POLARIZATION

NOTES BY L. G. MOLINARI

1. The effective interaction

In a many body system the "bare" interaction energy of two particles in empty space, U_0, is "dressed" by polarization insertions. The resulting effective interaction is

$$U_{\mu\mu',\nu\nu'}(1,2) = U^0_{\mu\mu',\nu\nu'}(1,2) + U^0_{\mu\mu',\rho\rho'}(1,3)\Pi_{\rho\rho',\sigma\sigma'}(3,4)U^0_{\sigma\sigma',\nu\nu'}(4,2) \tag{1}$$

where repeated variables are summed or integrated. The polarization $\Pi_{\rho\rho',\sigma\sigma'}(3,4)$ is the sum of all topologically distinct connected diagrams with a particle being created with spin ρ and one destroyed with spin ρ' at the space-time point 3 and another pair of events occurring at 4. Such diagrams are generated by the perturbative expansion of the two-point connected correlator:

$$iD^{\mathsf{T}}_{\mu\mu',\nu\nu'}(1,2) = \langle E|\mathsf{T}\ \delta\left(\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)\right)\ \delta\left(\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)\right)|E\rangle \tag{2}$$

where $\delta A(t) = A(t) - \langle E|A|E\rangle$ (the expectation value is independent of time) and $\langle E|T\delta A(t)\delta B(t')|E\rangle = \langle E|TA(t)B(t')|E\rangle - \langle E|A|E\rangle\langle E|B|E\rangle$. By the reduction formula:

$$\begin{aligned} iD^{\mathsf{T}}_{\mu\mu',\nu\nu'}(1,2) &= \langle E_0|\mathsf{T}S\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)|E_0\rangle_* \\ &-\langle E_0|\mathsf{T}S\psi^\dagger_\mu(1^+)\psi_{\mu'}(1)|E_0\rangle_*\langle E_0|\mathsf{T}S\psi^\dagger_\nu(2^+)\psi_{\nu'}(2)|E_0\rangle_* \end{aligned}$$

The zero-order term ($S=1$) is evaluated by Wick's theorem (fermions).

$$\begin{aligned} iD^{T(0)}_{\mu\mu',\nu\nu'}(1,2) &= \langle E_0|\mathsf{T}\psi_{\mu'}(1)\psi^\dagger_\nu(2)|E_0\rangle\langle E_0|\mathsf{T}\psi^\dagger_\mu(1)\psi_{\nu'}(2)|E_0\rangle \\ &= G^{(0)}_{\mu'\nu}(1,2)G^{(0)}_{\nu'\mu}(2,1) \end{aligned}$$

On the other hand, the first diagram of polarization is (fermions):

$$\Pi^{(0)}_{\mu\mu',\nu\nu'}(1,2) = \frac{i}{\hbar}(-1)G^{(0)}_{\mu'\nu}(1,2)G^{(0)}_{\nu'\mu}(2,1) \tag{3}$$

This establishes the correct factor for the important identity that holds at any order of perturbation theory:

$$\boxed{D^{\mathsf{T}}_{\mu\mu',\nu\nu'}(x,y) = \hbar\Pi_{\mu\mu',\nu\nu'}(x,y)} \tag{4}$$

It shows that the polarization is symmetric: $\Pi_{\rho\rho',\sigma\sigma'}(x,y) = \Pi_{\sigma\sigma'\rho\rho'}(y,x)$. The symmetry implies that *the exchange symmetry of the bare interaction U^0 is inherited by the effective potential*:

$$\boxed{U_{\mu\mu',\nu\nu'}(x,y) = U_{\nu\nu',\mu\mu'}(y,x)} \tag{5}$$

Date: revised Oct 2019.

Exercise 1.1. *Show that if the bare interaction does not modify the spin of the particles, i.e.* $U^0_{\mu\mu'\nu\nu'}(x,y) = \delta_{\mu\mu'}\delta_{\nu\nu'}U^0(x,y)$*, then the same form holds for the effective interaction:* $U_{\mu\mu'\nu\nu'}(x,y) = \delta_{\mu\mu'}\delta_{\nu\nu'}U(x,y)$*.*

If U^0 does not change spins, eq.(1) involves the scalar polarization:

$$\Pi(x,y) =: \sum_{\rho\sigma} \Pi_{\rho\rho\sigma\sigma}(x,y) \tag{6}$$

The corresponding two-point function $D(x,y) = \sum_{\rho\sigma} D_{\rho\rho\sigma\sigma}(x,y)$ is the density-density connected correlator:

$$\boxed{iD(x,y) = i\hbar\,\Pi(x,y) = \langle E|T\delta n(x)\delta n(y)|E\rangle} \tag{7}$$

Exercise 1.2. *Show that* $\int dx\Pi(x,y) = 0$*. Does this imply that* $\int dx\Pi^*(x,y) = 0$*?*

2. Proper polarization

The polarization diagrams may be reordered as $\Pi = \Pi^\star + \Pi^1 + \Pi^2 \ldots$, where $\Pi^\star$ is the sum of *proper* or *irreducible* polarization diagrams, i.e. diagrams that cannot be disconnected into two simpler polarisation diagrams by removal of a single U^0 line. Π^1 is the sum of polarization diagrams that may be disconnected (into polarization diagrams) in a unique way, i.e. there is just one line U^0 whose removal disconnects the diagram into two proper ones ($\Pi^1 = \Pi^\star U^0 \Pi^\star$), and so on. Therefore:

$$\begin{aligned}\Pi =&\Pi^\star + \Pi^\star U^0 \Pi^\star + \Pi^\star U^0 \Pi^\star U^0 \Pi^\star + \ldots \\ =&\Pi^\star + \Pi^\star U^0 (\Pi^\star + \Pi^\star U^0 \Pi^\star + \ldots) \\ =&\Pi^\star + \Pi^\star U^0 \Pi.\end{aligned}$$

In the same way one obtains $\Pi = \Pi^\star + \Pi U^0 \Pi^\star$. They are the Dyson's equations for the polarization Π, in terms of the proper polarization. One of them is:

$$\boxed{\Pi(1,2) = \Pi^\star(1,2) + \int d34\; \Pi^\star(1,3)U^0(3,4)\Pi(4,2)} \tag{8}$$

As a consequence, one obtains a Dyson equation for the effective interaction in terms of the proper polarization:

$$\boxed{U(1,2) = U^0(1,2) + \int d34\; U^0(1,3)\Pi^\star(3,4)U(4,2)} \tag{9}$$

Exercise 2.1. *Show that* $U(1,2) = U^0(1,2) + U(1,3)\Pi^\star(3,4)U^0(4,2)$ *and* $\int d3\; U^0(1,3)\Pi(3,2) = \int d3\; U(1,3)\Pi^\star(3,2)$*.*

3. Space-time translation invariance

If both U^0 and Π are space-time translation-invariant functions (i.e. $f(x+y, x'+y) = f(x,x')$ for all y), then also U and $\Pi^\star$ are invariant. It is convenient to expand the functions in $k = (\mathbf{k}, \omega)$ space: $f(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} f(k)$, with $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$. The Dyson equations become algebraic in $k-$space:

$$U(k) = U^0(k) + U^0(k)\Pi^\star(k)U(k)$$
$$\Pi(k) = \Pi^\star(k) + \Pi^\star(k)U^0(k)\Pi(k)$$

They are matrix equations in spin variables. If they are scalar equations, the solutions are:

$$\boxed{U(k) = \frac{U^0(k)}{1 - U^0(k)\Pi^\star(k)}\ , \qquad \Pi(k) = \frac{\Pi^\star(k)}{1 - U^0(k)\Pi^\star(k)}} \tag{10}$$

For a static two-particle potential $U^0(x, x') = v(\mathbf{x} - \mathbf{x}')\delta(t - t')$, it is $U^0(k) = v(\mathbf{k})$. Then:

$$U(\mathbf{k}, \omega) = \frac{v(\mathbf{k})}{\epsilon(\mathbf{k}, \omega)}, \qquad \Pi(\mathbf{k}, \omega) = \frac{\Pi^*(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \tag{11}$$

$$\boxed{\epsilon(\mathbf{k}, \omega) = 1 - v(\mathbf{k})\Pi^\star(\mathbf{k}, \omega)} \tag{12}$$

where $\epsilon(k)$ is the *generalised dielectric function*. Despite the bare interaction being static, the effective interaction is time-dependent through the dielectric function, which describes the response of the medium. For the Coulomb interaction,

$$v(\mathbf{k}) = \frac{4\pi e^2}{|\mathbf{k}|^2} \quad \Rightarrow \quad U(\mathbf{k}, \omega) = \frac{4\pi e^2}{|\mathbf{k}|^2 - 4\pi e^2 \Pi^\star(\mathbf{k}, \omega)}$$

the long-range Coulomb interaction is modified by the screening produced by the polarized medium.

4. Retarded correlators

The definitions so far introduced for U and ϵ involve the time-ordered correlator

$$iD^{\mathsf{T}}(x, x') = \langle E|\mathsf{T}\delta\hat{n}(x)\,\delta\hat{n}(x')|E\rangle$$

The correlator admits a simple reduction formula that allows for a diagrammatic expansion. In the theory of linear response, the retarded density-density correlator occurs:

$$iD^{\mathsf{R}}(x, x') = \langle E|[\hat{n}(x), \hat{n}(x')]|E\rangle\ \theta(t - t') \tag{13}$$

The two correlators are simply related in their Lehmann's representation.
The properties $\Pi^{\mathsf{T}}(x, y) = \Pi^{\mathsf{T}}(y, x)$ and $\Pi^{\mathsf{R}}(x, y) = \Pi^{\mathsf{R}}(x, y)^*$ imply:

$$\Pi^{\mathsf{T}}(k) = \Pi^{\mathsf{T}}(-k), \quad \text{and} \quad \Pi^{\mathsf{R}}(k)^* = \Pi^{\mathsf{R}}(-k).$$

5. Lehmann's representation

For simplicity we consider the density-density correlators. The time-dependence in $t - t'$ is made explicit by writing the Heisenberg evolution and by inserting a complete set of eigenstates of the exact Hamiltonian,

$$\hat{H}|a\rangle = \hbar\omega_a|a\rangle$$

The ground state is $\hat{H}|E\rangle = \hbar\omega_0|E\rangle$. The theta functions are replaced by their Fourier representation. The correlators in frequency space are obtained:

$$D(\mathbf{x}, \mathbf{y}; \omega) = \sum_a{}' \frac{\langle E|\hat{n}(\mathbf{x})|a\rangle\,\langle a|\hat{n}(\mathbf{y})|E\rangle}{\omega - (\omega_a - \omega_0) + i\eta} - \frac{\langle E|\hat{n}(\mathbf{y})|a\rangle\,\langle a|\hat{n}(\mathbf{x})|E\rangle}{\omega + (\omega_a - \omega_0) \mp i\eta}$$

where hereafter the upper sign applies to the T-ordered correlator, and the lower sign applies to the retarded one. The primed sum does not include the ground state. Since $\omega_a > \omega_0$, the poles have a precise pattern:

Remark 5.1. *The retarded function D^{R} is analytic in Im $\omega > 0$. If $\omega_1 + i\omega_2$ is a pole of D^{T}, then ω_1 and ω_2 have opposite signs.*

The Lehmann representation is written in the integral form:

$$D(\mathbf{x},\mathbf{y};\omega) = \int_0^\infty d\omega' \, \frac{d(\mathbf{x},\mathbf{y},\omega')}{\omega - \omega' + i\eta} - \frac{d(\mathbf{y},\mathbf{x},\omega')}{\omega + \omega' \mp i\eta}$$

$$d(\mathbf{x},\mathbf{y};\omega) = {\sum_a}' \langle E|\hat{n}(\mathbf{x})|a\rangle\langle a|\hat{n}(\mathbf{y})|E\rangle \delta(\omega - \omega_a + \omega_0) = d(\mathbf{y},\mathbf{x};\omega)^*$$

Finally, the introduction of the weight function (*spectral function*)

$$\Delta(\mathbf{x},\mathbf{y},\omega) = \theta(\omega) d(\mathbf{x},\mathbf{y},\omega) - \theta(-\omega) d(\mathbf{y},\mathbf{x},-\omega)$$

gives the simple representation, that exhibits the relation:

$$D^{\mathsf{T}}(\mathbf{x},\mathbf{y},\omega) = \int_{-\infty}^{\infty} d\omega' \frac{\Delta(\mathbf{x},\mathbf{y},\omega')}{\omega - \omega' + i\eta\,\mathrm{sign}\,\omega'}, \tag{14}$$

$$D^{\mathsf{R}}(\mathbf{x},\mathbf{y},\omega) = \int_{-\infty}^{\infty} d\omega' \frac{\Delta(\mathbf{x},\mathbf{y},\omega')}{\omega - \omega' + i\eta} \tag{15}$$

The connection become simpler if the correlators are translation-invariant.

5.1. **Space translation invariance.** Lehmann's representation is based on translation invariance in time. If H commutes with the total momentum $\mathbf{P}$, the eigenstates of H can be chosen as eigenvectors of $\mathbf{P}$ and labelled as $|a,\mathbf{k}\rangle$:

$$H|a,\mathbf{k}\rangle = \hbar\,\omega_{a\mathbf{k}}|a,\mathbf{k}\rangle, \qquad \mathbf{P}|a,\mathbf{k}\rangle = \hbar\mathbf{k}|a,\mathbf{k}\rangle$$

The frame of reference is chosen so that the system in the ground state is at rest: $\mathbf{P}|E\rangle = 0$. Under the action of the translation group, field operators transform as scalars:

$$\boxed{e^{-\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{P}}\psi_\mu(\mathbf{0})e^{\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{P}} = \psi_\mu(\mathbf{x})} \tag{16}$$

As a consequence: $\langle E|\hat{n}(\mathbf{x})|a,\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{x}}\langle E|\hat{n}(\mathbf{0})|a,\mathbf{k}\rangle$ and $d(\mathbf{x},\mathbf{y};\omega) = \frac{1}{V}\sum_{\mathbf{k}} d(\mathbf{k},\omega)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$

$$d(\mathbf{k},\omega) = V{\sum_a}' |\langle E|\hat{n}(\mathbf{0})|a,\mathbf{k}\rangle|^2 \delta(\omega - \omega_{a,\mathbf{k}} + \omega_0)$$

$$\Delta(\mathbf{k},\omega) = \theta(\omega) d(\mathbf{k},\omega) - \theta(-\omega) d(-\mathbf{k},-\omega) \tag{17}$$

Since the spectral function $\Delta(\mathbf{k},\omega)$ is real, by means of the fundamental identity:

$$\boxed{\frac{1}{x - y \pm i\eta} = \frac{P}{x-y} \mp i\pi\delta(x-y)} \tag{18}$$

we separate real and imaginary parts:

$$D^{\mathsf{T}}(\mathbf{k},\omega) = P\int_{-\infty}^{\infty} d\omega' \frac{\Delta(\mathbf{k},\omega')}{\omega - \omega'} - i\pi\ \mathrm{sign}(\omega)\ \Delta(\mathbf{k},\omega) \tag{19}$$

$$D^{\mathsf{R}}(\mathbf{k},\omega) = P\int_{-\infty}^{\infty} d\omega' \frac{\Delta(\mathbf{k},\omega')}{\omega - \omega'} - i\pi\ \Delta(\mathbf{k},\omega) \tag{20}$$

the comparison of real and imaginary parts gives the important relations that allow for the simple transition from one correlator to the other:

$$\boxed{\mathrm{Re}\ D^{\mathsf{R}}(\mathbf{k},\omega) \ = \ \mathrm{Re}\ D^{\mathsf{T}}(\mathbf{k},\omega), \quad \mathrm{Im}\ D^{\mathsf{R}}(\mathbf{k},\omega) \ = \ \mathrm{sign}(\omega)\mathrm{Im}\ D^{\mathsf{T}}(\mathbf{k},\omega)} \tag{21}$$

Remark 5.2. *Note that being* $\mathrm{Im}\, D^{\mathsf{R}}(\mathbf{k},\omega) = -\pi\Delta(\mathbf{k},\omega)$*, by eq.*(17) *the sign of Im* $D^{\mathsf{R}}(\mathbf{k},\omega)$ *is opposite to the sign of* ω*.*

In accordance with this, let us define the retarded polarizations

$$\Pi^{\mathsf{R}}(\mathbf{k},\omega) = \mathrm{Re}\,\Pi(\mathbf{k},\omega) + i\,\mathrm{sign}(\omega)\,\mathrm{Im}\Pi(\mathbf{k},\omega)$$
$$\Pi^{\star\mathsf{R}}(\mathbf{k},\omega) = \mathrm{Re}\,\Pi^{\star}(\mathbf{k},\omega) + i\,\mathrm{sign}(\omega)\,\mathrm{Im}\Pi^{\star}(\mathbf{k},\omega)$$

Proposition 5.3. *The full and proper retarded polarizations are related in the same way as* T*-ordered ones:*

$$\Pi^{\mathsf{R}}(\mathbf{k},\omega) = \frac{\Pi^{\star\mathsf{R}}(\mathbf{k},\omega)}{1 - v(\mathbf{k})\Pi^{\star\mathsf{R}}(\mathbf{k},\omega)} \tag{22}$$

Proof. By the exchange symmetry $v(\mathbf{k})$ is real. Let $\Pi^{\star} = \Pi_1^{\star} + i\Pi_2^{\star}$, then:

$$\begin{aligned}\Pi^{\mathsf{R}} =& [\mathrm{Re} + i\,\mathrm{sign}\,(\omega)\,\mathrm{Im}]\frac{\Pi_1^{\star} + i\Pi_2^{\star}}{1 - v\,\Pi_1^{\star} - iv\,\Pi_2^{\star}} = \frac{\Pi_1^{\star} - v(\Pi_1^{\star})^2 - v(\Pi_2^{\star})^2 + i\,\mathrm{sign}(\omega)\,\Pi_2^{\star}}{(1 - v\Pi_1^{\star})^2 + (\Pi_2^{\star})^2}\\ =& \frac{\Pi^{\star\mathsf{R}} - v|\Pi^{\star\mathsf{R}}|^2}{|1 - v\Pi^{\star\mathsf{R}}|^2} = \frac{\Pi^{\star\mathsf{R}}}{1 - v\Pi^{\star\mathsf{R}}}\end{aligned}$$

Note that Im Π^{R} and Im $\Pi^{\star ret}$ have *the same sign*, opposite to that of ω. □

5.2. **Dielectric function.**

The *retarded* generalized dielectric function of a translation-invariant system is

$$\epsilon^{\mathsf{R}}(\mathbf{k},\omega) = 1 - v(\mathbf{k})\Pi^{\star\mathsf{R}}(\mathbf{k},\omega) \tag{23}$$

In linear response, a perturbation $\varphi(x)$ coupled to the density causes a density variation

$$\delta n(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\frac{\Pi^{\star\mathsf{R}}(\mathbf{k},\omega)}{\epsilon^{\mathsf{R}}(\mathbf{k},\omega)}\varphi(\mathbf{k},\omega)$$

The frequency integral is strongly influenced by the zeros of the retarded dielectric function that are near the real axis. Such zeros are associated to *collective excitations* of the system, such as plasmons for the electron gas, or zero sound in ^{3}He.
Since $\Pi^{\mathsf{R}} = \Pi^{\star\mathsf{R}}/\epsilon^{\mathsf{R}}$, a zero of ϵ^{R} is a pole of Π^{R}, and poles occur in the lower half $\omega-$plane. Therefore, if $\omega_1(\mathbf{k}) + i\omega_2(\mathbf{k})$ is a zero of ϵ^{R} then it is always $\omega_2(\mathbf{k}) < 0$. This imaginary part of the zero describes a *damping process* of the collective excitation, while $\omega_1(\mathbf{k})$ is the dispersion relation.
Interesting zeros are those with $|\omega_2| \ll |\omega_1|$, that describe long lasting excitations. A linear expansion in ω_2 of the complex equation $\epsilon^{\mathsf{R}}(\mathbf{k},\omega_1 + i\omega_2) = 0$ gives two real equations: one for the real part ω_1 (dispersion relation):

$$1 - v(\mathbf{k})\,\mathrm{Re}\,\Pi^{\star}(\mathbf{k},\omega_1(\mathbf{k})) = 0. \tag{24}$$

the other one gives the damping:

$$\omega_2(\mathbf{k}) = -\frac{\mathrm{Im}\Pi^{\star\mathsf{R}}(\mathbf{k},\omega)}{\frac{\partial}{\partial\omega}\mathrm{Re}\Pi^{\star}(\mathbf{k},\omega)}\Big|_{\omega=\omega_1(\mathbf{k})} \tag{25}$$

6. The ring diagram

The simplest approximation to the proper polarization is the ring diagram

$$\Pi^0(x,x') = \sum_{\mu\nu} \Pi^0_{\mu\mu,\nu\nu}(x,x') = -\frac{i}{\hbar}\sum_{\mu\nu} G^0_{\mu\nu}(x,x')G^0_{\nu\mu}(x',x) \tag{26}$$

The effective interaction is the sum of insertions of rings, i.e. particle-hole excitations. This is known as "ring approximation" or "random phase approximation" (R.P.A.). We evaluate the ring diagram for non-interacting fermions with single-particle Hamiltonian h and eigenstates $h|a\rangle = \hbar\omega_a|a\rangle$. The Green function is:

$$iG^0_{\mu\mu'}(x,x') = \sum_a \langle \mathbf{x}\mu|a\rangle\langle a|\mathbf{x}'\mu'\rangle e^{-i\omega_a(t-t')}[\theta(t-t')\theta(\omega_a-\omega_F) - \theta(t'-t)\theta(\omega_F-\omega_a)]$$

where $\hbar\omega_F$ is the highest filled energy. Then:

$$\Pi^0(x,x') = -\frac{i}{\hbar}\sum_{\mu\mu'}\sum_{ab}\langle \mathbf{x}\mu|a\rangle\langle a|\mathbf{x}'\mu'\rangle\langle \mathbf{x}'\mu'|b\rangle\langle b|\mathbf{x}\mu\rangle e^{-i(\omega_a-\omega_b)(t-t')}$$
$$\times\,[\theta(t-t')\theta(\omega_a-\omega_F)\theta(\omega_F-\omega_b) + \theta(t'-t)\theta(\omega_F-\omega_a)\theta(\omega_b-\omega_F)]$$

The theta functions show that the ring diagram describes the propagation of a particle-hole pair. In ω space:

$$\Pi^0(\mathbf{x},\mathbf{x}',\omega) = \frac{1}{\hbar}\sum_{\mu\mu'}\sum_{ab}\langle \mathbf{x}\mu|a\rangle\langle a|\mathbf{x}'\mu'\rangle\langle \mathbf{x}'\mu'|b\rangle\langle b|\mathbf{x}\mu\rangle$$
$$\times\left[\frac{\theta(\omega_a-\omega_F)\theta(\omega_F-\omega_b)}{\omega-(\omega_a-\omega_b)+i\eta} - \frac{\theta(\omega_F-\omega_a)\theta(\omega_b-\omega_F)}{\omega-(\omega_a-\omega_b)-i\eta}\right] \tag{27}$$

this is already the Lehmann representation. The retarded polarisation is readily obtained, by changing $-i\eta$ to $+i\eta$ in the second term.

6.1. **Ideal Fermi gas.** The ring diagram for the ideal Fermi gas can be computed analytically (Jens Lindhard, 1954):

$$\Pi^0(\mathbf{q},\omega) = \frac{2}{\hbar}\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{\theta(k-k_F)\theta(k_F-|\mathbf{k}-\mathbf{q}|)}{\omega-(\omega_k-\omega_{|\mathbf{k}-\mathbf{q}|})+i\eta} - \frac{\theta(k_F-k)\theta(|\mathbf{k}-\mathbf{q}|-\omega_F)}{\omega-(\omega_k-\omega_{|\mathbf{k}-\mathbf{q}|})-i\eta}\right]$$

The real and imaginary parts are extracted, and depend on $q=|\mathbf{q}|$.

$$\mathrm{Re}\,\Pi^0(q,\omega) = \frac{2}{\hbar}⨍\frac{d\mathbf{k}}{(2\pi)^3}\frac{\theta(k-k_F)\theta(k_F-|\mathbf{k}-\mathbf{q}|)-\theta(k_F-k)\theta(|\mathbf{k}-\mathbf{q}|-k_F)}{\omega-\omega_k+\omega_{|\mathbf{k}-\mathbf{q}|}}$$

The replacement $\theta(k-k_F) = 1-\theta(k_F-k)$ and a change $\mathbf{k}-\mathbf{q}\to-\mathbf{k}$ give:

$$\mathrm{Re}\,\Pi^0(q,\omega) = \frac{2}{\hbar}⨍\frac{d\mathbf{k}}{(2\pi)^3}\theta(k_F-k)\left[\frac{1}{\omega+\omega_k-\omega_{|\mathbf{k}-\mathbf{q}|}} - \frac{1}{\omega-\omega_k+\omega_{|\mathbf{k}-\mathbf{q}|}}\right] \tag{28}$$

The calculations can be done to the end. The result for $\omega=0$ is useful:

$$\mathrm{Re}\,\Pi^0(q,0) = -\frac{mk_F}{(2\pi\hbar)^2}g(q/k_F), \qquad g(x) = \frac{1}{2} - \frac{4-x^2}{8x}\log\left|\frac{2-x}{2+x}\right| \tag{29}$$

$g(0)=1$, g' is singular in $x=2$, $g(x)\to 0$ for $x\to\infty$

$$\mathrm{Im}\,\Pi^0(q,\omega) = -\frac{2\pi}{\hbar}\int\frac{d\mathbf{k}}{(2\pi)^3}\delta(\omega-\omega_k+\omega_{|\mathbf{k}-\mathbf{q}|})$$
$$\times\,[\theta(k-k_F)\theta(k_F-|\mathbf{k}-\mathbf{q}|)+\theta(k_F-k)\theta(|\mathbf{k}-\mathbf{q}|-k_F)] \tag{30}$$

The imaginary part of the polarization is an integral in $\mathbf{k}$-space with support fixed by delta and theta functions that have simple interpretation as energy conservation for a particle-hole excitation/annihilation.
Consider a process where a particle-hole pair is created from the unperturbed Fermi ground state by absorbing energy $\hbar\omega$ (then $\omega > 0$) and momentum $\hbar\mathbf{q}$.
The momentum of the filled Fermi sphere is zero, and the energy is E. A particle with momentum $\hbar(\mathbf{k}-\mathbf{q})$ below the Fermi surface ($\omega_{|\mathbf{k}-\mathbf{q}|} < \omega_F$) is excited to a state of momentum $\hbar\mathbf{k}$ above the Fermi surface. The result is a particle-hole state if conservation of energy is fulfilled: $E + \hbar\omega = E + \hbar\omega_k - \hbar\omega_{|\mathbf{k}-\mathbf{q}|}$. Then:

$$\hbar\omega = -\frac{\hbar^2 q^2}{2m} + \frac{\hbar}{m}\mathbf{q}\cdot\mathbf{k} = \frac{\hbar^2 q^2}{2m} + \frac{\hbar}{m}\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})$$

Since $qk_F \le \mathbf{q}\cdot(\mathbf{k}-\mathbf{q}) = q|\mathbf{k}-\mathbf{q}|\cos\theta \le qk_F$, we obtain a boundary in (q,ω) space outside which the process of exciting a particle-hole pair is forbidden as it violates energy conservation:

$$\boxed{\frac{\hbar^2 q^2}{2m} - \frac{\hbar^2 k_F}{m} q \le \hbar\omega \le \frac{\hbar^2 q^2}{2m} + \frac{\hbar^2 k_F}{m} q} \tag{31}$$

Outside this region the imaginary part of the polarization is zero.