THE HAAR MEASURE OF A LIE GROUP
a simple construction

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$\mathcal{G}$ is representation of a Lie Group, with elements U that are unitary matrices of size N. In the exponential form $U = e^{iH}$, the Hermitian $N \times N$ matrix H belongs to a Lie algebra. Let n be the linear dimension of the real algebra. The n generators are chosen such that

$$[T_a, T_b] = i f_{abc} T_c, \qquad \mathrm{tr} T_a T_b = \delta_{ab} \tag{1}$$

The orthogonality relation for the generators implies the total antisymmetry of the structure constants f_{abc}. We have the parametrization $H(\mathbf{x}) = x_a T_a$, $x_a = \mathrm{tr}(H T_a)$.
The volume element for the Haar measure is constructed by means of the metric, which enters in the invariant measure:

$$ds^2 = -\mathrm{tr}[U^{-1} dU U^{-1} dU] = g_{ab}(\mathbf{x}) dx_a dx_b \tag{2}$$

g is a real symmetric $n \times n$ matrix. The parametrization of the group introduces an explicit construction of the Haar measure

$$\int_{\mathcal{G}} dU f(U) = \int d^n x \sqrt{\det g} f(\mathbf{x}) \tag{3}$$

We shall now determine the eigenvalues of the metric matrix $g(\mathbf{x})$. Let us recall the formula for the differential of the exponential of an operator

$$d(e^A) = \int_0^1 dt e^{(1-t)A} (dA) e^{tA} \tag{4}$$

and evaluate:

$$U^{-1} dU = e^{-iH} \int_0^1 e^{i(1-t)H} (idH) e^{itH} = i dx_a \int_0^1 dt e^{-itH} T_a e^{itH}$$

$$\begin{aligned} g_{ab} &= \int_0^1 dt_1 \int_0^1 dt_2 \mathrm{tr}[e^{-it_1 H} T_a e^{i(t_1 - t_2)H} T_b e^{it_2 H}] = \\ &= \int_0^1 dt_1 \int_0^1 dt_2 \mathrm{tr}[e^{i(t_2 - t_1)H} T_a e^{-i(t_2 - t_1)H} T_b] = \\ &= \int_{-1}^1 dt (1 - |t|) \mathrm{tr}[e^{itH} T_a e^{-itH} T_b] \end{aligned}$$

We used the cyclic property of the trace and made the change of variables $t = t_2 - t_1$, $2s = t_1 + t_2$ and integrated in s. We note the expansion

$$e^{itH(\mathbf{x})} T_a e^{-itH(\mathbf{x})} = c_{ab}(t\mathbf{x}) T_b \tag{5}$$

To evaluate $c_{ab}(t\mathbf{x}) = \mathrm{tr}[e^{itH}T_a e^{-itH}T_b]$ we write a differential equation for it

$$\begin{aligned}\frac{d}{dt}c_{ab}(t\mathbf{x}) =& - i\mathrm{tr}(e^{itH}T_a e^{-itH}[H,T_b]) = \\ =& f_{cbd}x_c \mathrm{tr}(e^{itH}T_a e^{-itH}T_d) = \\ =& f_{cbd}x_c c_{ad}(t\mathbf{x}) = \\ =& c_{ac}(t\mathbf{x})M_{cb}(\mathbf{x})\end{aligned}$$

where we introduced the real antisymmetric matrix $M_{ab}(\mathbf{x}) = x_c f_{cba}$, of size $n \times n$. The differential equation, with the initial condition $c_{ab}(0) = \delta_{ab}$, has the solution $c_{ab}(t\mathbf{x}) = [\exp tM(\mathbf{x})]_{ab}$. Therefore, we conclude with the matrix identity

$$g(\mathbf{x}) = \int_{-1}^{1} dt(1-|t|)e^{tM(\mathbf{x})} \tag{6}$$

Being $M = -M^\dagger$, the non-zero eigenvalues of M come in pairs $\pm i\lambda$, with real λ. The matrices g and M are diagonalized by the same unitary matrix, therefore if λ_i is an eigenvalue of M, the corresponding eigenvalue of g is:

$$g_i = \int_{-1}^{1} dt(1-|t|)e^{it\lambda_i} = \frac{\sin^2(\lambda_i/2)}{(\lambda_i/2)^2} \tag{7}$$

It follows that, because the eigenvalues λ_i come with opposite signs, we need only consider the positive ones:

$$\sqrt{\det g} = \prod_{\lambda_i>0} \frac{\sin^2(\lambda_i/2)}{(\lambda_i/2)^2} \tag{8}$$

This is a general result. The problem is then to obtain the dependence of (8) upon the coordinate $\mathbf{x}$ of the Lie algebra, which is of course specific of the definition of the group.

Let us make some remarks on the spectrum of the matrix M. The eigenvalue equation in C^n: $M(\mathbf{x})\mathbf{v} = i\lambda\mathbf{v}$ corresponds to

$$[V, H(\mathbf{x})] = \lambda V \qquad V = v_a T_a \tag{9}$$

This equation has the obvious solutions $V = |h_i\rangle\langle h_i|$, with eigenvalue $\lambda = 0$.
Since the matrix H is Hermitian, it is diagonalized by a $SU(N)$ matrix: $H = S^\dagger h S$. The matrix h is diagonal with elements h_i, the N eigenvalues of H. The equation translates into:

$$(S^\dagger V S)_{ij}(h_i - h_j - \lambda) = 0 \tag{10}$$

where it is clear that the eigenvalues $\lambda_a(\mathbf{x})$ that are needed to evaluate the Haar measure, are just differences $h_i - h_j$ of eigenvalues of $H(\mathbf{x}) = x_a T_a$.

For the Lie group $SU(N)$ of special unitary matrices of size N, $U^\dagger U = I$ and $\det U = 1$, the Lie algebra $su(N)$ is the set of all traceless Hermitian matrices, and has linear dimension $n = N^2 - 1$. Given the real eigenvalues $h_1, \ldots, h_N$ of a generic matrix H in the Lie algebra, the matrix M corresponding to H has n eigenvalues among which N are equal to zero and $n - N$ are given by differences $h_i - h_j$ $(i \neq j)$. The invariant measure of $SU(N)$ is then

$$dx_1 \ldots dx_n \prod_{i<j} \frac{\sin^2(h_i/2 - h_j/2)}{(h_i/2 - h_j/2)^2}$$