THE GELL-MANN AND LOW THEOREM AND THE REDUCTION FORMULA

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In the appendix of their paper *Bound States in Quantum Field Theory*, Murray Gell-Mann and Sir Francis Low [1] proved a fundamental theorem that bridges the ground states $|E_0\rangle$ and $|E\rangle$ of Hamiltonians H_0 and $H = H_0 + gV$ by means of time propagators, and makes the transition of time-ordered correlators from the Heisenberg to the interaction picture possible (reduction formula):

$$\langle E|\mathsf{T}O_1(t_1)...O_n(t_n)|E\rangle = \frac{\langle E_0|\mathsf{T}S\, O_1(t_1)...O_n(t_n)|E_0\rangle}{\langle E_0|S|E_0\rangle} \tag{1}$$

The single operator S contains all the effects of the interaction. The theorem borrows ideas from the scattering and the adiabatic theories and makes use of the concept of adiabatic switching of the interaction through the time-dependent operator

$$H_\epsilon(t) = H_0 + g\, e^{-\epsilon|t|}V \tag{2}$$

that interpolates between the operators of interest, H at $t = 0$ and H_0 at $|t| \to \infty$. The adiabatic limit is obtained for $\epsilon \to 0^+$. With the operator H_0 singled out, the theorem requires the time propagator in the interaction picture,

$$U_{\epsilon,I}(t,s) = e^{it/\hbar H_0} U_\epsilon(t,s)\, e^{-i/\hbar s H_0} \tag{3}$$

where $U_\epsilon(t,s)$ is the full propagator.

1. The Gell-Mann and Low theorem

Theorem 1.1 (Gell-Mann and Low, 1951)**.** *Let $|E_0\rangle$ be an eigenstate of H_0 then, if the limit vectors exist*

$$|\Psi^\pm\rangle = \lim_{\epsilon\to 0^+} \frac{U_{\epsilon,I}(0,\pm\infty)|E_0\rangle}{\langle E_0|U_{\epsilon,I}(0,\pm\infty)|E_0\rangle} \tag{4}$$

they are eigenvectors of $H = H_0 + gV$.

The original proof, given also in the textbooks by Fetter and Walecka, or Gross and Heinonen, employs the formal Dyson's perturbative expansion of the interaction picture propagator, with long and cumbersome calculations. In ref. [2] I presented a different proof, based on Schrödinger's equation for the propagators. It is much simpler, and perhaps more intelligible. I begin with a simple identity:

Proposition 1.2. *Let $U_\epsilon(t,s)$ be the propagator for the Hamiltonian* (2)*. Then, for all $\epsilon > 0$:*

$$\pm\epsilon\, g\frac{\partial}{\partial g}U_\epsilon(t,s) = \frac{\partial}{\partial t}U_\epsilon(t,s) + \frac{\partial}{\partial s}U_\epsilon(t,s) \tag{5}$$

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where sign plus is for $0 \geq t \geq s$ *and sign minus is for* $0 \leq t \leq s$.

Proof. In the following $g = \exp(\epsilon\theta)$. Let us begin with the case $0 \geq t \geq s$. The propagator solves the equation

$$U_\epsilon(t,s) = 1 + \frac{1}{i\hbar}\int_s^t dt'(H_0 + e^{\epsilon(t'+\theta)}V)U_\epsilon(t',s) \tag{6}$$

Consider the g-independent Hamiltonian $\tilde{H}(t) = H_0 + e^{\epsilon t}V$. Its propagator $\tilde{U}(t,s)$ solves $\tilde{U}(t,s) = 1 + \frac{1}{i\hbar}\int_s^t dt''(H_0 + e^{\epsilon t''}V)\tilde{U}(t'',s)$. A shift by θ of times gives:

$$\tilde{U}(t+\theta, s+\theta) = 1 + \frac{1}{i\hbar}\int_{s+\theta}^{t+\theta} dt''(H_0 + e^{\epsilon t''}V)\tilde{U}(t'', s+\theta);$$

with $t'' = t' + \theta$ it becomes:

$$\tilde{U}(t+\theta, s+\theta) = 1 + \frac{1}{i\hbar}\int_s^t dt'(H_0 + e^{\epsilon(t'+\theta)}V)\tilde{U}(t'+\theta, s+\theta).$$

Comparison with (6) and unicity of the solution give the identity $U_\epsilon(t,s) = \tilde{U}(t+\theta, s+\theta)$. Therefore:

$$\epsilon\, g\frac{\partial}{\partial g}U_\epsilon(t,s) = \frac{\partial}{\partial\theta}\tilde{U}(t+\theta,s+\theta) = \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right]\tilde{U}(t+\theta,s+\theta)$$

and (5) is obtained. In the case $s \geq t \geq 0$, the auxiliary Hamiltonian is $\tilde{H}(t) = H_0 + Ve^{-\epsilon t}$. The result is the same up to a sign. □

Proposition 1.3. *For* $0 \geq t \geq s$ *or* $s \geq t \geq 0$ *(upper and lower sign in the equation), and* $\forall\epsilon > 0$*:*

$$\pm i\hbar\epsilon\, g\frac{\partial}{\partial g}U_{\epsilon,I}(t,s) = H_{\epsilon,H_0}(t)U_{\epsilon,I}(t,s) - U_{\epsilon,I}(t,s)H_{\epsilon,H_0}(s) \tag{7}$$

where $O_{H_0}(t) = e^{\frac{i}{\hbar}tH_0}O(t)e^{-\frac{i}{\hbar}tH_0}$ *is the Heisenberg evolution of an operator* $O(t)$.

Proof. Il (3) is placed in eq.(5), a straightforward calculation gives an equation for the propagator in interaction picture:

$$\pm\epsilon\, g\frac{\partial}{\partial g}U_{\epsilon,I}(t,s) = \frac{\partial}{\partial t}U_{\epsilon,I}(t,s) + \frac{\partial}{\partial s}U_{\epsilon,I}(t,s) - \frac{i}{\hbar}[H_0, U_{\epsilon,I}(t,s)]$$

Schrödinger's equation $i\hbar\partial_t U_{\epsilon,I}(t,s) = ge^{-\epsilon|t|}V_{H_0}(t)U_{\epsilon,I}(t,s)$, and the adjoint equation $-i\hbar\partial_s U_{\epsilon,I}(t,s) = ge^{-\epsilon|s|}V_{H_0}(s)U_{\epsilon,I}(t,s)$ are used for time derivatives. □

In particular, for $t = 0$ and $s = \mp\infty$:

$$\pm i\hbar\epsilon\, g\frac{\partial}{\partial g}U_{\epsilon,I}(0,\mp\infty) = HU_{\epsilon,I}(0,\mp\infty) - U_{\epsilon,I}(0,\mp\infty)H_0 \tag{8}$$

If $|E_0\rangle$ is an eigenstate of H_0 we obtain:

$$\pm i\hbar\epsilon\, g\frac{\partial}{\partial g}U_{\epsilon,I}(0,\mp\infty)|E_0\rangle = (H - E_0)U_{\epsilon,I}(0,\mp\infty)|E_0\rangle \tag{9}$$

Clearly we cannot put $\epsilon = 0$, as E_0 would remain an eigenvalue of H. The left-hand-side should provide a finite correction to the eigenvalue, i.e. there must be a $1/\epsilon$ compensation. The inner product with $|E_0\rangle$ gives:

$$\pm i\hbar\epsilon\, g\frac{\partial}{\partial g}\langle E_0|U_{\epsilon,I}(0,\mp\infty)|E_0\rangle = g\langle E_0|VU_{\epsilon,I}(0,\mp\infty)|E_0\rangle \tag{10}$$

Let us consider the ratios:

$$|\Psi_\epsilon^\mp\rangle = \frac{U_{\epsilon,I}(0,\mp\infty)|E_0\rangle}{\langle E_0|U_{\epsilon,I}(0,\mp\infty)|E_0\rangle} \tag{11}$$

Then, with the aid of (9) and (10), and noting that $\langle E_0|\Psi^\mp\rangle = 1$, we have:

$$\begin{aligned}\pm i\hbar\epsilon\, g\frac{\partial}{\partial g}|\Psi_\epsilon^\mp\rangle =& (H - E_0 - \, g\langle E_0|V|\Psi_\epsilon^\mp\rangle)|\Psi_\epsilon^\mp\rangle \\ =& (H - \, \langle E_0|H|\Psi_\epsilon^\mp\rangle)|\Psi_\epsilon^\mp\rangle\end{aligned}$$

The limit $\epsilon \to 0^+$ can now be taken, and shows that $H|\Psi^\mp\rangle = E^\mp|\Psi^\mp\rangle$, with

$$E^\mp = \langle E_0|H|\Psi^\mp\rangle \tag{12}$$

The question arises about conditions for the eigenvalues to be equal. For this, we need discussing time-reversal. The time-reversal operator is anti unitary[1], with action $TU_\epsilon(t,s)T^\dagger = U_\epsilon(-t,-s)$ for any value $\epsilon \geq 0$. If H_0 is invariant under time-reversal it is also

$$TU_{\epsilon,I}(t,s)T^\dagger = U_{\epsilon,I}(-t,-s). \tag{13}$$

Proposition 1.4. *If both H_0 and H commute with time-reversal, and if $T|E_0\rangle = |E_0\rangle$, then $E^+ = E^-$.*

Proof. If $T|E_0\rangle = |E_0\rangle$ then (13) with $t=0$ and $s=-\infty$ and (11) give:

$$T|\Psi_\epsilon^-\rangle = \frac{\langle E_0|U_{\epsilon,I}(0,+\infty)|E_0\rangle}{\langle E_0|U_{\epsilon,I}(0,-\infty)|E_0\rangle^*}|\Psi_\epsilon^+\rangle \tag{14}$$

Therefore the vectors are the same up to a phase factor, for all ϵ. Next, if $[H,T]=0$ it is $0 = \langle\Psi^+|TH - HT|\Psi^-\rangle = (E^- - E^+)\langle\Psi^+|T|\Psi^-\rangle$, and we obtain $E^- = E^+$. $\square$

Suppose that $|E_0\rangle$ is the non-degenerate ground state of H_0. If we assume that in the adiabatic switching on and off of the interaction the ground state remains such and non-degenerate, then the normalised ground state of H is proportional to the vectors $|\Psi^+\rangle = e^{i\delta}|\Psi^-\rangle$. A ground state average may be written as follows:

$$\langle E|O|E\rangle = \frac{\langle\Psi^+|O|\Psi^-\rangle}{\langle\Psi^+|\Psi^-\rangle} = \lim_{\epsilon\to 0^+}\frac{\langle E_0|U_{\epsilon,I}(\infty,0)OU_{\epsilon,I}(0,-\infty)|E_0\rangle}{\langle E_0|S_\epsilon|E_0\rangle} \tag{15}$$

where $S_\epsilon = U_{\epsilon,I}(\infty,-\infty)$ is the *scattering operator*.

2. The reduction formula

The reduction formula is an important consequence of the Gell-Mann and Low theorem. It enables to evaluate the ground state average $\langle E|\mathsf{T}O_{1H}(t_1)\cdots O_{nH}(t_n)|E\rangle$ where operators are evolved with H at different times, in interaction picture.
The action of time-ordering is to permute the operators, producing a sign factor for fermions. We may thus assume that the operators are time-ordered (for convenience we keep $t_1 > \cdots > t_n$). Next, note that $O_H(t) = U_{\epsilon,I}(0,t)O_{H_0}(t)U_{\epsilon,I}(t,0)$.

[1]An operator is antiunitary if it is a bijection, antilinear ($T\lambda u = \lambda^* Tu$) and $(Tu|Tu') = (u'|u)$.

Then the average of the time ordered product is:

$$\begin{aligned}
&\langle E|O_{1H}(t_1)\cdots O_{nH}(t_n)|E\rangle \\
&= \frac{\langle E_0|U_{\epsilon,I}(\infty,0)O_{1H}(t_1)\cdots O_{nH}(t_n)U_{\epsilon,I}(0,-\infty)|E\rangle}{\langle E_0|S_\epsilon|E_0\rangle} \\
&= \frac{\langle E_0|U_{\epsilon,I}(\infty,t_1)O_{1H_0}(t_1)U_{\epsilon,I}(t_1,t_2)O_{2,H_0}(t_2)\cdots O_{nH_0}(t_n)U_{\epsilon,I}(t_n,-\infty)|E_0\rangle}{\langle E_0|S_\epsilon|E_0\rangle}
\end{aligned}$$

A $\mathsf{T}-$ordering can be inserted, because the time sequences are time-ordered. After this insertion, the operators $U_{\epsilon,I}$ can be permuted and joined to yield the scattering operator:

$$\langle E|\mathsf{T}O_{1H}(t_1)\cdots O_{nH}(t_n)|E\rangle = \frac{\langle E_0|\mathsf{T}S_\epsilon O_{1H_0}(t_1)O_{2,H_0}(t_2)\cdots O_{nH_0}(t_n)|E_0\rangle}{\langle E_0|S_\epsilon|E_0\rangle}$$

The formula remains valid for arbitrary times, and is the reduction formula. This, the Dyson expansion for S and Wick's theorem give rise to the diagrammatic expansion of correlators.

References

[1] M. Gell-Mann and F. Low, *Bound states in quantum field theory*, Phys. Rev. 84, 350 (1951).

[2] L. G. Molinari, *Another proof of Gell-Mann and Low's theorem*, J. Math. Phys. 48, 052113 (2007) (4 pages).