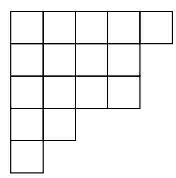
INTRODUCTION TO YOUNG TABLEAUX

In this note we present (without proofs) the rules necessary to construct irreducible representations (irreps) of the group $SU(N)$ i.e. the group of $N \times N$ complex unitary matrices ($UU^\dagger = 1$) with unit determinant ($det(U) = 1$).

- The complex multiplet ψ_i ($i = 1, .., N$) which belong to the fundamental representation of $SU(N)$ (i.e. the lower dimension non trivial representation, $\psi_i \to U_{ij}\psi_j$) is represented by a box:

  $\psi_i \equiv$  $\equiv \mathbf{N}$.

- A Young tableau is a diagram of *left-justified* rows of boxes where any row is *not longer* than the row on top of it, e.g.



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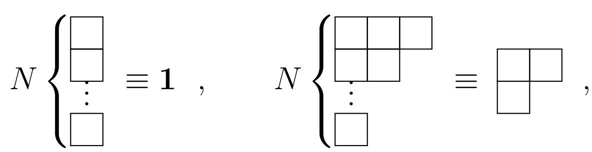
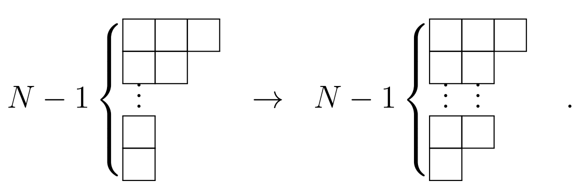
- Any column cannot contain more than N boxes.

- Any column with exactly N boxes can be crossed out since it correspond to the trivial representation (the singlet),

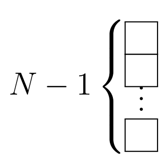


- Any Young tableau which respects the rules above correspond to an irrep of $SU(N)$ (and vice-versa).

- The complex conjugate of a given irrep is represented by a tableaux obtained by switching any column of k boxes with a column of $(N-k)$ boxes , e.g.



- From the previous rule: the complex conjugate multiplet $\overline{\psi}_i$ ($i = 1, .., N$) ($\overline{\psi}_i \to \overline{\psi}_j U^\dagger_{ji} = U^*_{ij}\overline{\psi}_j$) is represented by a column of $N-1$ boxes:

  $\overline{\psi}_i \equiv$  $\equiv \overline{\mathbf{N}}$.

- The dimension d of a Young tableau (i.e. the dimension of the associated irrep) can be obtained by the following ratio: $d = num/den$.

  - *Numerator:* start writing the number N in the top left box of the Young tableau. Moving to the right, write the number *increased* by a unit at each step. Moving to the bottom, write the number *decreased* by a unit at each step. The numerator is obtained by the product of the entries in each box. E.g.

$$\begin{array}{|c|c|c|}\hline N & N{+}1 & N{+}2 \\ \hline N{-}1 & N & \\ \cline{1-2} N{-}2 & N{-}1 & \\ \cline{1-2} N{-}3 & & \\ \cline{1-1} \end{array} \quad , \qquad num{=}N(N{+}1)(N{+}2)(N{-}1)N(N{-}2)(N{-}1)(N{-}3) \quad .$$

  - *Denominator:* write in each box the number of boxes being to its right plus the number of boxes being below it plus a unit (*the hook length*). The denominator is obtained by the product of the entries in each box. E.g.

$$\begin{array}{|c|c|c|}\hline 6 & 4 & 1 \\ \hline 4 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array} \quad , \qquad den{=}6{\times}4{\times}4{\times}2{\times}3 \quad .$$

- Any irrep of $SU(N)$ can be constructed starting from the fundamental irrep. The *direct product* of irreps can be decomposed in a *direct sum* of irreps with the following rules.

  - Write the two tableaux which correspond to the direct product of irreps and label successive rows of the second tableau with indices $a, b, c, \ldots$, e.g.

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & b & \\ \cline{1-2} c & & \\ \cline{1-1} \end{array}$$

  - Attach the boxes from the second to the first tableau, one a time following the order $a, b, c, \ldots$, in all the possible way. The resulting diagrams should be valid Young tableaux with no two (or more) a in the same column (neither b or c or $\ldots$).
  - Two generated tableaux with the same shape but labels *distributed differently* have to be kept. If two tableaux are *identical* only one has to be kept.
  - Counting the labels from the first row from *right to left*, then the second row (from right to left) and so on, at any given box position there should be no more b than a, more c than b and so on (if it is not the case discard the tableau). E.g. the tableau $\begin{array}{|c|c|c|}\hline & a & b \\ \hline & \\ \cline{1-1} \end{array}$ has to be discarded.

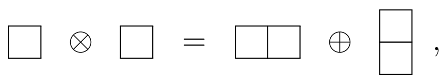
- The adjoint representation is the irrep with dimension equal to the dimension of the group (i.e. N^2-1) and can be constructed by a direct product of the fundamental representation and its complex conjugate:

$$\overline{\mathbf{N}}\otimes\mathbf{N} \equiv N-1\left\{\begin{array}{l}\Box\\ \Box\\ \vdots\\ \Box\end{array}\right. \otimes \Box = N-1\left\{\begin{array}{l}\Box\Box\\ \Box\\ \vdots\\ \Box\end{array}\right. \oplus N\left\{\begin{array}{l}\Box\\ \Box\\ \vdots\\ \Box\end{array}\right. = (\mathbf{N^2-1}) \oplus \mathbf{1}\,.$$

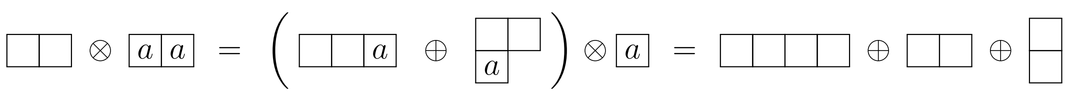
From the conjugation rule above it is clear that the adjoint representation is self conjugate $\overline{\mathbf{N^2-1}} = \mathbf{N^2-1}$

Examples

- SU(2) (any irrep is self conjugate)



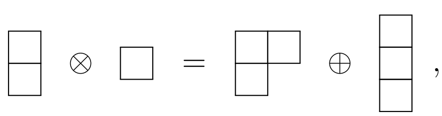
$$(\mathbf{2} \;\otimes\; \mathbf{2} \;=\; \mathbf{3} \;\otimes\; \mathbf{1}) \;;$$



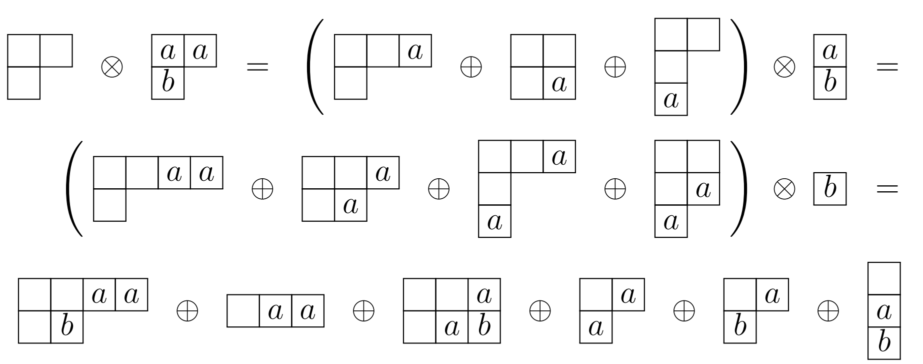
$$(\mathbf{3} \;\otimes\; \mathbf{3} \;=\; \mathbf{5} \;\oplus\; \mathbf{3} \;\oplus\; \mathbf{1}) \;;$$

$$(\mathbf{2\,j_1+1}) \;\otimes\; (\mathbf{2\,j_2+1}) \;=\; \oplus_{J=|j_1-j_2|}^{j_1+j_2}\; (\mathbf{2\,J+1}) \;;$$

- SU(3)



$$(\overline{\mathbf{3}} \;\otimes\; \mathbf{3} \;=\; \mathbf{8} \;\oplus\; \mathbf{1}) \;;$$



$$(\mathbf{8} \;\otimes\; \mathbf{8} \;=\; \mathbf{27} \;\oplus\; \mathbf{10} \;\oplus\; \overline{\mathbf{10}} \;\oplus\; \mathbf{8} \;\oplus\; \mathbf{8} \;\oplus\; \mathbf{1}) \;;$$