VORTICES

PRELIMINARY NOTES BY L. G. MOLINARI

Let us summarise the main formulas of the theory by Ginzburg and Landau for superconductivity. The free energy and the current density are

$$F = \int d\mathbf{x} \frac{\hbar^2}{2m^*}|(-i\,\mathbf{grad} + \frac{e^*}{\hbar c}\mathbf{A})\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{\mathbf{B}^2}{8\pi} + F_n^0 \tag{1}$$

$$\mathbf{J} = -\frac{e^{*2}}{m^* c}|\psi|^2(\frac{\phi_0}{2\pi}\mathbf{grad}\,\omega + \mathbf{A}) \tag{2}$$

where $\psi = |\psi|e^{i\omega}$, $a = a'(T - T_c)$, $b > 0$ and $\phi_0 = hc/e^*$ is the unit of magnetic flux. The single-valuedness of ψ implies this integral identity on a closed circuit:

$$\frac{m^* c}{e^{*2}} \oint_C \frac{\mathbf{J}\cdot d\ell}{|\psi|^2} + \int_S da\; \mathbf{n}\cdot\mathbf{B} = m\phi_0, \quad m \text{ integer} \tag{3}$$

where S is encircled by the line C. The (squared) coherence and London lengths, the GL ratio, the bulk value of the order parameter are:

$$\xi^2 = \frac{\hbar^2}{2m^*|a|}, \quad \lambda^2 = \frac{m^* c^2 b}{4\pi e^{*2}|a|}, \quad \kappa = \frac{\lambda}{\xi} = \frac{m^* c}{e^*\hbar}\sqrt{\frac{b}{2\pi}}, \quad \psi_\infty^2 = \frac{|a|}{b} \tag{4}$$

$$\frac{H_c(T)^2}{8\pi} = \frac{a^2}{2b}, \qquad H_{c2}(T) = \kappa\sqrt{2}H_c(T) \tag{5}$$

The following relations are useful:

$$\phi_0 = 2\pi\xi^2 H_{c2}, \quad \xi\lambda H_c = \frac{\phi_0}{2\pi\sqrt{2}} \tag{6}$$

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1. London vortices

The study of the G.L. equations is simpler in the regime $\kappa \gg 1$, where $f = 1$, i.e. $\psi = \psi_\infty e^{i\omega}$ in bulk regions, and drops to zero in contact with normal regions. The free energy and the current density (2) are approximated by

$$F = \int_V d\mathbf{x} \frac{e^{*2}}{2m^* c^2}\psi_\infty^2 |\frac{\phi_0}{2\pi}\mathbf{grad}\,\omega + \mathbf{A}|^2 + a\psi_\infty^2 + \frac{b}{2}\psi_\infty^4 + \frac{\mathbf{B}^2}{8\pi} + F_n^0$$

$$\mathbf{J} = -\frac{e^{*2}}{m^* c}\psi_\infty^2(\frac{\phi_0}{2\pi}\mathbf{grad}\,\omega + \mathbf{A}) \tag{7}$$

where the volume *excludes regions where f differs from 1.*
With the II GL equation (Maxwell's equation), $\mathbf{J} = \frac{c}{4\pi}\mathrm{rot}\mathbf{B}$, the free energy becomes:

$$F = \frac{1}{8\pi}\int_V d\mathbf{x}\,(\mathbf{B}^2 + \lambda^2|\mathrm{rot}\mathbf{B}|^2) - \frac{H_c^2}{8\pi} + F_{n,0} \tag{8}$$

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Remark 1.1. *The same expression results in London's theory for a superfluid with uniform mass density* $m^* n_s$*, velocity* $\mathbf{v}_s$*, supercurrent* $\mathbf{J}_s = -e^\star n_s \mathbf{v_s}$*. Maxwell's equation gives the kinetic energy*

$$\frac{1}{2} m^* n_s v_s^2 = \frac{1}{2} m^* n_s \frac{c^2}{16\pi^2} \frac{|\mathrm{rot} B|^2}{e^{\star 2} n_s^2} = \frac{\lambda^2}{8\pi} |\mathrm{rot} B|^2$$

The rotation of (7) and Maxwell's equation give: rot rot $\mathbf{B} = -\frac{4\pi e^{*2}}{m^* c^2} \psi_\infty^2 \mathbf{B}$, i.e.

$$\mathbf{B} - \lambda^2 \nabla^2 \mathbf{B} = 0 \tag{9}$$

In the following we consider vortex solutions $\mathbf{B} = B(\mathbf{x})\mathbf{k}$. Then: $|\mathrm{rot}\,\mathbf{B}|^2 = (\partial_x B)^2 + (\partial_y B)^2 = -B\nabla^2 B + \frac{1}{2}\nabla^2 B^2$. The free energy per unit length is

$$\epsilon = \frac{F}{L} = \frac{1}{8\pi} \int_S da\, B(B - \lambda^2 \nabla^2 B) + \frac{\lambda^2}{16\pi} \int_S da\ \nabla^2 B^2$$

where the surface S excludes the vortex cores. The first integral is zero for a solution of (9). Since $\nabla^2 =$ div grad we obtain an integration along the boundary:

$$\boxed{\epsilon = \frac{\lambda^2}{16\pi} \sum_k \oint d\ell\ \mathbf{n} \cdot \mathbf{grad} B^2} \tag{10}$$

The sum is on all vortex cores, and the integrals are on circles of radius ξ centered in each core, with normal vector $\mathbf{n}$ pointing to the center of the core. In a core the field is almost constant (supercurrents vanish). The contribution to the total energy of the cores is small.

1.1. **1-vortex solution.** For a single vortex, $B(r)$ solves (9) outside the core. In polar coordinates it is Bessel's equation

$$B'' + \frac{1}{r} B' - \frac{1}{\lambda^2} B = 0 \qquad r > \xi \tag{11}$$

The solution is Hankel's function $K_0(r/\lambda)$, up to a multiplicative constant c. The function is always positive and decreasing, with limit behaviours:

$$K_0(x) = \begin{cases} \sqrt{\frac{\pi}{2x}}\ \exp(-x) & x \gg 1 \\ -\log x + \log 2 - \gamma & x \to 0 \end{cases} \tag{12}$$

$\log 2 - \gamma \approx 0.12$ (γ is Euler's constant). $K_0(1) = 0.421$, $K_0(2) = 0.114$. The derivative is $K_0'(x) = -K_1(x)$.

Since $x = r/\lambda$, the limit $x \to 0$ is achieved for $\xi/\lambda \ll 1$ i.e. a type II superconductor.

To obtain the constant c let us evaluate the flux through a ring $\xi < r < R$, where R is arbitrary:

$$\Phi(R) = 2\pi \int_\xi^R r dr B(r) = 2\pi c \lambda^2 \int_{\xi/\lambda}^{R/\lambda} x dx \frac{1}{x} \frac{d}{dx}(-x K_1)$$
$$= 2\pi c \lambda [\xi K_1(\xi/\lambda) - R K_1(R/\lambda)]$$

For $\xi \ll \lambda$ we approximate $K_1(x) \simeq 1/x$; the total flux is collected within a distance of few screening London lengths λ. Then $R \gg \lambda$:

$$\Phi(R) \approx 2\pi \lambda^2 c - c(\pi\lambda)^{3/2} \sqrt{2R}\, e^{-R/\lambda}$$

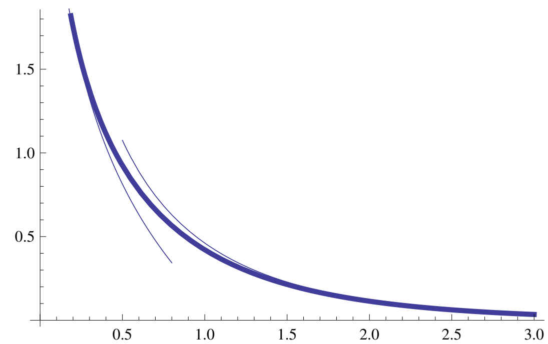

FIGURE 1. The function $K_0(x)$. The thin lines describe the limit functions in eq.(12)

We thus determine c, and put in the solution B:

$$B(r) = \frac{\Phi}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right) \tag{13}$$

The field at the core for a unit quantum flux, is

$$B(\xi) = \frac{2\pi\xi^2 H_{c2}}{2\pi\lambda^2} \log\kappa = H_c\sqrt{2}\frac{\log\kappa}{\kappa}$$

The actual field H_{c1} at which the flux first penetrates when leaving the pure diamagnetic phase is about half of it (the term 0.12 is omitted)

$$H_{c1} = H_c \frac{\log\kappa}{\kappa\sqrt{2}} \tag{14}$$

It is the value at which difference of the Gibbs potentials $G_s - G_n$ becomes negative. In a type II superconductor, nothing happens as $H = H_c$ (the thermodynamic critical field).

The free energy per unit length of a vortex is evaluated on the circle $r = \xi$:

$$\begin{aligned}\epsilon_1 &= -\frac{\lambda^2}{16\pi}\oint d\ell\,\frac{d}{dr}B^2(r) = -\frac{\lambda^2}{16\pi}\frac{d}{dr}B^2(r)\Big|_{r=\xi} 2\pi\xi \\ &= \left(\frac{\Phi}{4\pi\lambda}\right)^2 K_0\left(\frac{\xi}{\lambda}\right) K_1\left(\frac{\xi}{\lambda}\right)\frac{\xi}{\lambda}\end{aligned}$$

In the limit $\kappa = \lambda/\xi \gg 1$, it is:

$$\epsilon_1 = \left(\frac{\Phi}{4\pi\lambda}\right)^2 (\log\kappa + 0.12) \tag{15}$$

The constant value 0.12 will be neglected. The current density circulates around the core and fades within few London lengths:

$$\mathbf{J}(r) = \frac{c}{4\pi}\mathrm{rot}\mathbf{B} = \frac{c}{4\pi}[\mathbf{i}\partial_y B - \mathbf{j}\partial_x B] = -\frac{c}{4\pi}\frac{dB}{dr}\boldsymbol{\theta} = \frac{c\Phi}{8\pi^2\lambda^3}K_1\left(\frac{r}{\lambda}\right)\boldsymbol{\theta} \tag{16}$$

Example 1.2. The superconducting alloy Nb_3Sn has $T_c = 18.3K$, $\kappa \approx 40$, $\xi_0 = 3.3$nm, $\lambda = 135$nm, $H_{c1} = 0.038$T. It can attain $H_{c2} = 30$T [6]. Evaluate the free energy per unit length of a vortex.

$$\xi^2 = \frac{\phi_0}{2\pi H_{c2}} = \frac{2.07\times 10^{-7}\text{Oe}\cdot\text{cm}^2}{2\pi\cdot 30\times 10^{-4}\text{T}} = 10.9\times 10^{-14}\text{cm}^2, \quad \xi \approx 3.3\,\text{nm}$$

With $mc^2 = 0.51$ MeV, $a_0 = \hbar^2/(me^2) = 5.29\times 10^{-2}$nm, the energy per unit length of a vortex with 1 elementary flux $\phi_0 = hc/2e$ is:

$$\epsilon = \left(\frac{hc}{8e\pi\lambda}\right)^2 (\log\kappa + 0.12) = \frac{mc^2}{16}\frac{a_0}{\lambda^2}(\log\kappa + 0.12) \approx 3.5\frac{\text{MeV}}{\text{cm}} = 5.6\times 10^{-6}\frac{\text{erg}}{\text{cm}}$$

Example 1.3. Show that the ratio $\epsilon_1/\epsilon_{core} = 4\log\kappa$, where $\epsilon_{core} = \frac{1}{8}\xi^2 H_c^2$ is an estimate of the energy per unit length stored in the normal core of the vortex.

1.2. **The 2-vortex solution.** Since the equation is linear, a 2-vortex solution is the superposition a two 1-vortex solutions with a flux Φ in the origin and another in $\mathbf{R}$:

$$B(\mathbf{x}) = \frac{\Phi}{2\pi\lambda^2}\left[K_0\left(\frac{|\mathbf{x}|}{\lambda}\right) + K_0\left(\frac{|\mathbf{x}-\mathbf{R}|}{\lambda}\right)\right] \tag{17}$$

with $R \gg \xi$. The free energy per unit length is

$$\begin{aligned}
\epsilon_2 &= \frac{\lambda^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,\mathbf{n}\cdot\mathbf{grad}(B_1+B_2)^2 \\
&\approx \frac{\lambda^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,\mathbf{n}\cdot\mathbf{grad}B_j^2 + 2\frac{\lambda^2}{16\pi}\sum_{j=1,2}\oint_{C_j} d\ell\,\mathbf{n}\cdot\mathbf{grad}(B_1B_2) \\
&= 2\epsilon_1 + \frac{4\lambda^2}{16\pi}\left(\frac{\Phi}{2\pi\lambda^2}\right)^2\oint_{|\mathbf{x}|=\xi} d\ell(-\frac{d}{dr})K_0\left(\frac{|\mathbf{x}|}{\lambda}\right)K_0\left(\frac{|\mathbf{x}-\mathbf{R}|}{\lambda}\right) \\
&\approx 2\epsilon_1 + \frac{\lambda^2}{4\pi}\left(\frac{\Phi}{2\pi\lambda^2}\right)^2\frac{1}{\lambda}K_1\left(\frac{\xi}{\lambda}\right)K_0\left(\frac{R}{\lambda}\right)2\pi\xi
\end{aligned}$$

We neglected the contribution of B_1^2 to the hole 2 and of B_2^2 to hole 1 (i.e. $\lambda/\xi = \kappa \gg 1$), and a term arising from the derivative (it is order $1/\kappa^2$ of the first one).
The interaction energy per unit length among the two parallel and equal vortices at distance R is $\epsilon_{\text{int}} = \epsilon_2 - 2\epsilon_1$,

$$\boxed{\epsilon_{int}(R) = \frac{\Phi^2}{8\pi^2\lambda^2}K_0\left(\frac{R}{\lambda}\right)} \tag{18}$$

Since K_0 is monotonically decreasing, the force per unit length between vortices is *repulsive*: $f_{12}(R) = -\epsilon'_{int}(R) > 0$.
A more accurate expression, valid for $R \gg \xi$, is [5]

$$\epsilon_{int}(R) = c^2 K_0\left(\frac{R}{\lambda}\right) - \frac{d^2}{\kappa^2}K_0\left(\sqrt{2}\frac{R}{\xi}\right) \tag{19}$$

with parameters c, d. It is attractive for type I superconductors, and always attractive for vortex antivortex pairs. It is zero for $\kappa = 1/\sqrt{2}$.

Exercise 1.4. Show that the free energy per unit length of a single vortex with flux 2Φ is greater than the free energy of two vortices each carrying a flux Φ.

Exercise 1.5. (from [7]). Find the attractive force exerted on a vortex by the surface of a flat superconductor if the vortex is parallel to the surface at a distance $\ell = 50$nm and $\lambda = 300$nm.

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