A DERIVATION OF HEDIN'S EQUATIONS

LUCA GUIDO MOLINARI

ABSTRACT. In these pedagogical notes, I present a derivation of Hedin's equations for the evaluation of the propagator, the proper self-energy, the effective potential, the proper polarization and the vertex in a many-body theory with two-body interaction. I then discuss the Ward identities for the vertex. The solution of Hedin's equations in $d = 0$ allows to enumerate Feynman diagrams in various resummation schemes.

1. Introduction

We consider the many-body problem for interacting fermions with Hamiltonian $H = H_0 + U$, $H_0 = \sum_i h(\mathbf{x}_i, \mathbf{p}_i)$, $U = \sum_{i<j} v(\mathbf{x}_i, \mathbf{x}_j)$. h is a one-particle Hamiltonian and v is the two-body interaction. For simplicity we assume spin independence of the Hamiltonian (its inclusion is straightforward but makes notation heavy).

In 1965 Lars Hedin [1] derived the following formally closed set of equations for the propagator G, the proper self-energy $\Sigma^\star$, the effective potential W, the proper polarization $\Pi^\star$ and the dressed vertex Γ. Four of them are integral equations[1]:

$$G(1,2) = g(1,2) + \int d1'2' \; g(1,1')\Sigma^\star(1',2')G(2',2) \tag{1}$$

$$W(1,2) = v(1,2) + \int d1'2' \; v(1,1')\Pi^\star(1',2')W(2',2) \tag{2}$$

$$\Sigma^\star(1,2) = \frac{i}{\hbar}\int d34 \; \Gamma(4;1,3)G(3,2)W(4,2) \tag{3}$$

$$\Pi^\star(1,2) = -2\frac{i}{\hbar}\int d34 \; \Gamma(1;3,4)G(2,3)G(4,2) \tag{4}$$

where $v(1,2) = v(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2)$ and $g(1,2)$ is the time-ordered Green function of the interacting system in the Hartree approximation, with the *exact* particle density (Hartree-type insertions -tadpoles- are then already accounted for). The fifth equation contains a functional derivative:

$$\Gamma(1;2,3) = \delta(1,2)\delta(1,3) + \int d4567 \; \Gamma(1;4,5)G(6,4)G(5,7)\frac{\delta\Sigma^\star(2,3)}{\delta G(6,7)} \tag{5}$$

Eqs. (1) and (2) are the Dyson's equations that define the self-energy and the polarization as 1-particle irreducible insertions for the propagator and the effective potential[2]. The next two give the skeleton structure of the irreducible self-energy

Date: 18 oct 2017.

[1]Here, $1 = (\mathbf{x}_1, t_1)$, $1^+ = (\mathbf{x}_1, t_1 + \epsilon)$, and $\int d1 = \int_{-\infty}^{+\infty} dt_1 \int d\mathbf{x}_1$.

[2]The diagrams for G and W have the structure $G(1,2) = g(1,2) + g(1,1')\Sigma(1',2')g(2',2)$ and $W(1,2) = v(1,2) + v(1,1')\Pi(1',2')v(2',2)$, where Σ and Π are the self-energy and polarisation insertions, i.e. diagrams with two fermion or two "photon" slots.

The self-energy or polarisation diagrams that may not be disconnected by removal, respectively, and polarization. These four equations can be understood and derived by considering Feynman's rules and the diagrams' topology [2, 3]. They only involve integrals. The vertex equation (5) contains a functional derivative, which constitutes the main difficulty of the many-body problem [4, 5, 6]. It can be put also in the form [15],

$$\Gamma(1;2,3) = \delta(1,2)\delta(1,3) + \int d45\; g(4,1)g(1,5)\frac{\delta\Sigma^\star(2,3)}{\delta g(4,5)} \tag{6}$$

which is less suitable for non-perturbative calculations, but it offers a transparent diagrammatic interpretation: vertex diagrams are self-energy diagrams with one Hartree propagator $g(4,5)$ removed and replaced by the product $g(4,1)g(1,5)$ that introduces a new *bare* vertex at 1. Therefore, each self-energy diagram of order n produces $2n-1$ vertex diagrams of the same perturbative order by pinching a bare vertex in any of its $2n-1$ propagator lines.

If all corrections to the bare vertex, $\Gamma^{(0)}(1;2,3) = \delta(1,2)\delta(1,3)$, are ignored, the four eqs.(1-4) become a closed set of integral equations, which is numerically tractable [7, 8, 9]. This is the GW approximation (GWA):

$$\Sigma^\star(1,2) \approx \frac{i}{\hbar}G(1,2)W(1,2), \qquad \Pi^\star(1,2) \approx -2\frac{i}{\hbar}G(1,2)G(2,1).$$

.

2. Preliminaries

To derive Hedin's equations we add a source term to the Hamiltonian $\hat H$, that couples the particle density operator $\hat n(\mathbf{x}) = \sum_\mu \psi_\mu^\dagger(\mathbf{x})\psi_\mu(\mathbf{x})$ to a classical space-time field:

$$\hat H(t) = \hat H + \int dx\, \varphi(\mathbf{x},t)\, \hat n(\mathbf{x}) \tag{7}$$

The field will allow us to compute functional derivatives. At the end, we shall put $\varphi(\mathbf{x},t) = 0$. A different functional approach was formulated by Kleinert et al. [11, 12].

Because of the time-dependence in $\hat H(t)$, time-ordered correlators in presence of the source are defined in the interaction picture. The time-ordered propagator and the full polarization are:

$$i\delta_{\mu\nu}\, G(1,2) = \frac{\langle E|\mathsf{T}\hat S\hat\psi_\mu(1)\hat\psi_\nu^\dagger(2)|E\rangle}{\langle E|\hat S|E\rangle} \tag{8}$$

$$i\hbar\, \Pi(1,2) = \frac{\langle E|\mathsf{T}\hat S\hat\delta\hat n(1)\delta\hat n(2)|E\rangle}{\langle E|\hat S|E\rangle} \tag{9}$$

where $\delta\hat n(1) = \hat n(1) - n(1)$ is the density fluctuation operator, with the average density $n(1) = \langle E|T\hat S\hat n(1)|E\rangle/\langle E|\hat S|E\rangle = -2iG(1,1^+)$. The averages are taken in

of one line g or one line v are named 1-particle *irreducible* self-energy or polarization diagrams. Their sum is $\Sigma^\star$ or $\Pi^\star$. Reducible diagrams can be obtained by connecting irreducible ones with g or v lines. Then: $\Sigma(1,2) = \Sigma^\star(1,2) + \Sigma^\star(1,1')g(1',2')\Sigma^\star(2',2) + \Sigma^\star g\Sigma^\star g\Sigma^\star + \ldots$. A partial resummation gives Dyson's equation $\Sigma(1,2) = \Sigma^\star(1,2) + \Sigma^\star(1,1')g(1',2')\Sigma(2',2)$. In the same way: $\Pi(1,2) = \Pi^\star(1,2) + \Pi^\star(1,1')v(1',2')\Pi(2',2)$.

the ground state $|E\rangle$ of $\hat H$, the time-evolution of operators is driven by $\hat H$, and the source term is relegated to the scattering operator

$$\hat S = \mathsf{T}\exp\frac{1}{i\hbar}\int d^4x\,\varphi(x)\hat n(x) \tag{10}$$

The Hartree propagator solves the equation of motion with the source and the exact Hartree potential $V_H(1) = \int d2 v(1,2)n(2)$:

$$(i\hbar\partial_{t_1} - h(\mathbf{x}_1) - \varphi(1) - V_H(1))g(1,2) = \hbar\delta(1,2) \tag{11}$$

When, in the end, $\varphi = 0$, $\hat S = 1$, the correlators for H or H_0 are recovered. In particular, the average density and the Hartree potential are time-independent.

Proposition 2.1. *Let $\hat\Omega$ be a product of field operators evolved at different times with $\hat H$, that do not contain the source φ. Then:*

$$\boxed{\frac{\delta}{\delta\varphi(1)}\frac{\langle E|\mathsf{T}\hat S\hat\Omega|E\rangle}{\langle E|\hat S|E\rangle} = \frac{1}{i\hbar}\frac{\langle E|\mathsf{T}\hat S\hat\Omega\delta\hat n(1)|E\rangle}{\langle E|\hat S|E\rangle}} \tag{12}$$

Proof. Consider the Dyson expansion for the time-ordered product $\mathsf{T}[\hat S\hat\Omega]$. The functional derivative of a product has the Leibnitz property: $\delta[\varphi(1')...\varphi(k')]/\delta\varphi(1) = \sum_{\ell'}\varphi(1')...\delta(\ell'-1)...\varphi(k')$. Since primed variables are integrated, the k terms at order k are equal:

$$\frac{\delta\mathsf{T}[\hat S\hat\Omega]}{\delta\varphi(1)} = \sum_{k=1}^{\infty}\frac{k}{(i\hbar)^k k!}\int d1'...k'\mathsf{T}\hat n(1')...\hat n(k')\delta(1-1')\varphi(2')...\varphi(k') = \frac{1}{i\hbar}T[\hat S\hat\Omega\hat n(1)]$$

Note that under time-ordering the operator $\hat n(1)$ commutes with any operator.

$$\frac{\delta}{\delta\varphi(1)}\frac{\langle E|\mathsf{T}\hat S\hat\Omega|E\rangle}{\langle E|\hat S|E\rangle} = \frac{1}{i\hbar}\frac{\langle E|\mathsf{T}[\hat S\hat\Omega\hat n(1)]|E\rangle}{\langle E|\hat S|E\rangle} - \frac{1}{i\hbar}\frac{\langle E|\mathsf{T}\hat S\hat n(1)|E\rangle\langle E|\mathsf{T}\hat S\hat\Omega|E\rangle}{\langle E|\hat S|E\rangle^2}$$

and the resul follows. □

3. Polarization and effective potential

With the choice $\Omega = \hat n$ we get:

$$\frac{\delta}{\delta\varphi(2)}\frac{\langle E|\mathsf{T}\hat S\hat n(1)|E\rangle}{\langle E|\hat S|E\rangle} = \frac{1}{i\hbar}\frac{\langle E|\mathsf{T}\hat S\hat n(1)\delta\hat n(2)|E\rangle}{\langle E|\hat S|E\rangle} = \frac{1}{i\hbar}\frac{\langle E|\mathsf{T}\hat S\delta\hat n(1)\delta\hat n(2)|E\rangle}{\langle E|\hat S|E\rangle}$$

The right-hand side is the full polarization $\Pi(1,2)$ in presence of the source, eq.(9). Note the symmetry $\Pi(1,2) = \Pi(2,1)$. We obtained:

$$\Pi(1,2) = -2i\frac{\delta G(1,1^+)}{\delta\varphi(2)} \tag{13}$$

Setting $\varphi = 0$ (then $\hat S = 1$) we recover the familiar formula $i\hbar\Pi(1,2) = \langle E|\mathsf{T}\delta\hat n(1)\delta\hat n(2)|E\rangle$ (the time-ordered counterpart of the density-density response function).
Let us introduce the local potential

$$V(1) = \varphi(1) + V_H(1) = \varphi(1) - 2i\int d2\, v(1,2)G(2,2^+) \tag{14}$$

where V_H is the Hartree or mean field potential. A functional derivative gives:

$$\frac{\delta V(1)}{\delta\varphi(2)} = \delta(1,2) - 2i\int d3\, v(1,3)\frac{\delta G(3,3^+)}{\delta\varphi(2)} = \delta(1,2) + \int d3 v(1,3)\Pi(3,2) \tag{15}$$

Integration of (15) with $v(2,3)$ gives the two-body **effective potential**

$$W(1,3) \equiv \int d2 \frac{\delta V(1)}{\delta\varphi(2)} v(2,3) = v(1,3) + \int d24\, v(1,4)\Pi(4,2)v(2,3) \tag{16}$$

It is the bare interaction $v(1,3)$ dressed by all polarisation insertions. Symmetry of Π and v in exchange of variables implies $W(1,3) = W(3,1)$.

Exercise 3.1. Obtain the following identities:

$$\frac{\delta}{\delta\varphi(1)} = \frac{\delta}{\delta V(1)} + \int d23\, v(2,3)\Pi(3,1)\frac{\delta}{\delta V(2)} \tag{17}$$

$$\int d1 v(4,1)\frac{\delta}{\delta\varphi(1)} = \int d1\, W(4,1)\frac{\delta}{\delta V(1)} \tag{18}$$

4. The irreducible polarisation

Use of the chain rule (17) in (13) gives:

$$\Pi(1,2) = -2i\frac{\delta G(1,1^+)}{\delta V(2)} - 2i\int d34\, \frac{\delta G(1,1^+)}{\delta V(3)} v(3,4)\Pi(4,2)$$

We recognize Dyson's equation for the polarization in terms of the irreducible, or **proper polarization**:

$$\boxed{\Pi(1,2) = \Pi^\star(1,2) + \int d34\, \Pi^\star(1,3)v(3,4)\Pi(4,2)} \tag{19}$$

$$\Pi^\star(1,2) = -2i\frac{\delta G(1,1^+)}{\delta V(2)} \tag{20}$$

The symmetries of Π and v imply $\Pi^\star(1,2) = \Pi^\star(2,1)$. Eq. (18) implies:

$$\int d4\; v(5,4)\Pi(4,2) = \int d4\; W(5,4)\Pi^\star(4,2) \tag{21}$$

and equation (16) becomes the Dyson equation (2) in Hedin's set:

$$\boxed{W(1,3) = v(1,3) + \int d24\, v(1,4)\Pi^\star(4,2)W(2,3)} \tag{22}$$

Exercise 4.1.

$$\frac{\delta}{\delta V(1)} = \frac{\delta}{\delta\varphi(1)} - \int d23\, v(2,3)\Pi^\star(3,1)\frac{\delta}{\delta\varphi(2)} \tag{23}$$

$$\int d1 \Pi^\star(4,1)\frac{\delta}{\delta\varphi(1)} = \int d1\, \Pi(4,1)\frac{\delta}{\delta V(1)} \tag{24}$$

Let us introduce the functional inverse of the propagator:

$$\int d3\, G^{-1}(5,3)G(3,2) = \hbar\delta(5,2)$$

A functional derivative gives:

$$0 = \int d3 \frac{\delta G^{-1}(5,3)}{\delta V(4)} G(3,2) + \int d3\, G^{-1}(5,3)\frac{\delta G(3,2)}{\delta V(4)}$$

Contraction with $\int d5\, G(1,5)$ gives:

$$0 = \int d35\, G(1,5)\frac{\delta G^{-1}(5,3)}{\delta V(4)}G(3,2) + \hbar\frac{\delta G(1,2)}{\delta V(4)} \tag{25}$$

If $2 = 1^+$ it becomes $\hbar\Pi^\star(1,4) = 2i\int d23\, G(1,3)\frac{\delta G^{-1}(3,2)}{\delta V(4)}G(2,1)$. This provides the skeleton structure of the 1P-irriducible polarisation

$$\boxed{\Pi^\star(1,2) = -2\frac{i}{\hbar}\int d34\; \Gamma(2;3,4)G(1,3)G(4,1)} \tag{26}$$

with the identification of the **vertex** function:

$$\Gamma(1;2,3) = -\frac{\delta G^{-1}(2,3)}{\delta V(1)} \tag{27}$$

Remark 4.2. *Comparison of the expressions* (20) *and* (26) *for* $\Pi^\star$ *suggests that a functional derivative in* V *introduces a dressed vertex. This can be seen by using* (25) *and considering that* G *is determined by* V *in a one-to-one correspondence (Hohenberg-Kohn):*

$$\frac{\delta}{\delta V(5)} = \int d12\,\frac{\delta G(1,2)}{\delta V(5)}\frac{\delta}{\delta G(1,2)} = -\frac{1}{\hbar}\int d1234\, G(1,4)\frac{\delta G^{-1}(4,3)}{\delta V(5)}G(3,2)\frac{\delta}{\delta G(1,2)}$$

$$\frac{\delta}{\delta V(5)} = \frac{1}{\hbar}\int d1234\, G(1,4)\Gamma(5;4,3)G(3,2)\frac{\delta}{\delta G(1,2)} \tag{28}$$

5. The vertex equation

The vertex function (27) is the sum of **vertex** diagrams, characterised by one photon line in 1 a fermion entering in 3 and leaving from 2. We now obtain an equation for it by replacing $G^{-1}(2,3) = g^{-1}(2,3) - \hbar\Sigma^\star(2,3)$.
The functional inverse of the Hartree propagator in presence of the source is

$$g^{-1}(1,2) = \delta(1,2)[i\hbar\partial_{t_2} - h(\mathbf{x}_2) - V(2)]$$

so that $\delta g^{-1}(1,2)/\delta V(3) = -\delta(1,2)\delta(2,3)$. Eq.(27) becomes

$$\Gamma(1;2,3) = \delta(1,2)\delta(1,3) + \hbar\,\frac{\delta\Sigma^\star(2,3)}{\delta V(1)} \tag{29}$$

The insertion of a vertex in the self-energy is made explicit by using eq.(28) in the second term. One obtains the final form of Hedin's vertex equation:

$$\boxed{\Gamma(5;6,7) = \delta(5,6)\delta(6,7) + \int d1234\, G(1,4)\Gamma(5;4,3)G(3,2)\frac{\delta\Sigma^\star(6,7)}{\delta G(1,2)}} \tag{30}$$

Exercise 5.1. Since V determines g, prove that

$$\frac{\delta}{\delta V(5)} = \int d12\,\frac{\delta g(1,2)}{\delta V(5)}\frac{\delta}{\delta g(1,2)} = \frac{1}{\hbar}\int d12\, g(1,5)g(5,2)\frac{\delta}{\delta g(1,2)} \tag{31}$$

$$\boxed{\Gamma(1;2,3) = \delta(1,2)\delta(1,3) + \int d45\, g(4,1)g(1,5)\frac{\delta\Sigma^\star(2,3)}{\delta g(4,5)}} \tag{32}$$

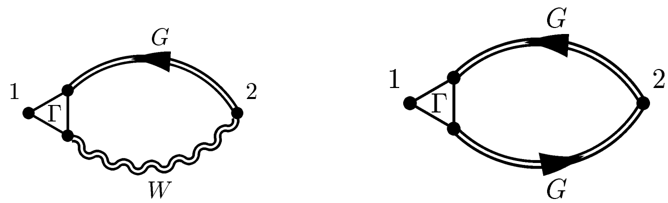

FIGURE 1. The dressed self-energy and polarization diagrams, with exact vertex Γ, propagator G and effective potential W.

6. The self-energy

The equation of motion for the propagator

$$[i\hbar\partial_{t_1} - h(\mathbf{x}_1) - V(1)]G(1,2) = \hbar\delta(1,2) + \frac{i}{2}\sum_\mu \int d3\, v(1,3)\frac{\langle E|T\hat{S}\hat{\psi}_\mu(1)\hat{\psi}^\dagger_\mu(2)\delta\hat{n}(3)|E\rangle}{\langle E|\hat{S}|E\rangle}$$

contains a four point function that may be expressed as a functional derivative

$$\frac{\delta G(1,2)}{\delta\varphi(3)} = -\frac{1}{2\hbar}\sum_\mu \frac{\langle E|T\hat{S}\hat{\psi}_\mu(1)\hat{\psi}^\dagger_\mu(2)\delta\hat{n}(3)|E\rangle}{\langle E|\hat{S}|E\rangle}$$

The equation of motion becomes

$$\boxed{[i\hbar\partial_{t_1} - h(\mathbf{x}_1) - V(1)]G(1,2) = \hbar\delta(1,2) + i\hbar\int d3 v(1,3)\frac{\delta G(1,2)}{\delta\varphi(3)}} \tag{33}$$

The equation has its own interest, as it leads to approximations that were investigated by Reining et al. [13]. If we insert Dyson's relation $G = g + g\Sigma^\star G$ into it, we obtain:

$$\int d3\, \Sigma^\star(1,3)G(3,2) = i\int d3\, v(1,3)\frac{\delta G(1,2)}{\delta\varphi(3)} = i\int d3\, W(1,3)\frac{\delta G(1,2)}{\delta V(3)}$$

For the last equality we used the property (18). To isolate $\Sigma^\star$ we use the functional inverse of G

$$\begin{aligned}\hbar\Sigma^\star(1,2) =& i\int d34\, W(1,3)\frac{\delta G(1,4)}{\delta V(3)}G^{-1}(4,2)\\ =& -i\int d34\, W(1,3)G(1,4)\frac{\delta G^{-1}(4,2)}{\delta V(3)}\end{aligned}$$

The last derivative is the full vertex. Then, Hedin's equation (3) is obtained:

$$\boxed{\Sigma^\star(1,2) = \frac{i}{\hbar}\int d34\, W(1,3)\Gamma(3;4,2)G(1,4)} \tag{34}$$

It gives the skeleton graph of the self-energy.

7. Identities

Hedin's equations imply identities. This one may also be checked at the level of skeleton diagrams:

$$\int d12\, \Pi^\star(1,2)W(1,2) = -2\int d12\, \Sigma^\star(1,2)G(2,1) \tag{35}$$

Other simple identities are $\int d1\, \Pi(1,2)=0$, $\int d1\, \Pi^\star(1,2)=0$.
The vertex equation implies Ward's identities, as in electrodynamics.

Lemma 7.1. *Let $g(1,2)$ be a propagator for independent particles (Hartree propagators are of this sort), with time-independent Hamiltonian. Then:*

$$\int d3\, g(1,3)g(3,2) \;=\; -i(t_1-t_2)\, g(1,2) \tag{36}$$

Proof. If $h|a\rangle = e_a|a\rangle$, the time-ordered Green function is:
$ig(\mathbf{x}t,\mathbf{x}'t') = \sum_a \langle \mathbf{x}|a\rangle\langle a|\mathbf{x}'\rangle e^{-\frac{i}{\hbar}e_a(t-t')}[\theta(t-t')\theta(e_a-e_F)-\theta(t'-t)\theta(e_F-e_a)]$.
After the integration in $\mathbf{x}_3$ one remains with: $-\sum_a\langle \mathbf{x}_1|a\rangle\langle a|\mathbf{x}_2\rangle e^{-\frac{i}{\hbar}e_a(t_1-t_2)}\times$
$\int dt_3[\theta(t_1-t_3)\theta(t_3-t_2)\theta(e_a-e_F)+\theta(t_2-t_1)\theta(t_2-t_3)\theta(e_F-e_a)]$. The time integral is $(t_1-t_2)[\theta(t_1-t_2)\theta(e_a-e_F)-\theta(t_2-t_1)\theta(e_F-e_a)]$. The result follows. □

The lemma holds true with the replacement

$$g(1,2)\to g_k(1,2) = e^{ik\cdot(1-2)}g(1,2)$$

where $k\cdot 1 = \mathbf{k}\cdot\mathbf{x}_1 - i\omega t_1$. In a loop the phases cancel, while in an open string it is $g_k(1,2)g_k(2,3)\dots g_k(n-1,n) = e^{ik\cdot(1-n)}g(1,2)\dots g(n-1,n)$. Therefore, upon replacement $g\to g_k$, W and $\Pi^\star$ remain unchanged, while $\Sigma^\star(2,3)$, $G(2,3)$ and $\Gamma(1;2,3)-\delta(1,2)\delta(1,3)$ gain a factor $e^{ik(2-3)}$.

Proposition 7.2 (Ward's identity)**.**

$$\boxed{\int d1\, \Gamma(1;2,3) \;=\; \delta(2,3) - i(t_2-t_3)\,\Sigma^\star(2,3)} \tag{37}$$

Proof. Let us consider Hedin's equations with $g\to g_k$.

$$\begin{aligned}\int d5\, \Gamma_k(5;3,4)-\delta(3,4) &= -i\int d12\,(t_1-t_2)g_k(1,2)\frac{\delta\Sigma^\star_k(3,4)}{\delta g_k(1,2)}\\ &= \int d12\, \frac{\partial g_k(1,2)}{\partial\omega}\,\frac{\delta\Sigma^\star_k(3,4)}{\delta g_k(1,2)} \;=\; \frac{\partial}{\partial\omega}\,\Sigma^\star_k(3,4) = -i(t_3-t_4)\,\Sigma^\star_k(3,4)\end{aligned}$$

by the chain rule. The auxiliary parameter k is now set to zero, and we gain a version of Ward's identity (it is eq.(7.22) in Strinati's report [5]). □

If we expand the vertex in frequency space:

$$\Gamma(1;2,3) = \int \frac{d\omega' d\omega''}{4\pi^2}\Gamma(\mathbf{x}_1;\mathbf{x}_2,\mathbf{x}_3;\omega',\omega'')e^{-i\omega'(t_2-t_1)-i\omega''(t_1-t_3)}$$

the integral in dt_1 gives a factor $2\pi\delta(\omega'-\omega'')$:

$$\int d1\,\Gamma(1;2,3) = \int dx_1\frac{d\omega}{2\pi}\Gamma(\mathbf{x}_1;\mathbf{x}_2,\mathbf{x}_3;\omega,\omega)e^{-i\omega(t_2-t_3)}$$

Ward's identity becomes

$$\int d\mathbf{x}\, \Gamma(\mathbf{x};\mathbf{x}_2,\mathbf{x}_3,\omega,\omega) = \delta(\mathbf{x}_2-\mathbf{x}_3) - \frac{\partial}{\partial\omega}\Sigma^\star(\mathbf{x}_1,\mathbf{x}_2;\omega) \tag{38}$$

If the theory is also invariant for space translations:

$$\boxed{\Gamma(q=0,\mathbf{p},\omega,\omega) = 1 - \frac{\partial}{\partial\omega}\Sigma^\star(\mathbf{p},\omega)} \tag{39}$$

8. Counting Feynman diagrams

In zero dimension of space-time, Hedin's equations allow for an easy enumeration of Feynman diagrams [15, 16, 17]. The four Hedin's equations (1-4) become algebraic, with parameters g and v, and the functional derivative in the vertex equation (6) becomes an ordinary one. After removing imaginary factors, $\hbar = 1$ and replacing the factor (-2) with a loop-counting parameter ℓ, Hedin's equations are:

$$G = g + g\Sigma^\star G \quad W = v + v\Pi^\star W \quad \Sigma^\star = GW\Gamma \quad \Pi^\star = \ell G^2\Gamma \quad \Gamma = 1 + g^2\frac{\partial\Sigma^\star}{\partial g}$$

By searching solutions as series expansions in $x = g^2 v$ and ℓ, one obtains coefficients that count the Feynman graphs that contribute to a perturbative order with a given number of loops. For example:

$$\begin{aligned}\Sigma^\star/vg &= 1 + (2+\ell)x + (10+9\ell+\ell^2)x^2 + (74+91\ell+23\ell^2+\ell^3)x^3\\ &\quad + (706+1063\ell+416\ell^2+46\ell^3+\ell^4)x^4\\ &\quad + (8162+14193\ell+7344\ell^2+1350\ell^3+80\ell^4+\ell^5)x^5+\dots\\ &= 1+3x+20x^2+189x^3+2232x^4+31130x^5+\dots\end{aligned}$$

The last line corresponds to $\ell = 1$. At order v^3 there are 20 self-energy diagrams: 10 with no loops, 9 with a single loop and 1 with two loops. The expansions for the vertex and the polarization are

$$\begin{aligned}\Gamma &= 1 + x + 3(2+\ell)x^2 + 5(10+9\ell+\ell^2)x^3 + 7(74+91\ell+23\ell^2+\ell^3)x^4\\ &\quad + 9(706+1063\ell+416\ell^2+46\ell^3+\ell^4)x^5+\dots\\ &= 1+x+9x^2+100x^3+1323x^4+20088x^5+\dots\\ \Pi^\star/g^2\ell &= 1+3x+(15+5\ell)x^2+(105+77\ell+7\ell^2)x^3+(945+1044\ell+234\ell^2+9\ell^3)x^4\\ &\quad + (10395+14784\ell+5390\ell^2+550\ell^3+11\ell^4)x^5+\dots\\ &= 1+3x+20x^2+189x^3+2232x^4+31130x^5+\dots\end{aligned}$$

The counting numbers $\ell = 1$ of the expansions of the self-energy and the polarisation are the same. This occurs also in Q.E.D., though with smaller numbers because of cancellations of diagrams with odd number of propagators in a fermionic loop (Furry's theorem).

While the numbers of Feynman diagrams grow factorially, it turns out that in the GW approximation (i.e. $\Gamma = 1$) they grow algebraically [15]. The enumeration of skeleton diagrams can be easily carried out from Hedin's equations, by proper choice of the expansion parameter [16].

References

[1] L. Hedin: *New method for calculating the one-particle Green's function with application to the electron-gas problem*, Phys. Rev. **139** (1965) A796.

[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw Hill (1965).

[3] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems*, (MacGraw Hill, New York, 1971).

[4] A. Schindlmayr and R. Godby, *Systematic vertex corrections through iterative solution of Hedin's equations beyond the GW approximation*, Phys. Rev. Lett. **80** (1998) 1702.

[5] G. Strinati, *Application of the Green's functions method to the study of optical properties of semiconductors*, Rivista del Nuovo Cimento **11**, (1988).

[6] G. Onida, L. Reining and A. Rubio, *Electronic excitations: density-functional versus many-body Green's-function approaches*, Rev. Mod. Phys. **74**, 601 (2002).

[7] W. G. Aulbur, L. Jönsson and J. W. Wilkins *Solid State Physics* vol.54 (Academic, New York, 2000).

[8] F. Aryasetiawan and O. Gunnarsson, *The GW method*, Rep. Progr. Phys. **61** (1998) 237.

[9] G. Fratesi, G. P. Brivio and L. G. Molinari, *Many-body method for infinite non-periodic systems*, Phys. Rev. B **69**, 245113 (2004).

[10] F. Bechstedt, *Many-body approach to electronic excitations, concepts and applications*, Springer Series in Solid-State Sciences 181, (Springer 2015).

[11] A. Pelster, H. Kleinert and M. Bachmann, *Functional closure of Schwinger-Dyson equations in QED*, Ann. Phys. **297**, 363 (2002).

[12] A. Pelster and K. Glaum, *Many-body vacuum diagrams and their graphical construction*, Phys. Status Solidi B **237**, 72 (2003).

[13] G. Lani, P. Romaniello and L. Reining, *Approximations for many-body Green's functions: insights from the fundamental equations*, New Journal of Physics **14** (2012) 013056.

[14] J. A. Berger, P. Romaniello, F. Tandetzky, B. S. Mendoza, C. Brouder and L. Reining, *Solution to the many-body problem in one point*, New Journal of Physics —bf 16 (2014) 113025

[15] L. G. Molinari, *Hedin's equations and enumeration of Feynman diagrams*, Phys. Rev. B **71**, 113102 (2005).

[16] L. G. Molinari and N. Manini, *Enumeration of many-body skeleton diagrams*, Eur. Phys. J. B **51** (2006) 331.

[17] Y. Pavlyukh and W. Hübner, *Analytic solution of Hedins equations in zero dimensions*, J. Math. Phys. **48** (2007) 052109