FUNCTIONAL INTEGRAL FOR THE THERMAL PARTITION FUNCTION FOR INTERACTING FERMIONS

LUCA G. MOLINARI

ABSTRACT. In the holomorphic representation, the creation and destruction operators of fermions are represented as multiplications and derivations on functions of anticommuting variables. The functional integral for the thermal partition function is then constructed by means of complete sewts of anticommuting coherent states.

1. Fock space for fermions

Given N orthonormal single-particle states $|1\rangle, \dots, |N\rangle$, the Fock space $\mathscr{F}_N$ for fermions supports an irreducible representation of the canonical anticommutation relations (CAR) with N pairs of linear operators $\hat{a}_r$, $\hat{a}_r^\dagger$, adjoint of each other, that destroy or create a particle in one of the single-particle states:

$$\boxed{\{\hat{a}_r, \hat{a}_s^\dagger\} = \delta_{rs}, \qquad \{\hat{a}_r, \hat{a}_s\} = 0, \qquad \{\hat{a}_r^\dagger, \hat{a}_s^\dagger\} = 0, \quad r, s = 1, \dots, N} \tag{1}$$

The rules imply $\hat{a}_r^2 = 0$ and $\hat{a}_r^{\dagger 2} = 0$. The relations are symmetric in the exchange of $\hat{a}_r$ with $\hat{a}_r^\dagger$ (particle - hole symmetry).
The destruction operators are defined by a common vacuum state $\hat{a}_r|0\rangle = 0$ for all r (the state is not the null state of the linear space, which has zero norm). The number vectors are obtained by adding particles to the vacuum:

$$|n_1, \dots, n_N\rangle = \hat{a}_1^{\dagger n_1} \cdots \hat{a}_N^{\dagger n_N} |0\rangle, \qquad n_r = 0, 1 \tag{2}$$

They are the eigenstates of the N commuting number operators $\hat{a}_r^\dagger \hat{a}_r$, with eigenvalues $n_r = 0, 1$, and form an orthonormal basis of $\mathscr{F}_N$. The linear dimension of the Fock space is thus 2^N, which is also the maximum number of fermions the system may contain. The state $|1, \dots, 1\rangle$ is annihilated by all creation operators.
It is useful to order the basis (**??**) in a sequence $|N_\alpha\rangle$, $\alpha = 1, \dots, 2^N$. Then:

$$\sum\nolimits_\alpha |N_\alpha\rangle\langle N_\alpha| = \hat{I}, \qquad \langle N_\alpha | N_{\alpha'}\rangle = \delta_{\alpha\alpha'}$$

The eigenvalue equation $\hat{a}_r|\psi\rangle = z_r|\psi\rangle$ has no solution in $\mathscr{F}_N$: the anticommutation rules would imply $z_r z_s + z_s z_r = 0$, i.e. the eigenvalues z_r would anticommute themselves. Coherent states prove very useful for bosons: it is desirable to obtain their analogue for fermions. This requires a Hilbert space larger than $\mathscr{F}_N$. Its construction is better understood through an explicit representation of the CAR rules (**??**) on a Hilbert space of functions of N anticommuting variables.

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2. Grassmann calculus

2.1. **Grassmann algebra.** The Grassmann algebra[1] Λ_N is constructed with N independent Grassmann units θ_k, that anticommute: $\theta_i\theta_j + \theta_j\theta_i = 0$. As the square of a Grassmann unit is zero, one can form 2^N independent products out of them: $\theta_1^{n_1}\cdots\theta_N^{n_N}$, $n_k = 0, 1$ ($\theta_i^0 = 1$). A criterion is to identify them as the terms of the following product, which includes the unit 1 (that commutes with all θ_a):

$$(1+\theta_1)(1+\theta_2)\cdots(1+\theta_N) = \Theta_1 + \Theta_2 + \cdots + \Theta_{2^n}$$

For $N = 2$ there are four such terms: 1, θ_1, θ_2, $\theta_1\theta_2$. For $N = 3$ there are 8 elements: 1, θ_1, θ_2, θ_3, $\theta_1\theta_2$, $\theta_1\theta_3$, $\theta_2\theta_3$, $\theta_1\theta_2\theta_3$.
Any product of different units $\theta_i\theta_j\ldots\theta_k$ is, up to a sign due to anticommutations that produce the desired order, an element Θ_α. The product of two elements $\Theta_\alpha\Theta_\beta$ is either zero (if they share one or more units) or another element $\pm\Theta_c$.

Λ_N is the real non commutative algebra of dimension 2^N with elements

$$f = \sum_\alpha \lambda_\alpha\Theta_\alpha, \quad \lambda_\alpha \in \mathbb{R}$$

Exercise 2.1. *Show that if M is a real matrix and $\theta'_r = \sum_{s=1}^N M_{rs}\theta_s$, then: $\theta'_1\ldots\theta'_N = (\det M)\,\theta_1\ldots\theta_N$.*

Besides the algebraic properties of addition and multiplication, one may introduce a "Grassmann calculus" on the algebra Λ_N.

2.2. **Grassmann derivative.** The (left) partial derivative of an element with respect to a Grassmann unit is defined by the requirement of linearity and by a modified Leibnitz property:

$$\frac{\partial}{\partial\theta_k}(\lambda f + g) = \lambda\frac{\partial f}{\partial\theta_k} + \frac{\partial g}{\partial\theta_k} \qquad \frac{\partial}{\partial\theta_k}(\theta_j f) = \delta_{jk}\, f - \theta_j\frac{\partial f}{\partial\theta_k} \tag{3}$$

where a Grassmann derivative anticommutes with a Grassmann unit (for example, $\partial_2\theta_1\theta_2\theta_7 = -\theta_1\theta_7$). The rule implies that Grassmann derivatives anticommute:

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0 \tag{4}$$

This operation justifies the terminology of θ_i as a *Grassmann variable* and of f as a *function* of such variables. Indeed we shall write $f(\theta) = \sum_\alpha \lambda_\alpha\Theta_\alpha(\theta)$.

2.3. **Grassmann integral.** The Grassmann integral of a function of Grassmann variables is defined by the rules:

1) linearity: $\int d\theta_k f(\theta) = \sum_\alpha \lambda_\alpha \int d\theta_k \Theta_\alpha(\theta)$;

2) if $\Theta_\alpha(\theta)$ does not contain θ_k it is $\int d\theta_k\Theta_\alpha(\theta) = 0$;

3) if a product $\theta_r\cdots\theta_s$ contains the factor θ_k, and there are n units at its left:

$$\int d\theta_k\,\theta_r\cdots\theta_k\cdots\theta_s = (-1)^n\theta_r\cdots 1\cdots\theta_s$$

[1]The axioms for an algebra of anticommuting variables (exterior algebra) were first given by Hermann Grassmann (1809 - 1877), a gymnasium professor.

Rules 2 and 3 correspond to $\int d\theta_k = 0$, $\int d\theta_k\,\theta_k = 1$ and $\{\theta_i, d\theta_k\} = 0$. The first two properties imply translation invariance in each "variable":

$$\int d\theta_k f(\theta_k + \eta) = \int d\theta_k f(\theta_k)$$

In multiple integrals by definition $\{d\theta_i, d\theta_j\} = 0$:

$$\int d\theta_1 \ldots d\theta_n f(\theta) = (-1)^P \int d\theta_{i_1} \cdots \int d\theta_{i_n}\, f(\theta).$$

A totally equivalent definition of the integral is:

$$\int d\theta_k f(\theta) = \frac{\partial}{\partial\theta_k} f(\theta) \tag{5}$$

3. Basic Grassmann integrals

Proposition 3.1. *If $\theta'_i = \sum_j M_{ij}\theta_j$ where M is an invertible $n \times n$ matrix, then:*

$$d\theta_1 \ldots d\theta_n = \det M\, d\theta'_1 \ldots d\theta'_n \tag{6}$$

Proof. Let $f(\theta) = \sum f_k \Theta_k(\theta)$, then $\int d\theta_1 \ldots d\theta_n\ f(\theta) = f_{2^n} \int d\theta_1 \ldots d\theta_n \theta_1 \cdots \theta_n$. Consider the integral: $\int d\theta_1 \ldots d\theta_n \quad f(\theta') \quad = \quad f_{2^n} \int d\theta_1 \ldots d\theta_n \quad \theta'_1 \cdots \theta'_n \quad = f_{2^n} M_{1i_1} \cdots M_{ni_n} \int d\theta_1 \ldots d\theta_n \quad \theta_{i_1} \cdots \theta_{i_n} \quad = \quad f_{2^n} \det M \int d\theta_1 \ldots d\theta_n \quad \theta_1 \cdots \theta_n \quad = \det M \int d\theta'_1 \ldots d\theta'_n f(\theta')$. □

Proposition 3.2. *Let η_i be Grassmann units,*

$$\int d\theta_1 \ldots d\theta_n\ \exp[\sum_{k=1}^n \theta_k \eta_k] = \eta_1 \cdots \eta_n \tag{7}$$

Proof. $\int d\theta_1 \ldots d\theta_n e^{\theta_1\eta_1 + \ldots + \theta_n\eta_n} \quad = \quad \int d\theta_1 \ldots d\theta_n (1 + \theta_1\eta_1) \cdots (1 + \theta_n\eta_n) \quad = \int d\theta_1 \ldots d\theta_n (\theta_1\eta_1) \cdots (\theta_n\eta_n) = \int d\theta_1 (\theta_1\eta_1) \cdots \int d\theta_n (\theta_n\eta_n)$. □

Proposition 3.3. *M is any $n \times n$ complex matrix and $\theta_k, \bar\theta_k$ are $2n$ Grassmann units:*

$$\boxed{\int d\theta_1 d\bar\theta_1 \ldots d\theta_n d\bar\theta_n\ \exp[\sum_{ij} \bar\theta_i M_{ij} \theta_j] \ = \ \det M} \tag{8}$$

Proof. Reorder the factors in the measure as follows $d\theta_1 d\bar\theta_1 \ldots d\theta_n d\bar\theta_n = \sigma d\theta_1 \ldots d\theta_n d\bar\theta_1 \ldots d\bar\theta_n$, where σ is a sign resulting from exchanges. Next put $\eta = M\theta$ and use propositions **??** and **??**. The Integral becomes:

$$\sigma \det M \int d\eta_1 \ldots d\eta_n \int d\bar\theta_1 \ldots d\bar\theta_n \exp(\sum_k \bar\theta_k \eta_k)$$
$$= \sigma \det M \int d\eta_1 \ldots d\eta_n \eta_1 \cdots \eta_n = \det M \int d\eta_1 \eta_1 \ldots \int d\eta_n \eta_n = \det M$$

□

Of course, the integral is equivalent to the one where θ_k and $\bar\theta_k$ are exchanged:

$$\int d\bar\theta_1 d\theta_1 \ldots d\bar\theta_n d\theta_n\, e^{-\sum_{ij} \bar\theta_i M_{ij} \theta_j} \ = \ \det M$$

Proposition 3.4. *ξ_i and $\bar\eta_j$ are Grassmann variables:*

$$\boxed{\int d\theta_1 d\bar\theta_1 \ldots d\theta_n d\bar\theta_n\, e^{\bar\theta M \theta + \bar\eta\theta + \bar\theta\xi} \ = \ e^{-\bar\eta [M^{-1}] \xi} \det M} \tag{9}$$

Proof. Make the shifts $\theta_i \to \theta_i + M^{-1}{}_{ij}\xi_j$ and $\overline{\theta}_i \to \overline{\theta}_i + \overline{\eta}_j M^{-1}{}_{ji}$ in (**??**). □

Proposition 3.5. *Let A be a real anti-symmetric $n\times n$ matrix:*

$$\int d\theta_1 \dots d\theta_n \, e^{\frac{1}{2}\sum_{ij} A_{ij}\theta_i\theta_j} = \sqrt{\det A} \quad (n \text{ even}) \tag{10}$$

(if n is odd it is always $\det A = 0$*).*

Proof. Consider the square of the integral

$$\int d\theta_1 \dots d\theta_n \int d\eta_1 \dots d\eta_n \, e^{\frac{1}{2}\sum_{ij} A_{ij}(\theta_i\theta_j + \eta_i\eta_j)}$$

and make the linear change of variables $\frac{1}{\sqrt{2}}(\theta_k + i\eta_k) = \xi_k$ and $\frac{1}{\sqrt{2}}(\theta_k - i\eta_k) = \overline{\xi}_k$ $(k = 1\dots n)$. The matrix is unitary. Then: $\frac{1}{2}\sum_{ij} A_{ij}(\theta_i\theta_j + \eta_i\eta_j) = \sum_{ij}\overline{\xi}_i A_{ij}\xi_j$. The squared integral is $\int d\xi_1 d\overline{\xi}_1 \dots d\xi_n d\overline{\xi}_n e^{\overline{\xi} A\xi} = \det A$. Note that $\det A \geq 0$ because the non-zero eigenvalues come in pairs $\pm i\lambda$. □

3.1. **Example: the partition function of non-interacting fermions** [?]. If the Hamiltonian is quadratic and diagonal, $\hat{K}_0 = \sum_j(\epsilon_j - \mu)\hat{a}_j^\dagger \hat{a}_j$, the partition function is given by the simple formula

$$Z_0 = \mathrm{tr}\, e^{-\beta \hat{K}_0} = \prod_j (1 + e^{-\beta(\epsilon_j - \mu)}) \tag{11}$$

where j enumerates states. We give a representation of each factor as a determinant. Consider the $n\times n$ matrix for a single mode of energy ϵ

$$S = \begin{bmatrix} 1 & 0 & & x \\ -x & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & -x & 1 \end{bmatrix}, \qquad x =: 1 - \beta\frac{\epsilon - \mu}{n} \tag{12}$$

The determinant is $1 + x^n$ and, for infinite n, it converges to $1 + e^{-\beta(\epsilon-\mu)}$. The determinant can be represented as a Grassmann integral on $2n$ variables:

$$\det S = \int d\overline{\theta}_1 d\theta_1 \dots d\overline{\theta}_n d\theta_n e^{-\overline{\theta} S \theta} \tag{13}$$

The quadratic form is

$$\sum_{ij} \overline{\theta}_i S_{ij}\theta_j = \sum_{i=1}^n \frac{\hbar\beta}{n}\overline{\theta}_i\left(\frac{\theta_i - \theta_{i-1}}{\hbar\beta/n} + \frac{\epsilon - \mu}{\hbar}\theta_{i-1}\right)$$

with the identification $\theta_0 = -\theta_n$. In the large n limit one interpolates the values θ_i with a function $\theta(\tau)$, $0 \leq \tau \leq \hbar\beta$, such that $\theta(i/\hbar\beta) = \theta_i$, with antiperiodic boundary condition $\theta(0) = -\theta(\hbar\beta)$. The finite sum becomes the integral

$$\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \overline{\theta}(\tau)(\hbar\partial_\tau + \epsilon - \mu)\theta(\tau)$$

With the measure $d\overline{\theta}(\tau)d\theta(\tau) = \prod_i d\overline{\theta}_i d\theta_i$ and summing all the modes we have:

$$Z_0 = \int \prod_a d\overline{\theta}_a(\tau) d\theta_a(\tau) e^{-\frac{1}{\hbar}\sum_a \int_0^{\hbar\beta} d\tau \overline{\theta}_a(\tau)(\hbar\partial_\tau + \epsilon_a - \mu)\theta_a(\tau)} \tag{14}$$

The result (**??**) for Z_0 is recovered with a change of variables

$$\theta_a(\tau) = \frac{1}{\sqrt{\hbar\beta}}\sum\nolimits_{n\in\mathbb{Z}} e^{-i\omega_n\tau}\theta_{a,n} \qquad \omega_n = \frac{n\pi}{\hbar\beta}$$

Antiperiodic b.c. require odd Matsubara frequencies. Since the transformation is unitary we obtain:

$$Z_0 = \int \prod_{a,n} d\overline{\theta}_{a,n} d\theta_{a,n} e^{-\overline{\theta}_{a,n}[-i\omega_n + \frac{1}{\hbar}(\epsilon_a-\mu)]\theta_{a,n}} = \prod_{a,n}\left[-i\omega_n + \frac{\epsilon_a-\mu}{\hbar}\right]$$

The sum $\sum_n e^{i\omega_n\eta}\log(-i\hbar\omega_n + \epsilon - \mu) = \log(1+e^{-\beta(\epsilon-\mu)})$ gives Ω_0.

In a representation which is not diagonal in energy, the path integral (**??**) is

$$\boxed{Z_0 = \int d\overline{\psi}(\mathbf{x},\tau) d\psi(\mathbf{x},\tau)\; e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\, \overline{\psi}(\mathbf{x},\tau)[\hbar\partial_\tau + k(\mathbf{x})]\psi(\mathbf{x},\tau)}} \tag{15}$$

with the boundary condition $\psi(\mathbf{x},\hbar\beta) = -\psi(\mathbf{x},0)$.

Now, let us introduce sources in the Grassmann integral for the partition function. The integral can be evaluated exactly, see (**??**) :

$$Z_0[\overline{\eta},\eta] = \int d\overline{\theta}_1 d\theta_1 \dots d\overline{\theta}_n d\theta_n\, e^{-\overline{\theta}S\theta - \overline{\eta}\theta - \overline{\theta}\eta} = \det S\, e^{\overline{\eta}S^{-1}\eta}$$

The inverse of S is the full matrix

$$S^{-1} = \frac{1}{1+x^n}\begin{bmatrix} 1 & -x^{n-1} & -x^{n-2} & & -x^2 & -x \\ x & \ddots & \ddots & \ddots & & -x^2 \\ x^2 & \ddots & & & \ddots & \\ & \ddots & & & & -x^{n-2} \\ x^{n-2} & & \ddots & \ddots & \ddots & -x^{n-1} \\ x^{n-1} & x^{n-2} & & x^2 & x & 1 \end{bmatrix}$$

$$[S^{-1}]_{ij} = \frac{x^{i-j}}{1+x^n}\vartheta_{ij} - \frac{x^{n+i-j}}{1+x^n}\vartheta_{ji} + \frac{1}{1+x^n}\delta_{ij} \tag{16}$$

$\vartheta_{ij} = 1$ if $i > j$ and zero otherwise. For large n put $\tau = \hbar\beta i/n$, $\tau' = \hbar\beta j/n$. Then

$$x^{i-j} = e^{-\frac{1}{\hbar}(\epsilon-\mu)(\tau-\tau')}, \qquad \vartheta_{ij} = \theta(\tau-\tau'), \qquad \frac{x^n}{1+x^n} = \frac{1}{e^{\beta(\epsilon-\mu)}+1} = f(\epsilon)$$

For the single mode: $\overline{\eta}S^{-1}\eta = -\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \overline{\eta}(\tau) g_0(\tau-\tau')\eta(\tau')$, with

$$-g_0(\tau-\tau') = e^{-\frac{1}{\hbar}(\epsilon-\mu)(\tau-\tau')}\{\theta(\tau-\tau')[1-f(\epsilon)] - \theta(\tau'-\tau)f(\epsilon)\}$$

Therefore: $\exp(\overline{\eta}S^{-1}\eta) = \exp[-\iint_0^{\hbar\beta} d\tau\, d\tau' \overline{\eta}(\tau) g_0(\tau-\tau')\eta(\tau')]$. Putting all modes together, in the position representation

$$Z[\overline{\eta},\eta] = Z_0\, e^{-\int dx dy \overline{\eta}(x)\mathscr{G}_0(x,y)\eta(y)} \tag{17}$$

4. The holomorphic representation

To introduce a conjugation, the number of Grassmann units must be even. In the algebra Λ_{2N}, N units are labelled $\theta_1, ..., \theta_N$ and N units are labelled $\overline{\theta}_1, ..., \overline{\theta}_N$. The unit 1 coincides with $\overline{1}$. The involutive operation is introduced, that exchanges θ_k with $\overline{\theta}_k$, with the rules

$$\overline{\overline{\theta}}_k = \theta_k, \quad \overline{\theta_i\theta_j\cdots\theta_r} = \overline{\theta}_r \cdots \overline{\theta}_j\overline{\theta}_i. \tag{18}$$

Consider the complex sub-algebra $\mathscr{A}_N$ with the N Grassmann units $\theta_1, \ldots, \theta_N$ ($\theta_a\theta_b + \theta_b\theta_a = 0$). It is useful to regard them as "Grassmann variables", although the θ_a have no range. An element of $\mathscr{A}_N$ is a "holomorphic" function of the Grassmann variables (but each unit can only have power 0 or 1):

$$f(\theta) = \sum_\alpha f_\alpha \Theta_\alpha(\theta), \qquad f_\alpha \in \mathbb{C}$$

The set of holomorphic functions is a linear space, and a non-commutative algebra. It is also a finite-dimensional Hilbert space with the inner product being defined through the basis elements:

$$\boxed{\langle \Theta_\alpha | \Theta_\beta \rangle = \delta_{\alpha\beta}} \tag{19}$$

and extended by linearity to functions: $\langle f|g\rangle = \sum_\alpha f_\alpha^* g_\alpha$.
The analogy with the theory of ordinary functions results in the following equivalent expression of the inner product, as a Grassmann integral. Here, the units formally appear as variables:

Proposition 4.1.

$$\boxed{\langle f|g\rangle = \int \prod_{a=1}^{N} d\theta_a d\overline{\theta}_a \; e^{\sum_a \overline{\theta}_a\theta_a} \overline{f(\theta)} g(\theta)} \tag{20}$$

Proof. The equivalence is checked on the basis functions. In the evaluation of the integral for $\langle\Theta_\alpha|\Theta_\beta\rangle$ with $\alpha = \beta$ we consider:

$$\int \prod_{a=1}^{N} d\theta_a d\overline{\theta}_a \; e^{\sum_a \overline{\theta}_a\theta_a} \overline{\theta_1\cdots\theta_k}\; \theta_1\cdots\theta_k$$

$$= \int \prod_{a=1}^{k} d\theta_a d\overline{\theta}_a (\overline{\theta_k}\cdots\overline{\theta_1}\;\theta_1\cdots\theta_k) \int \prod_{a=k+1}^{N} d\theta_a d\overline{\theta}_a e^{\sum_{a>k}\overline{\theta}_a\theta_a}$$

$$= \int \prod_{a=1}^{k} d\theta_a d\overline{\theta}_a (\overline{\theta_k}\theta_k)\cdots(\overline{\theta_1}\theta_1) \int \prod_{a=k+1}^{N} d\theta_a d\overline{\theta}_a \prod_{a>k}(\overline{\theta}_a\theta_a) = 1$$

If $\alpha \neq \beta$, there is always a variable θ or $\overline{\theta}$ or $d\theta$ or $d\overline{\theta}$ which remains unpaired, making the integral vanish. □

On the space $\mathscr{A}_N$ the canonical operators (**??**) are represented by:

$$\boxed{(\hat{a}_r f)(\theta) = \frac{\partial f(\theta)}{\partial\theta_r}, \qquad (\hat{a}_r^\dagger f)(\theta) = \theta_r f(\theta), \qquad r = 1\ldots N} \tag{21}$$

The symbol $\partial_r = \partial/\partial\theta_r$ is treated as a Grassmann variable: $\{\partial_r, \theta_s\} = \delta_{rs}$. It is checked that derivatives anticommute exactly among themselves: $\{\partial_r, \partial_s\} = 0$. The vacuum is defined by $\partial_r f = 0$ for all r, and is the function $f(\theta) = 1$.

Proposition 4.2. *The operators of multiplication by θ_r and derivation ∂_r are adjoint one to the other in the inner product: $\langle\partial_r f|g\rangle = \langle f|\theta_r g\rangle$.*

Proof. This is checked on the basis functions. Consider the integral for $\langle\Theta_\alpha|\theta_r\Theta_\beta\rangle$. If Θ_β does not contain θ_r among its factors, a non-zero integral is obtained if and only if (orthogonality) $\Theta_\alpha(\theta) = \sigma\theta_r\Theta_\beta(\theta)$ (σ is a sign, because of ordering of factors in the basis of Θ functions), i.e. $\partial_r\Theta_\alpha = \sigma\Theta_\beta - \sigma(\theta_r\partial_r\Theta_\beta)$ (the last term is zero). Therefore: $\langle\partial_r\Theta_\alpha|\Theta_\beta\rangle = \sigma\langle\Theta_\beta|\Theta_\beta\rangle = \sigma\langle\Theta_\alpha|\Theta_\alpha\rangle = \langle\Theta_\alpha|\theta_r\Theta_\beta\rangle$.
If Θ_β does contain θ_r their product is zero, and $0 = \langle\Theta_\alpha|\theta_r\Theta_\beta\rangle$. However it is also $\langle\partial_r\Theta_\alpha|\Theta_\beta\rangle = 0$ as the left hand side does not contain θ_r while the right hand side does. □

4.1. **Reproducing kernel.** The completeness of the basis gives the identity $f(\theta) = \sum_\alpha \Theta_\alpha(\theta)\langle\Theta_\alpha|f\rangle$ for all f. By writing the inner product explicitly, an integral representation of the unit operator is obtained:

$$f(\theta) = \int \prod_a d\theta'_a d\overline{\theta}'_a e^{\sum_a \overline{\theta}'_a\theta'_a} \left[\sum_\alpha \Theta_\alpha(\theta)\overline{\Theta_\alpha(\theta')}\right] f(\theta')$$

The sum in parenthesis is the *reproducing kernel* $I(\theta, \overline{\theta}')$, and it is:

$$\boxed{f(\theta) = \int \prod_a d\theta'_a d\overline{\theta}'_a e^{\sum_a \overline{\theta}'_a\theta'_a} I(\theta, \overline{\theta}') f(\theta')} \tag{22}$$

We prove the equivalent expressions:

Proposition 4.3.

$$\boxed{I(\theta, \overline{\eta}) =: \sum_\alpha \Theta_\alpha(\theta)\overline{\Theta_\alpha(\eta)} = \prod_a (1 + \theta_a\overline{\eta}_a) = \exp\sum_a \theta_a\overline{\eta}_a} \tag{23}$$

Proof. The defining expression of $I(\theta, \overline{\eta})$ corresponds to the sum of 2^N terms of the form $\theta_a\theta_b\cdots\theta_r\overline{\eta}_r\cdots\overline{\eta}_b\overline{\eta}_a$. The factor $\overline{\eta}_a$ is brought to the right of θ_a with an even number of anticommutations. This is done for all factors and gives pairs: $(\theta_a\overline{\eta}_a)(\theta_b\overline{\eta}_b)\cdots(\theta_r\overline{\eta}_r)$, and each pair is a c-number. The sum of all such products of pairs is the product in (**??**). Since $1 + \theta_a\overline{\eta}_a = \exp(\theta_a\overline{\eta}_a)$ and the exponents are c-numbers, the last representation in (**??**) results. □

Exercise 4.4. *Show the useful properties ($\partial_a = \partial/\partial\theta_a$):*

$$\begin{aligned} &1)\quad \overline{I(\theta, \overline{\eta})} = I(-\overline{\theta}, \eta), \\ &2)\quad I(\theta, \overline{\eta})I(\theta, \overline{\xi}) = I(\theta, \overline{\eta} + \overline{\xi}), \\ &3)\quad \partial_a\partial_b\cdots\partial_c I(\theta, \overline{\eta}) = \overline{\eta}_a\overline{\eta}_b\cdots\overline{\eta}_c I(\theta, \overline{\eta}). \end{aligned}$$

Remark 4.5. *Eq.(**??**) can be rewritten as follows:*

$$f(\theta) = \int \prod_a d\theta'_a d\overline{\theta}'_a I(\overline{\theta}', \theta')I(\theta, \overline{\theta}')f(\theta') = \int \prod_a d\theta'_a d\overline{\theta}'_a I(\overline{\theta}', \theta' - \theta)f(\theta').$$

Proposition 4.6.

$$\int \prod_a d\theta_a d\overline{\theta}_a e^{\sum_a \overline{\theta}_a \theta_a} I(\eta,\overline{\theta}) I(\theta,\overline{\xi}) = I(\eta,\overline{\xi}) \tag{24}$$

Proof. Because even products of Grassmann variables commute, it is: $I(\eta,\overline{\theta})I(\theta,\overline{\xi}) = \prod_a (1+\eta_a\overline{\theta}_a)(1+\theta_a\overline{\xi}_a) = \prod_a (1+\eta_a\overline{\theta}_a + \theta_a\overline{\xi}_a + \overline{\theta}_a\theta_a\eta_a\overline{\xi}_a)$. The integral is: $\prod_a \int d\theta_a d\overline{\theta}_a (1+\overline{\theta}_a\theta_a)(1+\eta_a\overline{\theta}_a + \theta_a\overline{\xi}_a + \overline{\theta}_a\theta_a\eta_a\overline{\xi}_a) = \prod_a (1+\eta_a\overline{\xi}_a)$. ☐

Remark 4.7. *The reproducing kernel $I(\eta,\overline{\theta})$, as a function of η, is not an element of $\mathscr{A}_N$, because the coefficients of its expansion in the basis $\Theta_\alpha(\eta)$ are Grassmann numbers. Note that:*

$$I(\eta,\overline{\theta}) = \sum_\alpha \Theta_\alpha(\eta)\overline{\Theta_\alpha(\theta)} = \sum_\alpha \overline{\Theta_\alpha(-\theta)}\Theta_\alpha(\eta)$$

Equation (**??**) may be written as follows:

$$f(\theta) = \int \prod_r d\eta_r d\overline{\eta}_r e^{\sum \overline{\eta}_r \eta_r} I(\theta,\overline{\eta}) f(\eta) = \int \prod_r d\eta_r d\overline{\eta}_r e^{\sum \overline{\eta}_r \eta_r} \overline{I(\eta,\overline{\theta})} f(\eta)$$

Although the kernel is not an element of the Hilbert space, the integral is well defined and is formally the inner product

$$f(\theta) = \langle I_{\overline{\theta}} | f \rangle$$

of $f(\eta) \in \mathscr{A}_N$ with the "improper function" $I_{\overline{\theta}}(\eta) =: I(\eta,\overline{\theta})$, which works as a delta function. We wish to export the notion and virtues of the reproducing kernel from $\mathscr{A}_N$ to the abstract Hilbert space $\mathscr{F}_N$.

5. The abstract Hilbert space

5.1. **Isomorphism.** The abstract Hilbert space $\mathscr{F}_N$ with an irreducible representation of N pairs of CAR, eq.(**??**), is isomorphic to the Hilbert space $\mathscr{A}_N$ of holomorphic functions of N Grassmann variables θ_a. The correspondence is established through the basis vectors $\Theta_\alpha \Leftrightarrow |N_\alpha\rangle$, $\alpha = 1 \dots 2^N$:

$$\Theta_\alpha(\theta) = \theta_1^{n_1} \cdots \theta_N^{n_N} \quad \Longleftrightarrow \quad |N_\alpha\rangle = (\hat{a}_1^\dagger)^{n_1} \cdots (\hat{a}_N^\dagger)^{n_N} |0\rangle, \quad (n_k = 0,1).$$

The operators of multiplication by θ_c and derivation ∂_c on holomorphic functions correspond to the operators $\hat{a}_c^\dagger$ and $\hat{a}_c$.
According to the isomorphism we associate to the improper function $I_{\overline{\eta}}(\theta) = I(\theta,\overline{\eta})$ the abstract vector $|\overline{\eta}\rangle$:

$$I_{\overline{\eta}} \quad \Longleftrightarrow \quad |\overline{\eta}\rangle = \sum_\alpha \overline{\Theta_\alpha(-\eta)} |N_\alpha\rangle \tag{25}$$

The equivalent expressions (**??**) are easily proven, with the rule that operators anticommute exactly with Grassmann units:

$$|\overline{\eta}\rangle = \prod_r (1 - \overline{\eta}_r \hat{a}_r^\dagger)|0\rangle = \exp\left[-\sum_r \overline{\eta}_r \hat{a}_r^\dagger \right] |0\rangle \tag{26}$$

The vector, having coefficients that are Grassmann numbers, does not belong to $\mathscr{F}_N$, but to an extension that is introduced below.

5.2. **Graded Hilbert space.** Given a Grassmann algebra Λ_N with N units θ_a, the graded Hilbert space $\mathscr{F}_{\Lambda,N}$ is the set of vectors

$$|\Psi\rangle = \sum\nolimits_\alpha \Theta_\alpha(\theta)|\psi_\alpha\rangle,$$

where $\Theta_\alpha(\theta) \in \Lambda_N$, and $|\psi_\alpha\rangle \in \mathscr{F}_N$.
This space is much larger than $\mathscr{F}_N$, and makes the formal steps involving Grassmann variables meaningful.

The inner product is $\langle\Psi'|\Psi\rangle = \sum_\alpha \langle\psi'_\alpha|\psi_\alpha\rangle$. The CAR operators act on vector components $|\psi_\alpha\rangle$ and anticommute by definition with the units of the algebra:

$$\hat{a}_r|\Psi\rangle = \sum\nolimits_\alpha (-1)^{\#\theta}\Theta_\alpha(\theta)\hat{a}_r|\psi_\alpha\rangle$$

where $\#\theta$ is the number of units in Θ_α.
A special vector in $\mathscr{F}_{\Lambda,N}$ gives the analogue of the reproducing kernel:

Proposition 5.1. *The vector $|\theta\rangle = \exp\left[-\sum_c \theta_c \hat{a}^\dagger_c\right]|0\rangle$ is a coherent state, i.e. it solves in $\mathscr{F}_{\Lambda,N}$:*

$$\hat{a}_c|\theta\rangle = \theta_c|\theta\rangle, \qquad c = 1\ldots N.$$

Proof. It is checked for $N = 1$: $\hat{a}(1-\theta\hat{a}^\dagger)|0\rangle = \theta\hat{a}\hat{a}^\dagger|0\rangle = \theta|0\rangle = \theta|\theta\rangle$. The extension to N modes is trivial, by factorization of commuting terms. □

Proposition 5.2. *If $|\psi\rangle \in \mathscr{F}_N$ and $\psi(\theta)$ is the holomorphic function that corresponds to it in the isomorphism, then:*

$$\boxed{\psi(\theta) = \langle\overline{\theta}|\psi\rangle} \tag{27}$$

Proof. It is sufficient to prove it for the basis vectors. Let's evaluate the inner product in $\mathscr{F}_{\Lambda,N}$: $\langle\overline{\theta}|N_\alpha\rangle = \left(\sum_\beta \langle N_\beta|\Theta_\beta(-\theta)\right)|N_\alpha\rangle$. The matrix element is typically: $\langle 0|\hat{a}\cdots\hat{a}\,\Theta_\beta(-\theta)\hat{a}^\dagger\cdots\hat{a}^\dagger|0\rangle$. Necessarily $\alpha = \beta$ or, by anticommuting ladder operators through Θ_β, they are not matched by adjoint operators on the other side, and kill the zero particle state. Then: $\langle\overline{\theta}|N_\alpha\rangle = \langle N_\alpha|\Theta_\alpha(-\theta)|N_\alpha\rangle = (-1)^r\langle 0|\hat{a}_r\cdots\hat{a}_1(\theta_1\cdots\theta_r)\hat{a}^\dagger_1\cdots\hat{a}^\dagger_r|0\rangle = (-1)^{r^2+r}\theta_1\cdots\theta_r\langle 0|(\hat{a}_1\hat{a}^\dagger_1)\cdots(\hat{a}_r\hat{a}^\dagger_r)|0\rangle = \Theta_\alpha(\theta)$ □

Proposition 5.3. *Completeness of coherent states on the subspace $\mathscr{F}_N$:*

$$\int\prod_a d\theta_a d\overline{\theta}_a e^{\sum_a \overline{\theta}_a\theta_a}|\overline{\theta}\rangle\langle\overline{\theta}| = \mathbb{I}$$

Proof. Let $|\psi\rangle$ and $|\phi\rangle$ be two vectors in $\mathscr{F}_N$, and let $\psi(\theta)$, $\phi(\theta)$ be their holomorphic representations. Isomorphism implies the equality of inner products. □

Remark 5.4. *By exchanging the variable θ with $\overline{\theta}$, the relation of completeness in $\mathscr{F}_N$ is:*

$$\boxed{\int\prod_a d\overline{\theta}_a d\theta_a e^{-\sum_a \overline{\theta}_a\theta_a}|\theta\rangle\langle\theta| = \mathbb{I}} \tag{28}$$

Proposition 5.5. *The following formula evaluates the trace of an operator $\hat{O}$: $\mathscr{F}_N \to \mathscr{F}_N$ (i.e. the operator commutes with the Grassmann units):*

$$\boxed{\int \prod_a d\overline{\theta}_a d\theta_a e^{-\sum_a \overline{\theta}_a\theta_a} \langle\theta|\hat{O}|-\theta\rangle \; = \; \mathrm{tr}\,\hat{O}} \tag{29}$$

Proof.

$$\begin{aligned}\mathrm{tr}\,\hat{O} = \sum_\alpha \langle N_\alpha|\hat{O}|N_\alpha\rangle &= \sum_\alpha \int \prod_a d\theta_a d\overline{\theta}_a e^{\sum_a \overline{\theta}_a\theta_a} \langle N_\alpha|\overline{\theta}\rangle\langle\overline{\theta}|\hat{O}|N_\alpha\rangle \\ &= \sum_{\alpha\beta} \langle N_\beta|\hat{O}|N_\alpha\rangle \int \prod_a d\theta_a d\overline{\theta}_a e^{\sum_a \overline{\theta}_a\theta_a} \langle N_\alpha|\overline{\theta}\rangle\langle\overline{\theta}|N_\beta\rangle\end{aligned}$$

Note that $\langle\overline{\theta}|N_\beta\rangle = \Theta_\beta(\theta)$, therefore the integral is different from zero only if $\alpha = \beta$. The factors are exchanged by means of the identity $\overline{\Theta_\alpha(\theta)}\Theta_\alpha(\theta) = \Theta_\alpha(-\theta)\overline{\Theta_\alpha(\theta)}$, i.e. $\langle N_\alpha|\overline{\theta}\rangle\langle\theta|N_\alpha\rangle = \langle -\theta|N_\alpha\rangle\langle N_\alpha|\theta\rangle$. Then:

$$\mathrm{tr}\,\hat{O} = \int \prod\nolimits_a d\theta_a d\overline{\theta}_a e^{\sum_a \overline{\theta}_a\theta_a} \langle -\overline{\theta}|\hat{O}|\overline{\theta}\rangle$$

The formula follows after the exchange of $\overline{\theta}_a$, θ_a with $-\theta_a$, $-\overline{\theta}_a$. □

6. Coherent states and functional integral

The formalism of fermionic coherent states allows to express the partition function of interacting fermions as a functional integral. Consider the Hamiltonian

$$\hat{K} = \sum_{mn} a_m^\dagger k_{mn} \hat{a}_n + \frac{1}{2} \sum_{mnpq} \hat{a}_m^\dagger \hat{a}_n^\dagger v_{mnpq} \hat{a}_q \hat{a}_p$$

where the labels denote a set of N Fermi oscillators. The partition function with no sources is evaluated first. Let $\hbar\beta = \tau_n > \ldots > \tau_0 = 0$ be a partition of the imaginary time interval, the trace is evaluated by (**??**):

$$\begin{aligned} Z &= \mathrm{tr}\,[e^{-\beta\hat{K}}] = \mathrm{tr}\left[e^{-\frac{1}{\hbar}(\tau_n-\tau_{n-1})\hat{K}} I_{n-1} \cdots e^{-\frac{1}{\hbar}(\tau_2-\tau_1)\hat{K}} I_1 e^{-\frac{1}{\hbar}(\tau_1-\tau_0)\hat{K}}\right] \\ &= \int \prod_k d\overline{\theta}_k(n) d\theta_k(n) e^{-\theta(n)^\dagger\theta(n)} \langle\theta(n)|e^{-\frac{1}{\hbar}(\tau_n-\tau_{n-1})\hat{K}} I_{n-1} \cdots I_1 e^{-\frac{1}{\hbar}(\tau_1-\tau_0)\hat{K}}|\theta(0)\rangle \end{aligned}$$

where we define $\theta(0) = -\theta(n)$. The identity operators $I_{n-1}, \ldots, I_1$ are replaced by resolutions of identity (**??**) with coherent states $|\theta(n-1)\rangle, \ldots, |\theta(1)\rangle$. At each time-step k, $|\theta(k)\rangle$ is a coherent state in $\mathscr{F}_{\Lambda,N}$: $\hat{a}_r|\theta(k)\rangle = \theta_r(k)|\theta(k)\rangle$, $r = 1\ldots N$. We need the matrix element among two coherent states:

$$\langle\theta(j+1)|\hat{K}|\theta(j)\rangle = \langle\theta(j+1)|\theta(j)\rangle\, K(\overline{\theta(j+1)}, \theta(j))$$

where

$$K(\overline{\theta}, \theta) = \sum_{mn} \overline{\theta}_m k_{mn} \theta_m + \frac{1}{2} \sum_{mnpq} \overline{\theta}_m \overline{\theta}_n v_{mnpq} \theta_q \theta_p \tag{30}$$

By the smallness of time intervals:

$$\begin{aligned}\langle\theta(j+1)|e^{-\frac{1}{\hbar}(\tau_{j+1}-\tau_j)\hat{K}}|\theta(j)\rangle &= \langle\theta(j+1)|\theta(j)\rangle\left[1-\frac{1}{\hbar}(\tau_{j+1}-\tau_j)K(\overline{\theta(j+1)},\theta(j))\right]\\ &= \langle\theta(j+1)|\theta(j)\rangle\,\exp\left[-\frac{1}{\hbar}(\tau_{j+1}-\tau_j)K(\overline{\theta(j+1)},\theta(j))\right]\\ &= \exp\left[\theta(j+1)^\dagger\theta(j)-\frac{1}{\hbar}(\tau_{j+1}-\tau_j)K(\overline{\theta(j+1)},\theta(j))\right]\end{aligned}$$

$$Z=\int\prod_{k=1}^{N}\prod_{j=1}^{n}d\overline{\theta}_k(j)d\theta_k(j)e^{-\sum_{j=0}^{n-1}(\tau_{j+1}-\tau_j)\left[\theta(j+1)^\dagger\frac{\theta(j+1)-\theta(j)}{\tau_{j+1}-\tau_j}+\frac{1}{\hbar}K(\overline{\theta(j+1)},\theta(j))\right]}$$

Finally, in a symbolical "continuum limit" we write

$$\boxed{Z=\int\prod_{k=1}^{N}\mathscr{D}[\overline{\theta}_k(\tau)\theta_k(\tau)]\exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\theta(\tau)^\dagger\hbar\frac{d}{d\tau}\theta(\tau)+K(\overline{\theta}(\tau),\theta(\tau))\right]}\tag{31}$$

with antiperiodic b.c.: $\theta_k(\hbar\beta)=-\theta_k(0)$.

Sources are added to the Hamiltonian through the term $\hbar\left(\overline{\eta}_k(\tau)\hat{a}_k+\hat{a}_k^\dagger\eta_k(\tau)\right)$ (summation is understood), and modify the partition function:

$$\begin{aligned}Z[\overline{\eta},\eta] = \int\prod_{k=1}^{N}\mathscr{D}[\overline{\theta}_k(\tau)\theta_k(\tau)]\exp\Big[-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\,\overline{\theta}_k(\tau)\hbar\frac{d}{d\tau}\theta_k(\tau)\\ +K(\overline{\theta}(\tau),\theta(\tau))+\hbar\overline{\eta}_k(\tau)\theta_k(\tau)+\hbar\overline{\theta}_k(\tau)\eta_k(\tau)\Big].\end{aligned}\tag{32}$$

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