Another proof of Gell-Mann and Low's theorem

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The theorem by Gell-Mann and Low is a cornerstone in quantum field theory and zero-temperature many-body theory. The standard proof is based on Dyson's time-ordered expansion of the propagator; a proof based on exact identities for the time propagator is here given. © *2007 American Institute of Physics.*
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I. INTRODUCTION

In the appendix of their paper "Bound States in Quantum Field Theory," Murray Gell-Mann and Francis Low[1] proved a fundamental relation that bridges the ground states $|\Psi_0\rangle$ and $|\Psi\rangle$ of Hamiltonians H_0 and $H=H_0+gV$ by means of time propagators, and makes the transition of time-ordered correlators from the Heisenberg to the interaction picture possible:

$$\langle\Psi|T\psi(1)\cdots\psi^\dagger(n)|\Psi\rangle=\frac{\langle\Psi_0|TS\psi(1)\cdots\psi^\dagger(n)|\Psi_0\rangle}{\langle\Psi_0|S|\Psi_0\rangle}. \tag{1}$$

The single operator $S=U_I(\infty,-\infty)$ contains all the effects of the interaction. The theorem borrows ideas from the scattering and the adiabatic theories and makes use of the concept of *adiabatic switching* of the interaction[2] through the time-dependent operator

$$H_\epsilon(t)=H_0+\mathrm{e}^{-\epsilon|t|}gV \tag{2}$$

that interpolates between the operators of interest, H at $t=0$ and H_0 at $|t|\to\infty$. The adiabatic limit is obtained for $\epsilon\to 0^+$. With the operator H_0 singled out, the theorem requires the time propagator in the interaction picture,

$$U_{\epsilon I}(t,s)=\mathrm{e}^{i/\hbar tH_0}U_\epsilon(t,s)\mathrm{e}^{-i/\hbar sH_0}, \tag{3}$$

where $U_\epsilon(t,s)$ is the full propagator.[3] The statement of Gell-Mann and Low's theorem is as follows.

Theorem: Let $|\Psi_0\rangle$ be an eigenstate of H_0 with eigenvalue E_0, and consider the vectors

$$|\Psi_\epsilon^{(\pm)}\rangle=\frac{U_{\epsilon I}(0,\pm\infty)|\Psi_0\rangle}{\langle\Psi_0|U_{\epsilon I}(0,\pm\infty)|\Psi_0\rangle}. \tag{4}$$

If the limit vectors $|\Psi^{(\pm)}\rangle$ for $\epsilon\to 0^+$ exist, then they are eigenstates of H.

The theorem is used to represent the ground state of an interacting system starting from a noninteracting one. For a time-dependent Hamiltonian, the eigenvalues evolve parametrically in time: if they do not cross and are not degenerate, eigenvectors can be traced univocally. According to adiabatic theory, the parametric evolution of eigenvectors is provided by time propagation and multiplication by a phase factor.[4,5] Then Gell-Mann and Low's theorem can be regarded as a

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statement concerning asymptotic states where the phase factor is properly dealt with. Adiabatic evolution of degenerate states[6] or with more general switching functions[7] has been considered.

In many-body theory the adiabatic switch of the interaction is smooth for Fermi liquids and takes free fermions into renormalized quasiparticles. It fails when symmetry changes: these systems require appropriate tools. In nonequilibrium theory the interaction is switched on in the past only, and time ordering is defined along a time loop beginning and ending in the past.[8] High energy physics emphasizes a scattering picture based on Lippmann-Schwinger equation.[2] The covariant realization of the adiabatic switch of the interaction in Lagrangian formalism was achieved by Bogoliubov and Shirkov.[9] The adiabatic switch is a tool to study interaction of quantum particles with time-periodic external fields $gV(t+T)=gV(t)$, with $\epsilon T \ll 1$.[10,11] The analytic properties in g of the quasienergy states become intricate as the size of the Hilbert space increases and avoided crossings coalesce.[12] The property that adiabatic evolution takes the eigenspaces of H_0 into eigenspaces of H is used in quantum field theory (QFT) to construct effective Hamiltonians for bound states in restricted Hilbert space.[13]

Despite the validity of the theorem beyond pertubation theory, in the original paper[1] and in textbooks[14–16] the proof makes use of Dyson's expansion of the interaction propagator, and is rather cumbersome. An elegant mathematical proof based on it was given by Hepp,[17] for the case where H_0 describes free particles and the interaction V is norm bounded. This ensures strong convergence of the Dyson series for the propagators $U_{I\epsilon}(0,\pm\infty)$, as discussed by Lanford.[18] Other mathematical proofs are based on versions of the adiabatic theorem.[19] They generally apply to a portion of the spectrum of $H_\epsilon(t)$ isolated from the rest at any time, but this gap condition can be relaxed.[20]

In this paper a simple equation for the propagator is derived, without use of Dyson's expansion. The equation can be used as intermediate nonperturbative result in the standard proof of Gell-Mann and Low's formula given in textbooks. This is described in the conclusion, where a short derivation of Sucher's formula is also given.

II. AN EQUATION FOR THE PROPAGATOR

Lemma: If $U_\epsilon(t,s)$ is the time propagator for $H_\epsilon(t)$ then, for all positive ϵ, the following relations hold:

$$i\hbar\epsilon g\frac{\partial}{\partial g}U_\epsilon(t,s) = H_\epsilon(t)U_\epsilon(t,s) - U_\epsilon(t,s)H_\epsilon(s) \quad \text{if } 0 \geq t \geq s, \tag{5}$$

$$= - H_\epsilon(t)U_\epsilon(t,s) + U_\epsilon(t,s)H_\epsilon(s) \quad \text{if } t \geq s \geq 0. \tag{6}$$

Proof: The trick is to make the g-dependence of the propagator explicit into the time dependence of some related propagator. Schrödinger's equation

$$i\hbar\partial_t U_\epsilon(t,s) = H_\epsilon(t)U_\epsilon(t,s), \quad U_\epsilon(s,s) = 1 \tag{7}$$

corresponds to the integral one, where we put $g=e^{\epsilon\theta}$:

$$U_\epsilon(t,s) = I + \frac{1}{i\hbar}\int_s^t dt'(H_0 + e^{\epsilon(\theta-|t'|)}V)U_\epsilon(t',s). \tag{8}$$

Consider the g-independent operators $H^{(\pm)}(t)=H_0+e^{\pm\epsilon t}V$, with corresponding propagators $U^{(\pm)}(t,s)$. For $0 \geq t \geq s$, a time translation in Eq. (8) gives

$$U_\epsilon(t,s) = I + \frac{1}{i\hbar}\int_{s+\theta}^{t+\theta} dt' H^{(+)}(t')U_\epsilon(t'-\theta,s). \tag{9}$$

Comparison with the equation for $U^{(+)}(t+\theta,s+\theta)$

$$U^{(+)}(t+\theta,s+\theta) = I + \frac{1}{i\hbar}\int_{s+\theta}^{t+\theta} dt' H^{(+)}(t')U^{(+)}(t',s+\theta)$$

and unicity of the solution imply the identification

$$U_\epsilon(t,s) = U^{(+)}(t+\theta,s+\theta). \tag{10}$$

Since θ enters in the operator $U^{(+)}(t+\theta,s+\theta)$ only in its temporal variables, we obtain

$$\partial_\theta U_\epsilon(t,s) = \partial_t U_\epsilon(t,s) + \partial_s U_\epsilon(t,s). \tag{11}$$

By using Eq. (7) and its adjoint, the first identity is proven.

If $t \geqslant s \geqslant 0$, the same procedure gives $U_\epsilon(t,s)=U^{(-)}(t-\theta,s-\theta)$ and therefore $\partial_\theta U_\epsilon(t,s) = -\partial_t U_\epsilon(t,s) - \partial_s U_\epsilon(t,s)$, which leads to the identity (6). An identity for $t \geqslant 0 \geqslant s$ can be obtained by writing $U_\epsilon(t,s)=U_\epsilon(t,0)U_\epsilon(0,s)$.

In the interaction picture, Eq. (3), the identities transform straightforwardly into the following ones:

$$\begin{aligned} i\hbar\epsilon g\frac{\partial}{\partial g}U_{\epsilon I}(t,s) &= H_{\epsilon I}(t)U_{\epsilon I}(t,s) - U_{\epsilon I}(t,s)H_{\epsilon I}(s) \quad \text{if } 0 \geqslant t \geqslant s, \\ &= - H_{\epsilon I}(t)U_{\epsilon I}(t,s) + U_{\epsilon I}(t,s)H_{\epsilon I}(s) \quad \text{if } t \geqslant s \geqslant 0, \end{aligned} \tag{12}$$

where $H_{\epsilon I}(t) = e^{i/\hbar t H_0} H_\epsilon(t) e^{-i/\hbar t H_0}$.

By applying Eqs. (12) with $s=-\infty$ or $t=\infty$ to an eigenstate $|\Psi_0\rangle$ of H_0, we obtain

$$\left(H - E_0 \pm i\hbar\epsilon g\frac{\partial}{\partial g}\right)U_{\epsilon I}(0,\pm\infty)|\Psi_0\rangle = 0. \tag{13}$$

This same equation is proven in the literature by direct use of Dyson's expansion. From now on, the proof of Gell-Mann and Low's theorem proceeds in the standard path, and is sketched for completeness in the next section.

III. CONCLUSION

The mathematical properties of the operators $U_{I\epsilon}(0,\pm\infty)$ were studied first by Dollard[21] for the case $H_0=-\Delta_2$ and square integrable or locally square integrable and asymptotically bounded potential $V(\vec{x})$, and extended to the many-particle Schrödinger equation. He showed that the operators are unitary and the Hamiltonians $H_\epsilon(t)$ do not have proper eigenstates. In the adiabatic limit, under further restrictions on the potential, they yield isometric Möller operators $\Omega^\pm = \lim_{\epsilon\to 0^+} U_{I\epsilon}(0,\pm\infty)$. The intertwining property $H\Omega^\pm = \Omega^\pm H_0$ implies that for scattering states the g-derivative term in Eq. (13) is zero. The emergence of a bound state from the adiabatic evolution of the unbounded states of H_0 was investigated by Suura *et al.*[22] Through the study of the potential $V(x)=-\delta(x)$, that allows for a single bound state, they conjectured that bound states are associated with nonanalytic behavior in ϵ of the Dyson series for $U_{I\epsilon}(0,\pm\infty)|\Psi_0\rangle$ when $E_0 < \epsilon$. A bound state requires a nontrivial adiabatic limit of Eq. (13) where the vector $U_{\epsilon I}(0,\pm\infty)|\Psi_0\rangle$ develops a phase proportional to $1/\epsilon$: this has been checked in diagrammatic expansion.[23,24] The singular phase is responsible of the energy shift and is precisely removed by the denominator in the definition of the vectors $|\Psi_\epsilon^{(\pm)}\rangle$, before the limit is taken.

The standard steps of the proof are as follows.

(1) For finite ϵ, the two identities, Eq. (13), are projected on the vector $|\Psi_0\rangle$, and yield a formula for the *energy shift*, where $E_\epsilon^{(\pm)} = \langle\Psi_0|H|\Psi_\epsilon^{(\pm)}\rangle$,

$$\mp i\hbar\epsilon g\frac{\partial}{\partial g}\log\langle\Psi_0|U_{\epsilon I}(0,\pm\infty)|\Psi_0\rangle = E_\epsilon^{(\pm)} - E_0. \tag{14}$$

(2) By eliminating E_0 in Eq. (13) with the aid of Eq. (14), with simple steps one obtains

$$\left(H - E_\epsilon^{(\pm)} \pm i\hbar\epsilon g \frac{\partial}{\partial g}\right)|\Psi_\epsilon^{(\pm)}\rangle = 0. \qquad (15)$$

The adiabatic limit $\epsilon \to 0^+$ is now taken, and the limit vectors $|\Psi^{(\pm)}\rangle$ obtained by pulling onward or backward in time the same asymptotic eigenstate $|\Psi_0\rangle$ are eigenvectors of $H = H_0 + gV$ with eigenvalues $E^{(\pm)}$.

(3) The time-reversal operator has the action $T^\dagger U_\epsilon(t,s) T = U_\epsilon(-t,-s)$. If H_0 commutes with T the relation extends to the interaction propagator and $T^\dagger U_{\epsilon I}(0,\infty) T = U_{\epsilon I}(0,-\infty)$. If $|\Psi_0\rangle$ is also an eigenstate of T, it follows that $T^\dagger|\Psi_\epsilon^{(+)}\rangle$ is parallel to $|\Psi_\epsilon^{(-)}\rangle$ and $E^{(+)} = E^{(-)}$. The proportionality factor equals 1, since $\langle E_0|\Psi^{(+)}\rangle = \langle E_0|\Psi^{(-)}\rangle$.

The formula for the energy shift, Eq. (14), can be recast in a form involving the S-operator. From Eqs. (12), the following relation follows:

$$-i\hbar\epsilon g \frac{\partial S_\epsilon}{\partial g} = H_0 S_\epsilon + S_\epsilon H_0 - 2U_{\epsilon I}(\infty,0) H U_{\epsilon I}(0,-\infty).$$

The expectation value on the eigenstate $|\Psi_0\rangle$ and use of the theorem give Sucher's formula[25]

$$E - E_0 = \lim_{\epsilon\to 0} \frac{i\hbar\epsilon}{2} g \frac{\partial}{\partial g} \log\langle\Psi_0|U_{\epsilon I}(\infty,-\infty)|\Psi_0\rangle. \qquad (16)$$

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