THE THERMAL FUNCTIONAL INTEGRAL FOR BOSONS

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1. The Bose oscillator

Consider a canonical pair $[\hat{b}, \hat{b}^\dagger] = 1$, irreducible on some Hilbert space (no operator exists that commutes with them, except unity). The number operator is self-adjoint and unbounded, with orthonormal eigenvectors $\hat{b}^\dagger \hat{b}|n\rangle = n\,|n\rangle$, $n = 0, 1, 2, ...$, that span the Hilbert space: $1 = \sum_n |n\rangle\langle n|$ (completeness).
The action of the operators is $\hat{b}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$. The vectors are obtained from the vacuum by

$$|n\rangle = \frac{1}{\sqrt{n!}} \hat{b}^{\dagger n}|0\rangle$$

A coherent state is an eigenvector of the destruction operator:

$$\boxed{\hat{b}|\alpha\rangle = \alpha|\alpha\rangle} \tag{1}$$

The equation has a normalized solution for any complex value α:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{2}$$

The series is summed as $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{b}^\dagger} e^{-\bar{\alpha}\hat{b}}|0\rangle$ (the extra factor acts as unit operator). With the formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$, that truncates the Baker Campbell Hausdorff expansion if $[A, B]$ commutes with A and B, one gets a unitary operator acting as translation in $\mathbb{C}$:

$$|\alpha\rangle = e^{\alpha\hat{b}^\dagger - \bar{\alpha}\hat{b}}|0\rangle$$

Two coherent states are never orthogonal, with overlap: $|\langle\alpha|\alpha'\rangle| = e^{-\frac{1}{2}|\alpha-\alpha'|^2}$.
Being a continuum, coherent states are an overcomplete set:

$$\boxed{\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1} \tag{3}$$

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Proof:

$$\begin{aligned}\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| &= \sum_{n,m}|n\rangle\langle m|\int\frac{d^2\alpha}{\pi}\langle n|\alpha\rangle\langle\alpha|m\rangle\\ &= \sum_{n,m}|n\rangle\langle m|\int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\bar\alpha^m}{\sqrt{m!}}\\ &= \sum_{n,m}|n\rangle\langle m|\delta_{n,m}\end{aligned}$$

One can extract countable subsets of coherent states that are still complete [1].

Exercise 1.1. *Find eigenvectors and eigenvalues of the shifted oscillator $\hat B^\dagger \hat B$, with $\hat B = \hat b - \beta$, $\hat B^\dagger = \hat b^\dagger - \beta^*$ (in terms of those of $\hat b^\dagger \hat b$).*

The Bose oscillator has two important representations:

Harmonic oscillator. In $L^2(\mathbb{R})$, on the dense subspace $\mathscr{S}(\mathbb{R})$ define:

$$(\hat b\varphi)(x) = \frac{1}{\sqrt2}[x\varphi(x)+\varphi'(x)], \qquad (\hat b^\dagger\varphi)(x) = \frac{1}{\sqrt2}[x\varphi(x)-\varphi'(x)]$$

$(\hat b^\dagger \hat b\,\varphi)(x) = -\frac12\varphi''(x) + \frac12(x^2-1)\varphi(x)$ with Hermite functions as number states:

$$u_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{2^n n!\sqrt\pi}}e^{-\frac12 x^2}H_n(x)$$

The coherent states are shifted Gaussians

$$\Phi_\alpha(x) = \langle x|\alpha\rangle = \frac{1}{\sqrt[4]{\pi}}e^{-\frac12(x-\alpha\sqrt2)^2}$$

Holomorphic representation. The Bargmann space [2] is the Hilbert space of entire functions such that $\int\frac{d^2z}{\pi}e^{-|z|^2}|f(z)|^2 < \infty$ ($d^2z = dx\,dy$) with the inner product

$$(f,g) = \int\frac{d^2z}{\pi}e^{-|z|^2}\overline{f(z)}g(z)$$

The monomials $u_n(z) = \frac{z^n}{\sqrt{n!}}$ are a complete orthonormal system[1]. The canonical operators are adjoints of each other, with action:

$$(\hat b f)(z) = f'(z), \qquad (\hat b^\dagger f)(z) = zf(z)$$

The monomials u_n are the normalized eigenfunctions of the number operator $z\frac{d}{dz}$. The coherent states are exponential functions:

$$\Phi_\alpha(z) = e^{-\frac12|\alpha|^2}\sum_{n=0}^\infty\frac{\alpha^n}{\sqrt{n!}}u_n(z) = e^{-\frac12|\alpha|^2+\alpha z}$$

If the basis functions $u_n(z)$ are replaced by the Hermite functions $u_n(x)$, the coherent state of the harmonic oscillator is obtained, which can be seen as a generating function of Hermite functions.

[1]The power expansion of an entire function has infinite radius of absolute convergence, and uniform convergence on any compact set in $\mathbb{C}$.

Appendix. Two useful integrals:

$$\int \frac{d^2z}{\pi} e^{-|z|^2}\, \bar{z}^n z^m = n!\,\delta_{nm} \tag{4}$$

$$\int \frac{d^2z}{\pi} e^{-|z|^2+\alpha\bar{z}} z^m = \alpha^m \tag{5}$$

The last one gives $\int \frac{d^2z}{\pi} e^{-|z|^2+\alpha\bar{z}} f(z) = f(\alpha)$ (the function $e^{-|z|^2+\alpha\bar{z}}$ is named "reproducing kernel").

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2. The thermal partition function for bosons

The thermal partition function with sources is the *generator* of Green functions. The log of it, is the generator of connected Green functions. We then give a path-integral representation of it, based on coherent states.

Consider a many-boson Hamiltonian, written in second quantization with CCR operators $\hat{b}^\dagger_r$, $\hat{b}_r$ where the index $r = 1, 2, ...$ enumerates an arbitrary 1-particle orthonormal complete basis $|r\rangle$:

$$K = \hat{H} - \mu\hat{N} = \sum_{rs} \hat{b}^\dagger_r (h_{rs} - \mu\delta_{rs})\hat{b}_s + \frac{1}{2}\sum_{rr'ss'} \hat{b}^\dagger_r \hat{b}^\dagger_s v_{rsr's'} \hat{b}_{s'} \hat{b}_{r'}$$

Time-ordered thermal averages can be evaluated with the help of auxiliary classical fields (sources) $\phi_r(\tau)$ and $\overline{\phi_r(\tau)}$:

$$K[\bar{\phi}, \phi] = K - \hbar\sum_r [\hat{b}_r \overline{\phi_r(\tau)} + \hat{b}^\dagger_r \phi_r(\tau)]$$

The τ-evolution with sources is given by the operator $\mathscr{U}(\tau, 0)$. In the interaction picture: $\mathscr{U}(\tau, 0) = \exp(-\frac{1}{\hbar}K\tau)\mathscr{U}_I(\tau, 0)$ where

$$\mathscr{U}_I(\tau, 0) = \mathsf{T}\exp\left[\sum_r \int_0^\tau d\tau' \hat{b}_r(\tau')\overline{\phi_r(\tau')} + \hat{b}^\dagger_r(\tau')\phi_r(\tau')\right]$$

with evolution of $\hat{b}_r$ and $\hat{b}^\dagger_r$ ruled by K. The partition function with sources is

$$\boxed{Z[\bar{\phi}, \phi] = \mathrm{tr}[\mathscr{U}(\hbar\beta, 0)] = Z\langle \mathscr{U}_I(\hbar\beta, 0)\rangle_K} \tag{6}$$

where $Z = \mathrm{tr}(e^{-\beta K})$.
The average of operators in presence of the sources is defined in the interaction picture:

$$\langle \mathsf{T} O_1(\tau_1)\ldots O_n(\tau_n)\rangle_{\phi,\bar{\phi}} = \frac{\langle \mathsf{T}\mathscr{U}_I(\hbar\beta, 0)O_1(\tau_1)\ldots O_n(\tau_n)\rangle_K}{\langle \mathscr{U}_I(\hbar\beta, 0)\rangle_K}$$

It is a thermal average when sources are turned off.
Sources are useful to perform functional derivatives that, in this case, produce T-ordered averages of creation and destruction operators, in presence of the sources:

$$\langle \hat{b}_r(\tau)\rangle_{\phi,\bar{\phi}} = \frac{\langle \mathsf{T}\mathscr{U}_I(\hbar\beta, 0)\hat{b}_r(\tau)\rangle_K}{\langle \mathscr{U}_I(\hbar\beta, 0)\rangle_K} = \frac{1}{Z[\bar{\phi}, \phi]}\frac{\delta Z[\bar{\phi}, \phi]}{\delta\overline{\phi_r(\tau)}} \tag{7}$$

$$\langle \mathsf{T}\hat{b}_r(\tau)\hat{b}^\dagger_{r'}(\tau')\rangle_{\phi,\bar{\phi}} = \frac{1}{Z[\bar{\phi}, \phi]}\frac{\delta^2 Z[\bar{\phi}, \phi]}{\delta\overline{\phi_r(\tau)}\delta\phi_{r'}(\tau')} \tag{8}$$

and so on. They become thermal averages (thermal Green functions) when the sources, in the end, are turned off.
In analogy with statistical mechanics, we introduce the functional

$$W[\bar\phi,\phi]=\log Z[\bar\phi,\phi] \tag{9}$$

The functional $W[\bar\phi,\phi]$ is the generator of connected correlators in presence of the sources. This is shown in the example:

$$\begin{aligned}\frac{\delta^2 W[\bar\phi,\phi]}{\delta\phi_s(\tau')\delta\overline{\phi_r(\tau)}} =& \frac{\delta}{\delta\phi_s(\tau')}\left(\frac{1}{Z[\bar\phi,\phi]}\frac{\delta Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}}\right)\\ =&\langle \mathsf{T}\hat b_r(\tau)\hat b^\dagger_{r'}(\tau')\rangle_{\phi,\bar\phi}-\langle \hat b_r(\tau)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger_{r'}(\tau')\rangle_{\phi,\bar\phi}\end{aligned}$$

$$\begin{aligned}&\frac{\delta^3 W[\bar\phi,\phi]}{\delta\phi(1)\delta\phi(2)\delta\overline{\phi(3)}} = \frac{\delta}{\delta\phi(1)}[\langle \mathsf{T}\hat b(3)\hat b^\dagger(2)\rangle_{\phi,\bar\phi}-\langle \hat b(3)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger(2)\rangle_{\phi,\bar\phi}]\\ &=\langle \mathsf{T}\hat b(3)\hat b^\dagger(2)\hat b^\dagger(1)\rangle_{\phi,\bar\phi}-\langle \mathsf{T}\hat b(3)\hat b^\dagger(2)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger(1)\rangle_{\phi,\bar\phi}-\langle \hat{\mathsf{T}} b(3)\hat b^\dagger(1)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger(2)\rangle_{\phi,\bar\phi}\\ &\quad-\langle \hat b(3)\rangle_{\phi,\bar\phi}\langle \mathsf{T}\hat b^\dagger(2)\hat b^\dagger(1)\rangle_{\phi,\bar\phi}+2\langle \hat b(3)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger(2)\rangle_{\phi,\bar\phi}\langle \hat b^\dagger(1)\rangle_{\phi\bar\phi}\end{aligned}$$

As sources are turned off, unless a broken symmetry occurs (BEC), the values that do not conserve the number, like $\langle \hat b_r\rangle$ and $\langle b^\dagger_r\rangle$, vanish.

3. Functional representation

We now give a functional representation of the partition function without sources. Sources will be added in the end.
The eigenvectors of $\hat b^\dagger_r\hat b_r$, $r=1,2,...$, are the number basis $|\mathbf{n}\rangle=|n_1,n_2,...,n_\infty\rangle$ of Fock space. An overcomplete basis are the coherent states $|\boldsymbol\alpha\rangle=|\alpha_1,\alpha_2,...,\alpha_\infty\rangle$, where $\hat b_r|\boldsymbol\alpha\rangle=\alpha_r|\boldsymbol\alpha\rangle$. They are very convenient because in second quantization the operators are normally ordered, then:

$$\langle\boldsymbol\alpha|\hat b^\dagger_r\hat b_s|\boldsymbol\alpha'\rangle=\langle\boldsymbol\alpha|\boldsymbol\alpha'\rangle\,\overline{\alpha_r}\alpha'_s,\quad \langle\boldsymbol\alpha|\hat b^\dagger_r\hat b^\dagger_s\hat b_{r'}\hat b_{s'}\hat |\boldsymbol\alpha'\rangle=\langle\boldsymbol\alpha|\boldsymbol\alpha'\rangle\,\overline{\alpha_r}\,\overline{\alpha_s}\alpha_{s'}\,\alpha_{s'}$$

Let us introduce the notation $\langle\boldsymbol\alpha|\hat K|\boldsymbol\alpha'\rangle=\langle\boldsymbol\alpha|\boldsymbol\alpha'\rangle\,K(\boldsymbol\alpha,\boldsymbol\alpha')$.

The partition function at temperature β is

$$Z=\mathrm{tr}(e^{-\beta K})=\int\prod_{k=1}^\infty\frac{d^2\alpha_k}{\pi}\langle\boldsymbol\alpha|e^{-\beta K}|\boldsymbol\alpha\rangle$$

In view of the next steps we introduce time labels $\tau\in[0,\hbar\beta]$ for each overcomplete resolution of the identity, and rewrite Z:

$$Z=\int\prod_{k=1}^\infty\frac{d^2\alpha_k(0)}{\pi}\langle\boldsymbol\alpha(\hbar\beta)|e^{-\beta K}|\boldsymbol\alpha(0)\rangle, \tag{10}$$

with label $\tau=0$ and with the boundary condition $\boldsymbol\alpha(\hbar\beta)=\boldsymbol\alpha(0)$.
The temperature interval $[0,\hbar\beta]$ is divided into subintervals (τ_i,τ_{i+1}) with $\tau_0=0$ and $\tau_{N+1}=\hbar\beta$, and the exponential is factored:

$$e^{-\beta K}=e^{-\frac{1}{\hbar}(\hbar\beta-\tau_N)K}e^{-\frac{1}{\hbar}(\tau_N-\tau_{N-1})K}...e^{-\frac{1}{\hbar}(\tau_1-0)K}$$

Between the factors, at τ_j, an identity $\int d^2\boldsymbol{\alpha}(\tau_j)|\boldsymbol{\alpha}(\tau_j)\rangle\langle\boldsymbol{\alpha}(\tau_j)|$ is inserted.

$$Z = \int \prod_{k=0}^{N} d^2\boldsymbol{\alpha}(\tau_k)\langle\boldsymbol{\alpha}(\hbar\beta)|e^{-\frac{1}{\hbar}(\hbar\beta-\tau_N)K}|\boldsymbol{\alpha}(\tau_N)\rangle\langle\boldsymbol{\alpha}(\tau_N)|e^{-\frac{1}{\hbar}(\tau_N-\tau_{N-1})K}|\boldsymbol{\alpha}(\tau_{N-1})\rangle$$
$$...\langle\boldsymbol{\alpha}(\tau_1)|e^{-\frac{1}{\hbar}(\tau_1-\tau_0)K}|\boldsymbol{\alpha}(0)\rangle$$

For small intervals, a factor is:

$$\begin{aligned}
&\langle\boldsymbol{\alpha}(\tau_{i+1})|e^{-\frac{1}{\hbar}(\tau_{i+1}-\tau_i)K}|\boldsymbol{\alpha}(\tau_i)\rangle \\
&\approx \langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle\left[1-\frac{1}{\hbar}(\tau_{i+1}-\tau_i)\frac{\langle\boldsymbol{\alpha}(\tau_{i+1})|K|\boldsymbol{\alpha}(\tau_i)\rangle}{\langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle}+...\right] \\
&\approx \langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle\exp\left[-\frac{1}{\hbar}(\tau_{i+1}-\tau_i)K(\boldsymbol{\alpha}(\tau_{i+1})\boldsymbol{\alpha}(\tau_i))\right]
\end{aligned}$$

The exponents of the prefactors $\langle\boldsymbol{\alpha}(\tau_{i+1})|\boldsymbol{\alpha}(\tau_i)\rangle$ are collected:

$$\begin{aligned}
&\sum_{i=0}^{N}\left[-\tfrac{1}{2}\|\boldsymbol{\alpha}(\tau_{i+1})\|^2-\tfrac{1}{2}\|\boldsymbol{\alpha}(\tau_i)\|^2+\overline{\boldsymbol{\alpha}(\tau_{i+1})}\boldsymbol{\alpha}(\tau_i)\right] \\
&=\sum_{i=0}^{N}\left[-\|\boldsymbol{\alpha}(\tau_{i+1})\|^2+\overline{\boldsymbol{\alpha}(\tau_{i+1})}\boldsymbol{\alpha}(\tau_i)\right] \\
&=-(\tau_{i+1}-\tau_i)\sum_{i=0}^{N}\overline{\boldsymbol{\alpha}(\tau_{i+1})}\frac{\boldsymbol{\alpha}(\tau_{i+1})-\boldsymbol{\alpha}(\tau_i)}{\tau_{i+1}-\tau_i}
\end{aligned}$$

The full exponent is

$$-\frac{1}{\hbar}\sum_{i=0}^{N}(\tau_{i+1}-\tau_i)\left[\hbar\overline{\boldsymbol{\alpha}(\tau_{i+1})}\frac{\boldsymbol{\alpha}(\tau_{i+1})-\boldsymbol{\alpha}(\tau_i)}{\tau_{i+1}-\tau_i}+K(\boldsymbol{\alpha}(\tau_{i+1}),\boldsymbol{\alpha}(\tau_i))\right]$$

In the (formal) limit $N\to\infty$ we write the action:

$$S=-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\left[\hbar\,\overline{\boldsymbol{\alpha}(\tau)}\frac{\partial\boldsymbol{\alpha}(\tau)}{\partial\tau}+K(\boldsymbol{\alpha}(\tau),\boldsymbol{\alpha}(\tau))\right]$$

The partition function is a functional integral on an infinite set of functions $\alpha_r(\tau)$

$$Z=\int\mathscr{D}^2\boldsymbol{\alpha}(\tau)e^{-\frac{1}{\hbar}S} \tag{11}$$

If the Bose operators are indexed by position (spin zero), i.e. $r=\mathbf{x}$, the complex vector $\boldsymbol{\alpha}(\tau)$ is replaced by a complex field $\psi(\mathbf{x},\tau)$ with periodic b.c. $\psi(\mathbf{x},\hbar\beta)=\psi(\mathbf{x},0)$.

$$Z=\int\mathscr{D}^2\psi(\mathbf{x},\tau)e^{-\frac{1}{\hbar}S} \tag{12}$$

$$\begin{aligned}
S=\int_0^{\hbar\beta}d\tau\Big[\int d\mathbf{x}\,\overline{\psi(\mathbf{x}\tau)}\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})-\mu\right)\psi(\mathbf{x}\tau)& \\
+\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\overline{\psi(\mathbf{x}\tau)}\psi(\mathbf{x}\tau)\,v(\mathbf{x}-\mathbf{y})\overline{\psi(\mathbf{y}\tau)}\psi(\mathbf{y}\tau)\Big]&
\end{aligned} \tag{13}$$

The ideal gas of bosons. The convenient choice is the basis $r = \mathbf{k}$. The field is Fourier expanded in bosonic Matsubara frequencies $\omega_n = \frac{n\pi}{\hbar\beta}$ (n even) that fulfill the condition $\psi(\mathbf{k}, 0) = \psi(\mathbf{k}, \hbar\beta)$.

$$\psi(\mathbf{k}, \tau) = \frac{1}{\sqrt{\hbar\beta}} \sum_n \psi(\mathbf{k}, i\omega_n) e^{-i\omega_n \tau} \tag{14}$$

The transformation is unitary and the measure does not change:

$$S_0 = \sum_{n,\mathbf{k}} \overline{\psi(\mathbf{k}, i\omega_n)} \left(-i\hbar\omega_n + \epsilon^0_k - \mu\right) \psi(\mathbf{k}, i\omega_n)$$

$$Z_0 = \prod_{\mathbf{k},n} \int \frac{d^2 z}{\pi} \exp\left[(i\omega_n - \frac{\epsilon^0_k - \mu}{\hbar})|z|^2\right] = \prod_{\mathbf{k},n} \frac{1}{i\omega_n - \frac{\epsilon^0_k - \mu}{\hbar}}$$

The factor $1/\pi$ for each mode has been recovered, and $\mu < 0$ for convergence of the Gaussian integral. The partition function $\Omega_0 = -\frac{1}{\beta} \log Z_0$ is[2]:

$$\Omega_0 = \sum_{\mathbf{k}} \frac{1}{\beta} \sum_n \log\left(i\omega_n - \frac{\epsilon^0_k - \mu}{\hbar}\right) e^{i\omega_n \eta} = \frac{1}{\beta} \sum_{\mathbf{k}} \log\left[1 - e^{-\beta(\epsilon^0_k - \mu)}\right].$$

4. Partition function with sources

The partition function with sources is the generator of Green functions. For this purpose we add a source-term in the action:

$$S[\bar\phi, \phi] = S_0 - \hbar \int_0^{\hbar\beta} d\tau \sum_r [\overline{\alpha_r(\tau)} \phi_r(\tau) + \overline{\phi_r(\tau)} \alpha_r(\tau)]$$

The functional derivatives of $Z[\bar\phi, \phi]$ in the sources produce time-ordered correlation functions in presence of the sources. For example:

$$\langle \hat b_r(\tau) \rangle = \frac{1}{Z[\bar\phi, \phi]} \frac{\delta Z[\bar\phi, \phi]}{\delta \overline{\phi_r(\tau)}} = \frac{1}{Z[\bar\phi, \phi]} \int \mathscr{D}^2 \boldsymbol{\alpha}(\tau) e^{-\frac{1}{\hbar} S[\bar\phi, \phi]} \alpha_r(\tau)$$

When the sources are turned off one obtains the thermal average.

The time-ordering is well understood at the defining level of the functional integral. The term $-\hbar \sum_r [\hat b^\dagger_r \phi_r(\tau) + \overline{\phi_r(\tau)} \hat b_r]$ is added to K. After the slicing of the interval $[0, \hbar\beta]$, one adds the term to the weight at exponent in each subinterval

$$-\hbar(\tau_{j+1} - \tau_j) \sum\nolimits_r [\overline{\alpha_r(\tau_{j+1})} \phi_r(\tau_j) + \overline{\phi_r(\tau_j)} \alpha_r(\tau_j)]$$

This explains why functional derivatives in the sources drop down fields into a time-ordered sequence.

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[2]Note: in the action we could include a factor $e^{-i\omega_n \eta}$ to stress the fact that the field in the left is at time $\tau = \hbar\beta = 0^+$. The factor then appears in the denominator of the final expression for Z_0.

5. Non-interacting bosons with sources

A shift of the fields in the measure does not change the measure and the functional integral. For the infinitesimal shift $\delta\overline{\boldsymbol{\alpha}}(\tau)$, the linear change in $Z[\bar\phi,\phi]$ is:

$$0 = -\frac{1}{\hbar}\int \mathscr{D}^2\boldsymbol{\alpha}(\tau)e^{-\frac{1}{\hbar}S}\sum_r \delta\overline{\alpha_r}(\tau)\left[\hbar\frac{\partial\alpha_r(\tau)}{\partial\tau} + \sum_s h_{rs}\alpha_s(\tau) - \mu\alpha_r(\tau) - \hbar\phi_r(\tau)\right]$$

The arbitrariness of the shift gives a Ward identity for $\langle\alpha_s(\tau)\rangle$:

$$\hbar\frac{\partial\langle\alpha_r(\tau)\rangle}{\partial\tau} + \sum_s h_{rs}\langle\alpha_s(\tau)\rangle - \mu\langle\alpha_r(\tau)\rangle = \hbar\phi_r(\tau) \tag{15}$$

The equation is solved with the Green function *without* sources:

$$\sum_s\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau} + h_{rs} - \mu\delta_{rs}\right]\mathscr{G}^0(s,\tau;r',\tau') = -\hbar\delta_{rr'}\delta(\tau-\tau')$$

$$\langle\alpha_r(\tau)\rangle = \frac{1}{Z[\bar\phi,\phi]}\frac{\delta Z[\bar\phi,\phi]}{\delta\overline{\phi_r(\tau)}} = -\int_0^{\hbar\beta}d\tau'\sum_{r'}\mathscr{G}^0(r,\tau;r',\tau')\phi_{r'}(\tau')$$

where we are assuming that the field is zero for $\phi = 0$ (the homogeneous solution). The functional equation has solution

$$\boxed{\log Z[\bar\phi,\phi] = \log Z_0 - \sum_{r,r'}\iint_0^{\hbar\beta}d\tau d\tau'\overline{\phi_r(\tau)}\mathscr{G}^0(r,\tau;r',\tau')\phi_{r'}(\tau')} \tag{16}$$

Wick theorem. The partition function of non interacting particles with sources can be written in two ways:

$$\int \mathscr{D}^2\boldsymbol{\alpha}(\tau)e^{-\frac{1}{\hbar}S+\sum_r\int_0^{\hbar\beta}d\tau(\bar\phi_r\alpha_r+\phi_r\bar\alpha_r)(\tau)} = Z_0 e^{-\sum_{rs}\iint\bar\phi_r(\tau)\mathscr{G}^0(r\tau;s\tau')\phi_s(\tau')}$$

$$\langle\prod_r e^{\int_0^{\hbar\beta}d\tau\bar\phi_r\alpha_r}\prod_s e^{\int_0^{\hbar\beta}d\tau\bar\alpha_r\phi_r}\rangle = \prod_{rs}e^{-\iint\bar\phi_r(\tau)\mathscr{G}^0(r\tau;s\tau')\phi_s(\tau')d\tau d\tau'}$$

The multilinear expansion of the left side in the sources gives coefficients $\langle\prod\alpha_r\prod\bar\alpha_s\rangle_0$. The expansion of the right side in the sources gives products of propagators. Every correlator in the left is either 0 or a product of propagators.
For example, in the expansions of both sides the coefficient $\bar\phi_{r_1}(\tau_1)\bar\phi_{r_2}(\tau_2)\phi_{r_3}(\tau_3)\phi_{r_4}(\tau_4)$ gives the equality:

$$\begin{aligned}\langle \mathsf{T}\hat b_{r_1}(\tau_1)\hat b_{r_2}(\tau_2)\hat b^\dagger_{r_3}(\tau_3)\hat b^\dagger_{r_4}(\tau_4)\rangle &= \mathscr{G}^0(r_1\tau_1;r_3\tau_3)\mathscr{G}^0(r_2\tau_2;r_4\tau_4)\\ &+\mathscr{G}^0(r_1\tau_1;r_4\tau_4)\mathscr{G}^0(r_2\tau_2;r_3\tau_3)\end{aligned}$$

6. Interacting bosons with sources

Ward identities originate in the same way for interacting particles. After a shift in the field $\alpha_r(\tau)$:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau} + \sum_s h_{rs} - \mu\delta_{rs}\right]\langle\alpha_s(\tau)\rangle + \sum_{srr'}v_{rsr's'}\langle\bar\alpha_s(\tau)\alpha_{r'}(\tau)\alpha_{s'}(\tau)\rangle = \hbar\phi_r(\tau)$$

A derivative in the source $\phi_v(\tau')$ gives an equation of motion for the propagator:

$$\left[\delta_{rs}\hbar\frac{\partial}{\partial\tau} - \mu\delta_{rs} + \sum_s h_{rs}\right]\mathscr{G}(s\tau;v\tau') - \sum_{srr'} v_{rsr's'}\langle\bar\alpha_s(\tau)\alpha_{r'}(\tau)\alpha_{s'}(\tau)\bar\alpha_v(\tau')\rangle$$
$$= -\hbar\delta_{rv}\delta(\tau-\tau')$$

where the average can now be taken with no sources.

Perturbation theory. In presence of an interaction among bosons, the perturbative approach in the position basis is as follows. Let $v(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau') \equiv U^0(x.x')$. The action is written as

$$S[\bar\phi,\phi] = S_0[\bar\phi,\phi] + \frac{1}{2}\iint dx\,dx'\;\overline{\psi(x)}\psi(x)\,U^0(x,x')\overline{\psi(x')}\psi(x')$$

$$Z[\bar\phi,\phi] = \int \mathscr{D}^2\psi(x)e^{-\frac{1}{\hbar}S_0[\bar\phi,\phi]}\sum_k \frac{(-1)^k}{k!(2\hbar)^k}\Big[\int dxdy\,\bar\psi(x)\bar\psi(y)U^0(x,y)\psi(y)\psi(x)\Big]^k$$

$$= Z_0\sum_k \frac{(-1)^k}{k!(2\hbar)^k}\left[\int dxdyU^0(x,y)\frac{\delta^4}{\delta\phi(x)\delta\phi(y)\delta\bar\phi(y)\delta\bar\phi(x)}\right]^k e^{-\iint dudv\bar\phi(u)\mathscr{G}^0(u,v)\phi(v)}$$

With sources turned off, this generates all vacuum graphs (connected and unconnected). The perturbative expansion for Green functions is done by applying further functional derivatives, whose arguments are the external points.

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