MATHEMATICAL METHODS FOR PHYSICS

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Preface

It is not completely obvious what a course named "Mathematical Methods for Physics" should include. Of course, an introduction to complex analysis, Fourier integral, series expansions ... the list continues but time is limited, and the rest is inevitably a matter of choice.
In preparing these notes I felt the need to present the selected topics with enough rigour and amplitude to offer methodological examples, interesting applications. I also grew in awareness of the beauty of the topics, true gems of intellectual achievement, and the giants who made them. Here and there I provide some historical notes.
The notes are still a "work in progress", as learning never ends. They cannot replace the vision and depth of books; the good student explores the library (and internet) and makes his own discoveries.

I mention some authors of textbooks that I found particularly useful and inspiring (others are specified in the footnotes). Complex analysis: *L. V. Ahlfors, J. Bak and D. J. Newman, R. P. Boas, P. Henrici, E. Hille, A. I. Markushevich, R. Remmert.* Functional analysis: *Ph. Blanchard and E. Brüning, A. Kolmogorov and S. Fomine, M. Reed and B. Simon, K. Schmüdgen,* A textbook with similarities with these notes is: W. Appel, *Mathematics for physics and physicists*, Princeton (2007).

I thank my colleague Mario Raciti for many useful comments.
The last line is devoted to my students: it is because of their participation and interest that these notes improved in the years, and are now collected in this volume.
Milano, 1 january 2014.

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Part I

COMPLEX ANALYSIS

Chapter 1

COMPLEX NUMBERS

1.1 Cubic equation and imaginary numbers.

Imaginary numbers appeared in algebra during the Reinassance, with the solution of the cubic equation[1 2]. The problem of solving the quadratic equation $x^2 + 1 = 0$ was considered meaningless, while a real cubic equation always has a real solution. However, the available method eventually provided the solution as a sum of terms with imaginary numbers.

The priority for the algebraic solution of the cubic equation is uncertain: it was probably known to Scipione del Ferro, a professor in Bologna, and Nicolò Fontana (Tartaglia). The general solution[3] was published in the book *Ars Magna* (1545) by Gerolamo Cardano (Pavia 1501, Rome 1576). It is based on the algebraic identity

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3$$

which he obtained by geometric construction[4]. After setting $x = t - u$, the identity becomes the reduced cubic equation

$$x^3 + 3px + q = 0 \tag{1.1}$$

with $tu = p$ and $t^3 - u^3 = -q$. Therefore, the solution $x = t - u$ of (1.1) is obtained by solving the quadratic equations for t^3 and u^3, in terms of p and q. Raffaello Bombelli (Bologna 1526, Rome? 1573) in his treatise *Algebra* was the

[1]Before the modern era, the solution was obtained by geometric means; the Persian poet and scientist Omar Khayyam (IX cent.) discussed it as the intersection of a parabola and a hyperbola, by methods that foreran Cartesian geometry.

[2]Refs: Jacques Sesiano, *An Introduction to the History of Algebra*, Mathematical World 27, AMS; Morris Kline, *Mathematical Thought from Ancient to Modern Times*, 3 voll, Oxford University Press 1972; Carl B. Boyer, *A History of Mathematics*, Princeton University Press 1985. A source of historical news and pictures is the *Mathematics Genealogy Project* (`https://www.genealogy.ams.org`).

[3]The use of letters to denote parameters of equations was introduced by Francois Viete few years later; Cardano solved examples of cubic equations, with all possible signs.

[4]Consider a cube with edge length t. If three concurring edges are partitioned in segments of lengths u and $t-u$, the cube is cut into two cubes and four parallelepipeds. The volume is $t^3 = u^3 + (t-u)^3 + 2tu(t-u) + u^2(t-u) + u(t-u)^2$; simple algebra gives the identity (W. Dunham, *Journey through Genius, the Great Theorems of Mathematics*, Wiley Science Ed. 1990).

first to regard imaginary numbers as a necessary detour to produce real solutions from real cubic equations. He studied the equation $x^3 - 15x - 4 = 0$, with real solution $x = 4$. Cardano's method works as follows: from $tu = -5$ and $t^3 - u^3 = 4$ obtain $t^6 - 4t^3 + 125 = 0$ with solutions $t^3 = 2 \pm \sqrt{-121}$. Bombelli showed that $2 \pm \sqrt{-121} = (2 \pm \sqrt{-1})^3$ so that $t_\pm = 2 \pm \sqrt{-1}$. With any choice of sign, a root is $x_1 = t_\pm + 5/t_\pm = 4$. The other two are then found $x_{2,3} = -2 \pm \sqrt{3}$.

Exercise 1.1.1. *Show that any cubic equation can be brought to the form* $z^3 \pm 3z + q = 0$ *by a linear transformation. For* q *real, find the roots through the substitution* $w = s \mp (1/s)$.

1.2 The quartic equation.

The *Ars Magna* also contains the solution of the quartic equation, by Cardano's disciple Ludovico Ferrari (1522, 1565). In modern language, a linear change of the variable puts the equation in the form $x^4 = ax^2 + bx + c$. The great idea is the introduction of an auxiliary parameter y in the equation:

$$(x^2 + y)^2 = (a + 2y)x^2 + bx + (y^2 + c).$$

The parameter is then chosen to make the right hand side (r.h.s.) of the equation the square of a binomial in x, so that square roots of both sides can be taken. The condition is the cubic equation for y, $0 = b^2 - 4(a + 2y)(y^2 + c)$, which may be solved by Cardano's formula. The value of y is entered in the quartic equation, $(x^2 + y)^2 = (a + 2y)[x + b/2(a + 2y)]^2$, and a square root brings it to a couple of quadratic equations in x.

Example 1.2.1. *To solve the quartic equation* $x^4 - 3x^2 - 2x + 5 = 0$, *rewrite it as* $(x^2 + y)^2 = (3 + 2y)x^2 + 2x - 5 + y^2$. *Choose* y *such that r.h.s. is a perfect square in* x, *i.e.* $2y^3 + 3y^2 - 10y - 16 = 0$ *with a solution* $y = -2$. *Then the equation is* $(x^2 - 2)^2 = -(x - 1)^2$, *i.e.* $x^2 - 2 = \pm i(x - 1)$. *The two quadratic equations give the four solutions of the quartic.*

The achievement started a great effort to solve higher order equations. Vandermonde and especially Giuseppe Lagrange (Torino 1736, Paris 1813) emphasized the role of the permutation group and of symmetric functions. In 1770 Lagrange obtained a new method of solution of the quartic equation,

$$z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0.$$

Since it is instructive, we give a brief description of it. The coefficients of the equation are symmetric functions of the roots,

$$a_1 = -\sum_i z_i, \quad a_2 = \sum_{i<j} z_i z_j, \quad a_3 = -\sum_{i<j<k} z_i z_j z_k, \quad a_4 = z_1 z_2 z_3 z_4.$$

Any other symmetric function of the roots is expressible in terms of them. The combination $s_1 = z_1 z_2 + z_3 z_4$ is not symmetric under all 4! permutations. By doing all permutations only two new combinations appear: $s_2 = z_1 z_3 + z_2 z_4$ and $s_3 = z_1 z_4 + z_2 z_3$.
The three quantities $A_1 = s_1 + s_2 + s_3$, $A_2 = s_1 s_2 + s_1 s_3 + s_2 s_3$ and $A_3 = s_1 s_2 s_3$ are symmetric in s_1, s_2 and s_3, and are invariant under permutations of the roots

z_i. As such, they may be expressed in terms of the coefficients a_i: $A_1 = a_2$, $A_2 = a_1a_3 - 4a_4$, and $A_3 = a_3^2 + a_1^2a_4 - 4a_2a_4$. The s_i are the roots of the cubic equation $s^3 - A_1s^2 + A_2s - A_3 = 0$, and are evaluated by Cardano's method. Next, the roots z_i are found.

1.3 Beyond the quartic.

For the fifth-order equation Lagrange eagerly tried to guess a polynomial combination of the roots that, under the 5! permutations, could produce at most 4 combinations s_1, ..., s_4 that would solve a quartic equation. His disciple Ruffini (Modena 1765, 1822) showed that such a polynomial should be invariant under 5!/4 permutations of the roots z_i, and does not exist.

The Norwegian mathematician Niels Henrik Abel (1802, 1829) put the last word in the memoir *On the algebraic resolution of equations* published in 1824. He proved that no rational solution involving radicals and algebraic expressions of the coefficients exists for general equations of order higher than four. The identification of the special equations that can be solved by radicals was done by Evariste Galois, by methods of group theory to which he much contributed. An interesting result is: an equation is solvable by radicals if and only if, given two roots, the others depend rationally on them (1830)[5].

In 1786 E. S. Bring, by exploiting an earlier method by Tschirnhausen (1683), showed that any equation of fifth degree can be brought to the amazingly simple form $z^5 + q_4z + q_5 = 0$, and then to $z^5 + 5z + a = 0$ if $q_4 \neq 0$ (in general, an equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ can be reduced to $y^n + q_4y^{n-4} + \cdots + q_n = 0$ by means of the variable change $y = p_0 + p_1x + \cdots + p_4x^4$ and solving equations of degree 2 and 3).

Charles Hermite succeeded in obtaining a solution of the quintic equation, in terms of elliptic functions (1858)[6]. Soon after Leopold Kronecker and Francesco Brioschi[7] gave alternative derivations[8].

In 1888 the solution of the general sixth order equation was obtained by Brioschi and Maschke, in terms of hyperelliptic functions.

Of course no one would solve even a quartic by the methods described, as efficient numerical methods yield the roots with the desired accuracy.

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[5]for a presentation of Galois theory, see V. V. Prasolov, *Polynomials*, Springer (2004)

[6]for example, the equation $y^3-3y+1 = 0$ can be solved with the aid of trigonometric tables: put $y = 2\cos x$ and obtain $0 = 2\cos(3x) + 1$; then $3x = \frac{2}{3}\pi$ and $3x = \frac{4}{3}\pi$ i.e. $y_1 = 2\cos(\frac{2}{9}\pi)$ and $y_2 = 2\cos(\frac{4}{9}\pi)$; the other solution is $y_3 = -y_1 - y_2$.

[7]F. Brioschi (1824, 1897) taught mechanics in Pavia. He then founded Milan's Politecnico (1863), where he taught hydraulics. He participated in Milan's insurrection, and became member of the Parliament. Among his students (in Pavia): Giuseppe Colombo (he inaugurated in 1883 in Milan the first thermo-electric generator in continental Europe, by lighting the lamps of the Scala theatre. The cables were manufactured by the newly born Pirelli. Colombo succeeded to Brioschi in the direction of the Politecnico), Eugenio Beltrami (non Euclidean geometry, singular values of a matrix, Laplace Beltrami operator in curved space) and Luigi Cremona (painter).

[8]A discussion of the quintic eq. is in J.V.Armitage and W.F.Eberlein, *Elliptic functions*, Lon. Math. Soc. Student Text 67 (2006). See also `http://wapedia.mobi/en/Bring_radical`, or V. Barsan, *Physical applications of a new method of solving the quintic equation*, arXiv:0910.2957v2; G. Zappa, *Storia della risoluzione delle equazioni di V e VI grado ...*, Rend. Sem. Mat. Fis. Milano (1995).

1.4 The complex field

Complex numbers were used in the early XVIII century by Leibnitz. Jean Bernoulli, Abraham De Moivre and by the genius Leonhard Euler (1707, 1783) who discovered several relations involving trigonometric, exponential and logarithmic functions with imaginary argument.
The great improvement in the perception of complex numbers as well defined entities was their visualisation as vectors or points, by the Norwegian cartographer Caspar Wessel (1797) and the mathematician Jean Robert Argand (1806). It were Gauss' authority and investigations since 1799, that gave complex numbers a status in analysis.

In 1833 the Irish mathematician William R. Hamilton (1805, 1865) presented before the Irish Academy an axiomatic setting of the complex field $\mathbb{C}$ as a formal algebra on pairs of real numbers. In 1867 Hankel proved that the algebra of complex numbers is the most general one that fulfils all fundamental laws of arithmetic (Boyer).

Figure 1.1: **Leonhard Euler (1707, 1783)** belongs to an impressive genealogy of mathematicians, rooted in Leibnitz and the Bernoullis. Euler spent many years in St. Petersburgh, at the dawn of the Russian mathematical school. He discovered several important formulae for complex functions, and established much of the modern notation. His student Joseph Lagrange was the advisor of Fourier and Poisson. Poisson's students Dirichlet and Liouville mentored illustrious mathematicians that contributed to the advancement of complex analysis in Paris (on the side of Dirichlet: Darboux, and then Borel, Cartan, Goursat, Picard, and then Hadamard, Julia, Painlevé ...; on the side of Liouville: Catalan and then Hermite, Poincaré, Padé, Stieltjes ...).

Figure 1.2: **Carl Friedrich Gauss (1777, 1855)** became a celebrity after computing the orbit of the first asteroid Ceres, discovered and lost of sight by padre Piazzi in Palermo (1801). To interpolate the best orbit from observed points, Gauss devised the Least Squares method. The orbital elements placed Ceres in the region were astronomers were searching for the fifth planet, that fitted in Titius and Bode's law. Gauss proved the "fundamental theorem of algebra": a polynomial of degree n has n zeros in the complex plane. He anticipated several results of complex analysis, which he did not publish. His genealogy contains venerable scientists as the astronomers Bessel and Encke, and mathematicians: Dedekind, Sophie Germain, Gudermann, and Georg Riemann. Among Gauss' "nephews" are Ernst Kummer and Karl Weierstrass.

Chapter 2

THE FIELD OF COMPLEX NUMBERS

2.1 The field $\mathbb{C}$

Definition 2.1.1. The field of complex numbers $\mathbb{C}$ is the set of pairs of real numbers $z = (x, y)$ with the binary operations of sum and multiplication:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{2.1}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \tag{2.2}$$

The sum is commutative, associative, with neutral element (called zero) $(0, 0)$ and opposite $(-x, -y)$ of (x, y). The multiplication is commutative, associative, with unity $(1, 0)$, inverse element of any nonzero number, and with the distributive property $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$.

Exercise 2.1.2. *Evaluate the inverse of a non-zero complex number* (x, y).

The subset of elements $(x, 0)$ is closed under both operations and identifies with the field of real numbers $\mathbb{R}$. Thus the field $\mathbb{C}$ is an extension of the field $\mathbb{R}$. If an element $(x, 0)$ is identified with x, a pair can be written as

$$(x, y) = (1, 0)(x, 0) + (0, 1)(y, 0) = 1x + iy = x + iy,$$

where i is Euler's symbol for the imaginary unit $(0, 1)$. This is the usual representation of complex numbers. Additions and multiplications are done by regarding i as a unit with the property $i^2 = (-1, 0) = -1$.

Proposition 2.1.3. $\mathbb{C}$ *is* not *an ordered field.*

Proof. Suppose that for any pair of different complex numbers it is either $z < w$ or $z > w$; we show that the rules $a < b \to a + c < b + c$ and $a < b,\ c > 0 \to ac < bc$ are violated. Fix $i > 0$ then $i^2 > 0$ i.e. $-1 > 0$. Add 1 to get $0 > 1$, multiply by $i^2 > 0$ and obtain $0 > -1$, which contradicts $-1 > 0$. □

2.2 Complex conjugation

The complex conjugate (c.c.) of $z = x + iy$ is $\overline{z} = x - iy$.
Properties: $\overline{\overline{z}} = z$ (c.c. is an involution), $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ and $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.
The real numbers x and y are respectively the *real* and *imaginary* parts of z:

$$x = \operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad y = \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

2.3 Modulus

The *modulus* of a complex number is $|z| = \sqrt{x^2 + y^2}$. It is $|z|^2 = z\overline{z}$ and $|\overline{z}| = |z|$, Re $z \le |z|$ and Im $z \le |z|$. The following properties qualify the modulus as a *norm* and $\mathbb{C}$ as a normed space:

$$|z| \ge 0, \quad |z| = 0 \text{ iff } z = 0, \tag{2.3}$$

$$|z_1 z_2| = |z_1||z_2|, \tag{2.4}$$

$$|z_1 + z_2| \le |z_1| + |z_2| \quad \text{(triangle inequality)} \tag{2.5}$$

Proof. The first property is obvious. The second one: use $|z_1 z_2|^2 = z_1 z_2 \overline{z}_1 \overline{z}_2$. Triangle inequality: $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + \overline{z}_1 z_2 + z_1 \overline{z}_2$; the last two terms are $2\operatorname{Re}(\overline{z}_1 z_2) \le 2|\overline{z}_1 z_2| = 2|z_1||z_2|$. Then $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$. □

If $1/z$ is the inverse of z it is $|z(1/z)| = 1$ i.e. $|1/z| = 1/|z|$.
The modulus simplifies the evaluation of the inverse of a complex number:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \quad i.e. \quad \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Remark 2.3.1. *Property (2.4) has two interesting consequences:*
1) The unit circle $|z| = 1$ *is closed for complex multiplication.*
2) For any four integers a, b, c, d *there are integers* $p = ad + bc$ *and* $q = |ac - bd|$ *such that* $(a^2+b^2)(c^2+d^2) = q^2+p^2$. *For example:* $(1^2+5^2)(1^2+7^2) = 12^2+34^2$.

Exercise 2.3.2. *Prove that* $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ *(in a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the sides).*

Exercise 2.3.3. *Prove the very useful inequalities:*

$$\tfrac{1}{\sqrt{2}}(|x| + |y|) \le |x + iy| \le |x| + |y| \tag{2.6}$$

$$\Big| |z| - |w| \Big| \le |z - w| \tag{2.7}$$

Hint: $x^2+y^2 \ge 2|xy|$, *then add* x^2+x^2; $|z-w|^2 = (|z|-|w|)^2+2|z||w|-2\operatorname{Re}(\overline{z}w)$.

2.4 Argument

A nonzero complex number $z = x + iy$ may be represented as $z = |z|(\cos\theta + i\sin\theta)$ (*polar representation*), where $\cos\theta = x/|z|$ and $\sin\theta = y/|z|$. The number

$\cos\theta + i\sin\theta$ belongs to the unit circle, where multiplication acts additively on phases:

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

De Moivre's formula follows: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

At this stage, Euler's representation $\cos\theta + i\sin\theta = e^{i\theta}$ is introduced as a definition. It is consistent with the rule $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$. Then we write:

$$\boxed{z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}} \tag{2.8}$$

The phase θ is the *argument* of z ($\theta = \arg\ z$) and is defined up to integer multiples of 2π. Note the property $\arg\,(z_1 z_2) = \arg z_1 + \arg z_2$ (mod. 2π).
The **principal argument** $\mathrm{Arg}\, z$ is the single determination such that

$$-\pi < \ \mathrm{Arg}\, z \le \pi.$$

With the choice $-\frac{\pi}{2} < \mathrm{Arctan}\, s \le \frac{\pi}{2}$, the geometric construction in fig.2.1 shows that

$$\mathrm{Arg}\, z = 2\,\mathrm{Arctan}\frac{y}{|z|+x} \tag{2.9}$$

The sign of $\mathrm{Arg}\, z$ is the same as $y = \mathrm{Im}\, z$. The real negative semi-axis is a line of discontinuity (branch cut). By definition its points have $\mathrm{Arg}\ z = \pi$.
It is not in general true that $\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b)$.

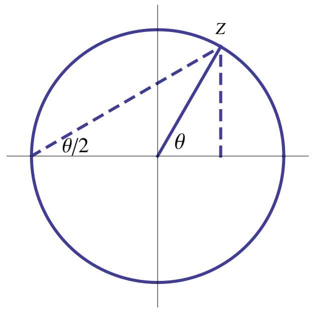

Figure 2.1: Given $z = x + iy$, the angle $\theta = \mathrm{Arg} z$ is twice the angle at the circumference. The latter is always in the interval $(-\frac{\pi}{2}, \frac{\pi}{2}]$, and $y = (|z| + x)\tan(\theta/2)$ with the correct sign.

Exercise 2.4.1. *Show that:*

$$\mathrm{Arg}(ab) = \mathrm{Arg}\, a + \mathrm{Arg}\, b + \begin{cases} -2\pi & \text{if } \pi < \mathrm{Arg}\, a + \mathrm{Arg}\, b \\ 0 & \text{if } -\pi < \mathrm{Arg}\, a + \mathrm{Arg}\, b \le \pi \\ 2\pi & \text{if } \mathrm{Arg}\, a + \mathrm{Arg}\, b \le -\pi \end{cases} \tag{2.10}$$

Other single determinations of the argument are possible, always having a line of discontinuity from the origin to infinity.
For example, if the line is chosen as the imaginary half line $z = ix$ ($x \ge 0$), the range of values of this arg is $(-\frac{3}{2}\pi, \frac{1}{2}\pi]$. The points on the cut, by definition, have argument $\frac{1}{2}\pi$; other values are: $\arg(-1+i) = -\frac{5\pi}{4}$, $\arg\,(-1) = -\pi$, $\arg 1 = 0$.

Exercise 2.4.2. *Write the numbers* $1 \pm i$ *in polar form.*

Exercise 2.4.3. *Show that* $e^{ia} + e^{ib} = 2\cos\frac{a-b}{2}\, e^{\frac{i}{2}(a+b)}, \quad a, b \in \mathbb{R}$

Exercise 2.4.4. *Show that the numbers*

$$z = \frac{1+is}{1-is}, \qquad s \in \mathbb{R}$$

belong to the unit circle. Is the map $s \to Arg\, z$ *invertible? If* H *is a Hermitian matrix, show that* $U = (1+iH)(1-iH)^{-1}$ *is a unitary matrix.*

Now an unexpected nice result:

Theorem 2.4.5 (Three Gap Theorem)**.** *If* $\frac{\theta}{2\pi}$ *is irrational, then* N *points* $e^{in\theta}$*,* $n = 0, 1, \dots, N-1$*, have at most 3 different gaps on the circle.*
(Vera T. Sós, 1957; see arXiv:2208.01680).

Note that for $N \to \infty$ the points $e^{in\theta}$ are uniformly distributed in the unit circle (Equidistribution theorem, H. Weyl).

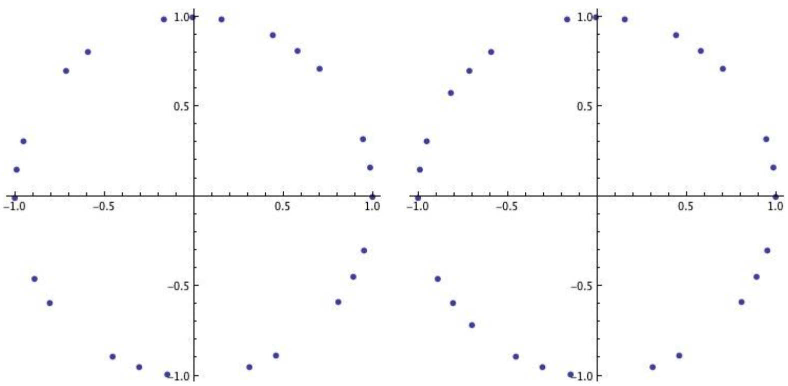

Figure 2.2: Left: the $N = 24$ points $\exp(in\sqrt{2})$, $n = 0, ..., 23$ are separated by three gaps. For $N = 26$ (right), the new points $\exp(i24\sqrt{2})$, $\exp(i25\sqrt{2})$ modify the gaps with neighbors, but still there are three gap-sizes (now the same as $N = 24$).

2.5 Exponential

The *exponential* of a complex number $z = x + iy$ is defined by the product

$$\boxed{e^z = e^x e^{iy} = e^x(\cos y + i \sin y)} \tag{2.11}$$

It exists for all z, with the fundamental property

$$\boxed{e^z e^\zeta = e^{z+\zeta}} \tag{2.12}$$

The exponential function is periodic, with period $2\pi i$: $e^{z+2\pi i} = e^z$ for all z. The trigonometric functions

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz})$$

are periodic with period 2π, and $\cos^2 z + \sin^2 z = 1$. Note that they are unbounded on the imaginary axis. One also defines $\tan z = \sin z/\cos z$ and $\cot z = 1/\tan z$. The hyperbolic functions

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \quad \sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

are periodic with period $2\pi i$, and $\cosh^2 z - \sinh^2 z = 1$. They are unbounded on the real axis. One defines $\tanh z = \sinh z/\cosh z$ and $\coth z = 1/\tanh z$.

Exercise 2.5.1.
1) Show that $|e^z| = e^x$, $\overline{e^z} = e^{\overline{z}}$.
2) Find the zeros of $\sinh(az + b)$, $a, b \in \mathbb{R}$.
3) Show that $|\cosh(x + iy)| \le \cosh x$.

Exercise 2.5.2 (Chebyshev polynomials)**.**
Show that $\cos(n\theta)$ *is a polynomial of degree* n *in* $t = \cos\theta$. *It is the Chebyshev polynomial of the first kind* $T_n(t)$. *Evaluate the first few polynomials.*
Show that they have the following properties of recursion and orthogonality:

$$T_{n+1}(t) - 2t\,T_n(t) + T_{n-1}(t) = 0 \tag{2.13}$$

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} T_m(t) T_n(t) = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \end{cases} \tag{2.14}$$

Prove similar properties for the Chebyshev polynomials of the second kind:

$$U_n(t) = \frac{\sin[(n+1)\theta]}{\sin\theta} \qquad (t = \cos\theta).$$

Hint: $2\cos(n\theta) = (\cos\theta + i\sin\theta)^n + c.c.$

2.6 Logarithm

The *logarithm* of a complex number z is the exponent that solves the equality $e^{\log z} = z$. Note that $\log z$ is not defined in $z = 0$. Because of the periodicity of the exponential function, there are an infinite number of solutions: if $\log z$ is a solution, also $\log z + i2k\pi$ is a solution for any integer k. All such solutions are denoted as $\log z$. The product rule of exponentials implies $\log(z_1 z_2) = \log z_1 + \log z_2$. In particular, with $z = |z|e^{i\theta}$ one obtains

$$\boxed{\log z = \log|z| + i\arg z + i\,2k\pi, \quad k \in \mathbb{Z}} \tag{2.15}$$

While the real part of $\log z$ is well determined as the log of a positive real number, the imaginary part reflects the same indeterminacy of the argument of a complex number.
It is natural to define the **principal logarithm** of a number as

$$\operatorname{Log} z = \log|z| + i\operatorname{Arg} z \tag{2.16}$$

The Log has a *cut of discontinuity* on the real negative axis: for vanishing $\epsilon > 0$ and $x < 0$: Log $(x + i\epsilon) = \log|x| + i\pi$ (the formula holds also for $\epsilon = 0$, Log $(-1) = i\pi$) and Log $(x - i\epsilon) = \log|x| - i\pi$.

Remark 2.6.1. *By ex.2.4.1, it is* $\mathrm{Log}(ab) = \mathrm{Log}\, a + \mathrm{Log}\, b$ *when* a *and* $\overline{b}$ *are in sight (i.e. the segment* $[a, \overline{b}]$ *does not cross the negative real axis). It is* $\mathrm{Log}(a/b) = \mathrm{Log}\, a - \mathrm{Log}\, b$ *when* a *and* b *are in sight.*

In full analogy with the argument, different single-valued determinations of the log are possible and always have a line of discontinuity (*branch cut*) from 0 (*branch point*) to ∞. For example, $\log(-i)$ is $-i\frac{\pi}{2}$ if the Log is used; it is $i\frac{3\pi}{2}$ if the log is chosen with cut on the real positive half-line. It is $-i\frac{\pi}{2}$ if the cut is on the positive imaginary axis.

Exercise 2.6.2. *Evaluate* $\mathrm{Log}\frac{1+i}{1-i}$, $\mathrm{Log}(ie^{i\theta})$.

Exercise 2.6.3. *What are* $\mathrm{Log}(-z)$, $\mathrm{Log}\,\overline{z}$, $\mathrm{Log}(1/z)$, *and* $\mathrm{Log}(1/\overline{z})$ *in terms of* $\mathrm{Log}\, z$*?*

Exercise 2.6.4. *Prove that, for* $0 < \theta < \frac{\pi}{2}$*:* $\mathrm{Log}\,(1 - e^{i\theta}) = \log[2\sin(\frac{1}{2}\theta)] + i\frac{1}{2}(\theta - \pi)$.

2.7 Power of a complex number

The power of a complex number is *defined* as

$$\boxed{z^{a+ib} = e^{(a+ib)\log z}} \tag{2.17}$$

In general it is multi-valued (such is the log).
If the exponent is real: $z^a = |z|^a \exp{(i\, a\, \mathrm{arg} z + i\, 2\pi a k)}, \quad k = 0, \pm 1, \pm 2, \dots$

- $a \in \mathbb{Z}$: the power $z^{\pm n}$ is single-valued.
- $a \in \mathbb{Q}$ ($a = \pm p/q$, with p and q coprime): the sequence of powers is periodic, and only q are distinct.
  Example: $(1 + i)^{2/3} = (\sqrt{2}e^{i\pi/4})^{2/3} = 2^{1/3}\exp(i\frac{\pi}{6} + i\frac{4}{3}k\pi)$, the values $k = 0, 1, 2$ give three distinct powers.
- a is irrational: the set of powers is infinite.
  Example: $(1+i)^\pi = (\sqrt{2}e^{i\pi/4})^\pi = 2^{\pi/2}\exp(i\pi^2/4+i2\pi^2 k)$, $k = 0, \pm 1, \pm 2, \dots$.

If the exponent is complex: $z^{a+ib} = |z|^a e^{-b(\mathrm{arg} z+2\pi k)} e^{i[b\log|z|+a(\mathrm{arg} z+2\pi k)]}$.
Examples: $1^i = e^{i\log 1} = \{e^{i(0+2\pi i k)}\}_{k\in\mathbb{Z}} = \{1, e^{\pm 2\pi}, e^{\pm 4\pi}, \dots\}$;
$(1+i)^{1-i} = \{e^{(1-i)(\log\sqrt{2}+i\frac{\pi}{4}+2\pi i k)}\}_{k\in\mathbb{Z}} = \{\sqrt{2}e^{\pi(\frac{1}{4}+2k)}e^{i(\frac{\pi}{4}+2\pi k-\log\sqrt{2})}\}_{k\in\mathbb{Z}}$

Exercise 2.7.1. *Evaluate:* $8^{1/3}$, $(-1)^{1/5}$, $i^{1/4}$, $(1-i)^{1/6}$.

Exercise 2.7.2. *Show that the square roots of* $z = a + ib$ *are:*

$$\pm\left[\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\frac{b}{2}\sqrt{\frac{2}{\sqrt{a^2+b^2}+a}}\right]$$

Exercise 2.7.3. *Show the properties:* $|z^a| = |z|^a$, $z^a z^b = z^{a+b}$, $(z\zeta)^a = z^a\zeta^a$ *(for multivalued powers, the sets in the two sides must coincide).*

2.8 The fundamental theorem of algebra.

The theorem states that in $\mathbb{C}$ *any polynomial of degree n with complex coefficients has precisely n zeros*[1]. Then, a polynomial $p(z) = a_0 z^n + \cdots + a_n$ has the unique factorization

$$p(z) = a_0(z - z_1)(z - z_2)\cdots(z - z_n), \tag{2.18}$$

where $z_1, \dots, z_n$ are the zeros (roots). A proof was found by Carl Friedrich Gauss, in his doctorate dissertation *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse* (1799). He showed that for any polynomial $p(x + iy) = u(x, y) + iv(x, y)$ (with real coefficients) the curves $u(x, y) = 0$ and $v(x, y) = 0$ necessarily intersect in the plane. Before him, Girard (1629), d'Alembert (1748) and Euler (1749), proved the weaker statement that any polynomial with real coefficients factors into real linear and quadratic terms.

A simple proof will be given with Liouville's theorem for entire functions.

2.9 The cyclotomic equation

The roots of the equation $z^n = 1$ are the corners of a regular n-polygon inscribed in the unit circle:

$$\zeta_k = \cos(\tfrac{2k\pi}{n}) + i\sin(\tfrac{2k\pi}{n}), \quad k = 0, \dots, n-1.$$

The equation $z^n - 1 = 0$ implies the sums $\zeta_0^p + \cdots + \zeta_{n-1}^p = 0$, $p = 1, \dots, n-1$.

Since $z^n - 1 = (z - 1)(z^{n-1} + \cdots + z + 1)$, the roots $\zeta_1, \dots, \zeta_{n-1}$ solve the *cyclotomic equation* $z^{n-1} + \cdots + z + 1 = 0$.

In 1796 Gauss, then a young student, two millennia after Euclid, announced the possibility to construct by ruler and compass the regular polygon with $n = 17$ sides. Five years later he gave a sufficient condition[2] for a polygon to be constructible by ruler and compass: $n = 2^k\, p_1^{m_1}\, p_2^{m_2} \cdots$, where the factor 2^k accounts for repeated duplications of the number of sides of a more basic polygon. The factors p are either 1 or a Fermat number (i.e. $p = 2^{2^q} + 1$) that is also a prime number[3]. The polygons $n = 3, 4, 5$ (i.e. $p = 1, 3, 5$) and duplications ($n = 6, 8, 10, 12, \dots$) were known since Euclid's time. The next Fermat prime number is $p = 2^{2^2} + 1 = 17$. Gauss was so proud of his discovery, that he asked for a 17-polygon to be carved on his gravestone (but the stonemason declined to do it)[4].

[1] This intuitive explanation is from T. Gowers, *The Princeton companion to Mathematics*, Princeton Univ. Press (2009). Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, with $a_n \neq 0$. For very large R the set $p(Re^{i\theta}) \approx R^n e^{in\theta}$, $\theta \in [0, 2\pi)$, approximates a circle of radius R^n run n times, that contains the origin. For very small R, the set $p(Re^{i\theta}) \approx a_{n-1} Re^{i\theta} + a_n$ is a circle does not contain the origin. By continuity, there must be a value R such that the set contains the origin i.e. an angle θ such that $p(Re^{i\theta}) = 0$.

[2] Wantzel (1836) showed that it is also necessary.

[3] Euler showed that the Fermat number $2^{32}+1$ ($q = 5$) is not a prime number. L. Anderson, J. S. Chahal, Jaap Top, *The last chapter of the Disquisitiones of Gauss*, `https://doi.org/10.48550/arXiv.2110.01355`.

[4] The tale of the $(2^8 + 1)-$gon is narrated in the nice book *Dr. Euler's fabulous formula* by Paul Nahin, Princeton Univ. Press (2006).

Example 2.9.1. *Solve the equation* $z^5 = 1$ *and obtain* $\cos\frac{2\pi}{5} = \frac{1}{4}(\sqrt{5}-1)$.
A: the sum of the zeros $\zeta_k = e^{i2\pi k/5}$, $k = 0, .., 4$ *is zero (the term* z^4 *is missing in the equation). Then the real part of the sum is zero:* $1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$. *Next use* $2\cos^2\frac{2\pi}{5} = 1 + \cos\frac{4\pi}{5}$ *and the result is found.*

Exercise 2.9.2. *Let* $\{\zeta_k\}_{k=0}^{n-1}$ *be the roots of* $z^n = 1$. *Show that:*

$$a^n - b^n = (a-b)(a-\zeta_1 b)\cdots(a-\zeta_{n-1}b) \tag{2.19}$$

Exercise 2.9.3. *Consider the polygon with corners at the* n *roots of unity* $1, \zeta, \dots, \zeta^{n-1}$ *and draw the diagonals connecting 1 to the other corners. Show that the product of their lengths is precisely* n: $|1-\zeta|\cdots|1-\zeta^{n-1}| = n$[5].

Exercise 2.9.4 (Discrete Fourier transform[6])**.** *Show that the following* $N \times N$ *matrix*

$$F_{rs} = \frac{1}{\sqrt{N}}\exp(i\frac{2\pi}{N}rs) \qquad r, s = 1...N$$

is unitary. Evaluate F^2 *and show that* $F^4 = 1$. *What are the eigenvalues of* F*? (Hint: you need the sum of powers of the* N *roots of unity).*

Exercise 2.9.5 (Madhava Math. competition, 2013)**.** *Let* $\zeta_0, \dots, \zeta_{N-1}$ *be the* N *roots of unity. Show that the sum of squared distances* $\sum_{k=0}^{N-1}|z-\zeta_k|^2$ *is the same for all* z *on the unit circle.*

Example 2.9.6. *Evaluate the characteristic polynomial of the* $n \times n$ *matrix*

$$H_n = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}.$$

A.: If $D_k(z) = \det[zI_k - H_k]$ *then* $D_{k+1}(z) = zD_k(z) - D_{k-1}(z)$ *with the initial conditions* $D_1(z) = z$ *and* $D_0(z) = 1$. *The Laplace expansion can be written as a first order one, with a "transfer matrix"* T:

$$\begin{bmatrix} D_{k+1}(z) \\ D_k(z) \end{bmatrix} = T \begin{bmatrix} D_k(z) \\ D_{k-1}(z) \end{bmatrix}, \quad T = \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}$$

Iteration gives the solution for any matrix size

$$\begin{bmatrix} D_{k+1}(z) \\ D_k(z) \end{bmatrix} = T^k \begin{bmatrix} z \\ 1 \end{bmatrix} = T^{k+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then $D_k(z) = [T^{k+1}]_{21}$. *The matrix* T^{k+1} *is evaluated with the Cayley-Hamilton theorem, that states that a square matrix solves its characteristic polynomial. In this case the polynomial is* $\det(\lambda I_2 - T) = \lambda^2 - \lambda z + 1$, *and* $T^2 - Tz + I_2 = 0$. *This implies* $T^{k+1} = a_{k+1}T + b_{k+1}I_2$, *with numbers* a_{k+1}, b_{k+1} *to be found. Since the eigenvalues of* T *are* $\lambda_\pm = \frac{1}{2}[z \pm \sqrt{z^2-4}]$, *it is* $\lambda_\pm^{k+1} = a_{k+1}\lambda_\pm + b_{k+1}$. *The result for* $k = n$ *is*

$$\det(zI_n - H_n) = [T^{n+1}]_{21} = a_{n+1} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}$$

[5] An amusing generalization to the ellipse is discussed in `https://doi.org/10.48550/arXiv.1810.00492`

[6] M.L.Mehta, Eigenvalues and eigenvectors of the finite Fourier transform, J. Math. Phys. 28 (1987) 781.

In particular, if $z = 2\cos\theta$ one finds $\lambda_\pm = e^{\pm i\theta}$:

$$\det(2\cos\theta I_n - H_n) = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta)$$

where U_n is a Chebyshev polynomial of the second kind.
The eigenvalues of H_n are easily found: $\epsilon_k = 2\cos(\frac{\pi}{n+1}k)$, $k = 1, ..., n$.

Exercise 2.9.7. *Consider the following $n \times n$ matrix:*

$$S = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} \tag{2.20}$$

Its action on a vector $u \in \mathbb{C}^n$ is a cyclic shift of the components: $(Su)_k = u_{k+1}$ and $(Su)_n = u_1$.
1) Show that $S^{n-1} = S^T$ (T means transposition).
2) Find the eigenvalues and the eigenvectors of S.
3) Find the spectrum of the periodic "Laplacian matrix" $\Delta = S + S^T - 2I_n$

$$\Delta = \begin{bmatrix} -2 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{bmatrix}$$

4) Find the characteristic polynomial $\det(zI_n - \Delta)$.
5) Write the circulant matrix $a_0 + a_1 S + \cdots + a_{n-1}S^{n-1}$, $a_i \in \mathbb{C}$, and evaluate its eigenvalues and eigenvectors.

Exercise 2.9.8. *A matrix $A = a_0 I_n + a_1 S_n + \dots a_{n-1}S_n^{n-1}$, where S_n is the shift matrix (2.20) is named "circulant". Find the eigenvalues.*
In this example $n = 4$:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}$$

the eigenvalues are: $\lambda_0 = a_0 + a_1 + a_2 + a_3$, $\lambda_1 = a_0 + ia_1 - a_2 - ia_3$, $\lambda_2 = a_0 - a_1 + a_2 - a_3$, $\lambda_3 = a_0 - ia_1 - a_2 + ia_3$.
Hint: the matrix A commutes with S, whose eigenvectors and eigenvalues are known.

Chapter 3

THE COMPLEX PLANE

The modulus of complex numbers defines an Euclidean distance between points $z = (x, y)$ and $z' = (x', y')$ in the complex plane:

$$\boxed{d(z, z') = |z - z'| = \sqrt{(x - x')^2 + (y - y')^2}} \tag{3.1}$$

Results of Cartesian geometry of $\mathbb{R}^2$ can be transposed to $\mathbb{C}$. Disks, circles and lines are important in complex analysis; let's review them in complex notation.

3.1 Straight lines and circles

Given two points a and b in $\mathbb{C}$, the oriented straight line ab has parametric equation $z(t) = (1 - t)a + tb$, $t \in \mathbb{R}$. The restriction $0 \leq t \leq 1$ traces the closed oriented segment $[a, b]$ from a to b.

Exercise 3.1.1.
1) Find the corners of the squares and of the equilateral triangles having one side on the segment $[a, b]$.
2) Describe the sets: $\operatorname{Arg}(z - i) = \frac{\pi}{3}$, $|\operatorname{Arg}(z - i)| < \frac{\pi}{3}$.
3) Give the conditions for a point z *to be inside the triangle with vertices* a, b, c.
Answer: $z = \alpha a + \beta b + \gamma c$, *where* $\alpha + \beta + \gamma = 1$ *and* $0 \leq \alpha, \beta, \gamma \leq 1$.

A circle $C(a, r)$ with center a and radius r has equation $|z - a| = r$.
The parametrization

$$z(\theta) = a + re^{i\theta}, \qquad 0 \leq \theta < 2\pi \tag{3.2}$$

endows the circle with the standard anticlockwise orientation.

Exercise 3.1.2.
1) Show that the locus $|z - a|^2 + |z - b|^2 = |a - b|^2$ *is a circle with diameter* $[a, b]$. *Find the parametric equation of the circle.*
2) Show that the locus $|z - a| = \lambda |z - b|$, $\lambda > 0$ *is a circle (Apollonius of Perga). The circle has radius* $r = \frac{\lambda}{|1-\lambda^2|}|a - b|$ *and center* $c = \frac{a-\lambda^2 b}{1-\lambda^2}$.
The value $\lambda = 1$ *corresponds to the limit case of a line (the axis of the segment* $[a, b]$*).*

3) Show that the circle through z_1, z_2, z_3 has equation

$$\det \begin{bmatrix} |z|^2 & z & \bar{z} & 1 \\ |z_1|^2 & z_1 & \bar{z}_1 & 1 \\ |z_2|^2 & z_2 & \bar{z}_2 & 1 \\ |z_3|^2 & z_3 & \bar{z}_3 & 1 \end{bmatrix} = 0$$

Exercise 3.1.3.
1) Find the equation of the ellipse with foci z_1 and z_2, major semiaxis length a.
2) Study the family of Cassini ovals[1] $|z^2 - 1| = r^2$ *as r changes. Show that it is a single closed line for $r \geq 1$.*

3.2 Simple maps

In this section we study simple maps $w = F(z)$, where $F : \mathbb{C} \to \mathbb{C}$. The subject is of considerable interest and will be fully appreciated after the discussion of analytic functions.
It is useful to introduce the **extended complex plane** $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ with the rules $z + \infty = \infty + z = \infty$ and $z\infty = \infty z = \infty$ $(z \neq 0)$. Moreover, one puts the conventions $z/\infty = 0$ $(z \neq \infty)$, and $z/0 = \infty$ $(z \neq 0)$.

3.2.1 The linear map

The linear map $w = az + b$ has one fixed point[2] $z^\star = b/(1 - a)$. By writing $w - z^\star = a(z - z^\star)$ one obtains

$$|w - z^\star| = |a||z - z^\star|, \qquad \arg(w - z^\star) = \arg a + \arg(z - z^\star)$$

Therefore the map is a *dilation* by a factor $|a|$ of all segments originating from $z^\star$ and a *rotation* of the plane by arg a around the fixed point.
Equivalently, the map can be viewed as a rotation by arg a around the origin and a dilation centred in the origin ($z' = az$), followed by a shift ($w = z' + b$).

Exercise 3.2.1. *Show that a linear map takes circles to circles and straight lines to straight lines.*

3.2.2 The inversion map

The map $w = 1/z$ transforms a circle $|z| = r$ centred in the origin into the circle $|w| = 1/r$. The interior of the unit circle is exchanged with the exterior.
By regarding straight lines as circles with a point at infinity, the inversion takes circles to circles (prove it). Then, a straight line that does not contain the origin is mapped to a circle through the origin; only straight lines through the origin (containing both 0 and ∞) are mapped to straight lines through the origin. Conversely, circles through the origin are mapped to straight lines, and circles

[1] A Cassini oval is the planar locus of points whose distances from two points have constant product: $|z - a| \cdot |z - b| = C^2$.

[2] A fixed point of a map is one that is mapped in itself, $z^\star = F(z^\star)$.

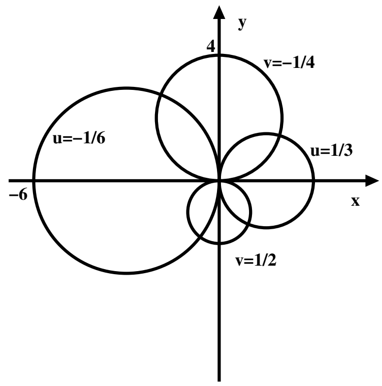

Figure 3.1: Inversion map. The circles through the origin of the z plane with centers on the real or the imaginary axis are pre-images of lines of constant u or v in the w plane. As the lines form an orthogonal grid, also the circles cross at right angles.

not through the origin are mapped to circles.
Set $w = u + iv$, the image of $x + iy$ has Cartesian coordinates

$$u = \frac{x}{x^2 + y^2}, \qquad v = -\frac{y}{x^2 + y^2}$$

A line $u(x, y) = U$ in $w-$plane is the image of a circle through the origin $z = 0$ with center $(\frac{1}{2U}, 0)$. A line $v(x, y) = V$ is the image of a circle again through the origin with center $(0, -\frac{1}{2V})$. All these circles, that are mapped to the orthogonal grid of $u - v$ lines, are orthogonal to each other (we'll give a general proof of this fact).

Exercise 3.2.2. *For the inversion map:*
1) Find the image of the circle $|z - i| = 1$.
2) Show that the circle with center z_0 *and radius* $|z_0|$ *is mapped to the axis of the segment* $[0, 1/z_0]$.
3) Obtain the image of the line $|z - a| = |z - b|$, *where* $a, b \in \mathbb{C}$.

3.2.3 Möbius maps

A Möbius map $w = M(z)$ is a linear fractional transformation

$$M(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \tag{3.3}$$

The map is unchanged if the complex parameters a, b, c, d are multiplied by the same nonzero value. It has at most two fixed points $z^\star = M(z^\star)$. If $c \neq 0$ $M(-d/c) = \infty$, and $M(\infty) = a/c$.
This is easily checked:

Proposition 3.2.3. *Möbius maps form a group.*

To the Möbius map (3.3) there corresponds an invertible matrix

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det L \neq 0. \tag{3.4}$$

Such matrices form the linear group $GL(2,\mathbb{C})$ of invertible complex 2×2 matrices. If $M_L(z)$ is the Möbius map with parameters specified by the matrix L, the composition of two Möbius maps is $M_{L'}(M_L(z)) = M_{L'L}(z)$.
The linear map and the inversion map are Möbius maps with matrices

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.5}$$

If $c \neq 0$ the factorization

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\frac{ad-bc}{c} & \frac{a}{c} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}. \tag{3.6}$$

shows that a Möbius map is a composition of two linear maps with an inversion between (the case $c = 0$ is a linear map). As a consequence:
1) *circles are mapped to circles* (where a straight line is a circle with point at ∞). More precisely, M takes every line and circle passing through $-d/c$ to a line, and every other line or circle into a circle.
2) Möbius maps are bijections from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ (actually, they are the most general bijections of the extended complex plane).

Example 3.2.4. *The Möbius maps of the upper half-plane* $\mathbb{H} = \{z : \, \mathrm{Im} z > 0\}$ *to the unit disk* $\mathbb{D} = \{w : \, |w| < 1\}$*, are*

$$w = e^{i\vartheta} \frac{z - z_0}{z - \overline{z_0}} \tag{3.7}$$

where z_0 *is the point with* $\mathrm{Im}\, z_0 > 0$ *that is mapped to* $w = 0$*, and the prefactor is a rotation of the disk.*

Proof. Let $w(z)$ have the form (3.3), with $c = 1$. Since a boundary is mapped to a boundary, the image of the real axis must be the unit circle. Then $|ax + b| = |x + d|$ for all real x. The limit cases $x \to \infty$ and $x = 0$ imply $|a| = 1$, $|b| = |d|$ i.e. $a = e^{i\vartheta}$, $b = -e^{i\vartheta} z_0$, $|d| = |z_0|$. Then $|x - z_0| = |x - d|$ for all x i.e.: $x^2 + |z_0|^2 - 2x\mathrm{Re}\, z_0 = x^2 + |z_0|^2 - 2x\mathrm{Re}\, d$. The equation is solved by $d = \overline{z_0}$. □

Exercise 3.2.5. *Show that the Möbius maps of the upper-half plane* $\mathbb{H}$ *on itself are represented by real matrices* $GL(2,\mathbb{R})$ *with positive determinant.*

A Möbius map can be specified by requiring that three points (z_1, z_2, z_3) are mapped (in the order) to prescribed points (w_1, w_2, w_3). The choice $(0, 1, \infty)$ for the image points gives the Möbius map

$$M(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}. \tag{3.8}$$

It maps the circle through z_1, z_2 and z_3 to the real axis (the circle that contains 0,1 and ∞).

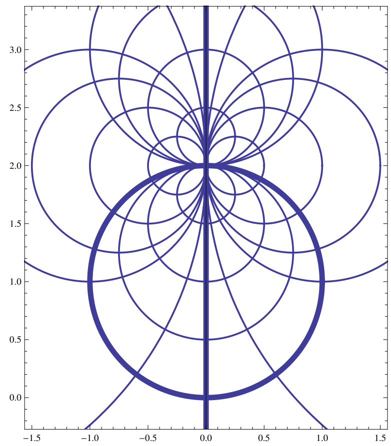

Figure 3.2: The Möbius map of example 3.2.7. The x and y axes are mapped to the thick circle and the v axis of the $w-$plane; 0 and i are fixed points. The circles with center on v axis are images of horizontal lines, while the other are images of vertical lines. The upper half plane is mapped to the interior of the thick circle.

Exercise 3.2.6. *Show that for any Möbius map, if $z' = M(z)$, then the cross-ratio with 4 points is invariant:*

$$\frac{z_1' - z_3'}{z_1' - z_4'}\frac{z_2' - z_4'}{z_2' - z_3'} = \frac{z_1 - z_3}{z_1 - z_4}\frac{z_2 - z_4}{z_2 - z_3}$$

Example 3.2.7. *The Möbius map $w(z) = \frac{2iz}{z+i}$ has fixed points 0 and i. The point $-i$ is mapped to infinity: any circle or line through it is mapped to a line. We know a priori that the parallel lines Im $z = Y$ $(z = x + iY)$ are mapped to circles parameterized by x. Elimination of x brings to the familiar expression:*

$$w = \frac{-2Y + 2ix}{x + i(1 + Y)} \;\Rightarrow\; \left|w - i\frac{2Y + 1}{Y + 1}\right| = \frac{1}{|Y + 1|}$$

The real axis $Y = 0$ is mapped to the circle $|w - 1| = 1$. The line $Y = -1$ $(z = x - iy)$ through the special point $-i$ is mapped to the line $w = (2/x) + 2i$ (not shown in fig.3.2). The lines with $Y > 0$ are mapped as circles in the disk $|w - 1| < 1$. The lines with $Y < 0$ are outside the disk. In any case the circles are tangent at $2i$. The lines Re $z = X$ are mapped to circles through $w = 2i$, orthogonal to the previous circles (see fig.3.2).

Exercise 3.2.8.
1) Find the images of $|z - 1| = 1$ and $|z - 1| = 2$ for the map $2 + (1/z)$.
2) Evaluate the Möbius map that takes (z_1, z_2, z_3) to (w_1, w_2, w_3).
3) Let 0 and 1 be the fixed points of a Möbius map $M(z)$. Write the general equation of circles through the two points, and evaluate their images under $M(z)$.

3.3 The stereographic projection

The stereographic projection is a bijection among the points of the extended complex plane and the points of the *Riemann sphere* S_2.
In a Cartesian frame XYZ consider the spherical surface $X^2+Y^2+(Z-\frac{1}{2})^2 = \frac{1}{4}$; it is tangent to the plane $Z = 0$ in its south pole[3]. Identify the plane $Z = 0$ with the complex plane z. A segment driven from the north pole $(0, 0, 1)$ to a point z in the complex plane intersects the spherical surface at the point

$$X = \frac{1}{2}\frac{z+\overline{z}}{|z|^2+1}, \quad Y = \frac{1}{2i}\frac{z-\overline{z}}{|z|^2+1}, \quad Z = \frac{|z|^2}{|z|^2+1} \tag{3.9}$$

The north pole corresponds to the point ∞ of the extended plane.

Exercise 3.3.1. *Find the coordinates (X', Y', Z') of the point antipodal to (X, Y, Z) on the Riemann's sphere. How are their images in $\mathbb{C}$ related?*
(Answer: $z' = -1/\overline{z}$)

The Euclidean distance $\|P - P'\|$ in $\mathbb{R}^3$ between two points of S_2 defines the **chordal distance** between the two corresponding points in $\mathbb{C}$:

$$d_c(z, z') = \frac{|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}} \tag{3.10}$$

The chordal distance differs from the distance established by the modulus. Its limit value 1 is achieved for the images of antipodal points: $(z, 1/\bar{z})$ and $(0, \infty)$.

Proposition 3.3.2. *The stereographic projection maps circles in S_2 to circles in $\mathbb{C}$. The circles through the north pole are mapped to straight lines.*

Proof. The locus of points of S_2 with Euclidean distance R from a point $C \in S_2$ is a circle. Its image in $\mathbb{C}$ is the locus $d_c(z, z_C) = R$, or $|z - z_C|^2 = R^2(1 + |z|^2)(1 + |z_C|^2)$: the equation of a circle. If the circle in S_2 goes through the north pole, the image contains $z = \infty$ and thus it must be $R^2(1 + |z_C|^2) = 1$ to balance infinities, i.e. the equation is linear in z (a straight line). □

Proposition 3.3.3. *The stereographic projection is conformal (angle-preserving).*

Proof. A triangle in $\mathbb{C}$ has vertices z, $z + \epsilon$ and $z + \eta$. The angle in z is given by Carnot's formula:

$$\cos\alpha = \frac{\overline{\epsilon}\eta + \epsilon\overline{\eta}}{2|\epsilon||\eta|}$$

The corresponding points on the Riemann's sphere form a triangle with side-lengths given by the chordal distances. The angle corresponding to α is

$$\begin{aligned}\cos\alpha' =& \frac{d_c(z, z+\epsilon)^2 + d_c(z, z+\eta)^2 - d_c(z+\epsilon, z+\eta)^2}{2d_c(z, z+\epsilon)d_c(z, z+\eta)}\\ =& \frac{|\epsilon|^2(1+|z+\eta|^2) + |\eta|^2(1+|z+\epsilon|^2) - |\epsilon-\eta|^2(1+|z|^2)}{2|\epsilon||\eta|\sqrt{(1+|z+\epsilon|^2)(1+|z+\eta|^2)}}\end{aligned}$$

In the limit $|\epsilon|, |\eta| \to 0$, α' identifies with the spherical angle and $\alpha' \to \alpha$. □

[3]The Riemann's sphere is often fixed to have unit radius and center in the origin; other choices are possible.

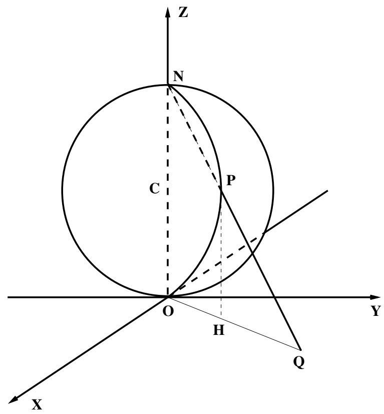

Figure 3.3: The stereographic projection maps a point $Q \in \mathbb{C}$ of coordinate z to a point $P = (X, Y, Z)$ of the Riemann sphere. The correspondence among coordinates is obtained through the similitudes: $1 : Z = |z| : (|z| - \sqrt{X^2 + Y^2})$ $(ON : HP = OQ : HQ)$; $\mathrm{Re}\, z : X = OQ : OH$, $\mathrm{Im}\, z : Y = OQ : OH$.

Example 3.3.4 (Rigid motions of the Riemann sphere)**.** *The Möbius maps that preserve the chordal distance, $d_c(M(z), M(z')) = d_c(z, z')$ for all $z, z' \in \mathbb{C}$, have coefficients*

$$|a|^2 + |b|^2 = |c|^2 + |d|^2 = |ad - bc|, \quad \bar{a}c + \bar{b}d = 0.$$

Via the stereographic map, such Möbius maps induce rigid transformations of the sphere S_2 into itself (rotations and reflections).
By choosing $|ad - bc| = 1$ (this is possible because coefficients can be rescaled by a common factor), they are represented by matrices where the conditions correspond to unitarity:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices form the group $U(2)$ of unitary matrices on $\mathbb{C}^2$. The (special) subgroup $SU(2)$ corresponds to $ad - bc = 1$ and induces pure rotations. The matrices U and $-U$ correspond to the same Möbius map and hence to the same rotation (see sect. 22.3.1).

Chapter 4

SEQUENCES AND SERIES

4.1 Topology

The modulus endows $\mathbb{C}$ with the structure of normed space[1] and defines a distance between points, $d(z, z') = |z - z'|$. Therefore $(\mathbb{C}, d)$ is also a metric space. Furthermore, at every point z_0 one may introduce a *basis of neighbourhoods*, which makes $\mathbb{C}$ a topological space. The elements of the basis are the disks centred in z_0 with radii $r > 0$:

$$D(z_0, r) = \{z : |z - z_0| < r\}.$$

The following definitions and statements are important and used thoroughly:

- A set S in $\mathbb{C}$ is *open* if for every point $z \in S$ there is a disk $D(z, r)$ wholly in S. The union of any collection of open sets is an open set; the intersection of two open sets is open.
- A point $z \in \mathbb{C}$ is an *accumulation point* of a set S if every disk $D(z, r)$ contains a point in S different from z.
- A *boundary point* of S is a point z such that every disk $D(z, r)$ contains points in S and points not in S. The boundary of S is the set ∂S of boundary points of S.
- A set is *closed* if it contains all its boundary points. The *closure* of a set S is the set $\overline{S} = S \cup \partial S$.
- A set S is disconnected if there are two disjoint *open* sets A and B such that $S \subseteq A \cup B$ but S is not a subset of A or B alone. A set is *connected* if it is not disconnected.

Proposition 4.1.1. *A set S is closed if and only if $\mathbb{C}/S$ is open.*

[1] Normed, metric and topological spaces are general structures that will be defined later.

Proof. Suppose that S is closed, then for any $z \notin S$ there is a disk that does not contain points in S (otherwise $z \in \partial S \subset S$), i.e. $\mathbb{C}/S$ is open. On the other hand, if $\mathbb{C}/S$ is open, every point in $\mathbb{C}/S$ cannot be in ∂S, i.e. S contains its frontier (S is closed). □

Definition 4.1.2. A **domain** is a set both open and connected.

Proposition 4.1.3. *Any two points in a domain can be joined by a continuous polygonal line in the domain (see Bak & Newman, Complex Analysis, Springer).*

4.2 Sequences

Complex sequences are maps $\mathbb{N} \to \mathbb{C}$, and are basic objects in mathematics. They arise in analysis, approximation theory, iteration of maps. Infinite series and infinite products are limits of sequences of partial sums and partial products. General statements about sequences are now presented.

Definition 4.2.1. A sequence z_n converges to z ($z_n \to z$) if $|z_n - z| \to 0$ i.e.

$$\forall\epsilon \quad \exists N_\epsilon \quad \text{such that} \quad |z_n - z| < \epsilon, \ \forall n > N_\epsilon. \tag{4.1}$$

Exercise 4.2.2. *Show that:*
1) if $z_n \to z$ and $w_n \to w$ then: i) $z_n + w_n \to z + w$, ii) $z_n w_n \to zw$, iii) $z_n/w_n \to z/w$ if $w_n, w \neq 0$, iv) $\overline{z}_n \to \overline{z}$.
2) z_n is convergent in $\mathbb{C}$ if and only if $\operatorname{Re} z_n$ and $\operatorname{Im} z_n$ are convergent in $\mathbb{R}$.
3) if $z_n \to z$, then $|z_n| \to |z|$. Hint: use inequality (2.7)*.*

Definition 4.2.3. A sequence z_n is a *Cauchy sequence* if

$$\forall\epsilon > 0 \quad \exists\, N_\epsilon \quad \text{such that :} \quad |z_n - z_m| < \epsilon, \quad \forall m, n > N_\epsilon. \tag{4.2}$$

A convergent sequence is always a Cauchy sequence, but the converse may not be true. When every Cauchy sequence is convergent to an element in the space, the space is *complete*. The Cauchy criterion is then an extremely useful tool to predict convergence, without the need to identify the limit.

Proposition 4.2.4. $\mathbb{C}$ is complete

Proof. The inequalities (2.6) imply that $\{x_n + iy_n\}$ is a Cauchy sequence if and only if both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since $\mathbb{R}$ is complete, they both converge. Let x and y be their limits, then $|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |y_n - y| \to 0$. □

4.2.1 Quadratic maps, Julia and Mandelbrot sets

The iteration of a map $z' = F(z)$, with a function $F : \mathbb{C} \to \mathbb{C}$ and initial value z_0, generates a sequence: z_0, $z_1 = F(z)$, $z_2 = F(F(z))$, ... The sequence depends on the initial value, and this dependence may be surprisingly interesting. The problem was studied by Pierre Fatou and Gaston Julia in the early 1900. The simplest non-trivial function to consider is

$$F(z) = z^2 + c, \qquad c \in \mathbb{C}.$$

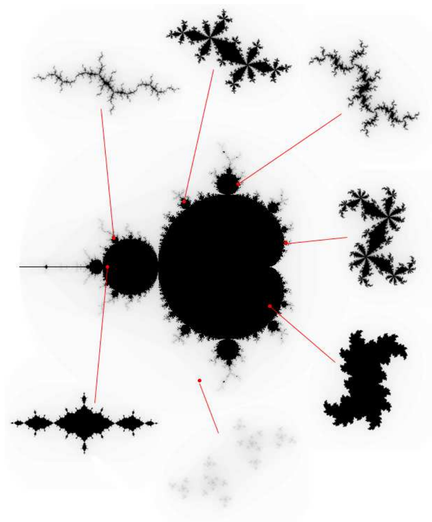

Figure 4.1: The Mandelbrot set is the locus of c-values such that the sequence $z_{n+1} = z_n^2 + c$ starting from $z_0 = 0$ remains bounded. The *Filled Julia sets* for some values of the parameter c are shown. An initial point z_0 picked in a filled Julia set generates a bounded sequence.

The quadratic map has two fixed points: $z^\star = z^{\star 2} + c$. Near a fixed point it is $F(z) \approx z^\star + 2z^\star(z - z^\star)$. The number $2|z^\star|$ depends on c and can be greater, less or equal to one. The linearized map is accordingly locally expanding, contracting or indifferent, i.e. the distance of images $|F(z) - z^\star|$ is greater, less or equal to the distance $|z - z^\star|$.

The sequences that escape to ∞ define a set of initial points $A_c(\infty)$ called the *attraction basin* of ∞. Of course, if $z \in A_c(\infty)$ also $F(z)$ and the whole sequence belongs to it. It was proven that it is an open and connected set.
The complementary set K_c is the *filled Julia set*. It contains the initial points of bounded sequences and it is a closed and bounded (i.e. compact) set. The boundary is the *Julia set* J_c.
The sets $A_c(\infty)$, K_c and J_c are left invariant by the action of the map.[2] Fatou and Julia proved the theorem: *If the sequence* $0, c, F(c), F(F(c)), \ldots$ *diverges to* ∞, *i.e.* $0 \in A_c(\infty)$, *then* J_c *is totally disconnected, whereas if* $0 \in K_c$, *then* J_c *is connected.*
While at IBM, in 1980, Benoit Mandelbrot (1924-2010) studied with the aid of a computer the properties of invariant Julia sets of the quadratic map and more complex ones, and disclosed the beauty of their fractal structures (see the book

[2]For $c = 0$ the sets are easily identified. Clearly $A_0(\infty)$ is the set $|z| > 1$, K_0 is the set $|z| \le 1$ (the filled Julia set) and J_0 (the Julia set) is the unit circle $|z| = 1$. The action of $z \to z^2$ on the points $e^{i2\pi\theta} \in J_0$ ($\theta \in [0, 1]$), is the map $\theta_{n+1} = 2\theta_n$ (mod 1). A point θ_0 after k iterations of the map is again θ_0 if $(2^k - 1)\theta_0$ is an integer. Then the initial point θ_0 generates a periodic sequence (*periodic orbit* of the map) of period k. Only rational angles give rise to periodic orbits. The irrational ones spread on the unit circle and the map is chaotic.

by Falconer for further reading, and Wikipedia for wonderful pictures). The *Mandelbrot set* (1980) is the set of parameters $c \in \mathbb{C}$ such that J_c is connected. Again, it is a wonderful fractal[3].

Exercise 4.2.5. *Consider the linear map $z_{n+1} = az_n + b$. Write the general expression of z_n in terms of z_0. When is the sequence bounded?*

Exercise 4.2.6. *Study the fixed points $z^\star$ of the exponential map $z' = e^z$. Show that they come in pairs $x \pm iy$ with $x > 0$, $|y| > 1$ (then the linearized map $z' = z^\star + z^\star(z - z^\star)$ is locally expanding)*[4].

4.3 Series

Infinite sums were studied long before sequences. The oldest known ones were obtained by Madhava (1350, 1425), the founder of the Kerala school of astronomy and mathematics. He developed series for trigonometric functions, including an error term. The one for arctan enabled him to evaluate π up to 11 digits. His work may have influenced European mathematics through transmission by the Jesuits. The same series were rediscovered by Gregory two centuries later.

Oresme in XIV century, Jakob & Johann Bernoulli (*Tractatus de seriebus infinitis*, 1689), and Pietro Mengoli (1625, 1686), discovered and rediscovered the divergence of the *Harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \dots$ In the *Tractatus*, the convergence of $\sum_k 1/k^2$ was also proven, but the sum (the Basel problem) was evaluated later (1734) by Johann's prodigious student Leonhard Euler. Euler also proved that the sum of the reciprocals of prime numbers is divergent[5].
Christian Huygens asked his student Leibnitz to evaluate the sum of reciprocals of triangular numbers[6]. The result (the sum is 2) was obtained after noting that $\frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1}$ (the sum is a *telescopic series*).

Series gained rigour after Cauchy, who defined convergence in terms of the sequence of partial sums.

Given a sequence of complex numbers a_k one constructs the partial sums $A_n = \sum_{k=0}^n a_k$. If the sequence A_n converges to a finite limit A, the limit is the *sum of the series*

$$A = \sum_{k=0}^{\infty} a_k.$$

If $\sum_k^\infty a_k = A$ and $\sum_k^\infty b_k = B$, where A and B are finite, the series $\sum_k^\infty (a_k \pm b_k)$ is convergent and the sum is $A \pm B$.

Proposition 4.3.1. *A necessary and sufficient condition for a series to con-*

[3]The boundary is the "Mandelbrot lemniscate". It is the limit of a sequence of level curves (lemniscates) $M_n = \{z \in \mathbb{C} : |p_n(z)| = 2\}$, where $p_n(z)$ is the sequence of polynomials: $p_1(z) = z$, $p_{n+1}(z) = p_n(z)^2 + z$.

[4]The map is chaotic, and is studied in `https://doi.org/10.48550/arXiv.1408.1129`.

[5]For nice accounts see P. Pollack, *Euler and the partial sums of the prime harmonic series*, `http://pollack.uga.edu/eulerprime.pdf`; M.Villarino, *Mertens Proof of Mertens Theorem*, arXiv:math/0504289.

[6]The triangular numbers are $n_1 = 1$, $n_{k+1} = 1 + 2 + \dots + k = \frac{(k+1)k}{2}$.

verge is that the sequence of partial sums is a Cauchy sequence:

$$\forall\epsilon > 0 \quad \exists N_\epsilon \quad \text{such that} \quad \forall m > N_\epsilon \text{ and } \forall n > 0 : \quad \left|\sum_{k=m+1}^{m+n} a_k\right| < \epsilon. \tag{4.3}$$

In particular for $n = 1$ *it is* $|a_{m+1}| < \epsilon$. *Therefore a necessary condition for convergence is* $|a_k| \to 0$.

4.3.1 Absolute convergence

The inequality $|\sum_{m+1}^{m+n} a_k| \leq \sum_{m+1}^{m+n} |a_k|$ implies that if $\sum_{k=1}^{\infty} |a_k|$ converges, then also $\sum_{k=1}^{\infty} a_k$ does.

Definition 4.3.2. A series $\sum_k a_k$ is *absolutely convergent* if $\sum_k |a_k|$ is convergent.

Absolute convergence is a sufficient criterion for convergence; moreover it deals with series in $\mathbb{R}$. We state but not prove the important property

Proposition 4.3.3. *The sum of an absolutely convergent series does not change if the terms of the series are permuted*[7]*:* $\sum_k a_k = \sum_k a_{\pi(k)}$.

Proof. The partial sums $S_m = \sum_{k=1}^{m} |a_k|$ are an increasing convergent sequence, with limit A. Let $T_n = a_{\pi(1)} + ... + a_{\pi(n)}$; if $N = \max_{k\leq n} \pi(k)$ it is $T_n \leq S_N$ i.e. $T_n \leq A$ for all n. Since T_n is an increasing bounded sequence, it is convergent to a limit $T \leq A$.
The sequence a_k is a rearrangement of $a_\pi(k)$, then $A = \lim_n \sum_k^n |a_k| \leq T$, i.e. $A = T$. (from `https://users.math.msu.edu/users/shapiro/Teaching/classes/320/Handouts/Series_Rearr.pdf`) □

Definition 4.3.4. Given an infinite set $A \subset \mathbb{R}$ bounded above, $\limsup A$ (or $\overline{\lim} A$) is the *largest* real number $\overline{a}$ such that $\forall\epsilon$ the set $\{a \in A : a > \overline{a} + \epsilon\}$ is finite and $\{a \in A : a > \overline{a} - \epsilon\}$ is infinite.

The following are sufficient conditions for absolute convergence of the series $\sum_k a_k$. They must hold for k greater than some N:

- **Comparison** (Gauss):

$$|a_k| < b_k, \quad \text{where} \quad \sum_k b_k < \infty$$

- **Ratio** (d'Alembert): for $a_k \neq 0$

$$\limsup_k \frac{|a_{k+1}|}{|a_k|} < 1$$

[7] A convergent series that is not absolutely convergent is *conditionally convergent*. The series $\sum_{k=1}^{\infty} i^k/k$ is convergent as it is the sum of two convergent series $\sum_{k\geq 1}(-1)^k/(2k) + i\sum_{k\geq 0}(-1)^k/(2k+1)$, and is conditionally convergent. Riemann proved the surprising result that by rearranging terms of a conditionally convergent real series one can obtain any limit sum in $\mathbb{R}$.

- **Root** (Cauchy-Hadamard):

$$\limsup_k \sqrt[k]{|a_k|} < 1$$

Proposition 4.3.5 (Cesaro)**.** *If the sequences* $|a_k|^{1/k}$ *and* $\frac{|a_{k+1}|}{|a_k|}$ *converge to finite limits, the limits are equal. The limit coincides with* lim sup.

4.3.2 Cauchy product of series

The Cauchy product of two series is the series of finite sums:

$$\left[\sum_{k=0}^{\infty} a_k\right]\left[\sum_{r=0}^{\infty} b_r\right] = \sum_{k=0}^{\infty}\left(\sum_{r=0}^{k} a_r b_{k-r}\right) \tag{4.4}$$

The definition is natural for power series, and extends the product of polynomials as a polynomial: $\left[\sum_{k=0}^{\infty} a_k x^k\right]\left[\sum_{r=0}^{\infty} b_r x^r\right] = \sum_{k=0}^{\infty} x^k \left(\sum_{r=0}^{k} a_r b_{k-r}\right)$

Proposition 4.3.6 (Franz Mertens, 1874[8])**.** *If a series is absolutely convergent to A and another is convergent to B, their Cauchy product is convergent to AB.*

Proposition 4.3.7. *If a series is absolutely convergent to A and another is absolutely convergent to B, their Cauchy product is absolutely convergent to AB.*

Proof.

$$\begin{aligned}\sum_{k=0}^{n}|c_k| &= \sum_{k=0}^{n}\left|\sum_{r=0}^{k} a_r b_{k-r}\right| \le \sum_{k=0}^{n}\sum_{r=0}^{k}|a_r||b_{k-r}| \\ &= \sum_{r=0}^{n}|a_r|\sum_{k=r}^{n}|b_{k-r}| = \sum_{r=0}^{n}|a_r|\sum_{k=0}^{n-r}|b_k| \le \sum_{k=0}^{\infty}|a_k|\sum_{l=0}^{\infty}|b_l|\end{aligned}$$

Since the sequence $\sum_{k=0}^{n}|c_k|$ is non decreasing and bounded above, it is convergent. Then $C_n = \sum_{k=0}^{n} c_k$ is absolutely convergent to a limit C. The limit does not depend on rearrangements of terms and is the sum of all possible products $a_r b_s$. It is $C = \lim_k (A_k B_k) = (\lim_k A_k)(\lim_k B_k) = AB$. □

4.3.3 The geometric series

The partial sum $S_n = 1 + z + \ldots + z^n$ is evaluated with the identities $S_{n+1} = S_n + z^{n+1}$ and $S_{n+1} = 1 + zS_n$:

$$\boxed{\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \qquad z \neq 1} \tag{4.5}$$

[8] see Boas, `http://www.math.tamu.edu/~boas/courses/617-2006c/sept14.pdf`

If $|z| < 1$, it is $\lim_{n\to\infty} z^n = 0$ and $S_n(z)$ converges to the simple but fundamental geometric series

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad |z| < 1} \tag{4.6}$$

The geometric series is useful for assessing convergence of series by the comparison test.

Exercise 4.3.8. *Evaluate the sums:* $1 + 2\cos x + \cdots + 2\cos nx$ *and* $\sin x + \sin 2x + \cdots + \sin nx$.

Exercise 4.3.9. *For* $a > 0$, b *real, obtain the sums:*

$$\sum_{k=0}^{\infty} e^{-ka}\cos kb = \frac{1}{2} + \frac{\sinh a}{2\cosh a - 2\cos b}, \qquad \sum_{k=0}^{\infty} e^{-ka}\sin kb = \frac{\sin b}{2\cosh a - 2\cos b}$$

(Hint: evaluate the partial sums $\sum_k e^{-k(a+ib)}$ *and separate* Im *and* Re *parts).*

Exercise 4.3.10. *Write* $(1-z^2)^{-1}$ *as a geometric series in* z^2*, as the product of two geometric series, as a linear combination of two geometric series.*

4.3.4 The exponential series

The sequence of partial sums $e_n(z) = 1 + z + \cdots + \frac{1}{n!}z^n$ is absolutely convergent for any z, because $e_n(|z|)$ converges. The limit is the complex exponential function:

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}} \tag{4.7}$$

Exercise 4.3.11. *Prove that the Cauchy product of two exponential series is the exponential series* $e^{z+z'} = e^z e^{z'}$*. Then show that* $e^{x+iy} = e^x(\cos y + i\sin y)$*, in accordance with* (4.7).

It is curious to observe that although e^z has no zeros, its polynomial truncations $e_n(z)$ have n of complex zeros[9].

Exercise 4.3.12 (Madhava math. competition 2013)**.** *Show that* $e_n(z)$ *has no real roots if* n *is even, and exactly one if* n *is odd*[10].

4.3.5 Riemann's Zeta function

The following series is of greatest importance in number theory[11], and is often encountered in physics:

$$\boxed{\zeta(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^z}, \qquad \mathrm{Re}\, z > 1} \tag{4.8}$$

[9]They fly to infinity. Szegö (1924) proved that for $n \to \infty$ the zeros of $e_n(z)$ divided by n distribute on the curve $|ze^{1-z}| = 1$ with $|z| \leq 1$.
(see `https://www.uvm.edu/~tdupuy/notes/partialsums.pdf`)

[10]see `http://www.madhavacompetition.com`, for texts and solutions. The contest is addressed to undergraduate students

[11]H. M. Edwards, *Riemann's Zeta Function*, Dover.

Since $|n^z| = |e^{z\log n}| = n^{\mathrm{Re} z}$, the series converges absolutely for Re $z > 1$. The values of the Riemann series are known at even integers (Euler). They result, for example, from the evaluation of certain Fourier series (see example 20.2.5). Two useful values are

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

Exercise 4.3.13. *Prove in the order:*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^z} = (1-2^{-z})\zeta(z), \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^z} = (1-2^{1-z})\zeta(z) \tag{4.9}$$

(since Riemann's series is absolutely convergent, the sum on even and odd n may be evaluated separately).
By collecting in the first series the terms $2n+1$ that are multiples of 3, obtain the sum on integers that are not divided by 2 and 3:

$$\sum_{n\neq 2k,3k} \frac{1}{n^z} = \left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\zeta(z)$$

The process outlined is iterated and gives the famous representation of Riemann's Zeta function as an infinite product on prime numbers *$p > 1$:*

$$\boxed{\frac{1}{\zeta(z)} = \prod_p \left(1-\frac{1}{p^z}\right)} \tag{4.10}$$

Exercise 4.3.14. *Show that* [12]

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{1}{1-xyz} = \zeta(3) \quad (\approx 1.202057)$$

(Hint: expand the fraction in geometric series)

[12] This type of integral representation was used to prove irrationality of $\zeta(2)$ and $\zeta(3)$ in a way simpler than Apery's proof of 1977 (see `https://doi.org/10.48550/arXiv.1308.2720`).

Chapter 5

COMPLEX FUNCTIONS

5.1 Differentiability and Cauchy-Riemann conditions

A complex function is a map from some set to $\mathbb{C}$. We shall focus on complex functions of a complex variable, $f : D \subseteq \mathbb{C} \to \mathbb{C}$ where the **domain** D, if not specified differently, will be an *open connected* set in $\mathbb{C}$.
To stress that the function only depends on the input variable $z = x + iy$, and not also on $\overline{z}$ (i.e. not freely on x and y), we often denote it as $f(z)$.
The real and imaginary parts of f are not themselves functions of the combination $x + iy$. For example: $z^2 = (x^2 - y^2) + i2xy$, $e^z = e^x \cos y + ie^x \sin y$. We then write:

$$f(z) = u(x, y) \; + \; i\, v(x, y)$$

Definition 5.1.1. A complex function $f(z)$ is continuous in z_0 if:

$$\forall \epsilon > 0 \quad \exists\, \delta > 0 \quad \text{such that} \quad |f(z) - f(z_0)| < \epsilon \quad \text{if } |z - z_0| < \delta. \tag{5.1}$$

Proposition 5.1.2. *$f(z)$ is continuous in $z_0 = x_0 + iy_0$ iff $u(x, y)$ and $v(x, y)$ are both continuous in (x_0, y_0).*

Proof. The inequality $|f(z) - f_0| \geq |u(x, y) - u_0|$ (and similar for v) implies continuity of u (and v) from continuity of f. The other way is proven by means of $|f(z) - f_0| \leq |u(x, y) - u_0| + |v(x, y) - v_0|$. ($u_0$ is short for $u(x_0, y_0)$ etc.) □

Definition 5.1.3. $f(z)$ is *differentiable* in $z_0 \in D$ if the following limit exists

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} \tag{5.2}$$

i.e. there is a number $f'(z_0)$ such that: $\forall \epsilon > 0$ there is a δ_ϵ such that

$$\Big| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \Big| < \epsilon \quad \forall z : \; |z - z_0| < \delta_\epsilon.$$

Then, in a neighbourhood of z_0 it is $f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z, z_0)(z - z_0)$, where $r(z, z_0)$ vanishes as $z \to z_0$.
It is clear that differentiability of f at z_0 implies continuity at z_0.

Exercise 5.1.4. *Show that the limit $z \to z_0$ of the incremental ratio for the function $|z|^2$ does not exist.*

The existence of the limit (5.2) is more demanding than in real analysis of one variable, as z may approach z_0 from all directions. It implies a strong constraint on the real and imaginary parts of f.

Proposition 5.1.5 (Cauchy-Riemann conditions). *If $f(z) = u(x,y) + iv(x,y)$ is differentiable in z_0, then the partial derivatives of u and v exist in (x_0, y_0) and the Cauchy-Riemann conditions hold in (x_0, y_0):*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{5.3}$$

Proof. The incremental ratio is evaluated with $z = z_0 + h$ and $z = z_0 + ih$ respectively (h real). By hypothesis the limits $h \to 0$ exist and coincide:

$$\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\frac{v(x_0+h,y_0)-v(x_0,y_0)}{h} \to f'(z_0)$$
$$\frac{u(x_0,y_0+h)-u(x_0,y_0)}{ih} + i\frac{v(x_0,y_0+h)-v(x_0,y_0)}{ih} \to f'(z_0)$$

Therefore, the real and imaginary parts exist separately as partial derivatives, and yield identities useful for the evaluation of f':

$$\mathrm{Re} f'(z_0) = \frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} = \frac{\partial v}{\partial y}\Big|_{(x_0,y_0)} \quad \mathrm{Im} f'(z_0) = -\frac{\partial u}{\partial y}\Big|_{(x_0,y_0)} = \frac{\partial v}{\partial x}\Big|_{(x_0,y_0)}$$

Bedides the Cauchy-Riemann conditions, we obtain the rule $f' = \partial_x(u+iv)$. □

The converse can be proven, but with further conditions on u and v:

Theorem 5.1.6. *If u and v have continuous partial derivatives in x and y in a disk centred in z_0, and if the Cauchy-Riemann conditions hold in z_0, then $f(z)$ is differentiable in z_0.*

Proof. By means of Taylor's expansion, and Cauchy-Riemann conditions at z_0:

$$\begin{aligned} f(z_0+h) - f(z_0) =& u(x_0+h_x, y_0+h_y) - u(x_0,y_0) + iv(x_0+h_x, y_0+h_y) - iv(x_0,y_0) \\ =& (\partial_x u + i\partial_x v)_0\, h_x + o(h_x) + (\partial_y u + i\partial_y v)_0\, h_y + o(h_y) \\ =& (\partial_x u + i\partial_x v)_0\, (h_x + ih_y) + o(h_x) + o(h_y) \end{aligned}$$

Divide by h. The limit $h \to 0$ exists and is $f'(z_0) = (\partial_x u + i\partial_x v)(z_0)$. □

The standard rules of derivation for functions of a real variable continue to hold for differentiable functions of a complex variable:

$$\begin{aligned} &(\lambda f + g)' = \lambda f' + g' && \text{(linearity)} \\ &(fg)' = f'g + f\,g' && \text{(Leibnitz property)} \\ &(1/f)' = -f'\,/\,f^2 && (f \neq 0) \\ &f(\,g(z)\,)' = f'(g(z))\,g'(z) && \text{(composite function)} \end{aligned}$$

Definition 5.1.7. A function $f(z)$ which is differentiable at all points of a domain D is **holomorphic** on D. A function holomorphic on $\mathbb{C}$ is **entire**.

Example 5.1.8. *The function $z \to z^n$, $n \in \mathbb{Z}$, is differentiable and*

$$\frac{d}{dz} z^n = n z^{n-1}.$$

For $n \geq 0$ it is entire, while for $n < 0$ it is holomorphic on $\mathbb{C}/\{0\}$.

Example 5.1.9. *The exponential function $e^z = e^x \cos y + i e^x \sin y$ has real and imaginary parts that are differentiable in x and y and solve the Cauchy-Riemann conditions. Then it is differentiable in z:*

$$\frac{d}{dz} e^z = \frac{\partial}{\partial x} e^{x+iy} = e^z$$

Since this holds at all points in $\mathbb{C}$, the exponential function is entire. Then, also $\sin z$, $\cos z$, $\sinh z$ and $\cosh z$ are entire functions and $\frac{d}{dz}\sin z = \cos z$, $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sinh z = \cosh z$ etc.
A direct check is $f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y$. The functions $u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$ solve the C.R. equations, and $f'(z) = \partial_x u + i \partial_y v = \cos z$.

Example 5.1.10. *$f(z) = \mathrm{Log}\, z$ with domain $\mathbb{C}/(-\infty, 0]$. The real part is $u(x,y) = \frac{1}{2}\log(x^2+y^2)$ and, with the single-valued $-\frac{\pi}{2} \leq \mathrm{Arctan} \leq \frac{\pi}{2}$, the imaginary part is:*

$$v(x,y) = 2\,\mathrm{Arctan} \frac{y}{\sqrt{x^2+y^2}+x}$$

The functions u and v are differentiable and solve the C.R. equations. Then

$$\frac{d}{dz}\mathrm{Log}\, z = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{z}.$$

Exercise 5.1.11. *Prove that if f is holomorphic on D and $f'(z) = 0$ everywhere on D, then f is constant on D,*

Remark 5.1.12. *A function $f(x,y)$ can be viewed as a function of the independent variables $z = x + iy$ and $\overline{z} = x - iy$, with partial derivatives*

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{5.4}$$

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{5.5}$$

$$4\,\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5.6}$$

The condition that a function does not depend on $\overline{z}$ yields the C.R. conditions:

$$0 = \frac{\partial f}{\partial \overline{z}} = \tfrac{1}{2}(\partial_x + i\partial_y)(u + iv) = \tfrac{1}{2}(\partial_x u - \partial_y v) + \tfrac{i}{2}(\partial_y u + \partial_x v).$$

  

Figure 5.1: **Augustin Luis Cauchy** (Paris 1789, Sceaux 1857) is the man who most contributed to the foundation of complex analysis. He introduced mathematical rigour and discovered the fundamental integral representation of holomorphic functions (the Cauchy integral formula): the values of a holomorphic function inside a region are fully determined by its values on the boundary. The whole theory descends from this powerful formula.

Figure 5.2: **Karl Weiertrass** (Ostenfelde 1815, Berlin 1897) was professor in Berlin. His fame as a lecturer attracted several students: Cantor, Frobenius, Fuchs, Grassmann, Killing, Mertens, Kovalewskaya, Runge, Schur, H. A. Schwarz. He studied analytic functions as power series. The possibility of constructing overlapping disks where series converge, extends the function to broader domains (*analytic continuation*). He did not bother much about priorities, and published many results late in his career.

Figure 5.3: **Bernhard Riemann** (Breselenz (Hannover) 1826, Verbania 1866). The young Riemann was oriented to become a pastor, and to study theology at Göttingen. Gauss encouraged him to study mathematics. In 1854 he completed his thesis *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the hypothesis which underlie geometry) which contains his ideas on Riemannian geometry, that generalize Gauss' results about surfaces. He succeeded to Dirichlet in the direction of the department, but died young of tuberculosis during a journey to lake Maggiore. He studied holomorphic functions as maps, and developed the theory of multi-sheet manifolds (Riemann surfaces) in order to investigate multivalued functions and to extend the theorems of single valued ones.

5.2 Conformal maps

What makes holomorphic functions very special is that they depend on a *single* variable z that spans a two-dimensional space.
To visualize $f = u + iv$ one may plot the real functions $u = u(x, y)$ and $v = v(x, y)$ in $\mathbb{R}^3$. However, it is far more interesting to view f as a *map* $w = f(z)$ from points (x, y) of a domain D to points (u, v) of the $w = u + iv$ plane (we already studied the linear map, the inversion and the Möbius map).
In this picture, the derivative f' of a function gains a simple geometric meaning. The following theorem shows that a holomorphic map *locally* performs a *dilation* by a factor $|f'(z)|$ and an anticlockwise *rotation* of angle arg $f'(z)$ (these facts were known to Gauss already in 1825).

Theorem 5.2.1. *A holomorphic map with $f' \neq 0$ preserves angles (it is a conformal map)*

Proof. Consider a differentiable function $\gamma : t \in [a, b] \to \mathbb{C}$, where $[a, b]$ is an interval of the real line. The range $\{\gamma(t),\ t \in [a, b]\}$ is a curve in the complex plane. The cartesian components of the tangent vector are the real and imaginary parts of $\dot\gamma(t)$, we thus identify $\dot\gamma(t)$ as the tangent vector. We require $\dot\gamma(t) \neq 0$ for all t. Let $z_0 = \gamma(t_0)$, with tangent vector $\dot\gamma(t_0)$.
A holomorphic map $w = f(z)$ takes γ to the curve $f(\gamma)$, with tangent vector $f'(z_0)\dot\gamma(t_0)$ at $w_0 = f(z_0)$. The direction $\theta = \arg\dot\gamma(t_0)$ of the tangent vector at z_0 is rotated to the direction $\theta' = \theta + \arg f'(z_0)$ at w_0, and the length of the tangent vector is scaled by the factor $|f'(z_0)|$.
If two curves intersect at z_0 with tangents forming a certain angle α, since the images of the tangent vectors in w_0 are rotated by the same angle arg $f'(z_0)$, the images of the curves continue to intersect at w_0 with the angle α. □

In the study of holomorphic maps it is often useful to evaluate how a grid of orthogonal lines (e.g. lines of constant x or y, u or v, r or θ) in some domain is mapped to, or back from, another grid of orthogonal lines.
The lines $u = U$ and $v = V$ in $w-$plane are orthogonal and are the images of the curves $u(x, y) = U$ and $v(x, y) = V$ in the z plane. The vectors grad$u = (\partial_x u, \partial_y u)$ and grad$v = (\partial_x v, \partial_y v)$ are respectively orthogonal to the level lines $u = U$ and $v = V$ and point towards increasing values of u and v. At a point of intersection of the curves in z plane, the vectors are orthogonal by the Cauchy-Riemann conditions:

$$\mathrm{grad}\, u \cdot \mathrm{grad}\, v = \partial_x u\, \partial_x v + \partial_y u\, \partial_y v = 0 \tag{5.7}$$

The same is true for the tangent vectors: the two vectors tangent to the curves at a point are orthogonal[1].

The map $w = f(z)$ of a domain where f is holomorphic, univalent, with $f'(z) \neq 0$, represents a local change of variables from (x, y) to (u, v). One

[1] The tangent and the normal vectors at a point of a curve are orthogonal. The vector $\vec t = (t_x, t_y)$ tangent to a curve $f(x, y) = c$ points in a direction of null variation of $f(x, y)$. This means that the directional derivative vanishes: $0 = \vec t \cdot \mathrm{grad} f$. If vectors are represented as complex numbers, the orthogonality means that $t_x + it_y = \pm i(\partial_x f + i\partial_y f)$.

evaluates:

$$\partial_u^2 + \partial_v^2 = \frac{1}{|f'(z)|^2}(\partial_x^2 + \partial_y^2) \tag{5.8}$$

$$du^2 + dv^2 = |f'(z)|^2(dx^2 + dy^2) \tag{5.9}$$

$$du \wedge dv = |f'(z)|^2 dx\, dy \tag{5.10}$$

Let us analyse some examples of maps. By restricting the domain, a map can be made one-to-one (univalent).

Example 5.2.2 (The quadratic map $\boldsymbol{w = z^2}$).

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

Since $\arg w = 2\arg z$, *the set* $-\frac{\pi}{2} < \mathrm{Arg} z \le \frac{\pi}{2}$ *(the half plane Re* $z > 0$ *with the negative imaginary axis included) is mapped onto the whole* $w-$*plane. The whole* $z-$*plane covers the* $w-$*plane twice, a point* w *being the image of two points* $\pm z$.

The derivative of the map is $2z$. *An infinitesimal square of area* $dx\,dy$ *centred in* z *is mapped to an infinitesimal "square" centred in* $w = z^2$ *with area* $dudv = 4|z|^2 dx\,dy$, *rotated by the angle arg* $(2z)$.
The lines $u = U$ *and* $v = V$ *in* w*-plane are the images of a grid of hyperbolas*

$$x^2 - y^2 = U, \qquad xy = \frac{V}{2}$$

that intersect orthogonally. The lines $x = X$ *or* $y = Y$ *are confocal parabolas in* $w-$*plane with focus in* $w = 0$, *which intersect orthogonally:*

$$u = -\frac{1}{4X^2}v^2 + X^2, \qquad u = \frac{1}{4Y^2}v^2 - Y^2.$$

Note that in the origin the angles are not conserved (actually they are doubled by the map): this is possible since the derivative is zero in $z = 0$.

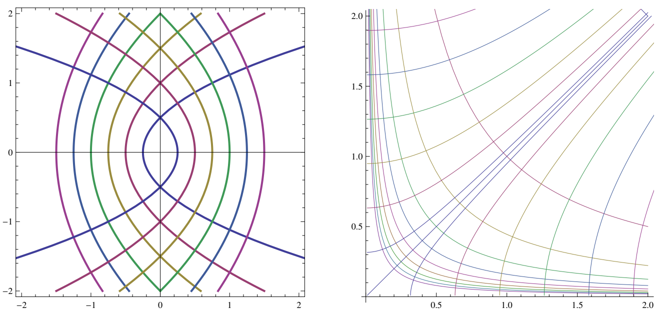

Figure 5.4: The quadratic map $w = z^2$. Left: The confocal parabolas in w-plane are the images of the coordinate square grid $x - y$ in z-plane. Right: The hyperbolas in z-plane are the pre-images of the coordinate square grid $u - v$ in w-plane (only a quadrant is shown).

Example 5.2.3 (The exponential map $\boldsymbol{w = e^z}$).

$$u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y$$

A point w that is image of z is also image of $z + 2\pi ik$. A choice of domain for univalency is the strip $-\pi < \mathrm{Im} z \le \pi$. Now exp is one-to-one from the strip to the w-plane $\mathbb{C}$ with $w = 0$ removed. The circles $u^2 + v^2 = r^2$ are images of $x = \log r$ and the radial lines Arg $w = \theta$ emanating from $w = 0$ are images of horizontal lines $y = \theta$.

Example 5.2.4 (The Joukowsky map[2] $\boldsymbol{w = \frac{1}{2}(z + \frac{1}{z})}$).
$z = \pm 1$ are fixed points, and $z = \pm i$ are mapped to $w = 0$. The map is not one-to-one z and $1/z$ have the same image. The pre-images of a point w solve the equation $z^2 - 2wz + 1 = 0$ and their product is one. Therefore the Jukowsky map is invertible on a domain that does not contain both z and $1/z$.

$$\text{If } z = e^{\xi + i\theta} \text{ it is: } \quad u = \cosh\xi \cos\theta \qquad v = \sinh\xi \sin\theta$$

The circles $|z| = e^\xi$ and the radial lines Arg $z = \theta$ are mapped respectively to confocal ellipses and hyperbolas with foci in ± 1

$$\frac{u^2}{\cosh^2\xi} + \frac{v^2}{\sinh^2\xi} = 1, \qquad \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1 \tag{5.11}$$

that cross at right angles. The angular directions of the asymptotes of the hyperbola are $\pm\theta$.

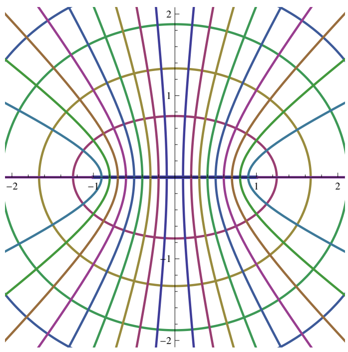

Figure 5.5: The Joukowsky map $w = \frac{1}{2}(z + 1/z)$ maps the unit circle to $[-1, 1]$ (degenerate ellipse), $|z| > 1$ to $\mathbb{C}$, circles to ellipses, radial lines to hyperbolas. The lines are confocal and intersect at right angles.

- *$|z| > 1$ is mapped to $\mathbb{C}/[-1, 1]$*
- *The wedge $\pi/2 - \theta < \mathrm{Arg} z < \pi/2 + \theta$ is mapped to the region between the branches of the hyperbola with parameter θ.*
- *Im $z > 0$ is mapped to $\mathbb{C}/(-\infty, -1] \cup [1, \infty)$.*

[2] The Russian mathematician, airplane designer, professor of mechanics Nikolay Y. Joukowsky (1847, 1921) used complex maps to study aerodynamics and flows. In 1918 he founded The Central Aero-Hydrodynamical Institute, which played a major role in the development of the aero-cosmic industry of the Soviet Union.

The boundary Im $z = 0$ is mapped to the cuts in w-plane as follows:
$(0, 1]$ *and* $[1, \infty)$ *are both mapped to the cut* $[1, \infty)$, $(-\infty, -1]$ *and* $[-1, 0)$ *are both mapped to* $(-\infty, -1]$ *(they contain x and $1/x$).*

Conformal maps have important applications in physics and engineering (see chapter on electrostatics). They are a powerful tool to solve differential equations in two real variables in domains with complicated boundaries, by mapping them to domains with simpler boundaries. The cornerstone is the following theorem, of remarkable generality:

Theorem 5.2.5 (Riemann mapping theorem). *Every simply connected domain in $\mathbb{C}$ (but not $\mathbb{C}$) can be mapped bi-olomorphically onto the unit disk (or the half plane).*

A useful class of conformal transformations are the Schwarz-Christoffel maps, from polygons to the half-plane (for the rectangle it is an elliptic function, prop.15.3.4)[3].

5.2.1 Harmonic functions

We shall prove that, unlike real functions, the existence of the derivative $f'(z)$ on a domain D (*holomorphism*) implies that $f(z)$ is *differentiable infinitely many times* on D and admits convergent Taylor series expansion in any disk contained in D (*analyticity*). Because of this, the words "holomorphic" and "analytic" are equivalent.
Analyticity implies that the real and imaginary parts u and v are functions $\mathscr{C}^\infty(D)$. The second partial derivatives and the Cauchy-Riemann property show that **the real and imaginary parts of a holomorphic function are harmonic functions** on the (open) domain: $\partial_x^2 u = \partial_x(\partial_y v) = \partial_y(\partial_x v) = -\partial_y^2 u$ and $\partial_x^2 v = -\partial_x(\partial_y u) = -\partial_y(\partial_x u) = -\partial_y^2 v$. Then:

$$\boxed{\nabla^2 u = 0, \quad \nabla^2 v = 0} \qquad (\nabla^2 = \partial_x^2 + \partial_y^2) \tag{5.12}$$

Given a real harmonic function $u(x, y)$ on a domain D, the Cauchy-Riemann equations may be solved to obtain $v(x, y)$ on D (the *conjugate* harmonic function) up to a constant. The function $f = u + iv$ is holomorphic on D.

Example 5.2.6. *To find the conjugate of the harmonic function $u(x, y) = x^3 y - xy^3$ use C.R. equations: $\partial_y v = \partial_x u = 3x^2 y - y^3$. An integral in y gives $v = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + \lambda(x)$ (λ is an unknown integrating function). The second C.R. equation is: $\partial_x v = -\partial_y u$ i.e. $3xy^2 + \lambda'(x) = -x^3 + 3xy^2$. Then $\lambda(x) = -\frac{1}{4}x^4 + \lambda_0$ and $v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}(y^4 + x^4)$ (up to λ_0). The associated holomorphic function is $u + iv = x^3y - xy^3 - \frac{i}{4}(x^4 - 6x^2y^2 + y^4) = -\frac{i}{4}z^4$*

A different approach gives f: $f'(z) = u_x - iu_y = 3x^2y - y^3 - ix^3 + i3xy^2 = 3xy(x+iy) - i(x^3 - iy^3) = 3xyz - i[(x+iy)^3 - 3ix^2y + 3xy^2] = iz^3$ (necessarily we end up with only z). An integral gives $f(z)$, and the imaginary part is $v(x, y)$.

Proposition 5.2.7. *If $\varphi(u, v)$ is harmonic on D_w (a domain in the plane $w = u + iv$) and if $f : D_f \to D_w$ is a holomorphic map, then $\varphi(u(x, y), v(x, y))$ is harmonic on D_f.*

[3]References: P. Henrici, *Applied and computational complex analysis*, vol. 3, Wiley (1976); T. Driscoll and L. Trefethen, *Schwarz-Christoffel mapping*, Cambridge Univ. Press. (2002); Z. Nehari, *Conformal Mapping*, reprinted by Dover

Proof. Direct evaluation of partial derivatives checks the statement. In alternative, φ is the real part of a holomorphic function $g(w)$. Then $g(f(z))$ is holomorphic in the variable z, and Re $g(f(z))$ is harmonic. $\square$

Example 5.2.8. *The function* $e^{x^2-y^2}\cos(xy)$ *is harmonic because it is the real part of the holomorphic function* e^{z^2}.
The function $1/z$ *is holomorphic on* $\mathbb{C}/\{0\}$*, then its real part* $x/(x^2+y^2)$ *is harmonic on* $\mathbb{C}/\{0\}$.

5.3 Inverse functions

Let f be a holomorphic function on a domain D; we discuss conditions for the existence of a holomorphic inverse function f^{-1}.

Definition 5.3.1. A function g is a *branch* of f^{-1} with domain $U \subseteq f(D)$ if:
1) g is continuous on U,
2) $f(g(w)) = w$ for all $w \in U$.

Remark 5.3.2. *The branch function* g *is univalent (injective) on* U *if:*

$$g(w_1) = g(w_2) \;\Rightarrow f(g(w_1)) = f(g(w_2)) \text{ and } w_1 = w_2$$

As a special case of the implicit function theorem in two real variables, we state the important result

Theorem 5.3.3 (Inverse function theorem)**.** *Suppose that* $f(z)$ *is holomorphic on a domain* D*, and* $g(w)$ *is a branch of* $f^{-1}(w)$ *with domain* U*. Let* $f(z_0) = w_0 \in U$*; if* $f'(z_0) \neq 0$ *then* g *is differentiable at* w_0 *and* $g'(w_0) = 1/f'(z_0)$*.*
Consequently, if $f'(z) \neq 0$ *in* $g(U)$*, then* g *is holomorphic on* U *and*

$$\boxed{g'(w) = \frac{1}{f'(g(w))}} \tag{5.13}$$

We now discuss the analytic structure of some relevant inverse functions. In these examples, the branch is chosen in order to reproduce the inverse function of real analysis, when restricted to real variable (*principal branch*).

5.3.1 The square root

The function $f(z) = z^2$ maps $\mathbb{C}$ to a double cover of the $w-$plane. It is useful to view the image as a two-sheet Riemann surface, i.e. two copies of the $w-$plane. In real analysis, the square root $u = \sqrt{x}$ is defined on the half-line $x \geq 0$, and maps the half-line monotonically to $u \geq 0$. Therefore, a convenient choice is to identify the first sheet as the *image* of the half-plane

$$-\frac{\pi}{2} < \operatorname{Arg} z < \frac{\pi}{2}$$

which includes the positive real axis. The first sheet has a cut: the negative real axis Arg $w = \pi$, which is the image of the imaginary axis in z plane.
The second sheet is the image of the half-plane $\pi > |\operatorname{Arg} z| > \frac{\pi}{2}$. The two sheets share an edge: the line Arg $w = \pi$.

By walking anticlockwise on a circle around the origin $z = 0$ starting at Arg $z = -\pi/2$, the image point walks in the first sheet starting at Arg $w = \pi$.
At Arg $z = \pi/2$ a full circle is run in the first sheet, and the walker enters the second sheet being at the other edge of the cut Arg $w = \pi$.
Another circle is completed as Arg $z = 3\pi/2$. At this point we are again at Arg $z = -\pi/2$, i.e. the image point is on the side of the cut where the walk in the first sheet started.

The principal branch of the square root has domain in the first sheet,

$$\sqrt{\ } \;:\; \rho e^{i\theta} \;\to\; \sqrt{\rho} e^{i\theta/2}, \quad |\theta| < \pi \tag{5.14}$$

The half line $(-\infty, 0] = \{w \ s.t. \ \mathrm{Arg}\, w = \pi\}$ is a *branch cut*, where the square root is discontinuous. Near the cut:

$$\sqrt{-|x| + i\epsilon} = i\sqrt{|x|}, \qquad \sqrt{-|x| - i\epsilon} = -i\sqrt{|x|}$$

On the domain $\mathbb{C}/(-\infty, 0]$ the principal branch is holomorphic and

$$\frac{d}{dw}\sqrt{w} = \frac{1}{2\sqrt{w}} \tag{5.15}$$

5.3.2 The logarithm

The function $f(z) = e^z$ is defined for all z but, being periodic, it is not invertible. On a strip $y_0 - \pi < \mathrm{Im}\, z < y_0 + \pi$ the map is one-to-one with the w-plane with a cut given by the half-line arg $w = y_0 + \pi$ removed. The whole z-plane is mapped to a *helix* (the Riemann surface of the log). It is a stack of an infinite number of sheets (copies of w-plane with cut); each sheet is a domain of a *branch* of the log: $\log w = \log |w| + i\,\mathrm{arg} w$ $(y_0 + (2k-1)\pi < \arg w < y_0 + (2k+1)\pi)$; it has discontinuity $2\pi i$ across the cut *if one remains on the same sheet*. The Riemann surface allows one to enter new sheets by crossing the cut: a 2π ascent round the helix axis moves from one sheet to another, where the new branch of $\log w$ differs by $2\pi i$ from the previous one.
A walk parallel to the imaginary axis in z-plane in the positive direction corresponds to a helicoidal path in the Riemann surface.
A value y_0 (modulo 2π) fixes the cuts; each sheet defines a branch of the log which is analytic with derivative

$$\frac{d}{dw}\log w = \frac{1}{e^{\log w}} = \frac{1}{w} \tag{5.16}$$

The principal log (Log) is the branch on the first sheet with the choice $y_0 = 0$. Its domain is $\mathbb{C}/(-\infty, 0]$, where $\mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg} z$, $(|\mathrm{Arg} z| < \pi)$.

Chapter 6

ELECTROSTATICS

Several two-dimensional problems (or three-dimensional ones, with a direction of translation invariance) in electrostatics, magnetostatics, fluid-dynamics, elasticity, may be elegantly solved in the complex plane, with proper conformal maps that take care of the geometry. In this chapter we consider electrostatics[1].

6.1 The fundamental solution

The fundamental solution or Green function[2] of 2D electrostatics is the solution of Poisson's equation for a unit point charge localized at $z' = x' + iy'$:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -\frac{1}{\epsilon_0}\,\delta(x-x')\delta(y-y') \tag{6.1}$$

The solution is easily obtained if the point charge is viewed as the section of a wire orthogonal to the plane, with unit linear charge. The 3D electric field is radial. Gauss theorem applied to a cylinder coaxial with the wire gives $\mathbf{E} = E\mathbf{n}$ with

$$E(x,y) = \frac{1}{2\pi\epsilon_0}\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}$$

The electrostatic potential in 2D of a point unit charge is $\mathbf{E}(x,y) = -\boldsymbol{\nabla}\varphi$ i.e.:

$$\boxed{\varphi\,(x,y) = -\frac{1}{2\pi\epsilon_0}\log\sqrt{(x-x')^2+(y-y')^2}.} \tag{6.2}$$

[1]A specialized text in this topic is: E. Durand, *Électrostatique*, 3 voll, Masson, Paris 1964.

[2]George Green (Nottingham 1793, 1841) was a miller. For more than forty years he worked hard in his father's windmill, while self-teaching mathematics and physics. In 1828 he published at his own expense his most important and astonishing paper: *Essay on the application of mathematical analysis to the theories of electricity and magnetism*, that contains the first exposition of the theory of potential. After his father's death, he applied to Cambridge's university, and graduated in mathematics in 1838 (the same year as Sylvester). After his death his achievements were rescued from obscurity by Lord Kelvin, and are nowadays an important tool in every area of physics, including quantum field theory (see J. Schwinger, *The Greening of quantum Field Theory: George and I.* `https://doi.org/10.48550/arXiv.hep-ph/9310283`).

By the superposition principle (linearity of Poisson's equation), the electrostatic potential generated by a charge distribution is:

$$\varphi(x,y) = -\frac{1}{2\pi\epsilon_0}\int dx'\,dy'\,\log\sqrt{(x-x')^2+(y-y')^2}\,\rho(x',y')$$

It solves Poisson's equation and it is *harmonic in empty space*. For this reason, it is very useful to regard it as the real part of a *complex potential* which, in empty space, is holomorphic and depends on $z = x + iy$:

$$\boxed{\Phi(z) = \varphi(x,y) + i\psi(x,y)}$$

The conjugated field $\psi(x,y)$ is harmonic. It can be obtained from φ by solving the Cauchy-Riemann equations and is defined up to a constant; sometimes it is found by good guess. For the point charge it is

$$\Phi(z) = -\frac{1}{2\pi\epsilon_0}\log(z-z') = -\frac{1}{2\pi\epsilon_0}[\log|z-z'| + i\arg(z-z')]$$

and the fields φ and ψ are harmonic in $\mathbb{C}/\{z'\}$. The cut of Φ can be chosen freely, to avoid the points of interest.

Being a function of z only, the complex potential is more manageable than the potential. For a charge distribution, $\Phi(z)$ may be obtained by superposition.

The physical meaning of the field ψ is now given. The function Φ maps the z plane to the plane $w = \varphi + i\psi$; the orthogonal straight lines $\varphi = U$ and $\psi = V$ correspond in the (x,y) plane to orthogonal curves that describe respectively *equipotential lines* and *lines of force*.
For the point charge, the equipotential lines are circles centred in the point source z', and the field lines are half-lines originating from z'.

The electric field $\mathbf{E} = -\boldsymbol{\nabla}\varphi$ is best evaluated as a complex field: $\overline{E} = E_x - iE_y = -\partial_x\varphi + i\partial_y\varphi$; by the Cauchy-Riemann equations $\partial_y\varphi = -\partial_x\psi$. Then:

$$\boxed{\overline{E} = -\frac{d}{dz}\Phi(z)} \tag{6.3}$$

The vector field $\mathbf{E}$ is orthogonal (by definition) to equipotential lines, and is tangent to the lines of force.

Example 6.1.1 (The uniformly charged segment)**.** *The electrostatic potential of a uniformly charged segment* $[a,b]$ *of the real axis with charge density* σ*,* $\varphi(x,y) = -\frac{\sigma}{4\pi\epsilon_0}\int_a^b ds\log[(x-s)^2+y^2]$*, is the real part of*

$$\Phi(z) = -\frac{\sigma}{2\pi\epsilon_0}\int_a^b ds\log(z-s)$$

To evaluate the electric field (6.3) *assume that* $-\frac{d}{dz}$ *commutes with integration and use* $-\frac{d}{dz}\log(z-s) = \frac{d}{ds}\log(z-s)$*:*

$$E_x - iE_y = -\frac{\sigma}{2\pi\epsilon_0}\log\frac{z-b}{z-a} = -\frac{\sigma}{2\pi\epsilon_0}\left[\log\left|\frac{z-b}{z-a}\right| - i\arg\frac{z-b}{z-a}\right]$$

At $z = x \pm i\epsilon$, $a < x < b$, it is Log $\frac{z-b}{z-a} = \log \frac{b-x}{x-a} \pm i\pi$. Then $E_y = \pm\sigma/2\epsilon_0$; E_x is non-zero because the uniformly charged segment is not equipotential. At each point of $[a, b]$ (and only there) there are two determinations $(E_x \pm iE_y)(x)$ that correspond to vectors pointing to opposite sides; it is singular at the end-points. Elsewhere, the vector field is unique.

Example 6.1.2. *Electrostatic potential of n charges q/n, equally spaced on a circle of radius R.*
Let $\zeta_1 \dots \zeta_n$ be the roots of unity, then $\Phi(z) = -\frac{q}{n}\frac{1}{2\pi\epsilon_0}\sum_k \log(z - R\zeta_k) = -\frac{q}{n}\frac{1}{2\pi\epsilon_0}\log(z^n - R^n)$. The potential is $\varphi(x, y) = -\frac{q}{n}\frac{1}{2\pi\epsilon_0}\log|z^n - R^n|$. In the continuum limit $n \to \infty$ (uniformly charged ring - or cylindrical surface in 3D):

$$\varphi(x, y) = -\frac{q}{2\pi\epsilon_0}\begin{cases}\log R & r < R \\ \log r & r > R\end{cases} \tag{6.4}$$

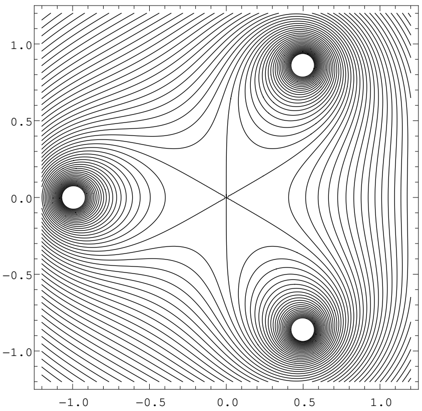

Figure 6.1: The equipotential lines of 3 unit charges.

Example 6.1.3 (The dipole)**.** *The complex potential of two point charges $\pm Q$ in $\pm i\delta/2$ is*

$$\Phi(z) = -Q\frac{1}{2\pi\epsilon_0}[\log(z - i\delta/2) - \log(z + i\delta/2)]$$

For $|z| \gg \delta$ one approximates $\log(z \pm i\delta/2) \approx \log z \pm i\delta/(2z)$. If $\delta \to 0$ and Q is rescaled such that $d = Q\delta$ is finite, we obtain the potential at a point z of a dipole placed in the origin and oriented as the imaginary axis:

$$\Phi(z) = i\frac{d}{2\pi\epsilon_0 z} = \frac{d}{2\pi\epsilon_0}\frac{y + ix}{x^2 + y^2}.$$

The equipotential lines are circles tangent to the real axis at the origin, the flux lines are circles through the origin and tangent to the imaginary axis. The electric field of the dipole is

$$\overline{E}(z) = i\frac{d}{2\pi\epsilon_0 z^2} = \frac{d}{2\pi\epsilon_0}\left[\frac{2xy}{(x^2 + y^2)^2} + i\frac{x^2 - y^2}{(x^2 + y^2)^2}\right] = E_x - iE_y.$$

The intensity of the field is $|E(z)| = d/[2\pi\epsilon_0(x^2 + y^2)]$.

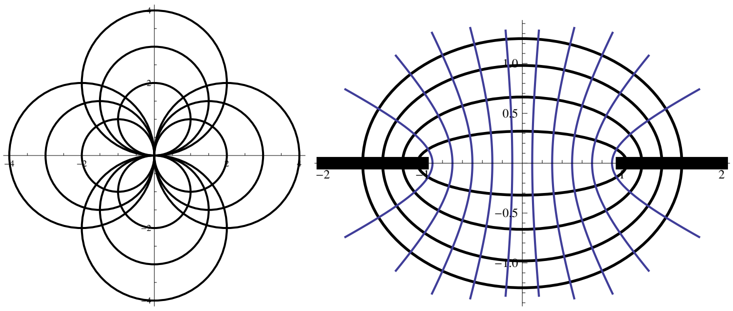

Figure 6.2: Equipotential lines and field lines for the dipole (left) and two coplanar thin metal plates (right).

A solution of an electrostatic problem (the complex potential $\Phi(w)$ in an empty region Ω_w with prescribed boundary conditions b.c.) can be used to solve more complex problems where the point charges are displaced, and the conducting surfaces are deformed, by some conformal map $f : \Omega_z \to \Omega_w$. The function $\Phi(f(z))$ is holomorphic in Ω_z, and its real part is the harmonic potential in Ω_z.
In the following examples, a solution is found for simple geometries in the $w-$plane, and related more complex problems are then solved by conformal mapping.

6.2 Thin metal plate

Consider in 3D a thin infinite metallic plate, with charge density σ, and at electrostatic potential φ_0. The translation-invariant problem is studied in 2D. If the section of the plate is the real axis of the $w = u + iv$ plane, the potential solves Laplace's equation with $\varphi(u, 0) = \varphi_0$ and $-(\partial_v\varphi)(u, 0) = \sigma/\epsilon_0$. The solution $\varphi(u, v) = -\frac{1}{\epsilon_0}\sigma\, v + \varphi_0$ is the real part of the holomorphic potential

$$\Phi(w) = \varphi_0 + \frac{i}{\epsilon_0}\sigma\, w + ic$$

where c is real and arbitrary.
This simple solution is useful for solving more difficult problems: if $f(z)$ is the conformal map of some empty domain to the half-plane Im $w > 0$, then

$$\Phi(z) = \varphi_0 + \frac{i}{\epsilon_0}\sigma\, f(z) + ic$$

is the complex potential in the new domain. Its real part is harmonic and takes the value φ_0 one the boundary of the domain, which f maps to the real axis $v = 0$. We consider two examples.

6.2.1 Field in a right-angle dihedral, with conducting walls

$f(z) = z^2$ maps the quadrant $\{z : x > 0, y > 0\}$ to Im $w > 0$. Then the complex potential $\Phi(z) = \varphi_0 + \frac{i}{\epsilon_0}\sigma\, z^2 + ic$ solves the electrostatic problem in the empty

quadrant with potential φ_0 at the boundary.
The electrostatic potential is the real part: $\varphi(x,y) = -(2\sigma/\epsilon_0)\,xy + \varphi_0$, with hyperbola as equipotential lines. The lines of force of the electric field are also hyperbola. The electric field is $\bar{E} = -i\frac{2\sigma}{\epsilon_0}z$, or $\vec{E}(x,y) = \frac{2\sigma}{\epsilon_0}(y\vec{i} + x\vec{j})$.
The linear charge at the boundary of the quadrant is not uniform: $\sigma(x) = 2\sigma\,x$ ($x \geq 0$) and $\sigma(y) = 2\sigma\,y$ ($y \geq 0$). Charge is depleted from the corner because of the electrostatic repulsion exerted by the other side. The total charge in $0 < x < L$ is σL^2, and equals the total charge in the image segment $0 < u < L^2$.

6.2.2 Field in a obtuse dihedral, with conducting walls

$f(z) = z^{2/3}$ maps the sector $0 < \arg z < \frac{3}{2}\pi$ to $\operatorname{Im} w > 0$. The complex potential in the sector is $\Phi(z) = \varphi_0 + i(\sigma/\epsilon_0)\,z^{2/3} + ic$. The electrostatic potential is $\varphi(x,y) = \varphi_0 - (\sigma/\epsilon_0)\,\rho^{2/3}\sin(\frac{2}{3}\theta)$. The linear charge on the boundary line $\theta = 0$ is:

$$\sigma(x) = \epsilon_0 E_y(x,0) = \epsilon_0 \mathrm{Im}\frac{d\Phi}{dz}\Big|_{y=0} = \tfrac{2}{3}\sigma x^{-1/3}.$$

The charge *accumulates near the edge* $x = 0$. The charge in the segment $0 < x < L$ is $\sigma L^{2/3}$ and equals the charge in the image segment $0 < u < L^{2/3}$.

6.3 Two adjacent thin metal plates

In 3D consider two adjacent thin metal plates at potentials 0 and V separated by an insulating line. The problem is 2D ($w = u + iv$ plane), with the two plates being the positive and negative parts of the real axis. The Laplace equation for the electrostatic potential with b.c. $\varphi(u,0) = 0$ if $u < 0$ and $\varphi(u,0) = V$ if $u > 0$, is found in polar coordinates:

$$\nabla^2\varphi(r,\theta) = 0, \qquad \varphi(r,0) = V, \quad \varphi(r,\pi) = 0 \quad r > 0$$

The solution is $\varphi(r,\theta) = \frac{1}{\pi}V(\pi - \theta)$ (indipendent of r) i.e. $\varphi(w) = \frac{1}{\pi}V(\pi - \operatorname{Arg} w)$. It is the real part of the complex potential, holomorphic in $v \neq 0$:

$$\Phi(w) = V + \frac{iV}{\pi}\mathrm{Log}w + ic$$

Again, this solution can be used to solve more difficult problems.

6.3.1 Semi-infinite plates at right angles, at potentials 0 and V

The complex potential is $\Phi(z) = V + \frac{iV}{\pi}\mathrm{Log}(z^2)$. The real part is the electrostatic potential of the new geometry: $\varphi(x,y) = V - 2\frac{V}{\pi}\mathrm{arctg}(y/x)$, with boundary values $\varphi(0,y) = 0$ and $\varphi(x,0) = V$. The electric field is

$$\overline{E} = -\frac{2iV}{\pi z} \to E_x = -\frac{2V}{\pi}\frac{y}{x^2+y^2},\; E_y = \frac{2V}{\pi}\frac{x}{x^2+y^2}$$

The charge on the metallic boundary $y = 0$, $x > 0$ is $\sigma(x) = \frac{1}{4\pi}E_y(x,0) = \frac{V}{2\pi^2 x}$.
On the boundary $x = 0$, $y > 0$ it is $\sigma(y) = \frac{1}{4\pi}E_x(0,y) = -\frac{V}{2\pi^2 y}$.

6.3.2 Two coplanar semi-infinite metallic plates, with gap

The thin plates have potentials 0 and V and are now separated by a gap of width 2. In the plane $z = x + iy$, the plates are the semi-infinite lines $x \le -1$ and $x \ge 1$ of the real axis. The electrostatic potential solves Laplace's equation with boundary conditions $\varphi(x, 0) = 0$ if $x \le -1$ and $\varphi(x, 0) = V$ if $x \ge 1$. The b.c. are implemented by a conformal map.
The domain where φ is harmonic (z-plane with two cuts in the real axis) is the image of the half-plane $\mathbb{H} = \{w : \operatorname{Im} w > 0\}$ for the Jukowski's map

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right)$$

In components Jukowski's map is ($w = re^{i\theta}$, $0 \le \theta \le \pi$):

$$x = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta, \qquad y = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

The half-line $\theta = 0$ is mapped to the cut $x \ge 1$, $y = 0$, while the half-line $\theta = \pi$ is mapped to the other cut $x \le -1$, $y = 0$. The complex potential of the problem with the gap is $\Phi(z) = V + \frac{iV}{\pi}\operatorname{Log} w(z)$, so that $\varphi(x, y) = \frac{1}{\pi}V[\pi - \theta(x, y)]$. With some algebra: $\cos\theta = \pm\frac{1}{2}[\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2}]$.
The equipotential lines are lines of constant $\theta(x, y)$, i.e. the images in z plane of radii in w plane: they are hyperbola with foci ± 1. The lines of force are the images of circles (constant r), i.e. ellipses with foci ± 1 (orthogonal to hyperbola and to the two cuts).

6.4 Point charge and semi-infinite conductor

Consider of a grounded semi-infinite conductor and a parallel wire at distance y', with linear charge density Q. The charged wire induces a surface charge that contributes to the electrostatic potential. In 2D the problem is that of a point charge Q in $z' = x' + iy'$ ($y' > 0$) in presence of the half plane Im $z \le 0$ at potential zero. For $y \to 0$, the lines of force are perpendicular to the real axis.

The evaluation of the field is done by replacing the semi-infinite conductor with an image charge $-Q$ in $\overline{z'}$. By symmetry, the real axis has constant (zero) potential. The complex field is:

$$\Phi(z) = -\frac{Q}{2\pi\epsilon_0}\log(z - z') + \frac{Q}{2\pi\epsilon_0}\log(z - \overline{z'}) + c \tag{6.5}$$

The real part is the electrostatic potential of the pair $\pm Q$, which is equivalent to the system "real charge and grounded conductor":

$$\varphi(x, y) = -\frac{Q}{4\pi\epsilon_0}\log\frac{(x - x')^2 + (y - y')^2}{(x - x')^2 + (y + y')^2} + \operatorname{Re} c$$

It is constant for $y = 0$, and equal to zero for Re $c = 0$. The complex electric field at the surface is

$$(E_x - iE_y)(x + i0) = \frac{Q}{2\pi\epsilon_0}\left[\frac{1}{x - z'} - \frac{1}{x - \overline{z'}}\right] = i\frac{Q}{\pi\epsilon_0}\frac{y'}{(x - x')^2 + y'^2}$$

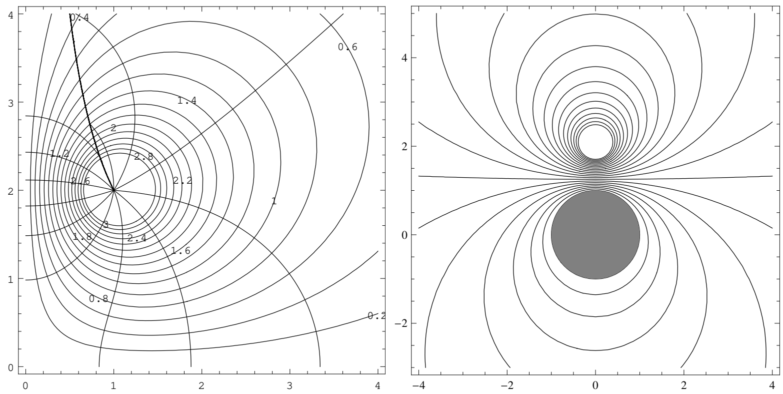

Figure 6.3: Left: Equipotential lines for a point charge in a right angle with conducting walls; right: equipotential lines of a point charge in (0,2) in presence of a conducting disk of unit radius.

Only E_y is non zero at the surface (as it has to be: field lines are orthogonal to equipotential lines). The surface charge density $\sigma(x) = \epsilon_0 E_y(x)$ is negative, and the total induced charge neutralizes the point charge:

$$Q_{ind} = \int_{-\infty}^{+\infty} dx\, \sigma(x) = -Q\frac{y'}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{(x-x')^2 + y'^2} = -Q.$$

The simple solution (6.5) is used to solve other problems, where a point charge is in presence of a different conducting boundary line. The trick is to map the different physical region Ω to the punctured upper half-plane $\mathbb{H}/\{z'\}$. The boundary of Ω is mapped to the half line. If the conformal map $\Omega_w \to \mathbb{H}$ is $z = z(w)$, the solution of the new electrostatic problem is

$$\Phi(w) = \Phi_{ref}(z(w))$$

where Φ_{ref} is (6.5). The new potential, being a composition of holomorphic functions, is holomorphic, and its real and imaginary parts are harmonic in Ω. The real part is the electrostatic potential $\varphi(u, v)$ of the new problem.

6.4.1 Point charge in an angle

Consider a point charge Q in w_0, in the first quadrant of the w-plane. The quadrant is bounded by conducting half-lines at potential $\varphi = 0$. The electrostatic problem is solved by mapping it to the solved problem in half-space.
The quadrant is flattened to the half-plane $\mathbb{H}$ by the map $z = w^2$. Therefore, the complex potential for the quadrant is $\Phi(w) = \Phi_{ref}(w^2)$, where $\Phi_{ref}(z)$ is (6.5). The electrostatic potential is the harmonic function $\varphi(u, v) = \varphi_{ref}(u^2 - v^2, 2uv)$:

$$\varphi(u, v) = -\frac{Q}{4\pi\epsilon_0}\log\frac{(u^2 - v^2 - u_0^2 + v_0^2)^2 + 4(uv - u_0 v_0)^2}{(u^2 - v^2 - u_0^2 + v_0^2)^2 + 4(uv + u_0 v_0)^2} \tag{6.6}$$

It is $\varphi(u, 0) = 0$ and $\varphi(0, v) = 0$. The equipotential lines are shown in fig.6.3.

6.4.2 Point charge and conducting half-line

In the z plane there are a point charge Q in z' and a conducting half-line (the real positive axis) at potential zero. Find the electrostatic potential and the induced charge density.

The map $z \to \sqrt{z}$ is a map from this region to the upper half-plane, where the reference model has been solved. The function $\sqrt{z}$ is holomorphic on $\mathbb{C}/[0,\infty)$. The solution of the problem is $\Phi(z) = \Phi_{ref}(\sqrt{z})$, where Φ_{ref} is eq.(6.5):

$$\Phi(z) = -\frac{Q}{2\pi\epsilon_0}\log\frac{\sqrt{z}-\sqrt{z'}}{\sqrt{z}-\sqrt{\overline{z'}}} \tag{6.7}$$

Being a composition of holomorphic functions, its real part (the electrostatic potential) is harmonic (save at z') and vanishes on the real line. The electric field is

$$\overline{E}(z) = \frac{Q}{4\pi\epsilon_0\sqrt{z}}\left[\frac{1}{\sqrt{z}-\sqrt{z'}} - \frac{1}{\sqrt{z}-\sqrt{\overline{z'}}}\right]$$

The charge density is $\sigma(x) = \epsilon_0[E_y(x+i\epsilon) + E_y(x-i\epsilon)]$ $(x>0)$; note that $\sqrt{x \pm i\epsilon} = \pm\sqrt{x}$. Then

$$\begin{aligned}\sigma(x) =& i\frac{Q}{4\pi\sqrt{x}}\left[\frac{1}{\sqrt{x}-\sqrt{z'}} - \frac{1}{\sqrt{x}-\sqrt{\overline{z'}}} + \frac{1}{\sqrt{x}+\sqrt{z'}} - \frac{1}{\sqrt{x}+\sqrt{\overline{z'}}}\right] \\ =& -Q\frac{1}{\pi}\frac{y'}{(x-x')^2+y'^2}\end{aligned}$$

The total induced charge is $-Q$.

6.4.3 Point charge and conducting disk

A point charge Q is at distance d from the center of a grounded disk of radius R $(d > R)$. The setting corresponds to a wire parallel to a conducting grounded cylinder. The problem of determining the complex field is solved by the conformal map that takes the solved reference problem in w-plane (point charge and half plane) to the present one in z-plane. Then, the solution is

$$\Phi(z) = \Phi_{\text{ref}}(w(z)) = -\frac{Q}{2\pi\epsilon_0}\log\frac{w(z)-w'}{w(z)-\overline{w'}}$$

Let id, $d > 0$ be the position of the point charge of the reference model. The map from the exterior of the disk $|z| > R$ to the upper half-plane Im $w > 0$, mapping $z = id$ to $w = id$, is the Möbius map:

$$w(z) = id\frac{d+R}{d-R}\frac{z-iR}{z+iR}$$

The complex potential generated by the disk and a point of charge Q in id is:

$$\Phi(z) = -\frac{Q}{2\pi\epsilon_0}[\log(z-id) - \log(z-iR^2/d) + \log(R/d)] \tag{6.8}$$

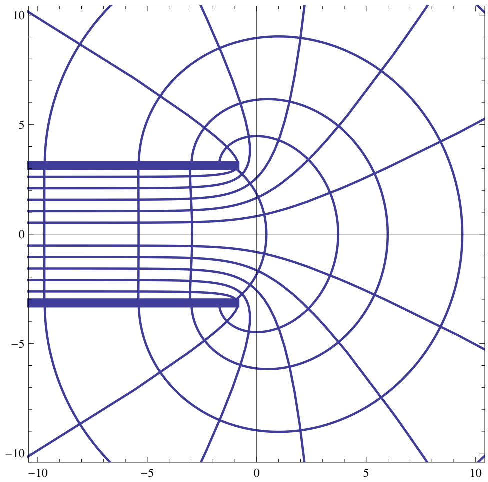

Figure 6.4: The semi-infinite planar capacitor.

It coincides with the potential of two point charges, (the actual charge Q at $z = id$ and an image charge $-Q$ at iR^2/d, inside the disk). The potential is

$$\varphi(x,y) = -\frac{Q}{4\pi\epsilon_0}[\log(x^2+(y-d)^2) - \log(x^2+(y-R/d)^2) + 2\log(R/d)$$

At the surface of the disk, $|z| = R$, the electrostatic potential φ is zero. The electric field is:

$$\overline{E}(z) = -\,\Phi'(z) = \frac{Q}{2\pi\epsilon_0}\left[\frac{x-i(y-d)}{x^2+(y-d)^2} - \frac{x-i(y-R^2/d)}{x^2+(y-R^2/d)^2}\right]$$

At the surface of the disk the electric field is radial, and proportional to a negative induced charge density.

6.5 The planar capacitor

Consider in 3D an infinite planar capacitor; its section in the complex $w = u+iv$ plane is the infinite strip between two conducting lines $v = \pm d/2$ at potentials $\pm V/2$. In the strip the electrostatic potential is linear, $\varphi(u,v) = v\,(V/d)$, and it is the real part of the complex potential

$$\Phi(w) = -i\,\frac{V}{d}\,w \qquad (6.9)$$

The complex electric field is uniform $\overline{E} = iV/d$, i.e. $E_y = -V/d$. The internal charge densities are $4\pi\sigma = -V/d$ on the lower plate and $4\pi\sigma = V/d$ on the upper one. This solution (infinite planar capacitor) can be used to study more complex geometries.

6.5.1 The semi-infinite planar capacitor

Consider the conformal map

$$z(w) = 2\pi(w/d) + e^{2\pi(w/d)}$$

It maps conjugate points $w, \overline{w}$ to conjugate points $z, \overline{z}$, then the x axis is a symmetry axis and we study the upper half. In components the map is

$$x = 2\pi\frac{u}{d} + e^{2\pi(u/d)}\cos\left(2\pi\frac{v}{d}\right), \qquad y = 2\pi\frac{v}{d} + e^{2\pi(u/d)}\sin\left(2\pi\frac{v}{d}\right).$$

The lines $(u, \pm d/2)$ are mapped to $x = \frac{2\pi}{d}u - e^{2\pi u/d}$, $y = \pm\pi$. Since $x'(u) = (2\pi/d)(1 - e^{2\pi u/d}) = 0$ in $u = 0$, the range of $x(u)$ is $(-\infty, -1]$. Therefore the map takes the lines $v = \pm d/2$ to the half-lines $x < -1$, $y = \pm\pi$.
The map takes the interior of an infinite capacitor (the strip $-d/2 < v < d/2$) to the interior and the exterior of a semi-infinite capacitor. If the plates have potentials $\pm V/2$, the complex potential of the semi-infinite capacitor is:

$$\Phi(z) = -i\frac{V}{d}\,w(z)$$

The solution is interesting for the study of the field near the aperture of a finite capacitor, and was obtained by Maxwell himself, by this conformal map.

The real axis $v = 0$ is mapped to $x = \frac{2\pi}{d}u + e^{2\pi u/d}$, $y = 0$, i.e. the real axis of the z plane. The lines parallel to the $u-$axis, $v = (d/2\pi)c$, $0 < c < \pi$, are mapped to lines that run inside the capacitor and, near the end of the capacitor, bend upward. If $c > \pi/2$ the lines fold back to the top of the condenser. The equation of such equipotential lines can be obtained by elimination of u in $x = 2\pi\frac{u}{d} + e^{2\pi(u/d)}\cos c$, $y = c + e^{2\pi(u/d)}\sin c$:

$$x = \log\frac{y - c}{\sin c} + (y - c)\cot c \tag{6.10}$$

Field lines are orthogonal to them and are the images of segments of constant u, $|v| < d/2$ inside the infinite capacitor: $2\pi u/d = c$; they are given parametrically (in $s \in [-\pi, \pi]$) by eqs. $x - c = e^c \cos s$ and $y - s = e^c \sin s$ (see note[3]).

The electric field is

$$\overline{E}(z) = i\frac{V}{d}\left(\frac{dz}{dw}\right)^{-1} = i\frac{V}{2\pi}\frac{1}{1 + e^{2\pi w(z)/d}}$$

Deep in the condenser the field is uniform: $E_x \approx 0$ and $E_y \approx -V/2\pi$. It diverges at $w = \pm id/2$, i.e. at the ends of the conducting boundaries of the capacitor $z = -1 \pm i\pi$. To avoid the occurrence of divergent fields near the tips of planar capacitors, one may deform the shape of the plates to trace equipotential lines $v = \pm(d/2\pi)c$, i.e. (6.10) (Rogowski's capacitor).

[3]Deep in the capacitor ($c \ll -1$) it is $x = c$, $-\pi < y < \pi$. For $c \gg 1$ the lines are circular arcs that end on the exterior of the capacitor's plates, $(x - c)^2 + y^2 = e^{2c}$ (s finite); for small s the lines $x - c \approx e^c - (1/2)y^2 e^{-c}$ are approximately straight segments inside the capacitor, near the aperture.

Chapter 7

COMPLEX INTEGRAL

7.1 Paths and Curves

A *path* is a continuous map of a real interval to the complex plane,

$$\gamma : [a, b] \to \mathbb{C}, \qquad (a < b)$$

The range of the map is a *curve* γ, and the path $\gamma(t)$ is its parametrization. The increasing value of the parameter assigns an orientation to the curve. Since intervals are compact sets in $\mathbb{R}$ and a path is a continuous function, the *curve* γ is a compact set[1] in $\mathbb{C}$.

We list some definitions that apply to paths, and imply geometric properties of the curves (that would be difficult to phrase otherwise):

- A path is *closed* if $\gamma(a) = \gamma(b)$ (we then say that the curve is closed).
- A path is *simple* if $\gamma(t_1) = \gamma(t_2) \Leftrightarrow t_1 = t_2$, $t_{1,2} \neq a, b$. The curve does not self-intersect, except possibly at the endpoints.
- A *Jordan curve* J is a simple closed curve. A theorem of topology (Jordan, Veblen) states that $\mathbb{C}/J$ is the union of two disjoint sets: one is bounded and the other is unbounded. They are the interior and the exterior of J.

The complex integral on a curve in $\mathbb{C}$, will be defined for "smooth curves" i.e. curves that are parametrised by *smooth* functions $\gamma(t)$:
$\gamma(t)$ is differentiable on $[a, b]$,
$\dot{\gamma}(t)$ is continuous on $[a, b]$ and $\dot{\gamma}(t) \neq 0$ on $[a, b]$.
The complex number $\dot{\gamma}(t) = \dot{x}(t) + i\dot{y}(t)$ gives the components of the vector tangent to the curve at $\gamma(t)$ (we shall often identify a complex number with a vector). The tangent vector never vanishes. The rotated vector $i\dot{\gamma}(t)$ is normal to the curve. If a curve is not simple, the geometric point where the curve self-intersects has two or more tangent vectors.

A path is *piecewise smooth* if it is continuous (it is a path) and is smooth on each subinterval of some finite partition of $[a, b]$.

[1]According to the Heine Borel theorem, a set in $\mathbb{C}$ is compact iff it is closed and bounded.

A curve can be given different parametrizations. For example the complex segment $[z_1, z_2]$ can be parameterized by $\gamma_1(t) = z_1 + t(z_2 - z_1)$ or by $\gamma_2(t) = z_1 + t^2(z_2 - z_1)$, $t \in [0, 1]$; both paths trace the same segment, but with different "time laws".
In general, let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ be a smooth path, and let $\tau(s)$ be a real map of $[a_2, b_2]$ to $[a_1, b_1]$ with $d\tau/ds > 0$, $\tau(a_2) = a_1$ and $\tau(b_2) = b_1$. The smooth path $\gamma_2(s) = \gamma_1(\tau(s))$ with $s \in [a_2, b_2]$ is a reparametrization of the same curve γ.

Two parametrizations $\gamma_1(\tau)$ and $\gamma_2(s)$ of a curve, at a geometric point of the curve with coordinates τ or s ($\tau(s) = \tau$), produce tangent vectors $\dot\gamma_1(\tau)$ and $\dot\gamma_2(s)$ with the same direction (if $d\tau/ds$ is positive):

$$\dot\gamma_2(s) = \dot\gamma_1(\tau)\,\frac{d\tau}{ds}, \qquad \tau(s) = \tau.$$

7.2 Complex integral

Given a **continuous complex function on a smooth oriented curve**, $f : \gamma \to \mathbb{C}$, the integral of f on γ is *defined* as:

$$\boxed{\int_\gamma f(z)\, dz = \int_a^b dt\, \dot\gamma(t) f(\gamma(t))} \tag{7.1}$$

where $\gamma(t)$ is a parametrization of the curve[2].
The value of the integral *does not depend on the parametrization of the curve.* If $\gamma_2(s) = \gamma_1(\tau(s))$ it is

$$\int_{a_2}^{b_2} ds\, \dot\gamma_2(s) f(\gamma_2(s)) = \int_{a_2}^{b_2} ds\, \dot\gamma_1(\tau)\dot\tau(s) f(\gamma_1(\tau(s))) = \int_{a_1}^{b_1} d\tau\, \dot\gamma_1(\tau) f(\gamma_1(\tau)).$$

Therefore, the integral of a function on an oriented curve is a geometric object. This is apparent in the equivalent construction of the integral sketched below.

Given an oriented curve γ from point z' to point z'', and a function f continuous on it, consider the sum

$$\sum_{k=0}^{n-1} (z_{k+1} - z_k)\, \frac{1}{2}[f(z_{k+1}) + f(z_k)]$$

where z_k are $n+1$ finitely spaced points of the curve, with $z_0 = z'$ and $z_n = z''$. Omitting technicalities, in the limits $n \to \infty$, $|z_{k+1} - z_k| \to 0$, the sum gives a parametrization-independent definition of an integral.
If a parametrization of the curve is used: $z_{k+1} - z_k = \gamma(t_{k+1}) - \gamma(t_k) \approx \dot\gamma(t_k)(t_{k+1} - t_k)$, and the sum yields the above defined integral.

If the curve γ is an interval of the real line $[a, b]$, a parameterisation is $\gamma(x) = x$, and the complex integral coincides with the Riemann integral of a continuous function $f : [a, b] \to \mathbb{C}$.

Note that the complex integral differs from the line integral:

$$\int_\gamma f(z)|dz| = \int_a^b dt \sqrt{\dot x^2 + \dot y^2}\, f(x(t) + iy(t))$$

[2]By expanding the product $\dot\gamma(t)\, f(\gamma(t)) = (\dot x + i\dot y)[u(x, y) + i\, v(x, y)]$ one obtains four real Riemann integrals that are well defined, since all functions are continuous on closed intervals.

Example 7.2.1. *Let's evaluate* $\int_\gamma z^2 dz$ *on the following curves:*
i) γ *is the real segment* $[a, b]$*. A parametrization is* $\gamma(x) = x$ *with* $a \le x \le b$*. Then* $\dot\gamma(x) = 1$ *and*

$$\int_{[a,b]} z^2 dz = \int_a^b dx\, x^2 = \tfrac{1}{3}(b^3 - a^3)$$

ii) γ *is the semicircle with diameter* $[a, b]$ *(from* a *to* b *counterclockwise). A parametrization is* $\gamma(\theta) = \frac{1}{2}(a + b) + \frac{1}{2}(b - a)e^{i\theta}$*,* $\pi \le \theta \le 2\pi$*. It is* $\gamma(\pi) = a$*,* $\gamma(2\pi) = b$*, and* $\dot\gamma = \frac{i}{2}(b - a)e^{i\theta}$

$$\int_\gamma z^2 dz = \tfrac{i}{2}(b - a) \int_\pi^{2\pi} d\theta e^{i\theta} \left[\tfrac{1}{2}(a + b) + \tfrac{1}{2}(b - a)e^{i\theta}\right]^2 = \tfrac{1}{3}(b^3 - a^3)$$

We use the integrals $I_k = \int_\pi^{2\pi} d\theta e^{ik\theta} = \int_\pi^{2\pi} d\theta[\cos(k\theta) + i \sin(k\theta)]$*. It is* $I_0 = \pi$*,* $I_k = 0$ *if* k *is even,* $I_k = -\frac{2i}{k}$ *if* k *is odd.*

Note that the two integrals give the same result, which only depends on the extrema of the curve.

If a curve γ is parametrized by $\gamma(t)$, $t \in [a, b]$, the curve with opposite orientation $-\gamma$ is parametrized by the path $\gamma(a+b-t)$ (on $[a, b]$). The integrals have opposite signs:

$$\begin{aligned}\int_{-\gamma} dz f(z) &= -\int_a^b dt\, \dot\gamma(a + b - t) f(\gamma(a + b - t)) \\ &= \int_b^a ds\, \dot\gamma(s) f(\gamma(s)) = -\int_\gamma dz f(z) \end{aligned} \tag{7.2}$$

7.2.1 Two useful inequalities

Proposition 7.2.2. *If the complex function* f *is integrable on a real interval* $I = [a, b]$*, then:*

$$\boxed{\left|\int_a^b dt f(t)\right| \le \int_a^b dt |f(t)|} \tag{7.3}$$

Proof. Let $\int_I f = e^{i\theta} |\int_I f|$, then

$$\left|\int_I f(t)dt\right| = \int_I e^{-i\theta} f dt = \int_I \mathrm{Re}(e^{-i\theta} f)\, dt \le \int_I |f| dt.$$

We used $\mathrm{Re} \int f dt = \mathrm{Re} \int (u + iv)dt = \int u\, dt = \int \mathrm{Re} f dt$. □

If a complex curve is parametrized as $\gamma(t)$, the real function $|f(\gamma(t))|$ is continuous on $[a, b]$ and has a maximum. The previous inequality gives:

Proposition 7.2.3 (Darboux's inequality). *If* f *is a complex continuous function on a smooth path* γ*, then:*

$$\boxed{\left|\int_\gamma dz f(z)\right| \le L(\gamma) \sup_{z\in\gamma} |f(z)|} \tag{7.4}$$

where $L(\gamma) = \int_a^b dt|\dot{\gamma}(t)|$ is the length of the path (it is a finite number as γ is required to have continuous derivative on $[a, b]$. Then $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are bounded).

Example 7.2.4. *Consider the integral $I = \int_{[0,2+i\pi]} dz\, e^z$. Darboux's inequality gives $|I| < \sqrt{4+\pi^2}\,\sup_{t\in[0,1]} |e^{t(2+i\pi)}| = e^2\sqrt{4+\pi^2}$. The exact value of the integral is obtained through the primitive (as explained below): $I = e^{2+i\pi} - e^0 = -e^2 - 1$, then $|I| = e^2 + 1$.*

7.3 Primitive

The evaluation of an integral is straightforward if the function has a primitive.

Definition 7.3.1. If $f(z)$ is continuous on a domain D, a primitive of f is a function $F(z)$ that is *holomorphic on D* such that

$$F'(z) = f(z), \qquad \forall z \in D$$

Proposition 7.3.2. *If f is continuous on D and has a primitive on it, and γ is a path in D, the integral of f on γ only depends on the endpoints of the path:*

$$\int_\gamma f(z)dz = \int_a^b dt\,\dot{\gamma}(t)\, f(\gamma(t)) = \int_a^b dt\,\dot{\gamma}(t) F'(\gamma(t))$$
$$= \int_a^b dt \frac{d}{dt} F(\gamma(t)) = F(\gamma(b)) - F(\gamma(a)). \qquad (7.5)$$

If the path is closed, $\gamma(b) = \gamma(a)$, then the integral is zero.

Example 7.3.3. *The function e^z is the primitive of e^z on $\mathbb{C}$; therefore $\int_\gamma dz e^z = e^b - e^a$ on any path from a to b.*

The existence of a primitive as a holomorphic function is a delicate issue, as it is related to global properties of the domain:

Theorem 7.3.4. *Let $f(z)$ be a continuous function on an open connected set D. The following statements are equivalent:*
1) for any two points a and b in D the integral of f along a (piecewise) smooth path with endpoints a and b does not depend on the path;
2) the integral of f along any closed (piecewise) smooth path in D is zero;
3) there is a function $F(z)$ holomorphic at all points of D such that $F'(z) = f(z)$ at all points of D.

Proof. To prove that $1 \to 2$, $2 \to 1$ and $3 \to 1$ is trivial. The only non-trivial statement to prove is 1 implies 3. Define $F(z)$ as the integral of f from an arbirtary point $a \in D$ to z along a path; by hypothesis 1 the function depends on z (and a) but not on the path.
To prove holomorphy of F consider a disk with center z contained in D (D is open), choose $z + h$ in the disk (we'll take the limit $h \to 0$) and consider the path from a to $z+h$ obtained by prolonging the path from a to z by the segment $\sigma = [z, z + h]$. Parameterize σ as $\zeta(t) = z + ht$ ($d\zeta = hdt$). Then:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_\sigma d\zeta f(\zeta) = \int_0^1 dt f(z+ht) = f(z) + \int_0^1 dt\,[f(z+ht) - f(z)].$$

Since f is continuous: $\forall\epsilon > 0\ \exists\delta$ such that $|f(z+ht) - f(z)| < \epsilon$ if $t|h| < \delta$, then:

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \le \int_0^1 dt|f(z+ht) - f(z)| \le \epsilon$$

if $|h| \le \delta$, i.e. $F'(z) = f(z)$. □

7.4 The Cauchy transform

Let γ be a (piecewise) smooth path *open or closed* and f a continuous function on γ. The following integral,

$$G_1(z) = \int_\gamma \frac{d\zeta}{2\pi i}\frac{f(\zeta)}{\zeta - z}\ , \qquad z \notin \gamma, \tag{7.6}$$

is the *Cauchy transform* of f. It will play an important role in Cauchy's theory of holomorphic function, together with the integrals[3]

$$G_n(z) = \int_\gamma \frac{d\zeta}{2\pi i}\frac{f(\zeta)}{(\zeta - z)^n}\ , \qquad z \notin \gamma, \quad n = 2, 3, ... \tag{7.7}$$

Since the functions $f(\zeta)/(\zeta - z)^n$ are continuous on γ, the integrals exist.

Theorem 7.4.1. *The functions $G_n(z)$, $n = 1, 2, \ldots$ are holomorphic in $\mathbb{C}/\gamma$ and $G'_n(z) = n\, G_{n+1}(z)$.*

Proof. The proof is first given for $n = 1$. Let $z \in \mathbb{C}/\gamma$. Being $\mathbb{C}/\gamma$ an open set, there is an open disk centred in z of radius δ_z that is not crossed by γ. Let $z+h$ belong to such disk and evaluate:

$$\begin{aligned} G_1(z+h) - G_1(z) &= \int_\gamma \frac{d\zeta}{2\pi i} f(\zeta) \left[\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right] \\ &= h\int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} \end{aligned}$$

Divide by h and subtract $G_2(z)$:

$$\begin{aligned} \frac{G_1(z+h) - G_1(z)}{h} - G_2(z) &= \int_\gamma \frac{d\zeta}{2\pi i} \left[\frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2}\right] \\ &= h\int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)^2} \end{aligned}$$

By the Darboux inequality with $M = \sup_{z\in\gamma}|f(z)|$:

$$\left|h\int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)^2}\right| \le |h|\frac{L(\gamma)M}{2\pi} \sup_{\zeta\in\gamma} \frac{1}{|\zeta - z - h||\zeta - z|}$$

For all $\zeta \in \gamma$ it is: $|\zeta - z| > \delta_z$ and $|\zeta - z - h| \ge \big||\zeta - z| - |h|\big| = |\zeta - z| - |h| \ge \delta_z - |h|$. Then:

$$\left|h\int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)^2}\right| \le \frac{L(\gamma)M}{2\pi}\frac{|h|}{\delta_z(\delta_z - |h|)} \to 0$$

[3]see Lars Ahlfors, Complex Analysis, 3rd ed. McGraw-Hill, p.121.

as $|h| \to 0$. Then $G_1(z)$ is holomorphic and $G_1' = G_2$.
For $n > 1$ the proof runs similarly, and can be led to the result:

$$\begin{aligned} G_n(z+h) - G_n(z) &= \int_\gamma \frac{d\zeta}{2\pi i} f(\zeta) \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] \\ &= \int_\gamma \frac{d\zeta}{2\pi i} f(\zeta) \frac{(\zeta - z)^n - (\zeta - z - h)^n}{(\zeta - z - h)^n (\zeta - z)^n} \\ &= nh \int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - z - h)^n (\zeta - z)} + \mathcal{O}(h^2), \end{aligned}$$

Now divide by h and subtract $nG_{n+1}(z)$. It remains to show that

$$n \int_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right]$$

vanishes for $h \to 0$, as in the case $n = 1$. □

7.5 Index of a closed curve

7.5.1 An instructive integral

Let C be the (anticlockwise) circle with center 0 and radius r. The following integral is computed in polar coordinates for any integer n, and is independent of the radius:

$$\oint_C \frac{dz}{z^n} = 2\pi i \, \delta_{n,1} \tag{7.8}$$

What is special about $1/z$ to give a non-zero result?
The functions $1/z$, $1/z^2$, $1/z^3$, ... are holomorphic in the punctured plane $\mathbb{C}_0 = \mathbb{C}/\{0\}$, but with a difference: a primitive in $\mathbb{C}_0$ (where the circle runs) exists for all of them, but not for $n = 1$.
$\log z$ is a local primitive of $1/z$, but it has branch cut that joins the origin to infinity, and C always crosses the cut.
If a ray from 0 (included) to ∞ is cut away from $\mathbb{C}$, then the log with branch cut along the ray is a primitive of $1/z$ in the new domain $\mathbb{C}$/ray.
Consider the open curve C_a obtained by removing the intersection (call it a) of the ray with C. The integral on C_a is the difference of primitives at the sides of the cut:

$$\int_{C_a} \frac{dz}{z} = \log a^+ - \log a^- = 2\pi i$$

$2\pi i$ is the discontinuity of $\log z$ across the cut. The result *does not depend* on the curve joining the two points: the curve C_a may be deformed to γ_a (with fixed a). Since only a point has been removed, where $1/z$ is continuous, the integrals on C, C_a and γ coincide. Therefore:

$$\oint_\gamma \frac{dz}{z} = 2\pi i$$

for any simple closed curve γ enclosing the origin. The contribution $2\pi i$ solely arises from the single crossing of the branch cut.
The result is invariant under a translation, and introduces the interesting topic of the following section.

7.5.2 The index of a closed curve

Definition 7.5.1. Let γ be a (piecewise) smooth closed oriented curve and $z \notin \gamma$. The Index (or winding number) of γ with respect to z is

$$\mathrm{Ind}(\gamma, z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{1}{\zeta - z} \tag{7.9}$$

The index is the Cauchy transform of $f(z) = 1$ on a closed curve. Therefore it is a holomorphic function on any open set not traversed by the curve. Indeed it is constant in each set:

$$\frac{d}{dz}\mathrm{Ind}(\gamma, z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{1}{(\zeta - z)^2} = 0$$

As $|z| \to \infty$, the index function vanishes and, since it is constant, it is zero everywhere in $\mathrm{Ext}(\gamma)$.

It can be easily understood that the constant is an integer that enumerates the number of windings of γ around the point z.
From the point z draw a half-line σ, with any direction that avoids self-intersections of the curve. The half-line crosses the curve in a number of points. Now, consider the domain $\mathbb{C}/\sigma$; in this domain the primitive $\frac{1}{2\pi i}\log(\zeta - z)$ (as a function of ζ) is holomorphic. The integral for the index is now evaluated for the curve with points removed along the chosen cut: it is the difference of the primitive at points at the two ends of each piece of the curve. Each difference is ± 1 (because of the discontinuity $\pm 2\pi i$ of the log), with $+1$ when the crossing is anticlockwise, and -1 for a clockwise crossing:

$$\boxed{\mathrm{Ind}(\gamma, z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{1}{\zeta - z} = N^+ - N^-} \tag{7.10}$$

where N^+ and N^- are the numbers of anticlockwise and clockwise turns of γ around z.

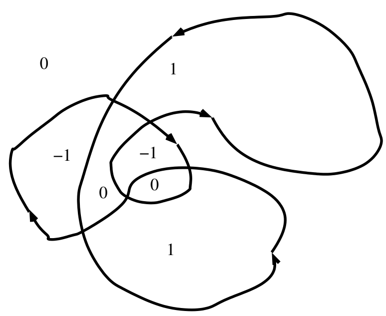

Figure 7.1: Indices of an oriented curve. To evaluate them, fix a point in a patch, draw a half-line left from it, and count the crossings from above (N_+) and from below (N_-).

7.5.3 2D Gauss theorem and Index function

The integral on a curve γ from a to b of the function $\overline{E}(z) = E_x - iE_y$ associated to the 2D electrostatic field $\mathbf{E} = E_x\mathbf{i} + E_y\mathbf{j}$ is:

$$\begin{aligned}\int_\gamma \overline{E}(z)dz &= \int dt(\dot{x} + i\dot{y})(E_x - iE_y) \\ &= \int dt(E_x\dot{x} + E_y\dot{y}) + i(E_x\dot{y} - E_y\dot{x}) \\ &= \int_\gamma \mathbf{E}(\mathbf{x})\cdot d\boldsymbol{\ell} + i\int_\gamma \mathbf{E}(\mathbf{x})\cdot\mathbf{n}\, d\ell \qquad (7.11)\end{aligned}$$

The real part is the work done by the electric field, the imaginary part is the flux of $\mathbf{E}$ through the curve, $\mathbf{n}$ is the vector normal to the curve.

Suppose that γ is contained in a domain where the complex potential Φ is holomorphic. Since $\overline{E} = -d\Phi/dz$, the top integral is $-\Phi(b) + \Phi(a)$. If the curve is closed, both the work and the flux are zero (no net charge is encircled).

Now a different situation: the closed curve encircles point-charges Q_i at z_i. The complex potential

$$\Phi(z) = -\frac{1}{2\pi\epsilon_0}\sum_i Q_i \log(z - z_i),$$

is holomorphic except for cuts joining z_i to ∞. Such discontinuities affect the imaginary part of the integral, while the real part of the integral (7.11) remains zero, i.e. the work done by the field on a closed line is zero. The evaluation of the integral gives an imaginary result (the flux):

$$\oint_\gamma dz\,\overline{E}(z) = -\oint_\gamma dz\frac{d\Phi}{dz} = \frac{1}{2\pi\epsilon_0}\sum_j Q_j \oint_\gamma \frac{dz}{z - z_j} = \frac{i}{\epsilon_0}\sum_j Q_j\, \mathrm{Ind}(\gamma, z_j)$$

Therefore

$$\int_\gamma \mathbf{E}(\mathbf{x})\cdot\mathbf{n}\, d\ell = \frac{1}{\epsilon_0}\sum_j Q_j\, \mathrm{Ind}(\gamma, z_j)$$

This is Gauss' theorem (the flux of the electric field through a closed curve is proportional to the total charge encircled).

Chapter 8

CAUCHY'S THEOREMS FOR RECTANGULAR DOMAINS

The fundamental theorem for the integral on closed curves of holomorphic functions was proven by Cauchy with the requirement that f' is continuous. The condition was relaxed by Goursat (1900), who proved Cauchy's theorem for a triangle. A simpler proof was then found for a rectangle, and is given below. The theorem will allow for the construction of a primitive and extend the theorem to arbitrary closed paths in a rectangular domain where the function is holomorphic. The Cauchy integral formula will then follow. For entire functions the theorems hold everywhere in $\mathbb{C}$.

Theorem 8.0.1. *If f is holomorphic on a domain, and R is a rectangle in the domain, with boundary ∂R:*

$$\boxed{\oint_{\partial R} dz f(z) = 0} \tag{8.1}$$

Proof. Let L be the length of the diagonal of R, and choose an orientation of the boundary. Halve the sides of R and obtain four rectangles R_{1k} (the index 1 stands for generation, $k = 1...4$); let $I(R_{1k})$ be the values of the integrals on the oriented boundaries of the rectangles. It is $I(R) = \sum_{k=1}^{4} I(R_{1k})$ because integrals on shared sides (they have opposite orientations) cancel. Denote I_1 the integral with largest modulus among the four, and R_1 the corresponding rectangle. Then $|I(R)| \leq \sum_k |I(R_{1k})| \leq 4|I_1|$.
Now repeat with R_1: a partition of R_1 into four rectangles R_{2k} gives $I_1 = \sum_{k=1}^{4} I(R_{2k})$. Select the rectangle R_{2k} with largest value $|I(R_{2k})|$, and let I_2 be the value of such integral, and R_2 the rectangle. It is $|I(R)| \leq 4^2|I_2|$.
By iterating the process, a sequence of rectangles $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ is selected. At generation n:

$$|I(R)| \leq 4^n |I_n|$$

The middle points of the rectangles form a Cauchy sequence whose limit point a belongs to the intersection of all the rectangles in the sequence.

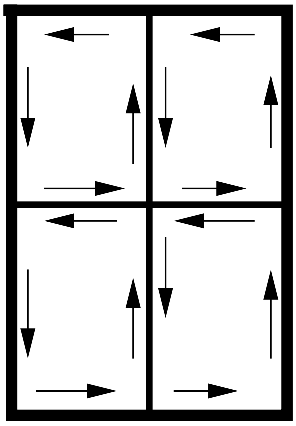

Figure 8.1: The integral on a rectangular path is the sum of four integrals on smaller rectangles; the contributions of oppositely oriented sides cancel.

Since $f(z)$ is holomorphic, it is true that given $\epsilon > 0$, there exists δ such that

$$\frac{f(z) - f(a)}{z - a} = f'(a) + r(z, a)$$

with remainder $|r(z, a)| < \epsilon$ for all z such that $|z - a| < \delta$. In R_n two points have separation at most $L/2^n$, therefore a rephrasing is: $\forall \epsilon \; \exists \, n$ (generation) such that $f(z) = f(a) + f'(a)(z - a) + r(z, a)(z - a)$ where $|r(z, a)| < \epsilon$ for all $z \in R_n$. This gives the estimate:

$$\begin{aligned} |I_n| &= \left| \oint_{\partial R_n} dz [f(a) + f'(a)(z - a) + r(z, a)(z - a)] \right| \\ &= \left| \oint_{\partial R_n} dz \, r(z, a)(z - a) \right| \leq \epsilon \times \text{perimeter of } R_n \times \sup_{z \in \partial R_n} |z - a| \\ &< \epsilon \frac{4L}{2^n} \frac{L}{2^n} \end{aligned}$$

Note that $\int_{\partial R} dz[f(a) + f'(a)(z - a)] = 0$ as a primitive exists. Since $|I(R)| \leq 4\epsilon \, L^2$ for arbitrary ϵ, it is $I(R) = 0$. $\square$

The hypothesis of the theorem can be weakened to allow for a function that is holomorphic up to a point (or a finite collection of points) in the domain, where the function remains continuous. This will be useful to prove the Cauchy integral formula.

Proposition 8.0.2. *Let $f(z)$ be a function that is continuous on a domain D and holomorphic on $D/\{a\}$. Then $\oint_{\partial R} dz f(z) = 0$ for any rectangle R in the domain.*

Proof. If $a \notin R$ Theorem 8.0.1 holds. If $a \in R$ decompose R as $R_a \cup R_1 \cup ... \cup R_k$ where R_a is a square with side ϵ that contains a (an interior or a boundary point of R), and $R_1 \dots R_k$ are rectangles that complete the partition. The contour integral is $I(R) = I(R_a) + \sum_i I(R_i)$. The integrals $I(R_i)$ are zero by Theorem

8.0.1. Since $|f(z)| < M$ on R (because f is continuous on the compact set R), Darboux's inequality gives $|I(R)| = |I(R_a)| \leq (4\epsilon)M$. □

Example 8.0.3 (Fourier transform of the Gaussian function).

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2-ikx} = \frac{1}{\sqrt{2}} e^{-\frac{1}{4}k^2}, \quad k \in \mathbb{R}} \tag{8.2}$$

Proof: consider the integral $\oint_{\partial R} dz e^{-z^2}$ *on the rectangle with corners* $-a$, b, $b+i(k/2)$ *and* $-a+i(k/2)$ $(a,b>0)$. *The integral is zero because the function is entire. The same integral is the sum of four integrals evaluated on the sides:*

$$0 = \int_{-a}^{b} dx\, e^{-x^2} + i\int_0^{\frac{k}{2}} dy\, e^{-(b+iy)^2} - \int_{-a}^{b} dx\, e^{-(x+\frac{1}{2}ik)^2} - i\int_0^{\frac{k}{2}} dy\, e^{-(-a+iy)^2}$$

For $a,b \to \infty$, *the integrals in* y *vanish, and the value of the first integral is* $\sqrt{\pi}$.

Exercise 8.0.4. *Evaluate the moments of the Gaussian distribution:*

$$\langle x^{2n} \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2} = \frac{(2n)!}{4^n n!} \tag{8.3}$$

(Hint: expand in power series of k *both sides of* (8.2) *and equate coefficients of equal powers). Prove the useful integral:*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2-xz} = e^{\frac{1}{4}z^2}, \qquad z \in \mathbb{C}. \tag{8.4}$$

8.1 Cauchy's theorem in rectangular domains

As a consequence of Cauchy's theorem for rectangles, within a rectangular domain R where a function f is holomorphic (up to a finite number of points where it remains continuous), a primitive F can be explicitly constructed.

The primitive in $z \in R$ is built as the integral of f along a curve joining a fixed reference point in R to z. The curve is made of two segments at right angles, parallel to the sides of the rectangle. Without loss of generality, we may assume that the rectangle has sides parallel to the axes and contains the origin. If $z = x+iy$ the path is $[0,x] \cup [x, x+iy]$ and

$$F(z) = \int_0^x dx' f(x') + i\int_0^y dy' f(x+iy')$$

Proposition 8.1.1. *$F(z)$ is holomorphic and $F'(z) = f(z)$ for all $z \in R$.*

Proof. Fix z and let $h = h_x + ih_y$; because of theorem 8.0.1, it is

$$\begin{aligned} F(z+h) &= \int_0^{x+h_x} dx' f(x') + i\int_0^{y+h_y} dy' f(x+h_x+iy') \\ &= F(z) + \int_x^{x+h_x} dx' f(x'+iy) + i\int_y^{y+h_y} dy' f(x+h_x+iy') \\ &= F(z) + \int_\gamma d\zeta f(\zeta) \end{aligned}$$

where the path γ is made of two segments at right angles: $[z, z+h_x] \cup [z+h_x, z+h]$. Divide by h and subtract $f(z)$ from both sides. Note that $\int_\gamma d\zeta = h$. Then:

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_\gamma d\zeta\, [\, f(\zeta) - f(z)\,]$$

By Darboux's inequality:

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \leq \sup_{z\in\gamma} |f(z) - f(\zeta)|\, \frac{|h_x|+|h_y|}{|h|} \leq 2\sup_{z\in\gamma}|f(z)-f(\zeta)|$$

Since f is continuous in z, the limit $h \to 0$ of the r.h.s. is zero. Then F' exists (i.e. F is holomorphic) in R and $F' = f$. □

With a primitive available at every point, we conclude that the integral of f on *any* closed (piecewise) smooth path in R is zero[1]:

Theorem 8.1.2 (Cauchy's theorem in rectangular domains)**.**

$$\boxed{\oint_\gamma dz f(z) = 0} \tag{8.5}$$

Exercise 8.1.3 (Fresnel integrals)**.** *Obtain the integrals:*

$$\int_0^\infty dx\, \cos(x^2) \;=\; \int_0^\infty dx\, \sin(x^2) \;=\; \frac{\sqrt{2\pi}}{4}$$

Hint: consider the null integral $\oint_C dz\, e^{iz^2}$ *on the closed path with sides* $[0, R]$, $[0, Re^{i\pi/4}]$ *and the circular arc* $\{Re^{i\theta},\ 0 \leq \theta \leq \pi/4\}$. *The integral on the arc is zero for* $R \to \infty$ (*use the inequality* $\sin\theta \geq 2\theta/\pi$ *valid for* $0 \leq \theta \leq \pi/2$).

Exercise 8.1.4. *Let* $F(z)$ *be the primitive of an entire function. Show that for any two points in* $\mathbb{C}$*:*

$$\left|\frac{F(b)-F(a)}{b-a}\right| \leq \sup_{z\in[a,b]} |F'(z)|$$

8.2 Cauchy's integral formula

Theorem 8.2.1 (Cauchy's integral formula). *If* f *is a holomorphic function on a rectangle* R, γ *is a (piecewise) smooth closed curve in* R *and* $a \in R/\gamma$, *then*

$$\boxed{\oint_\gamma \frac{d\zeta}{2\pi i}\frac{f(\zeta)}{\zeta - a} \;=\; f(a)\,\mathrm{Ind}(\gamma, a), \qquad a \notin \gamma} \tag{8.6}$$

where Ind (γ, a) *is the index of the curve at the point* a.

[1] This property was known to Gauss, who announced it in a letter to Bessel in 1811 [Boyer]. Gauss refrained from publishing several results he obtained, if not fully developed. His motto was *pauca sed matura*.

Proof. Consider the function

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(a)}{\zeta - a} & \zeta \neq a, \\ f'(a) & \zeta = a. \end{cases}$$

g is continuous on $\mathbb{C}$ and holomorphic in $\mathbb{C}/\{a\}$. Cauchy's theorem on rectangles holds (a may belong to the boundary), and enables the introduction of a primitive in $\mathbb{C}/\{a\}$. Therefore the integral of g on a closed path is zero:

$$0 = \oint_\gamma d\zeta \; g(\zeta) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - a} \; - \; f(a) \, \mathrm{Ind}(\gamma, a)$$

□

The Cauchy's integral formula is remarkable: it shows that the values of a function holomorphic on a bounded region are determined by the values at the boundary of this region!
It also shows that the Cauchy transform on a closed path of a holomorphic function coincides with the function itself, times the Index function. Since we proved in general that the Cauchy transform of a continuous function is holomorphic on $\mathbb{C}/\gamma$, the following important theorem follows:

Theorem 8.2.2. *The derivative f' of a function f holomorphic on a rectangle R, is holomorphic on the same rectangle. In the same way, all derivatives $f^{(n)}$ exist on R and*

$$\frac{1}{n!} \mathrm{Ind}(\gamma, z) \; f^{(n)}(z) = \oint_\gamma \frac{dz'}{2\pi i} \frac{f(z')}{(z' - z)^{n+1}}, \qquad z \in R/\gamma \tag{8.7}$$

The following theorem is the converse of Cauchy's theorem 10.2, and is here rephrased for rectangles:

Theorem 8.2.3 (Morera[2]). *If the Cauchy integral of a continuous function f vanishes for all rectangular paths in R, then f is holomorphic on R.*

Proof. The property of vanishing integral on all rectangular paths implies that a primitive exists, i.e. a function F holomorphic on R such that $F'(z) = f(z)$ for all $z \in R$. Being F holomorphic, also F' i.e. f are holomorphic. □

Proposition 8.2.4 (Mean value theorem). *$f(z)$ is the mean value of f on any circle in R centred in z:*

$$f(z) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(z + re^{i\theta}) \tag{8.8}$$

Proof. Apply Cauchy's formula to a circle with center z and radius r. □

Example 8.2.5. *Let γ be the ellipse $|z-1| + |z+1| = 4$, consider the integral*

$$\oint_\gamma dz \frac{e^{\pi z}}{2z - 3i}$$

The point $\frac{3}{2}i$ is in the ellipse, therefore the integral is $\frac{2\pi i}{2} \, e^{\pi(3i/2)} = \pi$.

[2]Giacinto Morera (1856, 1907).

Exercise 8.2.6. *Let γ be the circle $|z| = 3$. Show that*

$$\oint_\gamma dz \frac{\sin z}{z^2 - 3z + 2} = 2\pi i(\sin 2 - \sin 1)$$

Chapter 9

ENTIRE FUNCTIONS

Entire functions are holomorphic on the whole complex plane. Polynomials, the exponential and its linear combinations are entire functions. An entire function admits primitives (which differ by constants) which are again entire functions. The mean value theorem (8.8) implies the inequality

$$|f(z)| \le \sup_{\theta\in[0,2\pi]} |f(z+re^{i\theta})|$$

for any circle of radius r centred on z. This means that $|f(z)|$ has no peaks. Indeed, the following remarkable theorem holds:

9.1 Liouville theorem

Theorem 9.1.1 (Liouville's theorem). *If $f(z)$ is an entire function and $|f(z)| \le M$ for all $z \in \mathbb{C}$, then $f(z)$ is constant.*

Proof. Given a point z and a circle C with center 0 and radius $R > |z|$, the Cauchy integral formula gives:

$$f(z) - f(0) = \oint_C \frac{d\zeta}{2\pi i} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta}\right] = \oint_C \frac{d\zeta}{2\pi i}\frac{zf(\zeta)}{(\zeta - z)\zeta} = z\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{f(Re^{i\theta})}{Re^{i\theta} - z}$$

In the inequality

$$|f(z) - f(0)| \le |z|\int_0^{2\pi}\frac{d\theta}{2\pi}\left|\frac{f(Re^{i\theta})}{Re^{i\theta} - z}\right| \le |z|\, M \max_\theta \frac{1}{|Re^{i\theta} - z|} = M\,\frac{|z|}{R - |z|}.$$

R can be arbitrarily large, then $f(z) - f(0) = 0$ for all z, i.e. $f(z)$ is constant. □

In the proof, the boundedness of f is needed only on the path C. This leads to several generalizations, like this one:

Corollary 9.1.2. *If $f(z)$ is entire and $\lim_{z\to\infty} |f(z)| = 0$, then $f(z) = 0$.*

Proof. By hypothesis: $\forall\epsilon > 0\, \exists\, R_\epsilon$ such that $|f(z)| < \epsilon$ if $|z| > R_\epsilon$. Then for $R > R_\epsilon$ the Darboux inequality in the Cauchy formula gives $|f(z)| \le R/(R-|z|)\epsilon$. □

Suppose that f is entire and $f(z) - (az + b) \to 0$ for $|z| \to \infty$. Then $f(z) = az + b$ everywhere. More generally, if $f(z) - z^n \to 0$ at infinity, then $f(z)$ is a polynomial of degree n.

9.2 Picard's Little Theorem

The function $f(z) = z$ is entire and takes all complex values; the exponential function is entire and takes all complex values but one, the value zero. Is there a non-constant entire function that avoids two values? The answer is no:

Theorem 9.2.1 (Picard's little theorem[1]). *Let f be an entire function and a and b two distinct complex values such that, for all z, $f(z) \neq a$ and $f(z) \neq b$. Then $f(z)$ is constant.*

A sharpened version (Picard's great theorem) states that every transcendental[2] entire function f assumes every complex number *infinitely many times* with at most one exception[3].

9.3 Polynomials

Theorem 9.3.1 (The fundamental theorem of algebra). *A polynomial of order greater than zero has a zero.*

Proof. Suppose that a polynomial $p(z)$ of order greater than zero has no zeros. Then $1/p(z)$ is entire and vanishes for $z \to \infty$. By Cor.9.1.2 one gets the absurd result $1/p(z) = 0$ everywhere. □

The theorem implies that a polynomial of degree n with complex coefficients has exactly n zeros $z_1, \ldots, z_n$ in $\mathbb{C}$. Theorems for the location of zeros will be given in Sect.(14.4).
Let the coefficient of the leading power be $a_0 = 1$ (monic polynomial):

$$\begin{aligned} p(z) &= z^n + a_1 z^{n-1} + \ldots + a_n \\ &= (z - z_1)(z - z_2) \cdots (z - z_n). \end{aligned}$$

The two representations give useful relations among roots and coefficients:

$$\begin{aligned} \sigma_1 &= z_1 + z_2 + \ldots + z_n = -a_1 \\ \sigma_2 &= z_1 z_2 + z_1 z_3 + \ldots + z_{n-1} z_n = a_2 \\ &\ldots \\ \sigma_n &= z_1 z_2 \cdots z_n = (-1)^n a_n. \end{aligned}$$

The quantities σ_k are the elementary symmetric polynomials of $z_1, \ldots, z_n$. They are related to the sums $s_p = z_1^p + \ldots + z_n^p$ by *Newton's identities* ($s_1 = \sigma_1$, $s_2 = \sigma_1^2 - 2\sigma_2$, etc.)

Exercise 9.3.2. *Prove the following identity for polynomials with simple roots*

$$\boxed{\frac{1}{p(z)} = \sum_{k=1}^{n} \frac{1}{p'(z_k)} \frac{1}{z - z_k}} \tag{9.1}$$

[1] Charles E. Picard (1856, 1941).
[2] A function is transcendental if it's not algebraic. A function $\omega(z)$ is algebraic if it solves an equation $P(z, \omega) = 0$ where P is a polynomial both in z and ω.
[3] R. Remmert, *Classical topics in complex analysis*, GTM 172, Springer 1998.

Exercise 9.3.3. *If the roots are simple, show that:*

$$\frac{p''(z_k)}{p'(z_k)} = \sum_{j\neq k} \frac{2}{z_k - z_j}, \qquad \prod_k p'(z_k) = (-)^{\frac{1}{2}n(n-1)} \prod_{i>j}(z_i - z_j)^2. \tag{9.2}$$

Exercise 9.3.4. *Let $p_k(z)$ be monic polynomials of degree $k = 1, \dots, n-1$. Show that:*

$$\det \begin{bmatrix} 1 & p_1(z_1) & \cdots & p_{n-1}(z_1) \\ 1 & p_1(z_2) & \cdots & p_{n-1}(z_2) \\ \vdots & \vdots & & \vdots \\ 1 & p_1(z_n) & \cdots & p_{n-1}(z_n) \end{bmatrix} = \det \begin{bmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ 1 & z_2 & \cdots & z_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \cdots & z_n^{n-1} \end{bmatrix} = \prod_{i>j}(z_i - z_j)$$

The second matrix is the Vandermonde matrix, which is an important tool in linear algebra, matrix theory, and polynomial interpolation.

Exercise 9.3.5. *By expanding* (9.1) *in powers of* $1/z$ *and equating coefficients, obtain the identities for the roots:*

$$0 = \sum_{k=1}^{n} \frac{z_k^\ell}{p'(z_k)} \ (\ell = 0, \dots, n-2), \quad 1 = \sum_{k=1}^{n} \frac{z_k^{n-1}}{p'(z_k)}, \quad -a_1 = \sum_{k=0}^{n} \frac{z_k^n}{p'(z_k)}, \quad \dots$$

Exercise 9.3.6. *Prove the relations, valid for general polynomials:*

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}, \qquad \frac{p'(z)^2 - p''(z)p(z)}{p(z)^2} = \sum_{k=1}^{n} \frac{1}{(z - z_k)^2}$$

Exercise 9.3.7. *If $f(z)$ is an entire funtion and γ is a simple closed path enclosing the points $z = 0, 1, \dots, n$ show that:*

$$\oint_\gamma \frac{dz}{2\pi i} \frac{f(z)}{z(z-1)\cdots(z-n)} = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k)$$

Exercise 9.3.8. *Let the simple closed path γ contain the unit disk and let $f(z)$ be an entire function; show that:*

$$\oint_\gamma \frac{dz}{2\pi i} \frac{f(z)}{z^n - 1} = \frac{1}{n} \sum_{k=1}^{n} \omega^k f(\omega^k), \qquad \omega = \exp(i\tfrac{2\pi}{n})$$

Given a monic polynomial of order n with distinct zeros, the set $\{z : |p_n(z)| = c\}$ is a *lemniscate*. For $c = 0$ it consists of n points (the roots), by increasing c it evolves into n distinct ovals encircling the roots. As c is increased the ovals start to merge into fewer closed lines.
The length of the lemniscate $|p_n(z)| = 1$ is not greater than Kn, where $K \le 8\pi e$ (Borwein, 1995). The result was improved to $K \le 9.173$ (Eremenko and Hayman, 1999).

Cartan's Lemma: the inequality $|p_n(z)| > (R/e)^n$ holds outside at most n circular disks, the sum of the radii being at most $2R$.

Chapter 10

CAUCHY'S THEORY FOR HOLOMORPHIC FUNCTIONS

The extension of Cauchy's theory from rectangular domains to general domains requires care. If f is holomorphic on D and has singularities in $\mathbb{C}/D$, a closed curve in D may encircle them. In 1971 Dixon gave a proof of Cauchy's theorem and integral formula on general domains that solves the topological problem with the index function. The proof is elegant as it only requires Liouville's theorem for entire functions[1].

Theorem 10.0.1 (Dixon)**.** *Let γ be a piecewise smooth closed path in a domain D such that Ind $(\gamma, z) = 0$ $\forall z \notin D$. If f is holomorphic on D, $z \in D/\gamma$:*

$$\boxed{\operatorname{Ind}(\gamma, z)\, f(z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} \quad \text{(Cauchy's formula)}} \tag{10.1}$$

$$\boxed{0 = \oint_\gamma d\zeta\ f(\zeta) \qquad \text{(Cauchy's theorem)}} \tag{10.2}$$

Proof. Consider the function $g : D \times D \to \mathbb{C}$

$$g(\zeta, z) = \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{for } \zeta \neq z, \qquad g(z, z) = f'(z)$$

Since g is continuous in each variable, the following function exists $\forall z \in \mathbb{C}$:

$$h(z) = \oint_\gamma \frac{d\zeta}{2\pi i}\, g(\zeta, z) \qquad \text{if } z \in D \tag{10.3}$$

$$= \oint_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} \qquad \text{if } z \notin D \tag{10.4}$$

[1]John D. Dixon, *A brief proof of Cauchy's integral theorem*, Proc. Am. Math. Soc. 29 n.3 (1971) 625-26 (`https://www-users.cse.umn.edu/~brubaker/docs/8701-F13/dixon.pdf`). Peter A. Loeb, *A note on Dixon's proof of Cauchy's Integral Theorem*, Am. Math. Month. 98 n.3 (1991) 242-244. See the textbook by Lang for a presentation of Dixon's proof.

If $z \in D/\gamma$ the first expression is

$$h(z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} - f(z)\mathrm{Ind}(\gamma, z)$$

Where the index is zero, it coincides with the second expression of $h(z)$, which is the Cauchy transform of f. Therefore $h(z)$ is holomorphic wherever the index is zero, and $h(z) \to 0$ as $|z| \to \infty$.
The key point (not proven here) is that h is entire. Then, by Liouville's theorem (Cor. 9.1.2), it is $h(z) = 0$ on $\mathbb{C}$, and (10.1) is proven.

To obtain (10.2), choose $a \in D/\gamma$ and apply (10.1) to the function $(z-a)f(z)$:

$$\mathrm{Ind}(\gamma, z)\,(z - a)f(z) = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{\zeta - a}{\zeta - z} f(\zeta).$$

Now let $z = a$ and Cauchy's theorem follows. □

Corollary 10.0.2. *A holomorphic function can be differentiated indefinitely on its domain.*

Exercise 10.0.3. *Use Cauchy's formula to evaluate the integral (Hint: $z = e^{i\theta}$):*

$$\int_0^{2\pi} d\theta \frac{1}{1 - 2x\cos\theta + x^2} = \frac{2\pi}{1 - x^2}, \qquad 0 \le x < 1. \tag{10.5}$$

Chapter 11

POWER SERIES

11.1 Uniform and normal convergence

We consider sequences of functions f_n on some abstract set E to $\mathbb{C}$. At each point $P \in E$ there is a complex sequence $f_n(P)$, whose convergence is assessed by Cauchy's criterion.

• **Point-wise convergence on E**. Suppose that $\forall\, P \in E$ the sequence $f_n(P)$ converges to a finite complex number. The set of limit values defines a function $f : E \to \mathbb{C}$, and we say that $f_n \to f$ point-wise on E.
Since $\mathbb{C}$ is complete, we only need Cauchy's criterion for convergence:

$$\forall P \in E \;\; \forall \epsilon > 0 \;\; \exists N_{\epsilon,P} \text{ such that } |f_n(P) - f_m(P)| < \epsilon \;\; \forall\, m, n > N_{\epsilon,P}$$

• **Uniform convergence on E**. Suppose that the stronger condition occurs: $\forall\, P \in E$ the sequence $f_n(P)$ is Cauchy with N_ϵ *independent* of P. Then f_n converges on E, and the distance $|f_n(P) - f_m(P)|$ is bounded on E by the same ϵ for *all* $P \in E$, $n, m > N_\epsilon$. We say that $f_n \to f$ uniformly on E:

$$\forall \epsilon > 0 \;\; \exists N_\epsilon \text{ such that } |f_n(P) - f_m(P)| < \epsilon \;\; \forall\, m, n > N_\epsilon, \;\; \forall P \in E.$$

Definition 11.1.1. A series of functions $\sum_k f_k$ is uniformly convergent on E if the sequence of partial sums is uniformly convergent on E:

$$\forall \epsilon > 0 \quad \exists N_\epsilon : \left| \sum_{k=m+1}^{m+n} f_k(P) \right| < \epsilon \quad \forall m > N_\epsilon, \quad \forall n > 0, \quad \forall P \in E. \tag{11.1}$$

Theorem 11.1.2 (Weierstrass M-criterion). *If there are positive constants M_k such that $|f_k(P)| < M_k$ for all $P \in E$ and $\sum_k M_k$ is finite, then the series $\sum_k f_k$ is uniformly convergent on E.*

Proof. Let $S_m(P)$ be the sequence of partial sums. For all $P \in E$:

$$|S_{m+n}(P) - S_m(P)| \le \sum_{k=m+1}^{m+n} |f_k(P)| \le \sum_{k=m+1}^{m+n} M_k < \epsilon$$

for m large enough and all n, because the $M-$series converges. □

Proposition 11.1.3. *The geometric series* $\sum_k z^k$ *is uniformly convergent on the closed disk* $|z| \le 1-\eta$, $\eta > 0$.

Proof. Use the $M-$criterion with $M_k = (1-\eta)^k$. □

From now on, E is a set in $\mathbb{C}$. Uniform convergence of a series guarantees that integration can be made term by term:

Theorem 11.1.4 (Integral of series)**.** *Given a piecewise smooth curve* γ *in a domain* D*, a sequence* f_k *of functions continuous on* γ*, suppose that* $\sum_k f_k$ *is uniformly convergent on* γ*, then:*

$$\int_\gamma dz \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \int_\gamma dz\, f_n(z) \tag{11.2}$$

Proof. The partial sums $S_m(z)$ are continuous functions on γ which is a compact set, then uniform convergence implies that the limit series $S(z) = \sum_{k=0}^{\infty} f_n(z)$ is continuous on γ, and the integral exists. Uniform convergence on γ means that $|S_m(z) - S(z)| < \epsilon$ for all $m > N$ and for all $z \in \gamma$. Darboux's inequality shows that summation and integration commute:
$\left|\sum_{k=0}^{m} \int_\gamma dz f_k(z) - \int_\gamma dz\ \sum_{k=0}^{\infty} f_k(z)\right| \le \int_\gamma |dz|\, |S_m(z) - S(z)| < \epsilon\, L(\gamma)$. □

Since derivatives of holomorphic functions can be evaluated as Cauchy integrals, the theorem adds another important property:

Theorem 11.1.5 (Derivative of series)**.** *Let* f_n *be a sequence of holomorphic functions on a domain* D*. If* $S = \sum_n f_n$ *is uniformly convergent on* D *then* S *is holomorphic on* D *and* $S' = \sum_k f_k'$*.*

Proof. The function S, being a uniform limit of continuous partial sums S_n, is continuous on D. Then the integral of S on a curve in D exists. Since for any closed curve in D the integrals of S_n vanish, by the previous theorem, also the integral of S is zero. Therefore S is holomorphic by Morera's theorem.
The derivative of S_n in z is given by Cauchy's formula on a loop encircling z:

$$S'(z) = \oint_\gamma \frac{dz'}{2\pi i} \frac{S(z')}{(z'-z)^2} = \lim_{N\to\infty} \sum_{k=1}^{N} \oint_\gamma \frac{dz'}{2\pi i} \frac{f_k(z')}{(z'-z)^2} = \sum_{k=1}^{\infty} f_k'(z)$$

The same holds for higher order derivatives. □

11.1.1 Normal convergence

In some cases the requirement of uniform convergence on a domain is too restrictive. Theorem 11.1.4 and derivation term by term, can be proven under the weaker hypothesis of local uniform convergence (normal convergence).

Definition 11.1.6. A sequence of functions f_n converges *normally* to f on a domain D if it converges point-wise to f on D and, for any $z \in D$, there is a *closed* disk centred in z where convergence is uniform.

Theorem 11.1.7 (Integral of series)**.** *Given a piecewise smooth curve γ and a sequence of continuous functions f_k on D, suppose that the series $\sum_k f_k(z)$ is normally convergent on γ. Then*

$$\int_\gamma dz \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \int_\gamma dz f_k(z) \tag{11.3}$$

Theorem 11.1.8. *If a sequence f_k is normally convergent to f on a domain D and all functions f_k are holomorphic, then f is holomorphic and the sequence of n-th derivatives $f_k^{(n)}$ converges to $f^{(n)}$ normally.*

Theorem 11.1.9 (Derivative of series)**.** *If the series $\sum_k f_k$ is normally convergent on a domain D, and every term f_k is holomorphic on D, then the series is holomorphic on D. Moreover, the series can be differentiated term by term any number of times:*

$$\frac{d^n}{dz^n} \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \frac{d^n f_k}{dz^n}(z), \qquad n = 1, 2, \dots \tag{11.4}$$

for all $z \in D$, and each differentiated series is normally convergent.

11.2 Power series

A fundamental class of series of complex functions are power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \tag{11.5}$$

$a \in \mathbb{C}$ is the *center* of the power series, the coefficients c_n are complex numbers. Two notable examples are the geometric and exponential series. They converge absolutely, the first one in the open disk $|z| < 1$, the second one everywhere.
This is a fundamental theorem:

Theorem 11.2.1 (Abel, Weierstrass). *If the power series $\sum_k c_k(z-a)^k$ is convergent at a point z_0, then the series converges:*
1) absolutely for all z in the open disk $|z-a| < |z_0 - a|$,
2) uniformly in the closed disk $|z-a| \le (1-\eta)|z_0 - a|$, $\eta > 0$.

Proof. 1) We use the comparison criterion. Convergence of the series in z_0 implies that $|c_k(z_0-a)^k| \to 0$; then, there is a finite N such that $|c_k(z_0-a)^k| < 1$ for all $k > N$. This means that $|c_k(z-a)^k|$ is majorized by $|(z-a)/(z_0-a)|^k$. The sum of the latter terms converges (absolutely) for all z in the open disk $|z-a| < |z_0 - a|$.
2) If $|z-a| \le |z_0-a|(1-\eta)$ it follows that, for $k > N$, it is $|c_k(z-a)^k| \le (1-\eta)^k$. Then the convergence is uniform by the Weierstrass' M-criterion. □

The theorem shows that one out of three possibilities occurs:

- $z = a$ is the only point where the series converges.
- There are points $z \neq a$ where the series converges, but the series diverges at other points.

- The series converges everywhere in $\mathbb{C}$.

The Abel - Weierstrass theorem shows that convergence in z implies convergence in any open disk centred in a of radius $r < |z-a|$. Case two implies the existence of a radius R such that for $|z-a| < R$ the series converges absolutely (and uniformly in any closed disk strictly contained in it) and diverges in $|z-a| > R$. The value R is the **radius of convergence** of the series. If $R = \infty$ the series converges absolutely everywhere, and uniformly on any closed bounded set.
In general nothing can be said about the series on the circle $|z-a| = R$ [1].

We give an important formula for the radius R, which results from the sufficient criteria for absolute convergence of series.

Theorem 11.2.2 (Cauchy - Hadamard)**.**

$$\boxed{\frac{1}{R} = \limsup_k \sqrt[k]{|c_k|}} \tag{11.6}$$

If $\lim_{k\to\infty} \sqrt[k]{|c_k|}$ exists, it is equal to $1/R$.

Example 11.2.3. *In $\sum_{k=1}^{\infty}(2z)^{2k}$ odd powers are missing and the sequence $\sqrt[n]{c_n} = 2, 0, 2, 0, \ldots$ is not convergent, but its $\limsup$ is 2. Then the series converges in the disk $|z| < \frac{1}{2}$ (as a geometric series, it converges for $|4z^2| < 1$).*

Exercise 11.2.4. *Evaluate the radius of convergence of the series:*

$$\sum_{k=1}^{\infty} \frac{z^k}{k}, \qquad \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \qquad \sum_{k=1}^{\infty} (-1)^k z^{3k}, \qquad \sum_{k=0}^{\infty} z^{k^2}, \qquad \sum_{k=0}^{\infty} k^2 z^k$$

Theorem 11.2.5. *The power series*

$$S(z) = \sum_{k=0}^{\infty} c_k\,(z-a)^k \quad \textit{and} \quad S'(z) = \sum_{k=1}^{\infty} k\,c_k\,(z-a)^{k-1}$$

have the same radius of convergence.

Proof. In the proof we put $a = 0$. Suppose that S and S' have radii R and R' respectively. Since $|c_k z^k| < |z|\,|k c_k z^{k-1}|$ for all z, by the comparison test, the series S converges absolutely if S' does, i.e. $|z| < R'$ is a sufficient condition for S to absolutely converge. Then $R \geq R'$.
For z such that $|z| < R$ it is $k|z|^{k-1} < \frac{R^k}{R-|z|}$. Then it is $|k c_k z^{k-1}| < \frac{1}{R-|z|}|c_k R^k|$ and the series S' converges absolutely if S does, i.e. $R \leq R'$.
We used the inequality $\frac{1}{1-r} > 1 + r + \ldots r^{n-1} > n r^{n-1}$, $0 \leq r < 1$. ☐

Corollary 11.2.6. *A power series $\sum_n c_n(z-a)^n$ is a holomorphic function on the open disk of convergence (where it is also infinitely many times differentiable).*

Exercise 11.2.7. *Evaluate $\sum_{n=1}^{\infty} n^2 z^n$ (Note that $z\dfrac{d}{dz}z^n = nz^n$).*

[1] However, if the series converges in a point z_0 with $|z_0 - a| = R$, a theorem by Abel proves that on the radius $\zeta(t) = a + (z_0 - a)t$ ($t \in [0,1]$) the series $f(t) = \sum c_k (z_0 - a)^k t^k$ converges uniformly with respect to t and $\lim_{t\to 1} f(t) = f(1)$.

Theorem 11.2.8 (Power series for holomorphic functions). *Let f be holomorphic in a domain, and let D be an open disk centred in a of radius r in the domain, with (positively oriented) boundary C. Then, for any z in the disk, f has the power series expansion*

$$\boxed{f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k, \qquad c_k = \oint_C \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - a)^{k+1}} = \frac{1}{k!} f^{(k)}(a)} \tag{11.7}$$

Proof. For any z in the disk $D(a,r)$, $f(z)$ is given by the Cauchy integral

$$f(z) = \oint_C \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z}.$$

Since $r = |\zeta - a| > |z - a|$, the kernel admits an expansion in geometric series

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z-a)} = \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta - a)^{k+1}} \tag{11.8}$$

The sum can be taken out of the integral because the series converges uniformly in the interior of the disk. □

Corollary 11.2.9. *A holomorphic function can be differentiated indefinitely and*

$$f^{(k)}(a) = k! \oint_{C(a,r)} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \tag{11.9}$$

Darboux's inequality gives *Cauchy's Inequality*

$$|f^{(k)}(a)| \le \frac{k!}{r^k} \sup_{\theta \in [0,2\pi]} |f(a + re^{i\theta})| \tag{11.10}$$

Remark 11.2.10. *The radius of convergence of the power series centred in a of an analytic function is the largest admissible radius of a disk centred in a in the domain of analyticity: $R = |\xi - a|$ where ξ is the singular point or branch cut point nearest to a.*

Example 11.2.11. *The function $f(z) = [(z-2i)(z-3)]^{-1}$ is singular at $z = 2i$ and $z = 3$. A power expansion in the origin has radius 2. An expansion with center $a = 1$ has radius 2 because the singular point 3 has distance $|3-1| = 2$ smaller than the distance $|2i - 1| = \sqrt{5}$ of the singular point $2i$.*

Exercise 11.2.12. *Integrate term by term the geometric series to obtain:*

$$\boxed{\mathrm{Log}(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} \qquad |z| < 1} \tag{11.11}$$

Exercise 11.2.13. *From the expansion* (11.11) *obtain:*

$$\sum_{k=1}^{\infty} \frac{\rho^k}{k} \cos(k\theta) = -\frac{1}{2} \log(1 - 2\rho\cos\theta + \rho^2), \quad \rho < 1.$$

Example 11.2.14. *The function* $\operatorname{Log} z$ *has the branch cut* $(-\infty, 0]$*. A power series of Log z with center a is initially constructed in a disk* $D(a,r)$ *that avoids the cut, where Log z is holomorphic. If Re* $a > 0$ *the radius is* $r = |a|$ *(the least distance of a from the cut), if Re* $a < 0$ *this least distance is* $r = |\operatorname{Im} a|$*. Inside the disk* $D(a,r)$*:*

$$\operatorname{Log} z = \operatorname{Log} a + \operatorname{Log}\left(\frac{z}{a}\right) = \operatorname{Log} a + \operatorname{Log}\left(1 + \frac{z-a}{a}\right)$$

with no additional terms $\pm 2\pi i$ *(to check this requires some simple work). The expansion* (11.11) *gives:*

$$\operatorname{Log} z = \operatorname{Log} a - \sum_{k=1}^{\infty} \frac{(-1)^k}{ka^k}(z-a)^k$$

The right hand side is meaningful in a disk of convergence $D(a,|a|)$ *and does not distinguish the cases* $r = \operatorname{Im} a$ *or* $r = |a|$*. However, if z is taken such that* $|z| < |a|$ *and the line of sight* $[a,z]$ *is crossed by the cut, the left hand side differs from the right hand side by* $\pm 2\pi i$*. We are thus evaluating* $\log z$ *on a different sheet.*

Example 11.2.15. *Evaluate the coefficients of the power series*

$$\frac{e^{tz}}{1-z} = \sum_{n=0}^{\infty} c_n(t) z^n$$

It is convenient to compare the series $e^{tz} = (1-z)\sum_n c_n z^n = \sum_n z^n(c_n - c_{n-1})$*, with* $c_0 = 1$*,* $c_{-1} = 0$*. This gives the recursive rule* $c_n - c_{n-1} = t^n/n!$ *i.e.*

$$c_n(t) = 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}.$$

Example 11.2.16. *Evaluate the coefficients of the power series*

$$\frac{1}{z^2+z-1} = \sum_{n=0}^{\infty} c_n z^n$$

The radius of convergence of the series is dictated by the size of the smallest root of the binomial: $R = \frac{1}{2}(\sqrt{5}-1)$*. The quick way to obtain the coefficients is to write the l.h.s. as a combination of two geometric series. However, the following procedure is interesting.*
Multiply the series by the denominator to get $1 = \sum_n z^n(c_{n-2} + c_{n-1} - c_n)$ *i.e.* $c_n = c_{n-1} + c_{n-2}$ *and* $c_0 = 1$ *(*$c_{-1} = 0$*). The recursion generates the Fibonacci numbers* $1, 1, 2, 3, 5, 8, 13, \ldots$*. The general solution can be found:* $c_n = ax_1^n + bx_2^n$*, where* $x_{1,2}$ *solve the quadratic equation* $x^2 - x - 1 = 0$ *and a, b are specified by the initial conditions.*
If x_1 *is the root with largest modulus, Hadamard's criterion for the radius of the power series gives* $1/R = |x_1|$*, i.e.* $R = |x_2|$ *(note that* $x_1 x_2 = 1$*).*
The recursion can be written $(c_n/c_{n-1}) = 1 + (c_{n-2}/c_{n-1})$*. In the large n limit* $c_n/c_{n-1} \to \phi$ *and* $\phi = 1 + 1/\phi$ *(then, by the ratio criterion,* $R = 1/\phi$*). The number* $\phi = \frac{1}{2}(\sqrt{5}+1)$ *is the* golden mean*, and is the limit of ratios of Fibonacci numbers. Its continued fraction expansion is peculiar,* $\phi = 1 + 1/(1 + /(1 + /(1 + \ldots)$*, and makes this number the "most irrational" one (Hurwitz developed a theory for rational approximation of irrationals).*

Example 11.2.17 (Euler numbers). *They are the coefficients E_{2n} in the power series*

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{z^{2n}}{(2n)!}$$

The radius of convergence of the series is $\pi/2$ (the pole closest to the origin). This gives us the estimate $|E_{2n}| \approx (2n)!(2/\pi)^{2n}$ (up to factors that grow less than powers of n)[2]*. Multiplication by $\cos z$ gives a Cauchy product of series:*

$$1 = \sum_{k=0}^{\infty} z^{2k}\frac{(-1)^k}{(2k)!}\sum_{\ell=0}^{k}\binom{2k}{2\ell}E_{2\ell}$$

Comparison of powers in z gives, for the power $k=0$: $E_0 = 1$. Higher powers are absent in the l.h.s. therefore:

$$\sum_{\ell=0}^{k}\binom{2k}{2\ell}E_{2\ell} = 0, \quad k \geq 1.$$

Example 11.2.18 (Bernoulli numbers). *They are defined by the power series*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

The radius of convergence of the series is 2π. If we subtract the series evaluated at $-z$, we get $-z = 2\sum_{\text{odd}} B_n\, z^n/n!$. Therefore $B_1 = -\frac{1}{2}$ and $B_{odd>1} = 0$. Then the series can be rewritten as:

$$\frac{z}{2}\coth\frac{z}{2} = \sum_{n=0}^{\infty} B_{2n}\frac{z^{2n}}{(2n)!} \tag{11.12}$$

Multiplication by $\cosh(z/2)$ gives a Cauchy product of series, and recursive relations for the Bernoulli numbers[3].

Exercise 11.2.19 (Harmonic numbers). *Obtain the coefficients in*

$$\sum_{n=1}^{\infty} H_n z^n = -\frac{\log(1-z)}{1-z}. \tag{11.13}$$

11.2.1 The binomial series

The function $(1-z)^a$ has a pole in $z = 1$ if a is a negative integer. If $a \notin \mathbb{Z}$ the function has a branch cut from infinity to $z = 1$. In any case the function is analytic in the unit disk, where it admits the power expansion:

$$(1-z)^a = 1 - az + \tfrac{1}{2!}a(a-1)\,z^2 - \tfrac{1}{3!}a(a-1)(a-2)\,z^3 + \dots$$

[2] the actual behaviour is: $|E_{2n}| \approx 2^{2n+2}(2n)!\pi^{-2n-1}$ (NIST Handbook of Mathematical Functions, Cambridge 2010).

[3] The Bernoulli numbers were discovered almost at the same time in Japan by Seki Kowa (1640, 1708), the most eminent Wasan (Japanese calculator)

The useful *Pochhammer's symbol* a_k is introduced:

$$a_0 = 1, \quad a_1 = a, \quad a_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

The Gamma function $\Gamma(z)$ generalizes the factorial, and has the property $z\Gamma(z) = \Gamma(z+1)$; in particular, $\Gamma(n+1) = n!$ (see sect.12.3). We then obtain:

$$\boxed{(1-z)^a = \sum_{k=0}^{\infty} (-a)_k \frac{z^k}{k!}} \tag{11.14}$$

The sum truncates if a is a positive integer. The binomial symbol may be introduced, to recover the familiar Newton's expression:

$$\binom{a}{k} = \frac{(-1)^k}{k!}(-a)_k = \frac{a(a-1)\cdots(a-k+1)}{k!}$$

Exercise 11.2.20. *Show that*

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k}{k} z^k, \quad |z| < 1, \; n = 1, 2, \dots \tag{11.15}$$

11.2.2 Polilogarithms

Polilogarithms generalize the power series of the logarithm. They occur for example in the study of ideal quantum gases, or in quantum field theory.

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \qquad |z| < 1 \tag{11.16}$$

For $\mathrm{Re}s > 1$ it is $\mathrm{Li}_s(1) = \zeta(s)$. Being uniformly convergent in the disk, derivation and integration of the series term by term give

$$z\frac{d}{dz}\mathrm{Li}_s(z) = \mathrm{Li}_{s-1}(z), \quad \mathrm{Li}_{s+1}(z) = \int_0^z \frac{dz'}{z'}\mathrm{Li}_s(z')$$

With $\mathrm{Li}_1(z) = -\log(1-z)$, the dilogarithm is[4]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dz'}{z'}\log(1-z') = -\int_0^1 \frac{dt}{t}\log(1-zt)$$

The integral is well defined for $\mathbb{C}/[1,\infty)$, and is an analytic continuation of the power series.

Exercise 11.2.21. *Prove (using the series) the reflection rule:*
$\mathrm{Li}_s(-z) = -\mathrm{Li}_s(z) + 2^{1-s}\mathrm{Li}_s(z^2)$.
With the integral expression, for real x, prove:
$\mathrm{Li}_2(x) = \frac{\pi^2}{6} - \log x \log(1-x) - \mathrm{Li}_2(1-x)$, (Euler).

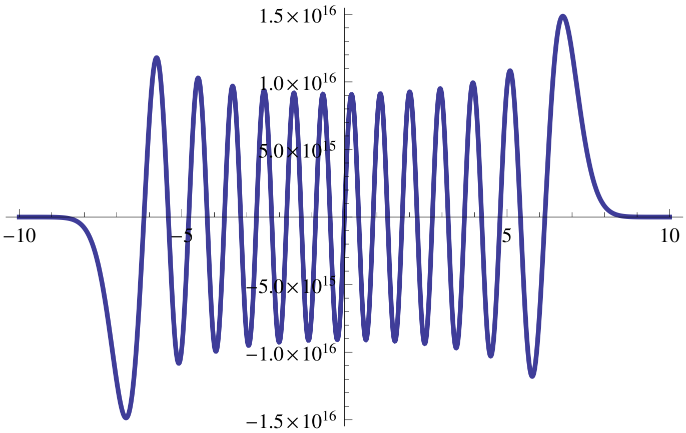

Figure 11.1: The Hermite polynomial $H_{25}(x)$ multiplied by $\exp(-x^2/2)$. Note the 25 real zeros and the wild oscillations in value.

11.3 Generating functions and polynomials

11.3.1 Hermite polynomials

The function $H(z,x) = e^{-z^2+2xz}$ is entire for any value $x \in \mathbb{R}$, and can be expanded in power series of z with center $z = 0$:

$$\boxed{e^{-z^2+2xz} = \sum_{k=0}^{\infty} H_k(x)\frac{z^k}{k!}} \tag{11.17}$$

The coefficients are functions of x. It is instructive to evaluate them by the methods introduced so far, as contour integrals around the origin:

$$H_k(x) = k!\oint \frac{d\zeta}{2\pi i}\frac{H(\zeta,x)}{\zeta^{k+1}} = k!\sum_{\ell=0}^{\infty}\frac{(2x)^\ell}{\ell!}\oint\frac{d\zeta}{2\pi i}e^{-\zeta^2}\zeta^{\ell-k-1}$$

Since terms with $\ell \geq k+1$ vanish because of Cauchy's theorem for entire functions, $H_k(x)$ turns out to be a polynomial of degree k (Hermite polynomial). The evaluation can proceed further.

$$= k!\sum_{\ell=0}^{k}\frac{(2x)^\ell}{\ell!}\oint\frac{d\zeta}{2\pi i}e^{-\zeta^2}\zeta^{\ell-k-1} = k!\sum_{\ell=0}^{k}\frac{(2x)^{k-\ell}}{(k-\ell)!}\oint\frac{d\zeta}{2\pi i}e^{-\zeta^2}\zeta^{-\ell-1}$$
$$= k!\sum_{\ell=0}^{[k/2]}\frac{(2x)^{k-2\ell}}{(k-2\ell)!}\oint\frac{d\zeta}{2\pi i}e^{-\zeta^2}\zeta^{-2\ell-1}$$

where $[k/2]$ is the integer part of $k/2$. The Cauchy integral is the coefficient of the term $z^{2\ell}$ of the power expansion of e^{-z^2}. The explicit expression of Hermite

[4] http://maths.dur.ac.uk/~dma0hg/dilog.pdf

polynomials is obtained:

$$H_k(x) = k! \sum_{\ell=0}^{[k/2]} (2x)^{k-2\ell} \frac{(-1)^\ell}{\ell!(k-2\ell)!} \tag{11.18}$$

$H(z,x)$ is the *generating function* of Hermite polynomials and encodes their analytic properties. We learn a lot from it with little effort:
1) the change z to $-z$ in (11.17) is compensated by x to $-x$, then Hermite polynomials have definite parity: $H_k(-x) = (-1)^k H_k(x)$.
2) the identities $\partial_x H(z,x) = 2zH(z,x)$ and $\partial_z H(z,x) = 2(-z+x)H(z,x)$, when translated to the power series expansion[5], give the recurrence relations

$$H_k'(x) = 2k\, H_{k-1}(x), \qquad H_{k+1}(x) = 2xH_k(x) - 2k\, H_{k-1}(x) \tag{11.19}$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$ that can be obtained by direct expansion of $H(z,x)$.
3) the identity $(2z\partial_z - 2x\partial_x + \partial_x^2)\, H(z,x) = 0$ corresponds to the second order equation (which has another non-polynomial independent solution)

$$H_k'' - 2x\, H_k' + 2k\, H_k = 0 \tag{11.20}$$

4) Hermite polynomials can be evaluated by Rodrigues' formula[6]:

$$\begin{aligned} H_k(x) = &\left[\frac{\partial^k}{\partial t^k} H(t,x)\right]_{t=0} = \left[e^{x^2} \frac{\partial^k}{\partial t^k} e^{-(t-x)^2}\right]_{t=0} \\ =&(-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \end{aligned} \tag{11.21}$$

5) The integral $\int_{-\infty}^{\infty} dx\, e^{-x^2} H_k(x)H_j(x)$ is symmetric in k and j. We evaluate it for $k \geq j$ by using Rodriguez's formula for $H_k(x)$, and doing k integration by parts:

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, e^{-x^2} H_k(x)H_j(x) &= (-1)^k \int_{-\infty}^{\infty} dx H_j(x) \frac{d^k}{dx^k} e^{-x^2} \\ &= \int_{-\infty}^{\infty} dx\, e^{-x^2} \frac{d^k}{dx^k} H_j(x) = 2^k k! \sqrt{\pi}\, \delta_{jk} \end{aligned} \tag{11.22}$$

because $H_k(x) = 2^k x^k + \dots$ (use eq.11.19). This is the *orthogonality* property of Hermite polynomials.
The Cauchy product $H(z,x)H(z,y) = H(z\sqrt{2}, \frac{x+y}{\sqrt{2}})$ gives an interesting summation formula (by equating the coefficients of equal powers of z):

$$\sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(x) H_{k-\ell}(y) = 2^{k/2} H_k\left(\frac{x+y}{\sqrt{2}}\right)$$

Exercise 11.3.1. *Evaluate the Cauchy product of the exponential series for* e^{-z^2} *and* e^{2tz} *and obtain the absolutely convergent power series of the generating function (11.17).*

[5] Derivation of the series term by term is possible because it is uniformly convergent both in z and x, in any compact set.
[6] Olinde Rodrigues (1795, 1851)

There are several other generating functions whose power series expansion yield special functions that are important in mathematical physics[7]. Some examples are briefly presented below, some will be considered in the study of orthogonal polynomials (Chapter on Hilbert spaces).

11.3.2 Laguerre polynomials

The polynomials arise in the study of the radial Schrödinger equation for the Hydrogen atom. The generating function is:

$$L(z,x) \equiv \frac{1}{1-z}\exp(-\frac{xz}{1-z}) = \sum_{k=0}^{\infty} L_k(x) z^k. \qquad (11.23)$$

It is holomorphic in $|z|<1$. If C is a circle in the unit disk around the origin:

$$L_k(z) = \oint_C \frac{dz}{2\pi i}\frac{L(z,x)}{z^{k+1}} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\oint_C \frac{dz}{2\pi i}\frac{z^{n-k-1}}{(1-z)^{n+1}}$$

The integral is zero if $n-k-1 \geq 0$ i.e. L_k is a polynomial of degree k. Use (11.15) to evaluate the integral.

11.3.3 Chebyshev polynomials (of the first kind)

The power series expansion in $z=0$ of $[(1-ze^{i\theta})(1-ze^{-i\theta})]^{-1}$ is easily done by means of the geometric series. It gives:

$$\frac{1-z\cos\theta}{1-2z\cos\theta+z^2} = \sum_{k=0}^{\infty} z^k\cos(k\theta), \qquad |z|<1 \qquad (11.24)$$

With $\cos\theta = x$, the same identity becomes:

$$\frac{1-xz}{1-2xz+z^2} = \sum_{k=0}^{\infty} z^k\, T_k(x), \qquad |z|<1 \qquad (11.25)$$

with functions $T_k(x)$. The identity $(1-xz) = (1-2xz+z^2)\sum_{k=0}^{\infty} T_k(x) z^k$ gives $T_0(z)=1$, $T_1(x)=x$ and the recurrence relation

$$0 = T_k(x) - 2xT_{k-1}(x) + T_{k-2}(x).$$

It appears that $T_k(x)$ is a polynomial of order k in x, and is the polynomial expansion of $\cos(k\theta)$ in $x=\cos\theta$. The k roots of $T_k(x)$ are real and known, and belong to the interval $[-1,1]$. On this interval $|T_k(x)|\leq 1$.
Chebyshev polynomials share the unique and important property: among all *monic* polynomials of degree k, the one that deviates the least from zero on $[-1,1]$ is the polynomial $2^{-k+1}T_k(x)$.

Exercise 11.3.2. *Study the Chebyshev polynomials of the second kind:*

$$\frac{1}{1-2xz+z^2} = \sum_{k=0}^{\infty} U_k(x) z^k, \quad |z|<1$$

[7] A beautiful little book on special functions is: N. N. Lebedev, *Special functions and their applications*, Dover Ed. A modern reference book is the *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, online at `https://dlmf.nist.gov`

11.3.4 Legendre polynomials

In his *Recherches sur l'attraction des spheroides homogènes* (1783), Adrien Marie Legendre introduced the important expansion in multipoles:

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{R}|} = \frac{1}{\sqrt{R^2 - 2Rr\cos\theta + r^2}} = \frac{1}{R}\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\frac{r}{R}\right)^\ell} \tag{11.26}$$

θ is the angle between the vectors, $R > r$, $P_k(\cos\theta)$ is a Legendre polynomial of order k in $\cos\theta$. The generating function

$$P(z,\cos\theta) = (1 - 2z\cos\theta + z^2)^{-1/2} = (1 - ze^{i\theta})^{-1/2}(1 - ze^{-i\theta})^{-1/2}$$

is analytic in the disk $|z| < 1$, where the two factors can be expanded in binomial series. The coefficients of the Cauchy product are the Legendre functions:

$$\begin{aligned}
\frac{1}{\sqrt{1 - 2z\cos\theta + z^2}} &= \sum_{n,m} \frac{(-1/2)_m}{m!}\frac{(-1/2)_n}{n!} e^{i(m-n)\theta} z^{m+n} \\
&= \sum_{k=0}^{\infty} z^k P_k(\cos\theta) \\
P_k(\cos\theta) &= \sum_{m=0}^{k} \frac{(-1/2)_m}{m!}\frac{(-1/2)_{k-m}}{(k-m)!}\cos(2m-k)\theta
\end{aligned}$$

As P_k is an expression of $\cos(n\theta)$ with $n = 0,\dots,k$, it may be rewritten as a polynomial of order k in the variable $\cos\theta$. In the variable x, $|x| \le 1$, it is:

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{k=0}^{\infty} z^k P_k(x) \tag{11.27}$$

From the identities $(1 - 2xz + z^2)\partial_z P(z,x) = (-z + x)P(z,x)$ and $(1 - 2xz + z^2)\partial_x P(z,x) = zP(z,x)$ one derives recurrence relations for the polynomials. Several more properties may be obtained by the above illustrated methods.

11.3.5 The Hypergeometric series

Several functions of mathematical physics correspond to special choices of the real parameters a, b and $c \neq 0, -1, -2, \dots$ of the hypergeometric function, which has a simple definition as a power series:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{a_n b_n}{c_n}\frac{z^n}{n!} = 1 + \frac{ab}{c}z + \frac{1}{2!}\frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \dots \tag{11.28}$$

Note that $F(a,b;c;z) = F(b,a;c;z)$. The series $F(-k,b;c,z)$ terminates, and is a polynomial of degree k in z. The hypergeometric series is convergent for $|z| < 1$ (ratio test).
The hypergeometric function $F(a,b;c;z)$ is a solution of the differential equation

$$z(z-1)F''(z) + [c - (a+b-1)z]F'(z) - abF(z) = 0$$

Exercise 11.3.3. *Show that:* $\partial_z F(a,b;c,z) = \frac{ab}{c}F(a+1,b+1;c+1;z)$.

11.4 Differential Equations

Power series are an effective representation of solutions of linear differential equations. The subject is vast, and we only illustrate it with a useful statement and an example.

Theorem 11.4.1. *In the linear second order differential equation,*

$$f''(z) + p(z)f'(z) + q(z)f(z) = 0$$

if $p(z)$ and $q(z)$ are analytic on a disk $|z| < R$, then any solution of the equation is analytic on the same disk.

11.4.1 Airy's equation

Airy's equation occurs in the study of a quantum particle in a uniform electric field[8], WKB theory, radio waves. Although the equation has a real variable, and one looks for real solutions, as a general rule it is useful to study it in the complex plane:

$$\boxed{f''(z) - zf(z) = 0} \tag{11.29}$$

The previous theorem assures that the solutions are entire functions and admit an absolutely convergent power series representation $f(z) = \sum_k c_k z^k$. Airy's equation

$$0 = \sum_{k\geq 0} c_k[k(k-1)z^{k-2} - z^{k+1}]$$

implies a recursion for the coefficients: $k(k-1)c_k = 0$ for $k = 0, 1, 2$ and $k(k-1)c_k = c_{k-3}$, $k \geq 3$. Therefore c_0 and c_1 are undetermined, while $c_2 = 0$. Starting from $c_0 \neq 0$ and $c_1 = 0$ one obtains $c_3 = c_0/(2\cdot 3)$, $c_6 = c_3/(5\cdot 6)$, $c_9 = c_6/(8\cdot 9)$ ... A good guess helps to obtain the expression for the general coefficient:

$$\begin{aligned}\frac{c_{3k}}{c_0} &= \frac{1}{3k(3k-1)\cdots 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} = \frac{(3k-2)\cdots 10\cdot 7\cdot 4\cdot 1}{(3k)!} \\ &= \frac{3^k}{(3k)!}\left(k-1+\tfrac{1}{3}\right)\cdots\left(2+\tfrac{1}{3}\right)\left(1+\tfrac{1}{3}\right)\tfrac{1}{3} = \frac{3^k}{\Gamma(3k+1)}\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\end{aligned}$$

With the aid of the triplication formula[9] for $\Gamma(3k+1)$:

$$c_{3k} = c_0 \frac{2\pi}{\Gamma\left(\frac{1}{3}\right)} \frac{1}{3^{2k+1/2} k!\,\Gamma\left(k+\frac{2}{3}\right)}$$

[8]The Hamiltonian for an electron in a uniform electric field E along the x axis is $H = \vec{p}^2/2m + eEx$. The eigenvalue equation is separable and for the x component is:

$$-\frac{\hbar^2}{2m}u''(x) + eEx\,u(x) = \lambda u(x)$$

The linear potential eEx equals the energy λ at $x_0 = \lambda/eE$, therefore the classical motion is confined in $x \leq x_0$. The problem has a natural length $\ell = \hbar^{2/3}(meE)^{-1/3}$ and the rescaling $x - x_0 = \ell\, s$ brings the eigenvalue equation to Airy's form. For a field $E = 1$ keV/cm it is $\ell \approx 9$ nm.

[9]$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$

By an appropriate choice of c_0 a solution is obtained:

$$f_0(z) = \sum_{k=0}^{\infty} \frac{z^{3k}}{9^k\, k!\, \Gamma(k+\frac{2}{3})} = \frac{1}{\Gamma(\frac{2}{3})}\left[1 + \frac{z^3}{6} + \frac{z^6}{180} + \dots\right]$$

Another independent solution is obtained with the choice $c_0 = 0$ and $c_1 \neq 0$. Then $c_4 = c_1/(4 \cdot 3)$, $c_7 = c_4/(7 \cdot 6)$ ...

$$f_1(z) = \sum_{k=0}^{\infty} \frac{z^{3k+1}}{9^k\, k!\, \Gamma(k+\frac{4}{3})} = \frac{1}{\Gamma(\frac{4}{3})}\left[z + \frac{z^4}{12} + \frac{z^7}{504} + \dots\right]$$

The two series have infinite radius and are entire functions. The standard solutions are the *Airy functions* of the first and second kind:

$$\mathrm{Ai}(z) = 3^{-2/3} f_0(z) - 3^{-4/3} f_1(z), \qquad \mathrm{Bi}(z) = 3^{-1/6} f_0(z) + 3^{-5/6} f_1(z)$$

For $z = x$ (real) both functions are oscillatory for $x \ll 0$; for $x \gg 0$ the function Ai (x) decays to zero exponentially while Bi (x) grows exponentially (see: Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, Springer; O. Vallée and M. Soares, *Airy functions and applications to physics*, World Scientific 2004.)

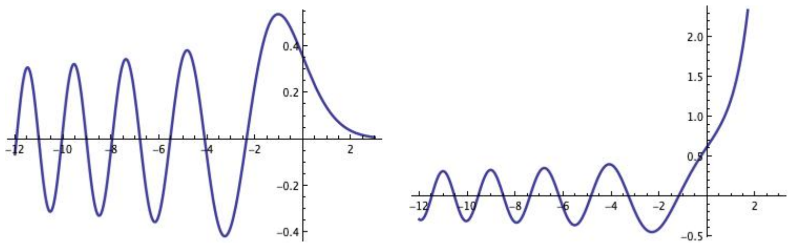

Figure 11.2: The Airy functions Ai(x) and Bi(x).

Chapter 12

ANALYTIC CONTINUATION

The power series representation of an analytic function implies that its zeros are *isolated*. This has important and unexpected consequences, as the powerful concept of analytic continuation. Some relevant theorems about analytic maps are then presented.

12.1 Zeros of analytic functions

Theorem 12.1.1. *Let $f(z)$ be analytic on a domain D.*
If $f(a) = 0$ at a point $a \in D$ and f is not a constant in a neighborhood of a, then there is a punctured disk centred in a where $f(z) \neq 0$.

Proof. If f is not a constant in a disk centred in a, the function has a power series centred in a, with some finite radius of convergence. If the zero is of order k, the coefficient c_k is nonzero and

$$f(z) = c_k(z-a)^k \left[1 + \frac{c_{k+1}}{c_k}(z-a) + \dots\right] = c_k(z-a)^k \varphi(z)$$

where $\varphi(z)$ is analytic in the disk, and $\varphi(a) = 1$. Since $\varphi(z)$ is continuous in a, $\forall\epsilon\ \exists\delta$ such that $|\varphi(z) - 1| < \epsilon$, i.e. there is no zero of φ in $D(a, \delta)$, and there is no zero of f on the same disk with point a removed. □

Corollary 12.1.2. *An analytic function on D that vanishes on a disk in D, or a line in D, or a set of points with an accumulation point in D, is zero on the whole set D. Two functions analytic on a domain D that take the same values on a line, or a sequence of points with an accumulation point in D, coincide.*

Example 12.1.3. *The function* $\sin(2z) - 2\sin z \cos z$ *is entire and vanishes on the real axis, therefore* $\sin(2z) - 2\sin z \cos z = 0\ \forall z \in \mathbb{C}$.

12.2 Analytic continuation

Suppose that f is an analytic function on a set D that contains at least a convergent sequence of points and its limit point. If $\tilde{f}$ is a function analytic on

$\tilde{D}$ such that $D \subset \tilde{D}$ and $\tilde{f} = f$ on D, then $\tilde{f}$ is the **analytic continuation** of f to $\tilde{D}$.

Theorem 12.2.1. *The analytic continuation of f to $\tilde{D}$ is unique.*

Proof. Suppose that f_1 and f_2 are two extensions. Then $f_1 - f_2$ is zero on D. This implies that $f_1 - f_2 = 0$ on the whole set $\tilde{D}$. □

Suppose that f_1 and f_2 are analytic on D_1 and D_2, and $f_1 = f_2$ on $D_1 \cap D_2$. If the intersection contains at least a convergent sequence of points, then the function $f = f_1$ on D_1 and $f = f_2$ on D_2 is analytic on $D_1 \cup D_2$.

Consider a series $f(z) = \sum_n a_n(z-a)^n$ that converges in the disk $|z-a| < R$. If in the disk we fix a point b and expand f with center b, we may find a radius $R' > R - |b-a|$ (the new disk leaks out of the first one). The two expansions coincide on the intersection, and together describe a single analytic function f on the union of the two disks. In the larger domain, one may build a new expansion and proceed, disk by disk, in building an analytic continuation of f by power series.

The following power series converges in the disk $|z| < 1$, but diverges at all points of the boundary $|z| = 1$. Then it cannot be continued analytically:

$$g(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + z^{16} + \dots$$

With coefficients 0 and 1, the Cauchy-Hadamard's criterion gives radius $R = 1$. The series clearly diverges at $z = \pm 1$. Note that $g(z^2) = g(z) - z$, therefore the series is divergent also at $z = \pm i$. Again: $g(z^4) = g(z) - z - z^2$ and the series $g(z)$ diverges at the roots of $z^4 = 1$. In this way one proves divergence at all points $z^{2^n} = 1$ ($n = 1, 2, \dots$); such points are dense in the unit circle, which is a *singular line* for $g(z)$.

12.3 Gamma function

The function was devised by Euler (1729) to extend the factorial of positive integers to real and complex numbers:

$$\boxed{\Gamma(z) = \int_0^\infty ds\, e^{-s}\, s^{z-1} \qquad \mathrm{Re}\, z > 0} \tag{12.1}$$

The meaning of s^z is $e^{z \log s}$. Note that:

$$|\Gamma(z)| \le \int_0^\infty ds\, e^{-s} |s^{z-1}| = \Gamma(\mathrm{Re} z)$$

To show that the Gamma function is holomorphic, let γ be an arbitrary closed path in the domain Re $z > 0$. The double integrals can be exchanged:

$$\oint_\gamma dz\, \Gamma(z) = \int_0^\infty ds\, e^{-s} \oint_\gamma dz\, e^{(z-1)\log s}$$

The last integral is zero by Cauchy's formula. Then Morera's theorem assures that $\Gamma(z)$ is holomorphic. The derivative $\Gamma'(z)$ is holomorphic on the same domain, and is related to the Digamma function (see sect.12.3.2).

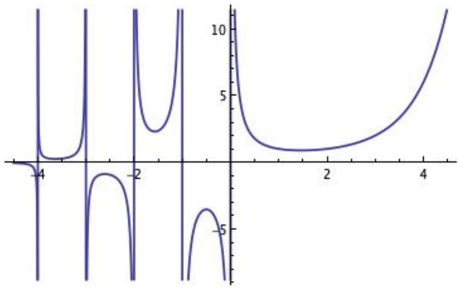

Figure 12.1: The Gamma function for real x. Note the poles at $0, -1, -2, \ldots$. For large x it diverges factorially (see Stirling formula).

For $z = x > 0$, integration by parts of (12.1) gives the typical property of the factorial: $\Gamma(x+1) = x\Gamma(x)$. Since $\Gamma(1) = 1$, it is $\Gamma(n+1) = n!$. The property remains valid in the complex domain: the function $\Gamma(z+1) - z\Gamma(z)$ is analytic and vanishes on the real positive axis, then it must vanish everywhere in its domain:

$$\boxed{\Gamma(z+1) = z\Gamma(z)} \tag{12.2}$$

Exercise 12.3.1. *Show that* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ *and* $\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\frac{(2n)!}{4^n n!}$ *(Hint: make the change* $s = t^2$ *in Euler's integral).*

Exercise 12.3.2. *Evaluate the volume* V *and the area* A *of the sphere of radius* r *in* $\mathbb{R}^n$. *(Hint: compare the integrals* $\int d^n x \exp(-\sum x_k^2)$ *in Cartesian and spherical coordinates).*

$$V = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})} r^n, \quad A = \frac{n}{r} V. \tag{12.3}$$

Exercise 12.3.3. Euler's Beta function. *Evaluate the double Euler integral for* $\Gamma(x)\Gamma(y)$, $x > 0$ *and* $y > 0$, *by changing to squared variables and then to polar coordinates, and prove the useful formula*[1]

$$\boxed{B(x,y) \equiv \int_0^1 dt\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}} \tag{12.4}$$

Euler's integral in (12.1) for the Gamma function is well defined for Re$z > 0$. However, by splitting the range of integration into $[0,1] \cup [1,\infty)$, and integrating on $[0,1]$ by series expansion, one obtains

$$\Gamma(z) = \int_1^\infty ds\, e^{-s}\, s^{z-1} + \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{z+k} \tag{12.5}$$

This expression provides the analytic continuation of Γ to Re $z \le 0$. It shows that $\Gamma(z)$ is the sum of an entire and a meromorphic function with simple poles

[1] Atle Selberg, winner of a Fields medal (with Laurent Schwartz, 1950) for his studies on prime numbers, obtained an important multi-dimensional extension of the Beta function (Selberg's integral).

at $-k$ with residue $(-1)^k/k!$, $k \in \mathbb{N}$. An alternative definition on the punctured complex plane was obtained by Gauss (1811):

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z\, n!}{z(z+1)(z+2)\cdots(z+n)} \qquad z \neq 0, -1, -2, \dots \tag{12.6}$$

Exercise 12.3.4. *Evaluate the following integral that, for large n, links the two definitions of the Gamma function:*

$$\int_0^n ds\, s^{z-1} \left(1 - \frac{s}{n}\right)^n = \frac{n^z n!}{z(z+1)\cdots(z+n)} \tag{12.7}$$

Hint: use integration by parts.

From Gauss' formula one easily obtains the useful duplication formula (as well as the triplication, or multiplication by n)

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z + \tfrac{1}{2}\right) \tag{12.8}$$

In 1854 Weierstrass gave a representation for the reciprocal of the Gamma function, which is an entire function:

$$\frac{1}{\Gamma(z)} = z\, e^{Cz} \prod_{m=1}^{\infty} \left[\left(1 + \frac{z}{m}\right) e^{-z/m}\right] \tag{12.9}$$

with Euler's constant

$$C = \lim_{n\to\infty} \left[1 + \tfrac{1}{2} + \ldots + \tfrac{1}{n} - \log n\right] = 0.5772156... \tag{12.10}$$

Another useful representation, by Hankel, is eq.(28.3.1). Several important formulae may be proven by the appropriate representation.

Exercise 12.3.5. *Prove the interesting formulae:*

$$\int_0^\infty dt\, \frac{t^{z-1}}{e^t - 1} = \Gamma(z)\zeta(z), \qquad \mathrm{Re}\, z > 1, \tag{12.11}$$

$$\int_0^\infty dt\, \frac{t^{z-1}}{e^t + 1} = (1 - 2^{1-z})\Gamma(z)\zeta(z), \quad \mathrm{Re}\, z > 0. \tag{12.12}$$

(Hint: multiply and divide by e^{-t} and expand in geometric series).
The integrals appear in the theory of free bosons and free fermions. The second one is an analytic extension of Riemann's $\zeta(z)$ from Re $z > 1$ *to* Re $z > 0$.

Exercise 12.3.6. *Show that* $\lim_{z\to 0} z\,\zeta(1-z) = -1$.

12.3.1 Stirling's formula

The well known formula for the growth of the factorial is a particular case of Stirling's expansion for $\Gamma(x+1)$, when x is real and large:

$$\boxed{\Gamma(x+1) = \sqrt{2\pi x}\, e^{x(\log x - 1)} \left[1 + \frac{1}{12\, x} + \frac{1}{288\, x^2} + \mathcal{O}\left(\frac{1}{x^3}\right)\right]} \tag{12.13}$$

Proof. Put $s = xt$ in the integral:

$$\Gamma(x+1) = \int_0^\infty ds\, e^{-s+x\log s} = x\, e^{x\log x} \int_0^\infty dt\, e^{-x(t-\log t)}$$

For $x \gg 1$, the neighborhood of the minimum of the exponent contributes most. The minimum is at $t = 1$, with expansion $t - \log t = 1 + \frac{1}{2}(t-1)^2 - \frac{1}{3}(t-1)^3 + \dots$ Therefore:

$$\begin{aligned}\Gamma(x+1) &= x\, e^{x\log x - x} \int_{-1}^\infty dt\, \exp\left[-x\left(\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3 + \tfrac{1}{4}t^4 - \dots\right)\right] \\ &= \sqrt{x}\, e^{x\log x - x} \int_{-\sqrt{x}}^\infty dt\, \exp\left(-\tfrac{1}{2}t^2 + \tfrac{1}{3\sqrt{x}}t^3 - \tfrac{1}{4x}t^4 + \dots\right) \\ &= \sqrt{x}\, e^{x\log x - x} \int_{-\sqrt{x}}^\infty dt\, e^{-\frac{1}{2}t^2}\left[1 + \tfrac{t^3}{3\sqrt{x}} + \tfrac{1}{x}\left(\tfrac{t^6}{18} - \tfrac{t^4}{4}\right) + \dots\right]\end{aligned}$$

The integrals of odd powers are neglected since they are the sum $\int_{-\sqrt{x}}^{\sqrt{x}} dt ... + \int_{\sqrt{x}}^\infty dt$ where the first integral is zero and the second one is exponentially small in x. For even powers the segment $(-\sqrt{x}, \sqrt{x})$ is where the function contributes for large x. The lower limit can be taken to $-\infty$ with an exponentially small error. Then:

$$\Gamma(x+1) = \sqrt{x}\, e^{x\log x - x} \int_{-\infty}^\infty dt\, e^{-\frac{1}{2}t^2}\left[1 + \tfrac{1}{x}\left(\tfrac{t^6}{18} - \tfrac{t^4}{4}\right) + \dots\right]$$

The result follows with the aid of $\int_{-\infty}^{+\infty} e^{-t^2} t^{2n} dt = \Gamma(n + \frac{1}{2})$. □

12.3.2 Digamma function

The logarithmic derivative of the Gamma function defines the Digamma function:

$$\boxed{\psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz} = \frac{1}{\Gamma(z)} \int_0^\infty ds\, e^{-s} s^{z-1} \log s} \qquad (12.14)$$

with the main property:

$$\psi(z+1) = \frac{1}{z} + \psi(z).$$

For integer values it gives $\psi(n+1) = \frac{1}{n} + \dots + \frac{1}{2} + 1 + \psi(1)$. The behaviour of the harmonic series (12.10) implies that, for large n: $\psi(n) - \psi(1) \approx \log n + C$. Stirling's formula gives the behaviour for large real x:

$$\psi(x+1) \approx \log x + \frac{1}{2x} - \frac{1}{12x^2} + \dots$$

then, $\psi(1) = -C$. Euler's constant (12.10) is deeply related to the properties of Riemann's zeta function[2].

[2]a nice book is: J. Havil, *Gamma, exploring Euler's constant*, Princeton Univ Press, 2003. See also the paper by Z. Silagadze: *Basel problem, a physicist's solution*, arXiv:1908.0751.

12.4 Analytic maps

Theorem 12.4.1 (**Open mapping theorem**). *If f is non-constant and analytic on a domain D, the image of an open subset in D is open (the converse is obviously true because f is continuous.)*

Proof. Consider an open subset O in D and a point $a \in O$. The function $g(z) = f(z) - f(a)$ vanishes in a; then there is a disk centred on a of radius δ and contained in O where $g(z) \neq 0$ i.e. $|f(z) - f(a)| > 0$. This means that the point $f(a)$ has a an open neighborhood of points contained in $f(O)$. □

Theorem 12.4.2 (**Maximum principle theorem**). *If f is non-constant and analytic on a domain D, the maximum of $|f(z)|$ is attained at the boundary ∂D.*

Proof. Suppose that $|f|$ attains its maximum at an interior point $z_0 \in D$, with image $w_0 = f(z_0)$. Then there is an open disk $D(z_0, r)$ centred in z_0 and contained in D. Since the image of the disk is open (Open mapping theorem 12.4.1) and contains w_0, there is a disk $|w - w_0| < r'$ contained in $f(D)$. Therefore there is a point w with $|w| > |w_0|$, and this contradicts the hypothesis. □

If $f(z) \neq 0$ on D, the same theorem applies to the holomorphic function $1/f$ to give the *minimum principle*: the minimum of $|f(z)|$ is attained at the boundary of D.

Exercise 12.4.3. *Find the maxima of $|\cosh z|$ in the square of vertices $0, \pi, \pi + i\pi, i\pi$.*

Proposition 12.4.4 (**Schwarz's Lemma**). *Suppose that f is holomorphic and maps the open unit disk $\mathbb{D}$ into itself, $f(\mathbb{D}) \subseteq \mathbb{D}$, with $f(0) = 0$. Then*

$$|f(z)| \leq |z|, \qquad \forall z \in \mathbb{D}.$$

Proof. The function $f(z)/z$ has a removable singularity in $z = 0$, and is holomorphic in the disk with the value $f'(0)$ at $z = 0$. Since it gains the maximum modulus at the boundary, it is: $|f(z)|/|z| \leq 1$. In particular $|f'(0)| \leq 1$. □

If the map is a bijection of the unit disk and $f'(z) \neq 0$ (f is a conformal map), Schwarz's lemma applies to the inverse map f^{-1} as well. Then $|f'(0)| = 1$, and $|f(z)| = 1$ if $|z| = 1$. For such functions, the power expansion is: $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$. *Bieberbach's conjecture* (1916) states that $|a_n| < n$ for all n. He only proved $|a_2| < 2$. Loewner proved $|a_3| < 3$ by means of Loewner's differential equation. The conjecture was proven by de Branges in 1984.
A limit case with real coefficients is the power expansion of Koebe's function:

$$z + 2z^2 + 3z^3 + 4z^4 + \ldots = z\frac{d}{dz}(1 + z + z^2 + \ldots) = \frac{z}{(1 - z)^2}$$

It maps univalently the unit disk to the w-plane with cut Re $w < -1/4$, Im$w = 0$. The cut is the image of the unit circle[3].

Here are some pearls from the theory of analytic functions:

[3]A magnificent reference is: R. Roy, *Sources in the development of mathematics*, Cambridge, 2011.

Theorem 12.4.5 (Bohr, 1914). *If $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is analytic on the unit disk, where $|f(z)| < 1$, then $\sum_{k=0}^{\infty} |c_k z^k| < 1$ on the disk $|z| < 1/3$. This radius is the best possible.*

Theorem 12.4.6 (Bloch, 1924). *If f is analytic on the unit disk $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$ then there is a number B (Bohr's constant) independent of f such that there is a subset $\Omega \subset \mathbb{D}$ where f is one-to-one, and such that $f(\Omega)$ contains a disk of radius B. ($B = 0.4469$,* arXiv:1702.01080*)*

Theorem 12.4.7 (MacDonald 1898, Whittaker 1935). *The number of zeros of a non-constant function f analytic in a region bounded by a contour $|f(z)| = c$ exceeds by unity the number of zeros of f' in the same region.* (arXiv:1702.03458)

Theorem 12.4.8 (Earle-Hamilton fixed point theorem). *Let f be a holomorphic function on a bounded domain D, such that the distance between $f(D)$ and $\mathbb{C}/D$ is greater than a positive constant. Then f has a unique fixed point.*

Chapter 13

LAURENT SERIES

13.1 Laurent's series of holomorphic functions

A Laurent series with center a is the bilateral sum

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{n=1}^{\infty} c_{-n}\frac{1}{(z-a)^n} \tag{13.1}$$

It exists if both one-sided series converge. The two series are respectively called the *analytic* and the *principal* parts of the series. The analytic part converges absolutely on a disk centred in a with radius R,

$$\limsup \sqrt[n]{|c_n(z-a)^n|} < 1, \quad \to \quad |z-a| < R, \quad \frac{1}{R} = \limsup \sqrt[n]{|c_n|}$$

The principal part (negative powers) converges absolutely for

$$\limsup \sqrt[n]{|c_{-n}(z-a)^{-n}|} < 1, \quad \to \quad |z-a| > r = \limsup \sqrt[n]{|c_{-n}|}$$

Therefore, the Laurent series is well defined in the *annulus*

$$A(a,r,R) = \{z \,:\, r < |z-a| < R\}.$$

If the inner radius is zero and the point a is avoided, the annulus is the *punctured disk* $D'(a,R) = D(a,R)/\{a\}$.

Exercise 13.1.1. *Discuss the convergence of the Laurent series:*

$$\sum_{k=-\infty}^{\infty} 3^{-|k|} z^k, \qquad \sum_{k=-\infty}^{\infty} \frac{z^k}{\cosh(3k)}.$$

Remark 13.1.2. *Because both the principal and the analytic parts are power series (in $z-a$ and its reciprocal) that are convergent in the annulus, convergence is absolute, and it is uniform in the closed annulus $r(1+\eta) \le |z-a| \le R(1-\eta')$ where $\eta, \eta' > 0$ are finite. Thus a Laurent series can be integrated term by term on a path in the annulus. In particular, for a closed path encircling the point a with index 1, the integral of the analytic part is zero, and the integral of the principal part is $2\pi i\, c_{-1}$:*

$$\oint_\gamma dz \sum_{n=-\infty}^{\infty} c_n(z-a)^n = 2\pi i\, c_{-1} \tag{13.2}$$

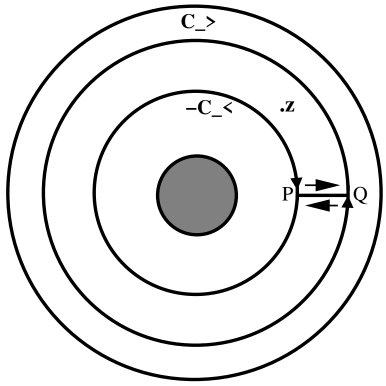

Figure 13.1: The integral on the closed path denoted by arrows equals the integral on the two circles, because the integrals on PQ and QP cancel.

The following fundamental theorem was proven by Pierre Laurent[1]:

Theorem 13.1.3 (Laurent). *If $f(z)$ is analytic in the (open) annulus $A(a, r, R)$, then it has a unique Laurent expansion in it, with center a:*

$$\boxed{f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k, \quad c_k = \oint_\gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta - a)^{k+1}}} \tag{13.3}$$

γ is any closed positive path in the annulus that encircles the center once.

Proof. Let z be a point in the annulus and draw in the annulus two circles with center a and positive orientation: $C_>$ with radius $r_>$ and $C_<$ with radius $r_<$, and $r_< < |z-a| < r_>$. Choose a point $P \in C_<$ and a point $Q \in C_>$. Consider the closed positively oriented path

$$\Gamma = [PQ] \cup C_> \cup [QP] \cup (-C_<)$$

where $-C_<$ is the inner circle with reversed orientation. Since the segment joining P and Q is covered twice in opposite directions, it is:

$$\oint_\Gamma \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} = \oint_{C_>} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} - \oint_{C_<} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z}$$

The path Γ encircles the point z; by Cauchy's integral formula, the first integral is $f(z)$. Therefore:

$$f(z) = \oint_{C_>} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z} - \oint_{C_<} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z}$$

Write $\zeta - z = (\zeta - a) - (z - a)$ in the Cauchy kernels of the two integrals and expand in Geometric series (where respectively it is $|\zeta - a| > |z - a|$ and

[1] He was an engineer in the army, and communicated the theorem to Cauchy in 1843, in a private letter. The proof was published after his death. Weierstrass arrived to the same theorem independently and, as he often did, he published the proof years later [Remmert]

$|z-a| < |\zeta - a|$). Uniform convergence allows to take the sums out of the integrals

$$= \oint_{C_>} \frac{d\zeta}{2\pi i} f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta-a)^{k+1}} + \oint_{C_<} \frac{d\zeta}{2\pi i} f(\zeta) \sum_{k=0}^{\infty} \frac{(\zeta-a)^k}{(z-a)^{k+1}}$$
$$= \sum_{k=0}^{\infty} (z-a)^k \oint_{C_>} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{(\zeta-a)^{k+1}} + \sum_{k=0}^{\infty} \frac{1}{(z-a)^{k+1}} \oint_{C_<} \frac{d\zeta}{2\pi i} f(\zeta)(\zeta-a)^k.$$

The two circles can be deformed into an arbitrary simple closed path γ in the annulus, without changing the values of the integrals (the coefficients c_k and c_{-k}).
Suppose that f admits another (uniformly convergent) Laurent expansion in the annulus: $f(z) = \sum_{k\in\mathbb{Z}} \tilde{c}_k (z-a)^k$. The coefficient c_n in (13.3) is evaluated:

$$c_n = \oint_\gamma \frac{dz}{2\pi i} \sum_{k\in\mathbb{Z}} \tilde{c}_k (z-a)^{k-n-1} = \sum_{n\in\mathbb{Z}} \tilde{c}_k \oint_\gamma \frac{dz}{2\pi i} (z-a)^{k-n-1} = \tilde{c}_n$$

because the integrals vanish if $k-n-1 \neq -1$. □

Remark 13.1.4. *The Laurent expansion of a function that is analytic on the whole disk with center a has no principal part, and the analytic part coincides with the power expansion in the disk.*

Remark 13.1.5. *In signal and image processing a useful tool is the* Z-transform *of a bilateral $\{x_n\}_{-\infty}^{\infty}$ or unilateral sequence $\{x_n\}_0^{\infty}$ of numbers. It is a function of the complex variable z defined by the Laurent series (note the sign of the exponent): $Z[\{x_n\}] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$. A sequence determines a domain of convergence. Operations on sequences determine operations on series.*

13.2 Bessel functions (integer order)

The function $J(z,x) = \exp[\frac{1}{2}x(z-1/z)]$ is analytic in the punctured complex plane $\mathbb{C}/\{0\}$, where it has the Laurent expansion

$$\boxed{\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right] = \sum_{k=-\infty}^{\infty} z^k J_k(x)} \tag{13.4}$$

It is the generating function of Bessel's functions of integer order, which arise in the study of Laplace's operator in cylindrical coordinates[2].

A change of sign of z is compensated by a change of sign of x, $J(-z,x) = J(z,-x)$, then $J_k(-x) = (-1)^k J_k(x)$. The exchange of z with $1/z$ amounts again to a change of sign of x, then

$$J_{-k}(x) = J_k(-x) = (-1)^k J_k(x)$$

[2]Friedrich Wilhelm Bessel (1784, 1846) was the director of Königsberg's astronomical observatory (Prussia) and measured the first stellar distance by parallax, after correctly interpreting the apparent motion of *61 Cygni* (discovered by Piazzi in Palermo) as due to Earth's annual motion. The same instrument (built by Fraunhofer) enabled him to discover the oscillations of Sirius, due to an invisible companion star (a white dwarf) to be observed a century later. He developed an accurate theory for solar eclipses, and introduced Bessel's function to solve Kepler's equation for planetary motion (see sect.20.3.2).

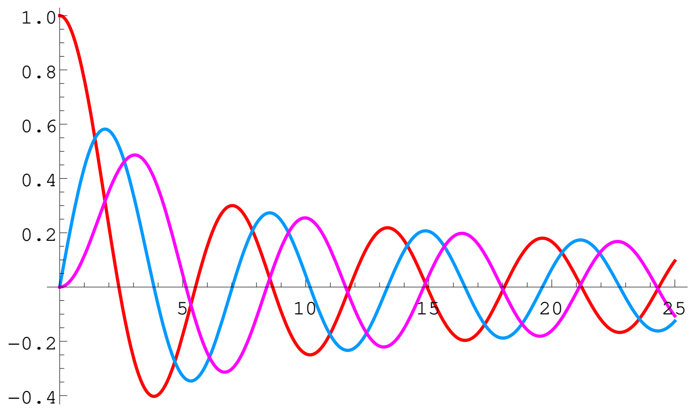

Figure 13.2: Bessel functions of the first kind J_0 (red), J_1 (blue), J_2 (fuchsia).

With $z = e^{i\theta}$ the expansion is $e^{ix\sin\theta} = \sum_{k=-\infty}^{\infty} e^{ik\theta} J_k(x)$. An integral expression for Bessel functions is readily obtained:

$$J_k(x) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(x\sin\theta - k\theta)} \tag{13.5}$$

The generating function allows for a simple derivation of several properties. From $\partial_x J(z,x) = \frac{1}{2}(z-1/z)J(z,x)$ and $z\partial_z J(z,x) = \frac{x}{2}(z+1/z)J(z,x)$ one obtains

$$2J_k'(x) = J_{k-1}(x) - J_{k+1}(x), \qquad \frac{2k}{x} J_k(x) = J_{k-1}(x) + J_{k+1}(x) \tag{13.6}$$

The two equations for $J(z,x)$ also yield an equation where z and ∂_z only appear in the combination $z\partial_z$: $(z\partial_z)^2 J(z,x) = (x^2\partial_x^2 + x\partial_x + 1)J(z,x)$. The equation gives Bessel's equation for integer order:

$$\boxed{\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{k^2}{x^2}\right] J_k(x) = 0} \tag{13.7}$$

Eq.(13.7) is written for another index, say m. Multiplicating the first by $xJ_m(x)$ and the other by $xJ_k(x)$ one obtains:

$$(1 + x\frac{d}{dx})(J_k' J_m - J_m' J_k) - \frac{k^2 - m^2}{x} J_k J_m = 0$$

Now integrate x on the positive reals. The first two terms cancel by integration by parts (all $J_k(x)$ decay as $1/\sqrt{x}$). The orthogonality property is:

$$\int_0^\infty \frac{dx}{x} J_k(x) J_m(x) = 0 \qquad k \neq m \tag{13.8}$$

Exercise 13.2.1. *By multiplication of generating functions prove*

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y) \tag{13.9}$$

Exercise 13.2.2. *By expanding the integral representation* (13.5) *show that* $J_k(x)$ *behaves as* x^k *for small* x *(*$k \geq 0$*).*

Exercise 13.2.3. *Write Helmholtz's equation* $(\nabla^2 + \lambda^2)u = 0$ *in polar coordinates* r, θ *and separate it with functions* $u(r, \theta) = R(r)Y(\theta)$*. Show how the radial equation becomes Bessel's equation.*

13.3 Fourier series

Suppose that f is analytic in an annulus centred in the origin, that contains the unit circle. In the annulus f has a Laurent expansion with coefficients f_k evaluated as integrals on the unit circle. In particular, on the unit circle the expansion of f is:

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k \, e^{ik\theta}, \qquad f_k = \int_0^{2\pi} \frac{d\theta}{2\pi} f(e^{i\theta}) e^{-ik\theta}$$

This Laurent's expansion is the *Fourier* expansion of the 2π-periodic function $f(e^{i\theta})$. It may be written in the form (with small change of notation):

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \, u_k(\theta), \qquad \hat{f}_k = \int_0^{2\pi} d\theta \, \overline{u_k(\theta)} \, f(\theta) \tag{13.10}$$

where $\{u_k\}_{k\in\mathbb{Z}}$ is the *Fourier basis* of orthonormal 2π-periodic functions

$$\boxed{u_k(\theta) = \frac{e^{ik\theta}}{\sqrt{2\pi}}, \qquad \int_0^{2\pi} d\theta \, \overline{u_m(\theta)} \, u_n(\theta) = \delta_{mn}} \tag{13.11}$$

The numbers $\hat{f}_k$ are the *Fourier coefficients* of the periodic function $f(\theta)$.
In this discussion f is the restriction of a holomorphic function to the unit circle, where it is continuous and differentiable. The problem of the minimal requirements for a periodic function to admit a Fourier expansion will be considered in section 20.1.

Chapter 14

THE RESIDUE THEOREM

14.1 Singularities.

Definition 14.1.1. When a function $f(z)$ fails to be analytic at a point a, but is analytic in a punctured disk $D'(a,r) = D(a,r)/\{a\}$ (a is removed from the disk), the point a is an **isolated singularity** of f.

Example 14.1.2. *The reciprocal $1/f$ of a holomorphic function has isolated singularities at the isolated zeros of f.*

The Laurent expansion of the function in the punctured disk $D'(a,r)$,

$$f(z) = \ldots + \frac{c_{-k}}{(z-a)^k} + \cdots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \ldots,$$

defines the principal and the analytic parts of f at its isolated singularity a. Since a punctured disk has inner radius equal to zero, the principal series exists at all points different from a, and converges uniformly on any compact set (for example, on any bounded curve) that does not contain the singular point.

According to the principal part being null, terminating, or containing an infinite number of terms, the singularity a is described by one of three types:

- The point a is a ***removable singularity*** if $c_{-k} = 0$ for all $k > 0$.
  Example: $f(z) = \sin(z-a)/(z-a)$.
- The point a is a ***pole of order*** k if there is $k > 0$ such that $c_{-k} \neq 0$ and $c_{-k-n} = 0$ for all $n > 0$. It follows that $\lim_{z\to a}(z-a)^k f(z) = c_{-k}$.
  Example: $f(z) = 1/(z-a)^3$ has a pole of order 3.
- The point a is an ***essential singularity*** if for any N there is $k > N$ such that $c_{-k} \neq 0$.
  Example: $f(z) = \exp[(z-a)^{-1}]$

Remark 14.1.3. *There is a substantial difference between a pole and an essential singularity. If a is a pole for f, then $\lim_{z\to a} f(z) = \infty$. However, if a is an essential singularity, then $\lim_{z\to a} f(z)$ is undefined. Indeed, a theorem by Picard states that if a is an essential singularity of f, the image $f(D')$ of the punctured disk contains all complex points with at most one exception.*

Exercise 14.1.4. *Study the behaviour of* $|1/z|$ *and* $|e^{1/z}|$ *for* $z \to 0$.

Definition 14.1.5. A function is **meromorphic** on a domain D if it is analytic in D up to a set of isolated *poles* in D.

Theorem 14.1.6 (Picard's little theorem for meromorphic function)**.** *Every function meromorphic on* $\mathbb{C}$ *that omits three distinct complex values* a, b *and* c *is constant*[1].

14.2 Residues and their evaluation

Definition 14.2.1. The **residue** of a function f at the *isolated singularity* a is the coefficient c_{-1} of the Laurent expansion in the punctured disk,

$$\boxed{\mathrm{Res}[f,a] = c_{-1} = \oint_\gamma \frac{dz}{2\pi i} f(z)} \tag{14.1}$$

where γ is any (piecewise) smooth simple path encircling the singularity a anticlockwise inside the punctured disk of analyticity.

In view of their importance, we give rules to calculate residues that avoid the evaluation of the contour integral.

- If $f(z)$ has a simple pole in a (a pole of order 1), its Laurent expansion is $f(z) = c_{-1}(z-a)^{-1}$ + analytic part. Therefore the residue is

$$\boxed{c_{-1} = \lim_{z\to a}(z-a)f(z)} \tag{14.2}$$

- If $f(z)$ has a pole of order k in a, it is $(z-a)^k f(z) = c_{-k} + c_{-k+1}(z-a) + \cdots + c_{-1}(z-a)^{k-1} + \ldots$. Then $(k-1)$ derivatives give $(k-1)!c_{-1}+$ terms that vanish in $z = a$. The residue of f in a is

$$\boxed{c_{-1} = \frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]} \tag{14.3}$$

- If a is an essential singularity, the residue is computed with (14.1).

Example 14.2.2. *In simple cases one may evaluate the residue by known expansions. From* $e^{1/z} = 1 + 1/z + \cdots + 1/(4!z^4) + \ldots$, *it is* $\mathrm{Res}\,[z^3 e^{1/z}, 0] = 1/4!$. $\mathrm{Res}\,[\tan z/z^5, 0] = 0$ *because the function is even, and its Laurent expansion in* $z = 0$ *only contains even powers.*

Theorem 14.2.3 (**The Residue Theorem**)**.** *Let* f *be an analytic function on* D/S*, where* $S = \{z_1, \ldots, z_n\}$ *is the set of its isolated singularities in the domain* D*. If* γ *is a closed piecewise smooth path in* D/S *such that* $\mathrm{Ind}\,(\gamma, z) = 0$ *for all* $z \notin D$*, then:*

$$\boxed{\oint_\gamma dz\, f(z) = 2\pi i \sum_{k=1}^{n} \mathrm{Ind}(\gamma, z_k)\,\mathrm{Res}[f, z_k].} \tag{14.4}$$

[1]see: R. Remmert, *Classical topics in complex function theory*, GTM 172, Springer 1998.

Proof. Each singularity z_k is the center of a punctured disk in D where f is analytic and can be expanded in Laurent's series: $f(z) = P_k(z) + A_k(z)$. While the analytic part A_k converges in the full disk, the principal part P_k converges in $\mathbb{C}/z_k$.
The function $g(z) = f(z) - \sum_k P_k(z)$ is analytic on D. By Cauchy's theorem:

$$0 = \oint_\gamma dz\, g(z) = \oint_\gamma dz\, f(z) - \sum_k \oint_\gamma dz\, P_k(z)$$

The principal series can be integrated term by term because it converges uniformly on γ. The integrals $\oint_\gamma dz(z - z_k)^{-\ell}$ are zero for $\ell \neq 1$; the integral $\ell = 1$ is by definition $2\pi i$ Ind (γ, z_k). Then $\oint_\gamma dz\, P_k(z) = 2\pi i \,\text{Res}[f, z_k]\,\text{Ind}(\gamma, z_k)$. □

14.3 Evaluation of integrals

The Residue Theorem is a fundamental tool for evaluating integrals. In all applications, one has to arrange the integral as an integral on a closed path in the complex plane, either by a change of variable, or by closing the set of integration with extra curves. Usually, the method is successful if the integral on such additional curves is known (zero), or is proportional to the initial integral. The choice of the correct closed path is a matter of wisdom. However, some cases are typical and are here illustrated by examples.

14.3.1 Trigonometric integrals

Integrals in $x \in [0, 2\pi]$ that only involve trigonometric functions may be attacked by writing the functions in terms of $e^{\pm ix}$ or powers, and putting $z = e^{ix}$. The new variable runs the unit circle C, and the Residue Theorem may apply.

Example 14.3.1.

$$\int_0^{2\pi} dx\, \frac{1}{2 + \cos x} = \oint_C \frac{dz}{iz} \frac{2}{4 + z + 1/z} = \oint_C dz\, \frac{-2i}{z^2 + 4z + 1}$$

The roots $z_\pm = -2 \pm \sqrt{3}$ *are simple poles, and* $|z_+| < 1$. *Then:*

$$= 2\pi i \lim_{z \to z_+} (z - z_+) \frac{-2i}{(z - z_+)(z - z_-)} = \frac{2\pi}{\sqrt{3}}$$

Example 14.3.2.

$$\int_0^{2\pi} dx \frac{\cos(4x)}{3 + \sin^2 x} = \text{Re} \int_0^{2\pi} dx \frac{e^{i4x}}{3 + \sin^2 x} = \text{Re} \oint_C \frac{dz}{iz} \frac{4z^4}{12 - (z - 1/z)^2}$$
$$= -4\text{Re} \oint_C \frac{dz}{i} \frac{z^5}{(z^2 - a)(z^2 - 1/a)} \qquad (a = 7 - 4\sqrt{3} > 0)$$
$$= -4\text{Re}2\pi i[\text{Res}(\sqrt{a}) + \text{Res}(-\sqrt{a})] = \frac{\pi}{\sqrt{3}}(7 - 4\sqrt{3})^2$$

By taking the real part, one avoids the evaluation of two integrals. The poles are all simple, and only $\pm\sqrt{a}$ *are in the unit disk.*

Exercise 14.3.3.

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\cos(k\theta)}{\cosh\xi - \cos\theta} = \frac{e^{-k\xi}}{\sinh\xi}, \quad \xi > 0,\ k = 0, 1, \dots \tag{14.5}$$

$$\int_0^{2\pi} dx \frac{\cos(2x)}{1+\sin^2 x} = \pi(3\sqrt{2} - 4) \tag{14.6}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \frac{1}{y-x} = \pi \frac{\text{sign}(y)}{\sqrt{y^2-1}}, \quad |y| > 1 \tag{14.7}$$

14.3.2 Integrals on the real line

Several integrals on the whole real line are attacked by *closing the interval* $[-R, R]$ with a semicircle in the upper or lower half-plane and promoting the real variable x to a complex variable $z = x + iy$ (then $dz = dx$ on the real axis). If the Residue Theorem applies to the closed path γ, it provides the value of the integral if *if the contribution of the semicircle vanishes* in the limit $R \to \infty$.

Example 14.3.4.

$$\int_{\mathbb{R}} dx \frac{x^2}{x^4+1} = \lim_{R\to\infty} \int_{-R}^{R} dx \frac{x^2}{x^4+1} = \lim_{R\to\infty} \oint_\gamma dz \frac{z^2}{z^4+1}$$

γ is the closed path $[-R, R] \cup \sigma$, where σ is the semicrcle $\{Re^{i\theta}, 0 \le \theta \le \pi\}$. Since the function decays as R^{-2} in every direction, for $R \to \infty$ the contribution of the semicircle vanishes (Darboux's inequality) (a semicircle in the lower half plane would be equally admissible). The path γ encircles the simple poles $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$. Then the integral is

$$= 2\pi i \lim_{z\to z_1} \frac{(z-z_1)z^2}{(z^4+1)} + 2\pi i \lim_{z\to z_2} \frac{(z-z_2)z^2}{(z^4+1)} = \frac{\pi}{\sqrt{2}}$$

Remark 14.3.5. *The asymptotic behaviour $|f(Re^{i\theta})|R \to 0$ as $R \to \infty$ is a sufficient condition for the integral $\int_\sigma dz f(z)$ on the semicircle σ to vanish.*

There is an important class of integrals where the choice of half-plane is not free, and the following result is useful:

Lemma 14.3.6 (Jordan). *Let f be a complex function, continuous on the semicircle $\sigma = \{Re^{i\theta},\ \theta \in [0, \pi]\}$, and let $M(R) = \max_{\theta\in[0,\pi]} |f(Re^{i\theta})|$. Then:*

$$\left| \int_\sigma dz f(z) e^{iaz} \right| \le \frac{\pi}{a} M(R), \qquad a > 0 \tag{14.8}$$

Proof.

$$\left| \int_\sigma dz f(z) e^{iaz} \right| = \left| iR \int_0^\pi e^{i\theta} d\theta\, f(Re^{i\theta})\, e^{iaRe^{i\theta}} \right|$$
$$\le 2R\, M(R) \int_0^{\frac{\pi}{2}} d\theta e^{-Ra\sin\theta} \le 2R\, M(R) \int_0^{\frac{\pi}{2}} d\theta e^{-2Ra\theta/\pi} \le \frac{\pi}{a} M(R)$$

The inequality $\sin\theta \ge 2\theta/\pi$ is used, for $0 \le \theta \le \pi/2$.
If $a < 0$ the semicircle σ for convergence is in the lower half plane. In any case $a \ne 0$. □

If the maximum $M(R)$ of $|f|$ on σ vanishes for $R \to \infty$, no matter how fast, then the semicircle contribution is zero. Here's an important example:

$$\boxed{\int_{-\infty}^{\infty} dx \, \frac{e^{-ikx}}{x^2 + a^2} \;=\; \frac{\pi}{a} e^{-a|k|}, \quad a > 0, k \in \mathbb{R}} \tag{14.9}$$

In this example, for $k = 0$ one is free to close in the upper or lower half plane. If $k \neq 0$, since $e^{-ik(x+iy)}$ must not blow up, one is compelled to close with a semicircle in the lower half plane if $k > 0$ or in the upper half plane if $k < 0$.

Example 14.3.7. *Evaluate:*

$$\int_0^{\infty} dx \frac{\cos(kx)}{(x^2+1)^2} = \frac{\pi}{4}(|k|+1)e^{-|k|} \tag{14.10}$$

The function is even, then the integral is $1/2$ *the integral on the real axis;* $\cos(kx) = \cos(|k|x) = \mathrm{Re}\exp(i|k|x)$*. The path is closed in the upper half plane, where it encircles the double pole* $z = i$

$$= \frac{1}{2}\mathrm{Re}\, 2\pi i \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{e^{i|k|z}}{(z^2+1)^2}\right] = \mathrm{Re}\, \pi i \lim_{z\to i} \frac{d}{dz} \frac{e^{i|k|z}}{(z+i)^2} = \dots$$

Exercise 14.3.8.

$$\int_0^{\infty} dx \frac{x\sin(\pi x)}{(x^2+1)^2} = \frac{\pi^2}{4}e^{-\pi} \tag{14.11}$$

Remark 14.3.9. *Ordinary integrals on the real line are tacitly defined with limits to infinity being taken independently:*

$$\int_{\mathbb{R}} f(x)dx = \lim_{u,v\to\infty} \int_{-u}^{v} f(x)dx$$

The examples presented above use a weaker definition (Cauchy's principal value):

$$P\int_{\mathbb{R}} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

where the limits are taken simultaneously. If the ordinary integral exists, it coincides with the principal value integral.

For the important class of Fourier integrals the following statement holds:

Theorem 14.3.10 (Jordan)**.** *Let* $f(z)$ *be analytic, save for isolated singularities. If* $a > 0$ *and if* $f(z) \to 0$ *as* $z \to \infty$ *in the upper half plane then:*

$$\int_{-\infty}^{\infty} dx\, f(x)\, e^{iax} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z)e^{iaz},\, p_k] \tag{14.12}$$

where $p_1, \dots, p_n$ *are the singularities in the upper half plane.*
If $a < 0$ *and* $f(z) \to 0$ *as* $z \to \infty$ *in the lower half plane, the sum involves the poles in the lower half plane, with a change of sign.*

Proof. Consider the rectangular path with corners $(-u, 0)$, $(v, 0)$, (v, iw) and $(-u, iw)$, $u, v > 0$, $w = u + v$. The rectangle is large enough to accomodate all singularities in the upper half plane. The integral of $f(z)e^{iaz}$ on this closed path is $2\pi i$ times the sum of residues. Let us show that integration on all sides but the interval $[-u, v]$ give zero for $u, v \to \infty$: the integral on the segment from v to $v + iw$ is

$$\left|\int_0^w idy f(v+iy)e^{iav-ay}\right| \le \sup_y |f(v+iy)| \int_0^w e^{-ay} \le \frac{1}{a} \sup_y |f(v+iy)|,$$

the sup factor vanishes for $v \to \infty$ (and u is left free). The opposite side behaves similarly for $u \to \infty$ and all v. The integral on the side from $-u + iw$ to $v + iw$ is

$$\left|\int_{-u}^{v} dx f(x+iw)e^{iax-aw}\right| \le e^{-aw}(v+u) \sup_x |f(x+iw)|,$$

and vanishes in the independent limits u and $v \to \infty$.
The integral on the whole real axis is obtained by two independent limits

$$\lim_{u\to\infty} \int_{-u}^{0} dx\, f(x)e^{iax} + \lim_{v\to\infty} \int_0^v dx f(x)e^{iax}$$

and equals the integral on the closed rectangle. □

14.3.3 Principal value integrals

When a real function has one or more simple poles on the real axis, we may still give meaning to the integral as a principal value integral (Cauchy), not to be confused with the principal value integral on $[-R, R]$, $R \to \infty$.
This is an example: the integral $\int_a^c (x - b)^{-1}$ is singular, for $a < b < c$. However, one may isolate the singularity by removing an infinitesimal interval, and evaluate:

$$\int_a^{b-\epsilon} \frac{dx}{x-b} + \int_{b+\eta}^{c} \frac{dx}{x-b} = \log\frac{\epsilon}{b-a} + \log\frac{c-b}{\eta}$$

The independent limits $\epsilon, \eta \to 0$ do not give a well defined result. However, a finite result is obtained with the "principal value" prescription: $\epsilon = \eta$, i.e. a symmetric interval, and $\epsilon \to 0$. The result is:

$$⨍_a^c \frac{dx}{x-b} \equiv \lim_{\epsilon\to 0}\left[\int_a^{b-\epsilon} \frac{dx}{x-b} + \int_{b+\epsilon}^{c} \frac{dx}{x-b}\right] = \log\frac{c-b}{b-a} \tag{14.13}$$

In this section we evaluate principal value integrals on the whole real line as follows:
1) for each singularity a_i of the real line remove a real interval $[a_i - \epsilon_i, a_i + \epsilon_i]$;
2) close the interval $[-R, R]$ (R big enough to include all gaps) with a semicircle in the appropriate half-plane (we assume that for infinite radius the contribution of the semicircle is zero);
3) close each gap centred in the singularity a_i by a semicircle of radius ϵ_i in the same half plane as the big semicircle;
4) the principal part integral (real axis with gaps) is the sum of two contributions: the integral on the closed path (evaluated by residues) with $R \to \infty$, and the integrals on the semicircles with opposite orientation (in order to cancel the contributions that were added to close the contour) in the limits $\epsilon_i \to 0$.

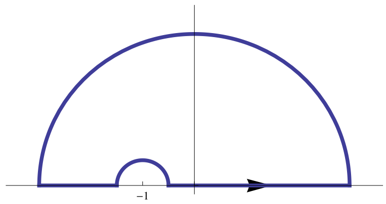

Figure 14.1: The contour for the principal part evaluation of $\int_{-\infty}^{\infty} dx(1+x^3)^{-1}$. The large semicircle of radius R is added to use the Residue theorem, the small semicircle isolates the singularity in $x=-1$.

Example 14.3.11.

$$\fint_{\mathbb{R}} dx \frac{1}{1+x^3} = \lim_{\epsilon\to\infty}\lim_{R\to 0}\left[\int_{-R}^{-1-\epsilon} + \int_{-1+\epsilon}^{R} dx\frac{1}{1+x^3}\right]$$

The singularity in $x=-1$ is a simple pole. The integral does not exist as an ordinary one, but as a principal part integral.
The contour is closed in the upper half plane and encircles the simple pole $e^{i\pi/3}$. The small semicircle centred in $x=-1$ is parameterized by $z=-1+\epsilon e^{i\theta}$. The integral is evaluated as the integral on the closed contour (residue theorem) minus the clock-wise *integral on the small semicircle:*

$$\begin{aligned}
&= 2\pi i \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{1+z^3} + \lim_{\epsilon\to 0}\int_0^{\pi} d\theta \frac{i\epsilon e^{i\theta}}{1+(-1+\epsilon e^{i\theta})^3} \\
&= \frac{2\pi i}{(e^{i\pi/3}+1)(e^{i\pi/3}-e^{-i\pi/3})} + i\lim_{\epsilon\to 0^+}\int_0^{\pi} d\theta\frac{\epsilon e^{i\theta}}{3\epsilon e^{i\theta}+\dots} \\
&= \frac{\pi}{2}\frac{e^{-i\pi/6}}{\cos(\pi/6)\sin(\pi/3)} + i\frac{\pi}{3} = \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

Exercise 14.3.12.

$$\fint_{\mathbb{R}} \frac{dx}{(x-a)(x-b)(x^2+1)} = \pi\frac{ab-1}{(a^2+1)(b^2+1)} \tag{14.14}$$

$$\fint_{\mathbb{R}} \frac{dx}{(2+x)(x^2+4)} = \frac{\pi}{8} \tag{14.15}$$

$$\fint_{\mathbb{R}} dx\frac{e^{ix}}{(x-a)(x-b)} = i\pi\frac{e^{ia}-e^{ib}}{a-b}. \tag{14.16}$$

$$\fint_{\mathbb{R}} dx\frac{e^{ikx}}{x-y} = i\pi e^{iky}\operatorname{sign} k \tag{14.17}$$

Example 14.3.13. *Evaluation of the integral:*

$$\boxed{\int_{-\infty}^{\infty} dx\,\frac{\sin x}{x} = \pi} \tag{14.18}$$

Consider the Cauchy integral $\oint_\gamma dz\frac{e^{iz}}{z}$ on the closed path made of segments $[-R,-\epsilon]$, $[\epsilon,R]$ and two semicircles of radii R (anticlockwise) and ϵ (clockwise) *in the upper half plane, centred in $z = 0$. The path avoids the pole $z = 0$, and the integral is zero:*

$$0 = \left[\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right] \frac{e^{ix}}{x} - i\int_0^{\pi} d\theta e^{i\epsilon e^{i\theta}} + \int_{\sigma(R)} dz \frac{e^{iz}}{z} \to \fint_{\mathbb{R}} dx \frac{e^{ix}}{x} - i\pi$$

The integral on the large half-circle vanishes for large R (Jordan's Lemma).

Exercise 14.3.14.

$$\int_{-\infty}^{\infty} dx\, \frac{\sin^2 x}{x^2} \;=\; \frac{1}{2}\mathrm{Re} \fint_{-\infty}^{\infty} dx \frac{1 - e^{i2x}}{x^2} = \pi \tag{14.19}$$

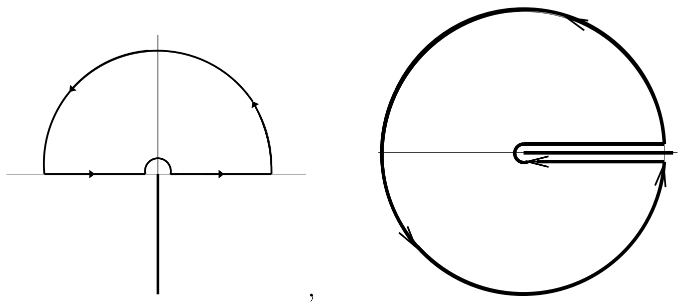

Figure 14.2: Left: Contour path with radii R and ϵ. The branch cut of the log is chosen as the negative imaginary axis. Right: the "keyhole" path, with branch cut in the positive real axis.

14.3.4 Integrals with branch cut

Certain integrals with log or non-integer powers can be evaluated by residues. We illustrate this by examples:

Example 14.3.15.

$$\int_0^{\infty} dx\, x^{a-1} \frac{\log x}{x^2 + 1}, \quad 0 < a < 2 \tag{14.20}$$

Use the contour γ shown in Fig.14.2; the log *function is chosen with the cut on the imaginary half-line. The integral is evaluated with the residue at the simple pole $z = i$:*

$$\oint_{\gamma} dz\, e^{(a-1)\log z} \frac{\log z}{z^2 + 1} = 2\pi i \lim_{z \to i} (z - i) e^{(a-1)\log z} \frac{\log z}{z^2 + 1} = \frac{\pi^2}{2} e^{ia\pi/2}$$

The same integral is the sum of two integrals on the real half-lines and the integrals on the semicircles, which are zero for $R \to \infty$ and $\epsilon \to 0$. Then

$$\begin{aligned} &= \int_{-\infty}^{0} dx\, e^{(a-1)(\log|x| + i\pi)} \frac{\log|x| + i\pi}{x^2 + 1} + \int_0^{\infty} dx x^{a-1} \frac{\log x}{x^2 + 1} \\ &= (1 - e^{ia\pi}) \int_0^{\infty} dx x^{a-1} \frac{\log x}{x^2 + 1} - i\pi e^{ia\pi} \int_0^{\infty} dx \frac{x^{a-1}}{x^2 + 1} \end{aligned}$$

Separation of real and imaginary parts gives two integrals:

$$\int_0^\infty dx \frac{x^{a-1}}{x^2+1} = \frac{\pi}{2\sin(a\pi/2)}, \quad \int_0^\infty dx \frac{x^{a-1}}{x^2+1} \log x = -\frac{\pi^2}{4} \frac{\cos(\pi a/2)}{\sin^2(\pi a/2)} \tag{14.21}$$

Example 14.3.16.

$$\int_0^\infty dx\, x^{a-1} \frac{\log x}{x+1}, \quad 0 < a < 1 \tag{14.22}$$

The path of the previous example does not help. The appropriate path is the keyhole path (see Fig.14.2), with the log branch cut chosen as the real positive half-line. The integral is evaluated with the residue at the simple pole $z = -1$:

$$\oint_\gamma dz e^{(a-1)\log z} \frac{\log z}{z+1} = 2\pi i \lim_{z\to -1} (z+1) e^{(a-1)\log z} \frac{\log z}{z+1} = 2\pi^2 e^{ia\pi}$$

The same integral is the sum of two integrals on two half-lines: the first one is just above the branch cut ($z = x + i\epsilon$, arg $z = 0$), the second one is just below the branch cut and with opposite orientation ($z = x - i\epsilon$, arg $z = i2\pi$). The large and small circles do not contribute. Then:

$$\begin{aligned} &= \int_0^\infty dx\, x^{a-1} \frac{\log x}{x+1} - \int_0^\infty dx\, e^{(a-1)(\log|x| + i2\pi)} \frac{\log|x| + i2\pi}{x+1} \\ &= (1 - e^{i2a\pi}) \int_0^\infty dx\, x^{a-1} \frac{\log x}{x+1} - i2\pi e^{i2a\pi} \int_0^\infty dx \frac{x^{a-1}}{x+1} \end{aligned}$$

By separating real and imaginary parts we obtain two integrals:

$$\int_0^\infty dx \frac{x^{a-1}}{x+1} = \frac{\pi}{\sin(a\pi)}, \quad \int_0^\infty dx \frac{x^{a-1}}{x+1} \log x = -\pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)} \tag{14.23}$$

Note that the integral with the log (and higher powers of the log) may be obtained by a derivative in the parameter a of the first integral.

Exercise 14.3.17.

$$1) \int_0^\infty dx \frac{\sqrt{x}}{x^3+1} = \frac{\pi}{3}, \quad \int_0^\infty dx \frac{x^{-1/3}}{x^2+a^2} = \frac{\pi}{\sqrt{3}} a^{-4/3} \quad (a > 0) \tag{14.24}$$

$$2) \int_0^\infty dx \frac{x^{-1/3}}{x^2+a^2} \cos(kx) = \frac{\pi}{\sqrt{3}} a^{-4/3} \cosh(ka), \quad k \in \mathbb{R} \tag{14.25}$$

$$3) \int_0^\infty dx \frac{x^{-1/3}}{x^2+a^2} \sin(kx) = \pi a^{-4/3} \sinh(ka), \quad k \in \mathbb{R} \tag{14.26}$$

$$4) \int_0^\infty dx \frac{x^{-1/3}}{x+1} = \frac{2\pi}{\sqrt{3}}, \quad \int_0^\infty dx \frac{x^{3/4}}{(x+1)^3} = \frac{3\sqrt{2}}{32}\pi \tag{14.27}$$

$$5) \int_0^\infty dx \frac{x^{1/4}}{(x+1)^2} = \frac{\pi}{2\sqrt{2}}, \quad \int_0^\infty dx \frac{x^{-1/n}}{(x+1)^2} = \frac{\pi/n}{\sin(\pi/n)} \tag{14.28}$$

$$6) \int_0^\infty dx \frac{\sqrt{x}}{x^3+x^2+x+1} = \frac{\pi}{2}(\sqrt{2}-1) \tag{14.29}$$

$$7) \int_0^\infty dx \frac{x^\mu}{x^2 - 2x\cos\theta + 1} = \pi \frac{\sin\mu(\pi-\theta)}{\sin\theta \sin(\mu\pi)}, \quad |\mu| < 1. \tag{14.30}$$

Integrals 2,3: first evaluate the integral with $\exp(ikx)$, then separate even/odd powers of k.

14.3.5 Integrals of hyperbolic functions

The exponential and the hyperbolic functions are periodic on the imaginary axis, so the trick of closing $[-R, R]$ with a semicircle at infinity does not work (the extra integral does not vanish). One rather exploits periodicity to close the path by a rectangle, such that the function on the new side $[-R+ih, R+ih]$ has a simple relation with the function on the real interval. The trick was used for the Fourier transform of the Gaussian function, (8.2). Here is another example:

Example 14.3.18.

$$\int_{\mathbb{R}} dx\, \frac{\cos(xy)}{\cosh x} = \pi \cosh(\frac{\pi}{2}y) \tag{14.31}$$

The integral on $[-R, R]$ *is closed by the rectangle with vertices* $(\pm R, 0)$, $(\pm R, i\pi)$, *where* $\cosh(x + i\pi) = -\cosh x$ *and the integrals on the short sides vanish for* $R \to \infty$. *The rectangle encloses the simple pole* $i\pi/2$. *The residue gives:*

$$2\pi i \lim_{z\to i\frac{\pi}{2}} \left(z - i\frac{\pi}{2}\right) \frac{\cos(zy)}{\cosh z} = 2\pi \cosh(\pi y)$$

the same integral evaluated on the boundary is:

$$= \int_{\mathbb{R}} dx \frac{\cos xy}{\cosh x} - \int_{\mathbb{R}} dx \frac{\cos y(x + i\pi)}{\cosh(x + i\pi)}$$
$$= [1 + \cosh(\pi y)] \int_{\mathbb{R}} dx \frac{\cos xy}{\cosh x} - i \sinh(\pi y) \int_{\mathbb{R}} dx \frac{\sin xy}{\cosh x}$$

The last integral is zero (odd function and symmetric domain).

Exercise 14.3.19.

$$\int_{-\infty}^{+\infty} dx \frac{x^2}{\cosh x} = \frac{\pi^3}{4}, \quad \int_0^{\infty} dx \frac{\cos(xy)}{\cosh^2 x} = \frac{\pi y/2}{\sinh(\pi y/2)} \tag{14.32}$$

Example 14.3.20.

$$\int_{-\infty}^{\infty} dk \frac{\sinh(ky)}{\sinh k}\, e^{ikx} = \frac{\pi \sin(\pi y)}{\cosh(\pi x) + \cos(\pi y)} \quad (0 < y < 1). \tag{14.33}$$

The integral, with a change $k \to -k$, *becomes:*

$$\int_{-\infty}^{\infty} \frac{dk}{2} \left[\frac{e^{k(y+ix)}}{\sinh k} + \frac{e^{k(y-ix)}}{\sinh k}\right] = \mathrm{Re} \fint_{-\infty}^{\infty} dk\, \frac{e^{k(y+ix)}}{\sinh k}.$$

Since $\sinh(z + i\pi) = -\sinh z$, *one considers a rectangle with corners* $-u$, v, $v + i\pi$, $-u + i\pi$, *where* $u, v \to \infty$. *Two sides are deformed by small half-circles of radius* ϵ *to exclude the singular points* $k = 0, i\pi$ *from the interior of the rectangle. By Cauchy's formula the loop-integral is zero, and the integrals on the sides parallel to the imaginary axis vanish in the limit. Then:*

$$0 = \left[1 + e^{i\pi(y+ix)}\right] \fint_{-\infty}^{\infty} dk \frac{e^{k(y+ix)}}{\sinh k} - \int_0^{\pi} i\epsilon e^{i\theta} d\theta \frac{e^{\epsilon e^{i\theta}(y+ix)}}{\sinh(\epsilon e^{i\theta})}$$
$$- \int_{\pi}^{2\pi} i\epsilon e^{i\theta} d\theta \frac{e^{(i\pi+\epsilon e^{i\theta})(y+ix)}}{\sinh(i\pi + \epsilon e^{i\theta})}$$

Therefore:

$$\fint_{-\infty}^{\infty} dk\, \frac{e^{k(y+ix)}}{\sinh k} = i\pi \frac{e^{\pi x} - e^{i\pi y}}{e^{\pi x} + e^{i\pi y}} \tag{14.34}$$

The real part gives the integral.

14.3.6 Other examples

Example 14.3.21.

$$\int_0^\infty dx \frac{1}{x^5+1} = \frac{\pi/5}{\sin(\pi/5)}, \quad \int_0^\infty dx\, \frac{\log x}{x^5+1} = -\left(\frac{\pi}{5}\right)^2 \frac{\cos(\pi/5)}{\sin^2(\pi/5)} \tag{14.35}$$

The denominator remains unchanged if x is replaced by $xe^{i2\pi/5}$. This suggests to close the real positive line with an arc of amplitude $2\pi/5$ and the radial line $z = xe^{i2\pi/5}$ (the second integral requires the first one).

Example 14.3.22.

$$\int_0^\infty dx\, \frac{\log x}{x^2-1} = \frac{\pi^2}{4} \tag{14.36}$$

The cut is chosen away from the real positive axis, and the point $x = 1$ is a removable singularity. If x is replaced by ix the integral is simple to evaluate. Theferore, integrate on the closed path formed by $[0, R]$, the circle $Re^{i\theta}$ $0 < \theta < \frac{\pi}{2}$ and the segment $[iR, 0]$, $R \to \infty$.

Example 14.3.23.

$$\int_0^\infty dx \frac{\log(x^2+1)}{x^2+1} = \pi \log 2 \tag{14.37}$$

Evaluate $\oint_\gamma dz\, \log(z+i)/(z^2+1)$ where γ is the real axis closed by a semicircle in the upper half plane. The cut of $\log(z+i)$ is any line connecting $-i$ to infinity, for example the line $[-i, -i\infty)$.

Example 14.3.24.

$$\int_0^\infty dx \frac{\log x}{(x+a)(x+b)} = \frac{1}{2} \frac{(\log b)^2 - (\log a)^2}{b-a} \tag{14.38}$$

If the keyhole path is considered, the integral is cancelled by the integral on the other side of the cut. The trick here is to evaluate the keyhole integral of the function $(\log z)^2/(z+a)(z+b)$ (see P. Nahin, *Inside interesting integrals*, Springer 2015*).*

The Mathematics Stack Exchange is a website devoted to questions and answers at any level. In particular the following pages contain hundreds of questions on contour integrals `http://math.stackexchange.com/questions/tagged/contour-integration`.

14.4 Enumeration of zeros and poles

Theorem 14.4.1. *Let $f(z)$ be meromorphic in D and γ a Jordan curve in D. If $z_1, \dots, z_n$ are the zeros (of order $k_1, \dots, k_n$) and $p_1, \dots p_m$ are the poles (of order $q_1, \dots, q_m$) of f encircled by γ, then:*

$$\boxed{\oint_\gamma \frac{dz}{2\pi i}\frac{f'(z)}{f(z)} = \sum_i k_i - \sum_j q_j.} \tag{14.39}$$

Proof. A zero or a pole of f is a pole for f'/f. In a neighbourhood of a zero z_i, $f(z) = A_i(z-z_i)^{k_i}\varphi_i(z)$ with φ_i analytic. Then

$$\frac{f'(z)}{f(z)} = \frac{k_i}{(z-z_i)} + \frac{\varphi_i'(z)}{\varphi_i(z)}$$

with residue k_i. In a neighbourhood of a pole, $f(z) = (z-p_j)^{-q_j}\phi_j(z)$ with ϕ_j analytic. Then

$$\frac{f'(z)}{f(z)} = \frac{-q_j}{(z-p_j)} + \frac{\phi_j'(z)}{\phi_j(z)}$$

and the residue is $-q_j$. The integral is the sum of the residues of all zeros and singular points encircled by γ. □

Exercise 14.4.2. *Show that:* $\int_{|z|=k\pi} \frac{dz}{2\pi i} \tan z = -2k$.

This beautiful theorem is useful for studying the location of zeros of analytic functions:

Theorem 14.4.3 (Rouché)**.** *Lef f and g be holomorphic functions on and inside a simple closed curve γ. If $|g(z)| < |f(z)|$ for all $z \in \gamma$, then f and $f+g$ have the same number of zeros inside γ.*

Proof. Since $|f| > |g|$ on γ it is $|f| \neq 0$ on γ and the variable $w = [f(z) + g(z)]/f(z)$ is well defined on γ. By the inequality $|w(z) - 1| < 1$, $z \in \gamma$, the image of the curve γ does not encircle the origin. By eq.(14.39)

$$\#(\text{zeros of } f+g) - \#(\text{zeros of } f) = \oint_\gamma \frac{dz}{2\pi i}\frac{w'(z)}{w(z)} = \frac{\Delta \arg w}{2\pi} = 0$$

(the zeros and poles of w are respectively the zeros of $f+g$ and of f). □

Example 14.4.4. *How many solutions of $e^z - 3z^4 + 5z = 0$ are in the disk $|z| < 2$? Choose $f(z) = -3z^4 + 5z$ and $g(z) = e^z$. On the boundary $|g(z)| = e^{2\cos\theta}$, $|f(z)| = |48e^{4i\theta} - 10e^{i\theta}| > 38$. Since $|f| > |g|$, the number of solutions equals that of $3z^4 - 5z = 0$ in $|z| < 2$, which is 4.*

14.5 Evaluation of sums

The theorem of residues can be used to evaluate infinite sums by reducing them to sums on a finite number of residues.

The functions $\pi\cot(\pi z)$ and $\pi\text{cosec}(\pi z)$ are meromorphic with simple poles on the real axis at $z_n = n \in \mathbb{Z}$, and residues respectively equal to 1 and $(-1)^n$. If $f(z)$ is analytic in the neighbourhood of the integer n, then:

$$\text{Res}[f(z)\pi\cot(\pi z), n] = f(n), \quad \text{Res}[f(z)\pi\text{cosec}(\pi z), n] = (-1)^n f(n).$$

Proposition 14.5.1. *Let $f(z)$ be meromorphic with a finite set of poles $\mathcal{P} = \{p_1, \ldots, p_m\}$ and suppose that there is $K > 0$ such that $|z^2 f(z)| < K$ for all z with $|z|$ larger than some constant R. Then*

$$0 = \sum_{n\in\mathbb{Z}/\mathcal{P}} f(n) + \sum_{k=1}^{n} \mathrm{Res}\,[f(z)\pi\cot(\pi z), p_k] \tag{14.40}$$

$$0 = \sum_{n\in\mathbb{Z}/\mathcal{P}} (-1)^n f(n) + \sum_{k=1}^{m} \mathrm{Res}\,[f(z)\pi\mathrm{cosec}(\pi z), p_k] \tag{14.41}$$

Proof. To apply the theorem of residues, consider a square path $\square$ with corners $\pm[n+\frac{1}{2} \pm i(n+\frac{1}{2}]$ big enough to include all poles p_k. Then:

$$\oint_\square \frac{d\zeta}{2\pi i} f(\zeta)g(\zeta) = \sum_a \mathrm{Res}[f(z)g(z), a]$$

where $a \in \{0, \pm 1, \ldots, \pm n\} \cup \{p_1, \ldots p_m\}$, and $g(z)$ is either $\pi\cot(\pi z)$ or $\pi\mathrm{cosec}(\pi z)$. On the sides of the squares it is $|\cot(\pi z)| \leq 1$ and $|\mathrm{cosec}(\pi z)| < 1$. Then Darboux's inequality and the bound on f ensure that the contour integral vanishes for $n \to \infty$:

$$\left|\oint_\square \frac{d\zeta}{2\pi i} f(\zeta)g(\zeta)\right| \leq \frac{4(2n+1)}{2\pi} \frac{K}{(n+1/2)^2}$$

$\square$

Corollary 14.5.2. *If f is meromorphic with poles in $p_1, \ldots, p_k \notin \mathbb{Z}$, then*

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \mathrm{Res}\,[f(z)\pi\cot(\pi z), p_k] \tag{14.42}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \mathrm{Res}\,[f(z)\pi\mathrm{cosec}(\pi z), p_k] \tag{14.43}$$

Example 14.5.3. *The function $f(\zeta) = 1/(\zeta^2 + z^2)$ has simple poles $\pm iz$. Then*

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+z^2} = \sum_{p=\pm iz} \mathrm{Res}\left[\frac{\pi\cot(\pi\zeta)}{\zeta^2+z^2}, p\right] = \frac{\pi}{z}\coth(\pi z)$$

The result gives a representation for $\coth(\pi z)$:

$$\coth(\pi z) = \frac{1}{\pi z} + 2\frac{z}{\pi}\sum_{n=1}^{\infty} \frac{1}{z^2+n^2} \tag{14.44}$$

For $|z| < 1$ one may expand in geometric series: $\frac{1}{n^2+z^2} = \sum_\ell (-1)^\ell \frac{z^{2\ell}}{n^{2+2\ell}}$. The exchange of sums is allowed because the series are absolutely convergent:

$$\coth(\pi z) = \frac{1}{\pi z} + \frac{2}{\pi}\sum_{\ell=0}^{\infty} (-1)^\ell z^{2\ell+1}\zeta(2+2\ell) \tag{14.45}$$

Comparison with eq.(11.12) reveals the relationship of Riemann's Zeta at even integers with Bernoulli numbers:

$$\zeta(2\ell) = \frac{1}{2}\frac{(2\pi)^{2\ell}}{(2\ell)!}|B_{2\ell}| \tag{14.46}$$

The replacement $z \to iz$ *gives:*

$$\cot(\pi z) = \frac{1}{\pi z} + 2\frac{z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2}. \tag{14.47}$$

As $\cot(\pi z) = \frac{1}{\pi}\frac{d}{dz}\log\sin(\pi z)$ *and* $\frac{2z}{z^2-n^2} = \frac{d}{dz}\log(z^2 - n^2)$*, the famous product formula for the sine function results* (Euler, 1734)*:*

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \tag{14.48}$$

Example 14.5.4. *Consider* $f(z) = z^{-2}$*, with a double pole at the origin.* $\sum_{n\neq 0} n^{-2} = -\mathrm{Res}[\frac{\pi\cot(\pi z)}{z^2}, 0]$*. The residue is evaluated with the rule for poles of order* 3*:*

$$2\zeta(2) = -\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\pi z\cos(\pi z)}{\sin(\pi z)}$$

A different way to evaluate the residue is to produce the Laurent series directly, from known series, and read c_{-1}*:* $\frac{\pi\cos(\pi z)}{z^2\sin(\pi z)} = \frac{\pi[1-(\pi z)^2/2!+\ldots]}{z^3\pi[1-(\pi z)^2/3!+\ldots]} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \ldots$*. Then* $c_{-1} = -\pi^2/3$ *and*

$$\boxed{\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

Chapter 15

ELLIPTIC FUNCTIONS

The theory of elliptic functions, shaped by the masters Abel, Jacobi and Weierstrass, *is a threasurehouse of results whose variety, aesthetic appeal, and capacity of arousing our astonishment, has not since been equaled by research in any other area. But the circumstance that this theory can be applied to solve problems arising in many departments of science and engineering graces the topic with an additional aura ...*[1].
This is a short list: the motion of the pendulum, of a rigid body, geodesics in Schwarzschild metric[2], Korteweg de Vries equation[3], the area of the ellipsoid[4], the potential of a homogeneous ellipsoid, motion on ellipsoid[5], the equation for $\lambda\varphi^4$ in 1d[6], conformal mapping of quadrangles and related problems of electrostatics, hydraulics; solitary waves[7].

Before entering this realm of complex analysis, let us linger on some functions of real variable. As circular functions were first studied on the unit circle, extended to the real line, and then to the complex plane, let us define the elliptic sine and cosine, as Cayley named them. Later, they will be extended to the complex plane.

15.1 The elliptic sine and cosine

The equation of the ellipse with $b \le a$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has parametric representation $x = a\cos\theta$ and $y = b\sin\theta$. The eccentricity $k = \sqrt{1-(b/a)^2}$, is hereafter named the *modulus*. The distance from the origin

[1]from the preface of Derek F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag 1989. Other nice books: F. Bowman, *Introduction to Elliptic Functions with applications*, Dover 1961; J. V. Armitage and W. F. Eberlein, *Elliptic Functions*, London Math. Soc. Student text 67, Cambridge 2006; A. L. Markushevich, *Theory of functions*, Chelsea 1985.
[2]G. Scharf, `https://doi.org/10.4236/jmp.2011.24036`
[3]B. G. Dimitrov, `https://arxiv.org/abs/2301.00643`
[4]NIST Handbook of Mathematical Functions, `https://dlmf.nist.gov/19.33`
[5]P. Erdös, `https://aapt.scitation.org/doi/10.1119/1.1285882`
[6]M. Frasca, `https://doi.org/10.1140/epjc/s10052-014-2929-9`
[7]S. Liu et al., `https://doi.org/10.1016/S0375-9601(01)00580-1`

is $r = \sqrt{x^2 + y^2} = a\sqrt{1 - k^2 \sin^2 \theta}$.
This periodic function of the angle is now introduced (it is an elliptic integral):

$$u = \int_0^\theta \frac{d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}} \tag{15.1}$$

Let $K = u(\frac{\pi}{2})$ (a complete elliptic integral):

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2 x^2}} \tag{15.2}$$

Both K and u depend on the modulus k. It is simple to show that

$$u(\theta + \frac{\pi}{2}) = K + u(\theta)$$

Modulo periods $4K$ and 2π, there is a one-to-one correspondence between u, θ and points (x, y) of the ellipse. Special triples are: $[0, 0, (a, 0)]$, $[K, \frac{\pi}{2}, (0, b)]$, $[2K, \pi, (-a, 0)]$.
The elliptic cosine and sine are introduced as a new parametrization of the ellipse with eccentricity k:

$$x = a\,\mathrm{cn}(u|k), \quad y = b\,\mathrm{sn}(u|k) \tag{15.3}$$

Of course, $\mathrm{cn}(u|k) = \cos\theta$ and $\mathrm{sn}(u|k) = \sin\theta$ when u is $u(\theta)$ in (15.1), and

$$\boxed{\mathrm{cn}^2(u|k) + \mathrm{sn}^2(u|k) = 1} \tag{15.4}$$

A useful function, with no circular analogue, is

$$\mathrm{dn}(u|k) = \sqrt{1 - k^2 \mathrm{sn}^2(u|k)} \tag{15.5}$$

Some values are simple to obtain. For example: $\mathrm{cn}(0|k) = \cos 0 = 1$, $\mathrm{cn}(K|k) = \cos\frac{\pi}{2} = 0$, $\mathrm{cn}(2K|k) = \cos\pi = -1$, and $\mathrm{dn}(0|k) = 1$, $\mathrm{dn}(K|k) = 1 - k^2$, $\mathrm{dn}(2K|k) = 1$.

The periodicity of $u(\theta)$ reflects in the relation

$$\mathrm{cn}(u + K|k) = \cos(\theta + \frac{\pi}{2}) = -\sin\theta = -\mathrm{sn}(u|k)$$

and similarly: $\mathrm{sn}(u + K|k) = \mathrm{cn}(u|k)$, $\mathrm{dn}(u + 2K|k) = \mathrm{dn}(u|k)$. Other relations follow. In particular, cn and sn have period $4K$ and dn has period $2K$.

15.1.1 Derivatives

Derivatives are easily obtained from $\mathrm{sn}(u(\theta)|k) = \sin\theta$ etc.

$$\frac{d}{du}\mathrm{sn}(u|k) = \frac{d\theta}{du}\cos\theta = \sqrt{1 - k^2 \sin^2\theta}\cos\theta = \mathrm{dn}(u|k)\mathrm{cn}(u|k) \tag{15.6}$$

$$\frac{d}{du}\mathrm{cn}(u|k) = -\mathrm{dn}(u|k)\mathrm{sn}(u|k) \tag{15.7}$$

$$\frac{d}{du}\mathrm{dn}(u|k) = -k^2\mathrm{sn}(u|k)\mathrm{cn}(u|k) \tag{15.8}$$

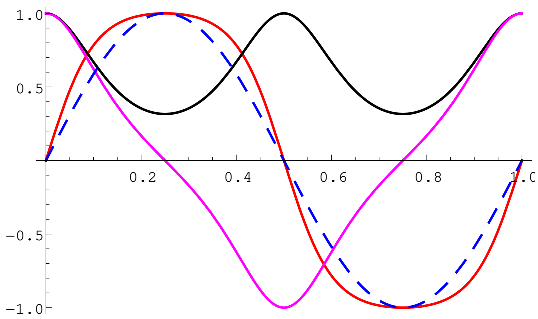

Figure 15.1: The Jacobi elliptic functions sn $(4Kx \,|\, k)$ (red), cn$(4Kx \,|\, k)$ (fuchsia), dn$(4Kx \,|\, k)$ (black) with $k = 0.9$, and the function $\sin(2\pi x)$ (dashed). Note the different aspect of sn and sin. As k increases, the maxima and minima of sn become more rounded.

They imply the interesting second-order differential equations:

$$\frac{d^2}{du^2}\mathrm{sn}(u|k) = -\,(1+k^2)\,\mathrm{sn}(u|k) + 2k^2\mathrm{sn}^3(u|k) \tag{15.9}$$

$$\frac{d^2}{du^2}\mathrm{cn}(u|k) = -\,(1-2k^2)\,\mathrm{cn}(u|k) - 2k^2\mathrm{cn}^3(u|k) \tag{15.10}$$

$$\frac{d^2}{du^2}\mathrm{dn}(u|k) = (2-k^2)\,\mathrm{dn}(u|k) - 2\,\mathrm{dn}^3(u|k) \tag{15.11}$$

Other equations are obtained by squaring the first derivatives; for example:

$$\left[\frac{d}{du}\mathrm{sn}(u|k)\right]^2 = 1 - (1+k^2)\mathrm{sn}^2(u|k) + k^2\mathrm{sn}^4(u|k) \tag{15.12}$$

15.1.2 Summation formulae

The elliptic functions have summation rules, more involved that their circular cousins ($k = 0$). The modulo k is omitted for brevity.

$$\mathrm{sn}(x_1 \pm x_2) = \frac{\mathrm{sn}(x_1)\mathrm{cn}(x_2)\mathrm{dn}(x_2) \pm \mathrm{sn}(x_2)\mathrm{cn}(x_1)\mathrm{dn}(x_1)}{1 - k^2\mathrm{sn}^2(x_1)\mathrm{sn}^2(x_2)} \tag{15.13}$$

$$\mathrm{cn}(x_1 \pm x_2) = \frac{\mathrm{cn}(x_1)\mathrm{cn}(x_2) \mp \mathrm{sn}(x_1)\mathrm{sn}(x_2)\mathrm{dn}(x_1)\mathrm{dn}(x_2)}{1 - k^2\mathrm{sn}^2(x_1)\mathrm{sn}^2(x_2)} \tag{15.14}$$

$$\mathrm{dn}(x_1 \pm x_2) = \frac{\mathrm{dn}(x_1)\mathrm{dn}(x_2) \mp k^2\mathrm{sn}(x_1)\mathrm{sn}(x_2)\mathrm{cn}(x_1)\mathrm{cn}(x_2)}{1 - k^2\mathrm{sn}^2(x_1)\mathrm{sn}^2(x_2)} \tag{15.15}$$

For $k = 0$ the familiar summation formula of the sine function is recovered.

Proof. Let $s_1(x) = \mathrm{sn}(x + x_1|k)$, $s_2(x) = \mathrm{sn}(x + x_2|k)$, etc. A prime denotes a derivative in x. Multiply (15.9) for s_1 by s_2, and subtract from it the equation with labels exchanged:

$$(s_2 s_1' - s_1 s_2')' = 2k^2 s_1 s_2 (s_1^2 - s_2^2)$$

Multiply (15.12) for s_1 by s_2^2 and subtract from it the equation with labels exchanged:

$$s_2^2(s_1')^2 - s_1^2(s_2')^2 = -(s_1^2 - s_2^2)(1 - k^2 s_1^2 s_2^2)$$

Divide the two equations and obtain:

$$\frac{(s_2 s_1' - s_1 s_2')'}{s_2 s_1' - s_1 s_2'} = -2k^2 \frac{s_1 s_2 (s_2 s_1' + s_1 s_2')}{1 - k^2 s_1^2 s_2^2} = \frac{(1 - k^2 s_1^2 s_2^2)'}{1 - k^2 s_1^2 s_2^2}$$

An integration gives $s_2 s_1' - s_1 s_2' = C(1 - k^2 s_1^2 s_2^2)$, where C is independent of x. With $s_i' = c_i d_i$ we obtain the algebraic relation valid for any x: $s_2 c_1 d_1 - s_1 c_2 d_2 = C(1 - k^2 s_1^2 s_2^2)$. For $x = -x_2$, it is $s_2 = 0$ and the identity gives $C = -\operatorname{sn}(x_1 - x_2|k)$. The first identity follows, the others are similarly obtained. □

Exercise 15.1.1. *Evaluate the special values (hint: put $x_1 = x_2 = K/2$):*

$$\operatorname{sn}\left(\frac{K}{2}\middle| k\right) = \frac{1}{\sqrt{1+k'}}, \quad \operatorname{cn}\left(\frac{K}{2}\middle| k\right) = \frac{\sqrt{k'}}{\sqrt{1+k'}}, \quad \operatorname{dn}\left(\frac{K}{2}\middle| k\right) = \sqrt{k'}. \quad (15.16)$$

where $k' = \sqrt{1-k^2}$ is the complementary modulus.

Let us show the connection with certain integrals that occur in physics and geometry.

15.2 Elliptic integrals

The following three classes of canonical integrals were studied by Legendre.
• The **elliptic integrals of the first kind** arise in the study of the pendulum. Let φ be the angular coordinate of a pendulum of length L, and φ_0 the maximal deviation of its motion from the vertical. Energy conservation gives: $\frac{1}{2}L^2\dot\varphi^2 = gL(\cos\varphi - \cos\varphi_0) = 2gL(\sin^2(\varphi_0/2) - \sin^2(\varphi/2))$. The time to swing from 0 to φ is:

$$t(\varphi) = \frac{1}{2}\sqrt{\frac{L}{g}} \int_0^{\varphi} \frac{d\varphi'}{\sqrt{\sin^2(\varphi_0/2) - \sin^2(\varphi'/2)}},$$

The period of the pendulum is $T = 4t(\varphi_0)$. For small oscillations ($\sin\varphi \approx \varphi$) $t(\varphi) = \sqrt{\frac{L}{g}} \int_0^{\varphi/\varphi_0} \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{L}{g}} \arcsin(\frac{\varphi}{\varphi_0})$. The period is $T_0 = 2\pi\sqrt{\frac{L}{g}}$ and inversion of $t(\varphi)$ gives $\varphi(t) = \varphi_0 \sin(\frac{2\pi t}{T})$.
In general, the change of variable $\sin(\varphi/2) = k\sin\theta$ with $k = \sin(\varphi_0/2)$, gives

$$t = \sqrt{\frac{L}{g}} \int_0^{\theta} \frac{d\theta'}{\sqrt{1 - k^2\sin^2\theta'}}, \quad T = T_0 \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta'}{\sqrt{1 - k^2\sin^2\theta'}}$$

The two integrals (forget the prefactors) are the incomplete and the complete elliptic integrals of the first kind.

$$F(\theta, k) = \int_0^{\theta} \frac{d\theta'}{\sqrt{1 - k^2\sin^2\theta'}}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta'}{\sqrt{1 - k^2\sin^2\theta'}} \qquad (15.17)$$

The incomplete integral is the function $u(\theta)$ introduced in (15.1). The period of the pendulum is $T = \frac{2}{\pi} T_0 K(k)$, $k = \sin\frac{\phi_0}{2}$. The motion of the pendulum is

$$\sin\frac{\varphi}{2} = k\sin\theta = k\,\mathrm{sn}(u|k) = k\,\mathrm{sn}(\frac{2\pi t}{T_0}|k)$$

• The **elliptic integrals of the second kind** arise in the evaluation of the arc-length of the ellipse. If $x = a\cos\theta$, $y = b\sin\theta$, then $ds^2 = dx^2 + dy^2 = a^2(1 - k^2\cos^2\theta)d\theta$. The integral for the arc-length is

$$L(\theta) = a\int_0^\theta d\theta'\sqrt{1 - k^2\cos^2\theta'} = a\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} d\theta'\sqrt{1 - k^2\sin^2\theta'}$$

The incomplete and complete elliptic integrals of the second kind are:

$$E(\theta, k) = \int_0^\theta d\theta'\sqrt{1 - k^2\sin^2\theta'}, \quad E(k) = \int_0^{\pi/2} d\theta'\sqrt{1 - k^2\sin^2\theta'} \tag{15.18}$$

Then $L(\theta) = aE(\frac{\pi}{2}, k) - aE(\frac{\pi}{2} - \theta, k)$. The perimeter is $4L(\frac{\pi}{2}) = 4aE(k)$.

• The **elliptic integral of the third kind** is:

$$\Pi(\theta, \alpha, k) = \int_0^\theta d\theta' \frac{1}{(1 + \alpha^2\sin^2\theta')\sqrt{1 - k^2\sin^2\theta'}} \tag{15.19}$$

The complete elliptic integrals K, E, $K' \equiv K(k')$ and $E' \equiv E(k')$ are linked by Legendre's relation:

$$KE' + K'E - KK' = \frac{\pi}{2} \tag{15.20}$$

The change of variable $x = \sin\theta'$ transforms the integrals F, E and Π into:

$$F(u|k) = \int_0^u \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} \tag{15.21}$$

$$E(u|k) = \int_0^u dx\sqrt{\frac{1 - k^2x^2}{1 - x^2}}, \tag{15.22}$$

$$\Pi(u|\alpha, k) = \int_0^u \frac{dx}{(1 + \alpha^2x^2)\sqrt{(1 - x^2)(1 - k^2x^2)}} \tag{15.23}$$

Elliptic integrals have the form $\int dx R\left(x, \sqrt{p_{3,4}(x)}\right)$ where R is a rational function of the real variable x and of the square root of a cubic or quartic polynomial with real coefficients (a higher degree defines hyperelliptic integrals). Legendre showed that any elliptic integral may be expressed as a combination of the three canonical elliptic integrals[8].

[8]A specialized book is P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd ed. 1971, Springer-Verlag.

A different treatment is based on the following *symmetric integrals of the I, II, and III kind*[9]

$$R_F(x,y,z) = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} \tag{15.24}$$

$$R_J(x,y,z,p) = \tfrac{3}{2}\int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}} \tag{15.25}$$

$$R_G(x,y,z) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, d\theta \sqrt{x\,n_x^2 + y\,n_y^2 + z\,n_z^2} \tag{15.26}$$

where $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a unit vector in spherical angles. The integrals are symmetric functions of $x, y, z \in \mathbb{C}/\{-\infty, 0]$. They are *complete* whenever one of the variables x, y, z is zero.

15.3 Jacobi Elliptic functions

In this section we extend the functions sn, cn and dn from real to complex variable. Their properties illustrate the general theory of elliptic functions.

The purely imaginary argument is introduced as follows. In the integral u, eq.(15.1), put $k^2 = 1 - k'^2$ (k' is the complementary modulus):

$$u = \int_0^\theta \frac{d\theta'}{\sqrt{\cos^2\theta' + k'^2\sin^2\theta'}} = \int_0^\theta \frac{d\theta'}{|\cos\theta'|}\frac{1}{\sqrt{1+k'^2\tan^2\theta'}}$$

Now make the change $\sinh x' = \tan\theta'$, and obtain another integral for u:

$$u = \int_0^x \frac{dx'}{\sqrt{1+k'^2\sinh^2 x'}}, \quad \sinh x = \tan\theta = \frac{\mathrm{sn}(x|k)}{\mathrm{cn}(x|k)} \tag{15.27}$$

This identity defines the Jacobi elliptic function with imaginary argument:

$$\boxed{\mathrm{sn}(ix|k') = i\,\frac{\mathrm{sn}(x|k)}{\mathrm{cn}(x|k)}} \tag{15.28}$$

Accordingly, the other functions with imaginary argument are:

$$\mathrm{cn}(ix|k') = \cosh x = \frac{1}{\mathrm{cn}(x|k)}, \tag{15.29}$$

$$\mathrm{dn}(ix|k') = \sqrt{1 - k'^2\mathrm{sn}^2(ix|k')} = \frac{\mathrm{dn}(x|k)}{\mathrm{cn}(x|k)} \tag{15.30}$$

The Jacobi functions with modulus k' change sign for a shift $x \to x+2K'$, where $K' = K(k')$, and have period $4K'$. The ratio of two of them is periodic with period $2K'$. Therefore, the functions with imaginary argument (and modulus k) are periodic: $\mathrm{sn}(ix + i2K'|k) = \mathrm{sn}(ix|k)$ etc.
One may check, they obey the same differential equations and addition rules as the functions with real argument.

[9] `https://dlmf.nist.gov/19`

The Jacobi elliptic functions with complex argument $z = x + iy$ are *defined* through the summation formulae:

$$\begin{aligned} \mathrm{sn}(z|k) &= \frac{\mathrm{sn}(x|k)\mathrm{cn}(iy|k)\mathrm{dn}(iy|k) + \mathrm{cn}(x|k)\mathrm{dn}(x|k)\mathrm{sn}(iy|k)}{1 - k^2\mathrm{sn}^2(x|k)\mathrm{sn}^2(iy|k)} \\ &= \frac{\mathrm{sn}(x|k)\mathrm{dn}(y|k') + i\,\mathrm{cn}(x|k)\mathrm{dn}(x|k)\mathrm{sn}(y|k')\mathrm{cn}(y|k')}{1 - \mathrm{dn}^2(x|k)\mathrm{sn}^2(y|k')} \end{aligned} \tag{15.31}$$

$$\mathrm{cn}(z|k) = \frac{\mathrm{cn}(x|k)\mathrm{cn}(y|k') - i\,\mathrm{sn}(x|k)\mathrm{dn}(x|k)\mathrm{sn}(y|k')\mathrm{dn}(y|k')}{1 - \mathrm{dn}^2(x|k)\mathrm{sn}^2(y|k')} \tag{15.32}$$

$$\mathrm{dn}(z|k) = \frac{\mathrm{dn}(x|k)\mathrm{cn}(y|k')\mathrm{dn}(y|k') - ik^2\,\mathrm{sn}(x|k)\mathrm{cn}(x|k)\mathrm{sn}(y|k')}{1 - \mathrm{dn}^2(x|k)\mathrm{sn}^2(y|k')} \tag{15.33}$$

The summation formulae (15.13), (15.14), (15.15), the expressions for derivatives and the differential equations remain valid, with z replacing x.
Some useful properties are listed:

$$\overline{\mathrm{sn}(z|k)} = \mathrm{sn}(\overline{z}|k), \quad \overline{\mathrm{cn}(z|k)} = \mathrm{cn}(\overline{z}|k), \quad \overline{\mathrm{dn}(z|k)} = \mathrm{dn}(\overline{z}|k) \tag{15.34}$$

.

$$\mathrm{sn}(z+K|k) = \frac{\mathrm{cn}(z|k)}{\mathrm{dn}(z|k)}, \qquad \mathrm{sn}(z+iK'|k) = \frac{1}{k\,\mathrm{sn}(z|k)} \tag{15.35}$$

$$\mathrm{cn}(z+K|k) = -k'\frac{\mathrm{sn}(z|k)}{\mathrm{dn}(z|k)}, \qquad \mathrm{cn}(z+iK'|k) = -i\frac{\mathrm{dn}(z|k)}{k\,\mathrm{sn}(z|k)} \tag{15.36}$$

$$\mathrm{dn}(z+K|k) = \frac{k'}{\mathrm{dn}(z|k)}, \qquad \mathrm{dn}(z+iK'|k) = -i\frac{\mathrm{cn}(z|k)}{\mathrm{sn}(z|k)} \tag{15.37}$$

$$\mathrm{sn}(z+2K|k) = -\mathrm{sn}(z|k), \qquad \mathrm{sn}(z+i2K'|k) = \mathrm{sn}(z|k) \tag{15.38}$$

$$\mathrm{cn}(z+2K|k) = -\mathrm{cn}(z|k), \qquad \mathrm{cn}(z+i2K'|k) = -\mathrm{cn}(z|k) \tag{15.39}$$

$$\mathrm{dn}(z+2K|k) = \mathrm{dn}(z|k), \qquad \mathrm{dn}(z+i2K'|k) = -\mathrm{dn}(z|k) \tag{15.40}$$

Proposition 15.3.1. *The Jacobi elliptic functions are doubly periodic:*

$$\mathrm{sn}(z + n_1\,4K + n_2\,2iK'|k) = \mathrm{sn}(z|k), \tag{15.41}$$

$$\mathrm{cn}(z + n_1\,4K + n_2\,(2K + i2K')|k) = \mathrm{cn}(z|k), \tag{15.42}$$

$$\mathrm{dn}(z + n_1\,2K + n_2\,4iK'|k) = \mathrm{dn}(z|k). \tag{15.43}$$

Proposition 15.3.2 (zeros and poles)**.**
1) In the rectangle $0 \leq \mathrm{Re}z < 4K$, $0 \leq \mathrm{Im}z < 2K'$, *the function* $\mathrm{sn}(z|k)$ *has two simple zeros at* $z = 0$, $z = 2K$, *and two simple poles at* $z = iK'$, $z = 2K + iK'$ *with residues* $1/k$, $-1/k$.
2) In the parallelogram with vertices 0, $4K$, $6K + i2K'$, $2K + i2K'$ *the function* $\mathrm{cn}(z|k)$ *has two simple zeros at* $z = K$, $z = 3K$, *and two simple poles at* $z = K + iK'$, $z = 3K + iK'$ *with residues* $1/k$, $-1/k$.
3) In the rectangle $0 \leq \mathrm{Re}z < 2K$, $0 \leq \mathrm{Im}z < 4K'$ *the function* $\mathrm{dn}(z|k)$ *has two simple zeros at* $z = K + iK'$, $z = K + i3K'$, *and two simple poles at* $z = iK'$, $z = 2K + iK'$ *with residues* $-i$, i.

Remark 15.3.3. *For the three elliptic functions:*
1) the sum of residues is zero,
2) the numbers of zeros equals the number of poles,
3) the sum of the zeros equals the sum of the poles modulo a lattice vector.

15.3.1 Conformal map for the rectangle

Jacobi Elliptic functions define useful conformal transformations, the basic one being *the map of the rectangle onto the upper half plane.*

Proposition 15.3.4. *The analytic function* $w \to z = \mathrm{sn}(w|k)$ *maps* the rectangle $R = \{w : 0 < \mathrm{Re}\, w < K,\ 0 < \mathrm{Im}\, w < K'\}$ *conformally* on the first quadrant *of the* z *plane (note that* R *is half of a fundamental parallelogram).*

Proof. Let $w = u + iv$ and $z = x + iy$. In components the map is:

$$x = \frac{\mathrm{sn}(u|k)\mathrm{dn}(v|k')}{1 - \mathrm{dn}^2(u|k)\mathrm{sn}^2(v|k')}, \quad y = \frac{\mathrm{cn}(u|k)\mathrm{dn}(u|k)\mathrm{sn}(v|k')\mathrm{cn}(v|k')}{1 - \mathrm{dn}^2(u|k)\mathrm{sn}^2(v|k')}$$

If $w \in R$ then: $x \geq 0$, $y \geq 0$, and $\mathrm{sn}'(w|k) = \mathrm{cn}(w|k)\,\mathrm{dn}(w|k) \neq 0$; therefore R is mapped in the first quadrant, and the map is injective. It is sufficient to obtain the image of the boundary of the rectangle (the image of R is the interior of it).
1) the segment $0 < u < K$, $v = 0$ is mapped to $0 < x < 1$, $y = 0$ ($x = \mathrm{sn}(u|k)$);
2) the segment $u = K$, $0 < v < K'$ is mapped to the segment $1 < x < 1/k$, $y = 0$ ($x = 1/\mathrm{dn}(v|k')$);
3) the segment $K > u > 0$, $v = K'$ is mapped to $1/k < x < \infty$ and $y = 0$ ($x = 1/[k\,\mathrm{sn}(u|k)]$); as the point $z = \infty$ is reached, the image of the rectangle's boundary descends to the origin along the imaginary axis:
4) the segment $u = 0$, $K' > v > 0$ is mapped to $x = 0$ and $\infty > y > 0$ ($y = \mathrm{sn}(v|k')/\mathrm{cn}(v|k')$). □

Remark 15.3.5. *The corners* 0, K, $K + iK'$ *and* iK' *are mapped to* 0, 1, $1/k$ *and* ∞*.*
A segment $(a, a + iK')$ *in* R *is mapped to a line with both ends on the real axis:* $z_1 = \mathrm{sn}(a|k) < 1$ *and* $z_2 = 1/[k\,\mathrm{sn}(a|k)] > 1/k$*. The line is parameterized as follows, by* $t = \mathrm{sn}(v|k') \in (0, 1)$*,*

$$x(t) = \frac{\mathrm{sn}(a|k)\sqrt{1 - k'^2 t^2}}{1 - t^2\mathrm{dn}^2(a|k)}, \quad y(t) = \frac{\mathrm{cn}(a|k)\mathrm{dn}(a|k)t\sqrt{1 - t^2}}{1 - t^2\mathrm{dn}^2(a|k)}$$

A segment $(ib, K + ib)$ *is mapped to a line with ends* $z_1 = i\,\mathrm{sn}(b|k')/\mathrm{cn}(b|k')$ *and* $1 < z_2 = \mathrm{dn}(b|k') < 1/k$*. The line is parameterized by* u*, and is orthogonal to the lines of constant* u*.*
At the corner $w = 0$ *the map is analytic with nonzero derivative; therefore the right angle of the rectangle is mapped to the right angle at* $z = 0$*.*

Corollary 15.3.6. *The map* $w \to z = \mathrm{sn}^2(w|k)$ *takes conformally the rectangle* $0 < u < K$, $0 < v < K'$ *to* $\mathbb{H}$ *(the half plane* $\mathrm{Im} z > 0$*).*
The images of 0, K, $K + iK'$ *and* iK' *are, in the order,* 0, 1, $1/k^2$, ∞*.*

15.4 Doubly periodic functions

A complex function f is *periodic* if there is a complex number ω (the period) such that $f(z+\omega) = f(z)$ for all z. The exponential and the hyperbolic functions are periodic with $\omega = 2\pi i$, the trigonometric functions are periodic with $\omega = 2\pi$.

A function is *doubly periodic* if there are *two* periods ω and ω', not proportional by a real number, such that[10]:

$$f(z+\omega) = f(z) \quad \text{and} \quad f(z+\omega') = f(z), \qquad \forall z \in \mathbb{C}$$

Although Gauss was aware of doubly periodic functions, the subject was disclosed in 1827 by Niels H. Abel, and developed further by Jacobi and Weierstrass.

The cell with corners 0, ω, $\omega+\omega'$, ω' is a fundamental parallelogram $\square$, with oriented boundary ∂. The values of f in $\square$ determine the function everywhere. The points $n\omega + n'\omega'$, $n, n' \in \mathbb{Z}$, form a lattice in $\mathbb{C}$.

A doubly periodic entire function is necessarily constant (being continuous on the compact set $\square$ it is bounded, but then it is bounded on $\mathbb{C}$ by periodicity, i.e. it is constant by Liouville's theorem). We then have to allow for isolated poles:

Definition 15.4.1. An elliptic function is a doubly periodic meromorphic function. The number of poles (counted with their order) inside a fundamental parallelogram, is the order of the elliptic function.

A famous example is the Weierstrass elliptic function (1872):

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq 0} \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 - n\omega_2)^2} \tag{15.44}$$

It has a second order pole at each site of the lattice $m\omega_1 + n\omega_2$. For review and applications see `https://arxiv.org/pdf/1706.07371.pdf`.

These properties hold in general for elliptic functions:

Proposition 15.4.2. *Let $f(z)$ be an elliptic function and consider the poles p_k and the zeros a_k of the function $f(z) - a$ inside a fundamental parallelogram:*
1) the sum of the residues is zero:

$$\sum_k \text{Res}\,(f, p_k) = 0$$

2) the number of zeros equals the number of poles (both counted with their order), i.e. an elliptic function takes each complex value a number of times equal to its order:

$$\#a_k = \#p_k$$

3) the sum of the zeros minus the sum of the poles is a lattice point, i.e. the sum of the points where the function takes a fixed value equals (modulo a lattice point) the sum of the poles:

$$\sum_k a_k = \sum_k p_k + n\omega + n'\omega'$$

4) If two elliptic functions f and g have the same periods, then there is a polynomial $P(z,z')$ with constant coefficients such that $P(f(z), g(z)) = 0$. In particular, f satisfies a differential equation of the type

$$P(f(z), f'(z)) = 0 \tag{15.45}$$

[10] Jacobi proved that a non-constant single-valued analytic function whose singularities do not have limit points at finite distance cannot have more than two periods, not proportional by a real factor

Proof. By the theorem of residues: $2\pi i \sum_k \text{Res}(f, p_k) = \oint_\partial dz f(z)$; as f takes the same values on opposite edges (with opposite orientations), the integral is zero. The function $f'/(f-a)$ has simple poles at the zeros a_k of $f-a$ (with residue equal to the order of the zero), and simple poles p_k at the poles of f (with residues equal to the opposite of their order). Then:

$$\oint_\partial \frac{dz}{2\pi i} \frac{f'(z)}{f(z)-a} = \# a_k - \# p_k$$

the integral vanishes because of the periodicity of f and f'. Next, consider the integral

$$\oint_\partial \frac{dz}{2\pi i} z \frac{f'(z)}{f(z)-a} = \sum_k a_k - \sum_k p_k$$

The integral is evaluated by parametrizing the four sides of the parallelogram with ordered vertices 0, ω, $\omega+\omega'$, ω'. Using the periodicity of f and f' on opposite sides, the contour integral is:

$$\begin{aligned}
&\frac{\omega}{2\pi i}\int_0^1 dt[t\omega - (\omega' + t\omega)]\frac{f'(t\omega)}{f(t\omega)-a} + \frac{\omega'}{2\pi i}\int_0^1 dt[(\omega + t\omega') - t\omega']\frac{f'(t\omega')}{f(t\omega')-a} \\
&= \frac{\omega\omega'}{2\pi i}\int_0^1 dt \left[-\frac{f'(t\omega)}{f(t\omega)-a} + \frac{f'(t\omega')}{f(t\omega')-a}\right] \\
&= -\frac{\omega'}{2\pi i}\log[f(t\omega)-a]_{t=0}^1 + \frac{\omega}{2\pi i}\log[f(t\omega')-a]_{t=0}^1
\end{aligned}$$

As $f-a$ is periodic, the real parts of log at $t=1$ and $t=0$ cancel, while arguments may only differ by an integer multiple of $2\pi i$. Therefore the contour integral is $n\omega + n'\omega'$, i.e. for any choice of a:

$$\sum_k a_k = \sum_k p_k + n\omega + n'\omega'.$$

The last proposition is proven for example in M. Lavrentiev and B. Chabat, *Méthodes de la Théorie des fonctions d'une variable complexe*, Éditions de Moscou 1972. □

15.5 Theta functions

Closely related to elliptic functions are the Jacobi Theta functions[11]. Let's begin with

$$\vartheta_3(z|\tau) = \sum_{m\in\mathbb{Z}} e^{i\pi(m^2\tau + 2mz)} = 1 + 2\sum_{m=1}^{\infty} e^{i\pi m^2\tau}\cos(2\pi m z) \tag{15.46}$$

For Im $\tau > 0$ the series is everywhere absolutely convergent and defines an entire function. Besides the obvious periodicity $\vartheta_3(z|\tau) = \vartheta_3(z+1|\tau)$, it is easy to show that

$$\vartheta_3(z+\tau|\tau) = e^{-i\pi(\tau^2+2z)}\vartheta_3(z|\tau)$$

[11] N. I. Akhiezer, *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs 79, AMS

The logarithmic derivative has the property

$$\frac{\vartheta_3'(z+\tau|\tau)}{\vartheta_3(z+\tau|\tau)} = -2i\pi + \frac{\vartheta_3'(z|\tau)}{\vartheta_3(z|\tau)}$$

The ratio has simple poles at the zeros of ϑ_3 (if p is the order of the zero, the ratio has a simple pole with residue p). The derivative cancels the term $-2i\pi$ with the result that the function

$$\varphi(z|\tau) = \frac{d}{dz}\frac{\vartheta_3'(z|\tau)}{\vartheta_3(z|\tau)}$$

is an elliptic function with double poles at the zeros of ϑ_3 and periods 1, τ.
Next, consider the Theta function:

$$\vartheta_0(z|\tau) = \vartheta_3\left(z + \frac{1}{2}\middle|\tau\right) = \sum_{m\in\mathbb{Z}} (-1)^m e^{i\pi(m^2\tau + 2mz)}. \tag{15.47}$$

$\vartheta_0(z+\tau|\tau) = \vartheta_3\left(z+\tau+\frac{1}{2}|\tau\right) = e^{-i\pi(\tau^2+2z+1)}\vartheta_3(z+\frac{1}{2}|\tau) = -e^{-i\pi(\tau^2+2z)}\vartheta_0(z|\tau)$, and $\vartheta_0(z+1|\tau) = \vartheta_0(z|\tau)$. The ratio

$$\varphi_3(z|\tau) = \frac{\vartheta_3(z|\tau)}{\vartheta_0(z|\tau)} \tag{15.48}$$

is meromophic with isolated double poles at the zeros of ϑ_0. Since $\varphi_3(z+1|\tau) = \varphi_3(z|\tau)$ and $\varphi_3(z+\tau|\tau) = -\varphi_3(z|\tau)$, it is an elliptic function with periods 1 and 2τ. Two other Theta functions are:

$$\vartheta_1(z|\tau) = ie^{-i\pi(z-\frac{\tau}{4})}\vartheta_3\left(z + \frac{1-\tau}{2}\middle|\tau\right), \tag{15.49}$$

$$\vartheta_2(z|\tau) = e^{-i\pi(z-\frac{\tau}{4})}\vartheta_3\left(z - \frac{\tau}{2}\middle|\tau\right). \tag{15.50}$$

Exercise 15.5.1. *Prove that the ratios*

$$\varphi_1(z|\tau) = \frac{\vartheta_1(z|\tau)}{\vartheta_0(z|\tau)}, \quad \varphi_2(z|\tau) = \frac{\vartheta_2(z|\tau)}{\vartheta_0(z|\tau)} \tag{15.51}$$

are elliptic functions, with periods $2, \tau$ *and* $2, 1+\tau$.

15.6 Reduction of elliptic integrals

Example 15.6.1.

$$\int_a^x \frac{dx'}{\sqrt{(x'-a)(b-x')(c-x')}} = \frac{2}{\sqrt{c-a}}F\left(\theta, \sqrt{\frac{b-a}{c-a}}\right), \quad \sin^2\theta = \frac{x-a}{b-a}$$

where $a \le x \le b < c$. *Similarly:*

$$\int_a^x \frac{x'\,dx'}{\sqrt{(x'-a)(b-x')(c-x')}}$$

$$= \int_a^x dx'\left[\frac{c}{\sqrt{(x'-a)(b-x')(c-x')}} - \sqrt{\frac{c-x'}{(x'-a)(b-x')}}\right]$$

$$= \frac{2c}{\sqrt{c-a}}F\left(\theta, \sqrt{\frac{b-a}{c-a}}\right) - 2\sqrt{c-a}\,E\left(\theta, \sqrt{\frac{b-a}{c-a}}\right)$$

In both cases the Legendre form is obtained with the replacement $x' = a + (b - a)\sin^2\theta'$. *Then* $x = a + (b-a)\sin^2\theta$.

Exercise 15.6.2.

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \tfrac{1}{\sqrt{2}} K\left(\tfrac{1}{\sqrt{2}}\right)$$

The integral is also amenable to Euler's Beta function, $\frac{1}{4}B(\frac{1}{4},\frac{1}{2})$.

Exercise 15.6.3. *(from Bowman. Put* $t = \tan\theta$*)*

$$\int_0^x \frac{dt}{\sqrt{t^4 \pm 2t^2\cos\varphi + 1}} = \frac{1}{2}F\left(2\,\mathrm{arctg}\,x, \sqrt{\frac{1\mp\cos\varphi}{2}}\right) \tag{15.52}$$

$$\int_0^\infty \frac{dt}{\sqrt{t^4 \pm 2t^2\cos\varphi + 1}} = K\left(\sqrt{\frac{1\mp\cos\varphi}{2}}\right) \tag{15.53}$$

Jacobi elliptic functions are useful in the evaluation of elliptic integrals. A standard change of variable is:

$$x = \frac{A_1 + A_2\mathrm{sn}^2(u|k)}{A_3 + A_4\mathrm{sn}^2(u|k)}, \quad 0 \le u \le K \tag{15.54}$$

The constants A_i and the modulus k are chosen in order that $dx/\sqrt{P(x)} = g\,du$, where g is a constant.

Example 15.6.4.

$$\int_0^x \frac{dx'}{\sqrt{x'(1-x')(c-x')(d-x')}} \qquad x \le 1 \le c \le d$$

As $0 \le x' \le 1$, *we require* $x = 0$ *for* $u = 0$ *and* $x = 1$ *for* $u = K$ *in* (15.54), *then*

$$x = \frac{(1+q)\mathrm{sn}^2(u|k)}{1+q\,\mathrm{sn}^2(u|k)}, \qquad dx = 2(1+q)\frac{\mathrm{sn}(u|k)\mathrm{cn}(u|k)\mathrm{dn}(u|k)}{[1+q\,\mathrm{sn}^2(u|k)]^2}$$

$$\frac{dx}{\sqrt{P(x)}} = 2\sqrt{\frac{1+q}{cd}}\frac{\mathrm{dn}(u|k)\,du}{\sqrt{1-(1/c+q/c-q)\mathrm{sn}^2(u|k)}\,\sqrt{1-(1/d+q/d-q)\mathrm{sn}^2(u|k)}}$$

Set $q = 1/(d-1)$ *and* $k^2 = 1/c + q/c - q = (d-c)/[c(d-1)]$ *to eliminate one of the square roots and cancel the other with* $\mathrm{dn}(u|k)$. *Then:*

$$\int_0^x \frac{dx}{\sqrt{P(x)}} = 2\frac{1}{\sqrt{c(d-1)}}u = 2\frac{1}{\sqrt{c(d-1)}}F\left(\sqrt{\frac{x(d-1)}{d-x}}\Big|\sqrt{\frac{d-c}{c(d-1)}}\right)$$

Chapter 16

QUATERNIONS AND BEYOND

16.1 Quaternions and vector calculus.

After the successful construction of $\mathbb{C}$ as the set of pairs of real numbers, $(a,b) = a+ib$, with vector sum and distributive product with the rule $i^2 = -1$, W. R. Hamilton eagerly tried to generalize the construction of new number fields by considering triplets (a,b,c). After years of efforts, in 1843, he realized that a consistent multiplication could be defined for quadruplets (a,b,c,d), which he called *quaternions*.
With three units $I = (0,1,0,0)$, $J = (0,0,1,0)$, $K = (0,0,0,1)$ (instead of a single $i = (0,1)$) a quaternion is a number $q = a + bI + cJ + dK$, where a,b,c,d are real numbers. Multiplications are done with the rules $I^2 = J^2 = K^2 = -1$ and

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J$$

The set $\mathbb{H}$ of quaternions is a non-commutative algebra with conjugation $q^\dagger = a - bI - cJ - dK$ (i.e. $I^\dagger = -I$ etc.) and $(q_1q_2)^\dagger = q_2^\dagger q_1^\dagger$.
$\mathbb{H}$ has the inner product $(q|p) = \frac{1}{2}(q^\dagger p + p^\dagger q)$. The norm of a quaternion is $\|q\| = \sqrt{(q|q)} = \sqrt{a^2+b^2+c^2+d^2}$, with the property $\|qp\| = \|q\|\,\|p\|$[1].

A realization of the quaternion basis is given by the 2×2 complex Pauli matrices: $1 = \sigma_0$ (the unit 2×2 matrix), $I = i\sigma_3$, $J = i\sigma_2$ and $K = i\sigma_1$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16.1}$$

Quaternions are then represented by complex matrices, with the ordinary matrix operations. The algebra of sigma matrices is summarized by the matrix identity[2]

$$\boxed{\sigma_i\sigma_j = \delta_{ij}\sigma_0 + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k} \tag{16.2}$$

[1] The rule implies a nontrivial identity: given integers m_i and n_i there are integers p_i $(i = 1,2,3,4)$ such that $(m_1^2+m_2^2+m_3^2+m_4^2)(n_1^2+n_2^2+n_3^2+n_4^2) = p_1^2+p_2^2+p_3^2+p_4^2$.

[2] ϵ_{ijk} is the totally antisymmetric symbol; $\epsilon_{123} = -\epsilon_{213} = 1$ and cyclic permutations, zero otherwise.

In modern language a quaternion with $b = c = d = 0$ is a (real) scalar, and a purely imaginary quaternion $bI + cJ + dK$ $(a = 0)$ is a vector. Then a quaternion can be viewed as a pair $(a, \vec{v})$, and this is where vector calculus originated from.
Hamilton introduced the operator $\nabla = I\partial_x + J\partial_y + K\partial_z$, and named it *nabla*[3]. If q is a scalar then ∇q is a vector. If q is a vector, ∇q is a quaternion with scalar part $-\mathrm{div}\, q$ and vector part $\mathrm{rot}\, q$. Quaternions influenced James Clerk Maxwell in the formulation of his fundamental *Treatise on Electricity and Magnetism* (1873), though he preferred the more familiar Cartesian form[4].
It is interesting to notice that modern 3D vector calculus stemmed from quaternions (but also from the works of Hermann Grassmann, less known at the time). The inauguration was made independently by Josiah Gibbs[5], professor of chemical physics at Yale's university, and Oliver Heaviside, the engineer who formulated Maxwell's equations in the present vector form[6].

A natural approach to quaternions is to define them as pairs of complex numbers (z_1, z_2) with sum $(z_1, z_2)+(w_1, w_2) = (z_1+w_1, z_2+w_2)$, multiplication

$$(z_1, z_2)(w_1, w_2) = (z_1 w_1 - \overline{w}_2 z_2, z_2\overline{w}_1 + w_2 z_1), \tag{16.3}$$

and conjugation $\overline{(z_1, z_2)} = (\overline{z}_1, -z_2)$. A pair $(a + ib, c + id)$ identifies with the real combination $a+Ib+cJ+dK$ with units $I = (i, 0)$, $J = (0, 1)$ and $K = (0, i)$.

16.2 Octonions

A generalization of quaternions is $\mathbb{O}$, the set of *octonions* or Cayley numbers[7]. They can be defined as pairs of quaternions (q_1, q_2) with multiplication and conjugation defined as above, for the pairs of complex numbers. The octonion algebra is both non commutative and non associative (because of non associativity, octonions cannot be represented as matrices). The conjugation acts as follows: $(O_1 O_2)^\dagger = O_2^\dagger O_1^\dagger$.
In alternative, one can introduce 8 units, I_j whose multiplication table $I_i I_j = f_{ijk} I_k$ is not given here, and write $O = \sum_j o_j I_j$. The norm

$$\|O\|^2 = O^\dagger O = o_0^2 + o_1^2 + \ldots + o_7^2$$

has the property $\|O_1 O_2\| = \|O_1\|\|O_2\|$. This makes $\mathbb{O}$ a division algebra (see below). Octonions enter in the construction of certain exceptional Lie algebras. Quaternions and Octonions are examples of Clifford algebras[8], which are relevant in the study of Dirac's equation for odd spin particles (fermions).

[3] The symbol ∇ recalls the nabla, an ancient Hebrew musical instrument.
[4] *I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use ... especially in electrodynamics ... can be expressed far more simply by a few words of Hamilton's, than the ordinary equations. One of the most important features of Hamilton's method is the division of quantities into Scalars and Vectors.*
[5] J. Gibbs and E. Wilson, *Vector analysis*, 1901.
[6] A detailed account is in M. J. Crowe, *A history of vector analysis*, Dover reprint of Notre Dame University Press, 1967.
[7] John Baez, *Octonions*, Bull. Am. Math. Soc. 39 (2001) 145-205.
[8] V. V. Prasolov, *Problems and theorems in linear algebra*, translations of mathematical monographs 134, Am. Math. Soc.

Continuation of the process with pairs of octonions brings to *Sedenions*, of dimension 16, with weaker algebraic properties.

Remark 16.2.1. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ *are real* division algebras *of dimensions 1, 2, 4 and 8, i.e. the equations* $ax = b$ *and* $ya = b$ *(where* $a \neq 0$ *and* b *are elements of the algebra) have unique solutions* x *and* y *in the algebra (for commutative algebras* $x = y$*).*

Theorem 16.2.2 (Bott-Milnor (1958), Kervair (1958))**.** *The only possible dimensions of a real division algebra are 1, 2, 4, 8.*

Proof. The proofs (for 2, 4, 8) are based on the topological assertion that the only spheres $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ that admit n linearly independent vector fields (parallelizable spheres) are S^1, S^3 and S^7.
No algebraic proof of the theorem is known. □

Algebras with anticommuting units $\xi_i\xi_j + \xi_j\xi_i = 0$ (where in particular $\xi_i^2 = 0$) were developed since 1844 by Hermann Grassmann, a gymnasium professor. Grassmann calculus is nowadays the basis for the path integral description of fermions, supersymmetry, and a tool in the theory of disordered systems and random matrices.

Exercise 16.2.3. *1) Evaluate the quaternion product of two vectors* $(aI + bJ + cK)(a'I + b'J + d'K)$ *and show that it coincides with the vector product.*
2) Represent quaternions as complex matrices and find the inverse of a quaternion. Show how addition and multiplication translate into operations on the four-vectors $(a, \vec{b})^t$*.*

Part II

FUNCTIONAL ANALYSIS

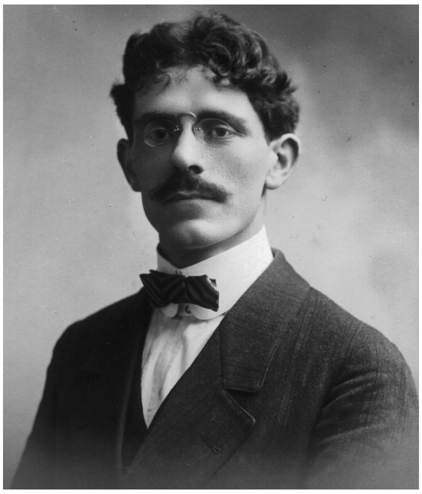

Figure 16.1: **Maurice Fréchet** (Maligny 1878, Paris 1973) had the fortunate chance of having the eminent mathematician Jacques Hadamard as teacher in secondary school. Hadamard noticed him and started an individual tutorship. In 1900 Maurice enrolled in mathematical studies at the École Normale Supérieure. His doctorate dissertation *Sur quelques points du calcul fonctionelle*, with Hadamard, contains the new concept of metric space. Fréchet served several institutions in France and abroad, with a period near the front-line during world war I.

Figure 16.2: **Stefan Banach** (Kraków 1892, Lviv 1945). The turn in his life happened in Kraków when the mathematician Hugo Steinhaus, during an evening walk, heard by chance the young engineer Banach and Otto Nikodym talking about Lebesgue measure. The three founded, with others, Kraków's (now Polish) Mathematical Society. The doctorate dissertation *Sur les opérations dans les ensembles abstraits et leur application aux équations integrales* (1920) contains the axioms of what Fréchet coined "Banach spaces". In 1924 he became full professor. His monograph *Théorie des Opérations linéaires* (1931) was very influential. After the German invasion in 1941 several collegues were murdered. He survived feeding lices in Weigl's institute for infectious diseases, but his health paid the toll. He died of cancer one year after the Soviets entered Lviv. Banach's main achievements are in functional analysis, measure theory, topological spaces, orthogonal series.

Chapter 17

METRIC SPACES

There are many contexts in mathematics where, given two items x and y (functions, matrices, sequences, operators, ...), one wishes to quantify how much they differ. In 1906 Maurice Fréchet, in his doctorate thesis, gave the axioms of a very general structure, the Metric Space, that allows to discuss topological concepts that arise in most situations of analysis.
Soon after, in 1914, Felix Hausdorff gave the axioms for Topological Spaces, the most general conceptualization of "nearness".

17.1 Metric spaces and completeness

Definition 17.1.1. A **Metric Space** (X, d) is a set X equipped with a distance between pairs of elements. A distance is characterized by the natural requirements of being symmetric, non negative, and constrained by the triangular inequality:

$$d(x, y) = d(y, x); \tag{17.1}$$

$$d(x, x) = 0, \quad d(x, y) > 0 \text{ if } x \neq y; \tag{17.2}$$

$$d(x, y) \leq d(x, z) + d(z, y) \qquad \forall x, y, z. \tag{17.3}$$

The distance defines a *topology* on X, i.e. a family of *neighbourhoods* at each point x. Such a family is the set of disks centred in x with radii $r > 0$:

$$D(x, r) = \{y \in X \,:\, d(x, y) < r\}.$$

With this definition, a metric space is a *Topological Hausdorff Space.* The following definitions are of great relevance:

Definition 17.1.2. A sequence x_n in X is *convergent* to $x \in X$ if the sequence of distances $d(x_n, x)$ converges to zero, i.e. $\forall \epsilon > 0\ \exists N_\epsilon$ such that $d(x_n, x) < \epsilon$ $\forall n > N_\epsilon$.

Definition 17.1.3. A sequence x_n is a **Cauchy sequence** (or a fundamental sequence) if $\forall \epsilon\ \exists\, N_\epsilon$ such that $d(x_m, x_n) < \epsilon\ \forall\, m, n > N_\epsilon$.

Exercise 17.1.4. *Prove that if x_n is a Cauchy sequence in a metric space, and x_{n_j} is a subsequence with limit x, then $x_n \to x$.*
Hint: use the triangle inequality $d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x)$.

Every convergent sequence is a Cauchy sequence (by the triangle inequality), but a Cauchy sequence may not converge. We are thus led to the important definition:

Definition 17.1.5. A metric space (X, d) is **complete** if every Cauchy sequence is convergent in X.

Completeness is a fundamental property: once a metric space is established to be complete, the Cauchy criterion ensures convergence of a sequence without knowledge of the limit.
A metric space that is not complete may be completed, in a manner similar to the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$.

Theorem 17.1.6. *If a metric space X is not complete, it is always possible to construct a metric space $\overline{X}$, the* completion *of X, which is complete.*

Proof. Consider the set of Cauchy sequences x_n in X. Two sequences are equivalent, $x_n \sim x'_n$, if they definitely approach:

$$\forall\epsilon \quad \exists\, N_\epsilon : \quad d(x_n, x'_m) < \epsilon \quad \forall\, m, n > N_\epsilon$$

The reflexive and symmetric properties are obvious, the transitive property is true by the triangle inequality: if $x_n \sim x'_n$ and $x'_n \sim x''_n$ then: $d(x_n, x''_m) \leq d(x_n, x'_p) + d(x''_m, x'_p) \leq 2\epsilon$ for all $m, n, p > \max(N_\epsilon, N'_\epsilon)$, i.e. $x_n \sim x''_n$.
Define $\overline{X}$ as the set of equivalence classes of Cauchy sequences in X. Such classes are of two types: for every $x \in X$ there is a class $[x]$ containing the constant sequence x and equivalent ones; the other type are the classes $[x_n]$ of Cauchy sequences with no limit in X.
$\overline{X}$ is a linear space by the rules: $[x_n + y_n] = [x_n] + [y_n]$ and $\lambda[x_n] = [\lambda x_n]$. It is a metric space with the distance

$$d([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n)$$

A finite limit exists because $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$: $|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_m, y_m) - d(x_n, y_m)| \leq d(y_n, y_m) + d(x_m, x_n) < 2\epsilon$ for $m, n > N_\epsilon$ because x_n and y_n are Cauchy sequences.
$\overline{X}$ is complete: let $[x_n]_m$ be a Cauchy sequence (a Cauchy sequence of equivalent Cauchy sequences): then $d([x_n]_p, [x_n]_q) = \lim_{n\to\infty} d(x_{n,p}, x_{n,q}) \leq \epsilon$ for $p, q > N_\epsilon$. The sequence $z_n = x_{n,n}$ is a Cauchy sequence: $d(z_n, z_m) = d(x_{n,n}, x_{m,m}) \leq d(x_{n,n}, x_{n,p}) + d(x_{n,p}, x_{m,p}) + d(x_{m,m}, x_{m,p}) < 3\epsilon$ for large enough n, m, p. The class $[z_n]$ is the limit in $\overline{X}$ of the Cauchy sequence: $d([x_n]_m, [z_n]) = \lim_{n\to\infty} d(x_{n,m}, x_{n,n}) \to 0$. □

17.2 Maps between metric spaces

Let $f : X \to Y$ be a map between metric spaces, with domain $\mathscr{D}(f)$. The image of the domain is the range $\mathrm{Ran}f = \{y \in Y : y = f(x),\ x \in \mathscr{D}(f)\}$.

Definition 17.2.1 (Continuity)**.** A map f is continuous at $x \in D(f)$ if:

$$\forall\epsilon\ \exists\,\delta_{\epsilon,x} \text{ such that } d(f(x'), f(x))_Y < \epsilon \quad \forall x' \in \mathscr{D}(f) \text{ with } d(x', x)_X < \delta_{\epsilon,x}.$$

A map f is sequentially continuous at $x \in \mathscr{D}(f)$ if for every sequence $\{x_n\}$ in $\mathscr{D}(f)$ with limit x it is $f(x_n) \to f(x)$ in Y

Theorem 17.2.2. *f is continuous at x if and only if f is sequentially continuous at x.*

Proof. Suppose that f is continuous and $x_n \to x$ in $\mathscr{D}(f)$. Then $\forall \epsilon > 0 \; \exists \, \delta_{\epsilon,x}$ such that $d(f(x_n), f(x)) < \epsilon$ for all x_n such that $d(x_n, x) < \delta_{\epsilon,x}$ i.e. for all $n > N_\delta$. This means precisely that $f(x_n) \to f(x)$.
To prove the converse, suppose that $x_n \to x$ and $x'_n \to x$ (same limit). Then $f(x_n)$ and $f(x'_n)$ converge by hypothesis. The limits coincide (suppose that they differ, then the sequence $x''_{2n} = x_n$ and $x''_{2n+1} = x'_n$ converges to x but $f(x''_n)$ has no limit, contrary to the hypothesis). Therefore all sequences that converge to x are mapped to sequences that converge to y. This implies continuity: $\lim_{z\to x} f(z) = y$ (otherwise a sequence of points z_n belonging to disks $d(z_n, x) < (1/n)$ for n large enough, would be mapped to a limit different from y). □

17.3 Contractive maps

Definition 17.3.1. A map $A : X \to X$ is a **contraction** if there is a constant $\alpha < 1$ such that $d(Ax, Ay) \le \alpha \, d(x, y)$ for every pair in X.

Exercise 17.3.2. *Show that a contraction is a continuous map.*

A point $x \in X$ is a fixed point for a map $A : X \to X$ if $Ax = x$. Contractions have this fundamental property:

Theorem 17.3.3 (**Fixed point theorem**)**.** *Every contraction on a complete metric space has a unique fixed point.*

Proof. Given an initial point x_0, let $x_k = A^k x_0$ be the sequence generated by iteration of the map. Let's show that x_k is a Cauchy sequence; for $n > m$:

$$\begin{aligned} d(x_n, x_m) &= d(Ax_{n-1}, Ax_{m-1}) \le \alpha d(x_{n-1}, x_{m-1}) \le \ldots \le \alpha^m d(x_{n-m}, x_0) \\ &\le \alpha^m [d(x_{n-m}, x_{n-m-1}) + \ldots + d(x_2, x_1) + d(x_1, x_0)] \\ &\le \alpha^m [\alpha^{n-m-1} + \ldots + \alpha + 1] \, d(x_1, x_0) \; \le \; \frac{\alpha^m}{1-\alpha} d(x_1, x_0) \end{aligned}$$

If m is large enough the estimate can be made smaller than any prefixed ϵ. Because of completeness, the Cauchy sequence converges to a limit $\overline{x}$. Since A is continuous: $A\overline{x} = A \lim_n x_n = \lim_n Ax_n = \lim_n x_{n+1} = \overline{x}$. Then a fixed point $\overline{x}$ exists (and is the limit of the sequence of iterates of a point).
Unicity is now proven. Suppose that a different fixed point $\overline{y}$ exists. The equality $d(\overline{x}, \overline{y}) = d(A\overline{x}, A\overline{y}) \le \alpha d(\overline{x}, \overline{y})$ implies $d(\overline{x}, \overline{y}) = 0$. □

Example 17.3.4 (Kepler's equation)**.** *A planet's orbit is an ellipse with semi-axis a and b and parametric equation $x = a \cos E$, $y = b \sin E$, where the angle E is the* eccentric anomaly. *The angle varies in time according to Kepler's equation, which translates the area law:*

$$M = E - e \sin E,$$

$e < 1$ is the eccentricity and the mean anomaly $M = 2\pi t/T$ *is measured by the time t from the perihelion passage, and T is the orbital period.*

The transcendental equation can be solved as a fixed point problem $E = K(E)$ *where* $K(E) = M + e\sin E$ *is a contraction on* $[0, 2\pi]$*:*

$$|K(E) - K(E')| = e|\sin E - \sin E'| = e|\cos E^*|\,|E - E'| \le e|E - E'|$$

The initial value $E_0 = M$ *can be used, then* $E_1 = K(E_0)$*, ... Convergence is faster the smaller is the eccentricity* e*.*

Example 17.3.5 (Volterra's equation)**.** *Consider the integral equation, with real* λ *and* $g \in \mathscr{C}[0,1]$*,* K *a continuous function on the unit square:*

$$u(x) = g(x) + \lambda \int_0^x dy K(x,y) u(y), \qquad x \in [0,1] \tag{17.4}$$

It can be written as a fixed point equation in $\mathscr{C}[0,1]$ *equipped with the sup norm:* $Tu = u$ *with* $(Tu)(x) = g(x) + \lambda \int_0^x dy K(x,y)u(y)$*. Let us evaluate:*

$$\begin{aligned} |(Tu_1)(x) - (Tu_2)(x)| \le & |\lambda| \int_0^x dy |K(x,y)||u_1(y) - u_2(y)| \\ \le & |\lambda| \sup_{y\in[0,1]} |u_1(y) - u_2(y)| \int_0^x dy |K(x,y)| \\ \le & |\lambda|\, K\, \|u_1 - u_2\|, \qquad K = \sup_{x\in[0,1]} \int_0^x dy |K(x,y)| \end{aligned}$$

Take the sup $x \in [0,1]$*:* $\|Tu_1 - Tu_2\| \le |\lambda| K\, \|u_1 - u_2\|$*. If* $|\lambda| K < 1$*,* T *is a contraction and the equation* $u = Tu$ *has a unique solution in* $\mathscr{C}[0,1]$*, which may be obtained by iteration:* $u_0 = g$*,* $u_1 = Tg$*,* $u_2 = Tu_1$ *....*

Chapter 18

BANACH SPACES

18.1 Normed and Banach spaces

Definition 18.1.1. A linear space X equipped with a *norm* is a **normed space**. A norm is defined by the properties ($x \in X$, $\lambda \in \mathbb{C}$):

$$\|\lambda x\| = |\lambda|\|x\|; \tag{18.1}$$

$$\|x\| \geq 0, \quad \|x\| = 0 \Leftrightarrow x = 0; \tag{18.2}$$

$$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|. \tag{18.3}$$

The definition of abstract normed space was given in the years 1920 - 22 by various authors, the most influential being the Polish mathematician Stefan Banach. It is straightforward to check that *a normed space is a metric space* with distance

$$d(x_1, x_2) = \|x_1 - x_2\| \tag{18.4}$$

A sequence $\{x_n\}$ converges to x if $\|x_n - x\| \to 0$.

Definition 18.1.2. A **Banach space** is a complete normed space (every Cauchy sequence is convergent).

Example 18.1.3. *The set $\mathscr{C}[a, b]$ of continuous functions $f : [a, b] \to \mathbb{C}$ is a Banach space with the sup-norm:*

$$\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$$

Proof. Completeness: let f_n be a Cauchy sequence: $\forall \epsilon > 0$ $\exists N_\epsilon$ such that for all $n, m > N_\epsilon$ it is $\sup_x |f_n(x) - f_m(x)| < \epsilon$. This implies that for each $x \in [a, b]$, the sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{C}$. Then it converges to a complex number which we name $f(x)$. This defines a function f. As $m \to \infty$, the Cauchy condition becomes $\sup_x |f_n(x) - f(x)| < \epsilon$, i.e. $f_n \to f$ in the sup-norm. It remains to show that f belongs to $\mathscr{C}[a, b]$. This amounts to prove that *a uniformly convergent sequence of continuous functions converges to a continuous function.* Continuity of f is proven as follows: $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f(y) - f_n(y)| + |f_n(x) - f_n(y)| \leq 3\epsilon$ for $|x - y|$ small and n large enough. □

The following linear spaces are Banach spaces in the sup norm:
$\mathscr{C}(\mathbb{R})$ (the set of continuous and bounded complex functions),
$\mathscr{C}_\infty(\mathbb{R})$ (the set of continuous functions that vanish at infinity). It is the completion of the normed space $\mathscr{C}_0(\mathbb{R})$ (the set of continuous functions with compact support).

18.2 The Banach spaces $\mathrm{L}^p(\Omega)$

Let Ω be $\mathbb{R}^n$ or a measurable subset, and consider the set of complex valued Lebesgue measurable functions $f : \Omega \to \mathbb{C}$. The sum of two functions and the product by a complex number λ are defined pointwise: $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x)$, and are measurable functions.

18.2.1 $\mathrm{L}^1(\Omega)$ (Lebesque integrable functions)

Consider the measurable functions on Ω such that

$$\int_\Omega dx\, |f(x)| < \infty \tag{18.5}$$

Since $|f(x) + g(x)| \le |f(x)| + |g(x)|$ and $|\lambda f(x)| = |\lambda||f(x)|$, it turns out that $f + g$ and λf are in $\mathscr{L}^1(\Omega)$ if f and g are. Then $\mathscr{L}^1(\Omega)$ is a linear space.

The integral (18.5) cannot define a norm because $\int |f|\, dx = 0$ implies $f = 0$ a.e. (almost everywhere) i.e. infinitely many functions, not the single null element $f = 0$ of the linear space. The remedy is to identify these functions by introducing equivalence classes: two functions $f, g \in \mathscr{L}^1$ are equivalent if $f = g$ a.e.
Let $[f]$ be the equivalence class containing f and all functions that differ from it on a set of measure zero. The set of equivalence classes of functions in $\mathscr{L}^1(\Omega)$ is a linear space with: $[f] + [g] = [f + g]$ and $\lambda[f] = [\lambda f]$. The linear space is $\mathrm{L}^1(\Omega)$, and it is a normed space with the norm[1]

$$\boxed{\|f\|_1 = \int_\Omega |f(x)|\, dx} \tag{18.6}$$

Now comes the beautiful proof of completeness:

Theorem 18.2.1 (Riesz - Fisher). $\mathrm{L}^1(\Omega)$ *is complete (is a Banach space).*

Proof. Let $\{f_n\}$ be a Cauchy sequence in L^1: $\forall\epsilon$ it is $\|f_n - f_m\|_1 < \epsilon$ for $n, m > N_\epsilon$. To prove convergence of the Cauchy sequence, it is sufficient to prove convergence of a subsequence, which is now produced.
For the choices $\epsilon = 1/2,\ 1/2^2, \dots, 1/2^k, \dots$ the Cauchy condition is satisfied for $N_\epsilon = N_1,\ N_2, \dots N_k, \dots$ The subsequence $f_{N_1},\ \dots,\ f_{N_k},\ \dots$ has the property $\|f_{N_{k+1}} - f_{N_k}\|_1 < 2^{-k}$.

Consider the sequence, with $f_{N_0} = 0$:

$$S_m(x) = \sum_{k=0}^{m-1} |f_{N_{k+1}}(x) - f_{N_k}(x)|$$

[1]as f can be any function in $[f]$, purists avoid writing $f(x)$, which can be any value by changing f within its class.

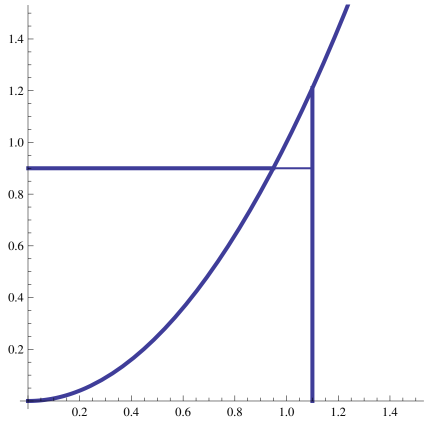

Figure 18.1: Construction for Hölder's inequality. The curve is $y = x^2$.

It has the properties: $S_m(x) \geq 0$; $S_{m+1}(x) \geq S_m(x)$ and $\|S_m\|_1 \leq \sum_{k=0}^{m} \frac{1}{2^k} \leq 2$. By Beppo Levi's theorem[2] the sequence S_m converges a.e. pointwise to a function S in $\mathrm{L}^1(\Omega)$ and $\int S_m dx \to \int S dx$.
Convergence of $S_m(x)$ is the absolute convergence of the sequence of partial sums $s_m(x) = \sum_{k=0}^{m-1} [f_{N_{k+1}}(x) - f_{N_k}(x)]$. Since the sum is telescopic, it is

$$s_m(x) = f_{N_m} \to f(x) \quad \text{a.e.}$$

i.e the subsequence f_{N_m} converges a.e. point-wise to a function f.
Since $|f_{N_m}(x)| = |s_n(x)| \leq S_n(x) \leq S(x)$ and S is integrable, it follows that f is integrable and $f_{N_m} \to f$ in L^1 (dominated convergence theorem)[3]. ☐

18.2.2 $\mathbf{L}^{\mathbf{p}}(\boldsymbol{\Omega})$ spaces

Consider the set $\mathscr{L}^p(\Omega)$ of measurable functions such that

$$\|f\|_p = \left(\int_\Omega |f|^p dx \right)^{\frac{1}{p}} < \infty \tag{18.7}$$

where $p \geq 1$. To show that $\mathscr{L}^p$ is a linear space one must show that it is closed for the sum of two functions. This follows from Minkowski's inequality, whose proof requires Hölder's inequality.

Proposition 18.2.2 (Hölder's inequality). *If $f \in \mathscr{L}^p(\Omega)$ and $g \in \mathscr{L}^q(\Omega)$, where $p, q > 1$ and $p^{-1} + q^{-1} = 1$ then: $fg \in \mathscr{L}^1(\Omega)$ and*

$$\boxed{\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad 1 = \frac{1}{p} + \frac{1}{q}} \tag{18.8}$$

[2] Monotone Convergence Theorem (B. Levi): Let f_n be a sequence of real integrable functions such that $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ a.e. in Ω and $\int_\Omega f_n dx \leq K$. Then the sequence $f_n(x)$ converges a.e. in Ω, $f_n(x) \to f(x)$, with $f \in \mathscr{L}^1(\Omega)$ and $\int_\Omega f_n \, dx \to \int_\Omega f \, dx$.

[3] Dominated Convergence Theorem: Given a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e $x \in \Omega$, and a function $g \in \mathscr{L}^1(\Omega)$ such that $|f_n(x)| \leq g(x)$ a.e. in Ω and for all n, then $f \in \mathscr{L}^1(\Omega)$ and $\int_\Omega f_n \, dx \to \int_\Omega f \, dx$.

Proof. Consider the curve $y = x^{p-1}$ in the positive quadrant. Inversion gives $x = y^{q-1}$. The area of the rectangle $0 < x < u$ and $0 < y < v$, is (see fig.18.1)

$$uv \le \int_0^u dx\, x^{p-1} + \int_0^v dy\, y^{q-1} = \frac{u^p}{p} + \frac{v^q}{q} \tag{18.9}$$

Put $u = |f(x)|/\|f\|_p$ and $v = |g(x)|/\|g\|_q$ and integrate x on Ω:

$$\int_\Omega dx \frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p}\int_\Omega dx \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\int_\Omega dx \frac{|g|^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q}.$$

The inequality is obtained. □

Remark 18.2.3. *If $m(\Omega) < \infty$, one may take $g = 1$ in Hölder's inequality, and note that if $f \in \mathscr{L}^p(\Omega)$ then $f \in \mathscr{L}^1(\Omega)$, i.e. $\mathscr{L}^p(\Omega) \subseteq \mathscr{L}^1(\Omega)$, for any $p > 1$.*

Proposition 18.2.4 (Minkowski's inequality). *If f, g in $\mathscr{L}^p(\Omega)$ then:*

$$\boxed{\|f + g\|_p \le \|f\|_p + \|g\|_p} \tag{18.10}$$

Proof. Inequality (18.9) with $u > 0$ and $v = z^p$, $\frac{1}{p} + \frac{1}{q} = 1$, is

$$uz^{p-1} \le \frac{u^p}{p} + \frac{z^p}{q}$$

Put $u = |f(x)|/\|f\|_p$, $z = |f(x) + g(x)|/\|f + g\|_p$ and integrate:

$$\int_\Omega dx\, |f(x)|\, |f(x) + g(x)|^{p-1} \le \|f\|_p \|f + g\|_p^{p-1}$$

Add the inequality with f and g exchanged:

$$\int_\Omega dx (|f(x)| + |g(x)|)\, |f(x) + g(x)|^{p-1} \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}$$

Since $|f(x) + g(x)| \le |f(x)| + |g(x)|$, the inequality becomes:

$$\int_\Omega dx |f(x) + g(x)|^p \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}$$

i.e. $\|f + g\|_p^p \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}$, and the result follows. □

The inequality implies that $\mathscr{L}^p(\Omega)$ is a linear space. To promote $\|f\|_p$ to a norm one must switch to the set of equivalence classes. Minkowski's inequality proves the triangle inequality, and $\mathrm{L}^p(\Omega)$ is a normed space. The Riesz-Fisher theorem states that it is complete (a Banach space) for all $p \ge 1$.

Hölder' inequality shows that the functions of $\mathrm{L}^q(\Omega)$ define linear continuous functionals on $\mathrm{L}^p(\Omega)$ if $1/p + 1/q = 1$. One actually proves that a space is the dual of the other.
The dual of $\mathrm{L}^1(\Omega)$ is the space $\mathrm{L}^\infty(\Omega)$, but the dual of L^∞ contains L^1 (see Reed and Simon, Functional Analysis, Academic Press).

18.2.3 $L^\infty(\Omega)$ space

$\mathscr{L}^\infty(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{C}$ that are a.e. bounded: given f, there is a constant M_f such that the set where $|f(x)| > M_f$ has Lebesgue measure zero. The inf of such bounds M_f is $\|f\|_\infty$, which becomes a norm for the equivalence classes of functions that are equal a.e. The space $L^\infty(\Omega)$ is a Banach space.
$\mathscr{C}(\mathbb{R})$ is a subspace of $\mathscr{L}^\infty(\mathbb{R})$. The norm coincides with the sup-norm.

18.3 Continuous and bounded maps

Hereafter $\hat{A} : X \to Y$ is a map (the term "operator" will be used) between normed spaces, with domain $\mathscr{D}(\hat{A})$. The image of the domain is the range $\mathrm{Ran}\hat{A} = \{y \in Y : y = \hat{A}x,\ x \in \mathscr{D}(\hat{A})\}$, the kernel is the set $\mathrm{Ker}\hat{A} = \{x \in X : \hat{A}x = 0\}$. We recall a definition and a result exported from the more general context of metric spaces:

Definition 18.3.1. A map $\hat{A}$ is continuous at $x \in \mathscr{D}(\hat{A})$ if:

$$\forall\epsilon\ \exists\,\delta_{\epsilon,x} \text{ such that } \|\hat{A}x' - \hat{A}x\|_Y < \epsilon \quad \forall x' \in \mathscr{D}(\hat{A}) \text{ with } \|x' - x\|_X < \delta_{\epsilon,x}.$$

The map is sequentially continuous at $x \in \mathscr{D}(\hat{A})$ if for any sequence $\{x_n\} \to x$ in $\mathscr{D}(\hat{A})$ it is $\hat{A}x_n \to \hat{A}x$.

Theorem 18.3.2. *$\hat{A}$ is continuous at x if and only if $\hat{A}$ is sequentially continuous in x (see metric spaces).*

Remark 18.3.3. *The kernel of a continuous operator is a closed set. (Suppose that x_n is a sequence in $\mathrm{Ker}\hat{A}$ and $x_n \to x \in \mathscr{D}(\hat{A})$. Then $\hat{A}x = \lim_Y \hat{A}x_n = 0$ i.e. $x \in \mathrm{Ker}\hat{A}$).*

Definition 18.3.4. $\hat{A}$ is **bounded** if there is a constant $C_A > 0$ such that $\|Ax\|_Y < C_A\|x\|_X$ for all $x \in \mathscr{D}(\hat{A})$.

18.3.1 Linear operators

Definition 18.3.5. $\hat{A}$ is **linear** if $\mathscr{D}(\hat{A})$ is a linear subspace and $\hat{A}(x + \lambda x') = \hat{A}x + \lambda\hat{A}x'$ for all $x, x' \in \mathscr{D}(\hat{A})$, $\lambda \in \mathbb{C}$.

Remark 18.3.6. *If $\hat{A}$ is linear, then $\hat{A}$ is continuous if and only if it is continuous at $x = 0$.*

Theorem 18.3.7. *A linear map $\hat{A} : \mathscr{D}(\hat{A}) \to Y$ is bounded if and only if it is continuous.*

Proof. Suppose that $\hat{A}$ is bounded and $x_n \to 0$ in $\mathscr{D}(\hat{A})$. Then $\|\hat{A}x_n\| \le C_A\|x_n\| \to 0$ i.e. $\hat{A}$ is sequentially continuous in the origin i.e. $\hat{A}$ is continuous. Let $\hat{A}$ be continuous. Then for $\epsilon = 1$ there is $\delta > 0$ such that $\|\hat{A}x\| < 1$ for all x in the domain with $\|x\| < \delta$. For any $y \in \mathscr{D}(\hat{A})$, put $x = y\delta/(2\|y\|)$. Then $\|x\| \le \delta$ so that $1 \ge \|\hat{A}x\| = \|\hat{A}y\|/\|y\|(\delta/2)$ i.e. $\hat{A}$ is bounded. □

Theorem 18.3.8. *Let X be a normed space and Y a Banach space. A linear bounded operator $\hat{A} : \mathscr{D}(\hat{A}) \subset X \to Y$ extends uniquely to a linear bounded operator on the closure of the domain: $\overline{A} : \overline{\mathscr{D}(\hat{A})} \to Y$.*

Proof. If $x \in \overline{\mathscr{D}}$ there is a convergent sequence x_n in $\mathscr{D}$ with limit x. A convergent sequence is Cauchy, therefore $\|\hat{A}x_n - \hat{A}x_m\|_Y \le C_A\|x_n - x_m\|_X \le \epsilon$ for $n, m > N_\epsilon$ i.e. $\hat{A}x_n$ is Cauchy in Y. Since Y is Banach, $\hat{A}x_n \to y$.
Define $\overline{A}x = y$. The domain $\overline{\mathscr{D}(A)}$ is a linear space: if $x_n \to x$ and $x'_n \to x'$ in $\overline{\mathscr{D}(A)}$, then $x_n + x'_n \to x + x'$ and $\lambda x_n \to \lambda x$. $\overline{A}$ is linear: $\overline{A}(x + x') = \lim_n \hat{A}x_n + \lim_n \hat{A}x'_n = \overline{A}x + \overline{A}x'$. $\overline{A}$ is bounded: since the norm is continuous, $\|\overline{A}x\| = \lim_n \|\hat{A}x_n\| \le C_A \lim_n \|x_n\| = C_A\|x\|$. □

18.3.2 The inverse operator

If $\hat{A} : X \to Y$ is an operator with domain $\mathscr{D}(\hat{A})$, the inverse operator $\hat{A}^{-1} : Y \to X$ exists if $\hat{A}$ is injective, i.e. $\hat{A}x = \hat{A}x'$ implies $x = x'$. Then, if $\hat{A}x = y$ it is $\hat{A}^{-1}y = x$, with $\mathscr{D}(\hat{A}^{-1}) = \mathrm{Ran}\,\hat{A}$, and $\mathrm{Ran}\,\hat{A}^{-1} = \mathscr{D}(\hat{A})$.

Proposition 18.3.9. *If $\hat{A}$ is linear, $\hat{A}^{-1}$ exists if and only if Ker $\hat{A} = \{0\}$. If $\hat{A}^{-1}$ exists, it is linear.*

Proof. The condition $\hat{A}x = \hat{A}y$ if and only if $x = y$ is equivalent to $\hat{A}(x-y) = 0$ iff $x - y = 0$ i.e. Ker $(A) = \{0\}$.
If $\hat{A}x = y$ and $\hat{A}x' = y'$ then $\hat{A}(\lambda x + x') = \lambda y + y'$. The inverse is such that $x = \hat{A}^{-1}y$, $x' = \hat{A}^{-1}y'$ and $\lambda x + x' = \hat{A}^{-1}(\lambda y + y')$, i.e. $\lambda\hat{A}^{-1}y + \hat{A}^{-1}y' = \hat{A}^{-1}(\lambda y + y')$, i.e. $\hat{A}^{-1}$ is linear. □

18.4 Linear bounded operators on X

The set $\mathscr{L}(X, Y)$ of linear operators on a normed space X, with domain X, to a normed space Y, is a linear space with the definitions

$$(\hat{A} + \hat{B})x = \hat{A}x + \hat{B}x, \quad (\lambda\hat{A})x = \lambda(\hat{A}x), \qquad x \in X,\ \lambda \in \mathbb{C}.$$

If two operators are bounded, their sum is bounded:

$$\|(\hat{A} + \hat{B})x\|_Y \le \|\hat{A}x\|_Y + \|\hat{B}x\|_Y \le (\|\hat{A}\| + \|\hat{B}\|)\|x\|_X, \qquad \forall x \in X \tag{18.11}$$

$\mathscr{B}(X, Y)$ is the linear space of *linear and bounded* operators from X to Y.

To be bounded is a very strong property of an operator $\hat{A}$. The best bound is the least constant C_A such that $\|\hat{A}x\|_Y/\|x\|_X \le C_A$ for all $x \in X$, $x \ne 0$. This constant is the **operator norm** of $\hat{A}$:

$$\boxed{\|\hat{A}\| = \sup_{x\in X, x\ne 0} \frac{\|\hat{A}x\|_Y}{\|x\|_X} = \sup_{\|x\|_X=1} \|\hat{A}x\|_Y} \tag{18.12}$$

As a consequence the best inequality is

$$\|\hat{A}x\|_Y \le \|\hat{A}\|\|x\|_X, \quad \forall x \in X.$$

Eq.(18.12) defines a norm and makes $\mathscr{B}(X, Y)$ a normed space: if $\|\hat{A}\| = 0$ it is $\|\hat{A}x\|_Y = 0$ for all x, i.e. $\hat{A} = 0$. The property $\|\lambda\hat{A}\| = |\lambda|\|\hat{A}\|$ is straightforward. The triangle inequality is obtained with (18.11): divide by $\|x\|_X \ne 0$ and take the sup: $\|\hat{A} + \hat{B}\| \le |\hat{A}\| + \|\hat{B}\|$.

.

Theorem 18.4.1. *If Y is a Banach space, then $\mathscr{B}(X,Y)$ is a Banach space.*

Proof. Let $\hat{A}_n$ be a Cauchy sequence of operators in $\mathscr{B}(X,Y)$:

$$\forall\epsilon \quad \exists N_\epsilon \text{ s.t. } \quad \|\hat{A}_n - \hat{A}_m\| < \epsilon \quad \forall n,m \geq N_\epsilon$$

It follows that also $\hat{A}_n x$ is a Cauchy sequence in Y by the inequality

$$\|\hat{A}_n x - \hat{A}_m x\|_Y \leq \|\hat{A}_n - \hat{A}_m\|\|x\|_X \tag{18.13}$$

Since Y is complete, the sequence $\hat{A}_n x$ converges. Define $\hat{A}$ as the operator $\hat{A}x = \lim_n \hat{A}_n x$. $\hat{A}$ is linear: $\hat{A}(x+\lambda y) = \lim_n \hat{A}_n(x+\lambda y) = \lim_n \hat{A}_n x + \lambda \lim_n \hat{A}_n y = \hat{A}x + \lambda\hat{A}y$.
By the inequality[4] $\Big|\|\hat{A}_n\| - \|\hat{A}_m\|\Big| \leq \|\hat{A}_n - \hat{A}_m\|$, the sequence of norms $\|\hat{A}_n\|$ is a Cauchy sequence in $\mathbb{R}$. As such, $\|A_n\|$ converges to a number α.
Eq. (18.13) implies: $\|\hat{A}_n x - \hat{A}_m x\|_Y \leq (\|\hat{A}_n\| + \|\hat{A}_m\|)\|x\|_X$ $m,n > N_\epsilon$. For infinite m: $\|\hat{A}_n x - \hat{A}x\|_Y \leq (\|\hat{A}_n\| + \alpha)\|x\|_X$. Therefore $\hat{A}_n - \hat{A}$ is bounded, i.e. $\hat{A}$ is bounded. $\hat{A}_n \to \hat{A}$ in the operator topology: $\|\hat{A}_n - \hat{A}\| = \sup_{\|x\|=1} \|\hat{A}_n x - \hat{A}x\|_Y \to 0$ because $\hat{A}_n x - \hat{A}x \to 0$ for all x. □

Remark 18.4.2. *Convergence $\hat{A}_n \to \hat{A}$ in $\mathscr{B}(X,Y)$ implies $\hat{A}_n x \to \hat{A}x$ in Y, for any x (norm convergence implies strong convergence).*

18.4.1 The dual space

In 1929 Banach introduced the important notion of **dual space** X^* of a Banach space X: it is the space of bounded linear functionals $\mathscr{B}(X,\mathbb{C})$. Since $\mathbb{C}$ is complete, X^* is a Banach space.

There are two ways by which a sequence may converge in X:
1) $x_n \to x$ *strongly* if $\|x_n - x\|_X \to 0$;
2) $x_n \to x$ *weakly* if $Fx_n \to Fx$ in $\mathbb{C}$ for all $F \in X^*$.

There are three ways by which a sequence of operators in $\mathscr{B}(X,Y)$ converges:
1) $\hat{A}_n$ converges in norm to $\hat{A}$ if $\|\hat{A}_n - \hat{A}\| \to 0$;
2) $\hat{A}_n$ converges strongly if $\hat{A}_n x$ converges in Y-norm for all $x \in X$;
3) $\hat{A}_n$ converges weakly if $F\hat{A}_n x$ converges in $\mathbb{C}$ for all $x \in X$, $F \in Y^*$.

18.5 The Banach algebra $\mathscr{B}(X)$

If $X = Y$, the space $\mathscr{B}(X) \equiv \mathscr{B}(X,X)$ is closed for the product. If $\hat{A}$ and $\hat{B}$ are linear and bounded operators $X \to X$, the operator $\hat{A}\hat{B}$ is defined by $(\hat{A}\hat{B})x = \hat{A}(\hat{B}x)$. The product is linear and bounded: $\|(\hat{A}\hat{B})x\| \leq \|\hat{A}\|\,\|\hat{B}x\| \leq (\|\hat{A}\|\|\hat{B}\|)\|x\|$. Therefore:

$$\boxed{\|\hat{A}\hat{B}\| \leq \|\hat{A}\|\,\|\hat{B}\|} \tag{18.14}$$

Then $\mathscr{B}(X)$ is an *associative algebra* (associative and distributive properties) with unit (the identity operator).

[4]By the triangle inequality $\|x\| \leq \|x-y\| + \|y\|$, it is always $\|x\| - \|y\| \leq \|x-y\|$. The same holds with x and y exchanged, so the modulus can be taken.

18.5.1 The inverse of a linear operator

Suppose that $\hat{A} \in \mathscr{B}(X)$ is invertible, and that $\operatorname{Ran}\hat{A} = X$, so that $\mathscr{D}(\hat{A}^{-1}) = X$. The inverse operator need not be bounded. From $\|x\|_X = \|\hat{A}\hat{A}^{-1}x\|_X \leq \|\hat{A}\|\|A^{-1}x\|_X$ for all x, it is

$$\frac{\|\hat{A}^{-1}x\|}{\|x\|} \geq \frac{1}{\|\hat{A}\|} \tag{18.15}$$

If A^{-1} is bounded, then $\|A^{-1}\| \geq \|A\|^{-1}$.

Theorem 18.5.1 (Neumann). *If X is a Banach space, $\hat{A} \in \mathscr{B}(X)$, and $\|\hat{A}\| < 1$, then $(1-\hat{A})^{-1}$ exists in $\mathscr{B}(X)$ and is given by the Neumann series:*

$$(1-\hat{A})^{-1} = \sum_{k=0}^{\infty} \hat{A}^k \tag{18.16}$$

Proof. Consider the partial sums $\hat{S}_n = I + \hat{A} + \hat{A}^2 + \cdots + \hat{A}^n$ in $\mathscr{B}(X)$. It is a Cauchy sequence: for any ϵ the norm $\|\hat{S}_{n+m} - \hat{S}_m\| = \|\hat{A}^{m+1} + \cdots + \hat{A}^{m+n}\| \leq \|\hat{A}\|^{m+1} + \cdots + \|\hat{A}\|^{m+n} \leq \frac{\|\hat{A}\|^{m+1}}{1-\|\hat{A}\|} \leq \epsilon$ for all n and all m greater that a suitable N. Therefore $\hat{S}_n$ converges to the geometric series $\sum_k \hat{A}^k$ in $\mathscr{B}(X)$.
Since $\|(1-\hat{A})\hat{S}_n - 1\| = \|\hat{A}^{n+1}\| \leq \|\hat{A}\|^{n+1} \to 0$ for $n \to \infty$, (18.16) is true. □

18.5.2 Power series of operators

The Neumann series is the operator analogue of the geometric series in complex analysis. Several important complex functions that are expressed as power series may be extended to functions of operators. Consider the two power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad f(\hat{A}) = \sum_{n=0}^{\infty} c_n \hat{A}^n$$

where $\hat{A} \in \mathscr{B}(X)$. If $\hat{S}_N$ are the operator partial sums, it is $\|\hat{S}_{N+p} - \hat{S}_N\| = \|\sum_{n=N+1}^{N+p} c_n \hat{A}^n\| \leq \sum_{n=N+1}^{N+p} |c_n|\,\|\hat{A}\|^n$. Therefore, the partial sums form a Cauchy sequence in $\mathscr{B}(X)$ if the partial sums of the power series $f(\|\hat{A}\|)$ are a Cauchy sequence in $\mathbb{C}$. A sufficient condition is

$$\|\hat{A}\| < R$$

where R is the radius of convergence of the power series $f(z)$.

Exercise 18.5.2. *Show that if $\hat{A}$ and $f(\hat{A})$ are in $\mathscr{B}(X)$, and if $\hat{A}x = \lambda x$, where $x \in X$, then $f(\hat{A})x = f(\lambda)x$.*

Example 18.5.3 (The exponential of an operator).
If $\hat{A} \in \mathscr{B}(X)$ and $z \in \mathbb{C}$, then the exponential series

$$e^{z\hat{A}} = \sum_{k=0}^{\infty} \frac{(z\hat{A})^k}{k!} \tag{18.17}$$

always converges in the operator norm to a bounded operator.

Proposition 18.5.4. *If $\hat{A}$ and $\hat{B}$ are commuting operators in $\mathscr{B}(X)$ then*

$$(\exp \hat{A})(\exp \hat{B}) = \exp(\hat{A} + \hat{B}) \tag{18.18}$$

Proof. Consider the product of partial sums:

$$\begin{aligned} S_n(\hat{A}) \cdot S_n(\hat{B}) =& (1 + \hat{A} + \tfrac{1}{2}\hat{A}^2 + ... + \tfrac{1}{n!}\hat{A}^n)(1 + \hat{B} + \tfrac{1}{2}\hat{B}^2 + ... + \tfrac{1}{n!}\hat{B}^n) \\ =& 1 + (\hat{A} + B) + \tfrac{1}{2}(\hat{A} + \hat{B})^2 + ... + \tfrac{1}{n!}(\hat{A} + \hat{B})^n + R_n(\hat{A}, \hat{B}) \\ R_n(\hat{A}, \hat{B}) =& \tfrac{\hat{A}\hat{B}^n}{n!} + \tfrac{\hat{A}^2}{2!}(\tfrac{\hat{B}^n}{n!} + \tfrac{\hat{B}^{n-1}}{(n-1)!}) + \tfrac{\hat{}A^3}{3!}(\tfrac{\hat{B}^n}{n!} + \tfrac{\hat{B}^{n-1}}{(n-1)!} + \tfrac{\hat{B}^{n-2}}{(n-2)!}) + ... \\ &+ \tfrac{\hat{A}^n}{n!}(\tfrac{\hat{B}^n}{n!} + \tfrac{\hat{B}^{n-1}}{(n-1)!} + \cdots + \hat{B}) \end{aligned}$$

Then $S_n(\hat{A})S_n(\hat{B}) - S_n(\hat{A} + \hat{B}) = R_n(\hat{A}, \hat{B})$. We prove that $\|R_n(\hat{A}, \hat{B})\| \to 0$ in $\mathscr{B}(X)$ as $n \to \infty$. The triangle inequality and $\|\hat{A}\hat{B}\| \leq \|\hat{A}\| \cdot \|\hat{B}\|$ give:

$$\begin{aligned} \|R_n(\hat{A}, \hat{B})\| \leq& \tfrac{\|\hat{A}\|}{1}\tfrac{\|\hat{B}\|^n}{n!} + \tfrac{\|\hat{A}\|^2}{2!}(\tfrac{\|\hat{B}\|^n}{n!} + \tfrac{\|\hat{B}\|^{n-1}}{(n-1)!}) + \tfrac{\|\hat{A}\|^3}{3!}(\tfrac{\|\hat{B}\|^n}{n!} + \tfrac{\|\hat{B}\|^{n-1}}{(n-1)!} + \tfrac{\|\hat{B}\|^{n-2}}{(n-2)!}) \\ &+ ... + \tfrac{\|\hat{A}\|^n}{n!}(\tfrac{\|\hat{B}\|^n}{n!} + \tfrac{\|\hat{B}\|^{n-1}}{(n-1)!} + \cdots + \tfrac{\|\hat{B}\|}{1}) = R_n(\|A\|, \|B\|) \end{aligned}$$

Note that $R_n(\|A\|, \|B\|) = S_n(\|A\|)S_n(\|B\|) - S_n(\|A\| + \|B\|)$. Since $S_n(\|A\|)$ converges to $\exp\|A\|$ in $\mathbb{R}$, then $R_n(\|A\|, \|B\|) \to 0$ as $n \to \infty$. □

In particular: $(\exp z\hat{A})(\exp w\hat{A}) = \exp(z+w)\hat{A}$ and $(\exp z\hat{A})^{-1} = \exp(-z\hat{A})$. For non-commuting operators, $(\exp \hat{A})(\exp \hat{B})$ is evaluated by the Baker - Campbell - Hausdorff formula.

Chapter 19

HILBERT SPACES

The theory of Hilbert spaces originates in the studies of David Hilbert and his student Erhard Schmidt on integral equations. The classification of linear integral equations bears the names of Vito Volterra (1860-1940) and Ivar Fredholm (1866-1927). The relevance of Hilbert spaces for the mathematical foundation of quantum mechanics led John von Neumann to give them an axiomatic setting, in the years 1929-30.

19.1 Inner product spaces

Definition 19.1.1 (Inner product). A linear space $\mathscr{H}$ (on $\mathbb{C}$) is an Inner Product Space (or pre-Hilbert space) if there is a map (the inner product) $(\ |\ ) : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ such that for all $x, y, z \in \mathscr{H}$ and $\lambda \in \mathbb{C}$:

$$(x|x) \geq 0 \quad \text{and} \quad (x|x) = 0 \quad \text{iff} \quad x = 0 \tag{19.1}$$

$$(x|y + z) = (x|y) + (x|z) \tag{19.2}$$

$$(x|\lambda y) = \lambda(x|y) \tag{19.3}$$

$$\overline{(x|y)} = (y|x). \tag{19.4}$$

The properties imply antilinearity in the first argument[1]:

$$(x + \lambda y|z) = (x|z) + \overline{\lambda}(y|z).$$

For real spaces the rule (19.4) is replaced by $(x|y) = (y|x)$, which implies bilinearity (linearity in both arguments of the inner product).
It will be shown that the inner product induces a norm; we begin by introducing the notation $\|x\| = \sqrt{(x|x)}$.

Exercise 19.1.2. *Show that: 1)* $(x|0) = 0$*; 2) if* $(x|y) = 0$ $\forall y$ *then* $x = 0$*.*

Definition 19.1.3. Two vectors x and x' are *orthogonal*, $x \perp x'$, if $(x|x') = 0$. A set of vectors $u_1, \ldots, u_n$ is an *orthonormal set* if $(u_i|u_j) = \delta_{ij}$.

Pythagoras' relation holds: if $x \perp x'$ then $\|x + x'\|^2 = \|x\|^2 + \|x'\|^2$.

[1] In the mathematical literature the inner product is often defined to be linear in the first and antilinear in the second argument. Physicists prefer the converse.

Theorem 19.1.4 (Bessel's inequality[2]). *Let $\{u_k\}_{k=1}^n$ be an orthonormal set, then for any x:*

$$\|x\|^2 \geq \sum_{k=1}^{n} |(u_k|x)|^2 \tag{19.5}$$

Proof. Set $x' = \sum_k (u_k|x)u_k$; since it is the sum of orthonormal vectors, $\|x'\|^2 = \sum_k |(u_k|x)|^2$. Next, let's show that x' and $x - x'$ are orthogonal: $(x'|x - x') = (x'|x) - \|x'\|^2 = \sum_k \overline{(u_k|x)}(u_k|x) - \|x'\|^2 = 0$. Then $\|x\|^2 = \|(x - x') + x'\|^2 = \|x - x'\|^2 + \|x'\|^2 \geq \|x'\|^2$. □

Bessel's inequality for $n = 1$ is $\|x\| \geq |(u|x)|$. With $u = y/\|y\|$ it gives a famous and fundamental inequality:

Proposition 19.1.5 (Schwarz's inequality). $\forall x, y \in \mathscr{H}$*:*

$$\boxed{|(x|y)| \leq \|x\|\|y\|} \tag{19.6}$$

Exercise 19.1.6. *Let $\|x\| = \|y\| = 1$. Show that if $(x|y) = 1$ then $x = y$, and if $|(x|y)| = 1$ then $x = (x|y)y$.*
Show that $|(x|y)| = \|x\|\,\|y\|$ if and only if $y = \lambda x$, $\lambda \in \mathbb{C}$.

19.2 The Hilbert norm

Proposition 19.2.1. *An inner product space is a normed space, with norm*

$$\boxed{\|x\| = \sqrt{(x|x)}} \tag{19.7}$$

Proof. We prove the triangle inequality: $\|x + y\|^2 = (x + y|x + y) = (x|x) + 2\text{Re}(x|y) + (y|y) \leq \|x\|^2 + 2|(x|y)| + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$. □

The existence of a norm implies notions such as continuity, convergent sequences, Cauchy sequences, etc.

Proposition 19.2.2. *Fix $x \in \mathscr{H}$, the function $(x|\cdot) : \mathscr{H} \to \mathbb{C}$ is continuous.*

Proof. Let $\{y_n\}$ be a convergent sequence, $y_n \to y$; we show that $(x|y_n) \to (x|y)$: $|(x|y_n) - (x|y)| = |(x|y_n - y)| \leq \|x\|\|y_n - y\| \to 0$. □

The following *Polarization formulae* express the inner product in terms of the Hilbert norm:

$$\text{Re}(x|y) = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \tag{19.8}$$
$$\text{Im}(x|y) = \tfrac{1}{4}(\|x - iy\|^2 - \|x + iy\|^2) \tag{19.9}$$

[2] Atle Selberg gave an extension of Bessel's inequality to non orthonormal vectors $y_1 ... y_n$ (for a proof see: *Inequalities in Hilbert spaces*, J. Wigestrand, 2008, `https://ntnuopen.ntnu.no/ntnu-xmlui/handle/11250/258406`):

$$\|x\|^2 \geq \sum_{k=1}^{n} \frac{|(y_k|x)|^2}{\sum_{j=1..n} |(y_j|y_k)|}$$

 

Figure 19.1: **David Hilbert** (Königsberg 1862, Göttingen 1943) continued the glorious tradition in mathematics of his predecessors Gauss, Dedekind and Riemann at the University of Göttingen, until the racial laws in 1933 dissolved his group. In 1899 he published the *Grundlagen der Geometrie* (The Foundations of Geometry) which contains the definitive set of axioms of Euclidean geometry. In his lecture on *The problems of mathematics* at the International Mathematical Congress in Paris (1900) he set forth a famous list of 23 problems for the mathematicians of the XX century. His studies on integral equations prepared for the important developments in functional analysis and quantum mechanics. He got interested in general relativity and derived the field equations from a variational principle. A partial list of famous students: Felix Bernstein, Richard Courant, Max Dehn, Erich Hecke, Alfréd Haar, Wallie Hurwitz, Hugo Steinhaus, Hermann Weyl, Ernst Zermelo.

Figure 19.2: **János von Neumann** (Budapest 1903, Washington 1957). Only few persons in XX century contributed to so many different fields. He made advances in axiomatic set theory, logic, theory of operators, measure theory, ergodic theory, the mathematical formulation of quantum mechanics, fluid dynamics, nuclear science, and is a founder of computer science. He started as a prodigiuos child: at the age of 15 he was tutored by the analyst Szegö; by the age of 19 he already published two major papers (providing the modern definition of ordinal numbers, which supersedes G. Cantor's definition). At 22 he received a PhD in mathematics and a diploma in chemical engineering (from ETH Zurich, to comply with his father's desire of a more practical orientation). He taught as Privatdozent at the University of Berlin, the youngest in its history. Since 1933 he was professor in mathematics at Princeton's Institute for Advanced Studies. He worked in the Manhattan Project.

The Hilbert norm has the *parallelogram property*, which can be directly checked out of its definition:

$$\boxed{\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2} \tag{19.10}$$

Proposition 19.2.3. *A norm is a Hilbert norm (i.e. there is a inner product such that the norm descends from it) if and only if it fulfils the parallelogram condition.*

Proof. We only need proving that a norm with the parallelogram property follows from an inner product. The formulae (19.8) and (19.9) suggest the following expression as a candidate for the inner product (for a real space the last two terms are absent):

$$\boxed{(x|y) \;=\; \tfrac{1}{4}\|x+y\|^2-\tfrac{1}{4}\|x-y\|^2-\tfrac{i}{4}\|x+iy\|^2+\tfrac{i}{4}\|x-iy\|^2} \tag{19.11}$$

The difficult task is to prove linearity in the second argument, the other properties being easy to prove. By repeated use of the parallelogram property,

$$\begin{aligned}\mathrm{Re}\,(x|y+z) &= \tfrac{1}{4}\|x+y+z\|^2-\tfrac{1}{4}\|x-y-z\|^2\\ &= (\tfrac{1}{2}\|x+y\|^2+\tfrac{1}{2}\|z\|^2-\tfrac{1}{4}\|x+y-z\|^2)-\tfrac{1}{4}\|x-y-z\|^2\\ &= \tfrac{1}{2}\|x+y\|^2+\tfrac{1}{2}\|z\|^2-\tfrac{1}{2}\|x-z\|^2-\tfrac{1}{2}\|y\|^2\end{aligned}$$

Now sum to it the expression with y and z exchanged, divide by 2 and obtain

$$\begin{aligned}\mathrm{Re}\,(x|y+z) &= \tfrac{1}{4}\|x+y\|^2-\tfrac{1}{4}\|x-z\|^2+\tfrac{1}{4}\|x+z\|^2-\tfrac{1}{4}\|x-y\|^2\\ &= \mathrm{Re}\,(x|y)+\mathrm{Re}\,(x|z).\end{aligned}$$

The same procedure works for the imaginary part: $\mathrm{Im}\,(x|y+z)=\mathrm{Im}\,(x|y)+\mathrm{Im}\,(x|z)$. Hence the additive property is proven.
Linearity for scalar multiplication: for $p\in\mathbb{N}$, additivity implies $(x|py)=p(x|y)$. For $q\in\mathbb{N}$: $p(x|y)=p(x|q(y/q))=pq(x|y/q)=q(x|p(y/q))\Rightarrow\frac{p}{q}(x|y)=(x|\frac{p}{q}y)$. Since the inner product is continuous and $(x|-y)=-(x|y)$ (see (19.11)), the property extends from rationals p/q to real numbers. Moreover, since $(x|iy)=i(x|y)$ holds (see (19.11)), the property holds for complex numbers. ☐

Exercise 19.2.4. *In a normed space, a vector x is orthogonal to y in the sense of Birkhoff-James if and only if $\|x\|\le\|x+\lambda y\|$ for all $\lambda\in\mathbb{C}$. Prove that in a inner-product space the definition coincides with $(x|y)=0$.*

Definition 19.2.5. A **Hilbert space** is a inner product space which is complete in the Hilbert norm topology.

Example 19.2.6. *The linear space $\mathbb{C}^n$ is a Hilbert space with the scalar product $(\mathbf{z}|\mathbf{w})=\sum_k\overline{z_k}w_k$.*

Exercise 19.2.7. *Show that the set of complex matrices $\mathbb{C}^{n\times n}$ is a Hilbert space with inner product $(A|B)=\mathrm{tr}(A^\dagger B)$. The corresponding norm*

$$\|A\|=\sqrt{\mathrm{tr}A^\dagger A}=\sqrt{\sum_{ij}|A_{ij}|^2}$$

is known as the Frobenius norm.
Show that also $(A|B) = \mathrm{tr}(PA^\dagger B)$ *is an inner product, if* $P > 0$ *(a matrix is positive if and only if* $P = M^\dagger M$*, with* M *invertible).*

Example 19.2.8. $\mathrm{L}^2(\Omega)$ *is the Banach space of (equivalence classes of) Lebesgue* square integrable *functions, with squared norm* $\|f\|_2^2 = \int_\Omega |f|^2 dx$*. The norm has the parallelogram property, and descends from the inner product*

$$\boxed{(f|g) = \int_\Omega \overline{f} g\, dx} \tag{19.12}$$

This is perhaps the most important space in functional analysis. Depending on the measurable set Ω*, one introduces important families of orthogonal functions (see section 19.4.1).*

Exercise 19.2.9. *Evaluate the norms of* $f(x) = 1/(1 + ix)$ *as an element in* $\mathrm{L}^1(0, 1)$ *and in* $\mathrm{L}^2(0, 1)$

Example 19.2.10. *The set* $\mathscr{C}[-1, 1]$ *of continuous complex-valued functions on* $[-1, 1]$ *is an inner product space with* $(f|g) = \int_{-1}^{1} dx\overline{f(x)}g(x)$*, but it is not a Hilbert space as this counter-example shows:*

$$f_n(x) = \begin{cases} -1 & -1 \le x \le -1/n \\ nx & -1/n < x < 1/n \\ 1 & 1/n \le x \le 1 \end{cases} \tag{19.13}$$

is a Cauchy sequence (i.e. $\forall \epsilon$ *there is* N *such that* $\int_{-1}^{1} dx|f_n(x) - f_m(x)|^2 < \epsilon^2$ *for all* $n, m > N$*) but the limit function (the step function) is not continuous.*

19.3 Isomorphism

Definition 19.3.1. Two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are *isomorphic* if there is a linear operator $\hat{U} : \mathscr{H}_1 \to \mathscr{H}_2$ that is a bijection among the two spaces, and conserves the norm $\|\hat{U}x\|_2 = \|x\|_1\ \forall x$. $\hat{U}$ is a **unitary** operator.

Remark. By the polarization formula (19.11), norm conservation and linearity imply the conservation of inner products: $(\hat{U}x|\hat{U}y)_2 = (x|y)_1\ \forall x, y \in \mathscr{H}_1$.

Any complex n-dimensional Hilbert space $\mathscr{H}_n$ is isomorphic to $\mathbb{C}^n$. Given an orthonormal basis $\{u_k\}_{k=1}^n$ in $\mathscr{H}_n$, the expansion $x = \sum_{k=1}^n x_k u_k$ has coefficients $x_k = (u_k|x)$ that define a vector $\mathbf{x} \in \mathbb{C}^n$. The map $\hat{U}x = \mathbf{x}$ is a unitary operator from $\mathscr{H}_n$ to $\mathbb{C}^n$.
Is there a canonical isomorphism for infinite-dimensional Hilbert spaces? The answer is affirmative, with a distinction about the cardinality of the linear basis. The first, and most important, case is that of countable basis set. We'll show that they are in this class:

Definition 19.3.2. A Hilbert space is **separable** if it has a countable dense subset.
The infinite dimensional analogue of $\mathbb{C}^n$ (or $\mathbb{R}^n$ for real Hilbert spaces) is the following.

19.3.1 Square summable sequences

In 1906 David Hilbert introduced the set $\ell^2(\mathbb{C})$ of complex sequences $\{a_k\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty |a_k|^2 < \infty$$

• $\ell^2(\mathbb{C})$ is a linear space. If $a = \{a_k\}$ and $b = \{b_k\}$, define:

$$a + b = \{a_k + b_k\}, \qquad \lambda a = \{\lambda a_k\}$$

The inequality $|a_k + b_k|^2 \le |a_k|^2 + |b_k|^2 + 2|a_k b_k| \le 2|a_k|^2 + 2|b_k|^2$ (use $0 < (x-y)^2 = x^2 + y^2 - 2xy$) implies that $a + b \in \ell^2(\mathbb{C})$.
• $\ell^2(\mathbb{C})$ is an inner product space with

$$(a|b) = \sum\nolimits_{k=1}^\infty \overline{a_k} b_k \tag{19.14}$$

The series converges absolutely: $2|a_k b_k| \le |a_k|^2 + |b_k|^2$. The formal properties of inner product are easily checked.
• $\ell^2(\mathbb{C})$ is complete (it is a Hilbert space).

Proof. Let $\{a_\nu\}$ be a Cauchy sequence of elements in ℓ^2 (a sequence of sequences $\{a_{\nu k}\}$), i.e. for any $\epsilon > 0$ there is N such that for all $\nu, \mu > N$ it is

$$\|a_\nu - a_\mu\|^2 = \sum\nolimits_{k=1}^\infty |a_{\nu k} - a_{\mu k}|^2 < \epsilon^2 \tag{19.15}$$

We show that a_ν converges in $\ell^2(\mathbb{C})$. The Cauchy condition implies that $|a_{\nu k} - a_{\mu k}| < \epsilon$ for all k and $\nu, \mu > N$, then each sequence $\{a_{\nu 1}\}$, $\{a_{\nu 2}\}$, ... is Cauchy and converges in $\mathbb{C}$ to limits a_1, a_2 ... Let $a = \{a_1, a_2, ...\}$ be the sequence of such limits.
Eq.(19.15) holds for all $\nu, \mu > N_\epsilon$; now let $\mu = \infty$: $\sum_k |a_{\nu k} - a_k|^2 < \epsilon^2$, i.e. $a_\nu - a \in \ell^2(\mathbb{C})$. Since a_ν belongs to the linear space, also a does. Moreover, $\|a_\nu - a\| \le \epsilon$, i.e. $a_n \to a$ in the ℓ^2 topology. □

• $\ell^2(\mathbb{C})$ is separable.

Proof. Consider the set ℓ' of sequences $q = \{q_0, q_1, ..., q_k, 0...\}$, where $\text{Re}q_j$ and $\text{Im}q_j$ are rational numbers for $j < k$, and $q_j = 0$ for $j > k$ (where each sequence has its own k). The set ℓ' is countable. We show that it is dense in $\ell^2(\mathbb{C})$, i.e. for any element a and any $\epsilon > 0$ there is an element $q \in \ell'$ such that $\|a - q\| < \epsilon$. Fix ϵ and choose n such that $\sum_{k=n+1}^\infty |a_k|^2 < \frac{1}{2}\epsilon^2$ and $q = \{q_1, \ldots, q_n, 0, \ldots\}$ such that $|a_k - q_k|^2 \le \epsilon^2/2n$ for all $k \le n$ (the numbers q_k are dense in $\mathbb{C}$). Then: $\|a - q\|^2 = \sum_{k=1}^n |a_k - q_k|^2 + \sum_{k>n} |a_k|^2 < n\frac{\epsilon^2}{2n} + \frac{1}{2}\epsilon^2 = \epsilon^2$. □

Example 19.3.3 (A non-separable Hilbert space[3])**.** *Consider the set* $\mathscr{E} = \{e_\omega\}_{\omega \in \mathbb{R}}$ *of functions* $e_\omega(t) = \exp(i\omega t)$, $t \in \mathbb{R}$. *The finite linear combinations* $f = \sum_\omega f_\omega e_\omega$ *with complex numbers* f_ω *form a linear space. The following "time average" is always well defined, and is an inner product:*

$$(f|g) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \overline{f(t)} g(t) dt \tag{19.16}$$

[3]N.I.Akhiezer and I.M.Glazman, "Theory of linear operators in Hilbert spaces", vol 1, par.13 (1961) (Dover reprint).

The completion of the linear space with respect to the norm is a Hilbert space. As $(e_\omega|e_{\omega'}) = \delta_{\omega,\omega'}$*, it is* $\|e_\omega - e_{\omega'}\|^2 = 2$ *if* $\omega \neq \omega'$*. Since* $\mathscr{E}$ *is uncountable, with elements separated by a finite distance, there cannot exist a countable set of functions such that any* e_ω *is approximated by a linear combination of such functions (we'd need an uncountable set of approximating functions!). Then the Hilbert space is non-separable.*

19.4 Orthogonal systems

Given n linearly independent points $x_1, \dots, x_n$ it is always possible to produce linear combinations $u_1, \dots, u_n$ that form an orthonormal set. The (Gram - Schmidt) orthonormalization procedure is:

$$\begin{aligned}
&y_1 = x_1, && \Rightarrow u_1 = y_1/\|y_1\|, \\
&y_2 = x_2 - (u_1|x_2)u_1, && \Rightarrow u_2 = y_2/\|y_2\|, \\
&y_3 = x_3 - (u_1|x_3)u_1 - (u_2|x_3)u_2, && \Rightarrow u_3 = y_3/\|y_3\|, \\
&\dots && \dots
\end{aligned}$$

Exercise 19.4.1. *Given vectors* $x_1, \dots, x_n$*, introduce Gram's matrix* $G_{ij} = (x_i|x_j)$*. Show that:*
1) the vectors are linearly dependent iff $\det G = 0$*;*
2) the matrix is non-negative ($G \geq 0$*) i.e.* $u^\dagger G u \geq 0 \ \forall u \in \mathbb{C}^n$*;*
3) $\det G \leq \|x_1\|^2 \cdots \|x_n\|^2$ *(use Hadamard's inequality for positive matrices).*

Exercise 19.4.2. *In* $\mathbb{C}^n$*, given n linearly independent vectors* $x_1, \dots, x_n$*, a vector has expansion* $u = \sum_{i=1}^n c_i x_i$*. The coefficients* c_i *solve the linear system* $(x_i|u) = \sum_j G_{ij} c_j$ *where G is Gram's matrix. Obtain the useful formula (Kramer):*

$$u = \frac{(-1)}{\det G} \det \begin{bmatrix} 0 & x_1 & \cdots & x_n \\ (x_1|u) & (x_1|x_1) & \cdots & (x_1|x_n) \\ \vdots & \vdots & & \vdots \\ (x_n|u) & (x_n|x_1) & \cdots & (x_n|x_n) \end{bmatrix} \tag{19.17}$$

19.4.1 Orthogonal polynomials

Let p_0, p_1, ... be a sequence of real *polynomials* of degree 0, 1, ..., that satisfy the *orthogonality condition*

$$\int_\sigma dx\, \omega(x)\, p_i(x) p_j(x) \;=\; h_j \delta_{ij} \tag{19.18}$$

where $\omega(x) \geq 0$ is a weight function, σ is a real (possibly unbounded) interval and $h_j > 0$ are constants.

The table lists some important sets of orthogonal polynomials, that may be obtained by orthogonalization of the monomials $1, x, x^2, \dots$:

σ	$\omega(x)$	p_k	
$\mathbb{R}$	e^{-x^2}	H_k	Hermite
$[0,\infty)$	e^{-x}	L_k	Laguerre
$[-1,1]$	1	P_k	Legendre
$[-1,1]$	$(1-x^2)^{-\frac{1}{2}}$	T_k	Chebyshev
$[-1,1]$	$(1-x^2)^{\alpha-\frac{1}{2}}$	C_k^α	Gegenbauer

Proposition 19.4.3. *Orthogonal polynomials satisfy a three-term recursion relation with real constants a_k, b_k and c_k:*

$$\boxed{xp_k(x) = a_k p_{k+1}(x) + b_k p_k(x) + c_k p_{k-1}(x)} \tag{19.19}$$

Proof. Suppose that the recursion contains the term $d_k p_{k-2}(x)$. Multiply the recursion by $\omega(x)p_{k-2}(x)$ and integrate. All terms but two vanish by orthogonality:

$$\int_\sigma dx\, \omega(x)\, xp_k(x)p_{k-2}(x) = d_k h_{k-2}.$$

Since $xp_{k-2} = a_{k-2}p_{k-1} + \dots$, also the left integral vanishes by orthogonality. Therefore $d_k = 0$. In the same way one proves the absence of all lower order terms in the recurrence. □

The constants a_k may be chosen positive. One finds $p_k(x) = x^k p_0/[a_{k-1}\dots a_0]+$ .... For monic orthogonal polynomials (the coefficient of the leading power is unity) a_k are all equal to 1.
The relations $c_k h_{k-1} = \int_\sigma dx\, \omega\, xp_k p_{k-1}$ and $xp_{k-1} = a_{k-1}p_k + \dots$ give

$$c_k h_{k-1} = a_{k-1} h_k \tag{19.20}$$

The recursion of polynomials may be written in multiplicative form:

$$\begin{bmatrix} p_{k+1}(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} (x-b_k)/a_k & -c_k/a_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_k(x) \\ p_{k-1}(x) \end{bmatrix}$$

Iteration provides p_k in terms of p_1 and p_0 via a product of k matrices (transfer matrix). In a different guise, the recursion corresponds to the evaluation of the determinant of a symmetric tridiagonal matrix. For monic polynomials:

$$p_{k+1}(x) = \det \begin{bmatrix} x-b_k & \sqrt{c_k} & & \\ \sqrt{c_k} & \ddots & \ddots & \\ & \ddots & x-b_1 & \sqrt{c_1} \\ & & \sqrt{c_1} & x-b_0 \end{bmatrix}$$

This has an important implication: the zeros of orthogonal polynomials are real.

Proposition 19.4.4 (Christoffel-Darboux summation formulae)**.**

$$\sum_{k=0}^n \frac{p_k(x)p_k(y)}{h_k} = \frac{a_n}{h_n}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \tag{19.21}$$

$$\sum_{k=0}^n \frac{p_k(x)^2}{h_k} = \frac{a_n}{h_n}\left[p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right] \tag{19.22}$$

Proof. Multiply the recursion (19.19) by $p_k(y)$, and subtract the same expression with x and y exchanged. Divide by h_k and sum on k. Cancellation of all but two terms occurs, because of (19.20). The second formula is the limit $y \to x$ of the first one. □

Exercise 19.4.5. *Show that $p_k(x)$ and $p'_k(x)$ cannot be zero at the same point, i.e. the* zeros of orthogonal polynomials are simple[4].

The books *Orthogonal Polynomials* by Szego and *Special functions* by Askey are standard references. Orthogonal polynomials arise in the theory of Jacobi operators (infinite Hermitian tridiagonal matrices), Sturm-Liouville differential equations, approximation theory, soluble models of statistical mechanics. They arise unexpectedly and beautifully in random matrix theory (see the books by Mehta, or Deift, or Forrester). Two important sets of orthogonal polynomials are discussed below.

Legendre polynomials

Legendre's polynomials $P_k(x)$ result from the orthogonalization of the monomials $1, x, x^2, \dots$ in $\mathrm{L}^2(-1,1)$:

$$\int_{-1}^{1} dx\, P_i(x)P_j(x) = h_j\delta_{ij}$$

The constants h_j are determined by the conditions $P_j(1) = 1$. Then $P_0(x) = 1$ and $P_1(x) = x$ (they are orthogonal). P_2 has no linear term to ensure orthogonality with P_1: $P_2(x) = C_2(x^2 + A_2)$; the conditions $1 = P_2(1)$ and $0 = \int_{-1}^{1} dx P_0 P_2$ give $P_2(x) = (3x^2-1)/2$. The odd polynomial $P_3(x) = C_3(x^3+A_3x)$ is orthogonal to even polynomials, and is evaluated by imposing $P_3(1) = 1$ and $0 = \int_{-1}^{1} dx P_3(x)P_1(x)$. The next one is even: $P_4(x) = C_4(x^4 + A_4x^2 + B_4)$, with parameters determined by three conditions: $P_4(1) = 1$, $P_4 \perp P_2$ and $P_4 \perp P_0$. The process is tedious to continue. Instead one can succeed in obtaining the recursive relation (where parity of polynomials is accounted for):

$$xP_k(x) = a_kP_{k+1}(x) + c_kP_{k-1}(x)$$

by evaluating h_k, a_k and c_k. For $x = 1$ it is $1 = a_k + c_k$, moreover $(1-a_k)h_{k-1} = a_{k-1}h_k$, by (19.20). Another equations is obtained by taking the derivative of the recursion, $P_k + xP'_k = a_kP'_{k+1} + c_kP'_{k-1}$, multiplication by P_k and integration on $[-1,1]$,

$$h_k + \tfrac{1}{2}\int_{-1}^{1} x\frac{d}{dx}P_k^2\, dx = a_k\int_{-1}^{1} P_kP'_{k+1}dx$$

and integration by parts: $\frac{1}{2}h_k + 1 = 2a_k$. The two equations for a_k and h_k give $h_kh_{k-1} + h_k - h_{k-1} = 0$ with initial condition $h_0 = 2$, i.e. $h_k^{-1} = h_{k-1}^{-1} + 1$. The solution is $h_k^{-1} = k + h_0^{-1}$ i.e. $h_k = \frac{2}{2k+1}$. Therefore, the orthogonality and

[4]Besides being real and simple, the zeros of orthogonal polynomials are all in the interval of orthogonality σ. Moreover, any zero of $p_k(x)$ is between two consecutive zeros of $p_{k+1}(x)$ (interlacing property).

recursive relations are:

$$\int_{-1}^{1} dx P_m(x)P_n(x) = \tfrac{2}{2n+1}\delta_{mn} \tag{19.23}$$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{19.24}$$

Hermite polynomials

Hermite polynomials are obtained by orthogonalization of the functions $1, x, x^2, \dots$ on the real line, with weight $\omega(x) = e^{-x^2}$:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2}\, H_i(x)H_j(x) \;=\; h_j\delta_{ij}$$

They are determined with the requirement on the leading coefficient $H_k(x) = 2^k x^k + \dots$. Because weight and domain are symmetric for $x \to -x$, Hermite polynomials have definite parity:

$$H_k(-x) = (-1)^k H_k(x).$$

This simplifies the recursion relation: $xH_k(x) = \frac{1}{2}H_{k+1}(x) + c_k H_{k-1}(x)$. To evaluate c_k and h_k note that $c_k h_{k-1} = \frac{1}{2}h_k$. The derivative of the recursion is multiplied by H_k and integrated with the weight:

$$h_k + \tfrac{1}{2}\int_{\mathbb{R}} dx\, e^{-x^2} x\frac{d}{dx}H_k^2(x) = \tfrac{1}{2}\int_{\mathbb{R}} dx\, e^{-x^2} H_k(x)H'_{k+1}(x)$$

Integrate by parts, $\frac{1}{2}h_k + \int_{\mathbb{R}} dx\, e^{-x^2}[xH_k(x)]^2 = \int_{\mathbb{R}} dx\, e^{-x^2} xH_k(x)H_{k+1}(x)$. Use the recursion to obtain $\frac{1}{2}h_k + \frac{1}{4}h_{k+1} + c_k^2 h_{k-1} = \frac{1}{2}h_{k+1}$, i.e. $\frac{h_{k+1}}{h_k} = \frac{h_k}{h_{k-1}} + 2$ with solution $\frac{h_{k+1}}{h_k} = 2k + \frac{h_1}{h_0}$. Since $h_1 = \int_{\mathbb{R}} dx\, 4x^2 e^{-x^2} = 2\sqrt{\pi}$ and $h_0 = \int_{\mathbb{R}} dx\, e^{-x^2} = \sqrt{\pi}$, one obtains $h_{k+1} = 2(k+1)h_k$ i.e. $h_k = 2^k k!\sqrt{\pi}$ and $c_k = k$. Therefore:

$$\int_{\mathbb{R}} dx\, e^{-x^2} H_i(x)H_k(x) = 2^k\, k!\, \sqrt{\pi}\, \delta_{ik} \tag{19.25}$$

$$H_{k+1}(x) = 2xH_k(x) - 2kH_{k-1}(x) \tag{19.26}$$

Hermite polynomials are related to *Hermite functions*, that are an orthonormal basis in $\mathrm{L}^2(\mathbb{R})$:

$$\boxed{h_k(x) = \frac{1}{\sqrt{2^k k!\sqrt{\pi}}}\, e^{-\frac{1}{2}x^2} H_k(x)} \tag{19.27}$$

A proof of completeness will be given in theorem 27.2.1, based on the theory of Fourier transform. The large-n distribution of the zeros of $H_n(x)$ is discussed in subsection 25.4.2.

19.4.2 Gauss quadrature formula

As a nice application of orthogonal polynomials is a method by Gauss to approximately evaluate integrals. The formula is

$$\boxed{\int_\sigma dx\, w(x)f(x) = \sum_{k=1}^n w_k f(x_k) + R_n} \tag{19.28}$$

$$w_k = \frac{1}{p_n'(x_k)}\int_\sigma dx \frac{p_n(x)}{x-x_k}, \quad R_n = \frac{h_n}{(2n)!}f^{(2n)}(\xi) \tag{19.29}$$

$\{x_k\}$ are the zeros of the monic polynomial $p_n(x)$, belonging to an orthogonal set on the interval σ with weight function w. The weights w_k and the roots x_k are tabulated (for the frequently used Chebyshev polynomials they are the zeros of $\cos n\theta$, with $x = \cos\theta$). The remainder R_n is here given for a smooth f, and ξ is a point in σ.

Proof. This proof gives a simple justification of the quadrature, but not of the remainder. The integral is equal to:

$$\begin{aligned}&\int_\sigma dx\, w(x)\frac{f(x_k)+f(x)-f(x_k)}{p_n(x)}p_n(x)\\ &=\sum_{k=1}^n \frac{1}{p_n'(x_k)}\int_\sigma dx\, w(x)\frac{f(x_k)+[f(x)-f(x_k)]}{x-x_k}p_n(x)\end{aligned}$$

One reads the quadrature approximation, and the remainder. Let us insert in the latter a truncated Taylor expansion where last $f^{(N)}$ is evaluated at $\xi \in \sigma$:

$$R_n^{(N)} = \sum_{k=1}^n \frac{1}{p_n'(x_k)}\sum_{\ell=1}^N \frac{f^{(\ell)}(x_k)}{\ell!}\int_\sigma dx\, w(x)(x-x_k)^{\ell-1}p_n(x)$$

By orthogonality, all terms $\ell < n$ give zero. If we stop at $N = n+1$ the integral is $h_n = \int dx w(x) p_n(x)^2$, and

$$R_n^{(n+1)} = h_n \frac{f^{(n+1)}(\xi)}{(n+1)!}\sum_{k=1}^n \frac{1}{p_n'(x_k)} = 0$$

because the sum is zero. If $N = n+2$:

$$R_n^{(n+2)} = \sum_{k=1}^n \frac{1}{p_n'(x_k)}\left[\frac{f^{(n+1)}(x_k)}{(n+1)!}h_n + \frac{f^{(n+2)}(\xi)}{(n+2)!}\int_\sigma dx\, w(x)(x-x_k)^{n+1}p_n(x)\right]$$

Consistently, expand $f^{(n+1)}(x_k) = f^{(n+2)}(\xi) + f^{(n+2)}(\xi)(x_k - \xi)$. Then:

$$R_n^{(n+2)} = h_n \frac{f^{(n+2)}(\xi)}{(n+2)!}\sum_{k=1}^n \frac{x_k}{p_n'(x_k)} = 0.$$

The sum $\sum_k x_k^j/p_n'(x_k)$ is zero for $j = 0, \dots, n-2$. It equals 1 for $j = n-1$. Therefore, the first non-zero result appears at $N = 2n$. The quadrature is exact for f being a polynomial of degree not higher that $2n-1$. □

19.5 Linear subspaces and projections

Definition 19.5.1. A linear subspace M is *closed* if any sequence in M that is convergent has limit in M. The *closure* $\overline{M}$ of a linear subspace is still a linear subspace.

Definition 19.5.2. If M is a linear subspace of $\mathscr{H}$, the *orthogonal complement* of M is the set $M^\perp$ of points that are orthogonal to M.

Proposition 19.5.3. *$M^\perp$ is a linear closed subspace.*

Proof. $M^\perp$ is a linear space: if x_1 and x_2 are in $M^\perp$, then $(z|x_1 + \lambda x_2) = (z|x_1) + \lambda(z|x_2) = 0$ if $z \in M$, i.e. $x_1 + \lambda x_2 \in M^\perp$. If x_n is a sequence in $M^\perp$, and $x_n \to x$ then, by the continuity of the inner product, $0 = (z|x_n) \to (z|x)$ for all $z \in M$, i.e. $x \in M^\perp$. □

Exercise 19.5.4. *Show that if $M_1 \subset M_2$ then $M_2^\perp \supseteq M_1^\perp$.*

The linear subspaces of $\mathbb{R}^3$ are planes and straight lines through the origin. A point $\mathbf{x}$ external to a line r of parametric equation $\mathbf{x}(t) = \vec{a}t$ has orthogonal projection $\vec{p} \in r$ which coincides with the point of minimal distance of $\mathbf{x}$ from the set r. Orthogonal projection means that the element $\mathbf{x} - \vec{p}$ is orthogonal to all points in r. This geometric property (minimal distance and orthogonality) is shared by Hilbert spaces.
The first non-trivial step is to prove that, given a *closed* linear subspace and a point x, there exists one and only one point p belonging to it, whose distance from x is minimal (the *set distance* of x).

Definition 19.5.5. The *distance* of a point x from a set M is $d(x, M) = \inf_{y \in M} \|x - y\|$.

Lemma 19.5.6. *Let M be a linear closed subspace in $\mathscr{H}$. Then, given $x \in \mathscr{H}$, there is a unique $p \in M$ such that $d(x, M) = \|x - p\|$.*

Proof. Let d be the distance of x from M. By definition, there is a sequence y_n in M such that $\|y_n - x\| \to d$; we show that it is a Cauchy sequence.
By the parallelogram law:

$$\|y_n - y_m\|^2 = \|(y_n - x) - (y_m - x)\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|2x - (y_n + y_m)\|^2$$

The vector $\frac{1}{2}(y_n + y_m)$ belongs to M, then $\|x - \frac{1}{2}(y_n + y_m)\| \geq d$ and

$$\|y_n - y_m\|^2 \leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4d^2$$

By hypothesis, for any $\epsilon > 0$ there is N_ϵ such that $\|y_n - x\|^2 - \delta^2 \leq \epsilon$ for all $n \geq N_\epsilon$. Then it is $\|y_n - y_m\|^2 \leq 4\epsilon$ for all $n, m \geq N_\epsilon$. The Cauchy sequence $\{y_n\}$ has a limit point $p \in M$ (M is closed); as the norm is a continuous map of $\mathscr{H}$ to $\mathbb{R}$, it follows that $\|x - p\| = \lim_n \|x - y_n\| = d$.
Suppose that there is another $p' \in M$ such that $\|x - p'\| = d$, then for the given x: $\|p - p'\|^2 = \|(p - x) - (p' - x)\|^2 = -\|p + p' - 2x\|^2 + 4d^2$. The mid-point $p'' = (p + p')/2$ belongs to M and $\|x - p''\| \geq d$. Therefore $\|p - p'\|^2 \leq 0$. □

Theorem 19.5.7 (Projection theorem). *Let M be a closed subspace in a Hilbert space $\mathscr{H}$. A vector in $\mathscr{H}$ has the unique decomposition*

$$x = p + w, \quad p \in M, \quad w \in M^\perp \tag{19.30}$$

Proof. Given x there is a unique vector p in M such that $\|x-p\| = d$. We have to show that $w \equiv x - p \in M^\perp$. For any λ and $y \in M$ it is $p + \lambda y \in M$ and

$$d^2 \le \|x - (p + \lambda y)\|^2 = d^2 - 2\mathrm{Re}[\lambda\,(w|y)] + |\lambda|^2\|y\|^2.$$

$-2\,\mathrm{Re}[\lambda(w|y)] + |\lambda|^2\|y\|^2 \ge 0$ for all complex λ if $(w|y) = 0$ for all y, i.e. $w \perp M$.
Suppose that $x = p + w = p' + w'$ with p, p' in M and w, w' in $M^\perp$. Then $(p - p') + (w - w') = 0$ with $p - p' \in M$ and $w - w' \in M^\perp$. The vanishing of the norm, $\|p - p'\|^2 + \|w - w'\|^2 = 0$, implies $p = p'$ and $w = w'$. □

Proposition 19.5.8. *If M is a linear subspace, then $M^{\perp\perp} = \overline{M}$.*

Proof. The statements: $x \in M^{\perp\perp} \Leftrightarrow (x|y) = 0\ \forall y \in M^\perp \Rightarrow x \in M$ imply that $M \subseteq M^{\perp\perp}$, and $\overline{M} \subseteq M^{\perp\perp}$. The other way, suppose that there is a vector $x \in M^{\perp\perp}$ with $x \notin \overline{M}$. Then $x \in M^\perp$: this means $x = 0$, as $M^\perp$ and $M^{\perp\perp}$ are orthogonal sets. □

Definition 19.5.9 (Orthogonal sum)**.** Given two orthogonal closed subspaces M_1 and M_2 in $\mathscr{H}$, their orthogonal sum is

$$M_1 \oplus M_2 = \{x_1 + x_2,\ x_1 \in M_1,\ x_2 \in M_2\}.$$

Exercise 19.5.10. *Prove that $M_1 \oplus M_2$ is closed. (Hint: show that if $x_{1j} + x_{2j} \to x$ then x_{1j} and x_{2j} are Cauchy sequences in M_1 and M_2).*

The projection theorem states that if M is closed then $\mathscr{H} = M \oplus M^\perp$.

Example 19.5.11. *Suppose that a subspace M is spanned by the orthonormal vectors $\{u_k\}_{k=1}^n$. Given a point x, its projection is the point $p = \sum_{k=1}^n p_k u_k$ in M that minimizes the squared distance of x from M:*

$$d^2 = \|x - p\|^2 = \|x\|^2 - \sum_{k=1}^n \Big[p_k (x|u_k) + \overline{p_k}(u_k|x) - \overline{p_k} p_k \Big]$$

Minimization in the coefficients p_k and $\overline{p}_k$ gives the projection of x,

$$\boxed{p = \sum_{k=1}^n (u_k|x) u_k} \tag{19.31}$$

and the (squared) minimal distance $\|x - p\|^2 = \|x\|^2 - \sum_{k=1}^n |(u_k|x)|^2$.

Example 19.5.12. *Consider the function*

$$f(x) = \frac{1}{a - x}, \quad x \in [-1, 1],\ |a| > 1$$

and the problem of finding its "best approximation" in terms of real polynomials of order n. In $\mathrm{L}^2(-1,1)$, this means to find the polynomial A_n of degree n with least $\mathrm{L}^2(-1,1)$-norm deviation from the function, i.e. to minimize the squared error

$$\|f - A_n\|^2 = \int_{-1}^{+1} dx \left[\frac{1}{a - x} - (a_n x^n + \ldots + a_1 x + a_0) \right]^2$$

with respect to the coefficients $a_n \dots a_0$. The minimal norm is realized by the distance of the function from the (closed) subspace $\mathscr{P}_n$ of polynomials of degree n, and the "best approximation" is precisely the projection of the function on $\mathscr{P}_n$. The Legendre polynomials P_k form an orthogonal basis of $\mathrm{L}^2(-1,1)$, and any polynomial of order n is a linear combination of them with $k \le n$. Taking into account normalization, the best polynomial of order n is:

$$A_n(x) = \sum_{k=0}^{n} C_k P_k(x), \quad C_k = \frac{2k+1}{2}\int_{-1}^{+1} dx\, \frac{P_k(x)}{a-x}$$

From the recursive relation (19.24) *of the polynomials, one may obtain a recursion scheme for C_k (in this case $C_k = (2k+1)Q_k(a)$, where $Q_k(x)$ is a Legendre functions of the second kind).*

Other norms give different approximations. In $\mathrm{L}^2([-1,1], dx/\sqrt{1-x^2})$ the Chebyshev polynomials are orthogonal (see also 2.5.2, 11.3.3). The projection on the subspace of polynomials of order n is:

$$\tilde{A}_n(x) = \sum_{k=0}^{n} \tilde{C}_k T_k(x), \quad \tilde{C}_k = \frac{1}{h_k}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}\frac{T_k(x)}{a-x}$$

with $h_0 = \pi$, and $h_k = \pi/2$. The coefficients may be evaluated with the Residue Theorem:

$$\tilde{C}_k = \frac{2}{\pi}\int_0^{\pi} d\theta \frac{\cos(k\theta)}{a-\cos\theta} = e^{-(k+1)\xi}$$

where $a = \cosh\xi$. The error can be computed exactly with the Christoffel-Darboux formula:

$$\Big|\frac{1}{\cosh\xi - x} - \tilde{A}_n(x)\Big| = \frac{e^{\xi}T_{n+1}(x) - T_n(x)}{(\cosh\xi - x)\sinh\xi} e^{-(n+1)\xi}$$

The property $|T_k(x)| \le 1$ on $[-1,1]$ allows for a uniform upper bound of the error:

$$\Big|\frac{1}{a-x} - \tilde{A}_n(x)\Big| \le \frac{e^{\xi}+1}{(a-1)\sinh\xi} e^{-(n+1)\xi}, \quad -1 < x < 1.$$

19.6 Complete orthonormal systems

Theorem 19.6.1. *Let $\{u_k\}_{k=1}^{\infty}$ be a countable set of orthonormal vectors in $\mathscr{H}$, and $\{c_k\}_{k=1}^{\infty}$ a complex sequence. Then*

$$\sum_{k=1}^{\infty} c_k u_k \in \mathscr{H} \Longleftrightarrow \sum_{k=1}^{\infty} |c_k|^2 < \infty$$

and, if the series converges to x, it is $c_k = (u_k|x)$.

Proof. The partial sums $x_n = \sum_{k=1}^{n} c_k u_k$ form a Cauchy sequence if and only if $\sum_{k=1}^{n} |c_k|^2$ is a Cauchy sequence in $\mathbb{C}$: this follows from the identity

$$\|x_n - x_m\|^2 = \Big\|\sum_{k=m+1}^{n} c_k u_k\Big\|^2 = \sum_{k=m+1}^{n} |c_k|^2.$$

Then $x_n \to x$ iff $c_k \in \ell^2(\mathbb{C})$.
Moreover $c_k = (u_k|x_n) \to (u_k|x)$ by continuity of the inner product, and $\|x\|^2 = \lim_n \|x_n\| = \lim_n \sum_{k=0}^n |c_k|^2 = \sum_{k=0}^\infty |c_k|^2$. □

Definition 19.6.2. An orthonormal set $\{u_a\}_{a\in A}$ of vectors, $(u_a|u_b) = 0$ if $a \neq b$ and $\|u_a\| = 1$, is *complete* if it is not a subset of another orthonormal set.
An equivalent statement is: an orthonormal set $\{u_a\}_{a\in A}$ is complete if

$$\boxed{(u_a|x) = 0 \quad \forall a \in A \ \Rightarrow x = 0} \tag{19.32}$$

A Hilbert space with a countable orthonormal complete set of vectors is separable, and it is isomorphic to $\ell^2(\mathbb{C})$.

Theorem 19.6.3 (Parseval's identity). *In a separable Hilbert space, if $\{u_k\}_{k=1}^\infty$ is an orthonormal complete basis, then:*

$$x = \sum_{k=1}^\infty (u_k|x)u_k, \qquad \|x\|^2 = \sum_{k=1}^\infty |(u_a|x)|^2, \quad \forall x \in \mathscr{H} \tag{19.33}$$

$$(x|y) = \sum_{k=1}^\infty (x|u_k)(u_k|y) \tag{19.34}$$

The numbers $(u_k|x)$ are the *Fourier coefficients* of the expansion.

19.7 Bargmann's space

Bargmann's space $\mathscr{B}(\mathbb{C})$ is the linear space of entire functions[5] such that

$$\int \frac{dz d\overline{z}}{\pi} e^{-|z|^2} |f(z)|^2 < \infty$$

where $dz\, d\overline{z} \equiv dx\, dy$ and $z = x + iy$. It is a Hilbert space with the inner product

$$(f|g) = \int \frac{d\overline{z} dz}{\pi} e^{-|z|^2} \overline{f(z)} g(z) \tag{19.35}$$

The functions $u_k(z) = \frac{1}{\sqrt{k!}} z^k$ are orthonormal (use polar coordinates):

$$\int \frac{d\overline{z} dz}{\pi} e^{-|z|^2} \overline{z}^n z^m = n!\, \delta_{nm}$$

Since every entire function has a power series expansion, the functions u_k form a complete set. On suitable domains one defines the linear operators

$$(\hat{a}f)(z) = f'(z), \qquad (\hat{a}^\dagger f)(z) = zf(z).$$

They act as *ladder operators* (lowering and raising operators) on the basis functions: $\hat{a}\, u_k = \sqrt{k}\, u_{k-1}$ and $\hat{a}^\dagger u_k = \sqrt{k+1}\, u_{k+1}$. The operator $\hat{N} \equiv \hat{a}^\dagger \hat{a} = z\frac{d}{dz}$ has action $\hat{N}\, u_k = k\, u_k$.

[5] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 (1961) 187.

The eigenstates of the lowering operator $(\hat{a}\phi_\xi)(z) = \xi\phi_\xi(z)$ are named *coherent states*, and exist for any value $\xi \in \mathbb{C}$. In Bargmann's space they are the functions

$$\phi_\xi(z) = e^{-\frac{1}{2}|\xi|^2+\xi z} = e^{-\frac{1}{2}|\xi|^2}\sum_{k=0}^{\infty} u_k(\xi)u_k(z)$$

The inner product of two coherent states vanishes exponentially in the distance of parameters $|(\phi_\xi|\phi_\eta)| = \exp(-|\xi-\eta|)$. The interest of such functions is in the following property: for any $f \in \mathscr{B}(\mathbb{C})$ it is true that

$$f(z) = \int \frac{d\bar{\xi}d\xi}{\pi} e^{-|\xi|^2+\bar{\xi}z} f(\xi) = \int \frac{d\bar{\xi}d\xi}{\pi} e^{-\frac{1}{2}|\xi|^2} f(\xi)\phi_{\bar{\xi}}(z) \tag{19.36}$$

(check it on the basis functions u_k). The first equality shows that $e^{-|\xi|^2+\bar{\xi}z}$ is a *reproducing kernel*. The second equality shows that coherent states are a continuous basis-set of normalized but not orthogonal functions (an *overcomplete set*). Can one remove coherent states from the basis without losing completeness? The answer is yes: one can select a countable set of ξ values that belong to a two dimensional lattice $\xi = n_1\omega_1 + n_2\omega_2$, and completeness survives if $\omega_1\omega_2 < 1$.

Coherent states are important in quantum mechanics, semiclassical dynamics of complex systems, quantum optics, path integral formulation of bosons[6]. A standard reference is: A. Perelomov, *Generalized Coherent States and their Applications*, (Springer-Verlag, Berlin 1986).

Exercise 19.7.1. *The set of functions holomorphic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and such that $\int_{\mathbb{D}} \frac{dzd\bar{z}}{\pi}|f(z)|^2 < \infty$ form a Hilbert space.*
i) Show that the functions $u_k(z) = \sqrt{k+1}\,z^k$ are orthonormal.
ii) Evaluate the norm of $(z-a)^{-1}$, $|a| > 1$.
iii) Evaluate the series: $\sum_{k=0}^{\infty} u_k(z)\overline{u_k(\zeta)}$ $(z, \zeta \in \mathbb{D})$.

[6] for fermions one needs a Hilbert space of anticommuting variables

Chapter 20

TRIGONOMETRIC SERIES

20.1 Fourier Series

Let f be a 2π-periodic real function and ask the following question: which are the real coefficients of an expansion in harmonics that "best" approximates f?

$$S_N(x) = \frac{1}{2}a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx) \tag{20.1}$$

We may require minimization of the total quadratic error

$$\delta^2 = \int_{-\pi}^{\pi} dx [f(x) - S_N(x)]^2$$

$$\begin{aligned}
0 &= \frac{\partial \delta^2}{\partial a_0} = -\int_{-\pi}^{\pi} dx\, [f(x) - S_N(x)] \\
0 &= \frac{\partial \delta^2}{\partial a_k} = -2\int_{-\pi}^{\pi} dx\, [f(x) - S_N(x)] \cos(kx) \\
0 &= \frac{\partial \delta^2}{\partial b_k} = -2\int_{-\pi}^{\pi} dx\, [f(x) - S_N(x)] \sin(kx)
\end{aligned}$$

Because of the "orthogonality" of the basis functions

$$\int_{-\pi}^{\pi} dx\, \cos(mx)\sin(nx) = 0, \tag{20.2}$$

$$\int_{-\pi}^{\pi} dx\, \cos(mx)\cos(nx) = \int_{-\pi}^{\pi} dx\, \sin(mx)\sin(nx) = \pi\delta_{mn} \tag{20.3}$$

the coefficients can be easily obtained (Euler):

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\cos(kx), \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\sin(kx). \tag{20.4}$$

They are well defined for integrable functions, and do not depend on N (a consequence of "orthogonality"). The quadratic error at minimum is evaluated:

$$\delta^2 = \int_{-\pi}^{\pi} dx f(x)^2 - \pi \left[\frac{a_0^2}{2} + a_1^2 + ... + b_N^2\right]$$

Since $\delta^2 \geq 0$ it is clear that the approximation improves by increasing the number of basis functions. If the error saturates to zero, one gets *Parseval's identity*:

$$\boxed{\int_{-\pi}^{\pi} dx f(x)^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2)} \tag{20.5}$$

By inserting the integral expressions for the coefficients (20.4) in the finite sum (20.1), one gets

$$S_N(x) = \int_{-\pi}^{\pi} dy f(y) \left[\frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{N} \cos k(x-y)\right] = \int_{-\pi}^{\pi} dy f(y) D_N(x-y)$$

with *Dirichlet's kernel*

$$\boxed{D_N(x-y) = \frac{1}{2\pi} \frac{\sin[(N+\frac{1}{2})(x-y)]}{\sin[\frac{1}{2}(x-y)]}} \tag{20.6}$$

Properties: $D_N(-x) = D_N(x)$, $D_N(0) = \frac{1}{\pi}(N+\frac{1}{2})$,

$$\int_{-\pi}^{\pi} dx D_N(x) = \int_{-\pi}^{\pi} \frac{dx}{\pi} \left(\tfrac{1}{2} + \cos x + \cos 2x + \ldots\right) = 1$$

We wish to study the large N behaviour of the difference

$$\begin{aligned} S_N(x) - f(x) &= \int_{-\pi}^{\pi} dy [f(y) D_N(x-y) - f(x) D_N(y)] \\ &= \int_{-\pi}^{\pi} dy [f(y+x) - f(x)] D_N(y) \end{aligned} \tag{20.7}$$

The main question is: under which conditions does the difference converge to zero point-wise, uniformly, or almost everywhere? Is the minimisation of the quadratic error the best criterion for constructing the trigonometric approximation? These problems require an appropriate setting in functional analysis[1].

Exercise 20.1.1. *1) Prove the orthogonality relations. 2) Evaluate Dirichlet's kernel (hint: use the complex representation).*

Remark. The Fourier expansion (20.1) can be rewritten as:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}} \tag{20.8}$$

[1] see for example: A. Kolmogorov and S. Fomine, *Éléments de la théorie des fonctions et de l'analyse fonctionelle*, Éditions de Moscou (1974), also a Dover reprint.

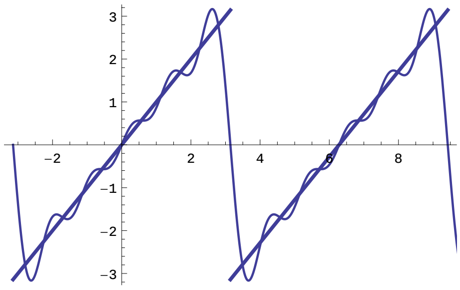

Figure 20.1: The Sawtooth function, and the Fourier approximation S_5

where $c_n = \sqrt{\frac{\pi}{2}}(a_n - ib_n)$ and $c_{-n} = \overline{c}_n$. For unrestricted coefficients c_n and c_{-n} the formula is the Fourier expansion of a complex function. The orthogonality relation and Parseval's identity (completeness) are:

$$\boxed{\int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{i(n-m)x} = \delta_{nm}\,, \qquad \int_{-\pi}^{\pi} dx |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2} \tag{20.9}$$

Example 20.1.2. *Consider the function $f(x) = x$ on $(-\pi, \pi]$, repeated periodically (sawtooth function). Being an odd function, the Fourier coefficients a_n are null, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x \sin(nx) = 2(-1)^{n+1}/n$. Then:*

$$x = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n} \tag{20.10}$$

Parseval's identity is: $\int_{-\pi}^{\pi} dx\, x^2 = 4\pi \sum_{n=1}^{\infty} 1/n^2$, which is true. Therefore the squared error is zero, $\delta^2 = 0$, meaning that, up to a zero-measure set, the series and the functions coincide.
At the point of discontinuity $x = \pi$ the periodic function has two limits: $f(\pi^-) = \pi$ and $f(\pi^+) = -\pi$. The Fourier series, being unable to choose which value to approximate, takes the value in the middle: $S_\infty(\pi) = 0$.
For $x = \frac{\pi}{2}$ one obtains the sum of Leibnitz's series: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.
Note that, being the function discontinuous, high frequency terms are needed to "fill the edges". Correspondingly, Fourier coefficients decay slowly, as $1/n$, and the series has slow convergence.

20.2 Pointwise convergence

We need the following lemma:

Lemma 20.2.1 (Riemann)**.** *If $f \in \mathscr{L}^1(a, b)$ then*

$$\boxed{\lim_{n\to\infty} \int_a^b dx f(x) \sin(nx) = 0} \tag{20.11}$$

Proof. If f' exists and is continuous, a partial integration gives

$$\int_a^b dx f(x)\sin(nx) = -\frac{1}{n}f(x)\cos(nx)\Big|_a^b + \frac{1}{n}\int_a^b dx f'(x)\cos(nx)$$

An arbitrary function in $\mathscr{L}^1(a,b)$ can be approximated by functions in $\mathscr{C}^1(a,b)$: for any $\epsilon > 0$ there is a function with continuous derivative such that $\|f-\varphi_\epsilon\|_1 = \int_{[a,b]} dx\,|f-\varphi_\epsilon| < \epsilon$. Then:

$$\begin{aligned}\left|\int_a^b dx f(x)\sin(nx)\right| &\leq \left|\int_a^b dx[f(x)-\varphi_\epsilon(x)]\sin(nx)\right| + \left|\int_a^b dx\,\varphi_\epsilon(x)\sin(nx)\right| \\ &\leq \int_a^b dx|f(x)-\varphi_\epsilon(x)| + \left|\int_a^b dx\,\varphi_\epsilon(x)\sin(nx)\right|\end{aligned}$$

The first term can be made arbitrarily small, the second decays to zero. □

Remark 20.2.2. *There is a relationship between the regularity properties of f and the decay properties of its Fourier coefficients (which dictate how fast the series converges, i.e. how many terms are needed to reproduce the function with small error). For a periodic function in $\mathscr{C}^1(-\pi,\pi)$ the boundary term in the proof of the Lemma is zero, therefore the Lemma proves that the Fourier coefficients a_n and b_n decay at least as $1/n$. However, if f is periodic and $\mathscr{C}^k$, further integrations by parts show that the coefficients decay at least as $1/n^k$.*

In the partial sum S_N of the Fourier series, the numerator of Dirichlet's kernel produces convergence for large N (the Lemma), but the zero in the denominator has to be neutralized; this is done in the theorem:

Theorem 20.2.3 (Dini[2]). *Let $f \in \mathscr{L}^1(-\pi,\pi)$ and suppose that for a fixed point x there is a $\delta > 0$ such that*

$$\int_{-\delta}^{\delta} dt \left|\frac{f(x+t)-f(x)}{t}\right| < \infty \tag{20.12}$$

then $S_N(x) \to f(x)$ at x.

Proof.

$$\begin{aligned}S_N(x) - f(x) &= \int_{-\pi}^{\pi} dt[f(x+t)-f(x)]D_N(t) \\ &= \int_{-\pi}^{\pi} dt \frac{f(x+t)-f(x)}{t}\,\frac{t}{2\pi\sin(\frac{1}{2}t)}\sin(N+\tfrac{1}{2})t\end{aligned}$$

If Dini's condition holds, the ratio $\frac{f(x+t)-f(x)}{t}\,\frac{t}{2\sin(t/2)}$ is integrable on $(-\pi,\pi)$. Then, by Riemann's lemma, $|S_N(x)-f(x)| \to 0$ as $N\to\infty$. □

[2]Ulisse Dini (1845, 1918) formerly a student in Pisa of Enrico Betti, went to Paris and got acquainted with Charles Hermite and J. L. Francoise Bertrand. He became professor in Pisa, member of the Parliament and senator, and for many years he directed the Scuola Normale. He was the advisor of Luigi Bianchi and Gregorio Ricci-Curbastro (with Betti) and Luigi Fubini. Other students of Betti in Pisa were Cesare Arzelá, Federigo Enriques and Vito Volterra.

Dini's condition can be replaced by existence of the left and right integrals:

$$\int_0^\delta dt \left|\frac{f(x+t)-f(x^+)}{t}\right|, \qquad \int_{-\delta}^0 dt \left|\frac{f(x+t)-f(x^-)}{t}\right|.$$

It implies the following sufficient condition for point-wise convergence:

Corollary 20.2.4. *If f is a bounded 2π-periodic function with only a finite number of discontinuities of the first kind (left and right finite limits exist at each discontinuity) on $[0,2\pi]$, and if the left and right derivatives exist everywhere, then the Fourier series is point-wise convergent where f is continuous and takes the value $\frac{1}{2}[f(x^+)+f(x^-)]$ at a discontinuity.*

Example 20.2.5. *$f(x)=x^2$ on $[-\pi,\pi)$ (periodic parabolic arc). Fourier coefficients: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\, x^2=\frac{2}{3}\pi^2$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\, x^2\cos(nx)=4(-1)^n n^{-2}$ and $b_n=0$ (even function). Then:*

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) \tag{20.13}$$

The function is continuous but its derivative is discontinuous at $x=-\pi$: note that its Fourier coefficients decay as $1/n^2$.
At $x=\pi$ one obtains $\zeta(2)=\sum_{n=1}^{\infty}1/n^2=\pi^2/6$, while at $x=0$: $\sum_{n=1}^{\infty}(-1)^n/n^2=-\pi^2/12$. Parseval's identity allows to obtain a further relation: $\int_{-\pi}^{\pi}dx\,x^4=2\pi\frac{\pi^4}{9}+16\pi\zeta(4)$, i.e. $\zeta(4)=\pi^4/90$.

Example 20.2.6. *$f(x)=(1-2a\cos x+a^2)^{-1}$ on $[-\pi,\pi)$ ($a^2<1$). The Fourier coefficients can be evaluated by the Residue Theorem in the unit circle ($\zeta=e^{ix}$):*

$$\begin{aligned} a_n &= \mathrm{Re}\frac{1}{\pi}\int_{-\pi}^{\pi}dx\,f(x)e^{inx}=\mathrm{Re}\int_{C(0,1)}\frac{d\zeta}{i\pi\zeta}\frac{\zeta^n}{1-a\zeta-a/\zeta+a^2} \\ &= -\mathrm{Re}\int_{C(0,1)}\frac{d\zeta}{i\pi a}\frac{\zeta^n}{(\zeta-a)(\zeta-1/a)} = \frac{2a^n}{1-a^2} \end{aligned}$$

$$\frac{1}{1-2a\cos x+a^2}=\frac{1}{1-a^2}+\frac{2}{1-a^2}\sum_{n=1}^{\infty}a^n\cos(nx) \tag{20.14}$$

The function is continuous with all its derivatives; the Fourier coefficients decay exponentially, as $e^{-n|\log a|}$.
Note that, with $\cos x=t$, it is $\cos(nx)=T_n(t)$, and the generating function of Chebyshev polynomials is recovered, eq.(11.25).

Exercise 20.2.7. *Show that*

$$\sum_{k=0}^{\infty}\frac{y^k}{k!}\cos(kx)=e^{y\cos x}\cos(y\sin x), \qquad \sum_{k=1}^{\infty}\frac{y^k}{k!}\sin(kx)=e^{y\cos x}\sin(y\sin x)$$

Trigonometric sums define functions, that may be rather extravagant. Consider the finite sum (it is not a Fourier series because of the modulus) $f_N(x)=\sum_{k=1}^{N}|\sin(k\pi x)|/k$. The function $f_N(x)$ has a strict local minimum at every rational p/q with $|q|\le\sqrt{N}$ (*An amusing sequence of functions*, S. Steinerberger, arXiv:1610.04090).

20.2.1 Gibbs' phenomenon

The Fourier series of a function with a jump discontinuity exhibits the *Gibbs phenomenon*, explained by Gibbs in 1899 in a letter to *Nature*. It is an over-shooting of the Fourier series caused by the high frequencies needed to describe the jump, that near the discontinuity pile up.

If a is a point of discontinuity and Δ_f is the jump of a 2π-periodic function, the partial sum S_N has the jump $\Delta_N = |S_N(a - \pi/N) - S_N(a + \pi/N)|$. Gibbs noted that in the limit it is not Δ_f:

$$\lim_{N\to\infty} \frac{\Delta_N}{\Delta_f} = 1.17898...$$

The error is about 18%, and has fixed size for any large N. It occurs in a region of width $\approx 1/N$ around the point of discontinuity.
Since it is a local effect, this is well illustrated by the following example.

Consider the periodic sawtooth function of Example 20.1.2. The jump at $a = \pi$ is 2π. The partial sums give:

$$S_N(\pi - \epsilon) - S_N(\pi + \epsilon) = 4\sum_{n=1}^{N} \frac{\sin(n\epsilon)}{n}$$

For $\epsilon \leq \pi/N$ all terms are positive and add up without interfering. At $\epsilon = \pi/N$:

$$\frac{\Delta_N}{\Delta_f} = \frac{2}{\pi}\sum_{k=1}^{N} \frac{\sin(N\pi/n)}{n} \to \frac{2}{\pi}\int_0^{\pi} dt\, \frac{\sin t}{t} = 1.17898...$$

(in the continuum limit $n\pi/N = t$, $dt = \pi/N$). The series overestimates the value of the function by 9%, at each sides of the discontinuity.

Exercise 20.2.8. *Show that the Fourier series of* $\cos(ax)$ *on* $[-\pi, \pi]$ *is:*

$$\cos(ax) = \frac{\sin(a\pi)}{\pi a} + \frac{\sin(a\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{2a}{a^2 - k^2}\cos(kx), \quad a \notin \mathbb{N}.$$

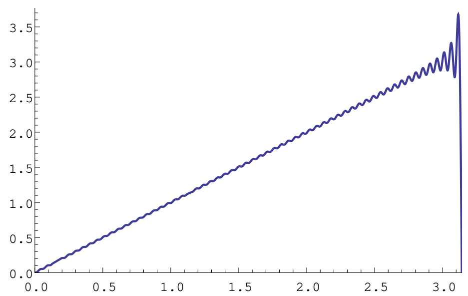

Figure 20.2: The Sawtooth function restricted to $[0, \pi]$ and the Fourier approximation S_{120}, showing the onset of the Gibbs phenomenon near the discontinuity.

In $x=\pi$ the function is continuous and the series converges. This gives

$$\operatorname{cotg}(a\pi) = \frac{1}{\pi a} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{2a}{a^2-k^2} \tag{20.15}$$

It is the logarithmic derivative of the famous Euler's product expansion for the sine (1784), analogous to the factorization of a polynomial in terms of the roots:

$$\boxed{\sin(\pi z) = \pi z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)} \tag{20.16}$$

20.2.2 Fourier series with different basis sets

Consider a function f with period 1. *As a periodic function* it has a Fourier series expansion on $[0,1]$, with basis functions $\cos(2k\pi x)$ and $\sin(2k\pi x)$:

$$f(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\alpha_k\cos(2k\pi x) + \beta_k\sin(2k\pi x)$$

However, *as a function on the interval* $[0,1]$ alone, other trigonometric series are possible. For example, the even/odd extensions of f on $[-1,1]$,

$$f_e(x) = \begin{cases} f(x) & x\in[0,1] \\ f(-x) & x\in[-1,0)\end{cases} \qquad f_o(x) = \begin{cases} f(x) & x\in[0,1] \\ -f(-x) & x\in[-1,0)\end{cases}$$

can be represented respectively as Fourier series with functions $\cos(k\pi x)$ or $\sin(k\pi x)$. The two series, restricted to the interval $[0,1]$, give two new trigonometric expansions of f:

$$f(x) = \frac{a_0}{2} + \sum_n a_n\cos(n\pi x) \;=\; \sum_n b_n\sin(n\pi x), \quad x\in[0,1].$$

The functions $\{1,\cos(k\pi x)\}$ and $\{\sin(k\pi x)\}$ form two independent sets of orthogonal functions on $[0,1]$:

$$\int_0^1 dx\,\cos(m\pi x)\cos(n\pi x) = \tfrac{1}{2}\delta_{mn}, \qquad \int_0^1 dx\,\sin(m\pi x)\sin(n\pi x) = \tfrac{1}{2}\delta_{mn}.$$

Example 20.2.9. *Consider the function with period 1, which is "x" on* $[0,1]$*. It has the Fourier expansion*

$$x = \tfrac{1}{2} - \sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{\pi k}, \quad 0<x<1 \tag{20.17}$$

The expansions on $[-1,1]$ *of the even extension* $|x|$ *and of the odd extension* x*, give two new representations of* x *on* $[0,1]$*:*

$$x = \tfrac{1}{2} - 4\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{\pi^2(2k+1)^2} = -2\sum_{k=1}^{\infty}(-1)^k\frac{\sin(k\pi x)}{\pi k}$$

Outside the interval $[0,1]$ *the three Fourier series describe different periodic functions. Note the faster convergence of the series for* $|x|$*, which is continuous. Parseval's identity gives:*

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

20.3 Applications

20.3.1 Heat Equation

In his fundamental treatise *Théorie analytique de la chaleur* (1822) Jean Baptiste Fourier discussed the transmission of heat in bodies of various shapes and different boundary conditions. By assuming that the transfer of heat between near regions is proportional to the temperature difference, he obtained the Heat Equation for the temperature field:

$$\frac{1}{D}\frac{\partial T}{\partial t} - \nabla^2 T = 0$$

D is the constant of thermal diffusion. Fourier solved the stationary problem of the temperature distribution in a rectangle with sides at different temperatures, by trigonometric series.

Consider the square $[0,\pi]\times[0,\pi]$ with three sides held at temperature $T=0$ and one at temperature $T(x,\pi)=x$ (at one corner there is a jump $\Delta T=\pi$). The stationary field $T(x,y)$ solves $T_{xx}+T_{yy}=0$ with the specified b.c. As Fourier noted, an elementary harmonic solution is $e^{\pm ay}\sin(ax)$. It is zero at $x=0,\pi$ if a is an integer. The linear combination $T(x,y)=\sum_{n=1}^\infty c_n\sinh(ny)\sin(nx)$ is a solution and is zero at three sides of the rectangle. The coefficients are determined by the b.c.: $x=\sum_{n=1}^\infty c_n \sinh(n\pi)\sin(nx)$. Fourier evaluated the infinite number of unknowns $x_n = c_n\sinh(n\pi)$ through the infinite linear system obtained by taking even derivatives of all order:

$$\begin{aligned}
0 &= x_1\sin x + 2^2 x_2\sin(2x) + 3^2 x_3 \sin(3x) + \dots \\
0 &= x_1\sin x + 2^4 x_2\sin(2x) + 3^4 x_3 \sin(3x) + \dots \\
&\dots
\end{aligned}$$

Instead, we exploit orthogonality: $\int_0^\pi dx x\sin(\ell x) = \sum_n x_n \int_0^\pi dx \sin(\ell x)\sin(nx)$ i.e. $x_\ell = -\frac{2}{\ell}(-1)^\ell$. The solution is:

$$T(x,y) = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\frac{\sinh(ny)}{\sinh(n\pi)}\sin(nx).$$

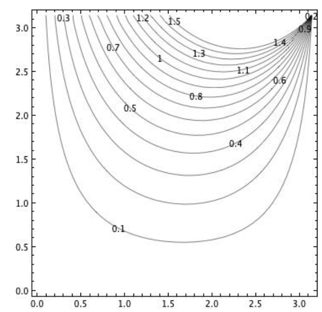

Figure 20.3: Fourier's solution for the temperature distribution in the square.

20.3.2 Kepler's equation

The elliptic orbit of a planet has semi-axis a and b ($a \geq b$) and eccentricity $e = \sqrt{1-(b/a)^2}$; the position of the Sun is the focus $(ae, 0)$.
The position of the planet can be parameterized as $x = a\cos E$, $y = b\sin E$, where $0 \leq E \leq 2\pi$ is the *eccentric anomaly*, $E = 0$ at perihelion and $E = \pi$ at aphelion. The distance from the Sun is $r = a(1 - e\cos E)$.
Since the areal velocity is constant (conservation of angular momentum), the area swept at time t after the passage at perihelion ($E = 0$ at $t = 0$) is $\pi ab(t/T)$, where T is the orbital period. The evaluation of the same area in terms of E gives *Kepler's law*:

$$E - e\sin E = M, \qquad M = \frac{2\pi}{T}t \tag{20.18}$$

M is the "mean anomaly", $E(M+2\pi) = E(M) = -E(-M)$. At each time one evaluates M and solves Kepler's equation to obtain E and the position of the planet.
The solution of Kepler's equation was first obtained by the astronomer Bessel in 1824 as a Fourier series (some terms were previously obtained by Lagrange, 1770) $E - M = \sum_{k=1}^{\infty} B_k \sin(kM)$ with coefficients that are Bessel's functions (see eq.(13.5))

$$\begin{aligned} B_k =& \frac{1}{\pi}\int_0^{2\pi} dM\,(E-M)\sin(kM) = \int_0^{2\pi} dM\left(\frac{dE}{dM} - 1\right)\frac{\cos kM}{\pi k} \\ =& \frac{2}{k}\int_0^{\pi}\frac{dE}{\pi}\cos(kE - ke\sin E) = \frac{2}{k}J_k(ke) \end{aligned}$$

Then: $E = M + 2J_1(e)\sin M + J_2(2e)\sin(2M) + \frac{2}{3}J_3(3e)\sin(3M) + \dots$

Exercise 20.3.1. *Consider the generating function for Bessel's functions of integer order with z on the unit circle:*

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta}J_n(x).$$

Prove the properties: $1 = J_0(x)^2 + 2\sum_{k=1}^{\infty} J_k(x)^2$ (Parseval's identity), and

$$e^{i\mathbf{k}\cdot\mathbf{r}} = J_0(kr) + 2\sum_{n=1}^{\infty} i^n \cos(n\theta)J_n(kr) \tag{20.19}$$

where θ is the angle formed by the vectors. The formula is useful in scattering theory.

20.3.3 Vibrating string

A thin stretched string is clamped at $x = 0$ and $x = L$. Its deviation from the straight configuration is a function $f(x,t)$ that solves the Wave Equation

$$f_{tt} - c^2 f_{xx} = 0,$$

where c is the speed of wave propagation. The Cauchy problem is specified by initial conditions $f(x, 0) = f_1(x)$ and $f_t(x, 0) = f_2(x)$. Boundary conditions

(b.c.) $f(0,t)=0$ and $f(L,t)=0$ are imposed at all times.
Multiplication by $\mu_0 f_t$ and integration by parts lead to a conservation law for the total energy (μ_0 is the linear mass density):

$$E=\frac{\mu_0}{2}\int_0^L dx\,\left(f_t^2+c^2 f_x^2\right)$$

E is time independent and is fixed by the initial conditions.
The wave equation is first solved for stationary states (i.e. factorized in time and space): $f(x,t)=A(t)u(x)$, with solutions $e^{\pm i\omega t}[\alpha e^{ikx}+\beta e^{-ikx}]$, $\omega=kc\geq 0$. The b.c. impose: $\alpha+\beta=0$ and $\alpha e^{ikL}+\beta e^{-ikL}=0$, i.e. $\alpha=-\beta$ and $k=\pi n/L$, $n=0,1,2,\dots$ (wave-lengths are quantized). Therefore, the stationary solutions (*normal modes*) are:

$$e^{\pm i\omega_n t}\sin\left(\frac{n\pi x}{L}\right),\qquad n=1,2,\dots$$

The general solution is a real linear superposition of normal modes

$$f(x,t)=\sum_{n=1}^{\infty}(c_n e^{i\omega_n t}+\overline{c}_n e^{-i\omega_n t})\,\sin\frac{n\pi x}{L},\quad \omega_n=\frac{\pi c}{L}n \tag{20.20}$$

The period of the n^{th} mode is $T_n=\frac{1}{n}(2L/c)$; the longest one is a global period: $f(x,t+T_1)=f(x,t)$. The coefficients c_n are determined by the initial conditions $\sum_n(c_n+\overline{c}_n)\sin(n\pi x/L)=f_1(x)$, $\sum_n i\omega_n(c_n-\overline{c}_n)\sin(n\pi x/L)=f_2(x)$. By means of the orthogonality relation $\int_0^L dx\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}=\frac{L}{2}\delta_{mn}$ one evaluates:

$$\mathrm{Re}\,c_n=\frac{1}{L}\int_0^L dx f_1(x)\sin\frac{n\pi x}{L},\quad \mathrm{Im}\,c_n=-\frac{1}{L\omega_n}\int_0^L dx f_2(x)\sin\frac{n\pi x}{L}.$$

The general solution, though describing oscillations of a string of length L, is $2L-$periodic. At $t=0$ it coincides with $f_1(x)$ on $[0,L]$ but on $[L,2L]$ it equals $-f_1(2L-x)$.

The total energy of the solution $f(x,t)$ is extensive (proportional to L) and is the *sum of the energies of the single modes*[3]:

$$E=L\,\frac{\mu_0}{2}\sum_{n=1}^{\infty}\omega_n^2[c_n\overline{c}_n+\overline{c}_n c_n] \tag{20.21}$$

As an example, let us suppose that the string is initially stretched as a triangle, $f_1(x)=\alpha x$ for $0<x\leq L/2$ and $f_1(x)=\alpha(L-x)$ for $L/2<x<L$, and left free to vibrate ($f_2=0$, no initial velocity on the whole length). The coefficients c_n and the solution are evaluated:

$$\begin{aligned}f(x,t)=&\frac{4\alpha L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cos(k_n ct)\sin(k_n x),\qquad k_n=n\frac{\pi}{L}\\ =&\frac{2\alpha L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left[\sin[k_n(x+ct)]+\sin[k_n(x-ct)]\right]\\ =&\tfrac{1}{2}\left[F_1(x+ct)+F_1(x-ct)\right]\end{aligned}$$

[3]The expression of the energy is ready to undergo “second quantization”, which yields an operator for a quantum description of vibrations in terms of elementary quanta called phonons.

Here, $F_1(x)$ is the $2L$-periodic function with value $F_1(x) = f_1(x)$ for $0 \le x \le L$ and $F_1(x) = -f_1(2L - x)$ for $L \le x \le 2L$.

20.3.4 The Euler - Mac Laurin expansion

Dirichlet's kernel $D_N(x - a)$ is 2π-periodic and peaked at the points $a + 2\pi n$. If we multiply it by a smooth function and integrate on an interval $[a, b]$, with $b = a + 2\pi M$, it is:

$$\int_a^b dx D_N(x-a) f(x) = \int_a^b \frac{dx}{2\pi} f(x) + \sum_{k=1}^N \int_a^b \frac{dx}{\pi} \cos[k(x-a)]\, f(x)$$

The integral in the left-hand-side is evaluated on sub-intervals $[a, a + \pi)$, $[a + \pi, a + 3\pi)$, $\dots [b - \pi, b]$. On sub-intervals of width 2π, as N goes to infinity, the kernel becomes a normalized delta function peaked at the center of the interval. On the intervals $[a, a + \pi)$ and $[b - \pi, b]$ only half of the kernel contributes. Therefore the integral is:

$$\sum_{k=0}^M f(a + 2\pi k) - \frac{f(b) + f(a)}{2}$$

In the right side we integrate by parts twice and obtain a recursive law:

$$C_k[f] = \int_a^b dx \cos[k(x-a)] f(x) = \frac{1}{k^2}[f'(b) - f'(a)] - \frac{1}{k^2} C_k[f'']$$

For large N:

$$\frac{1}{\pi} \sum_{k=1}^N C_k[f] = \frac{\zeta(2)}{\pi}[f'(b) - f'(a)] - \frac{\zeta(4)}{\pi}[f'''(b) - f'''(a)] + R$$

with remainder $R = \frac{1}{\pi}\sum_k \frac{1}{k^6} C_k[f^{iv}]$. The Euler - Mac Laurin formula gives the corrections to an integral approximating a sum:

$$\sum_{k=0}^M f(a + 2\pi k) = \int_a^b \frac{dx}{2\pi} f(x) + \frac{1}{2}[f(b) + f(a)] \tag{20.22}$$
$$+ \frac{\pi}{6}[f'(b) - f'(a)] - \frac{\pi^3}{90}[f'''(b) - f'''(a)] + R$$

20.3.5 Poisson's summation formula

This is a very useful tool in many areas of physics. Given an integrable function $f(t)$ on $\mathbb{R}$, the sum $F(t) = \sum_{k=-\infty}^{\infty} f(t+kT)$ (if convergent) defines a T-periodic function, which can be expanded in Fourier series. The result is Poisson's summation formula, which replaces the infinite sum on shifts of f with an infinite Fourier series:

$$\boxed{\sum_{k=-\infty}^{\infty} f(t + kT) \;=\; \frac{1}{T} \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell e^{i\frac{2\pi}{T}\ell t}} \tag{20.23}$$

where

$$\hat{f}_\ell = \int_0^T dt\, F(t)\, e^{-i\frac{2\pi}{T}\ell t} = \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)T} dt f(t) e^{-i\frac{2\pi}{T}\ell t} = \int_{-\infty}^{\infty} dt f(t) e^{-i\frac{2\pi}{T}\ell t}.$$

Example 20.3.2. $f(t) = e^{-at^2}$, $\hat{f}_\ell = \sqrt{\frac{\pi}{a}} e^{-\ell^2/4a}$.

$$\sum_{k=-\infty}^{\infty} e^{-a(t+2k\pi)^2} = \frac{1}{2\sqrt{\pi a}} \sum_{\ell=-\infty}^{\infty} e^{-\ell^2/4a + i\ell t} \tag{20.24}$$

In particular, for $t = 0$*:*

$$\sum_{k=-\infty}^{\infty} e^{-4\pi^2 a k^2} = \frac{1}{\sqrt{4\pi a}} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2}{4a}} \tag{20.25}$$

These series arise in the theory of Jacobi's theta functions.

Example 20.3.3. $f(t) = e^{-\omega|t|}$, $\hat{f}_\ell = \frac{2\omega}{\omega^2 + k^2}$.

$$\sum_{k=-\infty}^{\infty} e^{-\omega|t+2\pi k|} = \frac{\omega}{\pi} \sum_{\ell=-\infty}^{\infty} \frac{e^{i\ell t}}{\omega^2 + \ell^2} \tag{20.26}$$

For $t = 0$ *the l.h.s. is* $-1 + 2\sum_{k=0}^{\infty} e^{-2\pi\omega k} = \coth(\pi\omega)$*. Then:*

$$\coth(\pi\omega) = \frac{\omega}{\pi} \sum_{\ell=-\infty}^{\infty} \frac{1}{\omega^2 + \ell^2}. \tag{20.27}$$

Example 20.3.4. *Let us evaluate the partition function* Z *for non-interacting particles in a cubic box with side-length* L*. The energy levels are* $\epsilon_k = \hbar^2 k^2/2m$*, where* $\mathbf{k} = (2\pi/L)\mathbf{n}$*,* $\mathbf{n} \in \mathbb{Z}^3$*. It is:*

$$Z = \sum_{\mathbf{k}} e^{-\frac{\epsilon_k}{k_B T}} = \left[\sum_{n=-\infty}^{\infty} \exp\left(-\frac{\lambda^2}{L^2} \pi n^2 \right) \right]^3$$

where $\lambda = \sqrt{2\pi\hbar^2/mk_B T}$ *is the "thermal length" (the de Broglie length of a particle with energy* $k_B T$*). For large* L*, the terms are a slowly decreasing sequence. The sum can be computed via Poisson's formula (20.25):*

$$\sum_{n=-\infty}^{\infty} \exp(-\frac{\lambda^2}{L^2}\pi n^2) = \frac{L}{\lambda} \sum_{n=-\infty}^{\infty} \exp(-\frac{L}{\lambda}\pi n^2) = \frac{L}{\lambda}(1 + \text{negligible terms}).$$

The standard procedure in statistical mechanics to approximate the sum with an integral (for large L*) is legitimate:*

$$Z = L^3 \int \frac{d\mathbf{k}}{(2\pi)^3} \exp\left(-\frac{\hbar^2 k^2}{2mk_B T} \right) = \left(\frac{L}{\lambda} \right)^3$$

In general, the extension to 3D of the Poisson sum for a period $2\pi/L$ *is:*

$$\sum_{\mathbf{k}} f(\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} f(\frac{2\pi}{L}\mathbf{n}) = \frac{L^3}{(2\pi)^3} \sum_{\mathbf{m}\in\mathbb{Z}^3} \int_{\mathbb{R}^3} d\mathbf{s} f(\mathbf{s}) e^{iL(\mathbf{m}\cdot\mathbf{s})}$$

If the function $f(\mathbf{k})$ *has slow variation on the scale* $1/L$, *for large* L *the Fourier terms vanish except the term* $\mathbf{m} = 0$*:*

$$\boxed{\sum_{\mathbf{k}} f(\mathbf{k}) \approx L^3 \int \frac{d\mathbf{k}}{(2\pi)^3} f(\mathbf{k})}$$

Example 20.3.5. *Poisson's formula is used to extend Riemann's series* $\zeta(z)$ *on Re* $z > 1$ *to a function on Re* $z < 0$ *(the strip* $0 < \mathrm{Re} z < 1$ *is left out).*
Let $g(t) = \sum_{k=1}^{\infty} e^{-k^2\pi t}$ *and evaluate*

$$\int_0^\infty dt\, g(t) t^{z-1} = \sum_{k=1}^{\infty} \int_0^\infty dt\, e^{-k^2\pi t} t^{z-1} = \frac{1}{\pi^z}\, \zeta(2z)\ \Gamma(z)$$

Eq.(20.25) *shows the symmetry* $2g(t) + 1 = (1/\sqrt{t})[2g(1/t) + 1]$ *i.e.* $g(t) = -1/2 + 1/(2\sqrt{t}) + g(1/t) 1/\sqrt{t}$, *which is used to evaluate the integral in a different way:*

$$\begin{aligned}
\frac{1}{\pi^{z/2}} \zeta(z)\Gamma(z/2) &= \int_0^1 dt\, g(t)\, t^{z/2-1} + \int_1^\infty dt\, g(t)\, t^{z/2-1} \\
&= \int_0^1 dt \left[-\frac{1}{2} + \frac{1}{2\sqrt{t}} + \frac{1}{\sqrt{t}} g(1/t) \right] t^{z/2-1} + \int_1^\infty dt\, g(t) t^{z/2-1} \\
&= -\frac{1}{z} - \frac{1}{1-z} + \int_1^\infty \frac{dt}{t}\, (t^{(1-z)/2} + t^{z/2}) g(t)
\end{aligned}$$

The r.h.s. is invariant under the replacement $z \to 1 - z$, *therefore:* $\zeta(z)\Gamma(\frac{z}{2}) = \sqrt{\pi}\, \zeta(1-z)\Gamma(\frac{1}{2} - \frac{z}{2})$ *or*

$$\boxed{\zeta(1-z) = \zeta(z) \frac{2}{(2\pi)^z} \cos(\tfrac{1}{2}\pi z)\Gamma(z)} \tag{20.28}$$

For $z = -2, -4, \ldots$ *the function* $\Gamma(z)$ *has poles; however the left-hand side is finite. Then* $\zeta(z)$ *must be zero at* $z = -2, -4, \ldots$*: these are the "trivial zeros" of Riemann's zeta function. The famous Riemann's hypothesis states that the non-trivial zeros are all located on the line Re* $z = \frac{1}{2}$. *The statement is relevant for the study of the distribution of prime numbers*[4].

Exercise 20.3.6. *Show that* $\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$ *(see exercise 12.3.5).*

20.4 Fejér sums

The powerful theory developed by Lipót Fejér on trigonometric series allows to prove the "completeness" of the basis of trigonometric functions in the Banach spaces $\mathscr{C}([-\pi, \pi])$ of continuous functions on $[-\pi, \pi]$, and in the spaces $\mathrm{L}^1(-\pi, \pi)$

[4]Riemann's hypothesis is one of the seven problems that were selected by the Clay Mathematics Institute in 2000 (Millennium Prize). The single problem that has been solved so far is Poincaré's conjecture (every simply connected closed 3-manifold is homeomorphic to the 3-sphere) stated in 1904. The winner Grigorij Perelman declined the one-million $ prize, and a Fields medal.

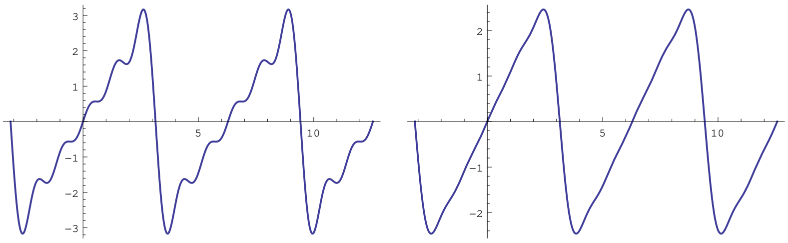

Figure 20.4: The trigonometric approximations S_5 and σ_5 of the Sawtooth function.

and $L^2(-\pi,\pi)$. It means that any function can be approximated arbitrarily well by a finite combination of trigonometric functions in the topology of the space.

A function that is continuous and 2π-periodic may not have a convergent Fourier series, and therefore may not be reproduced as the limit $N\to\infty$ of S_N. However, the arithmetic averages of the partial sums (*Fejér sums*) do the job in an excellent way. Consider the Fejér sum

$$\sigma_N(x) = \frac{1}{N}[S_0(x) + S_1(x) + \ldots + S_{N-1}(x)] = \int_{-\pi}^{\pi} dt\, f(x+t)\Phi_N(t) \quad (20.29)$$

where $\Phi_N(x)$ is *Fejér's kernel*:

$$\begin{aligned}\Phi_N(t) =& \frac{1}{N}\left[D_0(t) + D_1(t) + \ldots + D_{N-1}(t)\right] \\ =& \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\cos(kt) \\ =& \frac{1}{2\pi N}\frac{\sin^2(Nt/2)}{\sin^2(t/2)} \end{aligned} \quad (20.30)$$

The function is a finite sum of trigonometric functions, it is positive and has unit integral on $[-\pi,\pi]$. The Fejér sums σ_N are the *Cesaro means*[5] of the Dirichlet sums S_k. The interesting fact about them, is that they behave much better than Dirichlet sums, as the next theorems show.

Theorem 20.4.1 (I Fejér theorem, 1905)**.** *If f is real continuous and 2π-periodic, the sequence of Fejér sums σ_n converges to f uniformly on $\mathbb{R}$.*

Proof. Since f is real and continuous on $[-\pi,\pi]$ it is bounded, $|f(x)|<M$, and uniformly continuous: $\forall\epsilon\ \exists\delta : |f(x+t)-f(x)| < \epsilon/2\ \forall x,\ \forall t$ s.t. $|t|<\delta$.
Being periodic, f is bounded and uniformly continuous on the whole real line. Let us estimate the difference

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} dy[f(x+y)-f(x)]\Phi_N(y) = J_<(x) + J_0(x) + J_>(x)$$

[5]Ernesto Cesaro (1859, 1906) was professor in Naples. He proved that if a sequence a_n converges to a (or diverges) then: the sequence of arithmetic averages $s_n = \frac{1}{n}\sum_{k=1}^n a_k$ and the sequence of geometric averages $p_n = (a_1\cdots a_n)^{1/n}$ converge to the same limit (or diverge).

where integration is split on three intervals $[-\pi,-\delta)\cup[-\delta,\delta]\cup(\delta,\pi]$.

$$|J_0(x)| \le \int_{-\delta}^{\delta} dt|f(x+t)-f(x)|\Phi_N(t) < \frac{\epsilon}{2}\int_{-\delta}^{\delta} dt\,\Phi_N(t) \le \frac{\epsilon}{2}$$

$$|J_>(x)| \le \int_{\delta}^{\pi} dt|f(x+t)-f(x)|\Phi_N(t) < 2M\int_{\delta}^{\pi} dt\,\Phi_N(t)$$
$$\le \frac{M}{\pi N}\int_{\delta}^{\pi} dt\frac{1}{\sin^2(t/2)} \le \frac{M}{\pi N}\frac{\pi-\delta}{\sin^2(\delta/2)} \le \frac{M}{N\sin^2(\delta/2)}$$

The same estimate is valid for $J_<$. Now, given ϵ and δ, left's choose $\overline{N}$ such that $\frac{M}{N}\sin^{-2}(\delta/2) < \epsilon/4$. Then $|\sigma_N(x)-f(x)|<\epsilon$ for all $N>\overline{N}$. □

The theorem was proven by Fejér in 1905 at the age of 19, and strengthens the theorem by Weierstrass, that states that any continuous and periodic function is uniformly approximated by a sequence of trigonometric polynomials. Fejér has given an explicit expression (the Fejér sums) for such polynomials! A similar proof can be given for integrable functions:

Theorem 20.4.2 (II Fejér's theorem)**.** *If $f \in \mathscr{L}^1(-\pi,\pi)$, the sequence of Fejér's sums converges to f in $\mathrm{L}^1(-\pi,\pi)$.*

Proof. Given f in $\mathscr{L}^1$, its Fejér sum $\sigma_N(x) = \int_{-\pi}^{\pi} dy\, f(x+y)\Phi_N(y)$ is well defined, because Φ_N is a finite sum of trigonometric functions. We show that $\|f-\sigma_N\|_1 = \int_{-\pi}^{\pi} dx\,|f(x)-\sigma_N(x)| \to 0$ as $N\to\infty$.

$$\|f-\sigma_N\|_1 = \int_{-\pi}^{\pi} dx\left|\int_{-\pi}^{\pi} dy[f(x+y)-f(x)]\Phi_N(y)\right|$$
$$\le \int_{-\pi}^{\pi} dx\int_{-\pi}^{\pi} dy|f(x+y)-f(x)|\Phi_N(y)$$

The integrals can be exchanged by Fubini's theorem:

$$= \int_{-\pi}^{\pi} dy\,\Phi_N(y)\int_{-\pi}^{\pi} dx|f(x+y)-f(x)|$$

The integral in y is split on the intervals $[-\pi,-\delta]\cup(-\delta,\delta)\cup[\delta,\pi]$ where δ is by now unspecified but small. Because of the 2π-periodicity of the functions, the first interval is shifted by 2π and added to the third. Then:

$$= \int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta} dy\,\Phi_N(y)\int_{-\pi}^{\pi} dx|f(x+y)-f(x)|$$

The first integral is:

$$\le \left[\sup_{|y|<\delta}\int_{-\pi}^{\pi} dx\,|f(x+y)-f(x)|\right]\int_{-\delta}^{\delta} dy\,\Phi_N(y) \le \sup_{|y|<\delta}\int_{-\pi}^{\pi} dx\,|f(x+y)-f(x)|$$

because Fejér's kernel is normalized; the sup-term can be made smaller than ϵ, for $\delta\le\delta_\epsilon$. The second integral is:

$$\le \left[\sup_{\delta_\epsilon<y<2\pi-\delta_\epsilon}\int_{-\pi}^{\pi} dx\,|f(x+y)-f(x)|\right]\int_{\delta_\epsilon}^{2\pi-\delta_\epsilon} dy\,\Phi_N(y) \le \frac{2\|f\|_1}{N\sin^2(\delta_\epsilon/2)}$$

It is smaller than ϵ provided that N is large enough, once the δ_ϵ of the previous integral is fixed. □

Figure 20.5: **Lipót Fejér** (Pécs 1880, Budapest 1959) studied in Budapest and Berlin, as a student of Hermann Schwarz. In Budapest he led a relevant school of analysis, and was thesis advisor of John von Neumann, Paul Erdös, George Pólya, Pál Turán, Marcel Riesz, Gábor Szego, Michael Fekete, and others.

Figure 20.6: **Andrey N. Kolmogorov** (Tambov 1903, Moscow 1987) is the founder of axiomatic probability theory (1933). Independently with Chapman, he developed the basic equations for stochastic processes. He gave fundamental contributions to the theory of turbulence and of dynamical systems. Among his doctoral students are: Vladimir Arnold, Roland Dobrushin, Eugene Dynkin, Israel Gelfand, Yakov Sinai.

Corollary 20.4.3. *If $f \in \mathscr{L}^1(-\pi,\pi)$ has all Fourier coefficients $a_n = b_n = 0$, then $f = 0$ a.e.*

Proof. If $a_n = b_n = 0$ for all n, the Dirichlet's sums $S_N(x)$ and Fejér's sums $\sigma_N(x)$ vanish for all N. But II Fejér's theorem shows that $\|f - \sigma_N\|_1 \to 0$ as $N \to \infty$. Then $\|f\|_1 = 0 \Rightarrow f = 0$ a.e. □

20.5 Convergence in the mean

Proposition 20.5.1. *The trigonometric functions*

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\cos(nx), \quad \frac{1}{\sqrt{\pi}}\sin(nx), \qquad n = 1, 2, \dots$$

are a complete orthonormal system in $\mathrm{L}^2(-\pi,\pi)$.

Proof. If $f \in \mathrm{L}^2(-\pi,\pi)$ then $f \in \mathrm{L}^1(-\pi,\pi)$ (Schwarz's inequality: $\|f\|_1 = (1||f|) \leq \sqrt{2\pi}\|f\|_2$). f orthogonal to all basis elements ($a_n = b_n = 0$ $\forall n$) implies $f = 0$ a.e. by Corollary 20.4.3. □

By a change of scale, the Fourier basis in $\mathrm{L}^2(a,b)$ is ($n = 1, 2, \dots$):

$$\boxed{\frac{1}{\sqrt{b-a}}, \quad \sqrt{\frac{2}{b-a}}\cos\left(\frac{2\pi nx}{b-a}\right), \quad \sqrt{\frac{2}{b-a}}\sin\left(\frac{2\pi nx}{b-a}\right)} \tag{20.31}$$

If complex functions are used, the following form is more convenient:

$$\boxed{u_n(x) = \frac{1}{\sqrt{b-a}} \exp\left[i\frac{2\pi n}{b-a}x\right], \quad n \in \mathbb{Z}} \tag{20.32}$$

Since $\{u_n\}$ is a complete orthonormal set, any function in $L^2(a,b)$ has the Fourier expansion

$$f = \sum_{n=-\infty}^{\infty} f_n u_n, \qquad f_n = (u_n|f) = \int_a^b dx\, \overline{u_n(x)} f(x) \tag{20.33}$$

where convergence is in L^2 norm (*in the mean*): if S_N is the partial sum (Dirichlet's sum) $S_N = \sum_{n=-N}^{N} (u_n|f) u_n$, then $\|S_n - f\|^2 = \int_a^b dx\, |S_N - f|^2 \to 0$ as $N \to \infty$. Moreover:

$$\|f\|_2^2 = \int_a^b dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |f_n|^2 \qquad \text{(Parseval)} \tag{20.34}$$

What can be said about *point-wise convergence*? Luzin[6] conjectured (1906) that L^2 convergence implies almost everywhere convergence: $S_N(x) \to f(x)$ a.e. The conjecture was proven by Lennart Carleson[7] in 1966 and extended by Hund to spaces L^p for $p > 1$. The important case $p = 1$ has a different story: in 1923 and at the age 19, Andrei Kolmogorov provided a Lebesgue integrable function such that the sequence of partial sums is a.e. divergent. Two years later he sharpened it by showing that divergence is everywhere, and became a celebrity.

20.6 From Fourier series to Fourier integrals

Consider the Fourier expansion of a function on the interval $[-\frac{L}{2}, \frac{L}{2}]$:

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{e^{i2\pi kx/L}}{L} f_k, \qquad f_k = \int_{-L/2}^{L/2} dy\, e^{-i2\pi ky/L} f(y)$$

where for convenience the two normalization factors $\sqrt{L}$ are replaced by L in the sum. In view of the limit $L \to \infty$, introduce the new variable $s = 2\pi k/L$, with spacings $\delta s = 2\pi/L$:

$$f(x) = \sum_s \delta s\, \frac{e^{isx}}{2\pi} \tilde{f}(s), \qquad \tilde{f}(s) = \int_{-L/2}^{L/2} dy\, e^{-isy} f(y)$$

[6] Nikolai Luzin and the older Dimitri Egorov were influential mathematicians of Moscow's Mathematical Society during stalinian purges. They were both attacked and censured as reactionaries. Egorov was arrested and died one year after. Luzin, a specialist in real analysis, was processed but rehabilitated. No longer were important papers published on foreign journals. Luzin had important students, like Aleksandr Khinchin, Andrei Kolmogorov, Mickail Lavrentiev, Aleksei Lyapunov, Pavel Uryson.

[7] L. Carleson, *On the convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966).

If f is integrable, the function $\tilde{f}(s)$ exists for $L \to \infty$. If $\tilde{f}$ is sufficiently regular, it does not change on the scale δs, and the sum may be replaced by an integral:

$$f(x) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{isx} \int_{-\infty}^{\infty} dy\, e^{-isy} f(y)$$

According to this formula a function is a continuous superposition of Fourier components, weighted by a function of the continuous index:

$$f(x) = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{2\pi}} e^{isx} (\mathscr{F} f)(s) \tag{20.35}$$

$$(\mathscr{F} f)(s) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, e^{-isy} f(y) \tag{20.36}$$

The second line defines the Fourier integral of f (the factors 2π are often distributed differently).

Chapter 21

BOUNDED LINEAR OPERATORS ON HILBERT SPACES

21.1 Linear functionals

The space of linear bounded functionals $\mathscr{B}(\mathscr{H},\mathbb{C})$ is $\mathscr{H}^*$, the dual space of $\mathscr{H}$. The inner product with a fixed vector $(x|\cdot)$ is a linear functional and is bounded, by Schwarz's inequality. The following theorem proves that all functionals act as inner products:

Theorem 21.1.1 (Riesz's lemma). *For each $F \in \mathscr{H}^*$ there is a unique $x_F \in \mathscr{H}$ such that $Fx = (x_F|x)$ for all $x \in \mathscr{H}$. In addition $\|F\| = \|x_F\|$.*

Proof. If Ker F=$\mathscr{H}$ then $x_F = 0$. If Ker$F$ is a proper subspace, then there is a vector y orthogonal to it. Any vector x can be decomposed as $x = (x - y\frac{Fx}{Fy}) + y\frac{Fx}{Fy}$, where the first term belongs to Ker$F$ and then is orthogonal to y. The evaluation $(y|x) = Fx\frac{(y|y)}{Fy}$ shows that $x_F = y\frac{(Fy)^*}{\|y\|^2}$. The vector is unique, for suppose that $Fx = (x_F|x) = (x'_F|x)$ for all x, then $0 = (x_F - x'_F|x)$ i.e. $x_F = x'_F$.
Because of Schwarz's inequality: $|Fx| = |(x_F|x)| \le \|x_F\|\|x\|$; equality is attained at $|Fx_F| = \|x_F\|^2$. Therefore: $\|F\| = \sup_x \frac{|Fx|}{\|x\|} = \|x_F\|$. □

Because of the identification of bounded functionals with vectors, Hilbert spaces are *self-dual*.

21.2 Bounded linear operators

The linear bounded operators with domain $\mathscr{H}$ and range in $\mathscr{H}$ form the Banach space $\mathscr{B}(\mathscr{H})$. The set is closed for the involutive action of adjunction:

Theorem 21.2.1. *For any operator $\hat{A} \in \mathscr{B}(\mathscr{H})$ there is an operator $\hat{A}^\dagger \in \mathscr{B}(\mathscr{H})$ (the adjoint of $\hat{A}$) such that:*

$$\boxed{(x|\hat{A}y) = (\hat{A}^\dagger x|y) \qquad \forall x, y \in \mathscr{H}} \tag{21.1}$$

and $\|\hat{A}^\dagger\| = \|\hat{A}\|$.

Proof. Fix $x \in \mathscr{H}$, the map $y \to (x|\hat{A}y)$ is a linear bounded functional. Then, by Riesz' theorem, there is a unique vector v such that $(x|\hat{A}y) = (v|y)$ for all y. The correspondence $x \to v$ is linear, i.e. it defines a linear operator on $\mathscr{H}$: $v = \hat{A}^\dagger x$.
Equality of norms follows from Schwarz's inequality and boundedness of $\hat{A}$: $|(\hat{A}^\dagger x|y)| = |(x|\hat{A}y)| \le \|x\|\,\|\hat{A}\|\|y\|$, if $y = \hat{A}^\dagger x$ then $\|\hat{A}^\dagger x\| \le \|x\|\|\hat{A}\|$ i.e. $\hat{A}^\dagger$ is bounded and $\|\hat{A}^\dagger\| \le \|\hat{A}\|$. The proof of equality is left to the reader. □

Exercise 21.2.2. *Show that:*

$$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger \quad (\lambda\hat{A})^\dagger = \overline{\lambda}\hat{A}^\dagger \tag{21.2}$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \quad (\hat{A}^\dagger)^\dagger = \hat{A} \tag{21.3}$$

$$\hat{A}_n \to \hat{A} \Rightarrow \hat{A}_n^\dagger \to \hat{A}^\dagger \tag{21.4}$$

Proposition 21.2.3. *Let $\hat{A} \in \mathscr{B}(\mathscr{H})$, then* Ker $\hat{A}$ *is closed and*

$$\boxed{\mathscr{H} = \text{Ker}\,\hat{A} \oplus \overline{\text{Ran}\hat{A}^\dagger} = \text{Ker}\,\hat{A}^\dagger \oplus \overline{\text{Ran}\hat{A}}} \tag{21.5}$$

Proof. Suppose that x_n is a sequence in Ker $\hat{A}$, and $x_n \to x$. Then $\|\hat{A}x_n - \hat{A}x\| \le \|\hat{A}\|\|x_n - x\| \to 0$ i.e. $\hat{A}x = \lim_n \hat{A}x_n = 0$ i.e. $x \in$ Ker $\hat{A}$.
Since Ker $\hat{A}$ is a closed linear subspace, it is $\mathscr{H} = \text{Ker}\hat{A} \oplus (\text{Ker}\hat{A})^\perp$.

$$\begin{aligned}(\text{Ran}\hat{A}^\dagger)^\perp =& \{x\,:\,(x|y) = 0\ \forall y \in \text{Ran}\hat{A}^\dagger\}\\ =& \{x\,:\,(x|A^\dagger x') = 0\ \forall x' \in \mathscr{H}\}\\ =& \{x\,:\,(\hat{A}x|x') = 0\ \forall x' \in \mathscr{H}\}\\ =& \{x\,:\,\hat{A}x = 0\} = \text{Ker}\hat{A}\end{aligned}$$

Therefore $(\text{Ker}\hat{A})^\perp = \overline{\text{Ran}\hat{A}^\dagger}$. The second equality results after exchanging the operators. □

This statement is of practical utility. Consider the equation $\hat{A}x = y$; a solution x exists in $\mathscr{H}$ if y belongs to the range of $\hat{A}$. Suppose that the range is a closed set; then y belongs to it if it is orthogonal to all vectors that solve $\hat{A}^\dagger x' = 0$.

Exercise 21.2.4. *Show that if $\hat{A} \in \mathscr{B}(\mathscr{H})$ is invertible with bounded inverse, then also $\hat{A}^\dagger$ is invertible with bounded inverse, and $(\hat{A}^\dagger)^{-1} = (\hat{A}^{-1})^\dagger$.*

If the equation $\hat{A}x = \lambda x$ has a nonzero solution $x \in \mathscr{H}$, then λ and x are respectively an eigenvalue and an eigenvector of $\hat{A}$, and $|\lambda| \le \|A\|$.

Definition 21.2.5. An operator $\hat{A} \in \mathscr{B}(\mathscr{H})$ is **self-adjoint** if $\hat{A} = \hat{A}^\dagger$:

$$(\hat{A}x|y) = (x|\hat{A}y), \qquad \forall x, y \in \mathscr{H}$$

Theorem 21.2.6. *The eigenvalues of a self-adjoint operator in $\mathscr{B}(\mathscr{H})$ are real, and the eigenvectors corresponding to different eigenvalues are orthogonal.*

Proof. Let x and x' be eigenvectors corresponding to eigenvalues $\lambda \neq \lambda'$. From $(\hat{A}x|x) = (x|\hat{A}x)$ one obtains $\overline{\lambda} = \lambda$. From $(\hat{A}x|x') = (x|\hat{A}x')$ one obtains $(\lambda - \lambda')(x|x') = 0$ i.e. $x \perp x'$ if $\lambda \neq \lambda'$. □

Exercise 21.2.7. *Show that if $\hat{A}$ is* normal *(i.e. $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A}$), then eigenvectors corresponding to different eigenvalues of $\hat{A}$ are orthogonal.*

Exercise 21.2.8. *1) Let $\hat{A} \in \mathscr{B}(\mathscr{H})$; show that $\|\hat{A}^\dagger\hat{A}\| = \|\hat{A}\|^2$.*
2) If $\hat{A}$ and $\hat{B}$ are bounded and self-adjoint then $\|\hat{A}\hat{B}\| = \|\hat{B}\hat{A}\|$.

Exercise 21.2.9. *Suppose that $\hat{A} \in \mathscr{B}(\mathscr{H})$, and $\mathscr{H}$ is complex. Prove that:*
if $(x|\hat{A}x) = 0$ for all x then $\hat{A} = 0$;
if $(x|\hat{A}x)$ is real for all x, then $\hat{A} = \hat{A}^\dagger$.
(if $\mathscr{H}$ is real, the first condition implies $\hat{A} = -\hat{A}^\dagger$).

Exercise 21.2.10. *Show that for a bounded and self-adjoint operator it is:*

$$\|\hat{A}\| = \sup_{x \neq 0} \frac{|(x|\hat{A}x)|}{\|x\|^2} \tag{21.6}$$

21.2.1 Orthogonal projections

Given a closed subspace M and a vector x, the projection theorem states that $x = p + w$, where $p \in M$ and $w \in M^\perp$, and the decomposition is unique. Therefore the projection operator on M, $\hat{P} : x \to p$, is well defined (sometimes we write $\hat{P}(M)$ to identify the subspace).

Proposition 21.2.11. *The orthogonal projector $\hat{P}$ is linear, bounded, self-adjoint, and idempotent ($\hat{P}^2 = \hat{P}$).*

Proof. - Linearity. Let $x = p + w$ and $x' = p' + w'$, then $\alpha x + \beta x' = (\alpha p + \beta p') + (\alpha w + \beta w')$, where $\alpha p + \beta p' \in M$ and $\alpha w + \beta w' \in M^\perp$, $\forall \alpha, \beta$. As the decomposition is unique, necessarily it is $P(\alpha x + \beta x') = \alpha p + \beta p' = \alpha\hat{P}x + \beta\hat{P}y$.
- Boundedness. Since $x = \hat{P}x + w$, $\|x\|^2 = \|\hat{P}x\|^2 + \|w\|^2 \geq \|\hat{P}x\|^2$ then $\|\hat{P}x\| \leq \|x\|$. Equality holds for $x \in M$. Then $\hat{P} \in \mathscr{B}(\mathscr{H})$ and

$$\|\hat{P}\| = 1 \tag{21.7}$$

- Self-adjointness. $(x - \hat{P}x|y - \hat{P}y) = (x - \hat{P}x|y)$ (because $x - \hat{P}x \in M^\perp$), for analogous reason $(x - \hat{P}x|y - \hat{P}y) = (x|y - \hat{P}y)$; then $(\hat{P}x|y) = (x|\hat{P}y)$.
- Idempotency. $\hat{P}^2x = \hat{P}p = p = \hat{P}x$, then $\hat{P}^2 = \hat{P}$. □

These properties uniquely characterise an orthogonal projection operator. It is convenient to reverse the approach and introduce such operators through the definition:

Definition 21.2.12. $\hat{P} \in \mathscr{B}(\mathscr{H})$ is an orthogonal projector if $\hat{P}^2 = \hat{P}$, $\hat{P}^\dagger = \hat{P}$.

The definition recovers the properties of the original geometric characterization:
1) The subspace of projection is $M = \text{Ran}\hat{P}$.
If $x \in M$ then $x = \hat{P}y$ and $\hat{P}x = \hat{P}^2 y = x$.
2) $1 - \hat{P}$ is an orthogonal projector if $\hat{P}$ is. Since $(1-\hat{P})(\hat{P}y) = 0$, it is Ker $(1-\hat{P}) = M$. Therefore M is closed.
3) Let $y = x - \hat{P}x$, then $\hat{P}y = 0$ and $(y|\hat{P}x) = 0\,\forall x$, i.e. $\text{Ker}\hat{P} = M^\perp$.
4) $\|\hat{P}x\|^2 = (\hat{P}x|\hat{P}x) = (\hat{P}^2x|x) \le \|\hat{P}x\|\|x\|$, then $\|\hat{P}x\| \le \|x\|$. Equality holds for $x \in M$, then $\|\hat{P}\| = 1$.
5) Since $M \oplus M^\perp = \mathscr{H}$, a projector $\hat{P}$ has only the eigenvalues 1 and 0, with eigenspaces M and $M^\perp$.

Examples: 1) In $L^2(\mathbb{R})$ the multiplication operator $f \to \chi_{[a,b]}f$, where $\chi_{[a,b]}$ is the characteristic function of an interval $[a,b]$, is an orthogonal projector. The invariant subspace M is given by functions that vanish a.e. for $x \notin [a,b]$.
2) The operators $(P_\pm f)(x) = \frac{1}{2}[f(x) \pm f(-x)]$ are orthogonal projectors on the orthogonal subspaces of (a.e.) even and odd functions.

Exercise 21.2.13. *Let $\hat{P}$ and $\hat{P}'$ be projectors on subspaces M and M', then:*
1) $\hat{P} + \hat{P}'$ is a projector iff $M \perp M'$. In this case $\hat{P} + \hat{P}' = \hat{P}(M \oplus M')$.
2) $\hat{P}\hat{P}'$ is a projector iff $\hat{P}$ and $\hat{P}'$ commute. In this case $\hat{P}\hat{P}' = \hat{P}(M \cap M')$.

Example 21.2.14. *If $\{u_k\}_{k=1}^N$ is a s.o.n. the operator $\hat{P}x = \sum_{k=1}^N (u_k|x)u_k$ is the orthogonal projector on the linear subspace spanned by the vectors u_k.*
This is true also for $N = \infty$. For a s.o.n.c. $\hat{P} = 1$ (identity operator).

Exercise 21.2.15. *Let $x_1 \dots x_n$ be linearly independent vectors in a Hilbert space. Write the projection operator on the linear subspace spanned by the n vectors.*
Solution: $\hat{P}y = \sum_{ij} x_i [G^{-1}]_{ij}(x_j|y)$, where $G_{ij} = (x_i|x_j)$ is Gram's matrix.

Exercise 21.2.16. *Let $(\hat{D}_N f)(x) = \int_0^{2\pi} dy D_N(x-y)f(y)$, where $D_N(x)$ is the Dirichlet kernel* (20.6)*, and $f \in L^2(0,2\pi)$. Show that $\hat{D}_N$ is a projector, and identify the subspace. Are the sequences $\hat{D}_N$ and $\hat{D}_N f$ both convergent?*

21.2.2 Integral operators

Linear integral operators are important, and arise for example in the study of linear differential equations. Consider the inhomogeneous equation

$$f(x) = g(x) + \lambda \int_\Omega dy\, k(x,y) f(y),$$

where λ is a parameter, k is the *kernel* of the integral operator, g is assigned and f is the unknown function. In operator form it is $f = g + \lambda \hat{K} f$,

$$(\hat{K}f)(x) = \int_\Omega dy\, k(x,y)\, f(y) \tag{21.8}$$

The appropriate functional setting is suggested by the properties of the function g and of the kernel k. A solution exists if g belongs to the range of $1 - \lambda\hat{K}$.

Let Ω be a bounded interval ($[0,1]$ for definiteness). If $k \in \mathscr{L}^2(Q)$ (Q is the unit square) then $\hat{K}$ is a bounded operator on $L^2(0,1)$: by Fubini's theorem the function $y \to |k(x,y)|$ is measurable and belongs to $\mathscr{L}^2(0,1)$; by Schwarz's inequality $|(\hat{K}f)(x)| \le \|k(x,\cdot)\|\|f\|_2$. Squaring and integration in x gives:

$$\|\hat{K}f\|_2 \le \|k\|\,\|f\|_2$$

where $\|k\|^2 = \int_Q dxdy|k(x,y)|^2$. The adjoint operator is now evaluated[1]:

$$\begin{aligned}(h|\hat{K}f) &= \int_0^1 dx\,\overline{h(x)}\left[\int_0^1 dy\,k(x,y)f(y)\right] \\ &= \int_0^1 dy\overline{\left[\int_0^1 dx\overline{k(x,y)}h(x)\right]}f(y) = (\hat{K}^\dagger h|f)\end{aligned}$$

$$(\hat{K}^\dagger h)(x) = \int_0^1 dy\overline{k(y,x)}h(y) \tag{21.9}$$

The integral operator is self-adjoint if $k(x,y) = \overline{k(y,x)}$.

Let us enquire about the solution of the integral equation $(I - \lambda\hat{K})f = g$. A solution exists if $g \in \text{Ran}(I - \lambda K)$ i.e. $g \perp \text{Ker}(I - \overline{\lambda}\hat{K}^\dagger)$. This means that g must be orthogonal to the eigenvectors $\hat{K}^\dagger u = (1/\overline{\lambda})u$.
In particular, if $|\lambda|\|K\| < 1$, it is both Ker $(1 - \overline{\lambda}\hat{K}^\dagger) = \{0\}$ and Ker $(1 - \lambda\hat{K}) = \{0\}$, and the operator $(1 - \lambda\hat{K})^{-1}$ exists with domain $\mathscr{H}$. The operator can be expanded in a geometric series that converges for any g:

$$f = g + \lambda\hat{K}g + \lambda^2\hat{K}^2 g + \dots$$

The powers of the operator are integral operators with kernels $k_1(x,y) = k(x,y)$, $k_{n+1}(x,y) = \int_0^1 du\,k(x,u)\,k_n(u,y)$.

Example 21.2.17. *Consider the integral equation*

$$f(x) = g(x) + \int_0^x f(y)dy, \quad a.e.\, x \in [0,1],\ g \in \mathrm{L}^2(0,1)$$

It corresponds to the Cauchy problem $f' = f + g'$ with $f(0) = g(0)$. The kernel $k(x,y) = \theta(x-y)$ has L^2 norm $1/\sqrt{2}$, therefore the iterative solution of the integral equation converges, with kernels $k_{n+1}(x,y) = \theta(x-y)\,(x-y)^n/n!$. The convergent Neumann series gives

$$f(x) = g(x) + \int_0^x dy\,e^{x-y}g(y)$$

Example 21.2.18. *In $\mathrm{L}^2(0,2\pi)$ (real functions) consider the equation*

$$f = \lambda\hat{K}f + g, \quad (\hat{K}f)(x) = \int_0^{2\pi} dy \sin(x+y)f(y), \quad \lambda \in \mathbb{R}$$

[1] the steps are justified by Fubini's theorem for the exchange of order of integration: let $f(x,y)$ be measurable on $M \times N$, then $\int_M dm(\int_N dm|f|) < \infty$ iff $\int_N dm(\int_M dm|f|) < \infty$ and, if they are finite it is:

$$\int_M dm\left(\int_N dmf\right) = \int_N dm\left(\int_M dmf\right)$$

It is $(\hat{K}f)(x) = f_1 \sin x + f_2 \cos x$, *where* $f_1 = \int dx \cos(x) f(x)$ *and* $f_2 = \int dx \sin(x) f(x)$. *The operator* $\hat{K}$ *is rank 2 and self-adjoint. The problem is then two-dimensional and the solution, if existent, has the form:*

$$f(x) = \lambda(f_1 \sin x + f_2 \cos x) + g(x)$$

By taking the inner products with $\cos x$ *and* $\sin x$ *we get:*

$$\begin{bmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

where $g_1 = (\cos|g)$ *and* $g_2 = (\sin|g)$. *The solution exists for any* g *and is unique if* $1 - \lambda^2\pi^2 \neq 0$.
- If $\lambda \neq \pm\frac{1}{\pi}$, *matrix inversion gives* f_1 *and* f_2, *and:*

$$f(x) = g(x) + \lambda \int_0^{2\pi} dy \frac{\sin(x+y) + \lambda\pi\cos(x-y)}{1 - \lambda^2\pi^2} g(y)$$

- If $\lambda = \frac{1}{\pi}$, *a solution exists if* $g \in \mathrm{Ran}(I - \frac{1}{\pi}\hat{K})$ *i.e.* $g \perp \mathrm{Ker}(I - \frac{1}{\pi}\hat{K})$ $(\hat{K} = \hat{K}^\dagger)$. *The functions* $\hat{K}u = \pi u$, *up to a pre-factor, are:* $u(x) = \sin x + \cos x$. *Then:* $\int_0^{2\pi} dx(\sin x + \cos x) g(x) = g_1 + g_2 = 0$. *The solution is*

$$f(x) = \frac{f_1}{\pi}\sin x + \frac{f_1 - g_1}{\pi}\cos x + g(x)$$

where f_1 *is an arbitrary constant.*
- If $\lambda = -\frac{1}{\pi}$ *the case is treated similarly.*

Exercise 21.2.19. *Show that the following linear operator belongs to* $\mathscr{B}(\mathrm{L}^2(\mathbb{R}))$ *and it is self-adjoint:*

$$(\hat{T}f)(x) = \int_{-\infty}^{\infty} dy \, \frac{f(y)}{x^2 + y^2 + 1}$$

($\hat{T}$ *is not invertible, as all odd-parity functions belong to the kernel).*

21.2.3 The position operator

On $\mathrm{L}^2[a,b]$ define the "position" operator that multiplies a function f by the function x ($x(t) = t$): $\hat{Q}f = xf$ i.e. $(\hat{Q}f)(t) = tf(t)$.
The operator is bounded, $\|\hat{Q}f\| \leq \max(|a|,|b|)\,\|f\|$, and it is self-adjoint:

$$(g|\hat{Q}f) = \int_a^b dt\, \overline{g(t)} t\, f(t) = \int_a^b dt\, \overline{tg(t)} f(t) = (\hat{Q}g|f).$$

The eigenvalue equation $\hat{Q}f = \lambda f$ has no solution in $\mathrm{L}^2[a,b]$: the equation $(t-\lambda)f(t) = 0$ a.e. $t \in [a,b]$ implies that $f = 0$ a.e. We may ask if there are "approximate eigenfunctions": $\|\hat{Q}u - \lambda u\| < \epsilon\|u\|$. Given λ, consider the normalized functions $u_\eta = \frac{1}{\sqrt{2\eta}}\chi_{[\lambda-\eta,\lambda+\eta]}$ and evaluate

$$\|\hat{Q}u_\eta - \lambda u_\eta\|^2 = \int_a^b dt(t-\lambda)^2 u_\eta(t)^2 = \frac{\eta^2}{3} \to 0$$

The limit $\eta \to 0$ of u_η does not exist in Hilbert space[2]. λ is a generalized eigenvalue, and u_η is a generalized eigenfunction. The set of such eigenvalues is the continuum spectrum; in this case $\sigma_c(\hat{Q}) = [a, b]$.

21.2.4 The linear momentum operator

In $L^2[a, b]$ the operator acting as a derivative requires that a function admits derivative and is square-integrable. We introduce a suitable set.
A function f is *absolutely continuous* on $[a, b]$, f is $AC[a, b]$, if there is a function $h \in \mathscr{L}^1[a, b]$ such that:

$$f(x) = f(a) + \int_a^x dx' h(x'), \quad x \in (a, b).$$

The functions $AC[a, b]$ are continuous and differentiable: $f' = h$ a.e.. They form a linear subset in $\mathscr{L}^p(a, b)$ for all $p \geq 1$. If h is a continuous function, by the theorem of the mean: $f'(x) = h(x)$.
The derivative is the "linear momentum" operator[3]

$$(\hat{P}f)(x) = -if'(x) \tag{21.10}$$

with domain of functions $AC[a, b]$ such that $f' \in \mathscr{L}^2[a, b]$. If f_1 and f_2 are such functions, integration by parts gives:

$$(f_1|\hat{P}f_2) = -i \int_a^b dx \overline{f_1(x)} f_2'(x) = -i\overline{f_1} f_2 \Big|_a^b + (\hat{P}f_1|f_2).$$

The operator is symmetric provided that the boundary terms cancel. This is achieved if the domain of $\hat{P}$ is restricted to AC functions with appropriate boundary conditions (b.c.): $f(b) = f(a) = 0$ (Dirichlet b.c.), or $f(b) = \pm f(a)$ (periodic/antiperiodic b.c.) or $f(b) = e^{i\theta} f(a)$ (Bloch b.c. with $\theta \in \mathbb{R}$). They produce three different definitions of $\hat{P}$ (same action but different domains where it is symmetric).
With periodic b.c. the operator $\hat{P}$ has a complete orthonormal set of eigenfunctions $\hat{P}u_k(x) = ku_k(x)$, $k \in \mathbb{Z}$, given by the Fourier basis (20.32) .

21.3 Unitary operators

Definition 21.3.1. A linear operator $\hat{U}$ with domain $\mathscr{H}$ and range in $\mathscr{H}$ is an **isometry** if $\|\hat{U}x\| = \|x\|$ for all x. If also Ran $\hat{U} = \mathscr{H}$ the operator is **unitary**.

1) The conservation of the norm implies that Ker $\hat{U} = \{0\}$ i.e. $\hat{U}^{-1}$ exists and

$$\boxed{(\hat{U}x|\hat{U}y) = (x|y) \quad \forall x, y} \tag{21.11}$$

2) The operator is bounded with norm $\|\hat{U}\| = 1$.
3) $(x|y) = (\hat{U}x|\hat{U}y) = (\hat{U}^\dagger \hat{U}x|y)$ $\forall x, y$, then $\hat{U}^\dagger \hat{U}x = x$ for all x i.e.

$$\boxed{\hat{U}^\dagger \hat{U} = I} \tag{21.12}$$

[2]the functions do not form a Cauchy sequence: for $\delta < \eta$ it is $\|u_\eta - u_\delta\|^2 = 2(1 - \sqrt{\delta/\eta})$ which may not be small as $\eta, \delta \to 0$.

[3]physicists introduce Planck's constant, $\hat{P}f = -i\hbar f'$.

Therefore, $\hat{U}^\dagger = \hat{U}^{-1}$, and it is a unitary operator: $\|x\| = \|\hat{U}(\hat{U}^\dagger x)\| = \|\hat{U}^\dagger x\|$ for all x.
4) If λ is an eigenvalue of $\hat{U}$, then $|\lambda| = 1$. Eigenvectors with different eigenvalues are orthogonal.
5) The product of unitary operators is unitary.

Exercise 21.3.2. *If $\hat{U}$ and $\hat{V}$ are unitary then $\|\hat{U}\hat{A}\hat{V}\| = \|\hat{A}\|$ $\forall \hat{A} \in \mathscr{B}(\mathscr{H})$.*

Exercise 21.3.3. *If $\{u_k\}_{k=1}^\infty$ is a complete orthonormal system and $\{q_k\}_{k=1}^\infty$ are real numbers, then*

$$\hat{U}x = \sum_{k=1}^{\infty} e^{iq_k}(u_k|x)u_k$$

is a unitary operator.

Proof. If $\hat{E}_n x$ is the partial sum, for all x the sequence $\hat{E}_n x$ is Cauchy:

$$\|\hat{E}_{n+p}x - \hat{E}_n x\|^2 = \sum\nolimits_{k=n+1}^{n+p} |(u_k|x)|^2$$

because $\sum_{k=1}^n |(u_k|x)|^2$ is convergent (Parseval). Then $\hat{E}_n x$ converges and $\hat{U}x$ exists for all x. $\|\hat{E}_n x\|^2 = \sum_{k=1}^n |(u_k|x)|^2$. As the norm is continuous, in the limit and by Parseval's theorem: $\|\hat{U}x\| = \|x\|$ $\forall x$ (i.e. $\hat{U}$ is isometric).
The vectors $\hat{U}u_k = e^{iq_k}u_k$ are a complete orthonormal system contained in $\mathrm{Ran}\,\hat{U}$. Then $\mathrm{Ran}\,\hat{U} = \mathscr{H}$ ($\hat{U}$ is unitary).
If $|q_k| \le Q$ for all k, then $\hat{U} = \exp(i\hat{H})$ where $\hat{H}x = \sum_{k=1}^\infty q_k(u_k|x)u_k$ is bounded and self-adjoint, and $\|\hat{H}\| \le Q$. □

Example 21.3.4. *If $\hat{H} \in \mathscr{B}(\mathscr{H})$ and $\hat{H} = \hat{H}^\dagger$, then $\hat{U}_t = e^{-it\hat{H}}$, $t \in \mathbb{R}$ are unitary operators and form a one-parameter strongly continuous group*

$$\hat{U}_t\hat{U}_s = \hat{U}_{t+s}$$
$$\lim_{t\to 0} \hat{U}_t x \to x \quad \forall x \in \mathscr{H}$$

$\hat{H}$ is the generator of the group.

Proof. The sequence $\hat{E}_n(t) = \sum_{k=0}^n (-it\hat{H})^k/k!$ converges in $\mathscr{B}(\mathscr{H})$ for all t.
- It is $\hat{E}_n(t)^\dagger = \hat{E}_n(-t)$; the adjunction is a continuous map, then $\hat{U}_t^\dagger = \hat{U}_{-t}$.
- The operators form a 1-parameter group: $\hat{U}_t\hat{U}_s = \hat{U}_{t+s}$.
- $\hat{U}_t$ is isometric: $(\hat{U}_t x|\hat{U}_t y) = (\hat{U}_t^\dagger \hat{U}_t x|y) = (x|y)$.
- Given $y \in \mathscr{H}$ $\exists x$ s.t. $\hat{U}_t x = y$? It is $\hat{U}_{-t}y$. Then $\mathrm{Ran}\,\hat{U}_t = \mathscr{H}$ ($\hat{U}_t$ is unitary).
- Strong continuity: $\|\hat{E}_n(t)x - x\| \le \sum_{k=1}^n \frac{|t|^k}{k!}\|H\|^k\|x\| \le (e^{|t|\|H\|} - 1)\|x\|$ $\forall n$.
For $t \to 0$ one obtains that $\|\hat{E}_n(t)x - x\| \to 0$ uniformly in n. □

Exercise 21.3.5. *Let $\hat{H} \in \mathscr{B}(\mathscr{H})$ be self-adjoint. Show that the Cayley transform of $\hat{H}$, $(1 - i\hat{H})(1 + i\hat{H})^{-1}$, is well defined and is a unitary operator. What is the Cayley transform of a projector?*

Exercise 21.3.6. *Show that if $\hat{H}$ is bounded and self-adjoint, the operators $\hat{U}_t = e^{-it\hat{H}}$, $t \in \mathbb{R}$, are unitary and form a group. $\hat{H}$ is the generator of the group (see Sect.18.5.3).*

Exercise 21.3.7. *$\hat{P}$ is a projector; evaluate $e^{i\theta\hat{P}}$ and $(z - \hat{P})^{-1}$ ($z \ne 0, 1$).*

21.4 Notes on spectral theory

Discrete spectrum: $z \in \sigma_p(\hat{A})$ if z is an eigenvalue:

$$\exists u \in \mathscr{H} \text{ such that } \hat{A}u = zu$$

i.e. $\text{Ker}(z - \hat{A}) \neq \{0\}$ i.e. $(z - \hat{A})^{-1}$ does not exist.
Continuous spectrum: $z \in \sigma_c(\hat{A})$ if z is a generalized eigenvalue:

$$z \notin \sigma_p(\hat{A}), \quad \exists \{u_n\} \subset \mathscr{H}, \|u_n\| = 1, \text{ such that } \lim_{n\to\infty} \|\hat{A}u_n - zu_n\| = 0.$$

For a bounded operator, the set $\sigma = \sigma_p \cup \sigma_c$ is the spectrum of the operator.

Remark 21.4.1.
1) If $\hat{A} = \hat{A}^\dagger$, then $\sigma_p(\hat{A}) \subseteq \mathbb{R}$.
2) Continuous spectrum: $\{u_n\}$ is not a Cauchy sequence. Convergence $u_n \to u$ and continuity of $\hat{A}$ would imply that u is an eigenvector and $z \in \sigma_p(\hat{A})$.
3) If $z \in \sigma_c(\hat{A})$ the operator $(z - \hat{A})^{-1}$ exists, but it cannot belong to $\mathscr{B}(\mathscr{H})$.
Proof: Suppose that it does (then the domain is $\mathscr{H}$ and it is bounded):

$$1 = |(u_n|(z - \hat{A})^{-1}(z - \hat{A})u_n)| \leq \|(z - \hat{A})^{-1}\| \|zu_n - \hat{A}u_n\|$$

since for $n \to \infty$ the last factor is zero, the other factor cannot be finite.
It turns out that either $(z - \hat{A})^{-1}$ has domain $\mathscr{H}$ without being bounded, or its domain is strictly contained in $\mathscr{H}$.
4) Since $|(u_n|(\hat{A} - \lambda)u_n)| \leq \|\hat{A}u_n - \lambda u_n\|$, it is $\lambda = \lim_{n\to\infty}(u_n|\hat{A}u_n)$. Then, for self-adjoint operators, it is $\sigma_c(\hat{A}) \subseteq \mathbb{R}$ and $|\lambda| \leq \|\hat{A}\|$ (use continuity of the modulus).
5) If $z \in \sigma_c(\hat{A})$ then $\overline{z} \in \sigma_c(\hat{A}^\dagger)$ and $\{0\} = \text{Ker}\,(\overline{z} - \hat{A}^\dagger) = \text{Ran}\,(z - \hat{A})^\perp$ i.e. $\mathscr{H} = \overline{\mathscr{D}\,(z - \hat{A})^{-1}}$. Therefore, either $\mathscr{D}(z - \hat{A})^{-1} = \mathscr{H}$ (but the resolvent is not bounded) or $\mathscr{D}(z - \hat{A})^{-1}$ is a proper subset, but dense.

Proposition 21.4.2. *If $z \notin \sigma$ then $(z - \hat{A})^{-1} \in \mathscr{B}(\mathscr{H})$*

Proof. Since $z \notin \sigma$ the resolvent exists. The domain of the resolvent $\text{Ran}\,(z - \hat{A})$ is dense in $\mathscr{H}$: $\{0\} = \text{Ker}(z - \hat{A})^\dagger = \text{Ran}(z - \hat{A})^\perp$. It is also:

$$\exists \delta \text{ such that } \|(z - \hat{A})u\| \geq \delta \|u\| \quad \forall u \in \mathscr{H}$$

The vectors $v = (z - \hat{A})u$ span the domain of the resolvent, then:

$$\exists \delta \text{ such that } \|v\| \geq \delta \|(z - \hat{A})^{-1}v\| \quad \forall v \in \mathscr{D}(z - \hat{A})^{-1}$$

Then the resolvent is bounded by $1/\delta$ on its domain. By theorem 18.3.8 the resolvent extends to a bounded operator on the closure of the domain, $\mathscr{H}$. □

Chapter 22

UNITARY GROUPS

The Hilbert space is the mathematical stage for quantum mechanics, continuous symmetries are main actors in the play, and correspond to unitary operators[1]. Of special importance are groups that depend continuously on one parameter, like rotations around a given axis, or translations along a given direction. They correspond to special families of unitary operators. The following theorems are very important and useful (they are stated without proof[2]).

22.1 Stone's theorem

Definition 22.1.1. A family of unitary operators $\{\hat{U}_s,\ s \in \mathbb{R}\}$, is a **strongly continuous one-parameter unitary group** if:

$$\hat{U}_s \hat{U}_{s'} = \hat{U}_{s+s'}$$
$$\lim_{s\to 0} \|U_s x - x\| = 0 \quad \forall x \in \mathscr{H}.$$

It is clear that the group is Abelian, $\hat{U}_0 = I$ and $\hat{U}_s^{-1} = \hat{U}_{-s}$. For such groups, the following fundamental theorem holds:

Theorem 22.1.2 (Stone's theorem). *Let $\hat{U}_s$ be a strongly continuous one-parameter unitary group on a Hilbert space. Then there is a self-adjoint operator $\hat{H}$ such that*

$$\boxed{\hat{U}_s = e^{-is\hat{H}}} \tag{22.1}$$

$\hat{H}$ is the **generator** *of the group, with domain*

$$\mathscr{D}(\hat{H}) = \{x \in \mathscr{H} \text{ such that } \lim_{s\to 0} \frac{\hat{U}_s x - x}{-is} \text{ exists}\},$$

and $\hat{H}x$ is precisely the above limit.

[1]Discrete symmetries may correspond also to anti-unitary (i.e. antilinear and norm conserving) operators. An example is time-reversal.

[2]see for example: K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer 2012.

Theorem 22.1.3 (Lie-Trotter formula). *Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators. If $\hat{A}+\hat{B}$ is self-adjoint on $\mathscr{D}(\hat{A})\cap\mathscr{D}(\hat{B})$, then:*

$$e^{it(\hat{A}+\hat{B})}x = \lim_{n\to\infty}\left[e^{it\hat{A}/n}e^{it\hat{B}/n}\right]^n x \tag{22.2}$$

22.2 Weyl operators

Two important families of unitary operators on $\mathrm{L}^2(\mathbb{R}^n)$ are:
translations by vectors $\mathbf{a}\in\mathbb{R}^n$:

$$(\hat{U}_\mathbf{a} f)(\mathbf{x}) = f(\mathbf{x}-\mathbf{a}), \quad \hat{U}_\mathbf{a}\hat{U}_{\mathbf{a}'} = \hat{U}_{\mathbf{a}+\mathbf{a}'} \tag{22.3}$$

multiplication by a phase factor with vector $\mathbf{p}\in\mathbb{R}^n$:

$$(\hat{V}_\mathbf{p} f)(\mathbf{x}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x}), \quad \hat{V}_\mathbf{p}\hat{V}_{\mathbf{p}'} = \hat{V}_{\mathbf{p}+\mathbf{p}'} \tag{22.4}$$

Planck's constant $\hbar$ is here introduced to give $\mathbf{x}$, $\mathbf{a}$ the physical dimension of length, and $\mathbf{p}$ the dimension of momentum.
The translations along a fixed direction $\hat{U}_{a\mathbf{n}}$, $a\in\mathbb{R}$, and phase multiplications $\hat{V}_{p\mathbf{n}}$, $p\in\mathbb{R}$, are strongly continuous one-parameter unitary groups. Stone's theorem implies the existence of self-adjoint generators.
On the dense set $\mathscr{S}(\mathbb{R}^3)$ ($\mathscr{C}^\infty$ rapidly decreasing functions) a Taylor expansion gives:

$$(\hat{U}_{a\mathbf{n}} f)(\mathbf{x}) = f(\mathbf{x}-a\mathbf{n}) = f(\mathbf{x}) + (-a\mathbf{n}\cdot\nabla)f(\mathbf{x}) + \tfrac{1}{2!}(-a\mathbf{n}\cdot\nabla)^2 f(\mathbf{x}) + \dots$$

Therefore, the generator of translations with direction $\mathbf{n}$ is the self-adjoint operator whose restriction to analytic functions is $-i\hbar\,\mathbf{n}\cdot\nabla = \sum_i n_i\hat{P}_i$, and

$$\hat{U}_\mathbf{a} = \exp\left[-\tfrac{i}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}\right], \qquad \hat{P}_k f = -i\hbar\frac{\partial f}{\partial x_k}$$

$\hat{P}_k$ is the generator of translations along the direction k. Momentum operators are self-adjoint on a dense domain in $\mathrm{L}^2(\mathbb{R}^3)$ and act as derivatives on the subset of analytic functions. The property $\hat{U}_\mathbf{a}\hat{U}_{\mathbf{a}'} = \hat{U}_{\mathbf{a}'}\hat{U}_\mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{a}'$ implies, for infinitesimal vectors, the relation $[\hat{P}_i,\hat{P}_j]=0$.

The generator of the one-parameter group $(\hat{V}_{p\mathbf{n}}f)(\mathbf{x}) = \exp(-\frac{i}{\hbar}p\mathbf{n}\cdot\mathbf{x})f(\mathbf{x})$ is the (unbounded) self-adjoint operator $\mathbf{n}\cdot\hat{\mathbf{Q}}$, and

$$\hat{V}_\mathbf{p} = \exp\left[-\frac{i}{\hbar}\mathbf{p}\cdot\hat{\mathbf{Q}}\right], \qquad (\hat{Q}_i f)(\mathbf{x}) = x_i f(\mathbf{x}).$$

The n operators $\hat{Q}_i$ are self-adjoint and commute on a dense domain in $\mathrm{L}^2(\mathbb{R}^3)$.
The generators $\hat{Q}_i$ and $\hat{P}_i$ $i=1\dots n$ form *Heisenberg's algebra*:

$$[\hat{Q}_i,\hat{Q}_j]=0, \quad [\hat{P}_i,\hat{P}_j]=0, \quad [\hat{Q}_i,\hat{P}_j]=i\hbar\delta_{ij}$$

The first two commutation relations correspond to the fact that the unitary groups are Abelian. The mixed commutator, proportional to the identity operator, reflects a simple relation among the groups (Weyl's commutation relation):

$$\hat{V}_\mathbf{p}\,\hat{U}_\mathbf{a} \,=\, e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}}\,\hat{U}_\mathbf{a}\,\hat{V}_\mathbf{p} \tag{22.5}$$

Proof. $(\hat V_{\mathbf p}\hat U_{\mathbf a} f)(\mathbf x) = e^{\frac{i}{\hbar}\mathbf p\cdot\mathbf x}(\hat U_{\mathbf a} f)(\mathbf x) = e^{\frac{i}{\hbar}\mathbf p\cdot\mathbf x} f(\mathbf x - \mathbf a) = e^{\frac{i}{\hbar}\mathbf p\cdot\mathbf a}(\hat V_{\mathbf p} f)(\mathbf x - \mathbf a)$
$= e^{\frac{i}{\hbar}\mathbf p\cdot\mathbf a}(\hat U_{\mathbf a}\hat V_{\mathbf p} f)(\mathbf x)$. □

Exercise 22.2.1. *Show that the translation operator* $(\hat U_a f)(x) = f(x-a)$ *does not have eigenvectors in* $\mathrm{L}^2(\mathbb R)$. *(Hint: a periodic function cannot be integrable).*

Exercise 22.2.2. *Let* $[\hat A, \hat B] = iI$, $\hat A = \hat A^\dagger$ *and* $\hat B = \hat B^\dagger$. *Show that* $\hat A$ *and* $\hat B$ *cannot be both bounded.*
Sketch of the proof: suppose that $\hat A$ *and* $\hat B$ *are bounded, then* $[\hat A^n, \hat B] = in\hat A^{n-1}$ *and* $[\exp(-it\hat A), \hat B] = t\exp(-it\hat A)$*; conclude that* $\|\hat B\|$ *is larger that any real number.*

Exercise 22.2.3 (Scale transformations). *On* $L^2(\mathbb R)$ *the operators*

$$(\hat D_\lambda f)(x) = e^{-\lambda/2} f(e^{-\lambda}x), \quad \lambda \in \mathbb R \tag{22.6}$$

are a strongly continuous one-parameter unitary group. On the dense subspace $\mathscr S(\mathbb R)$ *(see chapter on Schwartz space):*
1) obtain the generator $\hat D = \frac12(\hat Q\hat P + \hat P\hat Q)$ *i.e.* $\hat D_\lambda = \exp(-i\lambda\hat D)$*;*
2) show that $\hat D_\lambda^\dagger \hat Q\hat D_\lambda = e^\lambda\hat Q$ *and* $\hat D_\lambda^\dagger\hat P\hat D_\lambda = e^{-\lambda}\hat P$.

Scale transformations are related to the **Virial property**. This is illustrated for the anharmonic oscillator, $\hat H = \frac{1}{2m}\hat P^2 + \frac12 k\hat Q^2 + b\hat Q^4$ $(b>0)$.
Let $\hat H\psi = E\psi$, then $E = \frac{1}{2m}\langle\hat P^2\rangle + \frac12 k\langle\hat Q^2\rangle + b\langle\hat Q^4\rangle$, where $E = \langle\hat H\rangle = (\psi|\hat H\psi)$ etc. are the average values. The action of a scale transformation is: $\langle\hat D_\lambda^\dagger\hat H\hat D_\lambda\rangle = e^{-2\lambda}\frac{1}{2m}\langle\hat P^2\rangle + e^{2\lambda}\frac12 k\langle\hat Q^2\rangle + be^{4\lambda}\langle\hat Q^4\rangle$. The linear term of an expansion in small λ is: $i(\psi|[\hat D,\hat H]\psi) = -2\frac{1}{2m}\langle\hat P^2\rangle + 2\frac12 k\langle\hat Q^2\rangle + 4b\langle\hat Q^4\rangle$. Since the left hand side is zero for an eigenvector, an identity is obtained among terms that contribute to the total energy. If $b=0$, one gets the equality of kinetic and potential energy terms of the harmonic oscillator, $\langle\frac{1}{2m}\hat P^2\rangle = \langle\frac12 k^2\hat Q^2\rangle$, that is used, for example, in the theory of specific heats.

22.3 Space rotations, SO(3)

Space rotations act on vectors in $\mathbb R^3$ as 3×3 real matrices R that preserve lengths, $RR^t = I_3$, and orientation, $\det R = 1$. They form the group $SO(3)$. Being unitary, rows or colums of R are orthogonal and normalized, eigenvalues are on the unit circle.
The characteristic polynomial $\det(zI_3 - R)$ is real cubic, then it necessarily has a positive real zero (equal to one) corresponding to a real eigenvector of R: the invariant vector $R\mathbf n = \mathbf n$.
The other two eigenvalues may be $1,1$ or $-1,-1$ (then R is the identity matrix or a diagonal matrix describing two reflections) or $e^{\pm i\varphi}$ with eigenvectors $\mathbf u \pm i\mathbf v$ making with $\mathbf n$ an orthogonal set in $\mathbb C^3$. This implies that the vectors $\mathbf n$, $\mathbf u$ and $\mathbf v$ are orthogonal in $\mathbb R^3$. Note that $\mathrm{tr}\, R = 1 + 2\cos\varphi$.
The eigenvalue equation $R(\mathbf u \pm i\mathbf v) = e^{\pm i\varphi}(\mathbf u \pm i\mathbf v)$ i.e. the pair

$$R\mathbf u = \mathbf u\cos\varphi - \mathbf v\sin\varphi$$
$$R\mathbf v = \mathbf u\sin\varphi + \mathbf v\cos\varphi$$

shows that two vectors rotate under the action of R in the plane perpendicular to $\mathbf{n}$. For the rotation to be anticlockwise, the orientation $\mathbf{n} = \mathbf{v} \times \mathbf{u}$ is chosen. Every vector $\mathbf{x}$ can be expanded in the orthonormal basis: $\mathbf{x} = x_n \mathbf{n} + x_u \mathbf{u} + x_v \mathbf{v}$ where $x_n = \mathbf{x} \cdot \mathbf{n}$ etc. The action of R is:

$$\begin{aligned} R\mathbf{x} &= x_n \mathbf{n} + x_u(\mathbf{u}\cos\varphi - \mathbf{v}\sin\varphi) + x_v(\mathbf{u}\sin\varphi + \mathbf{v}\cos\varphi) \\ &= x_n\mathbf{n} + (x_u\mathbf{u} + x_v\mathbf{v})\cos\varphi + (x_v\mathbf{u} - x_u\mathbf{v})\sin\varphi \\ &= x_n\mathbf{n} + (\mathbf{x} - x_n\mathbf{n})\cos\varphi + \mathbf{n}\times\mathbf{x}\,\sin\varphi \\ &= \mathbf{x}\cos\varphi + (1-\cos\varphi)(\mathbf{n}\cdot\mathbf{x})\mathbf{n} + \mathbf{n}\times\mathbf{x}\,\sin\varphi \end{aligned}$$

Exercise 22.3.1. *Write the rotation as a matrix on the column vector* $(x, y, z)^t$.

The invariant unit vector $\mathbf{n}$ that identifies the rotation axis and the rotation angle φ measured anticlockwise, are a useful parametrization of rotations. We'll soon discover that the parameters $\mathbf{n}$ and φ combine in a vector $\mathbf{n}\,\varphi$.
The expansion for an infinitesimal angle provides the neighbourhood of the identity matrix, which is extremely instructive in the study of groups with analytic dependence on parameters (Lie groups):

$$R\mathbf{x} = \mathbf{x} + \mathbf{n}\times\mathbf{x}\,\delta\varphi + \ldots = (I_3 + \delta\varphi\, A + \ldots)\mathbf{x}$$

$$A = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = \mathbf{n}\cdot\mathbf{A} \tag{22.7}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rotations with same invariant vector form a commutative subgroup:

$$R(\mathbf{n}\varphi_1)R(\mathbf{n}\varphi_2) = R(\mathbf{n}(\varphi_1 + \varphi_2)) \tag{22.8}$$

If one angle is infinitesimal: $R(\mathbf{n}\varphi)(I_3 + \delta\varphi\,\mathbf{n}\cdot\mathbf{A} + \ldots) = R(\mathbf{n}(\varphi + \delta\varphi))$. Since the factors can be exchanged, one concludes that the matrices of the subgroup $R(\mathbf{n}\varphi)$ commute with $\mathbf{n}\cdot\mathbf{A}$ (then, they have the same eigenvectors). Moreover, by taking the limit $\delta\varphi$ to zero:

$$\frac{d}{d\varphi}R(\mathbf{n}\varphi) = (\mathbf{n}\cdot\mathbf{A})R(\mathbf{n}\varphi) \tag{22.9}$$

with the initial condition $R(0) = I_3$. Since factors commute, the solution is

$$\boxed{R(\mathbf{n}\varphi) = e^{\varphi\,\mathbf{n}\cdot\mathbf{A}}} \tag{22.10}$$

The matrix $\mathbf{n}\cdot\mathbf{A}$ is the *generator* of rotations along the direction $\mathbf{n}$. The three matrices A_i are the generators along the three coordinate directions, and are a *basis* for antisymmetric matrices.
Real antisymmetric matrices form a linear space that is closed under the operation $A, A' \to [A, A']$. The commutator is a Lie product[3], and the linear space

[3] In a linear space X, a Lie product is a bilinear map $*: X\times X \to X$ such that $x*x = 0$, $x*y = -y*x$, $(x*y)*z + (y*z)*x + (z*x)*y = 0$ (Jacobi property).

with Lie product is the *Lie algebra* $so(3)$ of the $SO(3)$ group. Because A_i are a basis, the Lie product of two basis matrices is expandable in the basis itself,

$$\boxed{[A_i, A_j] = \epsilon_{ijk} A_k} \tag{22.11}$$

The coefficients[4] ϵ_{ijk} are the *structure constants* of $so(3)$ and are typical of the rotation group.
The characteristic polynomial of a 3×3 matrix M is $\det(z\,I_3 - M) = z^3 + az^2 + bz + c = 0$. Cayley-Hamilton's theorem states that $M^3 + aM^2 + bM + cI_3 = 0$. Then it is possible to expand the exponential of a 3×3 matrix into a sum of powers:

$$R = e^{\varphi(\mathbf{n}\cdot\mathbf{A})} = \alpha\, I_3 + \beta(\mathbf{n}\cdot\mathbf{A}) + \gamma(\mathbf{n}\cdot\mathbf{A})^2$$

If this identity is acted on the three eigenvectors of R (that are also eigenvectors of $\mathbf{n}\cdot\mathbf{A}$ with eigenvalues 0, $\pm i$) a linear system is obtained: $1 = \alpha$, $e^{\pm i\varphi} = \alpha \pm i\beta - \gamma$. The coefficients are evaluated and one re-obtains the explicit general form of a rotation matrix:

$$R(\mathbf{n}\varphi) = I_3 + \sin\varphi(\mathbf{n}\cdot\mathbf{A}) + (1-\cos\varphi)(\mathbf{n}\cdot\mathbf{A})^2 \tag{22.12}$$

Each rotation is identified by a vector $\mathbf{n}\varphi$ in the ball of radius π. A diameter corresponds to the subgroup of rotations with same rotation axis; the two opposite points $\pm\mathbf{n}\pi$ at the surface give the same rotation and must be identified. The diameter is then a circle, the center of the sphere is the unit of the group. Therefore, the manifold of parameters is a sphere with opposite points at the surface being identified (a bundle of circles through the origin). The identification makes the manifold doubly connected: two rotations (two points A, B of the manifold) may be connected by two inequivalent paths (one path cannot be continuously deformed into the other). A choice is: the chord AB, the path that joins A to the surface at a point C = C′ (antipode) and continues to B.

Example 22.3.2. *Find the angle and the invariant vector of the rotation*

$$R = e^A\,, \qquad A = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}.$$

Note that $A = -A^t$ *is normal and shares with* R *the same eigenvectors. Therefore if* z *is an eigenvalue of* A, e^z *is an eigenvalue of* R. *The invariant vector solves* $A\mathbf{n} = 0$ *i.e.*

$$\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \Rightarrow \mathbf{n} = \pm\frac{1}{\sqrt{35}} \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} \tag{22.13}$$

The eigenvalues of A *solve* $0 = \det(zI_3 - A) = z^3 + 35z$. *They are* 0, $\pm i\sqrt{35}$. *The angle of rotation is therefore* $\sqrt{35}$ *radiants (mod.* 2π*).*
Since the rotation angle is positive if anticlockwise with respect to the direction of $\mathbf{n}$, *we must decide the sign of* $\mathbf{n}$. *This can be done by checking the infinitesimal rotation of a conveniently chosen vector: a rotation with same axis but*

[4] $\epsilon_{123} = 1$ and cyclic, $\epsilon_{213} = -1$ and cyclic, zero otherwise. Note that the vector product of two real vectors is $(\vec{a}\times\vec{b})_i = \epsilon_{ijk} a_j b_k$.

infinitesimal angle is $I_3 + \frac{\epsilon}{\sqrt{35}} A + \dots$. *The vector* $\mathbf{v} = (0,0,1)^t$ *transforms to* $\mathbf{v} + \delta\mathbf{v}$ *where*

$$\delta\mathbf{v} = \frac{\epsilon}{\sqrt{35}} \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} = \epsilon\,\mathbf{n} \times \mathbf{v}$$

provided that $\mathbf{n}$ *is chosen with the upper sign in* (22.13).

22.3.1 SU(2)

More fundamental than SO(3) for the theory of rotations is the group SU(2) of unitary 2×2 complex matrices, $U^\dagger U = I$ with $\det U = 1$:

$$U = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}, \qquad |z|^2 + |w|^2 = 1 \tag{22.14}$$

The parameters span the surface of the unit sphere in $\mathbb{R}^4$, which has no boundary, and is simply connected. Another parametrization of SU(2) is obtained by solving the constraints:

$$U(\mathbf{n}\varphi) = \cos\frac{\varphi}{2} - i\mathbf{n}\cdot\boldsymbol{\sigma}\,\sin\frac{\varphi}{2} \tag{22.15}$$

σ_i are the Pauli matrices. It is simple to check that the expression corresponds to the exponential representation:

$$\boxed{U(\mathbf{n}\varphi) = e^{-\frac{i}{2}\varphi\mathbf{n}\cdot\boldsymbol{\sigma}}} \tag{22.16}$$

The generators $\sigma_i/2$ are a basis for traceless Hermitian 2×2 matrices, which is the Lie algebra $su(2)$ of SU(2), with Lie product $H, H' \to -i[H, H']$. The structure constants are the same of $su(3)$:

$$\boxed{-i\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \epsilon_{ijk}\frac{\sigma_k}{2}} \tag{22.17}$$

The exponential map takes the Lie algebra to the Lie group:

$$\exp : su(2) \to \mathrm{SU}(2).$$

Despite the diversity of SO(3) and SU(2), their exponential representations are formally identical: only a replacement of the basis matrices changes one into another, the structure constants being the same.
For SU(2) the parameter space is the ball in $\mathbb{R}^3$ of radius 2π. However, all "surface" points correspond to the single matrix of inversion: $U(\pm\mathbf{n}\,2\pi) = -1$. Therefore, the manifold is a bundle of circles (the diameters are the one parameter subgroups) with points (matrices) $\pm\mathbf{n}\,2\pi$ in common.
This parameter space, or the surface of the unit sphere in $\mathbb{R}^4$ ($|z|^2 + |w|^2 = 1$) are *simply connected* manifolds: this makes SU(2) more fundamental (covering group) than SO(3).

Exercise 22.3.3. *Show that* $U^\dagger\boldsymbol{\sigma}U = R\boldsymbol{\sigma}$, $R \in \mathrm{SO}(3)$. *Therefore* $\pm U$ *correspond to the same* R *matrix.*

22.3.2 Representations

From the point of view of physics, space rotations are a subgroup of the Galilei and of the Lorentz groups of symmetries for physical laws.
An observer O is specified by an orthonormal frame $\mathbf{e}_k$, which can be identified materially by rigid rulers for measuring positions, at right angles at a point (the origin). A rotated observer O' is an orthonormal frame $\mathbf{e}'_k$ with same origin and orientation. A point P has coordinates x'_k linked by a rotation matrix to the coordinates measured by O: $x'_k = R_{kj}x_j$.
Suppose that O and O' measure in every point some quantity (a field). In the simplest case the quantity is a number (a scalar) that is a property of the point. They measure two scalar fields f and f', and the values of the two fields at the same physical point P (of coordinates x and x') must be the same:

$$f(x) = f'(x'), \quad \text{i.e.} \quad \boxed{f'(x) = f(R^{-1}x)} \tag{22.18}$$

If they measure a vector field (for example a force field), they measure a physical arrow at each point. Since they use rotated frames, the components of the arrow at a point are different:

$$\sum_k V_k(x)\mathbf{e}_k = \sum_k V'_k(x')\mathbf{e}'_k, \quad \Rightarrow \quad \boxed{V'_k(x) = R_{kj}V_j(R^{-1}x)} \tag{22.19}$$

The two laws describe the transformation of a scalar field and a vector field under rotations. A spinor 1/2 field is a two component complex field that transforms with a SU(2) rotation:

$$\boxed{\psi'_a(x) = U(R)_{ab}\psi_b(R^{-1}x)} \tag{22.20}$$

If the (scalar, spinor, vector or whatever) fields F and F' measured by O and O' are thought of as points in the *same* functional space, the rotation of coordinates R induces a map $U_R : F \to F'$. Since fields can be linearly combined, the map is asked to be linear. Moreover, if two rotations R and R' are done in the order, the observer O'' is linked to O by $x'' = R'Rx$, and thus $F'' = U_{R'R}F$ i.e. maps form a **linear representation** of the rotation group:

$$\boxed{U_{R'R} = U_{R'}U_R} \tag{22.21}$$

Let us consider some relevant cases.
• Think of a **scalar field** f as a “point” in the function space $L^2(\mathbb{R}^3)$. A rotation induces the linear map $f' = \hat{U}_R f$ defined by $(\hat{U}_R f)(x) = f(R^{-1}x)$. The operators $\hat{U}_R$ are unitary because Lebesgue’s measure is rotation invariant, then $\hat{U}_{R^{-1}} = \hat{U}_R^\dagger$.
The rotations with fixed direction $\mathbf{n}$ correspond to a commutative subgroup parameterized by the angle: $\hat{U}_\mathbf{n}(\varphi)\hat{U}_\mathbf{n}(\varphi') = \hat{U}_\mathbf{n}(\varphi+\varphi')$ (they are a strongly continuous one-parameter group). By Stone’s theorem there is a self-adjoint (unbounded) generator such that $\hat{U}_\mathbf{n}(\varphi) = e^{-\frac{i}{\hbar}\varphi\hat{L}(\mathbf{n})}$. The self-adjoint operator $\hat{L}(\mathbf{n})$ is the generator of the one parameter group, and is conveniently found by

considering the action of an infinitesimal rotation on the set of analytic functions

$$\begin{aligned}(\hat{U}_{\mathbf{n}}(\delta\varphi)f)(\mathbf{x}) =& f(\mathbf{x} - \delta\varphi\, \mathbf{n} \times \mathbf{x} + \ldots) \\ =& f(\mathbf{x}) - \delta\varphi\, (\mathbf{n} \times \mathbf{x})_k \frac{\partial f}{\partial x_k}(x) + \ldots \\ =& f(\mathbf{x}) - \frac{i}{\hbar}\delta\varphi\, (\mathbf{n} \cdot \mathbf{L} f)(\mathbf{x}) + \ldots\end{aligned}$$

The generator is found to be $\hat{L}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{L}$, the projection along $\mathbf{n}$ of the *orbital angular momentum*

$$\boxed{\mathbf{L} = \mathbf{Q} \times \mathbf{P}} \tag{22.22}$$

The three operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ satisfy the Lie algebra

$$\boxed{\frac{1}{i\hbar}\, [\hat{L}_i, \hat{L}_j] = \epsilon_{ijk} \hat{L}_k} \tag{22.23}$$

and generate the unitary subgroups representing rotations around the three coordinate axis.

• For a 1/2 **spinor field** ψ_a $(a = 1, 2)$, the expansion near unity of (22.20) gives

$$\begin{aligned}[U(\mathbf{n}\,\delta\varphi)\psi]_a(\mathbf{x}) &= \left[I - i\delta\varphi\, \mathbf{n} \cdot \frac{\boldsymbol{\sigma}}{2} + \ldots\right]_{ab} \left(I - \frac{i}{\hbar}\delta\varphi\, \mathbf{n} \cdot \mathbf{L} + \ldots\right) \psi_b(\mathbf{x}) \\ &= \psi_a(\mathbf{x}) - \frac{i}{\hbar}\, \delta\varphi\, (\mathbf{n} \cdot \mathbf{J})_{ab} \psi_b(\mathbf{x}) + \ldots\end{aligned}$$

σ^k_{ab} are the components of the Pauli matrix σ^k. The generator is the projection along $\mathbf{n}$ of the total angular momentum, which is the vector sum of *spin* and *orbital* angular momentum operators, acting independently on spin variables (components a, b) and on position variables:

$$\boxed{\mathbf{J} = \mathbf{S} + \mathbf{L}} \tag{22.24}$$

where $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ are the spin operators (matrices) for spin 1/2 particles.
Also in this case the algebra is that of angular momentum (rotation group):

$$\boxed{\frac{1}{i\hbar}\, [\hat{J}_i, \hat{J}_j] \;=\; \epsilon_{ijk} \hat{J}_k.} \tag{22.25}$$

Exercise 22.3.4. *Show that* $[\hat{J}^2, \hat{J}_i] = 0$, *where* $\hat{J}^2 = \sum_i \hat{J}_i^2$.

Chapter 23

UNBOUNDED LINEAR OPERATORS

Many operators of interest are unbounded, and are defined on a subset of the Hilbert space. The domain ($\mathscr{D}$), the range (Ran) and the kernel (Ker) of a linear operator are linear subspaces. If the kernel only contains the null vector, the operator is injective and the inverse operator exists (and is linear).

23.1 The graph of an operator

The graph of a real function f is the set (x, y) in $\mathbb{R}^2$, where x is in the domain of f and $y = f(x)$. A very useful tool is the graph of a linear operator, introduced by Von Neumann: the *graph* of a linear operator is the set

$$\mathcal{G}(\hat{A}) = \{(x, y) : x \in \mathscr{D}(\hat{A}),\ y = \hat{A}x\} \tag{23.1}$$

It is a subset of $\mathscr{H}^2 = \mathscr{H} \times \mathscr{H}$, with elements $X = (x, y)$. $\mathscr{H}^2$ is a linear space with the rules $\lambda X = (\lambda x, \lambda y)$ and $X + X' = (x + x', y + y')$, and a Hilbert space with inner product

$$(X|X') = (x|x') + (y|y').$$

The norm is $\|X\|^2 = \|x\|^2 + \|y\|^2$. Completeness is readily proven: if X_n is a Cauchy sequence, $\|X_n - X_m\|^2 = \|x_n - x_m\|^2 + \|y_n - y_m\|^2 < \epsilon$ for all $n, m > N_\epsilon$, then x_n and y_n are both Cauchy sequences, and then converge to x and y. The pair $X = (x, y)$ is the limit of X_n in $\mathscr{H}^2$: $\|X_n - X\|^2 = \|x_n - x\|^2 + \|y_n - y\|^2 \to 0$.

The graph of a linear operator is a linear subspace in $\mathscr{H}^2$. Conversely, a linear subspace $\mathcal{S}$ is the graph of a linear operator if $(x, y) \in \mathcal{S}$ and $(x, y') \in \mathcal{S}$ imply $y = y'$ (the assignment $x \to y$ is unique). This is equivalent to: $(0, y) \in \mathcal{S} \Rightarrow y = 0$.

If the inverse of the operator $\hat{A}$ exists, its graph is

$$\mathcal{G}(\hat{A}^{-1}) = \{(y, x) : (x, y) \in \mathcal{G}(\hat{A})\} \tag{23.2}$$

Definition 23.1.1. A linear operator $\hat{A}'$ is an *extension* of a linear operator $\hat{A}$ if $\mathcal{G}(\hat{A}) \subset \mathcal{G}(\hat{A}')$ (in other words: $\mathscr{D}(\hat{A}) \subset \mathscr{D}(\hat{A}')$ and $\hat{A}' = \hat{A}$ on $\mathscr{D}(\hat{A})$).

23.2 Closed operators

A graph is a closed set if every Cauchy sequence X_n in the graph converges to a point X in the graph (a Cauchy sequence X_n is convergent, by the completeness of $\mathscr{H}^2$; the point is that the limit X must be in the graph).

Definition 23.2.1. A linear operator $\hat{A}$ is **closed** if its graph is a closed set.

This definition is an example of how useful the concept of graph is. Without it, the definition is: $\hat{A}$ is closed if $\forall x_n$ in $\mathscr{D}(\hat{A})$ such that $x_n \to x$ and $\hat{A}x_n \to y$ $\Rightarrow$ $x \in \mathscr{D}(\hat{A})$ and $y = \hat{A}x$.

Remark 23.2.2.
1) "$\hat{A}$ is closed" does not mean that $\mathscr{D}(\hat{A})$ is closed nor that $\hat{A}$ is continuous: it only means that $\mathscr{D}(\hat{A})$ contains the limit points of convergent sequences such that also the sequence $\hat{A}x_n$ converges, and converges to the image of the limit point (the graph is very useful to visualize this).
2) If $\hat{A}$ is closed, Ker $\hat{A}$ is a closed subspace.
3) The inverse of a closed linear operator is closed.

If the graph $\mathcal{G}(\hat{A})$ of a linear operator is not closed, the set can be always extended to its closure $\overline{\mathcal{G}(\hat{A})}$ by adding the frontier. The closure is still a linear subspace[1] and it is the smallest closed extension of the set. However, it might not be a graph of a linear operator, as it may fail to have the property $(0, y) \in \overline{\mathcal{G}(\hat{A})} \Rightarrow y = 0$.

Definition 23.2.3. If $\overline{\mathcal{G}(\hat{A})}$ is the graph of a linear operator $\overline{\hat{A}}$, the operator $\overline{\hat{A}}$ is the *closure* of $\hat{A}$, and it is the minimal closed extension of $\hat{A}$. It is $\overline{\mathcal{G}(\hat{A})} = \mathcal{G}(\overline{\hat{A}})$.

Proposition 23.2.4. *If $\overline{\mathcal{G}(\hat{A})}$ is* not *a graph of a linear operator, then $\hat{A}$ has no closed extensions.*

Proof. If $\overline{\mathcal{G}(\hat{A})}$ is not a graph, it contains a point $(0, y)$ with $y \neq 0$. Suppose that $\hat{A}$ has a closed extension $\hat{A}^c$; then $\overline{\mathcal{G}(\hat{A})}$ is necessarily a proper subset of $\mathcal{G}(\hat{A}^c)$. But if $(0, y) \in \mathcal{G}(\hat{A}^c)$, the set cannot be the graph of an operator. □

The property of being a closed operator is important. In this presentation it will be a prerequisite for the introduction of the spectrum.
The following theorem due to Banach is important:

Theorem 23.2.5 (closed graph theorem)**.** *Let $\hat{A}$ be a linear operator with a closed domain. Then $\hat{A}$ is continuous if and only if $\hat{A}$ is closed.*

Proof. If $\hat{A}$ is continuous and $x_n \to x$ is a convergent sequence in $\mathscr{D}(\hat{A})$, then $x \in \mathscr{D}(\hat{A})$ (by hypothesis) and $\hat{A}x_n \to \hat{A}x$. This means that $\mathcal{G}(\hat{A})$ is closed, i.e. the operator is closed.
Suppose that $\hat{A}$ is closed, with a closed domain. Then for any convergent sequence x_n in $\mathscr{D}(\hat{A})$ such that $\hat{A}x_n$ is also convergent (to y), it is $x_n \to x \in \mathscr{D}(\hat{A})$ and $y = \hat{A}x$. Since the graph is closed, it is $y = \hat{A}x_n$, i.e. $\hat{A}$ is continuous. □

By the general theorem 18.3.7, if $\hat{A}$ has a closed domain, $\hat{A}$ is continuous if and only if $\hat{A}$ is bounded.

[1] If two limit points X and X' belong to the frontier there are two sequences X_n and X'_n convergent to them. $X + \lambda X'$ belongs to the closure because it is the limit of $X_n + \lambda X'_n$ that belongs to the graph, a linear space.

23.3 The adjoint operator

Let us enquire about the existence of the adjoint operator. According to the definition given for bounded operators, the adjoint operator should map $x \in \mathscr{D}(\hat{A}^\dagger)$ to an element x' such that $(x'|y) = (x|\hat{A}y)$ for all $y \in \mathscr{D}(\hat{A})$. The vector x' corresponding to x must be unique; suppose that there are two such vectors, then $(x' - x''|y) = 0$ for all $y \in \mathscr{D}(\hat{A})$, i.e. $x' - x'' \in \overline{\mathscr{D}(\hat{A})}^\perp$. Since we need $x' = x''$, it must be $\{0\} = \mathscr{D}(\hat{A})^\perp$ i.e. $\mathscr{H} = \mathscr{D}(\hat{A})^{\perp\perp} = \overline{\mathscr{D}(\hat{A})}$ i.e. $\hat{A}$ must be *densely defined.*

Proposition 23.3.1. *If $\hat{A}$ is a densely defined linear operator, then the adjoint $\hat{A}^\dagger$ exists, it is a linear operator, and the domain is the maximal set*

$$\mathscr{D}(\hat{A}^\dagger) = \{x \in \mathscr{H} \,:\, \forall y \in \mathscr{D}(\hat{A}) \,\exists x' \text{ s.t. } (x'|y) = (x|\hat{A}y)\},$$
$$A^\dagger x = x'.$$

This is the relation between $\hat{A}$ and its adjoint:

$$\boxed{(\hat{A}^\dagger x|y) = (x|\hat{A}y), \quad \forall x \in \mathscr{D}(A^\dagger), \forall y \in \mathscr{D}(\hat{A})} \tag{23.3}$$

The requirement that the domain of the adjoint is maximal, implies the following statement:

Proposition 23.3.2. *Let $\hat{A}$ be densely defined. If $\hat{A}'$ is an extension of $\hat{A}$, then the adjoint of $\hat{A}$ is an extension of the adjoint of $\hat{A}'$:*

$$\hat{A} \subseteq \hat{A}' \;\Rightarrow\; \hat{A}'^\dagger \subseteq \hat{A}^\dagger \tag{23.4}$$

The graphs of $\hat{A}$ and $\hat{A}^\dagger$ are closely related. Let us introduce the involution $V(x, y) = (y, -x)$. It has the properties: $(VX|X') = (X|VX')$, $V(\mathcal{S}^\perp) = (V\mathcal{S})^\perp$, $V^2X = -X$, $V^2\mathcal{S} = \mathcal{S}$ ($\mathcal{S}$ is a linear subspace).

Proposition 23.3.3 (Graph of the adjoint operator)**.**

$$\boxed{\mathcal{G}(\hat{A}^\dagger) = (V\mathcal{G}(\hat{A}))^\perp} \tag{23.5}$$

Proof. If $(x, x') \in \mathcal{G}(\hat{A}^\dagger)$ then $x \in \mathscr{D}(\hat{A}^\dagger)$ and, for all $y \in \mathscr{D}(\hat{A})$,

$$(x|\hat{A}y) = (x'|y) \Leftrightarrow (x|\hat{A}y) + (x'|-y) = 0 \Leftrightarrow (\,(\hat{A}y, -y)\,|\,(x, x')\,) = 0$$

in the inner product of $\mathscr{H}^2$. Then: $X \in \mathcal{G}(\hat{A}^\dagger) \Leftrightarrow (VX'|X) = 0, \forall X' \in \mathcal{G}(\hat{A})$. This gives $(X'|VX) = 0$ i.e. $V\mathcal{G}(\hat{A}^\dagger) = \mathcal{G}(\hat{A})^\perp$ i.e. $\mathcal{G}(\hat{A}^\dagger) = V(\mathcal{G}(\hat{A})^\perp)$ and the statement is obtained. □

This characterization of the graph of the adjoint has an important corollary. Since $V\mathcal{G}(\hat{A})$ is a linear subspace (not necessarily the graph of a linear operator), its orthogonal complement $\mathcal{G}(\hat{A}^\dagger)$ is closed, and so is the operator:

Corollary 23.3.4. *The adjoint of a linear operator is a closed operator.*

Proposition 23.3.5. *If $\hat{A}$ is densely defined, then* $\text{Ker}\,\hat{A}^\dagger = (\text{Ran}\hat{A})^\perp$

Proof. $(\text{Ran}\hat{A})^\perp = \{x \in \mathscr{H} : (x|\hat{A}x') = 0, \forall x' \in \mathscr{D}(\hat{A})\} = \{x \in \mathscr{H} : (\hat{A}^\dagger x|x') = 0, \forall x' \in \mathscr{D}(\hat{A})\}$. Since $\hat{A}$ is densely defined the set is $\{x \in \mathscr{H} : \hat{A}^\dagger x = 0\} = \text{Ker}\hat{A}^\dagger$. □

Suppose that also $\hat{A}^\dagger$ is densely defined: then $\hat{A}^{\dagger\dagger}$ exists and is closed (because it is the adjoint of $\hat{A}^\dagger$). We show that it is the closure of $\hat{A}$:

Proposition 23.3.6. $\hat{A}^{\dagger\dagger} = \overline{\hat{A}}$.

Proof. $\mathcal{G}(\hat{A}^{\dagger\dagger}) = V(\mathcal{G}(\hat{A}^\dagger))^\perp = (\mathcal{G}(\hat{A}))^{\perp\perp} = \overline{\mathcal{G}(\hat{A})} = \mathcal{G}(\overline{\hat{A}})$. □

Corollary 23.3.7. $\hat{A}^{\dagger\dagger\dagger} = (\hat{A}^\dagger)^{\dagger\dagger} = \overline{\hat{A}^\dagger} = \hat{A}^\dagger$ *because* $A^\dagger$ *is closed.*

23.3.1 Self-adjointness

In applications one often encounters operators that are densely defined and symmetric. The issue is then to establish a self-adjoint extension for them. Self-adjointness is a requirement for an operator to represent an observable in quantum mechanics.

Definition 23.3.8. A densely defined linear operator $\hat{A}$ is **symmetric** if

$$(\hat{A}x|y) = (x|\hat{A}y), \quad \forall x, y \in \mathscr{D}(\hat{A}) \tag{23.6}$$

The definition is equivalent to the statement $\hat{A} \subseteq \hat{A}^\dagger$. Then $\hat{A}^\dagger$ is a closed extension of the symmetric operator $\hat{A}$. However, the smallest closed extension of $\hat{A}$ (if it exists) is $\overline{\hat{A}} = \hat{A}^{\dagger\dagger}$. Then, in general (the last inclusion holds for a symmetric operator), $\hat{A} \subseteq \overline{\hat{A}} = \hat{A}^{\dagger\dagger} \subseteq \hat{A}^\dagger$.

Exercise 23.3.9. *Show that the eigenvalues of a symmetric operator are real, and the eigenvectors with different eigenvalues are orthogonal.*

Definition 23.3.10. A densely defined operator $\hat{A}$ is **self-adjoint** if $\hat{A} = \hat{A}^\dagger$.

1) If $\hat{A}$ is self-adjoint then it is closed, and $\hat{A} = \overline{\hat{A}} = \hat{A}^{\dagger\dagger} = \hat{A}^\dagger$.
2) If $\hat{A}$ is symmetric and $\hat{A}'$ is a self-adjoint extension of it, then $\hat{A} \subset \hat{A}' = \hat{A}'^\dagger \subseteq \hat{A}^\dagger$. Therefore the self-adjoint extension of $\hat{A}$, if it exists, is "between" $\hat{A}$ and $\hat{A}^\dagger$.
3) If $\hat{A}$ is self-adjoint it is $\text{Ker}\hat{A} = (\text{Ran}\hat{A})^\perp$, i.e.

$$\boxed{\mathscr{H} = \text{Ker}\hat{A} \oplus \overline{\text{Ran}\hat{A}}} \tag{23.7}$$

An intermediate situation is $\hat{A} \subset \hat{A}^\dagger$ and $\overline{\hat{A}} = \overline{\hat{A}^\dagger}$. Since $\overline{\hat{A}^\dagger} = \hat{A}^\dagger$, it is $\overline{\hat{A}} = \hat{A}^\dagger$. This occurs frequently in practice, and deserves a definition:

Definition 23.3.11. A densely defined operator is **essentially self-adjoint** if $\hat{A}$ is symmetric and $\overline{\hat{A}}$ is self-adjoint.

If $\hat{A}$ is essentially self-adjoint: $\hat{A} \subset \overline{\hat{A}} = \hat{A}^{\dagger\dagger} = \hat{A}^\dagger$. The closure is the unique self-adjoint extension of $\hat{A}$ (suppose that $\hat{B}$ is another self-adjoint extension and $\overline{\hat{A}} \subset \hat{B}$; then $\hat{B} = \hat{B}^\dagger \subset \overline{\hat{A}}$, which is impossible).

Example 23.3.12. *The* position *operator* $(\hat{Q}_0 f)(x) = xf(x)$ *is well defined on the domain* $\mathscr{S}(\mathbb{R})$ *of* $\mathscr{C}^\infty$ *functions of rapid decay (see chapter), which is a linear space dense in* $\mathrm{L}^2(\mathbb{R})$*. The domain is invariant under the action of* $\hat{Q}_0$*, and* $\hat{Q}_0$ *is symmetric on it:*

$$(\hat{Q}_0\varphi|\psi) = (\varphi|\hat{Q}_0\psi), \quad \forall\varphi,\psi \in \mathscr{S}$$

The domain of $\hat{Q}_0^\dagger$ *is the set of functions* $f \in \mathrm{L}^2(\mathbb{R})$ *such that there is* $g \in \mathrm{L}^2$ *such that* $(g|\varphi) = (f|\hat{Q}_0\varphi)$ $\forall\varphi \in \mathscr{S}(\mathbb{R})$*. This is:*

$$\int_{\mathbb{R}} \overline{(g - xf)}\,\varphi\, dx = 0, \quad \forall\varphi \in \mathscr{S}(\mathbb{R}).$$

Since $\mathscr{S}$ *is dense in* L^2*, this means* $g = xf$ *a.e., i.e.*

$$\mathscr{D}(\hat{Q}_0^\dagger) = \{f \in \mathrm{L}^2(\mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} |xf|^2 dx < \infty\}, \qquad \hat{Q}_0^\dagger f = xf$$

This domain contains $\mathscr{S}$*. What about* $\hat{Q}_0^{\dagger\dagger}$*? The iteration of the above construction shows that the domain is unchanged and* $\hat{Q}_0^\dagger = \hat{Q}_0^{\dagger\dagger}$*. Therefore* $\hat{Q}_0$ *is essentially self-adjoint, and the operator* $\hat{Q} = \hat{Q}_0^\dagger$ *is the self-adjoint extension of* $\hat{Q}_0$*.*

23.4 Spectral theory (for closed operators)

Consider a $n\times n$ matrix A. The resolvent set $\rho(A)$ is the set of complex numbers z such that $z - A$ is invertible. This means that Ker $(z - A) = \{0\}$, i.e. the eigenvalue equation $Au = zu$ has only the trivial solution $u = 0$. The matrix $(z - A)^{-1}$ is called the resolvent of A at z.
The set $\sigma(A) = \mathbb{C}/\rho(A)$ is the spectrum; for finite matrices it consists of at most n eigenvalues z_i for which the equation $Au = zu$ has a non-trivial solution.
Let us review the spectral theory of operators on Hilbert spaces.

23.4.1 The resolvent and the spectrum

Hereafter $\hat{A}$ is a closed linear operator. If $z - \hat{A}$ is invertible, the operator

$$\hat{R}(z) = (z - \hat{A})^{-1} \tag{23.8}$$

is closed and it is named the **resolvent** of $\hat{A}$ at $z \in \mathbb{C}$. The domain of the resolvent is Ran $(z - \hat{A})$. The resolvent set is defined as follows:

Definition 23.4.1. If $\hat{R}(z)$ exists with domain $\mathscr{H}$, then z belongs to the *resolvent set* $\rho(\hat{A})$.

Being closed with domain $\mathscr{H}$ the resolvent $\hat{R}(z)$ is bounded by the *closed graph theorem* 23.2.5. Therefore, $z \in \rho(\hat{A}) \Leftrightarrow \hat{R}(z) \in \mathscr{B}(\mathscr{H})$. We state without proof:

Proposition 23.4.2.
1) $\rho(\hat{A})$ *is an open set in* $\mathbb{C}$*.*
2) $\hat{R}(z)$ *is an analytic function of* $z \in \rho(\hat{A})$*, i.e. it admits a norm-convergent power expansion on any disk in the resolvent set:* $\hat{R}(z) = \sum_{n=0}^\infty \hat{C}_n (z - z_0)^n$*.*

Exercise 23.4.3. *For $z_1, z_2 \in \rho(\hat{A})$ prove the "resolvent identity":*

$$\hat{R}(z_1)\hat{R}(z_2) = -\frac{\hat{R}(z_2) - \hat{R}(z_1)}{z_2 - z_1} \tag{23.9}$$

Hint: start from $(z_2 - z_1 + z_1 - \hat{A})\hat{R}(z_2) = 1$.

Definition 23.4.4. The closed set $\mathbb{C}/\rho(\hat{A}) = \sigma(\hat{A})$ is the **spectrum** of $\hat{A}$. It may be decomposed into the *pure*, the *continuous* and the *residual* spectrum:

$$\sigma(\hat{A}) = \sigma_p(\hat{A}) \cup \sigma_c(\hat{A}) \cup \sigma_r(\hat{A})$$

$$\sigma_p(\hat{A}) = \{z : \nexists (z - \hat{A})^{-1}\}, \tag{23.10}$$

$$\sigma_c(\hat{A}) = \{z : \exists (z - \hat{A})^{-1} \ \overline{\text{Ran}(z - \hat{A})} = \mathscr{H}\}, \tag{23.11}$$

$$\sigma_r(\hat{A}) = \{z : \exists (z - \hat{A})^{-1}, \ \overline{\text{Ran}(z - \hat{A})} \neq \mathscr{H}\}. \tag{23.12}$$

Les us comment on this partition:

$z \in \sigma_p$ means that Ker $(z - \hat{A}) \neq \{0\}$, i.e. the equation $\hat{A}u = zu$ has solution in $\mathscr{D}(\hat{A})$. Therefore σ_p is the set of eigenvalues.

$z \notin \sigma_P(\hat{A})$ means that the resolvent $(z - \hat{A})^{-1}$ exists. There are three disjoint possibilities: the resolvent has domain $\mathscr{H}$ (and is necessarily bounded, $z \in \rho(\hat{A})$), the domain of the resolvent is dense in $\mathscr{H}$ (the resolvent cannot be bounded, $z \in \sigma_c$), the closure of the domain of the resolvent is a subset of $\mathscr{H}$ ($z \in \sigma_r$).

Proposition 23.4.5. *The spectrum of a self-adjoint operator is real: $\sigma(\hat{A}) \subseteq \mathbb{R}$.*

Proof. If $\lambda \in \sigma_P(\hat{A})$¡ then λ is real. Suppose that $\lambda \in \sigma_{c,r}(\hat{A})$, and $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_2 \neq 0$. Then $(\lambda - \hat{A})^{-1}$ exists with domain $\text{Ran}(\lambda - \hat{A})$ that is dense in $\mathscr{H}$. For $\hat{A}$ self-adjoint we obtain the inequality

$$\|(\lambda - \hat{A})x\|^2 = \|(\lambda_1 - \hat{A})x\|^2 + |\lambda_2|^2\|x\|^2 > |\lambda_2|^2\|x\|^2$$

for all $x \in \mathscr{D}(\hat{A})$. It implies that $(\lambda - \hat{A})^{-1}$ is bounded. But then $\lambda \in \rho(\hat{A})$. □

Definition 23.4.6. A **resolution of the identity** (or **spectral family**) is a family of projection operators $\{\hat{E}_t, \, t \in \mathbb{R}\}$ such that:

$$\hat{E}_t\hat{E}_s = \hat{E}_{min(t,s)} \tag{23.13}$$

$$\lim_{t \to s^+} \|\hat{E}_t x - \hat{E}_s x\| \to 0, \quad \forall x \in \mathscr{H} \tag{23.14}$$

$$\lim_{t \to -\infty} \hat{E}_t x = 0, \quad \lim_{t \to +\infty} \hat{E}_t x = x, \quad \forall x \in \mathscr{H} \tag{23.15}$$

1) If $\hat{E}_t$ and $\hat{E}_s$ belong to the spectral family, property (23.13) implies that they commute: $\hat{E}_t\hat{E}_s = \hat{E}_s\hat{E}_t$.
2) The closed subspaces $M_t = \text{Ran}\hat{E}_t$ have the properties:

$$M_s \subseteq M_t \text{ if } s \leq t, \quad M_s = \bigcap_{t>s} M_t, \quad \bigcap_{t \in \mathbb{R}} M_t = \{0\}, \quad \overline{\bigcup_{t \in \mathbb{R}} M_t} = \mathscr{H}$$

3) For $t \geq s$, define $\hat{E}_{(s,t]} = \hat{E}_t - \hat{E}_s$. The operator is a projector. Property (23.14) says that $\|\hat{E}_{(t,t+\delta]}x\| \to 0$, as $\delta \to 0^+$. Of interest is the weak limit

$$\lim_{\delta \to 0^+} \frac{1}{\delta}(\hat{E}_{(t,t+\delta]}x|y)$$

4) For generic x, introduce the positive measure $\mu_x(s,t) = (x|\hat{E}_{(s,t]}x) = \|\hat{E}_{(s,t]}x\|^2$, $s \leq t$. In particular $\mu_x(-\infty, s) \leq \mu_x(-\infty, t) \leq \mu_x(\mathbb{R}) = \|x\|^2$.
The complex measures $\mu_{xy}(s,t) = (x|\hat{E}_{(s,t]}y)$ can be expressed via the polarization formula, as a combination of positive measures with vectors $x \pm y$, $x \pm iy$.
5) The *support of the spectral family* is the closure of the set of t-values such that $\hat{E}_t \neq 0$ and $\hat{E}_t \neq 1$.

Example 23.4.7. *In* $\mathrm{L}^2(\mathbb{R})$ *consider the projection operators*

$$(\hat{E}_t\, f)(x) = \chi_{(-\infty,t]}(x) f(x), \quad t \in \mathbb{R}.$$

$\{\hat{E}_t\}$ *is a spectral family: property (23.13) follows from* $\chi_{(-\infty,t]}\chi_{(-\infty,s]} = \chi_{(-\infty,u]}$ *where* $u = \min(s,t)$*; property (23.14): for* $t \to s^+$ *it is*

$$\|\hat{E}_t f - \hat{E}_s f\|^2 = \int \chi_{(s,t]}(x)|f(x)|^2 dx \to 0$$

by Lebesgue's dominated convergence theorem. The same theorem proves properties (23.15). The spectral measure is $\mu_{f,g}(-\infty,t) = \int_{-\infty}^t \overline{f} g\, dx$*. By the general theory of Lebesgue integral,* $\mu_{f,g}$ *is a continuous function of* t *and is a.e. differentiable:*

$$d\mu_{f,g}(t) = (f|d\hat{E}_t g) = \overline{f(t)}\, g(t)\, dt$$

If F *is a bounded real continuous function, it is*

$$(f|F(\hat{Q})g) = \int_{\mathbb{R}} \overline{f(t)}\, F(t) g(t)\, dt \; = \int_{\mathbb{R}} F(t)\, d\mu_{fg}(t) \Longrightarrow F(\hat{Q}) = \int_{\mathbb{R}} F(t) d\hat{E}_t$$

This is the spectral theorem for multiplication operators.

Theorem 23.4.8 (The spectral theorem). *If* $\hat{A}$ *is a self-adjoint operator, there is a unique spectral family* $\hat{E}_t$ *such that*

$$\mathscr{D}(\hat{A}) = \left\{ x \in \mathscr{H} \; : \; \int_{\sigma(\hat{A})} t^2\, (x|d\hat{E}_t x) < \infty \right\} \tag{23.16}$$

$$\hat{A}\, x = \int_{\sigma(A)} t\, d\hat{E}_t\, x. \tag{23.17}$$

Chapter 24

SCHWARTZ SPACE AND FOURIER TRANSFORM

24.1 Introduction

As Laurent Schwartz recalls, his intuition of distributions came one night in 1944, just after the liberation of France. The two volumes *Théorie des distributions* were published in 1950, and the same year he received the "Fields medal" with Atle Selberg[1].
Similar work on *generalized functions* was done by the Russian mathematicians Israel M. Gel'fand and his collaborator E. Shilov. Earlier contributors to the ideas were S. Bochner, J. Leray, K. Friedrichs and S. Sobolev. Distributions are the natural frame for the study of partial differential equations, and play a dominant role in the formulation of quantum field theories.

Distributions are defined on the basis of a set of well-behaved *test functions* φ that decay fast to zero at infinity, with some notion of convergence. The linear sequentially continuous functionals are the *distributions.* They include integral functionals $\varphi \to \int dx f(x)\varphi(x)$ where f belongs to a large set, because of the good properties of φ. Some operations on test functions may be extended to such general(ized) functions through duality. For example, the derivative f' can be defined as the functional $\varphi \to -\int dx f(x)\varphi'(x)$, which coincides with $\int dx f'\varphi$ if integration by parts can be done. In this manner one obtains a calculus for generalized functions, always to be understood in the weak sense.

Test functions are often taken in two sets: the space $\mathscr{D}(\mathbb{R}^n)$ of $\mathscr{C}^\infty$ functions with compact support in $\mathbb{R}^n$, and the larger Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing $\mathscr{C}^\infty$ functions. The dual spaces are respectively the space of distributions $\mathscr{D}'(\mathbb{R}^n)$ and the smaller space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$.
Because of their relevance in quantum physics and Fourier analysis, we privilege the study of the Schwartz space (in $n = 1$) and its dual. There are other motivations: $\mathscr{S}$ is left invariant by the important multiplication and derivation

[1]The Fields golden medal is assigned every four years by the International Mathematical Union to mathematicians not older than 40. It was established by the Canadian professor J. C. Fields. The Italians in the Fields list are Enrico Bombieri (1947) and Alessio Figalli (2018).

operators, and the Fourier transform is a bijection on it[2].

24.2 The Schwartz space

Definition 24.2.1. The Schwartz space $\mathscr{S}(\mathbb{R})$ of *rapidly decreasing* functions is the set of $\mathscr{C}^\infty$ functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that for all $m, n \in \mathbb{N}$:

$$\|\varphi\|_{m,n} = \sup_{x\in\mathbb{R}} |x^m (D^n \varphi)(x)| < \infty \tag{24.1}$$

Hereafter $D = d/dx$. The functions and their derivatives fall off at infinity more quickly than the inverse of any polynomial. By the obvious properties:
1) $\|\lambda\varphi\|_{m,n} = |\lambda| \, \|\varphi\|_{m,n}$,
2) $\|\varphi_1 + \varphi_2\|_{m,n} \le \|\varphi_1\|_{m,n} + \|\varphi_2\|_{m,n}$,
the Schwartz space is a linear space, and $\|\cdot\|_{m,n}$ is a family of *seminorms* for it (actually each one is a norm on $\mathscr{S}$).

$\mathscr{S}(\mathbb{R}) \subset \mathscr{C}(\mathbb{R})$ (Banach space of bounded continuous functions with sup norm):

$$\|\varphi\|_\infty = \sup_{x\in\mathbb{R}} |\varphi(x)| = \|\varphi\|_{0,0} < \infty$$

$\mathscr{S}(\mathbb{R}) \subset \mathscr{L}^1(\mathbb{R})$. The following trick is frequenly used:

$$\|\varphi\|_1 = \int_{\mathbb{R}} dx\, |\varphi(x)| \le \sup_x |(1+x^2)\varphi(x)| \int_{\mathbb{R}} \frac{dx}{1+x^2} \le (\|\varphi\|_{0,0} + \|\varphi\|_{2,0})\pi$$

$\mathscr{S}(\mathbb{R}) \subset \mathscr{L}^p(\mathbb{R})$. Use the inequality:

$$\|\varphi\|_p^p = \int_{\mathbb{R}} dx |\varphi|^p = \int_{\mathbb{R}} dx |\varphi|^{p-1} \, |\varphi| \le \|\varphi\|_\infty^{p-1} \|\varphi\|_1.$$

The space contains the functions $e^{-\frac{1}{2}x^2} x^n$ and the Hermite functions:

$$h_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \, e^{-\frac{1}{2}x^2} \, H_n(x). \tag{24.2}$$

24.2.1 Seminorms and convergence

Definition 24.2.2. A sequence of functions φ_r converges to φ in $\mathscr{S}(\mathbb{R})$ if:

$$\|\varphi_r - \varphi\|_{m,n} \to 0, \qquad \forall m, n.$$

Convergence in $\mathscr{S}(\mathbb{R})$ is very restrictive, as this example illustrates: the sequence $\varphi_r(x) = \frac{1}{2r} e^{-(rx)^2}$ is uniformly convergent to zero ($\|\varphi_r\|_{0,0} \to 0$), but it does not converge for all seminorms: $\|\varphi_r\|_{0,1} = \sup_x |xr\, e^{-(rx)^2}| = \sup_y |ye^{-y^2}| = 1/\sqrt{2e} \neq 0$.

Definition 24.2.3. A sequence φ_r in $\mathscr{S}(\mathbb{R})$ is a Cauchy sequence if it is Cauchy for every seminorm, i.e.

$$\forall \epsilon,\ m,\ n\ \ \exists N_{\epsilon,m,n} \text{ such that } \|\varphi_r - \varphi_s\|_{m,n} < \epsilon \quad \forall r, s > N_{\epsilon,m,n}.$$

[2] Bibliography: Reed and Simon *Functional Analysis*, Academic Press; Blanchard and Bruning, *Mathematical Methods in Physics*, Birkhauser (2003).

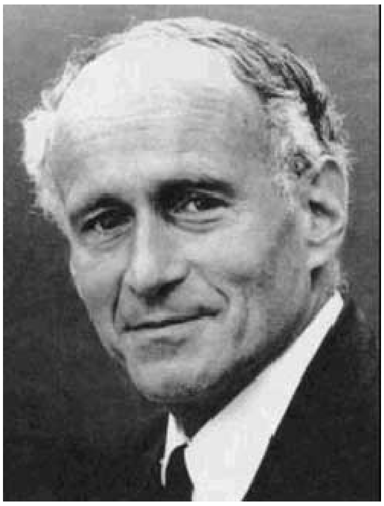 

Figure 24.1: **Laurent Schwartz** (Paris 1915, Paris 2002) was nephew of Jacques Hadamard; his wife Marie-Helene Levy was daughter of probabilist Paul Levy. Because of their Jewish origins, during the second world war they worked at the university of Strasbourg with different names. After the war he taught in Grenoble and Nancy and, from 1958 to 1980, at the École Politechnique. He was politically engaged and was suspended for two years for his opposition to the Algerian war. In 1951 he earned the Fields medal for the theory of distributions. Among his students are Jacques-Louis Lyon, Alexander Grothendieck, Francois Bruhat.

Figure 24.2: **Israel Gelfand** (Odessa 1913, New Brunswick 2009). At the age of 16 Gelfand already attended lectures at Moscow State University, and when he was 19 he was admitted directly to the graduate school. He completed a doctorate on abstract functions and linear operators in 1935 under Kolmogorov. He started a famous weekly Mathematics Seminar, and devoted much effort to education of young mathematicians. In 1990 he moved to USA. He won three Orders of Lenin and the first Wolf prize in mathematics (1978), with Siegel, for his important contributions to functional analysis and the theory of representations of groups.

Theorem 24.2.4. *$\mathscr{S}(\mathbb{R})$ is complete in the seminorm topology*[3] *(every Cauchy sequence is seminorm-convergent).*

Proof. If φ_r is a Cauchy sequence in $\mathscr{S}(\mathbb{R})$ then, for all (m, n), the sequences $x^m D^n \varphi_r$ are Cauchy sequences in the sup −norm and converge to functions $\psi_{m,n} \in \mathscr{C}(\mathbb{R})$ (which is complete in the sup-norm). In particular $\varphi_r \to \psi_{0,0}$. If we show that $\psi_{m,n}(x) = x^m(D^n\psi_{0,0})(x)$ for all x, it follows that $\|\psi_{0,0}\|_{m,n} = \sup_x |\psi_{m,n}(x)| < \infty$ and therefore $\psi_{0,0} \in \mathscr{S}(\mathbb{R})$ and $\varphi_r \to \psi_{0,0}$ also in the seminorm topology.

Consider the identity $(x^m D^k\varphi_r)(x) = (x^m D^k\varphi_r)(0) + \int_0^x D(y^m D^k\varphi_r)(y)dy$. Convergence in r is uniform, so we may take the limit $r \to \infty$ also in the integral: $\psi_{m,k}(x) = \psi_{m,k}(0) + \int_0^x [m\psi_{m-1,k}(y) + \psi_{m,k+1}(y)]dy$. Therefore $\psi_{m,k}$ is differentiable for all m, k and

$$D\psi_{m,k} = m\psi_{m-1,k} + \psi_{m,k+1}.$$

For $m = 0$: $D\psi_{0,k} = \psi_{0,k+1}$, i.e. $\psi_{0,k} = D^k\psi_{0,0}$; this also implies that $\psi_{0,0}$ can be differentiated any number of times resulting in continuous functions that decay to zero at infinity. The previous identity gives the following one:

$$\psi_{m,k+1} - x^m\psi_{0,k+1} = D(\psi_{m,k} - x^m\psi_{0,k}) - m(\psi_{m-1,k} - x^{m-1}\psi_{0,k})$$

Iteration provides $\psi_{m,k+1} - x^m\psi_{0,k+1}$ as a linear combination of powers of derivatives of the simpler functions $\psi_{m,0} - x^m\psi_{0,0}$. To show that $\psi_{m,k} = x^m D^k\psi_{0,0}$ it is then sufficient to prove that $\psi_{m,0} = x^m\psi_{0,0}$. This is done here:

$$\begin{aligned}
&|\psi_{m,0} - x^m\psi_{0,0}| \le |\psi_{m,0} - x^m\varphi_r| + |x^m\varphi_r - x^m\psi_{0,0}| \\
&\le \sup_x |\psi_{m,0} - x^m\varphi_r| + |x|^m \sup_x |\varphi_r - \psi_{0,0}| \\
&\le \epsilon(1 + |x|^m), \quad \forall r > N_\epsilon
\end{aligned}$$

where both ϵ and N_ϵ are independent of x (uniform convergence). □

Exercise 24.2.5.
1) Show that $\|\cdot\|_{0,2}$ is a norm in $\mathscr{S}(\mathbb{R})$.
2) Show that convergence in $\mathscr{S}(\mathbb{R})$ implies L^2 convergence.

Proposition 24.2.6. *The linear operators $(\hat{Q}_0\varphi)(x) = x\varphi(x)$ and $(\hat{P}_0\varphi)(x) = -i\varphi'(x)$ on $\mathscr{S}(\mathbb{R})$ are continuous in the seminorm topology.*

Proof. The identity $D^n(x\varphi) = xD^n\varphi + nD^{n-1}\varphi$ implies $\|Q_0\varphi\|_{m,n} = \sup_x |x^m D^n(x\varphi)| \le \|\varphi\|_{m+1,n} + n\|\varphi\|_{m,n-1}$. Similarly: $\|P_0\varphi\|_{m,n} = \sup_x |x^m D^n\varphi'| = \|\varphi\|_{m,n+1}$. If $\varphi_k \to 0$ then $Q_0\varphi_k \to 0$ and $P_0\varphi_k \to 0$. □

Exercise 24.2.7. *Show that the operator $\hat{P}_0$ is invertible. What is the domain of the inverse operator?*

[3] such spaces are named *Frechét* spaces.

24.3 The Fourier Transform in $\mathscr{S}(\mathbb{R})$

A main motivation to introduce the Schwartz space is the special behaviour of the Fourier transform in this space.

Definition 24.3.1. The Fourier transform and antitransform of a function in $\mathscr{S}(\mathbb{R})$ are:

$$(\mathscr{F}\varphi)(k) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\varphi(x) \tag{24.3}$$

$$(\mathscr{F}^{-1}\varphi)(k) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{ikx}\varphi(x) \tag{24.4}$$

The notation anticipates that $\mathscr{F}\mathscr{F}^{-1} = 1$: this will be a main result to prove. The transforms are related by $(\mathscr{F}^{-1}\varphi)(k) = (\mathscr{F}\varphi)(-k)$ and are well defined, as $\mathscr{S}(\mathbb{R}) \subset \mathscr{L}^1(\mathbb{R})$.

Theorem 24.3.2. *The Fourier transform $\mathscr{F}$ and antitransform $\mathscr{F}^{-1}$ are continuous maps of $\mathscr{S}(\mathbb{R})$ on itself.*

Proof. We must show that all seminorms of $(\mathscr{F}\varphi)(k)$ are finite:

$$\begin{aligned} k^m \frac{d^n}{dk^n}(\mathscr{F}\varphi)(k) &= \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} (-ix)^n \varphi(x) \left(-i\frac{d}{dx}\right)^m e^{-ikx} \\ &= \ldots = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ikx} \left(i\frac{d}{dx}\right)^m [(-ix)^n\varphi(x)] \end{aligned}$$

Multiply and divide by $1+x^2$ and note that $D^m(x^n\varphi) = \sum_{p=0}^m c_p x^{n-p} D^{m-p}\varphi$ ($c_p = 0$ if $n-p<0$. The precise values of c_p have no relevance here). Then:

$$\|\mathscr{F}\varphi\|_{m,n} \le \sqrt{\pi/2}\sum_{p=0}^m |c_p|(\|\varphi\|_{n-p,m-p} + \|\varphi\|_{n-p+2,m-p}) < \infty$$

Then $\mathscr{F}\varphi \in \mathscr{S}(\mathbb{R})$. The bound implies that the Fourier transform is continuous (a sequence seminorm-convergent to zero is mapped to a sequence seminorm-convergent to zero). The same conclusions hold for $\mathscr{F}^{-1}$. □

Theorem 24.3.3 (Inversion theorem).

$$\boxed{\varphi(x) = \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{ikx} (\mathscr{F}\varphi)(k)} \tag{24.5}$$

Proof. The double integral is meaningful because $\mathscr{F}\varphi \in \mathscr{S}(\mathbb{R})$:

$$I(x) = \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{ikx} \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}} e^{-iky}\varphi(y)$$

To circumvent the difficulty that the two integrals cannot be exchanged, let us replace the function $(\mathscr{F}\varphi)(k)$ with $e^{-\epsilon k^2}(\mathscr{F}\varphi)(k) \in \mathscr{S}(\mathbb{R})$. The product function is in $\mathscr{S}(\mathbb{R})$ and converges to $\mathscr{F}\varphi$ as $\epsilon \to 0$ point-wise and in the seminorm topology. Since $\mathscr{F}^{-1}$ is continuous:

$$I(x) = \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{ikx} e^{-\epsilon k^2} \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}} e^{-iky}\varphi(y)$$

The two integrals can now be exchanged

$$I_\epsilon(x) = \int_{\mathbb{R}} dy\, \varphi(y) \int_{\mathbb{R}} \frac{dk}{2\pi} e^{-\epsilon k^2 + ik(x-y)} = \int_{\mathbb{R}} dy\, \varphi(y) \frac{1}{\sqrt{4\epsilon\pi}} e^{-(x-y)^2/4\epsilon}$$

The function φ multiplies the Heat kernel, whose integral is one. Then:

$$I_\epsilon(x) - \varphi(x) = \int_{\mathbb{R}} dy\, [\varphi(x+y) - \varphi(x)] \frac{e^{-y^2/4\epsilon}}{\sqrt{4\epsilon\pi}}$$

Take the modulus and use Lagrange's theorem:

$$\begin{aligned} |I_\epsilon(x) - \varphi(x)| &\le \int_{\mathbb{R}} dy\, |\varphi(x+y) - \varphi(x)| \frac{e^{-y^2/4\epsilon}}{\sqrt{4\epsilon\pi}} \\ &\le \sup_\xi |\varphi'(\xi)| \int_{\mathbb{R}} dy\, |y| \frac{e^{-y^2/4\epsilon}}{\sqrt{4\epsilon\pi}} = 2\sqrt{\frac{\epsilon}{\pi}}\, \|\varphi\|_{0,1} \end{aligned}$$

As $\epsilon \to 0$ the limit is zero, for all x. □

Corollary 24.3.4. *$(\mathscr{F}^2\varphi)(x) = \varphi(-x)$; $\mathscr{F}^3 = \mathscr{F}^{-1}$, $\mathscr{F}^4 = 1$.*

Proof. $(\mathscr{F}^2\varphi)(x) = (\mathscr{F}^{-1}\mathscr{F}\varphi)(-x) = \varphi(-x)$; $(\mathscr{F}^3\varphi)(x) = (\mathscr{F}^2\mathscr{F}\varphi)(x) = (\mathscr{F}\varphi)(-x)$. □

Proposition 24.3.5. *The Hermite functions* (24.2) *are eigenfunctions of the Fourier transform:*

$$\boxed{(\mathscr{F}h_n)(k) = (-i)^n h_n(k)} \tag{24.6}$$

Proof. The generating function of the Hermite functions is obtained from the generator of Hermite polynomials:

$$\frac{1}{\sqrt[4]{\pi}} e^{-t^2 + 2tx - \frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{(t\sqrt{2})^n}{\sqrt{n!}} h_n(x) \tag{24.7}$$

By acting with $\mathscr{F}$ (integrate the variable x):

$$\frac{1}{\sqrt[4]{\pi}} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-t^2 + 2tx - \frac{1}{2}x^2} e^{-ikx} = \sum_{n=0}^{\infty} \frac{(t\sqrt{2})^n}{\sqrt{n!}} (\mathscr{F}h_n)(k)$$

The integral in the left hand side is

$$\frac{e^{-t^2}}{\sqrt[4]{\pi}} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{(2t-ik)x - \frac{1}{2}x^2} = \frac{1}{\sqrt[4]{\pi}} e^{t^2 - 2ikt - \frac{1}{2}k^2} = \sum_{n=0}^{\infty} \frac{(-it\sqrt{2})^n}{\sqrt{n!}} h_n(k)$$

The equality of coefficients of powers t^n of the two series gives the result. □

Proposition 24.3.6. *In the inner product of* $\mathrm{L}^2(\mathbb{R})$*:*

$$(\varphi|\mathscr{F}\psi) = (\mathscr{F}^{-1}\varphi|\psi), \qquad \forall\varphi,\, \psi \in \mathscr{S}(\mathbb{R}) \tag{24.8}$$

$$(\mathscr{F}\psi|\mathscr{F}\varphi) = (\psi|\varphi), \qquad \|\mathscr{F}\psi\|_2 = \|\psi\|_2 \tag{24.9}$$

Proof.

$$\int_{\mathbb{R}} dk\overline{\varphi(k)}(\mathscr{F}\psi)(k) = \int_{\mathbb{R}} dk\overline{\varphi(k)} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\psi(x)$$
$$= \int_{\mathbb{R}} dx\,\psi(x) \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}}\overline{\varphi(k)}e^{-ikx} = \int_{\mathbb{R}} dx\,\psi(x)\overline{(\mathscr{F}^{-1}\varphi)(k)}$$

The other relations follow. □

Example 24.3.7 (Fourier transform of $x^n e^{-a^2x^2}$).

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\, x^n e^{-a^2x^2} e^{-ikx} = i^n \frac{\partial^n}{\partial k^n} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-a^2x^2-ikx} = \frac{i^n}{a\sqrt{2}} \frac{\partial^n}{\partial k^n} e^{-\frac{k^2}{4a^2}}$$

Rodrigues' formula (11.21) *is used to evaluate the derivatives, and the result contains the Hermite polynomial of degree* n*:*

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\, x^n e^{-a^2x^2} e^{-ikx} = \frac{1}{(2ia)^n} \frac{1}{a\sqrt{2}}\, e^{-\dfrac{k^2}{4a^2}} H_n\left(\frac{k}{2a}\right) \tag{24.10}$$

The "unitarity" (24.9) *of the Fourier transform gives the integral identity:*

$$\int_{\mathbb{R}} dx\, e^{-(a^2+b^2)x^2} x^{n+p} = \frac{2i^{n-p}}{(2a)^{n+1}(2b)^{p+1}} \int_{\mathbb{R}} dk e^{-k^2\frac{a^2+b^2}{4a^2b^2}} H_n\left(\frac{k}{2a}\right) H_p\left(\frac{k}{2b}\right)$$

Note that n *and* p *must have the same parity (or the result is zero). The left hand side is evaluated with the change* $x^2 = t$ *and gives a Gamma function*[4]*.*

Proposition 24.3.8.

$$\hat{Q}_0\mathscr{F} = \mathscr{F}\hat{P}_0, \qquad \hat{P}_0\mathscr{F} = -\mathscr{F}\hat{Q}_0, \tag{24.11}$$

Proof. Integrate by parts. Boundary terms cancel because φ is of rapid decrease:

$$(\hat{Q}_0\mathscr{F}\varphi)(k) = k(\mathscr{F}\varphi)(k) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\varphi(x)\, i\frac{d}{dx}e^{-ikx}$$
$$= \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ikx}(-i)\frac{d}{dx}\varphi(x) = (\mathscr{F}\hat{P}_0\varphi)(k)$$

The other equality is proven similarly. □

Exercise 24.3.9. *Show that the operators* $\hat{P}_0^2 + \hat{Q}_0^2$ *and* $\mathscr{F}$ *commute.*

24.4 Convolution product

Definition 24.4.1. The **convolution product** of two functions in $\mathscr{S}(\mathbb{R})$ is

$$\boxed{(\psi * \varphi)(x) = \int_{\mathbb{R}} dy\, \psi(x-y)\; \varphi(y)} \tag{24.12}$$

[4] A useful source of tabulated integrals is Gradshteyn - Ryzhik, *Tables of Integrals, Products and Series*, Academic Press.

Exercise 24.4.2. *Show that the convolution product is commutative.*

Theorem 24.4.3. *$\psi * \varphi \in \mathscr{S}(\mathbb{R})$ if ψ and φ are in $\mathscr{S}(\mathbb{R})$. The map $\psi \to \psi * \varphi$ is linear and continuous,*

$$\boxed{\mathscr{F}(\psi * \varphi) = \sqrt{2\pi}\,(\mathscr{F}\psi)(\mathscr{F}\varphi) \quad , \quad (\mathscr{F}\psi) * (\mathscr{F}\varphi) = \sqrt{2\pi}\,\mathscr{F}(\psi\varphi)} \tag{24.13}$$

The convolution product is commutative, associative and distributive.

Proof. The convolution of two functions in $\mathscr{S}$ is in $\mathscr{S}$:

$$\begin{aligned}
|x^m D^n(\psi * \varphi)(x)| &= \Big| \int_{\mathbb{R}} dy\, x^m D^n_x \psi(x-y)\varphi(y) \Big| \\
&\le \sum_{k=0}^{m} \binom{m}{k} \int_{\mathbb{R}} dy |(x-y)^{m-k} D^n_x \psi(x-y)|\, |y^k \varphi(y)| \\
&\le \sum_{k=0}^{m} \binom{m}{k} \sup_z |z^{m-k} D^n_z \psi(z)| \int_{\mathbb{R}} dy |y^k \varphi(y)|
\end{aligned}$$

let us introduce factors $(1+y^2)$ in the integral and take the sup:

$$\|\psi * \varphi\|_{m,n} \le \pi \sum_{k=0}^{m} \binom{m}{k} \|\psi\|_{m-k,n} (\|\varphi\|_{0,0} + \|\varphi\|_{2,0})$$

Then $\psi * \varphi$ is in $\mathscr{S}(\mathbb{R})$ is ψ and φ are. Moreover, if $\varphi_r \to \varphi$ it follows that $\psi * \varphi_r \to \psi * \varphi$ (continuity).
The Fourier transform:

$$\begin{aligned}
\mathscr{F}(\psi * \varphi)(k) &= \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} \int dy \psi(x-y)\phi(y) \\
&= \int dy \phi(y) e^{-iky} \int \frac{dx}{\sqrt{2\pi}} e^{-ik(x-y)} \psi(x-y) = \sqrt{2\pi} (\mathscr{F}\varphi)(k)\,(\mathscr{F}\psi)(k)
\end{aligned}$$

The other relation follows: replace ψ, φ by $\mathscr{F}\psi$, $\mathscr{F}\varphi$; then $\mathscr{F}(\mathscr{F}\psi * \mathscr{F}\varphi)(x) = \sqrt{2\pi}\psi(-x)\varphi(-x)$. Apply $\mathscr{F}^{-1} = \mathscr{F}^3$ and obtain $\mathscr{F}\psi * \mathscr{F}\varphi = \sqrt{2\pi}\mathscr{F}(\psi\varphi)$.
The identity $\mathscr{F}[(\varphi_1 * \varphi_2) * \varphi_3] = \sqrt{2\pi}\mathscr{F}(\varphi_1 * \varphi_2)\mathscr{F}(\varphi_3) = 2\pi\,\mathscr{F}(\varphi_1)\mathscr{F}(\varphi_2)\mathscr{F}(\varphi_3)$ shows that the product is associative. □

24.4.1 The Heat Equation

In one space dimension the heat (or diffusion) equation is

$$\boxed{\left(\frac{1}{D}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0} \tag{24.14}$$

The *Heat Kernel* is the function of time $t > 0$ and space coordinate x:

$$K_t(x-y) = \frac{1}{\sqrt{4Dt\pi}} \exp\left[-\frac{(x-y)^2}{4Dt} \right] \tag{24.15}$$

As a function of x it is peaked in y, with height decreasing in time; its space integral is one at all times t. The heat kernel solves the heat equation with initial condition $K_0(x-y)=\delta(x-y)$. It describes the diffusive spread in time of a quantity (temperature, pollutant density ...) that is initially delta-localized in $x=y$. The width of the Gaussian grows with the square root law $\approx \sqrt{Dt}$.

Since the equation is linear, the linear superposition of heat kernels with parameters y weighted by a function $\varphi(y)$ is also a solution; it is the convolution:

$$u(x,t)=(K_t*\varphi)(x)=\int_{\mathbb{R}} dy K_t(x-y)\varphi(y) \tag{24.16}$$

It solves the Heat equation with the initial condition $u(x,0)=\varphi(x)$.
The same result is obtained with the aid of Fourier integral. Put

$$u(x,t)=\int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{ikx}\tilde{u}(k,t)$$

in the Heat equation, and obtain the first order equation

$$\frac{1}{D}\frac{d}{dt}\tilde{u}(k,t)+k^2\tilde{u}(k,t)=0 \quad \Rightarrow \quad \tilde{u}(k,t)=C(k)e^{-Dtk^2}$$

The initial condition imposes $C(k)=\tilde{u}(k,0)=\tilde{\varphi}(k)$. The solution in $x-$space is then obtained:

$$u(x,t)=\int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{ikx-k^2Dt}\tilde{\varphi}(k)$$

This is the Fourier antitransform of the product of two Fourier transforms: $\tilde{\varphi}(k)$ and e^{-k^2Dt}. Then it coincides (up to numerical factor) with the convolution product (24.16) of the Heat kernel and the initial condition, i.e. (24.16).

24.4.2 Laplace equation in the strip

Consider the Laplace equation in the strip $-\infty<x<\infty$, $0\le y\le 1$ with boundary conditions (b.c.):

$$u_{xx}+u_{yy}=0, \qquad u(x,0)=u_0(x), \quad u(x,1)=u_1(x) \tag{24.17}$$

For simplicity, $u_0,u_1\in\mathscr{S}(\mathbb{R})$. With the Fourier representation

$$u(x,y)=\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{ikx}\tilde{u}(k,y) \tag{24.18}$$

Laplace's equation gives: $-k^2\tilde{u}+\tilde{u}_{yy}=0$, i.e. $\tilde{u}(k,y)=A_k\exp(ky)+B_k\exp(-ky)$. The functions A_k and B_k are determined by the b.c. At $y=0$: $\tilde{u}_0(k)=A_k+B_k$; at $y=1$: $\tilde{u}_1(k)=A_ke^k+B_ke^{-k}$. Then:

$$\tilde{u}(k,y)=\tilde{u}_0(k)\frac{\sinh[k(1-y)]}{\sinh k}+\tilde{u}_1(k)\frac{\sinh(ky)}{\sinh k}$$

To evaluate $u(x,y)$ by (24.18) we exploit the properties of the convolution. Let's identify $\tilde{u}$ as $\tilde{u}(k,y)=\tilde{u}_0(k)\tilde{S}_{1-y}(k)+\tilde{u}_1(k)\tilde{S}_y(k)$ where $\tilde{S}_y(k)$ is evaluated with the Residue theorem (see ex.14.3.20):

$$S_y(k)=\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikx}\frac{\sinh(ky)}{\sinh k}=\sqrt{\frac{\pi}{2}}\frac{\sin(\pi y)}{\cosh(\pi x)+\cos(\pi y)}$$

The Fourier transform (24.18) becomes the sum of two convolutions, and the solution of the Laplace equation in the strip is obtained:

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2\pi}}[(S_y * u_1)(x) + (S_{1-y} * u_0)(x)] \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dx' \left[\frac{\sin(\pi y)\, u_1(x')}{\cosh[\pi(x-x')] + \cos(\pi y)} + \frac{\sin(\pi y)\, u_0(x')}{\cosh[\pi(x-x')] - \cos(\pi y)}\right]. \end{aligned}$$

Chapter 25

TEMPERED DISTRIBUTIONS

Depuis l'introduction par Dirac de la fameuse fonction $\delta(x)$, qui serait nulle partout sauf pour $x = 0$ et serait infinie pour $x = 0$ de telle sort que $\int_{-\infty}^{+\infty} dx\, \delta(x) = +1$, les formules du calcul symbolique sont devenues plus inacceptables pour la rigueur des mathématiciens. Ecrire que la fonction d'Heaviside $\theta(x)$ égale a 0 pour $x < 0$ et á 1 pour $x \geq 0$ a pour dérivé la fonction de Dirac $\delta(x)$ dont la définition même est mathématiquement contradictoire, et parler de dérivées $\delta'(x)$, $\delta''(x)$ ... de cette fonction denuée d'existence réelle, c'est dépasser les limites qui nous sont permises ... (L. Schwartz, "Théorie des Distributions", Hermann & C. Paris, 1950)

25.1 Introduction

The linear continuous functionals on Schwartz's space provide a generalization of ordinary functions that is very useful in many applications, like Green functions in the theory of differential equations, and spectral theory of operators.

A linear continuous functional on $\mathscr{S}(\mathbb{R})$ is a linear map

$$f : \varphi \in \mathscr{S}(\mathbb{R}) \to f\varphi \in \mathbb{C}$$

such that $f\varphi_k \to 0$ for any seminorm-convergent sequence $\varphi_k \to 0$.

Definition 25.1.1. The space of linear continuous functionals on $\mathscr{S}(\mathbb{R})$ is the space $\mathscr{S}'(\mathbb{R})$ of *tempered distributions.*

It is a linear space with the linear operations $(f + \lambda g)\varphi = f\varphi + \lambda\,(g\varphi)$ (if f and g are sequentially continuous, so are their linear combinations)[1].

A sufficient condition for continuity is boundedness with respect to a seminorm (or a finite sum of seminorms): there is a constant C_f and a seminorm such that:

$$|f\varphi| \leq C_f \|\varphi\|_{k,m} \quad \forall \varphi \in \mathscr{S}(\mathbb{R}).$$

[1]A nice and instructive presentation is: I. Richards and H. Youn, *Theory of distributions: a non-technical introduction*, Cambridge University Press (1990).

The **regular distributions** are the integral functionals

$$f\varphi = \int_{\mathbb{R}} dx f(x)\varphi(x)$$

where the function f is locally integrable (i.e. integrable on any compact subset of the line) and algebraically bounded for large x: there are constants $C > 0$, $R > 0$, and an integer $n > 0$ such that $|f(x)| \le C|x|^n$ for all $|x| > R$. Then:

$$\begin{aligned} |f\varphi| &\le \int_{-R}^{R} dx|f(x)||\varphi(x)| + C\int_{|x|>R} dx|x|^n|\varphi(x)| \\ &\le \|\varphi\|_{00}\int_{-R}^{R} dx|f(x)| + C\pi\sup_x[|x|^n(1+x^2)|\varphi(x)|] \\ &\le K(\|\varphi\|_{00} + \|\varphi\|_{n,0} + \|\varphi\|_{2+n,0}) \end{aligned}$$

where K depends on f but not on the test function.

Definition 25.1.2 (convergence)**.** $f_n \to f$ in $\mathscr{S}'(\mathbb{R})$ if $f_n\varphi \to f\varphi$ $\forall\varphi \in \mathscr{S}(\mathbb{R})$.

Example 25.1.3. *The sequence $f_n(x) = \cos(nx)$ has no limit as a function. However, in the distributional sense it has limit zero: $f_n\varphi = \int_{\mathbb{R}} dx\, \cos(nx)\varphi(x) = \frac{1}{n}\int_{\mathbb{R}} dx\, \sin(nx)\varphi'(x)$. Then $|f_n\varphi| \le \frac{1}{n}\int_{\mathbb{R}} dx\,|\varphi'(x)| \le \frac{\pi}{n}(\|\varphi\|_{0,1} + \|\varphi\|_{2,1}) \to 0$ as $n \to \infty$, for all φ, i.e. $f_n \to 0$ in $\mathscr{S}'(\mathbb{R})$.*

Remark 25.1.4. *We adopt Dirac's* bra *and* ket *notation to write the action of* any *tempered distribution on a test function:*

$$f\varphi = < f|\varphi >$$

Regular distributions extend functionals in the dual of L^2. Show that if $f \in \mathscr{L}^2(\mathbb{R})$ then $< f|\varphi >= (\overline{f}|\varphi)$ is a regular distribution on $\mathscr{S}(\mathbb{R})$.

25.2 Special distributions

25.2.1 Dirac's delta function

The Dirac's delta at a point $a \in \mathbb{R}$ is the functional

$$\boxed{< \delta_a|\varphi >= \varphi(a)} \tag{25.1}$$

Since $| < \delta_a|\varphi > | \le \|\varphi\|_{0,0}$, the functional δ_a is continuous. It is customary (and convenient) to write the action of the functional as if it were a regular one, with a generalized function: $< \delta_a|\varphi >= \int_{\mathbb{R}} dx\, \delta(x-a)\varphi(x)$.
There are several approximations of Dirac's delta by regular distributions; this one is particularly important:

Proposition 25.2.1. *The following function converges to δ_a in $\mathscr{S}'(\mathbb{R})$ as $\epsilon \to 0^+$ (Lorentzian or Cauchy distribution):*

$$\frac{\epsilon}{\pi}\frac{1}{(x-a)^2+\epsilon^2} \tag{25.2}$$

Proof. For any ϵ the function has unit integral, and defines a regular distribution. Let's show that for any test function φ, the action on φ converges to $\varphi(a)$ as $\epsilon \to 0$. This amounts to the vanishing of

$$I = \int_{\mathbb{R}} dx \frac{\epsilon}{\pi} \frac{\varphi(x) - \varphi(a)}{(x-a)^2 + \epsilon^2}$$

shift the variable by a and take the symmetric part of numerator

$$= \int_{\mathbb{R}} dx \frac{\epsilon}{2\pi} \frac{\varphi(a+x) + \varphi(a-x) - 2\varphi(a)}{x^2 + \epsilon^2}$$

$$|I| \le \frac{\epsilon}{2\pi} \int_{\mathbb{R}} dx \frac{|\varphi(a+x) + \varphi(a-x) - 2\varphi(a)|}{x^2}$$

The integral is a finite number, as the function is finite in $x = 0$ and decays at least as $|x|^{-2}$ at infinity. Then the limit $\epsilon \to 0^+$ is zero. $\square$

The following regular distributions also converge to δ_a in $\mathscr{S}'(\mathbb{R})$ as $\epsilon \to 0^+$ (proofs are simple and left as exercise)

$$\frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{1}{\epsilon}(x-a)^2}, \quad \frac{1}{2\epsilon} \chi_{[a-\epsilon, a+\epsilon]}(x) \tag{25.3}$$

Exercise 25.2.2. *Show that the sequences converge to* δ_0*:*

$$\frac{2}{\pi\epsilon^2} \sqrt{\epsilon^2 - x^2}\, \theta(\epsilon^2 - x^2), \quad \frac{\sin(nx)}{\pi x}.$$

(Hint for the second case: the function $[\varphi(x) - \varphi(0)\chi_{[-1,1]}(x)]/x$ *is in* $\mathscr{L}^1(\mathbb{R})$ *for* $\varphi \in \mathscr{S}(\mathbb{R})$*. Use the Riemann-Lebesgue theorem 27.1.3).*

25.2.2 Heaviside's theta function

Consider the functional θ_a, $a \in \mathbb{R}$,

$$\boxed{< \theta_a | \varphi > = \int_a^\infty dx\, \varphi(x)} \tag{25.4}$$

It is a regular distribution, with function $\theta(x - a) = \chi_{[a,\infty)}(x)$.

Exercise 25.2.3.
Prove the distributional limits ($n \to \infty$*)*[2]

$$1) \quad \chi_{[0,n]} \to \theta(x),$$

$$2) \quad f_n(x) = \begin{cases} 0 & x < 0 \\ x^n & x = [0,1] \\ x = 1 & x > 1 \end{cases} \to \theta(x-1)$$

$$3) \quad \frac{1}{e^{nx}+1} \to \theta(-x).$$

[2]The function 3) with $n = \frac{\mu}{k_B T}$ and $x = \frac{E-\mu}{\mu}$ is the Fermi-Dirac distribution, which gives the average number of fermions with energy E, in thermal equilibrium at temperature T and chemical potential $\mu > 0$. The limit distribution, $\theta(\mu - E)$, is the Fermi distribution (the ground state, $T = 0$).

25.2.3 Principal value of $\frac{1}{\mathbf{x-a}}$

The function $\frac{1}{x-a}$ has a non-integrable singularity in $x = a$ and thus it does not yield a regular distribution. One then defines the "principal value of $\frac{1}{x-a}$" as the functional

$$\boxed{< P\tfrac{1}{x-a}| \varphi >= \fint_{\mathbb{R}} dx \, \frac{\varphi(x)}{x-a}} \tag{25.5}$$

The principal value is:

$$\lim_{\epsilon\to 0^+} \left(\int_{-\infty}^{a-\epsilon} dx + \int_{a+\epsilon}^{\infty} dx \right) \frac{\varphi(x)}{x-a} = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\infty} dx \, \frac{\varphi(a+x) - \varphi(a-x)}{x}; \tag{25.6}$$

the limit $\epsilon \to 0^+$ exists. Now, use Lagrange's formula on the interval $\epsilon \le x \le 1$, and the triangle inequality on $x \ge 1$:

$$\begin{aligned} \left|< P\tfrac{1}{x-a}|\varphi >\right| &\le 2(1-\epsilon) \sup_x |\varphi'(x)| + \int_1^{\infty} dx (|\varphi(a+x)| + |\varphi(a-x)|) \\ &\le 2\|\varphi\|_{0,1} + 2\|\varphi\|_{\mathrm{L}^1} \end{aligned}$$

The L^1 norm is majored by seminorms, therefore the principal part is a tempered distribution.

25.2.4 The Sokhotski-Plemelj formulae

The following identities among distributions are extremely useful in the study of Green functions (differential equations, potential theory, linear response theory, spectral theory of operators)[3]:

$$\boxed{\lim_{\epsilon\to 0^+} \frac{1}{x-a\pm i\epsilon} = P\frac{1}{x-a} \mp i\pi\delta(x-a)} \tag{25.7}$$

a is real and the limit is distributional, i.e.

$$\boxed{\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} dx \, \frac{\varphi(x)}{x-a\pm i\epsilon} = \fint_{\mathbb{R}} dx \, \frac{\varphi(x)}{x-a} \mp i\pi \, \varphi(a), \quad \varphi \in \mathscr{S}(\mathbb{R})}$$

Proof. The real part is the distribution

$$\int_{\mathbb{R}} dx \frac{(x-a)\varphi(x)}{(x-a)^2+\epsilon^2} = \int_{\mathbb{R}} dx \frac{x\varphi(x+a)}{x^2+\epsilon^2} = \int_0^{\infty} dx \frac{x}{x^2+\epsilon^2} [\varphi(a+x) - \varphi(a-x)]$$

[3]The identities were discovered by Sokhotski in 1873, and proven rigorously about thirty years later by Josip Plemelj. Given a Cauchy integral on a curve (it can be open or closed)

$$F(z) = \int_{\gamma} \frac{d\zeta}{2\pi i} \frac{f(\zeta)}{\zeta - z}, \quad z \notin \gamma$$

and the Hölder condition $|f(\zeta) - f(\zeta_0)| \le A|\zeta - \zeta_0|^{\alpha}$ (A, $\alpha > 0$) for all $\zeta, \zeta_0 \in \gamma$, the identities establish the limits $F^{\pm}(\zeta_0)$ as z approaches a point $\zeta_0 \in \gamma$ different than endpoints, from the left or the right sides of γ.
The identities are frequently used in theoretical physics with the path γ given by the real axis.

which, for $\epsilon \to 0$, converges to $⨍ \frac{\varphi}{x-a}$ (see eq.(25.6)). The imaginary part is a regular distribution that converges to the delta function:

$$\int_{\mathbb{R}} dx \frac{\mp\epsilon}{(x-a)^2+\epsilon^2}\varphi(x) \to \mp\pi\varphi(a).$$

□

Exercise 25.2.4. *Prove the identity, for real x, y, and $\epsilon, \eta > 0$:*

$$\int_{\mathbb{R}} \frac{dx'}{\pi^2} \mathrm{Re}\left[\frac{1}{x'-x+i\epsilon}\right] \mathrm{Re}\left[\frac{1}{x'-y+i\eta}\right] = \frac{1}{\pi}\frac{\epsilon+\eta}{(x-y)^2+(\epsilon+\eta)^2} \tag{25.8}$$

Therefore, for $x \neq y$ and in the limit $\epsilon + \eta \to 0^+$:

$$\int \frac{dx'}{\pi^2}\frac{P}{x'-x}\frac{P}{x'-y} = \delta(x-y) \tag{25.9}$$

Exercise 25.2.5 (Hilbert transform). *The Hilbert transform of a function is*

$$(\mathscr{H} f)(x) = ⨍_{\mathbb{R}} \frac{dx'}{\pi}\frac{f(x')}{x-x'} \tag{25.10}$$

By means of (25.9) *show that* $(\mathscr{H}^2\varphi)(x) = -\varphi(x)$. *Therefore, the solution of the integral equation $\mathscr{H} f = g$ is $f = -\mathscr{H} g$.*

Exercise 25.2.6. *Show that this is a tempered distribution:*

$$< P\frac{1}{x^2}|\varphi >= \lim_{\epsilon\to 0^+}\Big[\int_{-\infty}^{-\epsilon} dx\, \frac{\varphi(x)-\varphi(0)}{x^2} + \int_{\epsilon}^{\infty} dx\, \frac{\varphi(x)-\varphi(0)}{x^2}\Big] \tag{25.11}$$

Can you suggest a generalization to define $P(1/x^n)$?

25.3 Linear response and Kramers-Krönig relations

The theory of **linear response** evaluates the variation in time of an observable of a system that is coupled to a weak time-dependent field $\delta\varphi(t)$, in the linear approximation. The physical requirement of *causality*, i.e. the effect on the system may only depend on the field's values at earlier times, implies the general Kramers and Kronig relations for the response function.

Let $g(t)$ be the value at time t of a measurable quantity of the system in presence of the perturbation, and $g_0(t)$ be the value of the same quantity in absence of the perturbation. In the linear approximation, the variation $\delta g(t) = g(t) - g_0(t)$ is linearly related to the external perturbation through a *response function* $R(t,t')$ that only depends on the variables of the unperturbed system:

$$\delta g(t) = \int_{-\infty}^{t} R(t,t')\,\delta\varphi(t')dt'$$

The integral involves the history of the system at times less than t because of the physical requirement of causality.

If the properties of the unperturbed system are time-independent (as in thermal equilibrium), the response function depends on $t - t'$. If the upper limit t is replaced by $+\infty$ with the insertion of a theta function, the integral becomes the convolution:

$$\delta g(t) = \int_{-\infty}^{+\infty} \theta(t-t')R(t-t')\,\delta\varphi(t')dt' \tag{25.12}$$

Hereafter we consider this very common situation.
In frequency space the convolution (25.12) becomes the product of the Fourier transforms[4], and the linear response takes the simple form of a direct proportionality between response and driving field *at the same frequency*:

$$\boxed{\delta g(\omega) = \chi(\omega)\delta\varphi(\omega)} \tag{25.13}$$

Notable examples are: $J = \sigma E$, $M = \chi H$, $D = \epsilon E$ (where each quantity depends on ω, with possible further dependence on wave-vector). The response function $\chi(\omega)$ is the *generalized susceptibility*:

$$\chi(\omega) = \int_{\mathbb{R}} dt\, e^{i\omega t}\,\theta(t)R(t) = \int_0^{\infty} dt\, e^{i\omega t}R(t) \tag{25.14}$$

According to Paley and Wiener's theorem, if $R \in \mathscr{L}^2(0,\infty)$, the function $\chi(\omega)$ is analytic on Im $\omega \geq 0$. This is a consequence of causality and, in turn, it implies the *Kramers - Kronig relations*. They were obtained independently in 1926 and 1927, and relate the real and imaginary parts of the response in ω space:

Proposition 25.3.1. *Suppose that $\chi(\omega) \to 0$ if $|\omega| \to \infty$; for real ω it is:*

$$\boxed{\mathrm{Re}\,\chi(\omega) = -⨍_{-\infty}^{\infty} \frac{d\omega'}{\pi}\frac{\mathrm{Im}\,\chi(\omega')}{\omega-\omega'}, \quad \mathrm{Im}\,\chi(\omega) = ⨍_{-\infty}^{\infty} \frac{d\omega'}{\pi}\frac{\mathrm{Re}\,\chi(\omega')}{\omega-\omega'}} \tag{25.15}$$

Proof. Since $\chi(\omega')$ is analytic in Im $\omega' > 0$ it is:

$$\int_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega-\omega'-i\epsilon} = 0, \qquad \omega \in \mathbb{R}$$

(check: close the path of integration with a semicircle in the upper half plane). The Plemelj-Sokhotski formula gives:

$$0 = ⨍_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega-\omega'} + i\pi\chi(\omega)$$

Separation of real and imaginary parts gives the relations. □

25.4 Eigenvalues of random matrices

A random matrix is an element of an ensemble of matrices defined by choosing the matrix elements with some probability distribution. For example, the ensemble of $n \times n$ matrices X with $X_{ij} = \pm 1$ with equal probability, is a finite set

[4] In physics it is customary to define the Fourier transform of a function of time as $\tilde{f}(\omega) = \int_{\mathbb{R}} dt f(t)e^{i\omega t}$, with inverse $f(t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}$. In this section we use this convention.

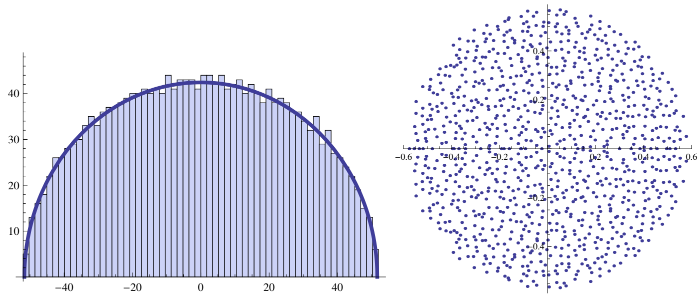

Figure 25.1: Left: the histogram of the eigenvalues of two Hermitian matrices of size 1000 with real and imaginary parts of matrix elements chosen uniformly in [-1,1]. The Semicircle Law by Wigner is a general feature of eigenvalues of large random real-symmetric or complex-Hermitian matrices with independent identically distributed (i.i.d.) matrix elements. Right: the complex eigenvalues of two non-Hermitian matrices of size 1000 with i.i.d. matrix elements (Circle Law by Ginibre). If the real and imaginary parts have different variance, the distribution is the Elliptic Law.

of 2^{n^2} equally probable matrices. Another choice are Gaussian and identically distributed matrix elements. The random matrices may be constrained to be Hermitian, unitary, positive, banded, ... according to the problem at hand. An interesting question is: what are the statistical properties of the eigenvalues?

Random matrices in physics were introduced by Wigner in the fifties as a reference model for the statistical properties of sequences of nuclear resonances[5]. Since then, they found many applications: the study of quantum systems with chaotic classical motion (quantum chaos), transport in mesoscopic structures with impurities, spectra of Dirac matrices in QCD, molecular spectra, statistical mechanics on random graphs, random surfaces, etc. The subject is still evolving in several unexpected directions, with beautiful mathematics[6].

25.4.1 Semicircle law of GUE random matrices

The *Gaussian Unitary Ensemble* (GUE) of matrices consists of Hermitian matrices H_{ij} where H_{ii}, ReH_{ij} and ImH_{ij} $(i < j)$ are independent random numbers with Gaussian distribution. The probability measure on GUE matrices is

$$p(H)dH \propto \prod_i e^{-nH_{ii}^2} dH_{ii} \prod_{i<j} e^{-2n|H_{ij}|^2} d^2 H_{ij} = e^{-n\mathrm{tr} H^2} dH$$

[5]It was noted that the energy separations s of resonances normalized by an average value, obey the same statistical laws $P(s) \approx s^\beta e^{-ks^2}$ of the normalized separations of eigenvalues of Gaussian random matrices (k is a constant, β is 1 for real symmetric matrices, 2 for complex Hermitian matrices and 4 for quaternionic self-dual). Note the feature of "level repulsion": small spacings are rare.

[6]see: *The Oxford Handbook of Random Matrix Theory*, G. Akemann, J. Baik and P. Di Francesco Editors, Oxford University Press, 2011; M. L. Mehta, *Random Matrices*, 3rd Ed. Elsevier, 2004; F. Haake, *Quantum signatures of chaos*, 3rd Ed. Springer, 2010. See the book by G. Livan. online at `https://arxiv.org/pdf/1712.07903.pdf`.

The set GUE and the measure are invariant under the action of unitary matrices $H \to U^\dagger H U$. Unitary invariance implies the factorization of the matrix measure dH into a (Haar) measure on the unitary group dU and a measure for the eigenvalues. The joint probability density for the eigenvalues is found to be:

$$p(\lambda_1 \dots \lambda_n) = \frac{1}{Z_n} \prod_{i<j} (\lambda_j - \lambda_j)^2 e^{-n\sum \lambda_i^2} \equiv \frac{1}{Z_n} e^{-n^2 U(\lambda_1 \dots \lambda_n)}$$
$$U(\lambda_1 \dots \lambda_n) = \frac{1}{n} \sum_i \lambda_i^2 - \frac{2}{n^2} \sum_{i<j} \log |\lambda_i - \lambda_j|.$$

with normalization constant $Z_n = \int d\lambda_1 \dots d\lambda_n \, \exp(-n^2 U)$. For GUE the joint probability density vanishes quadratically when two eigenvalues approach (degenerate eigenvalues are rare, with level repulsion exponent $\beta = 2$)[7].
For large n, the largest contribution to the probability density comes from the set of eigenvalues that minimize the "potential energy" U. This configuration determines the statistical properties of eigenvalues of GUE matrices for $n \to \infty$. It has the interpretation of minimal energy state of a bidimensional gas of n charged particles (log potential) in a harmonic potential. The minimum solves

$$\frac{\partial U}{\partial \lambda_i} = 0 \text{ i.e. } \quad \lambda_i = \frac{2}{n} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \quad \to x_i = \sum_{j\neq i} \frac{1}{x_i - x_j}, \quad x_i = \sqrt{\frac{n}{2}} \lambda_i$$

The solution is found by a nice trick (Stieltjes): define the polynomial $p(x) = (x - x_1) \cdots (x - x_n)$. By eq.(9.2) the n conditions for minimum become: $x_i = p''(x_i)/p'(x_i)$ i.e. the polynomial $p''(x) - xp'(x)$ must be zero at $x = x_1, \dots, x_n$. Since it is of degree n, it is proportional to $p(x)$ itself. Then $p''(x) - xp'(x) + np(x) = 0$, with solution $p(x) = c_n H_n(x\sqrt{2})$. Therefore, for $n \to \infty$ the eigenvalues of a random GUE matrix (with probability one) have a *semicircle distribution* with radius determined by the variance of the Gaussian (in this case the radius in $\frac{1}{2}\sqrt{n}$).

25.4.2 Zeros of large-n Hermite polynomials

Let $x_1 \dots x_n$ be the zeros of the Hermite polynomial $H_n(x)$. They are real and simple, and we wish to evaluate their distribution when n is large. The polynomial solves the differential equation (11.20), $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0$, which becomes Weber's equation (the equation of the harmonic oscillator in quantum mechanics):

$$-h_n''(x) + (x^2 - 2n - 1) h_n(x) = 0$$

for the Hermite function (19.27). If $x^2 > 2n + 1$, $h_n(x)$ and $h_n''(x)$ have the same sign, and vanish at infinity. Therefore, as $|x|$ decreases from ∞, $|h_n(x)|$

[7] A simple argument for level repulsion. Let x, y be the real eigenvalues of the 2×2 GUE matrix ($\beta = 2$):

$$\begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix}, \quad |x - y| = \sqrt{(a-d)^2 + 4b^2 + 4c^2}$$

To have $x = y$ we need that the random numbers $a - d$, b and c are zero simultaneously: this is very unlikely. For random real symmeric matrices (GOE) it is $c = 0$ and level repulsion is weaker ($\beta = 1$).

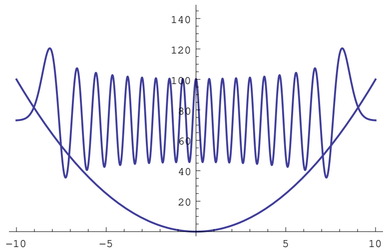

Figure 25.2: The oscillatory part of the Hermite function h_{36} with its 36 real zeros, is confined in the interval $|x| < \sqrt{73}$, between the extreme zeros of h''_{36}. The plot of the function is shifted upwards by 73 to show that the zeros are confined in the parabolic well.

increases. For a zero to occur the curvature h''_n must change sign, and then the function may start to point to the real axis. As a consequence, the zeros of $H_n(x)$ are confined in the interval $x^2 < 2n+1$.

Rescale the zeros $x_i = s_i\sqrt{2n}$, $(i = 1\dots n)$; if a limit distribution exists the rescaled zeros s_i are described by a density $\rho(s) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \delta(s-s_i)$ with support in $\sigma = [-1,1]$. To evaluate the density we introduce the useful function $F_n(z)$, and the limit function:

$$F_n(z) = \frac{1}{n}\sum_{i=1}^n \frac{1}{z-s_i} \qquad F(z) = \int_\sigma ds\, \frac{\rho(s)}{z-s} \quad z \notin \sigma$$

For large $|z|$, $F(z)$ behaves as $1/z$ and, by the Sokhotski-Plemelj identity:

$$\boxed{\rho(s) = \frac{1}{\pi}\mathrm{Im}\, F(s-i\epsilon)} \tag{25.16}$$

The limit function $F(z)$ is obtained from $F_n(z)$ by noting that

$$\frac{H'_n(x)}{H_n(x)} = \sum_{i=1}^n \frac{1}{x-\sqrt{2n}\, s_i} = \sqrt{\frac{n}{2}}F_n\left(\frac{x}{\sqrt{2n}}\right)$$

A derivative in x and the equation $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$ give the Riccati equation

$$\frac{1}{n}F'_n(z) + F_n^2(z) - 4zF_n(z) + 4 = 0$$

In the large-n limit the derivative is neglected: $F^2(z) - 4zF(z) + 4 = 0$. The solution with correct large z asymptotics is: $F(z) = 2z - 2\sqrt{z^2-1}$.
Eq.(25.16) gives the *semicircle law*:

$$\rho(s) = \begin{cases} \frac{2}{\pi}\sqrt{1-s^2} & \text{if } |s| < 1 \\ 0 & \text{if } |s| > 1 \end{cases} \tag{25.17}$$

It is a remarkable fact that the semicircle law also describes the distribution of eigenvalues of Hermitian matrices with random matrix elements, in the large n limit.

25.5 Distributional calculus

Important operations such as conjugation, derivative, Fourier transform, may be extended simply and naturally from well behaved functions (such as test functions or functions associated to regular distributions) to distributions (generalized functions, which may be as awkward as a delta function). The procedure reproduces the steps of this first example.

Complex conjugation has a natural definition if the distribution is regular: if f is the regular distribution $< f|\varphi >= \int dx\, f(x)\varphi(x)$, its complex conjugate is the distribution f^* with action $< f^*|\varphi >= \int dx\, \overline{f(x)}\varphi(x) = \overline{< f|\overline{\varphi} >}$. As the last equality does not depend on f being a regular distribution, it provides the extension: The complex conjugate of a distribution f is the distribution

$$< f^*|\varphi >= \overline{< f|\overline{\varphi} >} \tag{25.18}$$

Example 25.5.1. *The Dirac's delta is real:* $\delta_a^* = \delta_a$.

Along the same line one defines the multiplication of a tempered distribution by a $\mathscr{C}^\infty$ function g that is algebraically bounded with all its derivatives[8]. It is the tempered distribution

$$< gf|\varphi >=< f|g\,\varphi > \tag{25.19}$$

25.5.1 Derivative

Consider a regular distribution with a function $f \in \mathscr{C}^1(\mathbb{R})$ such that f' still defines a regular distribution. Then:

$$< f'|\varphi >= \int_{-\infty}^{\infty} dx\, f'\varphi = -\int_{-\infty}^{\infty} dx\, f\varphi' = -< f|\varphi' > .$$

This evaluation suggests the definition of the derivative of *any* distribution:

Definition 25.5.2. The derivative of a distribution f is the distribution f' with action

$$\boxed{< f'|\varphi > = -< f|\varphi' >} \tag{25.20}$$

As derivation is a continuous operator on $\mathscr{S}(\mathbb{R})$, the derivative of tempered distributions gives tempered distributions, and is linear and continuous on $\mathscr{S}'(\mathbb{R})$. One can evaluate as many derivatives of a distribution as wanted.

Example 25.5.3. *Derivative of Heaviside's functional* θ_a.
By definition: $< \theta_a'|\varphi >= -< \theta_a|\varphi' >= -\int_a^\infty \varphi'(x)dx = \varphi(a)$. *Therefore:* $\theta_a' = \delta_a$, *or*

$$\boxed{\frac{d}{dx}\theta(x-a) = \delta(x-a)}$$

Example 25.5.4. *Derivative of* δ_a. *By definition:* $< \delta_a'|\varphi >= -\varphi'(a)$.

Exercise 25.5.5. *Show that:* $\frac{d}{dx}|x| = \mathrm{sign}x$, $< \frac{d}{dx}\delta_a|\varphi >= -\frac{d}{da} < \delta_a|\varphi >$.

[8] $g(x) = \cos(e^x)$ is bounded and smooth, but $g'(x)$ is unbounded

Example 25.5.6. *The derivative of* $F(x) = \theta(x^2 - m^2)$*,* $m > 0$*, is by definition:* $< F'|\varphi >= -\int_{-\infty}^{-m} dx\, \varphi'(x) - \int_m^\infty dx\, \varphi'(x) = -\varphi(-m) + \varphi(m) =< \delta_m - \delta_{-m}|\varphi >$*. As an identity among generalized functions one writes:*

$$\frac{d}{dx}\theta(x^2 - m^2) = \delta(x - m) - \delta(x + m).$$

Remark 25.5.7. *In ordinary calculus one would write:* $\frac{d}{dx}\theta(x^2 - m^2) = 2x\,\delta(x^2 - m^2)$*, leading to the conclusion*

$$\delta(x^2 - m^2) = \frac{1}{2x}[\delta(x - m) - \delta(x + m)] = \frac{1}{2m}[\delta(x - m) + \delta(x + m)]$$

This can be generalized; the evaluation of the derivative of $\theta(f(x))$*, where* f *is a smooth real function with isolated zeros* $x_1, \ldots, x_n$*, leads to the useful formula:*

$$\boxed{\delta(f(x)) = \sum_{k=1}^{n} \frac{\delta(x - x_k)}{|f'(x_k)|}} \tag{25.21}$$

Example 25.5.8. *It is instructive to evaluate the derivative of the regular distribution* $f(x) = \log|x|$*. By definition it is*

$$< f'|\varphi >= -\int_{\mathbb{R}} dx \log|x|\varphi'(x)$$

An integration by parts would give a non-integrable factor $1/x$*. We make progress by removing a neighborhood of* $x = 0$*, noting that the integral coincides with the following one:*

$$= -\lim_{\epsilon\to 0^+}\left[\int_{-\infty}^{-\epsilon} dx \log(-x)\varphi'(x) + \int_{\epsilon}^{\infty} dx \log x\, \varphi'(x)\right]$$
$$= \lim_{\epsilon\to 0^+} \log\epsilon[\varphi(\epsilon) - \varphi(-\epsilon)] + \lim_{\epsilon\to 0^+}\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} dx\frac{1}{x}\varphi(x) = ⨍_{-\infty}^{+\infty} dx\frac{1}{x}\varphi(x)$$

The lesson is that the introduction of an appropriate ϵ *may surmount difficulties. Here we can identify* f' *with* $P\frac{1}{x}$*.*

We state but not prove the following characterisation of tempered distributions:

Theorem 25.5.9. *Every tempered distribution is the distributional derivative of finite order of a continuous algebraically bounded function:* $f \in \mathscr{S}'(\mathbb{R}) \Leftrightarrow$ *there is a continuous function* ξ *such that* $|\xi(x)| \le C(1 + |x|^n)$ *for some* C *and* n*, and* $k \ge 0$ *such that*

$$< f|\varphi >=< \xi^{(k)}|\varphi >= (-1)^k \int_{-\infty}^{\infty} dx\, \xi(x)\, \varphi^{(k)}(x).$$

25.5.2 Fourier transform

To extend the Fourier transform to distributions, start from regular functionals. Suppose that both f and $\mathscr{F}f$ are functions that produce regular distributions (for example f is in $\mathscr{S}$). Then:

$$< \mathscr{F} f|\varphi >= \int_{\mathbb{R}} dx (\mathscr{F} f)(x)\varphi(x) = \int_{\mathbb{R}} dx\, \varphi(x) \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{-ikx} f(k)$$
$$= \int_{\mathbb{R}} dk\, f(k) \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\varphi(x) =< f|\mathscr{F}\varphi >$$

The extension to all distributions is:

Definition 25.5.10. The Fourier transform of a tempered distribution f is the distribution $\mathscr{F} f$ with action

$$\boxed{< \mathscr{F} f|\varphi >=< f|\mathscr{F}\varphi >} \tag{25.22}$$

Proposition 25.5.11. *The Fourier transform $\mathscr{F} : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is invertible ($\mathscr{F}\mathscr{F}^{-1} f = f$) and continuous.*

Proof. Continuity and the inversion property are exported from the same properties in $\mathscr{S}(\mathbb{R})$: $< \mathscr{F}^{-1}\mathscr{F} f|\varphi >=< \mathscr{F} f|\mathscr{F}^{-1}\varphi >=< f|\mathscr{F}\mathscr{F}^{-1}\varphi >=< f|\varphi >$.
Since the Fourier transform is linear, it is enough to check continuity in the origin: let $f_n \to 0$ in $\mathscr{S}'(\mathbb{R})$ then $< \mathscr{F} f_n|\varphi >=< f_n|\mathscr{F}\varphi >\to 0$ i.e. $\mathscr{F} f_n \to 0$. □

Example 25.5.12. *Fourier transform of δ_a.*

$$< \mathscr{F}\delta_a|\varphi >=< \delta_a|\mathscr{F}\varphi >= (\mathscr{F}\varphi)(a) = \int_{\mathbb{R}} dx \frac{e^{-iax}}{\sqrt{2\pi}}\varphi(x).$$

The Fourier transform of Dirac's delta is the regular distribution with function:

$$\boxed{(\mathscr{F}\delta_a)(x) = \frac{e^{-iax}}{\sqrt{2\pi}}} \tag{25.23}$$

In particular: $1 = \sqrt{2\pi}\,\mathscr{F}\delta_0\,(= \sqrt{2\pi}\,\mathscr{F}^{-1}\delta_0)$.
The same result is obtained by exploiting the continuity of the Fourier transform, acting on a regular approximation of δ_a. For example:

$$u_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{(x-a)^2+\epsilon^2}, \qquad (\mathscr{F} u_\epsilon)(x) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon|x|-ixa} \tag{25.24}$$

As a family of regular distributions, for $\epsilon \to 0$ the sequence u_ϵ converges to δ_a. The sequence of Fourier transforms also converges in $\mathscr{S}'$ and the limit is (25.23).

Example 25.5.13. *The following approximation of the delta function is useful:*

$$\boxed{\lim_{N\to\infty} \frac{\sin(Nx)}{\pi x} = \delta_0} \tag{25.25}$$

A simple proof based on continuity of the Fourier transform: since $\chi_{[-N,N]} \to 1$ in $\mathscr{S}'(\mathbb{R})$, then $\mathscr{F}\chi_{[-N,N]} \to \sqrt{2\pi}\delta_0$.

Example 25.5.14.

$$\mathscr{F} x^n = i^n \sqrt{2\pi}\, \delta_0^{(n)} \tag{25.26}$$

$< \mathscr{F} x^n|\varphi > = < 1|x^n \mathscr{F}\varphi > = (-i)^n < 1|\mathscr{F}\varphi^{(n)} >$. *Since* $1 = \sqrt{2\pi}\mathscr{F}\delta_0$, *it is:* $< \mathscr{F} x^n|\varphi > = (-i)^n\sqrt{2\pi} < \delta_0|\varphi^{(n)} > = i^n\sqrt{2\pi} < \delta_0^{(n)}|\varphi >$

Example 25.5.15. *Fourier transform of* $P\,\frac{1}{x-a}$.

$$< \mathscr{F}\frac{P}{x-a}|\varphi > = \fint \frac{dx}{x-a}\int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} e^{-ikx}\varphi(k) = \lim_{R\to\infty}\int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}}\varphi(k) e^{-ika} \fint_{-R}^{+R}\frac{dx}{x}e^{-ikx}.$$

The interval $[-R, R]$ *is specified to exchange the integrals. The principal-valued integral is evaluated in* $\mathbb{C}$ *with appropriate contour:*

$$\fint_{-R}^{+R}\frac{dx}{x}e^{-ikx} = \text{sign}(k)\left[-i\pi + i\int_0^\pi d\theta e^{i|k|Re^{i\theta}}\right].$$

The second term, coming from the semi-circle, is bounded in modulus and yields an integral with $\varphi(k)$ *that vanishes for* $R \to \infty$. *Then:*

$$(\mathscr{F} P\frac{1}{x-a})(k) = -i\sqrt{\frac{\pi}{2}}\,\text{sign}(k) e^{-ika} \tag{25.27}$$

Example 25.5.16. *Fourier transform of* θ_a.

$$< \mathscr{F}\theta_a|\varphi > = < \theta_a|\mathscr{F}\varphi > = \int_a^\infty dx \int_{\mathbb{R}}\frac{dy}{\sqrt{2\pi}}e^{-ixy}\varphi(y)$$

Since integrals cannot be exchanged, introduce a convergence factor and consider the Fourier transform of the distributions $\theta_{a,\epsilon}(x) = e^{-\epsilon x}\theta_a(x)$ $(\epsilon > 0)$. *As* $\theta_{a,\epsilon} \to \theta_a$ *and the Fourier transform is continuous in* $\mathscr{S}'$, *it is*

$$\begin{aligned} < \mathscr{F}\theta_a|\varphi > &= \lim_{\epsilon\to 0^+}\int_a^\infty dx\, e^{-\epsilon x}\int_{\mathbb{R}}\frac{dy}{\sqrt{2\pi}}e^{-ixy}\varphi(y) \\ &= \int_{\mathbb{R}} dy\,\varphi(y)\int_a^\infty \frac{dx}{\sqrt{2\pi}}e^{-ixy-\epsilon x} = \int_{\mathbb{R}} dy \frac{1}{i\sqrt{2\pi}}\frac{e^{-iay}}{y-i\epsilon}\varphi(y) \end{aligned}$$

In the last line it is understood that the limit $\epsilon \to 0^+$ *is taken. The limit can be done* <u>*after*</u> *the action on a test function is evaluated (this is the meaning of convergence of distributions). We obtain* $\mathscr{F}\theta_a$ *as a limit of regular distributions:*

$$(\mathscr{F}\theta_a)(x) = \frac{1}{i\sqrt{2\pi}}\frac{e^{-iax}}{x-i\epsilon}. \tag{25.28}$$

Inversion gives a useful representation of Heaviside's function:

$$\boxed{\theta(x-a) = \int_{-\infty}^{\infty}\frac{dk}{2\pi i}\frac{e^{ik(x-a)}}{k-i\epsilon} = i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{-ik(x-a)}}{k+i\epsilon}} \tag{25.29}$$

Exercise 25.5.17 (Fourier transform of the sign-function)**.** *Show that*

$$(\mathscr{F}\text{sign})(x) = -i\sqrt{\frac{2}{\pi}}\,\mathrm{P}\frac{1}{x} \tag{25.30}$$

Exercise 25.5.18. *Evaluate the Fourier transform of the following generalized functions: 1)* $(x-a)$*; 2)* $\log|x|$*; 3)* $\exp(-i\omega x)$*,* $\omega \in \mathbb{R}$*.*

Exercise 25.5.19. *What is the Plemelj-Sokhotski identity after Fourier transform?*

25.6 Fourier series and distributions

Definition 25.6.1. A tempered distribution f is periodic with period τ if $< f|U_\tau \varphi >=< f|\varphi >$, where $(U_\tau \varphi)(x) = \varphi(x-\tau)$.

We state a generalization of Fourier series (but omit the proof), that is well defined as a distribution although it may not converge as an ordinary function:

Theorem 25.6.2. *f is a $\tau-$periodic tempered distribution if and only if there are constants c_n with $|c_n| < C(1+|n|^k)$ for some k, such that*

$$f = \sum_{n\in\mathbb{Z}} c_n \exp\left(i\frac{2\pi n}{\tau}x\right), \quad \text{i.e.} \quad < f|\varphi >= \sum_{n\in\mathbb{Z}} c_n \int_{\mathbb{R}} dx\, \varphi(x) \exp\left(i\frac{2\pi n}{\tau}x\right)$$

or, equivalently:

$$\mathscr{F} f = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} c_n \delta_{2\pi n/\tau}, \quad \text{i.e.} \quad < \mathscr{F} f|\varphi >= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} c_n \varphi\left(\frac{2\pi n}{\tau}\right)$$

Example 25.6.3. *Consider the periodic distribution $\Delta_x = \sum_{n\in\mathbb{Z}} \delta_{x+n\tau}$, with action $< \Delta_x|\varphi >= \sum_n \varphi(x+n\tau)$. The series is well defined because φ is in $\mathscr{S}(\mathbb{R})$, and defines a periodic function $g(x)$ with period τ. Then, g can be expanded in Fourier series: $g(x) = \sum_n g_n e^{i(2\pi/\tau)nx}$.*

25.7 Linear operators on distributions

We'll prove that the Hermite functions are a complete orthonormal system in $L^2(\mathbb{R})$, therefore the Schwartz space $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, i.e. any square-integrable function is the limit of a L^2norm-convergent sequence of rapidly decreasing functions.
By Riesz's theorem $L^2(\mathbb{R})$ is self-dual, i.e. each function f can be viewed as a linear sequentially continuous functional on $L^2(\mathbb{R})$ and thus as a tempered distribution (up to conjugation):

$$| < f|\varphi > |^2 = |(\bar{f}|\varphi)|^2 \le \|f\|_2^2 \|\varphi\|_2^2 \le \|f\|_2^2\, \pi \|\varphi\|_{00}(\|\varphi\|_{00} + \|\varphi\|_{20})$$

for all test functions. Therefore we have the relation (Gel'fand triplet):

$$\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}) \tag{25.31}$$

The structure is useful for extending operators with domain $\mathscr{S}(\mathbb{R})$ to operators on $\mathscr{S}'(\mathbb{R})$, and give meaning to expansions of L^2 functions in terms of generalized functions.

Suppose that the operator $A : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is linear and continuous (convergent sequences are mapped to convergent sequences, in the seminorm topology). If f is a tempered distribution, the functional $\varphi \to < f|A\varphi >$ is a tempered distribution. Therefore, the operator A induces an "adjoint" operator on tempered distributions: $A' : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$

$$\boxed{< A'f|\varphi >=< f|A\varphi >, \qquad f \in \mathscr{S}'(\mathbb{R}),\, \varphi \in \mathscr{S}(\mathbb{R})} \tag{25.32}$$

Being A densely defined in $\mathrm{L}^2(\mathbb{R})$, the adjoint $A^\dagger$ exists: $(A^\dagger f|\,\varphi) = (f|A\varphi)$, $\forall f \in \mathscr{D}(A^\dagger)$, $\forall \varphi \in \mathscr{S}(\mathbb{R})$. By viewing $A^\dagger f$ and f as elements of the dual space $\mathrm{L}^2(\mathbb{R})^*$, and thus as distributions, we rewrite it as $< (A^\dagger f)^*|\varphi >=< f^*|A\varphi >$, i.e. $< (A^\dagger f^*)^*|\varphi >=< f|A\varphi >$. Therefore, for distributions that are functions $f \in \mathscr{D}(A^\dagger)$ it is

$$\boxed{A'f = (A^\dagger f^*)^*} \tag{25.33}$$

With this rule, A' is an extension of $A^\dagger$ to $\mathscr{S}'$.

25.7.1 Generalized eigenvectors

The eigenvalue equation $A'f = \lambda f$ has solution in $\mathscr{S}'$ if

$$\exists\, f_\lambda \quad \text{such that} \quad < f_\lambda|A\varphi >= \lambda < f_\lambda|\varphi > \quad \forall \varphi \in \mathscr{S}(\mathbb{R}). \tag{25.34}$$

f_λ is a "generalized eigenvector".

Example 25.7.1. *The operator $(Q_0\varphi)(x) = x\varphi(x)$ leaves $\mathscr{S}$ invariant and is continuous; the corresponding operator Q_0' acts on distributions as multiplication by the algebraically bounded function x. The eigenvalue equation $Q_0' f_\lambda = \lambda f_\lambda$, i.e. $< f_\lambda|Q_0\varphi >= \lambda < f_\lambda|\varphi >$ $\forall\varphi$ has solution for any real λ: $f_\lambda = \delta_\lambda$.*
Note the "completeness" property with respect to the inner product of L^2:

$$(\varphi_1|\varphi_2) = \int_{\mathbb{R}} d\lambda\, \overline{< \delta_\lambda|\varphi_1 >} < \delta_\lambda|\varphi_2 > .$$

Example 25.7.2. *The operator $P_0\varphi = -i\varphi'$ leaves $\mathscr{S}$ invariant and is continuous. The operator P_0' on distributions is $< P_0' f|\varphi >=< f| - i\varphi' >= i < f'|\varphi >$, i.e. $P_0' f = if'$. The eigenvalue equation $P_0' e_\lambda = \lambda e_\lambda$ is solved by the ordinary functions $e_\lambda(x) = e^{-i\lambda x}/\sqrt{2\pi}$, $\lambda \in \mathbb{R}$ (Im $\lambda \neq 0$ gives exponentially divergent functions, which do not give regular distributions). The normalization is such that $< e_\lambda|\varphi >= (\mathscr{F}\varphi)(\lambda)$, which implies the completeness property:*

$$(\varphi_1|\varphi_2) = \int_{\mathbb{R}} d\lambda\, \overline{< e_\lambda|\varphi_1 >} < e_\lambda|\varphi_2 > .$$

Chapter 26

Green functions

In physics one often deals with inhomogeneous linear differential equations. Green functions are an important tool for solving them. Let us begin with examples.

26.1 The Yukawa equation

The inhomogeneous Yukawa equation $(\nabla^2_{\mathbf{x}} - m^2)\varphi(\mathbf{x}) = -4\pi\rho(\mathbf{x})$ contains a local linear operator and a source ρ. For $m = 0$ it is the Poisson equation for the electrostatic potential generated by a charge distribution.

The standard approach to the inhomogeneous equation is to exploit linearity and begin by solving the equation with a point-source[1]

$$(\nabla^2_{\mathbf{x}} - m^2)G(\mathbf{x}, \mathbf{y}) = -4\pi\delta(\mathbf{x} - \mathbf{y})$$

This is the *fundamental equation*, and a solution is a *Green function* of the operator. Evidently, this equation is meaningful in the space of distributions. Two solutions differ by a solution of the homogeneous equation.
With the Green function, a particular solution of the inhomogeneous problem is

$$\varphi(\mathbf{x}) = \int d\mathbf{y}\, G(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})$$

Since the operator is local and the delta-source is translation-invariant, the Green function depends on $\mathbf{x} - \mathbf{y}$, and can be found via Fourier transform. With the conventions of physicists for functions of space coordinates:

$$G(\mathbf{x} - \mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{k})$$
$$\delta(\mathbf{x} - \mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

In $\mathbf{k}-$space the equation is algebraic: $-(k^2 + m^2)G(\mathbf{k}) = -4\pi$. Back to coordi-

[1] $\delta(\mathbf{x}-\mathbf{y}) = \delta(x_1 - y_1)\delta(x_2 - y_2)\delta(x_3 - y_3)$, where x_i and y_i are the Cartesian components.

nates $\mathbf{r} = \mathbf{x} - \mathbf{y}$:

$$\begin{aligned} G(\mathbf{r}) &= \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{4\pi}{k^2+m^2} e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= 2\pi \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \frac{4\pi}{k^2+m^2} \int_0^\pi \sin\theta d\theta\, e^{ikr\cos\theta} \\ &= \int_0^\infty \frac{k^2 dk}{\pi} \frac{1}{k^2+m^2} \frac{e^{ikr}-e^{-ikr}}{ikr} = \frac{1}{r}\int_{-\infty}^\infty \frac{kdk}{i\pi} \frac{e^{ikr}}{k^2+m^2} \end{aligned}$$

with simple poles $k = \pm im$. Since $r > 0$ the path is closed in the upper half-plane, and

$$\boxed{G(r) = \frac{\exp(-mr)}{r}}$$

This is the Green function of the Yukawa operator (and the static Yukawa potential for a massive boson). It is non-unique, as we may add a solution of the homogeneous equation. It is the one that decays at infinity.
With $m = 0$ it is the familiar Coulomb potential (if $m = 0$ from the start, the poles would have been on the real axis but, in the sense of distributions, one may freely modify k^2 to $k^2 + \epsilon^2$, with $\epsilon = 0$ in the end).
The solution of the inhomogeneous problem is:

$$\varphi(\mathbf{x}) = \int d\mathbf{y} \frac{\exp(-m|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y}) \tag{26.1}$$

26.2 The forced undamped oscillator

$$\ddot{x}(t) + \Omega^2 x(t) = F(t)$$

The general solution is the sum of a solution of the homogeneous equation, $x_0(t) = A\cos(\Omega t) + B\sin(\Omega t)$, and a particular solution $x_P(t)$. The latter is obtained via a Green function, $x_P(t) = \int ds\, G(t,s)F(s)$ that solves

$$\frac{d^2}{dt^2}G(t,s) + \Omega^2 G(t,s) = \delta(t-s)$$

Here the Green function is a function of $t - s$. In Fourier space the variable conjugated to time is ω and the following sign convention is used:

$$G(t-s) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} G(\omega) e^{-i\omega(t-s)}$$

The equation for $G(\omega)$ is algebraic: $(-\omega^2 + \Omega^2)G(\omega) = 1$. Then:

$$G(t-s) = -\int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-s)}}{\omega^2-\Omega^2}$$

The poles $\pm\Omega$ are real. They have to be pushed off the real axis by adding imaginary parts $\pm i\epsilon$. Each sign combination gives a different Green function. They differ by solutions of the homogeneous equation.

The choice with all poles in the lower half-plane (i.e. the poles are $\pm\Omega - i\epsilon$) defines the **retarded Green function** (the reason will become clear):

$$G^R(t-s) = \int_{\mathbb{R}} \frac{d\omega}{4\pi\Omega} e^{-i\omega(t-s)} \left[\frac{1}{\omega+\Omega+i\epsilon} - \frac{1}{\omega-\Omega+i\epsilon}\right]$$

If $t < s$ the path is closed in Im $\omega > 0$ and the integral is zero. If $t > s$ the path encloses both poles:

$$\begin{aligned} G^R(t-s) &= \theta(t-s)\,\frac{-2\pi i}{4\pi\Omega}[e^{i\Omega(t-s)} - e^{-i\Omega(t-s)}] \\ &= \theta(t-s)\,\frac{1}{\Omega}\sin[\Omega(t-s)] \end{aligned}$$

The particular solution

$$x_P(t) = \frac{1}{\Omega}\int_{-\infty}^{t} ds\, \sin[\Omega(t-s)]\, F(s)$$

has the property of causality: its value at time t only depends on the forcing field at earlier times. This makes the retarded Green function of special importance in physics.

For example, if $F(t) = 0$ for $t < 0$ and $F(t) = F_0 \sin\omega t$ for $t > 0$ it is

$$x_p(t) = \frac{F_0}{\Omega^2 - \omega^2}[\sin(\omega t) - \frac{\omega}{\Omega}\sin(\Omega t)]$$

The motion is a superposition of oscillations with forcing frequency ω and natural frequency Ω. If $\omega = (1+\epsilon)\Omega$, for $\epsilon \to 0$ the expression becomes a resonance (an amplitude grows linearly in time):

$$x_p(t) = \frac{F_0}{2\Omega^2}\sin(\Omega t) - \frac{F_0}{2\Omega} t\cos(\Omega t).$$

26.3 Wave equation with source

The wave equation with source ρ is

$$\Box\varphi(x) = -\rho(x) \tag{26.2}$$

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{26.3}$$

where $x = (\mathbf{x}, ct)$, and $\Box$ is the d'Alembertian operator (or the wave operator). The Green function (or "fundamental solution") of the wave operator is the distribution that solves

$$\Box_x G(x, x') = -\delta_4(x - x') \tag{26.4}$$

Then, $\varphi(x) = \varphi_0(x) + \varphi_P(x)$, where φ_0 solves the homogeneous equation, $\Box_x\varphi_0 = 0$ and $\varphi_P(x) = \int d^4x'\, G(x, x')\rho(x')$ is a particular solution.

The Green function is not unique. Its determination can be dictated by physics. For example *causality* requires that the field φ_P at time t cannot depend on values of the source ρ at later times $t' > t$ (actually, because of the finite wave speed

c, the effect is further delayed). Being the equation (26.4) translation-invariant, we'll solve it by the Fourier expansion

$$G(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')}\, G(k)$$

where $k = (\mathbf{k}, \omega/c)$, $k\cdot x = \mathbf{k}\cdot\mathbf{x} - \omega t$. In Fourier space eq.(26.4) is algebraic: $-(k\cdot k)\, G(k) = -1$ i.e. $(|\mathbf{k}|^2 - \omega^2/c^2) G(\mathbf{k},\omega) = 1$. Therefore

$$G(x,x') = \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi c} e^{-i\omega(t-t')} \frac{(-c^2)}{\omega^2 - |\mathbf{k}|^2 c^2}$$

A general feature occurs: *the integral is singular* with poles at $\omega = \pm|\mathbf{k}|c$. This is no problem: we are dealing with distributions, and small epsilons may be introduced to shift poles off the real axis. As this can be done in different ways, there are different Green functions.
Causality determines one choice of the signs. If $t' > t$ the integration path closes in the upper half-plane (this ensures exponential decay on the semicircle); if we require that $G(x,x') = 0$ for $t' > t$, no poles must be encircled, i.e. the two poles are in the lower half plane. This choice defines the *retarded* Green function:

$$\begin{aligned} G^R(x,x') &= \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{(-c)}{(\omega+i\epsilon)^2 - |\mathbf{k}|^2c^2} \\ &= \theta(t-t') \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{-c}{2|\mathbf{k}|c} \left[e^{-i|\mathbf{k}|c(t-t')} - e^{i|\mathbf{k}|c(t-t')}\right] \\ &= \theta(t-t') \int_0^{\infty} \frac{k^2 dk}{(2\pi)^2} \frac{-1}{2k} \left[e^{-ikc(t-t')} - e^{ikc(t-t')}\right] \int_{-\pi}^{\pi} \sin\theta\, d\theta e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \\ &= \theta(t-t') \int_0^{\infty} \frac{dk}{8\pi^2} \left[e^{-ikc(t-t')} - e^{ikc(t-t')}\right] \left[e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}\right] \end{aligned}$$

A change $k \to -k$ gives the sum of two delta functions, one of which is identically zero, and a factor 2π. Then:

$$\boxed{G^R(x,x') = \theta(t-t') \frac{\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t'))}{4\pi|\mathbf{x}-\mathbf{x}'|}}$$

The Green function has support on the spherical surface centred in $\mathbf{x}'$ with radius $c(t-t') > 0$, increasing with t. The causal field with source $\rho(\mathbf{x}', t')$ is a continuous superposition of such spherical waves being emitted at every point of the source:

$$\begin{aligned} \varphi_P(\mathbf{x},t) &= \int d\mathbf{x}' \int_{-\infty}^{t} dt' \frac{\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t'))}{4\pi|\mathbf{x}-\mathbf{x}'|} \rho(\mathbf{x}', t') \\ &= \int d\mathbf{x}' \frac{\rho(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|)}{4\pi c\, |\mathbf{x}-\mathbf{x}'|} \end{aligned} \tag{26.5}$$

26.3.1 Green functions as distributions.

Let us consider the problem of inverting $\hat{A}f = g$ naively.
In the basis of position, the equation $\hat{A}\hat{A}^{-1} = 1$ is $\int dx' \langle x|\hat{A}|x'\rangle\langle x'|\hat{A}^{-1}|y\rangle =$

$\delta(x-y)$. If $\hat{A}$ is diagonal, namely if $\langle x|\hat{A}|x'\rangle = \delta(x-x')A_x$ where A_x acts with derivates and multiplication by functions of x, it is:

$$A_x G(x,y) = \delta(x-y)$$

where $G(x,y) = \langle x|\hat{A}^{-1}|y\rangle$ is the Green function of the local operator A_x. The problem $(\hat{A}f)(x) = g(x)$ has general solution:

$$f(x) = f_0(x) + \int dy\, G(x,y)g(y)$$

where f_0 solves the homogeneous problem $\hat{A}f_0 = 0$.

Now let's be more formal. Let $A : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ be a linear continuous operator. A Green function G_s of A, if it exists, is a distribution that solves the equation $< G_s|A\varphi >=< A'G_s|\varphi >= \varphi(s)$, for any test function i.e.

$$A'G_s = \delta_s, \quad s \in \mathbb{R}$$

In terms of generalized functions it is $\int dx\, G_s(x)(A\varphi)(x) = \varphi(s)$. The inhomogeneous equation $\hat{A}\varphi = \psi$ has the particular solution

$$\boxed{\varphi(s) = < G_s|\psi >= \int dx\, G_s(x)\,\psi(x)}$$

Proof. $\varphi(s) =< \delta_s|\varphi >=< A'G_s|\varphi >=< G_s|\hat{A}\varphi >=< G_s|\psi >$. □

A second Green function would produce a particular solution that differs by a solution of the homogeneous equation $A\varphi = 0$.

Example 26.3.1. *The Green functions of* $(P_0\varphi)(t) = i\varphi'(t)$ *solve* $-iG_t' = \delta_t$*:*

$$-i\int ds \frac{d}{ds}G_t(s)\varphi(s) = \varphi(t)$$

A solution is $G_t = i\theta_t$ *i.e.* $G_t(s) = i\theta(s-t)$*; it gives the particular solution for* $P_0\varphi = \psi$*:*

$$\varphi(t) =< G_t|\psi >= i\int_{\mathbb{R}} ds\,\theta(s-t)\psi(s) = i\int_t^\infty ds\,\psi(s)$$

The Green function is named "advanced" (the solution is built with the source at later *times). The Green function* $G_t(s) = -i\theta(t-s)$ *yields a retarded solution:*

$$\varphi(t) = -i\int_{\mathbb{R}} ds\,\theta(t-s)\psi(s) = -i\int_{-\infty}^t ds\,\psi(s)$$

The difference of the two solutions is a constant (solution of the homogeneous equation).

Chapter 27

FOURIER TRANSFORM II

The Fourier transform $\mathscr{F}$ and antitransform $\mathscr{F}^{-1}$ were studied in detail in $\mathscr{S}(\mathbb{R})$ and extended to the dual space. The same properties generalize to $\mathscr{S}(\mathbb{R}^n)$ and the dual, with

$$(\mathscr{F}u)(\mathbf{q}) = \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^{n/2}} e^{-i\mathbf{q}\cdot\mathbf{x}} u(\mathbf{x}) \tag{27.1}$$

$$(\mathscr{F}^{-1}u)(\mathbf{q}) = \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^{n/2}} e^{i\mathbf{q}\cdot\mathbf{x}} u(\mathbf{x}) \tag{27.2}$$

It is of great interest to investigate the properties of $\mathscr{F}$ as operators on the spaces $\mathrm{L}^p(\mathbb{R}^n)$. Here we restrict to one dimension ($n = 1$). Most of the properties continue to hold in higher dimension.

27.1 Fourier transform in $\mathrm{L}^1(\mathbb{R})$

By the inequality $|\mathscr{F}u(k)| \le \int_{\mathbb{R}} dx\, |u(x)|$ the Fourier transform is well defined on the whole space $\mathscr{L}^1(\mathbb{R})$, and $|\mathscr{F}u(k)| \le \frac{1}{\sqrt{2\pi}}\|u\|_1$ for all k. Therefore $\mathscr{F}$ is a linear operator from $\mathrm{L}^1(\mathbb{R})$ to $\mathrm{L}^\infty(\mathbb{R})$, and

$$\|\mathscr{F}u\|_\infty \le \frac{1}{\sqrt{2\pi}}\|u\|_1$$

The operator is *continuous*: if $u_n \to 0$ in $\mathrm{L}^1(\mathbb{R})$ then $\mathscr{F}u_n \to 0$ uniformly, i.e. $\mathscr{F}u_n \to 0$ in $\mathrm{L}^\infty(\mathbb{R})$.

Exercise 27.1.1. *What is the condition for an integrable real function to have a real Fourier transform?*

Example 27.1.2. *The Fourier transform of a characteristic function*

$$(\mathscr{F}\chi_{[a,b]})(k) = \int_a^b \frac{dx}{\sqrt{2\pi}} e^{-ik\,x} = \sqrt{\frac{2}{\pi}}\,\frac{\sin[\frac{b-a}{2}k]}{k}\, e^{-i\frac{b+a}{2}\,k}$$

is a continuous bounded function, i.e. $\in \mathscr{C}(\mathbb{R})$, *and vanishes for* $|k| \to \infty$.

The same properties hold true for ladder functions σ, which are finite linear combinations of χ functions, and are a dense subset of $L^1(\mathbb{R})$. This anticipates the fundamental theorem:

Theorem 27.1.3 (Riemann - Lebesgue). *If $f \in \mathscr{L}^1(\mathbb{R})$ then $\mathscr{F}f$ is bounded, continuous, and*

$$\boxed{\lim_{|k|\to\infty} |(\mathscr{F}f)(k)| = 0} \tag{27.3}$$

Proof. If σ_n is a sequence of ladder functions and $\sigma_n \to f$ in L^1, then $\mathscr{F}\sigma_n \to \mathscr{F}f$ uniformly. Since $\mathscr{F}\sigma_n$ are continuous, also $\mathscr{F}f$ is continuous[1].
$\forall\epsilon$ there is a ladder function σ such that $\|f-\sigma\|_1 < \epsilon$. Since $|\mathscr{F}\sigma|$ vanishes at infinity, there is R such that for all $|k| > R$ it is $|(\mathscr{F}\sigma)(k)| < \epsilon$. Then, for $|k| > R$ it is:

$$|(\mathscr{F}f)(k)| \le |\mathscr{F}(f-\sigma)(k)| + |(\mathscr{F}\sigma)(k)| \le \frac{1}{\sqrt{2\pi}}\|u-\sigma\|_1 + |(\mathscr{F}\sigma)(k)| < \epsilon$$

up to a constant. □

Remark 27.1.4. *The set where $\mathscr{F}\chi_{[a,b]}(x) \neq 0$ has infinite measure. This is true in general: if $f \in \mathscr{L}^1(\mathbb{R})$ and the Lebesgue measures of the sets where $|f| \neq 0$ and $|\mathscr{F}f| \neq 0$ are both finite, then $f = 0$ a.e. (M. Benedicks,* On Fourier transforms of functions supported on sets of finite Lebesgue measure*, Math. Anal. Appl.* **106** *(1985) 180–183). The theorem holds in any dimension.*

Since the Fourier transform is bounded, the following integrals exist and are equal (Fubini's theorem applies):

$$\int_{\mathbb{R}} dk\overline{(\mathscr{F}f)(k)}g(k) = \int_{\mathbb{R}} dk\overline{f(k)}(\mathscr{F}^{-1}g)(k) \tag{27.4}$$

Theorem 27.1.5 (Inversion)**.** *If $f \in \mathscr{L}^1(\mathbb{R}) \cap \mathscr{L}^\infty(\mathbb{R})$, then $\mathscr{F}\mathscr{F}^{-1}f = f$.*

Proposition 27.1.6 (Convolution product)**.** *The* **convolution product** *of $f, g \in \mathscr{L}^1(\mathbb{R})$*

$$(f*g)(x) = \int_{\mathbb{R}} dy\, f(x-y)g(y) \tag{27.5}$$

is a function in $\mathscr{L}^1(\mathbb{R})$, $\|f*g\|_1 \le \|f\|_1\|g\|_1$, and

$$\mathscr{F}(f*g) = \sqrt{2\pi}(\mathscr{F}f)(\mathscr{F}g) \tag{27.6}$$

The product $*$ is commutative, associative and distributive.

Proof. $\int_{\mathbb{R}} dx|(f*g)(x)| \le \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy|f(x-y)||g(y)| = \|f\|_1\|g\|_1$ after a shift $x' = x-y$. The Fourier transform of $f*g$ exists, and

$$\mathscr{F}(f*g)(k) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-ik(x-y)} \int_{\mathbb{R}} f(x-y)g(y)e^{iky} = \sqrt{2\pi}(\mathscr{F}f)(k)(\mathscr{F}g)(k)$$

□

[1] If f_n is a sequence of continuous functions that converges to f uniformly, then f is continuous. Proof: for any x it is $|f(x)-f(y)| \le |f(x)-f_n(x)| + |f(y)-f_n(y)| + |f_n(x)-f_n(y)| \le 3\epsilon$ for n large enough and $|y-x| < \delta$.

27.2 Fourier transform in $L^2(\mathbb{R})$

Hermite functions belong to $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. The following theorem has various interesting implications:

Theorem 27.2.1. *The Hermite functions h_m are an orthonormal and complete set in* $L^2(\mathbb{R})$.

Proof. From the generating function of Hermite polynomials (11.17) one obtains the generating function of Hermite functions:

$$h(x,z) = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} h_m(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}(x^2 - \sqrt{2}zx + z^2)}$$

The series converges in L^2 norm because the coefficients $z^m/\sqrt{m!}$ are a sequence in $\ell^2(\mathbb{C})$ (Parseval theorem). Suppose that there is $f \in L^2$ such that $(h_m|f) = 0$ for all m. This implies that $(h|f) = 0$ for all z. For $z = -ik/\sqrt{2}$, up to irrelevant factors, it is

$$0 = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 - ixk} f(x) = (\mathscr{F}g)(k), \qquad g(x) = e^{-\frac{1}{2}x^2} f(x)$$

The function g belongs to $\mathscr{L}^1(\mathbb{R})$ (Schwarz inequality) and its Fourier transform is zero. Then $g \in \mathrm{Ker}\mathscr{F}$, but $\mathrm{Ker}\,\mathscr{F} = \{0\}$ by the inversion theorem in L^1. Then $g = 0$ i.e. $f = 0$. □

Since $\mathscr{F}$ is isometric on $\mathscr{S}(\mathbb{R})$ ($\|\mathscr{F}\varphi\|_2 = \|\varphi\|_2$) and the Schwartz space is dense in L^2, the operator may be extended to a unitary operator $\hat{F}$ on $L^2(\mathbb{R})$. The explicit construction exploits the property of h_n to be eigenstates of $\hat{F}$ and to form an orthonormal complete set.
Consider the expansion $f = \sum_n (h_n|f)h_n$; then $f_N = \sum_{n\le N}(h_n|f)h_n$ is a Cauchy sequence of functions in $\mathscr{S}(\mathbb{R})$ that is norm-convergent to f. It is $\mathscr{F}f_N = \sum_{n\le N}(-i)^n u_n(u_n|f)$. This new sequence is again a Cauchy sequence. Its limit defines the **Fourier-Plancherel operator**

$$\boxed{\hat{F}f = \sum_{n=0}^{\infty} (-i)^n \, (h_n|f) \, h_n} \tag{27.7}$$

The inverse transform $\mathscr{F}^{-1}$ extends uniquely to the adjoint $\hat{F}^\dagger$.

Remark 27.2.2. *The Hermite functions are the eigenfunctions of the number operator $\hat{N} = \frac{1}{2}(\hat{P}^2 + \hat{Q}^2 - 1)$: $\hat{N}h_n = nh_n$. The operator is the generator in $L^2(\mathbb{R})$ of the one-parameter unitary group*

$$\hat{U}(\theta) = e^{-i\theta\hat{N}} = \sum_{n=0}^{\infty} e^{-in\theta}(h_n|\cdot)h_n$$

with domain $\mathscr{D}(\hat{N}) = \{f : \sum_{n=0}^{\infty} n^2|(h_n|f)|^2 < \infty\}$. The group describes the 2π-periodic evolution in time θ of the quantum oscillator. Both $\hat{N}$ and $\hat{U}(\theta)$ leave $\mathscr{S}(\mathbb{R})$ invariant. It is $\hat{U}(\frac{\pi}{2}) = \hat{F}$, $\hat{U}(-\frac{\pi}{2}) = \hat{F}^{-1} = \hat{F}^\dagger$. $\hat{U}(\pi)$ is the parity operator.

Another way to evaluate the Fourier transform of a function $f \in \mathscr{L}^2$ is to consider the functions $\chi_{[-R,R]} f \in \mathscr{L}^1(\mathbb{R})$. On such functions $\hat{F} = \mathscr{F}$ and, since $\hat{F}$ is continuous:

$$(\hat{F}f)(k) = \underset{R\to\infty}{\text{l.i.m.}} \int_{-R}^{R} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x)$$

where l.i.m. is "limit in the mean", i.e. in L^2 topology.

27.2.1 Completeness of the Fourier basis

The Fourier transform of a function in $\mathscr{S}(\mathbb{R})$ can be written as

$$(\mathscr{F}\varphi)(k) = \int_{\mathbb{R}} dx\, \overline{u_k(x)} \varphi(x) \; = < \overline{u}_k | \varphi >, \qquad u_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}},\; k \in \mathbb{R}$$

where $< \overline{u}_k | \;\; >$ is the action of a regular distribution. By the inversion theorem it is:

$$\boxed{\varphi(x) = \int_{\mathbb{R}} dk \;\, < \overline{u}_k | \varphi > \; u_k(x)} \tag{27.8}$$

Since Schwartz's space is dense in L^2, the relation shows the *completeness* of the continuous system of functions $\{u_k\}$. They are the eigenfunctions of the derivative operator on distributions, which extends the self-adjoint unbounded operator $\hat{P}$.

Proposition 27.2.3 (Vladimirov[2]).

$$\varphi \in \mathscr{S}(\mathbb{R}) \iff \sum_{n=0}^{\infty} n^{2k} |\varphi_n|^2 < \infty \quad \forall\, k \in \mathbb{N} \tag{27.9}$$

where $\varphi_n = (h_n|\varphi)$ *are the coefficients of the expansion in the Hermite basis.*

Similarly one proves the theorem: $f \in \mathscr{S}'(\mathbb{R})$ if and only if there are a positive constants C, p such that $| < f | u_n > | \le C(1+n)^p$, for all $n \in \mathbb{N}$.

[2] V. Vladimirov, *Le distribuzioni nella fisica matematica*, Mir (1981).

Chapter 28

THE LAPLACE TRANSFORM

28.1 The Laplace integral

The Fourier transform exists for functions that belong to $\mathscr{L}^1(\mathbb{R})$. This is very restrictive in applications, for it leaves out several important functions. However, a non-integrable function $f(x)$ multiplied by e^{-cx} may become integrable on $x \geq 0$ for suitable Re $c > 0$. At the same time, to avoid problems at the other end of integration, one requires $f(x) = 0$ for $x < 0$. Then, if f has the property

$$\int_0^\infty dx\, e^{-cx}\, |f(x)| < \infty \tag{28.1}$$

for a suitable value c, the Fourier integral of the function $\sqrt{2\pi}e^{-cx}f(x)$ for $x \geq 0$ and 0 for $x < 0$ exists, and is: $\int_0^\infty dx\, e^{-ikx}e^{-cx}f(x)$.
By setting $z = c + ik$, the integral defines the Laplace transform of f:

$$\boxed{(\mathscr{L}f)(z) = \int_0^\infty dx f(x) e^{-zx}} \tag{28.2}$$

The complex function is well defined for Re $z > c$, where c is some real number that depends on f, and is bounded by (28.1):

$$|(\mathscr{L}f)(z)| \leq \int_0^\infty dx\, e^{-x\mathrm{Re}z}|f(x)| \leq \int_0^\infty dx\, e^{-cx}|f(x)|.$$

Proposition 28.1.1.

$$\lim_{\mathrm{Re}\, z\to+\infty} |(\mathscr{L}f)(z)| = 0$$

Proof. For $x \geq 0$ and Re $z \to \infty$: $|e^{-zx}f(x)| \to 0$. Moreover:

$$|e^{-zx}f(x)| = e^{-x(\mathrm{Re}z-c)}e^{-cx}|f(x)| \leq e^{-cx}|f(x)| \in \mathscr{L}^1(\mathbb{R}^+)$$

By the dominated convergence theorem the limit Re$z \to \infty$ can be exchanged with integral of the Laplace transform. □

Lemma 28.1.2. *If* $|(\mathscr{L}f)(z)| < \infty$ *for* $\mathrm{Re}z > c + \epsilon$ *then* $|(\mathscr{L}x^n f)(z)| < \infty$ *for* $\mathrm{Re}z > c + \epsilon$.

Proof. $\left|\int_0^\infty dx\, e^{-zx} x^n f(x)\right| \le \sup_{x\ge 0}[x^n e^{-x(\mathrm{Re}z-c-\epsilon)}]\int_0^\infty dx\, e^{-(c+\epsilon)x}|f(x)| < \infty$ □

Theorem 28.1.3. $(\mathscr{L}f)(z)$ *is holomorphic in the half-plane Re* $z > c$ *and*

$$\frac{d}{dz}(\mathscr{L}f)(z) = -(\mathscr{L}xf)(z) \tag{28.3}$$

Proof. Let's show that the limit $h \to 0$ exists:

$$\begin{aligned}\left|\frac{(\mathscr{L}f)(z+h)-(\mathscr{L}f)(z)}{h} + (\mathscr{L}xf)(z)\right| &= \left|\int_0^\infty dx\, e^{-xz} f(x)\left[\frac{e^{-hx}-1}{h}+x\right]\right| \\ \le \int_0^\infty dx\, e^{-x\mathrm{Re}z}|f(x)|\left|\frac{e^{-hx}-1}{h}+x\right| &\le \frac{|h|}{2}\int_0^\infty dx\, e^{-xz}x^2|f(x)| \to 0\end{aligned}$$

where the following inequality is used

$$\left|\frac{e^{-hx}-1}{h}+x\right| = \left|\int_0^x dy(e^{-hy}-1)\right| \le \int_0^x dy|e^{-yh}-1| \le \tfrac{1}{2}|h|x^2$$

The last step is proven here: let $y > 0$, $h = h_1 + ih_2$, then: $|e^{-yh}-1| = |e^{-yh_1} - e^{iyh_2}| = [(e^{-yh_1}-1)^2 + 4e^{-yh_1}\sin^2(\tfrac{1}{2}yh_2)]^{1/2} \le \sqrt{(yh_1)^2+(yh_2)^2} = y|h|$ by means of Lagrange's formula and the bound $|\sin x/x| \le 1$. □

28.2 Properties

These properties are proven without difficulty:

$$\mathscr{L}(af+bg) = a\,\mathscr{L}f + b\,\mathscr{L}g \quad \text{(linearity)} \tag{28.4}$$

$$\overline{(\mathscr{L}f)(z)} = (\mathscr{L}\overline{f})(\overline{z}) \tag{28.5}$$

$$(\mathscr{L}f')(z) = -f(0) + z(\mathscr{L}f)(z) \tag{28.6}$$

$$(\mathscr{L}f'')(z) = -f'(0) - zf(0) + z^2(\mathscr{L}f)(z) \tag{28.7}$$

$$(\mathscr{L}\,xf)(z) = -(\mathscr{L}f)'(z) \tag{28.8}$$

$$(\mathscr{L}\,e^{-ax}f)(z) = (\mathscr{L}f)(z+a). \tag{28.9}$$

Some simple Laplace transforms:

$$(\mathscr{L}[e^{\alpha x}])(z) = \frac{1}{z-\alpha}, \qquad \mathrm{Re}\,z > \mathrm{Re}\,\alpha \tag{28.10}$$

$$(\mathscr{L}[\sin\omega x])(z) = \frac{\omega}{z^2+\omega^2}, \qquad \mathrm{Re}\,z > 0 \tag{28.11}$$

$$(\mathscr{L}[\cos\omega x])(z) = \frac{z}{z^2+\omega^2}, \qquad \mathrm{Re}\,z > 0 \tag{28.12}$$

Example 28.2.1.

$$(\mathscr{L}[x^{a-1}])(z) = \int_0^\infty dx\; e^{-zx}x^{a-1} = \frac{\Gamma(a)}{z^a}, \quad \mathrm{Re}\,a > 0,\ \mathrm{Re}\,z > 0 \tag{28.13}$$

Logs and powers can be included by replacing a *by* $a+\epsilon$*, and expanding in* ϵ*:*

$$x^{a-1+\epsilon} = x^{a-1}[1 + \epsilon \log x + \tfrac{1}{2}\epsilon^2 (\log x)^2 + \ldots]$$
$$\Gamma(a+\epsilon) = \Gamma(a)[1 + \epsilon\, \psi(a) + \tfrac{1}{2}\epsilon^2 \psi_2(a) + \ldots]$$

where $\psi(z)$ *is the digamma function etc. By equating equal powers of* ϵ *one obtains (*$\mathrm{Re}\, z > 0$*):*

$$(\mathscr{L}[x^{a-1}\log x])(z) = \frac{\Gamma(a)}{z^a}[\psi(a) - \mathrm{Log} z] \tag{28.14}$$
$$(\mathscr{L}[x^{a-1}\log^2 x])(z) = \frac{\Gamma(a)}{z^a}[\psi_2(a) - 2\psi(a)\mathrm{Log} z - \mathrm{Log}^2 z] \tag{28.15}$$

28.3 Inversion

The Laplace transform can be inverted. With the understanding that $f(x) = 0$ for $x < 0$, recall the correspondence:

$$(\mathscr{F}[\sqrt{2\pi} e^{-ax} f(x)])(k) = (\mathscr{L}f)(a+ik)$$

If, as a function of k, it is $(\mathscr{L}f)(a+ik) \in \mathscr{L}^1(\mathbb{R})$, the Fourier transform can be inverted:

$$\sqrt{2\pi} e^{-ax} f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} (\mathscr{L}f)(a+ik) e^{ikx}.$$

The formula transforms into: $f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{(a+ik)x} (\mathscr{L}f)(a+ik)$. Put $z = a+ik$, $dz = idk$ and the inversion formula is obtained:

$$\boxed{f(x) = \int_{a-i\infty}^{a+i\infty} \frac{dz}{2\pi i} e^{zx} (\mathscr{L}f)(z)} \tag{28.16}$$

The line of integration is parallel to the imaginary axis with $a > c$. The function $(\mathscr{L}f)(z)$ is analytic for Re $z > c$. If it is meromorphic for Re$z < c$, the computation of the integral can be done by the Residue Theorem, by closing the integration path by a semicircle in Re $z < c$ surrounding the poles (Bromwich's contour).

28.3.1 Hankel's representation of Γ

The antitransform of $(\mathscr{L}[x^{a-1}])(z)$ is

$$x^{a-1} = \Gamma(a) \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} e^{zx} z^{-a}$$

where $c > 0$ is arbitrary. For $x = 1$ Hankel's integral representation of the Gamma function is obtained:

$$\boxed{\frac{1}{\Gamma(a)} = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dz}{2\pi i} e^z z^{-a}} \tag{28.17}$$

The path of integration can be deformed to the loop shown in fig. 28.1. In

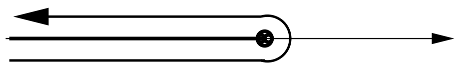

Figure 28.1: Hankel's loop for $1/\Gamma(a)$.

particular, if $a = n$ the singularity is a pole in the origin, the loop may be deformed to a circle, and the Residue Theorem gives $\Gamma(n) = (n-1)!$.

If $a \neq n$ the evaluation of the loop integral gives an important identity of the Gamma function. Let $z^a = e^{a\mathrm{Log}z}$; the loop runs along the cut of discontinuity of the Log: Log $(x \pm i\epsilon) = \log|x| \pm i\pi$ for $x < 0$. Then:

$$\frac{1}{\Gamma(a)} = \int_{loop} \frac{dz}{2\pi i} e^{z-a\mathrm{Log}z} = \int_{-\infty}^{0} \frac{dx}{2\pi i} \left[e^{x-i\epsilon-a\mathrm{Log}(x-i\epsilon)} - e^{x+i\epsilon-a\mathrm{Log}(x+i\epsilon)} \right]$$
$$= \frac{\sin(\pi a)}{\pi} \int_0^\infty dx\, e^{-x} x^{-a}$$

The identity is:

$$\boxed{\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}} \tag{28.18}$$

28.4 Convolution

The Fourier transform of the convolution of two functions is the product of their Fourier transforms. Because of the relation between Fourier and Laplace transforms, a similar property is expected for the Laplace transform.

Consider two functions of the type considered so far, $\sqrt{2\pi}\theta(x)e^{-ax}f(x)$ and $\sqrt{2\pi}\theta(x)e^{-ax}g(x)$, where the vanishing condition for $x < 0$ is written explicitly. Let's write their convolution product according to the theory of Fourier transform:

$$\int_{-\infty}^{\infty} dy[\sqrt{2\pi}\theta(y)f(y)e^{-ay}][\sqrt{2\pi}\theta(x-y)g(x-y)e^{-a(x-y)}]$$
$$= 2\pi e^{-ax} \int_0^x dy f(y)g(x-y)$$

This integral defines the convolution product of two Laplace-transformable functions:

$$(f * g)(x) = \int_0^x dy\, f(y)g(x-y) \tag{28.19}$$

Exercise 28.4.1. *Show that the convolution product is commutative.*

Theorem 28.4.2 (Convolution theorem)**.**

$$\boxed{\int_0^x dy\, f(y)g(x-y) = \int_{a-i\infty}^{a+i\infty} \frac{dz}{2\pi i} e^{zx} (\mathscr{L}f)(z)(\mathscr{L}g)(z)} \tag{28.20}$$

Proof. We use results from the theory of Fourier transform. By the convolution theorem for the Fourier transform, it is

$$\begin{aligned}
&2\pi e^{-ax}\int_0^x dy f(y)g(x-y)\\
&=\sqrt{2\pi}\mathscr{F}^{-1}\left(\mathscr{F}[\sqrt{2\pi}\theta(y)f(y)e^{-ay}]\mathscr{F}[\sqrt{2\pi}\theta(y)g(y)e^{-ay}]\right)(x)\\
&=\int_{-\infty}^{\infty} dk\, e^{ikx}(\mathscr{L}f)(a+ik)(\mathscr{L}g)(a+ik)
\end{aligned}$$

Therefore:

$$\int_0^x dy f(y)g(x-y)=\int_{\mathbb{R}}\frac{dk}{2\pi}e^{(a+ik)x}(\mathscr{L}f)(a+ik)(\mathscr{L}g)(a+ik)$$

Now put $z=a+ik$, $dz=idk$. □

Example 28.4.3. *Consider the inhomogeneous equation*

$$\begin{cases} f''(t)+\omega^2 f(t)=g(t)\\ f(0)=A, \quad f'(0)=B\end{cases}$$

The Laplace transform yields the algebraic equation $(z^2+\omega^2)(\mathscr{L}f)(z)-f'(0)-zf(0)=(\mathscr{L}g)(z)$*, with solution*

$$(\mathscr{L}f)(z)=\frac{(\mathscr{L}g)(z)+B+Az}{z^2+\omega^2}.$$

To obtain f *one inverts a Laplace transform:*

$$\begin{aligned}
f(t)=&\mathscr{L}^{-1}\left(\mathscr{L}[g]\frac{1}{\omega}\mathscr{L}[\sin\omega t]+\frac{B}{\omega}\mathscr{L}[\sin\omega t]+\frac{A}{\omega}\mathscr{L}[\cos\omega t]\right)\\
=&\int_0^t dt'\,\frac{\sin\omega(t-t')}{\omega}\,g(t')+f_{\text{hom}}(t),
\end{aligned}$$

where $f_{\text{hom}}(t)=\frac{B}{\omega}\sin\omega t+\frac{A}{\omega}\cos\omega t$ *solves the homogeneous equation. The particular solution exhibits the causality property: its value at time* t *is determined by the values of the forcing field at times* $t'<t$.

28.5 Mellin transform

In analogy with the construction of the Laplace transform, the Fourier identity

$$f(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikx}\int_{-\infty}^{\infty}dy\, e^{iky}f(y)$$

is formally modified to define the useful Mellin transform[1].
In the identity put $x=\log s$ and $y=\log t$:

$$f(\log s)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}s^{-ik}\int_0^{\infty}dt\, t^{ik-1}f(\log t)$$

[1]for a nice introduction see: `https://www.cs.purdue.edu/homes/spa/papers/chap9.ps`

Let $z = c + ik$, $dz = idk$: $s^{-c} f(\log s) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} s^{-z} \int_0^\infty dt\, t^{z-1} t^{-c} f(\log t)$; rename $f(\log s) s^{-c}$ as $f(s)$. Then: $f(s) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} s^{-z} \int_0^\infty dt\, t^{z-1} f(t)$.
The Mellin transform of a function is:

$$\boxed{(\mathscr{M} f)(z) = \int_0^\infty dx\, x^{z-1} f(x)} \tag{28.21}$$

It exists for $|(\mathscr{M} f)(z)| \leq \int_0^\infty dx |f(x)| x^{\mathrm{Re}z-1} < \infty$, which means that z is bounded in some strip of the complex plane. The inversion formula is

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} x^{-z} (\mathscr{M} f)(z) \tag{28.22}$$

The Mellin transform of e^{-x} is $\Gamma(z)$, and is defined for $\mathrm{Re}z > 0$.

Exercise 28.5.1. *Prove the properties:*

$$(\mathscr{M} x f)(z) = (\mathscr{M} f)(z+1) \tag{28.23}$$

$$(\mathscr{M} f')(z) = -(z-1)(\mathscr{M} f)(z-1) \tag{28.24}$$

$$(\mathscr{M} x^n D^n f)(z) = (-1)^n (z-n) \ldots (z-1)(\mathscr{M} f)(z) \tag{28.25}$$

$$\int_0^\infty dx |f(x)|^2 = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} |\mathscr{M} f(z)|^2 \quad \text{(Parseval identity)} \tag{28.26}$$

$$(\mathscr{M} fg)(z) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} (\mathscr{M} f)(z')(\mathscr{M} g)(z-z') \tag{28.27}$$

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