{18}$$

As such, they are completely computable, together with the weights $h_k > 0$. Beginning with $p_0(x) = 1$, one evaluates h_0. Next, orthogonalization gives $p_1(x) = x - \langle x \rangle$, where $\langle x \rangle = (1/h_0)\int dx\, e^{-nv} x$, and h_1 is then evaluated, etc.

Partition function. $Z_n = \int dx_1 \dots dx_n \Delta^2(x_1, \dots, x_n) \exp[-n\sum_k v(x_k)]$

$$= \epsilon_{r_1\dots r_n}\epsilon_{s_1\dots s_n}\prod_k \int dx_k e^{-nv(x_k)} p_{r_k}(x_k) p_{s_k}(x_k) = \epsilon^2_{r_1\dots r_n} h_{r_1}\cdots h_{r_n}$$

$$\boxed{Z_n = n!\, h_0 h_1 \cdots h_{n-1}} \tag{19}$$

One easily proves a general statement: if $f_a(x)$, $g_a(x)$ $a = 1...n$ are two sets of functions, $\mu(x)$ is a measure and the matrix $G_{ab} = \int d\mu(x) f_a(x) g_b(x)$ exists, then:

$$\int d\mu(x_1)...d\mu(x_n) \det[f_a(x_b)] \det[g_a(x_b)] = n! \det G$$

Eigenvalue density. $\rho_n(x) = \langle \frac{1}{n}\sum_{j=1}^n \delta(x - x_j)\rangle$

$$= \frac{1}{Z_n}\int d^n\mathbf{x}\, \Delta^2(\mathbf{x})\, \delta(x - x_n) e^{-n\sum_k v(x_k)}$$

$$= \frac{1}{Z_n}\epsilon_{r_1\dots r_n}\epsilon_{s_1\dots s_n}\prod_{k=1}^n \int dx_k e^{-nv(x_k)} p_{r_k}(x_k) p_{s_k}(x_k)\, \delta(x - x_n)$$

$$= \frac{1}{Z_n}\epsilon^2_{r_1\dots r_n}\, e^{-nv(x)}\, h_{r_1}\dots h_{r_{n-1}} p^2_{r_n}(x)$$

$$\boxed{\rho_n(x) = e^{-nv(x)}\frac{1}{n}\sum_{k=0}^{n-1}\frac{p_k^2(x)}{h_k}} \tag{20}$$

2-point correlation function.

$$\rho_n(x,y) = \langle \delta(x-x_n)\delta(y-x_{n-1})\rangle - \rho_n(x)\rho_n(y) \equiv -K_n(x,y)^2 \tag{21}$$

$$\boxed{K_n(x,y) = e^{-\frac{n}{2}[v(x)+v(y)]}\sum_{k=0}^{n-1}\frac{p_k(x)p_k(y)}{h_k}} \tag{22}$$

Note that, because of orthogonality:

$$K_n(x,y) = \int_{-\infty}^{+\infty} dz\; K_n(x,z)K_n(z,y) \tag{23}$$

We can view $\frac{1}{\sqrt{Z_n}}\exp[-\frac{n}{2}\sum_k v(x_k)]\Delta(\mathbf{x}) = \det[u_k(x_j)]$ as the Stater determinant of n orthonormal 1-particle states, $k = 0, ..., n-1$,

$$u_k(x) = \frac{1}{\sqrt{h_k}}\exp[-\tfrac{n}{2}v(x)]\, p_k(x) \tag{24}$$

Then the joint probability for the eigenvalues is:

$$\boxed{p(x_1,...,x_n) = \det([u_{r-1}(x_k)]_{r,k=1}^n)^2 = \det[K_n(x_i,x_j)]_{i,j=1}^n} \tag{25}$$

The following theorem by Dyson shows that all marginal distributions of eigenvalues (partial integrations of p) are in the form of a determinant built out of the same kernel. Therefore, the whole statistical information is contained in $K_n(x,y)$.

Theorem 5.1 (Dyson, 1970 [16]). *If a function has the properties:* $K(x,y) = K(y,x)^*$ *and* $\int dzK(x,z)K(z,y) = K(x,y)$, *then for* $p = 2, 3, ...$*:*

$$\int \det[K(x_i,x_j)]_1^p dx_p = (n-p-1)\,\det[K(x_i,x_j)]_1^{p-1} \tag{26}$$

$$\int \det[K(x_i,x_j)]_1^n dx_{p+1}...dx_n = (n-p)!\,\det[K(x_i,x_j)]_1^{p-1} \tag{27}$$

where $\int dxK(x,x) = n$.

In passing, note that the eigenvalue density and the two-point functions are:

$$\rho_n(x) = \frac{1}{n}\langle s\ell_n|\hat\psi^\dagger(x)\hat\psi(x)|s\ell_n\rangle$$
$$K(x,y) = \langle s\ell_n|\hat\psi^\dagger(x)\hat\psi(y)|s\ell_n\rangle$$

where $\hat\psi$ is a field operator. Connected density correlators of any order are evaluated by Wick's theorem. For example: $\frac{1}{n^2}\langle s\ell_n|\hat\psi^\dagger(x)\hat\psi(x)\hat\psi^\dagger(y)\hat\psi(y)|s\ell_n\rangle_{\rm conn} = -K(x,y)^2$.

5.1. **Properties of orthogonal polynomials.** As a consequence of orthogonality, with any weight $w(x)$ (not necessarily of the form $\exp[-nv(x)]$), the polynomials satisfy a three-term recurrence relation:

$$xp_k(x) = p_{k+1}(x) + S_kp_k(x) + R_kp_{k-1}(x) \tag{28}$$

Proof. Suppose that the recurrence contains a term T_kp_{k-2}. Multiply the recurrence with the extra term by $w(x)\,p_{k-2}(x)$ and integrate. Because of orthogonality one gets: $\int dxw(x)\,p_k(xp_{k-2}) = T_kh_{k-2}$. Since xp_{k-2} is orthogonal to p_k, then $T_k = 0$. □

If the polynomials have definite parity, it is $S_k = 0$. Multiplication by $p_{k-1}(x)w(x)$ and integration gives

$$\boxed{h_k = R_k h_{k-1}} \tag{29}$$

so that $R_k > 0$. The recurrence relation (28) corresponds to the evaluation of the characteristic polynomial of a real symmetric tridiagonal matrix, with initial condition $p_0(x) = 1$.

$$p_{k+1}(x) = \det \begin{bmatrix} x - S_k & \sqrt{R_k} & & \\ \sqrt{R_k} & \ddots & \ddots & \\ & \ddots & x - S_1 & \sqrt{R_1} \\ & & \sqrt{R_1} & x - S_0 \end{bmatrix}$$

This has an important implication: *the zeros of orthogonal polynomials are real and simple.* Moreover, the zeros of $p_k(x)$ and the zeros of $p_{k+1}(x)$ interlace (Cauchy theorem for interlacing of eigenvalues of a Hermitian matrix of size k and of a principal block of size $k-1$).

Exercise 5.2. Prove the Christoffel-Darboux formulae

$$\sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} = \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_{n-1}(x-y)} \tag{30}$$

$$\sum_{k=0}^{n-1} \frac{p_k^2(x)}{h_k} = \frac{p_n'(x)p_{n-1}(x) - p_{n-1}'(x)p_n(x)}{h_{n-1}} \tag{31}$$

Exercise 5.3. Prove the following formulae by Szëgo:

$$p_n(x) = \Big\langle \prod_{k=1}^{n} (x - x_k) \Big\rangle \tag{32}$$

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) \tag{33}$$

$$A_n(x) = \frac{n}{h_{n-1}} \int dy e^{-nv(y)} \frac{v'(x) - v'(y)}{x - y} p_n(y)^2$$

$$B_n(x) = \frac{n}{h_{n-1}} \int dy e^{-nv(y)} \frac{v'(x) - v'(y)}{x - y} p_n(y)p_{n-1}(y)$$

Hints: $\Delta(x_1 \dots x_n)\prod_k (x - x_k) = \Delta(x_1 \dots x_n x)$; $p_n'(x) = \sum_{k=0}^{n-1} c_k p_k(x)$ where c_k are obtained by orthogonality.

6. Spacing functions

[Mehta, 3rd Ed] The probability that exactly m eigenvalues fall in a set $\mathcal{I}$ and the other $n - m$ in $\sigma/\mathcal{I}$ is:

$$E(m, \mathcal{I}) = \frac{n!}{m!(n-m)!} \int_{\mathcal{I}} dx_1 ... dx_m \int_{\sigma/\mathcal{I}} dx_{m+1} ... dx_n \; p(x_1, ..., x_n)$$

It can be generated as $E(m, \mathcal{I}) = (-1)^m \frac{1}{m!} (\partial_z)^m E(z, \mathcal{I})|_{z=1}$, where

$$E(z, \mathcal{I}) = \int_{\sigma} dx_1 ... dx_n \prod_{k=1}^{n} (1 - z\chi_{\mathcal{I}}(x_k))\, p(x_1, ..., x_n) \tag{34}$$

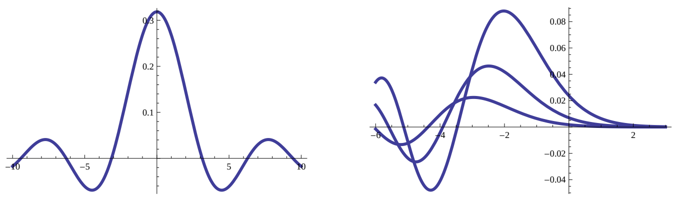

FIGURE 5. The bulk correlator $K(x,y)$ as a function of $|x-y| = t/n$. The edge correlator $K(t,s)$ for $t = 0.5, 1.0, 1.5$, as a function of $s-t$.

$\chi_{\mathcal{I}}$ is the characteristic function of the set $\mathcal{I}$. The probability that no eigenvalue is in $[a,b]$:

$$\begin{aligned} E(a,b) &= \int \prod_{1\le j\le n} dx_j [1-\chi_{[a,b]}(x_j)] p(x_1,\dots,x_n) \\ &= \int \prod_{1\le j\le n} dx_j [1-\chi_{[a,b]}(x_j)]^2 \det[u_k(x_j)]^2 \\ &= \int dx_1 \dots dx_n \det[(1-\chi_{[ab]}(x_j))u_k(x_j)]^2 \\ &= \epsilon_{r_1\dots r_n}\epsilon_{s_1\dots s_n} \prod_k \int dx (1-\chi_{[a,b]}(x)) u_{r_k}(x) u_{s_k}(x) \\ &= \det\left[\delta_{rs} - \int_a^b dx\, u_r(x) u_s(x)\right] \end{aligned}$$

Then $E(a,b) = \det[1-K_n^{a,b}]$ (Gaudin). The combination

$$E(a,b) - E(a-\delta a, b) - E(a, b+\delta b) + E(a-\delta a, b+\delta b) \approx -\delta a \delta b \frac{\partial^2}{\partial a \partial b} E(a,b)$$

is the probability that no eigenvalue is in $[a,b]$ given that at least one is in $[a-\delta a, b]$ and at least one is in $[a, b+\delta b]$. By sending δa and δb to zero, it becomes the probability of having an eigenvalue in a, another in b with no one between.

$$P(a,b) = -\frac{\partial^2}{\partial a \partial b} \det[1-K_n^{(a,b)}] \tag{35}$$

is the probability that (a,b) is exactly a gap between two eigenvalues.

Characteristic polynomials. Brézin and Hikami, Mehta and Normand, evaluated the following average on a $n\times n$ 1-matrix ensemble:

$$\langle \prod_{k=1}^m \det(x_k - H)\rangle = \frac{1}{\Delta(x_1\dots x_m)} \det[p_{n+j-1}(x_k)]_{j,k=1\dots m} \tag{36}$$

where $p_k(x)$ are the monic orthogonal polynomials for the measure $\exp(-nv(x))$. By exploiting an algebraic identity, based on the Cauchy-Littlewood formula, Fyodorov and Strahov [21] extended it to ratios of products of characteristic polynomials. In particular, if the numbers of factors are the same:

$$\Big\langle \prod_{k=1}^{m} \frac{\det(x_k - H)}{\det(y_k - H)} \Big\rangle = \frac{1}{\Delta(x_1 \dots x_m)\Delta(y_1 \dots y_m)} \frac{n!}{(n-m)!} \frac{Z_{n-m} Z_m}{Z_n}$$

$$\times \det \begin{bmatrix} h_1(y_1) & \cdots & h_1(y_m) & p_{n-m}(x_1) & \cdots & p_{n-m}(x_m) \\ h_2(y_1) & \cdots & h_1(y_m) & p_{n-m}(x_1) & \cdots & p_{n-m}(x_m) \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{2m}(y_1) & \cdots & h_{2m}(y_m) & p_{n+m-1}(x_1) & \cdots & p_{n+m-1}(x_m) \end{bmatrix} \tag{37}$$

where p_k are the orthogonal polynomials and

$$h_k(y) = \int dt\, e^{-nv(t)} \frac{p_k(t)}{t-y}$$

The analysis of characteristic polynomials proceeds with the distribution of the maximum value of $-\log|\det(x-H)|$, for $\mathsf{GUE}(n \to \infty)$, x in the support of the semicircle distribution [20].

7. GUE - Hermite ensemble

For $v(x) = \frac{1}{2}x^2$ the orthogonal polynomials are the Hermite polynomials. Then $p_k(x) = (\sqrt{2n})^{-k} H_k(x\sqrt{n/2})$ and $h_k = \int_{-\infty}^{+\infty} dx\; e^{-\frac{n}{2}x^2} (2n)^{-k} H_k^2(x\sqrt{n/2}) = \sqrt{\frac{2\pi}{n}} \frac{k!}{n^k}$.

$$Z_n^{\mathsf{GUE}} = \frac{(2\pi)^{n/2}}{n^{n^2/2}} \prod_{k=1}^{n} k! \tag{38}$$

The eigenvalue densities for finite n and $n \to \infty$ are:

$$\begin{aligned} \rho_n^{\mathsf{GUE}}(x) =& \frac{e^{-\frac{1}{2}nx^2}}{\sqrt{2n\pi}} \sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k^2(x\sqrt{\tfrac{n}{2}}) = \frac{1}{\sqrt{2n\pi}} \sum_{k=0}^{n-1} \int_0^\infty \frac{ds}{\sqrt{\pi}}\, e^{-\frac{s^2}{4}} L_k(\tfrac{s^2}{2}) \cos(sx\sqrt{\tfrac{n}{2}}) \\ =& \frac{1}{\sqrt{2n\pi}} \int_0^\infty \frac{ds}{\sqrt{\pi}} e^{-\frac{s^2}{4}} L^1_{n-1}(\tfrac{s^2}{2}) \cos(sx\sqrt{\tfrac{n}{2}}) = \int_0^\infty \frac{ds}{n\pi} e^{-\frac{s^2}{2n}} L^1_{n-1}(\tfrac{s^2}{n}) \cos(sx) \\ \to& \int_0^\infty \frac{ds}{\pi s} J_1(2s) \cos(sx) = \frac{1}{2\pi}\sqrt{4-x^2} \qquad \text{(semicircle law)} \end{aligned}$$

We used the integral representation of H_k^2 with Laguerre polynomials (Lebedev[2] p.95), the sum GR 8.974.3, the asymptotics $\frac{1}{n} L_n^1(s^2/n) \to J_1(2s)/s$ (GR 8.978.2), and the integral GR 6.693.2.

For the 2-point function, we use the Christoffel-Darboux formula:

$$K_n(x,y) = \frac{e^{-\frac{n}{4}(x^2+y^2)}}{n!\sqrt{\pi}2^{-n+1}} \frac{H_n(x\sqrt{\frac{n}{2}})H_{n-1}(y\sqrt{\frac{n}{2}}) - H_{n-1}(x\sqrt{\frac{n}{2}})H_n(y\sqrt{\frac{n}{2}})}{x-y}$$

In the oscillatory region the large n behaviour of Hermite polynomials is (Lebedev 4.14.9):

$$H_n(x) \approx 2^{\frac{1}{2}(n+1)} e^{\frac{1}{2}x^2 - \frac{1}{2}n(1-\log n)} \cos(x\sqrt{2n+1} - n\tfrac{\pi}{2})$$

$$\boxed{K_{\text{bulk}}^{\mathsf{GUE}}(x,y) = \frac{\sin[n(x-y)]}{\pi n(x-y)}} \tag{39}$$

[2] N. N. Lebedev, Special functions and their applications, Dover Ed.; I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series and products, Academic Press.

The large n behaviour of $H_n(x\sqrt{n/2})$ near the edge $x=2$ of the parabolic potential is read in the differential equation. A linearisation of the potential and proper rescaling give an Airy function. The function $u_n(x) = e^{-\frac{n}{4}x^2}H_n(x\sqrt{n/2})$ solves $-u_n''(x) + \frac{n^2}{4}x^2 u_n(x) = n(n+\frac{1}{2})u_n(x)$. With $\phi(t) = u_n(2+tn^{-\alpha})$ it becomes $\phi''(t) = (tn^{2-3\alpha} - \frac{1}{2}n^{1-2\alpha} + \dots)\phi(t)$. With $\alpha = \frac{2}{3}$ it is the Airy equation $\phi''(t) = (t - \frac{1}{2}n^{-1/3})\phi(t)$. Then

$$e^{-n^{1/3}t}H_n(\sqrt{2n} + \tfrac{n^{-1/6}}{\sqrt{2}}t) \approx C_n \mathrm{Ai}(t - \tfrac{1}{2}n^{-1/3}) \approx C_n[\mathrm{Ai}(t) - \tfrac{1}{2}n^{-1/3}\mathrm{Ai}'(t)]$$

The constant C_n is evaluated at $t=0$. The derivative in t and $H_n' = 2nH_{n-1}$ give: $H_{n-1}(\sqrt{2n} + \frac{n^{-1/6}}{\sqrt{2}}t) \approx \frac{1}{\sqrt{2n}}C_n[\mathrm{Ai}(t) + \frac{1}{2}n^{-1/3}\mathrm{Ai}'(t)]e^{n^{1/3}t}$. Near the edge:

$$\begin{aligned} K_{\text{edge}}^{\mathsf{GUE}}(t,s) &= \lim_{n\to\infty} K_n(\sqrt{2n} + \tfrac{t}{\sqrt{2}}n^{-1/6}, \sqrt{2n} + \tfrac{s}{\sqrt{2}}n^{-1/6}) \\ &\approx \frac{\mathrm{Ai}(t)\mathrm{Ai}'(s) - \mathrm{Ai}'(t)\mathrm{Ai}(s)}{t-s} \end{aligned} \tag{40}$$

7.1. **Universality.** Brézin and Zee (1993) [10] conjectured that for even potentials with large-n support of eigenvalues $(-a,a)$, the orthogonal polynomials behave for large k as

$$p_k(x) = \frac{e^{nv(x)/2}}{\sqrt[4]{a^2-x^2}}\cos[n\zeta(x) + (n-k+\tfrac{1}{2})\arccos(\tfrac{x}{a}) + \text{const.}]$$

where $\zeta'(x) = -\pi\rho(x)$. Eynard proved that the asymptotics remains true for any potential [18]. In the microscopic limit $|x-y| = \mathcal{O}(1/n)$, they yield a universal expression for the correlator in bulk

$$K(x,y) \to \frac{\sin[n\pi\rho(\frac{x+y}{2})|x-y|]}{n\pi|x-y|} \tag{41}$$

A proof not dependent on the asymptotics was given by Pastur and Shcherbina (arXiv:0705.1050).

7.2. **Riemann's zeta function.** The functional relation

$$\zeta(1-z)(2\pi)^z = 2\zeta(z)\Gamma(z)\cos(\tfrac{\pi}{2}z)$$

shows that $\zeta(z)$ has simple zeros where $\Gamma(z)$ has simple poles: these are the trivial zeros, $z = 0, -2, -4, \dots$. The other zeros are symmetric around the line $\mathrm{Re}z = \frac{1}{2}$. Riemann's hypothesis (1859) states that the nontrivial zeros are exactly on this line: $z = \frac{1}{2} + ib$. The number of such zeros with $0 \le b \le T$ is

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \mathcal{O}(T)$$

Thus, for large T, the mean spacing of zeros is $\bar{s} \approx 2\pi/\log T$. The precise counting is: $N(T) = \bar{N}(T) + N_{\text{osc}}(T)$ where:

$$\bar{N}(T) = 1 - \frac{T}{2\pi}\log\pi - \frac{1}{\pi}\mathrm{Im}\,\log\Gamma(\tfrac{1}{4} - i\tfrac{T}{2}), \qquad N_{\text{osc}}(T) = -\frac{1}{\pi}\mathrm{Im}\log\zeta(\tfrac{1}{2} - iT)$$

Montgomery showed that if $\rho(b)db = \#$zeros in $[b, b+db]$, then, for fixed $\eta > 0$:

$$\lim_{b\to\infty}\frac{\langle\rho(b)\rho(b+\eta)\rangle}{\langle\rho(b)\rangle\langle\rho(b+\eta)\rangle} = 1 - \frac{\sin^2(\pi\beta)}{(\pi\beta)^2} \tag{42}$$

where the average is in a window $[b, b+\Delta b]$ with $1 \ll \Delta b \ll b$, and $\beta = b/\bar{s}$ (mean spacing around b). When Montgomery told Freeman Dyson this formula, Dyson responded that it was the pair-correlation function for eigenvalues of GUE. The Montgomery-Odlyzko conjecture states that the correlation functions for the large zeros (scaled with spacing) of ζ are the same of the bulk eigenvalues of GUE. This has been tested on samples of billions of large zeros. This is consistent with the conjecture by Polya and Hilbert that the zeros are eigenvalues of some Hermitian operator.

Keating and Snaith [25] evaluated the moments for CUE(n), $\langle|\det(1-U)|^{2n}\rangle$, and its large n limit and conjectured the moments of the zeta function:

$$\frac{1}{T}\int_0^T dT|\zeta(\tfrac{1}{2}+it)|^{2n} \approx a_n \prod_{j=0}^{n-1}\frac{j!}{(j+n)!}[\log(\frac{T}{2\pi})]^{n^2} \tag{43}$$

7.3. **Longest increasing subsequence in a permutation.** A permutation π of $1,\dots,n$ is represented by the sequence of numbers $\pi(1),\dots,\pi(n)$. The sequence may contain increasing subsequences. For example, the permutation 76415823 contains the increasing subsequences 78, 68, 458, 15, 18, 123, 58, and 23, with lengths 2, 3: the maximum is $\ell_8(76415823)=3$. For the identity permutation $\ell_8(id)=8$. Erdös and Szekeres proved in 1938 that every permutation in $S(n)$ contains either an increasing or a decreasing subsequence of length $\geq\sqrt{n}$.
Assuming that each permutation in $S(n)$ has probability $1/n!$, consider the probability $p_{n,k}=\mathrm{Prob}(\pi:\ell_n(\pi)\leq k)$. Ulam studied the problem numerically in 1961, and realised that $\mathbb{E}(\ell_n)\approx 2\sqrt{n}$. Jinho Baik, Percy Deift and Kurt Johansson [3] proved that, for $n,k\to\infty$:

$$\lim_{n\to\infty}\mathrm{Prob}\left(\pi\in S(n):\frac{\ell_n(\pi)-2\sqrt{n}}{n^{1/6}}\leq t\right)=F(t) \tag{44}$$

$$F(t)=\exp\left[-\int_t^\infty (x-t)u^2(x)dx\right] \tag{45}$$

where u is the unique solution to the Painlevé II equation $u''=xu+2u^3$ that for large x tends to $\mathrm{Ai}(x)$. Tracy and Widom (1994) proved that

$$F(t)=\det(I-A)_{L^2(t,\infty)}\equiv\sum_{m=0}^\infty\frac{(-1)^m}{m!}\int_t^\infty dx_1\dots\int_t^\infty dx_m\det[A(x_i,x_j)]$$

where $A(x,y)=\int_0^\infty dt\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)$ is the Airy kernel. For this reason $F(t)$ is often referred to as the Tracy-Widom distribution.

From Mehta, 3rd Ed. Let $I(n,k)$ be the number of permutations in S_n in which the longest increasing subsequence is less than or equal to k. Gessel (1990) has shown that it coincides with the expectation value of $|\mathrm{tr}U|^{2k}$ in CUE(n):

$$\int dU_n\,|\mathrm{tr}U|^{2k}=\frac{1}{n!}\prod_{j=1}^n\int_0^{2\pi}\frac{d\theta_j}{2\pi}\prod_{p<q}|e^{i\theta_p}-e^{i\theta_q}|^2\left|\sum_{r=1}^n e^{i\theta_r}\right|^{2k}$$

The same function $F(t)$ is the distribution for the *largest* eigenvalue of a GUE random matrix in the edge-scaling limit [33] (see Deift's homepage).
The distribution can be very well approximated by a Gamma distribution [13]:

$$F(t)\approx\Gamma(k)^{-1}\gamma(k,\tfrac{x+\alpha}{\theta})$$

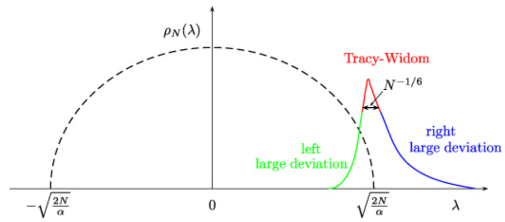

FIGURE 6. Average density of eigenvalues of GUE (Wigner's semicircle), with support $[-\sqrt{2N/\alpha}, \sqrt{2N/\alpha}]$. The largest eigenvalue has mean value $\sqrt{2N\alpha}$ (large N). Its distribution close to the mean value over a scale $\mathcal{O}(N^{-1/6})$, has the Tracy-Widom form (red). Over a scale $\sqrt{N}$ the distribution has large deviation tails (green and blue). (from Nadal and Majumdar [28]).

where $k = 4/s^2$, $\theta = s\sigma/2$, $\alpha = k\theta - \mu$ and μ, σ are the mean, the variance and $s = \langle(\frac{x-\mu}{\sigma})^3\rangle$ is the skewness of the Tracy-Widom distribution.

7.4. **String equations.** For the quadratic potential all is known: the orthogonal polynomials are Hermite. For non-quadratic potentials one needs two recursive relations for the coefficients R_k, S_k, called "string equations":
1) Multiplication of (28) by $e^{-nv}p_k'$ and integration give $\int dx e^{-nv} x(kp_{k-1}+...)p_k = R_k \int dx e^{-nv} p_{k-1}p_k'$. After integration by parts:

$$\frac{k}{n} = \frac{1}{h_{k-1}} \int dx e^{-nv(x)} v'(x) p_{k-1}(x) p_k(x). \tag{46}$$

If v' is a polynomial, the product $v'p_k$ is amenable to a sum of orthogonal polynomials and the integrals are done.
2) Multiplication of (28) by $e^{-nv}v'(x)p_k(x)$ and integration give $\int dx e^{-nv} v'(x) x p_k^2(x) = \int dx e^{-nv} v'(x)[p_{k+1} + S_k p_k + R_k p_{k-1}]p_k$. Integration by parts and (46) give:

$$0 = \frac{S_k}{h_k} \int dx e^{-nv} v'(x) p_k^2(x) \tag{47}$$

The string equations are solved with a number of initial conditions. It has been shown that the numerical sequence $\{R_k, S_k\}$ generated by the recursion is erratic but, for k of order n, it stabilises to one or more pairs of interpolating smooth functions. The case of a single pair of interpolating functions $R(x), S(x)$ corresponds to the 1-cut phase. For the 2-cut phase the initial conditions require two pairs of interpolating functions (the "doubling phenomenon" [27]).

Exercise 7.1. Show that $\frac{1}{n^2}\log Z_n = \frac{1}{n^2}\log n! + \frac{1}{n}\log h_0 + \frac{1}{n}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\log R_k$, which is appropriate for a large-n limit.

7.5. **Quartic potential.** $v(x) = \frac{1}{2}\mu x^2 + g_4 x^4$. The monic orthogonal polynomials have definite parity: $xp_k(x) = p_{k+1}(x) + R_k p_{k-1}(x)$. Eq.(46) gives the recurrence law:

$$\frac{k}{n} = \mu R_k + 4g_4 R_k(R_{k+1} + R_k + R_{k-1}) \tag{48}$$

with the initial conditions

$$R_1 = \frac{h_1}{h_0} = \frac{\int dx e^{-nv} x^2}{\int dx e^{-nv}}, \quad R_2 = \frac{h_2}{h_1} = \frac{\int dx e^{-nv}(x^2 - R_1)^2}{\int dx e^{-nv} x^2}$$

In the large n-limit, if v has a single minimum in $x = 0$, one postulates a single interpolating function $R_k = R(k/n) = R(x)$ with $R(0) = 0$ and (48) becomes:

$$x = \mu R(x) + 12 g_4 R(x)^2 + \frac{4g_4}{n^2} R(x) R''(x) + ... \tag{49}$$

Given the $1/n^2$ expansion $R(x) = r_0(x) + \frac{1}{n^2} r_2(x) + ...$, it is $x = \mu r_0(x) + 12 g_4 r_0(x)^2$,

$$r_2(x) = -4g_4 \frac{r_0(x) r_0''(x)}{\mu + 24 g_4 r_0(x)}, \quad \ldots$$

The free energy is evaluated via the Euler-MacLaurin formula, and partial integrations show that it *only depends on the value* $R(1) \equiv R$.
Let $R = r_0 + \frac{1}{n^2} r_2 + \ldots$ with $1 = \mu r_0 + 12 g_4 r_0^2$, $r_2 = -4g_4 r_0 r_0''/(\mu + 24 g_4 r_0)$. The term r_2/n^2 can be enhanced if r_2 diverges i.e. $\mu + 24 g_4 R_c = 0$ i.e. $48 g_4 + \mu^2 = 0$. This is precisely the value g_4^* for an edge singularity (which fixes the radius of convergence of the perturbative series in g). With the scaling variable $x = 1 - t/n^\alpha$, $R(t) = R_c + f(t) n^{-\beta}$ it is: $R'' = f''(t) n^{2\alpha - \beta}$. The string equation (49) becomes: $-n^{-\alpha} t = 12 g_4^* f(t)^2 n^{-2\beta} + 4 g_4^* n^{-2+2\alpha-\beta} R_c f''(t)$. With $\alpha = 4/5$ and $\beta = 2/5$ we get the equation $t = \frac{1}{4}\mu^2 f^2(t) + \frac{1}{6}\mu f''(t)$, which can be reduced to the standard Painlevé I type ($t = f^2 + f''$).

Exercise 7.2. Prove that $p_k'(x) = k p_{k-1}(x) + 4 g_4 (R_k R_{k-1} R_{k-2}) p_{k-3}(x)$

8. $1/n^2$ expansion of Free energy and diagram counting

The perturbative expansion of the free energy enumerates the connected vacuum diagrams (including symmetry factors) at various orders:

$$F(g_4) = -\log \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}\varphi^2 - g_4 \varphi^4} = \sum_k c_k g^k = 3g - 48 g^2 + 1584 g^3 - 78071 g^4 + ...$$

In the corresponding matrix model, the diagrams are the same but acquire a weight in n. The expansion of the free energy in powers $1/n^{2h}$ of the quartic model in $d = 0$ was evaluated by Bessis, Itzykson and Zuber [5] for $h = 0, 1, 2$:

$$F_n(g_4) = -\frac{\log a^2}{2} + \frac{(a^2-1)(9-a^2)}{24} + \frac{\log(2-a^2)}{12 n^2} + \frac{(1-a^2)^3(82 + 21 a^2 - 3a^4)}{6!(2-a^2)^5 n^4} + \ldots$$

where $a^2 = (\sqrt{1 + 48 g_4} - 1)/24 g_4$. Hereafter, F_n^{GUE} is being subtracted. The singularity $g_4^* = -1/48$ is the radius of convergence of the perturbative series for each term. The coefficients provide the counting numbers of diagrams with k vertices and weight n^{-2h}:

$$\begin{aligned} F_n(g_4) = & -\sum_{k=1}^{\infty} \frac{(-48 g_4)^k}{2k(k+2)(k+1)} \frac{(2k)!}{4^k (k!)^2} + \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{(-48 g_4)^k}{24k} \left[\frac{(2k)!}{4^k (k!)^2} - 1 \right] \\ & - \frac{1}{n^4} \sum_{k=3}^{\infty} \frac{(-48 g_4)^k}{135 \cdot 32} (k-1) \left[(28k+9) \frac{(2k)!}{4^k (k!)^2} - \frac{195}{8} \right] + \ldots \end{aligned} \tag{50}$$

g_4^k	$n=1$	$h=0$	$h=1$	$h=2$
$k=1$	3	2	1	0
2	48	18	30	0
3	1584	288	1056	240
4	78336	6048	40176	32112

g_4	F	F_0	F_1	F_2
0.1	0.170	0.122	0.029	0.0016
1	0.478	0.420	0.046	0.0031
10	0.952	0.881	0.054	0.0035
100	1.492	1.417	0.056	0.0035

Tables 1,2. The quartic matrix model in $d = 0$. The numbers of diagrams (left) and the free energy (right) for $h = 0, 1, 2$. Despite being a minority, planar diagrams account for a large fraction of the value of the total free energy; moreover, the smallness of F_1 and F_2 indicates that cancellations occur.

The planar free energy of the two-cut phase ($\mu = -1$, $g_4 < \frac{1}{16}$) is [11]:

$$F_0^{II}(g_4) = -\frac{1}{16g_4} - \frac{3}{8} + \frac{1}{4}\log(4g_4)$$

g_4	F	F_0^{II}
0.001	-62.850	-64.255
0.005	-12.863	-13.853
0.01	-6.634	-7.430
0.05	-1.718	-2.027
1/16	-1.4541	-1.7216

Table 3. Quartic model in $d = 0$ with double well ($\mu = -1$), two-cut solution.

For the cubic model ($\mu = 1$), diagram counting beyond the planar term is involved [7].

g_3^{2k}	$n=1$	$h=0$	$h=1$	$h=2$
$k=1$	15/2	6	3/2	0
2	405	216	189	0
3	89.505/2	13.608	26.892	8.505/2
4	7.413.930	1.119.744	4.076.568	2.217.618

Table 4. Cubic model in $d = 0$, one cut. The counting numbers for $h = 0, 1, 2$:

8.1. **Double scaling.** The asymptotics of the first terms of the $1/n^2$ expansion of free energies are polilogs:

$$F_0(g_3) \approx -\frac{1}{2\sqrt{6\pi}}\sum_k \frac{(108\sqrt{3}\,g_3^2)^k}{k^{7/2}} - \frac{1}{n^2}\sum_k \frac{(108\sqrt{3}\,g_3^2)^k}{48k} + \dots$$

$$F_0(g_4) \approx -\frac{1}{2\sqrt{\pi}}\sum_k \frac{(-48g_4)^k}{k^{7/2}} - \frac{1}{n^2}\sum_k \frac{(-48g_4)^k}{24k} - \frac{7}{1080\sqrt{\pi}n^4}\sum_k (-48g_4)^k k^{3/2} + \dots$$

They have the same finite radius of convergence, given by the "edge singularity" $g_3^* = \frac{18}{\sqrt[4]{3}}$ or $g_4^* = -\frac{1}{48}$, and the same exponents. Near the edge[3] and omitting regular terms: $F_n(g_4) \approx a_0(g_4^* - g_4)^{5/2} + \frac{a_2}{n^2}\log(g_4^* - g_4) + \frac{a_4}{n^4}(g_4^* - g_4)^{-5/2} + \dots$. In general, for all polynomial potentials:

$$F_n(g) \approx (g^* - g)^{5/2}\sum_h \frac{a_k}{[n^2(g^* - g)^{\frac{5}{2}}]^h}$$

In the *double scaling limit* $g \to g^*$ and $n \to \infty$ with finite $x = n^2(g^* - g)^{5/2}$ one resums all orders of the topological expansion.

[3] $\mathrm{Li}_s(z) = \sum_{k=1}^\infty z^k k^{-s}$. For $\mu \to 0$: $Li_s(e^\mu) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{k=0}^\infty \frac{1}{k!}\mu^k\zeta(s-k)$

8.2. **The topological expansion.** In 1974 Gerard 't Hooft showed that the $1/n^2$ expansion of the free energy of QCD based on the colour group $\mathsf{SU}(n)$ corresponds to a *genus* expansion of the Feynman diagrams [32]. This remains true for matrix models with $\mathsf{O}(n)$, $\mathsf{U}(n)$ or $\mathsf{Sp}(2n)$ symmetry, and is independent of the space-time dimension.

A matrix propagator is represented as a double line, each line carrying a matrix index (see Predrag Cvitanović, [14]), and a closed graph can be viewed as a polyhedron with faces given by loops of such lines.

Consider a vacuum diagram of the model $S = n\mathrm{tr}[\frac{\mu}{2}H^2 + g_3 H^3 + g_4 H^4]$, with V_3 cubic and V_4 quartic vertices, P propagators and L loops. The weight in n is:

$$n^L (n\mu)^{-P} (ng_3)^{V_3} (ng_4)^{V_4} = n^{L+V-P} \left(\frac{g_3}{\mu^{3/2}}\right)^{V_3} \left(\frac{g_4}{\mu^2}\right)^{V_4}$$

If loops are faces of a polyhedron, with P edges and $V = V_3 + V_4$ vertices, the exponent $L + V - P$ is the genus χ of the closed surface (Euler). It is a topological index that is left invariant by continuous deformations of the surface. For a sphere (a cube, a tetrahedron, ...) $\chi = 2$. It is $\chi = 2 - 2h$, where h is the number of handles.

The $1/n^{2h}$ expansion is the topological expansion:

$$F_n(g) = -\frac{1}{n^2} \log\left(\frac{Z_n}{Z_n^{\mathsf{GUE}}}\right) = \sum_{h=0}^{\infty} \frac{1}{n^{2h}} F_h(g) = \sum_{h,k} \frac{g^k}{n^{2h}} c_{h,k} \tag{51}$$

the coefficients $c_{h,k}$ count the Feynman diagrams of genus $2 - 2h$ with k vertices. The first term is the sum of planar diagrams, that tessellate the surface of a sphere. Next come the diagrams with the topology of a torus, and so on.

These aspects: finite radius of convergence, topological significance, the free energy in $d = 0$ as generating function for counting connected diagrams, made 1-matrix and multi-matrix models, a formidable tool for the statistical mechanics on random graphs and, in a continuum limit, random surfaces.

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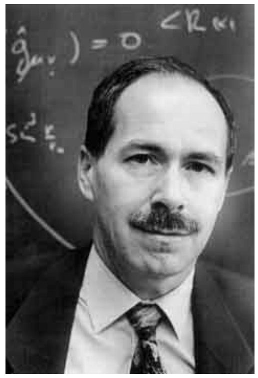 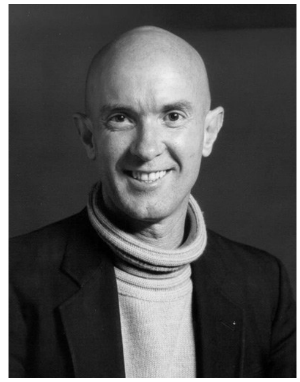

Figure 7. Gerard 't Hooft, born 1946. Now professor at Utrecht University. He shared the 1999 Nobel Prize in Physics with his thesis advisor M.J.G. Veltman "for elucidating the quantum structure of electroweak interactions". His research concentrates on gauge theory, black holes, quantum gravity and fundamental aspects of quantum mechanics.

Figure 8. Peter Forrester was a student of Rodney Baxter at the Australian Nat. Univ. (1985) and is now professor at Melbourne Univ. (with Paul Zinn-Justin). He is a renowned expert in RMT and related topics such as Jack polynomials, Ramanujan identities, Selberg integrals, exactly integrable statistical models, as the integrable $1/r^2$ many-body problem. He wrote the important book "Log-gases and random matrices".

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