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Simplicity constraints in Spin Foam models for Quantum Gravity

Relatore interno: Prof. Sergio Caracciolo

Relatore esterno: Dr. Daniele Oriti

Marco Celoria
matr. 826117

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To my family

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Chapter 1

Motivations and scope of the thesis

One of the most fascinating aspect of physics is the possibility of addressing fundamental questions such as what is time? what is space? what is matter?

Tentative answers date back to the very beginning of physics itself, due to the works of Galileo and Newton, and, despite the great results of the XX century, a complete description of nature at high energies, particularly the Planck energy, is still missing. The first aspect that should be pointed out is that a good physical theory should predict the results of possible experiments (i.e. there is a notion of determinism). The second aspect is that a notion of relation between different observers would be required (i.e. there is a principle of relativity). Indeed Galileo realized these issues in the XVII century formalizing the scientific method and the Galilean principle of relativity between inertial observers. For more than three centuries, nature has been investigated and many observed phenomena have been described in terms of mathematical equations. Particularly, in early '900 several experiments indicated that the fundamental description of nature is depicted by Quantum Mechanics. At the same time, a reconsideration of the gravitational theory inspired Einstein to formulate General Relativity, characterized by a new principle of relativity together with a new idea of space-time. Thus, a Quantum theory of Gravity would unravel the essential nature of space and time.

General Relativity

General Relativity is "probably the most beautiful of all existing theories" [1].
Indeed, it proposes a new description of space-time, clarifies the relation between space-time and matter in a pure geometric picture and generalizes the notion of relativity to all inertial as well as non-inertial frames. Actually, in the framework of General Relativity space-time is a four-dimensional Lorentzian manifold with a Levi-Civita connection, whose curvature is related to the energy-momentum tensor by the Einstein field equations [2]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

The metric $g_{\mu\nu}$, the curvature $R_{\mu\nu}$, and matter, described by the stress-energy tensor $T_{\mu\nu}$, are dynamical fields affecting each other: matter tells space-time how to curve, and curved space-time tells matter how to move [3]. Actually, since the metric is the gravitational field, General Relativity is the most advanced classical field theory that we have developed describing the gravitational interaction, with many experimental confirmation [1–3].

The tensorial nature of the Einstein field equations expresses the first fundamental symmetry of the theory, the general covariance, i.e. passive and active diffeomorphism invariance. Particularly, a physical state corresponds to an equivalence class under active diffeomorphism [4] of solutions of the Einstein field equations (1.1). As a consequence, coordinates have no physical meaning, whereas physical observables are diffeomorphism invariant quantities, relational concepts, such as the intersections of world-lines.

The second fundamental symmetry of the theory is local Lorentz invariance, related to the principle of equivalence. Specifically, in any gravitational field it is always possible to choose a coordinate system which is locally flat and such that the tidal effects of the gravitational acceleration may be ignored. In other words, "space-time curvature and tidal gravity are the same thing expressed in different languages, the former in the language of relativity, the latter in the language of Newtonian gravity" [5].

Importantly, the gravitational field (the space-time metric), has a dynamical nature, since is the solution of the Einstein field equations. The precise structure of space-time is background independent as it is the result of a subtle interaction between matter fields and the gravitation field [6]. Therefore, in General Relativity there is no absolute space nor absolute motion [4].

Quantum Mechanics

The second aspect of the revolution that characterized the beginning of the XX and that has deeply changed the notion of determinism in physics is Quantum Mechanics. While General Relativity was formulated in order to describe gravitational interactions in a coherent theoretical framework, and then experimentally verified, Quantum Mechanics was formulated in order to explain a wealth of phenomena in the microscopic world. In the new picture of Quantum Mechanics, the previous classical description could be only a macroscopic approximation, where quantum effects are neglected.

The main feature of Quantum Mechanics relies in its intrinsic probabilistic nature, since only the probability of possible outcomes of experiments can be predicted. The consequences are manifold involving both technical [7] and fundamental issues [8]. Mathematically, at microscopic scale (quantum regime) the classical notion of trajectory disappears, the phase space is replaced by a Hilbert space, physical states are represented by rays in the Hilbert space, and observables are represented by Hermitian operators acting on physical states. Particularly, the operatorial nature of observables and their non-commutativity lead to the uncertainty principle and the quantum regeneration of states. Also the role of the observer has changed, especially in the measurement process.

However, Quantum Mechanics is formulated in a non-covariant language: while position is a physical observable (an Hermitian operator), time is just a parameter, that parametrizes the evolution of states (in the Schrodinger picture) or observables (in the Heisenberg picture) [9].

Quantum Field Theory

Attempts to conciliate Quantum Mechanics and Special Relativity lead to the formulation of Quantum Field Theories [10], which are the mathematical framework of the Standard Model. Among the confirmations of the Standard model, we can mention the agreement between the predicted and measured value of the anomalous magnetic dipole moment of the electron, the prevision of Z and $W^{\pm}$ and Higgs bosons, and the predictions concerning the nucleon internal structures in deep inelastic scattering experiments.
Mathematically, the Standard model is a Yang-Mills theory (non-abelian gauge theory) describing strong and electro-weak interactions, characterized by global Lorentz invariance and internal gauge symmetries.

However, Quantum Field Theories are formulated on a fixed background, i.e. a fixed metric structure (the Minkowski space-time or, more generally, a curved space-time). Particularly, Quantum Field Theory in curved space-time [11] has been used to formulate semi-classical gravity, resulting in the predictions of black holes radiation [12], and the black hole entropy [13].

Nevertheless, the notion of particle, Fourier modes, vacuum, Poincaré invariance are essential tools that can only be constructed on a given space-time geometry [14]. Particularly, the fixed background is necessary in the decoupling of high energy modes and low energy modes, and thus in the standard renormalization process. Actually, we know that standard background dependent quantum field theories are characterized by divergences, which can be removed by tuning the parameters of the theory in order to match observations [14]. The remarkable fact is that if we try to formulate a background dependent quantum field theory of gravity it turns out that the theory is non-renormalizable [15]. Generally, it is considered that a non-renormalizable theory could be only an effective theory, a low-energy approximation of a more fundamental, renormalizable theory [14]. The reason is that in a non-renormalizable theory we have to fix an infinite number of quantities by comparison with experiment [16], and consequently it has very limited predictive power. Therefore, the fundamental background independent quantum theory of gravity is expected to be a renormalizable theory.

Quantum Gravity

As we have seen, both General Relativity and Quantum Field Theory have several experimental confirmations, the first describing macroscopic phenomena of cosmological scales, while the second microscopic phenomena, such as elementary particle physics; but they are not compatible, since they rely on a different notion of space-time, different mathematical assumptions, a different principle of relativity and a different notion of determinism.

Since fundamental aspects of nature are described by quantum physics, it seems that a quantum theory of gravitation, describing the microscopic degrees of freedom of space-time, is necessary. Moreover, Quantum Gravity is expected to solve several open questions. For example, the break-down of classical General Relativity in presence of space-time singularities; the need to explain the Bekenstein-Hawking entropy in terms of microscopic space-time degrees of freedom; the description of the early universe approaching the Big Bang; the question of the fate of black holes at the end of Hawking's evaporation and the information paradox [17].

Moreover, Quantum Gravity may provide a description of the interaction between quantum matter fields and the gravitational field; clarify the consequences of the uncertainty principle on General Relativity and the the role of UV divergences [17].

But what would be the principal features of a quantum theory of gravity?
From General Relativity insights we know that space-time is the gravitational field, and it is a dynamical entity.
From Quantum Field Theory insights we know that all dynamical fields (and thus also the gravitational field) are described within the framework of Quantum Mechanics, they are quantum fields [4].
Thus, from the great results of Quantum Field Theories and General Relativity we would expect that Quantum Gravity could be a background independent, renormalizable Quantum Field Theory. Recently, there have been proposed several background independent Quantum Field Theories that could describe Quantum Gravity [18] such as M-theory and Supergravity, Loop Quantum Gravity (LQG) and Spin Foams, Causal Dynamical Triangulations (CDT) and Simplicial path integrals, Tensor models and Group Field Theories (GFT).

1.1 The Spin Foam approach and Group Field Theories

In this thesis we consider the Spin Foam approach and, particularly, its formulation in terms of Group Field Theories.

Spin Foams models [14, 17, 19–21] were originally introduced as an attempt for a covariant quantization of space-time and geometry, charaterized by a fundamental discreteness. Particularly, continuum space-time is replaced by a cellular complex to which purely algebraic data are associated to encode geometric degrees of freedom. Indeed, the Spin Foam approach shares several features with lattice gauge theories [22], nevertheless the nature of the cellular complex is different in the two approaches [20].

Subsequently, it was argued that Spin Foams may be also seen as a covariant quantization of Loop Quantum Gravity [4, 21], whereas the latter is based on a canonical formalism [23–26]. Indeed, also the canonical quantization of gravity has been considered in past decays. However, the difference between Loop Quantum Gravity and canonical quantum gravity studied by DeWitt [27] is related to the choice of the canonically conjugate variables.

Finally, it has been understood that every Spin Foam model admits a dual formulation in terms of Group Field Theories, a generalization of matrix models for 2d gravity (proposed in a string theory context [28–31]) and a point of convergence of many approaches to quantum gravity, such as Simplicial path integrals, Loop Quantum Gravity, Tensor models and Non-commutative geometry [32].

Group Field Theories [32–35] are quantum field theories on Lie groups, characterized by combinatorial non-local terms in the action. The expansion of the partition functions in terms of Feynman diagrams defines a sum over all topologies and all geometries of simplicial complexes, corresponding to Spin Foams.
The reason of this duality is related to the fact that the classical phase space of these theories is connected to the cotangent bundle of Lie groups [36, 37]. The quantization process of these phase spaces is subtle, as it requires tools from algebraic canonical quantization, but it has been discovered that there are three representations on the Hilbert space: a spin representation (corresponding to Spin Foam models), a group representation (corresponding to GFT models) and a non-commutative metric representation (corresponding to simplicial path integrals). Remarkably, these representations are equivalent.

In four dimensions, model building has been mainly inspired by the classical formulation of General Relativity as a constrained BF theory, called the Holst-Plebanski formulation.
Importantly, BF theory is a topological field theory, and it is easy to quantize, both in the Spin Foam framework and in the corresponding Group Field Theory context.

Having quantized the BF part, the crucial step is the implementation of the constraints that reduce it to gravity, called simplicity constraints.
At the classical level, simplicity constraints have been extensively studied in recent years, both in the continuum as well as in the discrete context. Historically, the first simplicity constraints considered were the quadratic simplicity constraints, replaced recently by the stronger linear simplicity constraints, as the latter have several advantages [38–41].

Using tools from geometric quantization [42], simplicity constraints are turned into operator equations and then imposed on the quantum states. However, the correct imposition has not been fully understood yet, although several strategies have been developed and applied in the literature, based on different mathematical tools, and leading to different candidate models for quantum space-time.

In the thesis we investigate and compare some of these models, pointing out key differences and similarities. Specifically, we consider the EPRL model [43], based on the Master constraint criterion; the modification of the EPRL model suggested by Alexandrov [44], proposed to relate the spin connection of Spin Foam models and the Ashtekar-Barbero connection of Loop

Quantum Gravity; the Ding-Han-Rovelli solutions of the Gupta-Bleuler citerion [45]; the Freidel Krasnov model [38], based on Livine-Speziale coherent states [46, 47] and the Baratin Oriti model [48–50], based on a non-commutative representation of Group Field Theory. Actually, we implement all these models in the extended GFT formalism, recently developed in [49, 50].

Remarkably, we show that the Baratin Oriti model satisfies the Gupta-Bleuler criterion and the Master constraint criterion exactly, for every well-defined quantization map, with no restriction on the Immirzi parameter. This is the main result of the thesis, as it put the Baratin Oriti model into the foreground as one of the most promising model for four dimensional Quantum Gravity.
Moreover, we investigate the Baratin Oriti model in the large-j limit, which is supposed to be related to the semi-classical limit, although the continuum limit has not been fully understood. Particularly, we show that the function weighting the Baratin Oriti solutions peaks on the conditions that characterize the Freidel Krasnov (FK) model and (partially) the EPRL model. However, the Baratin Oriti model is more general, since it does not enforce the spin relations characteristic of the EPRL/FK models, and it does not impose any rationality condition on the Immirzi parameter, allowing more solutions. Also the Ding-Han-Rovelli model allows more solutions than the EPRL and Freidel Krasnov models, although these extra solutions have not been explored in the literature.
In the thesis we investigate these extra solutions, particularly in relation with the extra solutions characterizing the Baratin Oriti model. Our preliminary analysis shows that these solutions are good solutions, with a proper asymptotic behaviour (as the extra solutions with a strange large-j behaviour are suppressed by the weighting functions) and shouldn't be overlooked.

1.2 Plan of the thesis

Chapter 2 is dedicated to the presentation of the Palatini formulation of General Relativity, which is the starting point of most of the approaches to Quantum Gravity. Then, in Chapter 3, we introduce Loop Quantum Gravity, a canonical quantization of General Relativity, based on the introduction of the Ashtekar-Barbero connection.
The reason we introduce Loop Quantum Gravity is twofold. On one hand spin network states are the boundary states of Spin Foam models. On the other hand, Spin Foam models are a covariant quantization of General Relativity, and they might provide a solution for the problem of the dynamics, affecting Loop Quantum Gravity.

In Chapter 4 we introduce some mathematical tools that are extensively used in the thesis. Particularly, we consider the classical and quantum mechanics on $SO(3)$. The reason are manifold, since $T^*SO(3)$, the cotangent bundle of $SO(3)$, is related to the phase space of Loop Quantum Gravity, Spin Foam models and GFT models. As mentioned before, the corresponding Hilbert space $L^2(SO(3))$ has three equivalent representations, depicting the Spin Foam / GFT / Simplicial path integral correspondence: the spin representation (related to Spin Foam), the group representation (related to GFT), and the metric representation (related to simplicial path integrals). Moreover, these representations appear also in Loop Quantum Gravity as the holonomy representation, the spin network representation and the flux representation.

Then, in Chapter 5 we introduce Spin Foams, and their relations with Loop Quantum Gravity, lattice gauge theories and Group Field Theories. Particularly, we consider the Spin Foam and the GFT quantization of BF theories, as gravity can be seen as a constrained BF theory. The constraints that reduce BF theory to gravity are described in Chapter 6. In Chapter 7, we present the strategy to define a GFT model describing Quantum Gravity, and then we present the EPRL model in Chapter 8; the modification of the EPRL model suggested by Alexandrov in Chapter 9; the Ding-Han-Rovelli solutions of the Gupta-Bleuler citerion in Chapter 10; the Freidel Krasnov model in Chapter 11; and the Baratin Oriti model in Chapter

12.

Last sections are dedicated to the important results of this thesis, namely the proof that the Baratin Oriti model satisfies the Master constraint criterion and the Gupta-Bleuler exactly, without imposing any rationality condition on the Immirzi parameter, nor imposing any particular choice of the quantization map. Moreover, the Baratin Oriti model is compared, in the asymptotic limit, with the other modes presented in the thesis. Results are summarized in Chapter 13, together with discussions on the role of simplicity constraints and possible outlooks. Finally, in Appendix A we describe part of the differential geometry used in the thesis, as well as quantization procedures in Appendix B.

Chapter 2

General Relativity

In this chapter, following [4, 51], we present some topics in advanced General Relativity. Particularly, we start with the Palatini (or first order) formulation of General Relativity, as it is the starting point for modern approaches to quantum gravity. Then, following [20, 51] we present the Lagrangian and the Hamiltonian formulation of General Relativity in the first order formalism. Both path integral and canonical attempts to quantize General Relativity have been considered in the past decays, particularly after the formulation of Yang-Mills theory. However, significant improvements have been obtained with the introduction of new variables: holonomies and fluxes. Specifically, the last section is dedicated to the introduction of the holonomy-flux algebra, a particular example of the Poisson algebra of a system whose phase space is the cotangent bundle of a Lie group.

2.1 The Palatini formulation of General Relativity

In General Relativity, space-time is a four dimensional smooth manifold $\mathcal{M}$. Coordinates on $\mathcal{M}$ are denoted by $x = x^\mu$, where indices $\mu, \nu = 0, \ldots, 4$ are space-time tangent indices.

The gravitational field or tetrad e is a one form with values in Minkowski space:

$$e^I(x) = e^I_\mu(x) dx^\mu \tag{2.1}$$

where indices $I, J = 0, \ldots, 3$ label the components of a Minkowski vector, carry a representation of the Lorentz group, and are raised and lowered by the Minkowski metric η_{IJ}. The tetrad is symmetric under

- Diffeomorphism

$$e'^I_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^I_\nu(x) \tag{2.2}$$

- Local Lorentz transformation

$$e'^I_\mu(x) = \Lambda^I_J(x) e^J_\mu(x). \tag{2.3}$$

Mathematically, the tetrad provides an isomorphism between the tangent bundle of $\mathcal{M}$, $T\mathcal{M}$, and a Lorentz principal bundle $F = (\mathcal{M}, SO(3,1))$ [51].

On this bundle we have a connection ω^{IJ}_μ, that is a 1-form with values in the Lorentz algebra, which we can use to define a covariant differentiation on the fibres,

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega^I_{\mu J}(x) v^J(x). \tag{2.4}$$

and a gauge covariant exterior derivative D on forms. For instance, for a 1-form u^I:

$$Du^I = du^I + \omega^I_J \wedge u^J. \tag{2.5}$$

The torsion 2-form is defined by the first Cartan structure equation:

$$T^I \equiv T^I_{\mu\nu} dx^\mu \wedge dx^\nu = De^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J. \tag{2.6}$$

A tetrad field e determines uniquely a torsion free spin connection $\omega = \omega[e]$, called compatible with e, by

$$T^I = \mathrm{d}e^I + \omega[e]^I{}_J \wedge e^J = 0. \tag{2.7}$$

Given the connection, the curvature of ω is the Lorentz algebra valued 2-form defined by the second Cartan structure equation:

$$R^{IJ} \equiv R^{IJ}_{\mu\nu} dx^\mu \wedge dx^\nu = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}, \tag{2.8}$$

which shows that General Relativity is a gauge theory whose local gauge group is the Lorentz group, and the Riemann tensor is nothing but the field-strength of the spin connection.

In terms of the tetrad the Einstein-Hilbert action can be rewritten as

$$S(e^I_\mu) = \frac{1}{2}\epsilon_{IJKL} \int e^I \wedge e^J \wedge F^{KL}(\omega(e)). \tag{2.9}$$

Finally, we can consider the connection to be an independent variable. Namely,

$$S(e^I_\mu, \omega^{IJ}_\mu) = \frac{1}{2}\epsilon_{IJKL} \int e^I \wedge e^J \wedge F^{KL}(\omega). \tag{2.10}$$

The two actions are equivalent on-shell since the extra field equations, coming from varying the action with respect to ω, simply impose that the connection is the spin connection of General Relativity.

2.2 Hamiltonian analysis of tetrad formulation

In this section, following [24, 51], we consider the Hamiltonian analysis of the Palatini formulation of General Relativity, since it is the starting point for loop quantization. As usual, the first step is to rewrite the Palatini action in terms of canonically conjugated variables and then perform a Legendre transform.

To this end, we assume that the space-time manifold $\mathcal{M}$ is globally hyperbolic. Specifically, $\mathcal{M}$ has the topology $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, where Σ is a fixed three-dimensional manifold of arbitrary topology and space-like signature, and $\mathcal{M}$ has no causally disconnected region.

With this assumption, we can foliate $\mathcal{M}$ into a one-parameter family of hypersurfaces $\Sigma_t = X_t(\Sigma)$ embeddings of Σ in $\mathcal{M}$, with coordinates coordinates (t, x), where we identify the coordinate $t \in \mathbb{R}$ as a time parameter.
Due to diffeomorphism invariance, physical quantities are independent of the specific choice of the foliation [51].
Given a foliation X_t and the adapted coordinates (t, x), we can define the time flow vector:

$$\tau^\mu(x) \equiv \frac{\partial X^\mu_t(x)}{\partial t} = (1, 0, 0, 0). \tag{2.11}$$

Note that generally, this vector is not parallel to the unit normal vector to Σ, which we denote n^μ, despite the fact that they are both time-like.
Thus, we can decompose τ^μ into its normal and tangential parts,

$$\tau^\mu(x) = N(x) n^\mu(x) + N^\mu(x). \tag{2.12}$$

Finally, it is convenient to parametrize $n^\mu = (1/N, N^a/N)$, so that $N^\mu = (0, N^a)$, where N is called lapse function, and N^a shift vector.

Now, in the tetrad formalism, it it possible to show that [51]:

$$e_0^I = e_\mu^I \tau^\mu = N n^I + N^a e_a^I, \qquad \delta_{ij} e_a^i e_b^j = g_{ab},\ i, a = 1, 2, 3. \tag{2.13}$$

where we have introduced the triad e_a^i, i.e. the spatial part of the tetrad.

As mentioned before, in order to perform the Legendre transformation we need to identify canonically conjugated variables. Fortunately, Ashtekar found a particular choice of variables which simplifies the analysis, the famous Ashtekar variables, which are usually introduced in the "time gauge" $e_\mu^I n^\mu = \delta_0^I$.

The crucial change of variables is the following: we define the *densitized triad*

$$E_i^a = e e_i^a = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{abc} e_b^j e_c^k, \tag{2.14}$$

and the *Ashtekar-Barbero connection*

$$A_a^i = \gamma \omega_a^{0i} + \frac{1}{2}\varepsilon^i_{\ jk}\omega_a^{jk}. \tag{2.15}$$

where γ is the Immirzi parameter.

These variables are conjugated, with the fundamental Poisson bracket:

$$\left\{A_a^i(x), E_j^b(y)\right\} = \gamma \delta_a^b \delta_j^i \delta^3(x, y) \tag{2.16}$$

The new internal index i corresponds to the adjoint representation of $SU(2)$, should be seen as an auxiliary local symmetry group, since the connection with the original Lorentz group of the tetrad formulation is hidden in the change of variables (2.14)-(2.15).

Finally, we can rewrite the action in terms of the new variables as [23, 24, 51, 52]

$$S(A, E, N, N^a) = \frac{1}{\gamma}\int \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left[\dot{A}_a^i E_i^a - A_0^i \mathcal{G}_i - N\mathcal{S} - N^a \mathcal{C}_a\right], \tag{2.17}$$

where

$$\mathcal{G}_j \equiv D_a E_i^a = \partial_a E_j^a + \varepsilon_{jk\ell} A_a^j E^{a\ell}, \tag{2.18}$$

$$\mathcal{C}_a = \frac{1}{\gamma} F_{ab}^j E_j^b - \frac{1+\gamma^2}{\gamma} K_a^i G_i, \tag{2.19}$$

$$\mathcal{S} = \left[F_{ab}^j - \left(\gamma^2 + 1\right)\varepsilon_{jmn} K_a^m K_b^n\right] \frac{\varepsilon_{jk\ell} E_k^a E_\ell^b}{\det E} + \frac{1+\gamma^2}{\gamma} G^i \partial_a \frac{E_i^a}{\det E}. \tag{2.20}$$

Lapse and shift are Lagrange multipliers, and consistently we refer to $\mathcal{S}(A, E)$ and $\mathcal{C}_a(A, E)$ as the Hamiltonian or scalar and space-diffeomorphism or vector constraints. The last constraint, the Gauss constraint (2.18), generates $SU(2)$ gauge transformations.

Despite the fact that for the special case $\gamma = i$ we have several advantages [51], for example the $SU(2)$ subgroup of the Gauss constraint corresponds to the self-dual subgroup of the Lorentz group, as well as the Hamiltonian constraint (2.20) simplifies, in this case we would need to impose reality conditions on the complex variables (2.14)-(2.15). These reality conditions are difficult to deal with at the quantum level, and for this reason we focus on γ real and positive.

2.3 Smearing of the variables

In this section, following [32], we smear the algebra (2.16).
Being a 1-form A can naturally (i.e. without referring to a background metric) be integrated along 1-dimensional submanifolds of Σ, namely along embedded edges e.

$$\int_e A := \int_e A_a^j \tau_j dx^a \tag{2.21}$$

where τ_i are the generators of $SU(2)$.

Moreover, a brief look at (2.14) shows that the densitised triad is a 2-form. Hence, it has a natural associated 2-form $(*E)_{ab}^j(\sigma) = \epsilon_{abc}E^{cj}(\sigma)$ that can be integrated along submanifolds of codimension one, namely analytic 2-surfaces S (by means of appropriate parallel transports): it is natural to smear it on a surface,

$$E(S,f) := \int_S (*E)^j f_j, \tag{2.22}$$

where f is a smearing function with values in $\mathfrak{su}(2)^*$, the topological dual of $\mathfrak{su}(2)$ and f_j are its components in a local basis.

2.4 The holonomy-flux algebra

In order to get quantities with a nicer behaviour under $SU(2)$-transformations we introduce the holonomy

$$h_e(A) := \mathcal{P}\exp\left[-\int_e A\right] \tag{2.23}$$

where $\mathcal{P}$ denotes the path-ordering, that is,

$$h_e(A) = \sum_{n=0}^{\infty} \iiint\limits_{1>s_n>\cdots>s_1>0} A(e(s_1))\cdots A(e(s_n))ds_1\cdots ds_n \tag{2.24}$$

and we parametrized the line e with $x^a(s) : [0,1] \subset \mathbb{R} \to \Sigma$.

Let us list some important properties of the holonomy [24]:

- The definition of $h_e[A]$ is independent of the parametrization of the path e.
- The holonomy is a representation of the groupoid of oriented paths. Namely, the holonomy of a path given by a single point is the identity, and, given two oriented paths e_1 and e_2 such that the end point of e_1 coincides with the starting point of e_2, we have

$$h_{e=e_1e_2}[A] = h_{e_1}[A]h_{e_2}[A], \tag{2.25}$$

  where the multiplication on the right is the $SU(2)$ multiplication. We also have that

$$h_{e^{-1}}[A] = h_e^{-1}[A]. \tag{2.26}$$

- The holonomy has a very simple behaviour under gauge transformations. Particularly, under a gauge transformation generated by the Gauss constraint, the holonomy transforms as

$$h'_e[A] = g(x(0))\ h_e[A]\ g^{-1}(x(1)). \tag{2.27}$$

- The holonomy transforms in a very simple way under the action of spatial diffeomorphisms. Given $\phi \in \mathrm{Diff}(\Sigma)$ we have

$$h_e[\phi^* A] = h_{\phi^{-1}(e)}[A], \tag{2.28}$$

where $\phi^* A$ denotes the action of ϕ on the connection. In other words, transforming the connection with a diffeomorphism is equivalent to simply 'moving' the path with ϕ^{-1}.

For a graph γ with $|\gamma|$ edges the holonomy assigns an element $h_\gamma(A) \in SU(2)$ to every edge.

Importantly, as noted in [32] we can either specify the connection field at every point in the spatial manifold or provide the holonomies along all the paths embedded in the same manifold. In fact, they are two different parametrization of the same phase space.

Now, we can introduce the cylindrical functions, which play a crucial role in the definition of the configuration space of the theory.

Generically, a cylindrical function is a functional of a field that depends only on some subset of components of the field itself. In our case, the field is the connection, and the cylindrical functions are functionals that depend on the connection only through the holonomies along some finite set of paths e.

Thus, we can define the space:

$$Cyl^\gamma = \{C^\gamma : A \mapsto C^\gamma(A) \in \mathbb{C} | C^\gamma(A) := c(h_{e_1}(A), \ldots, h_{e_{|\gamma|}}(A)\} \tag{2.29}$$

of functions called cylindrical with respect to γ, i.e. that depend on A only through the holonomies $h_{e_i}(A)$ and $c : SU(2)^{|\gamma|} \to \mathbb{C}$ is a continuous complex valued function.

As mentioned, the configuration space of the theory is related to Cyl^γ, particularly is defined to be the space Cyl as the space of functions that are cylindrical with respect to some graph. Then, it is possible to show [32] that the fluxes $E(S, f)$ are vector fields on Cyl.

Finally, as mentioned in [32], in principle one can compute the classical Poisson algebra between cylindrical functions in full generality, finding commutativity among any two holonomies and non-commutativity among fluxes. However, the Poisson brackets are rather complicated, as they depend on the specific path and surface associated to the variables considered, to the point that the general commutator between fluxes is not known [53, 54].

Fortunately, we can compute the simplest case of a single link e and a surface and a single elementary dual surface S_e, also labelled by e intersecting it at a single point:

$$\begin{aligned} \{h_e, h_{e'}\}_\gamma &= 0 \\ \{E_i^e, h_{e'}\} &= \delta_{e'}^e \frac{\tau_i}{2} h_e \\ \{E_i^e, E_j^{e'}\} &= -\delta^{ee'} \epsilon_{ij}{}^k E_k^e \end{aligned} \tag{2.30}$$

with τ_i the generators of the $\mathfrak{su}(2)$ Lie algebra and the dual triad E_i^e is define as:

$$E_i^e = tr \left[\tau_i \int_{S_e} Ad(h_{e,x}) E(x) \right] \tag{2.31}$$

where $h_{e,x}$ is the holonomy along the path from the starting point of the edge e to the point x on the surface S_e.

Remarkably, this phase space has the algebraic structure of $T^*(SO(3)) \simeq SO(3) \times \mathfrak{so}(3)$, the cotangent bundle over $SO(3)$ [53], that we study in detail in Chapter 4. The interesting fact is that $T^*(SO(3))$ is also the kinematical phase space of discrete topological BF theory [55–57], with $SO(3)$ as gauge group, and thus also simplicial BF theory is based on a configuration space given by cylindrical functions for the gauge group.

For a more general graph Γ, we need to consider the direct product of phase spaces of the L individual links $l \in \Gamma$, modulo relations arising from the requirement of $SO(3)$ gauge invariance at the V vertices $v \in \Gamma$, $T^* \left(SO(3)^L / SO(3)^V \right)$ [37].

Chapter 3

Loop quantum gravity

General Relativity can be formulated in terms of canonically conjugated variables: the Ashtekar-Barbero connection and the densitized triad with Poisson brackets (2.16). In terms of these variables the Hamiltonian is a totally constrained system, specifically the sum of three sets of constraints: the Gauss constraint $\mathcal{G}_i$, the spatial diffeomorphism constraint $\mathcal{C}_a$ and the Hamiltonian constraint $\mathcal{S}$.

At this point, following [24, 51], we can consider the quantum theory defined using Dirac quantization, Loop Quantum Gravity. The first step is to represent (2.18), (2.19) and (2.20) as operators in an auxiliary Hilbert space $\mathcal{H}_{kin}$ and then solve the constraint equations

$$\hat{\mathcal{G}}_i \Psi = 0, \qquad \hat{\mathcal{C}}_a \Psi = 0, \qquad \hat{\mathcal{S}} \Psi = 0. \tag{3.1}$$

The Hilbert space of solutions is the so-called physical Hilbert space $\mathcal{H}_{phys}$. In Loop Quantum Gravity, being a totally constrained system, constraint equations govern the quantum dynamics and represent quantum Einstein's equations. Importantly, in the thesis we usually assume natural units $c = \hbar = 1$, although sometimes we restore $\hbar$ for the sake of clarity.

Since holonomies and fluxes are canonically conjugated variables, at the quantum level, we can choose to work in the holonomy (or group) representation or in the flux (or non-commutative metric) representation.
Here, we present the more traditional representation of Loop Quantum Gravity, based on holonomies and spin networks. However, for an interesting description of Loop Quantum Gravity in terms of fluxes the reader can refer to [54].

3.1 Cylindrical functions and the kinematical Hilbert space

In the previous chapter we have defined the space of functions called cylindrical with respect to γ, Cyl^γ. Now, this space of functionals can be turned into a Hilbert space if we equip it with a scalar product, defined by:

$$\left\langle \psi_{(\Gamma,f)} \mid \psi_{(\Gamma,f')} \right\rangle \equiv \int \prod_e \mathrm{d}h_e \overline{f(h_{e_1}[A], \ldots, h_{e_L}[A])} f'(h_{e_1}[A], \ldots, h_{e_L}[A]) \tag{3.2}$$

where $\mathrm{d}h$, is a unique gauge-invariant and normalized measure, called the Haar measure.

Next, we define the Hilbert space of all cylindrical functions for all graphs as the direct sum of Hilbert spaces on a given graph,

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma = L^2[A, \mathrm{d}\mu_{AL}] \tag{3.3}$$

which defines a Hilbert space over (suitably generalized, see [58–60] for details) gauge connections A on Σ.
The integration measure $\mathrm{d}\mu_{AL}$ over the space of connections is called the Ashtekar-Lewandowski measure. Finally, the scalar product (3.2) can be generalized on $\mathcal{H}_{kin}$ as a scalar product between cylindrical functionals of the connection with respect to the Ashtekar-Lewandowski measure:

$$\left\langle \psi_{(\Gamma_1,f_1)} \mid \psi_{(\Gamma_2,f_2)} \right\rangle \equiv \int \mathrm{d}\mu_{AL}\, \overline{\psi_{(\Gamma_1,f_1)}(A)}\psi_{(\Gamma_2,f_2)}(A). \tag{3.4}$$

Remarkably, the representation of the holonomy-flux algebra on $\mathcal{H}_{kin}$ is *unique*, as proved by Fleischhack and Lewandowski, Okolow, Sahlmann, Thiemann [61, 62].

Thus, we have defined a unique, well-behaved, kinematical Hilbert space for General Relativity, carrying a representation of the canonical Poisson algebra. Following Dirac, we now have a well-posed problem of reduction by the constraints:

$$\mathcal{H}_{kin} \xrightarrow{\hat{\mathcal{G}}_i = 0} \mathcal{H}^0_{kin} \xrightarrow{\hat{\mathcal{C}}_a = 0} \mathcal{H}_{Diff} \xrightarrow{\hat{\mathcal{S}} = 0} \mathcal{H}_{phys}. \tag{3.5}$$

3.2 Gauge-invariant Hilbert space

Given the kinematical Hilbert space, we need to impose the constraints. Particularly, we start by looking at the solutions of the quantum Gauss constraint, i.e. $SU(2)$ gauge invariant states in $\mathcal{H}_{kin}$. These solutions define a new Hilbert space, that we call $\mathcal{H}^0_{kin} = L^2[A, \mathrm{d}\mu_{AL}]/\mathcal{G}$.

Gauge transformations generated by the Gauss constraint act non-trivially at the endpoints of the holonomy:

$$h_e \to h'_e = \hat{U}_G h_e = g_{s(e)} h_e g^{-1}_{t(e)} \tag{3.6}$$

i.e. at nodes of graphs.
Particularly, it is possible to show [24] that cylindrical functions invariant under action of the group at the nodes are a particular class of Wilson loops (the trace of the holonomy around a closed loop), called spin network states, introduced by Penrose in [63] and independently used in lattice gauge theory [64, 65], which form an orthonormal basis of $\mathcal{H}^0_{kin}$.

Spin-networks [66–68] are defined by a graph γ in Σ, a collection of spins $\{j_\ell\}$, labelling unitary irreducible representations of $SU(2)$, associated with links $\ell \in \gamma$ and a collection of $SU(2)$ intertwiners $\{\iota_n\}$ associated to nodes $n \in \gamma$:

$$\Psi_{\gamma,\{j_\ell\},\{\iota_n\}}(A) = \prod_{n\in\gamma} \iota_n \prod_{\ell\in\gamma} D^{j_\ell}[h_\ell(A)], \tag{3.7}$$

where $D^{j_\ell}[h_\ell(A)]$ denotes the corresponding j_ℓ-representation matrix evaluated at corresponding link holonomy and the matrix index contraction is left implicit.

Particularly, we construct the spin network gauge invariant wave functional $\Psi_{\gamma,\{j_\ell\},\{\iota_n\}}(A)$ by first associating an $SU(2)$ matrix in the D^{j_ℓ}-representation to the holonomies $h_\ell(A)$ corresponding to the link ℓ, and then contracting the representation matrices at nodes with the corresponding intertwiners ι_n.

3.3 Solutions of the diffeomorphisms constraint

The next constraint we consider is the 3-diffeomorphisms constraint. The usual procedure [24] to solve the constraint uses group averaging techniques and the existence of the diffeomorphism invariant measure, and the result is that solutions correspond to generalized states in the topological dual space Cyl^* [60].

However, in the present discussion we would like to point out the intuitive idea that these generalized spin networks are given by equivalence classes of spin-network states up to diffeomorphism. In other words, two spin network states are considered equivalent if their underlying graphs can be deformed into each other by the action of a diffeomorphism [20].

As a consequence, we note that the spin-network states can be thought as pure combinatorial structures, completely characterized by the combinatorics of cellular decompositions of space (piecewise linear manifold [69]), rather than graphs embedded in a smooth manifold.

3.4 Quantum geometry

Before considering the dynamics, encoded in the scalar constraint, we give a look at the geometric operators that can be constructed in Loop Quantum Gravity, generally defined in terms of the fundamental triad operators $\hat{E}_i^a$.

At the quantum level it is possible to rigorously define the area operator $\hat{A}_S(E)$ [70–72]. Particularly, $\hat{A}_S(E)$ has a diagonal action on spin networks states ψ_Γ, associated to the graph Γ, and its spectrum is discrete. More precisely, the area of the surface S, with a finite number of punctures p caused by the intersect of links of Γ with S, is given by:

$$\hat{A}(S)\psi_\Gamma = \sum_{p\in S\cap\Gamma} \hbar\sqrt{\gamma^2 j_p(j_p+1)}\psi_\Gamma \tag{3.8}$$

or

$$\hat{A}(S)\psi_\Gamma = \sum_{p\in S\cap\Gamma} \hbar\gamma\left(j_p+\frac{1}{2}\right)\psi_\Gamma \tag{3.9}$$

according to the quantization map chosen [73].

Also the volume operator, $\hat{V}_\sigma(E)$, can be rigorously defined [70, 71, 74, 75]. Furthermore, it has been shown that its spectrum is discrete and volume is concentrated in nodes.
Therefore, spin networks describe a quantum geometry, where each face dual to a link has an area proportional to the spin j_e, and each region around a node has a volume determined by the intertwiner i_n as well as the spins of the link sharing the node.

Remarkably, in Loop Quantum Gravity the space geometry is discrete at the Planck scale, and this fact is not a built-in discretization, rather is a result of the quantum theory. As a consequence of this fundamental discreteness, Loop Quantum Gravity is expected to have no ultraviolet divergences, as well as resolving the breakdown of General Relativity in presence of singularities [51].

3.5 Quantum dynamics

The last constraint we need to impose to complete Dirac's program, reducing the kinematical Hilbert space to the physical Hilbert space $\mathcal{H}_{phys}$ is the scalar or Hamiltonian constraint, encoding the dynamics of the theory [24]:

$$\hat{\mathcal{S}}\Psi = 0. \tag{3.10}$$

Unfortunately, the quantization of the scalar constraint is more involved and subtle with respect to the Gauss and vector constraints, regularization choices have to be made and the result is not unique.
The reason of the complexity of the time re-parametrization constraint $\mathcal{S}$ is related to the $3+1$ splitting of the canonical formulation, whereas gravity has a manifest 4-dimensional symmetry. Several rigorously quantization procedures have been proposed [76–79], nevertheless the nature

of solutions and thus the dynamics seem to depend critically on different ambiguities present in the quantization schemes. At the same time, for each approach the complete characterization of $\mathcal{H}_{phys}$ is unknown as well as the full spectrum of $\hat{\mathcal{S}}$, and the question whether any of these theories is rich enough to reproduce General Relativity in the classical continuum limit has not been answered yet [51].

In order to solve the problem of the dynamics, other possibilities have been explored, particularly the Spin Foam formalism, a covariant approach to quantum gravity based on a path integral formulation of Loop Quantum Gravity. The motivation is that, in the generally covariant context, path integrals provide a tool to find solutions to the constraint equations, (see for example [80]). We will describe in details the Spin Foam approach in the next chapters.

Let us finish by stating some properties of $\hat{\mathcal{S}}$ that do not depend on the ambiguities mentioned above. Particularly, in [81,82] it was realized that smooth loop states naturally solve the scalar constraint operator. Unfortunately, this set of solutions is too small, as they span a zero volume sector of the physical Hilbert space. However, as a consequence it has been realized that $\hat{S}$ acts only on spin network nodes, by creating and/or annihilating new links at nodes, called exceptional edges, which are characterized by being invisible to subsequent actions of the constraint [20].

3.6 Projected spin networks

In this section, following [83], we introduce a generalization of the $SU(2)$ spin network states used in Loop Quantum Gravity. The reason is two-fold: on one hand they correspond to the boundary states of the extended Group Field Theory formalism, described in Section 5.7.3. On the other hand they are related to Covariant Loop Quantum Gravity, a loop quantization based on the full local Lorentz invariance of gravity.
In fact, in the definition of the Ashtekar-Barbero connection we have broken Lorentz covariance by means of the time gauge-fixing, thus reducing the initial Lorentz gauge group to the rotational $SU(2)$ subgroup corresponding to the rotational invariance of the triad frame.
Also note that Spin Foams are defined in terms of the spin connection, thus a Covariant Loop Quantum Gravity may clarify the relation between the two approaches.

If we want to reformulate Loop Quantum Gravity in a Lorentz-covariant way, we need a generalization of the usual spin network, called projected spin network [84,85].
Generally, projected spin networks can be introduced for any group G and its compact subgroup H. Actually, let $\mathcal{H}_G^{(\lambda)}$ be the representation space of an irreducible representation λ of G. This representation can be decomposed with respect to the subgroup H_x of $x \in X = G/H$ as the direct sum

$$\mathcal{H}_G^{(\lambda)} = \bigoplus_j \mathcal{H}_H^{(j)} \tag{3.11}$$

and we define the projected spin network function by:

$$\Psi^{(\Gamma,\vec{\lambda},\vec{\jmath}_s,\vec{\jmath}_t,\vec{\imath})}[g_\ell, x_n] = \prod_{\ell\in\gamma} D^{\lambda_\ell}(g_\ell) \prod_{n\in\gamma} \mathcal{I}^{\vec{\lambda}_n,\vec{\jmath}_n,\imath_n)}(x_n) \tag{3.12}$$

where we have defined class of intertwiners

$$\bigotimes_{k=1}^{L} \mathcal{H}_G^{(\lambda_k)} \ni \mathcal{I}_{p_1\ldots p_L}^{(\vec{\lambda},\vec{\jmath},i)}(x) = \sum_{m_{j_1}\ldots m_{j_L}} i^{(\vec{\jmath})}_{m_{j_1}\ldots m_{j_L}} \prod_{k=1}^{L} D^{(\lambda_k)}_{p_k m_{j_k}}(g_x) \tag{3.13}$$

Notations are the following. g_x is a representative of $x \in X$ in G, $\vec{\lambda}$ and $\vec{\jmath}$ are a set of representations of G and H, and

$$i^{(\vec{\jmath})} \in Inv\left(\bigotimes_{k=1}^{L} \mathcal{H}_H^{(j_k)}\right) \tag{3.14}$$

is an invariant H-intertwiner.
First, we observe that the projected spin networks depend on the group elements $g_\ell \in G$, but also on the elements of the factor space $x_n \in X$.
Second, we note that the intertwiner $\mathcal{I}$ is not invariant, rather under general G-transformations it transforms covariantly:

$$\sum_{q_1 \ldots q_L} \left(\prod_{k=1}^{L} D^{(\lambda_k)}_{p_k q_k}(g)\right) \mathcal{I}^{(\vec{\lambda},\vec{\jmath},i)}_{q_1 \ldots q_L} = \mathcal{I}^{(\vec{\lambda},\vec{\jmath},i)}_{p_1 \ldots p_L}(g \cdot x) \tag{3.15}$$

Nevertheless, it is invariant under tranformations from the stabilizer subgroup H_x, this invariance property was called in [86,87] relaxed closure condition, and appears also in the extended GFT formalism described in Section 5.7.3.

Finally, in the case of Riemannian four-dimensional gravity we are interested in, $G = SO(4)$ and $H = SU(2)$, and we have [87]:

$$\Psi^{(\lambda_f, j_{tf}, k_t)}[g_\ell, x_n] = \prod_f \left(C^{j_f^- j_f^+ j_{u(f)}}_{mm'\ell_{u(f)}} D^{(j_f^-)}_{mn}(G_f^-) D^{(j_f^+)}_{m'n'}(G_f^+) \overline{C^{j_f^- j_f^+ j_{d(f)}}_{nn'\ell_{d(f)}}} \right) \prod_t \iota^{\{j_{tf_i}\}}_{\{\ell_{tf_i}\}}(k_t) \tag{3.16}$$

where $G_f = (g_{x_{u(t)}})^{-1} g_f g_{x_{d(t)}}$, g_x is a representative of $x \in X$ in G, $C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ are $SU(2)$ Clebsch-Gordan coefficients, $\iota^{\{j_i\}}_{\{m_i\}}(k)$ are matrix elements of an invariant $SU(2)$ intertwiner between 4 representations with spins j_i, $i = 1, \ldots, 4$, and with intermediate spin k. The sum over repeated representation indices is implied.
The dependence of the normals x_t can be pushed from the matrix coefficients into the intertwiners, in this way the projection is done on the rotated subgroup H_{x_t}, as we will see in Section 5.7.3.

The projected spin networks appear in Covariant Loop Quantum Gravity describing kinematical Hilbert space of canonical loop quantization, and in extended GFT models as boundary states, where the variables x_n represent the normals of the tetrahedra, thus providing a link between the two formalisms.

Chapter 4

Some mathematical tools

In this chapter we, briefly, introduce several mathematical tools that we will use in this thesis, particularly the geometric structures of classical and quantum mechanics on Lie groups.
The reason is that the phase spaces of Loop Quantum Gravity and discrete topological BF theory are the cotangent bundle of Lie groups. Specifically, the classical phase space of four dimensional simplicial BF theory decomposes into $T^*SO(4) = T^*SU(2) \times T^*SU(2)/\mathbb{Z}_2$ for each triangle of the simplicial complex, and is the starting point for Group Field Theory and Spin Foams models for Quantum Gravity.

As we will see, already at the classical level, such phase spaces are characterized by a non-commutative structure, given by the non-commutativity of momenta. Thus, at the quantum level, we need to be careful when we consider (non-commutative) momentum representation and (non-commutative) Fourier transform.

Following [36, 37], in Section 4.1 we consider for simplicity the phase space $T^*SO(3)$, since results can be easily generalized to the double covering $T^*SU(2)$. Interestingly, in Loop Quantum Gravity, this is the phase space describing the simplest case of a link l and a surface S intersecting it at a single point. In Section 4.2 we introduce a set of coordinates parametrizing the phase space, and we canonically quantize these coordinates by choosing an opportune quantization map in Section 4.3.

We will see that there are three different representations acting on the Hilbert space, the group representation in Section 4.4, the spin representation in Section 4.5 and the algebra representation in Section 4.6. The latter is also called non-commutative momentum basis, or non-commutative metric representation, due to the fact that we need to introduce a non-commutative $\star$-product that depends on the choice of the quantization map $\mathcal{Q}$. In Section 4.7 we present the transformation relating these representations, called non-commutative Group Fourier transform, whose properties are described in Section 4.8. Particularly, in Section 4.9 we give an explicit example, the Freidel-Livine-Majid quantization map, which will be used in the following.

4.1 Structure of the phase space $T^*SO(3)$

Every Lie group G has a globally trivial cotangent bundle $T^*G \simeq G \times \mathfrak{g}$.
In the case of $SO(3)$, the cotangent bundle $T^*SO(3) \simeq SO(3) \times \mathfrak{so}(3)$ is parallelizable due to the existence of left-(right-) invariant 1-forms P_g induced by the group action as

$$P^L_{gh} = L^*_g P_h \in T^*_{gh}SO(3) \quad \text{and} \quad P^R_{hg} = R^*_g P_h \in T^*_{hg}SO(3) \ , \tag{4.1}$$

where L^* and R^* are the pull-back of the left- multiplication $L_g h = gh$ and right- multiplication $R_g h = hg$ on 1-forms.
We can consider left-invariant 1-forms $P^L_e =: P^L$ or right-invariant 1-forms $P^R_e =: P^R$ in the

cotangent space at the unit element $T_e^*SO(3) \simeq \mathfrak{so}(3) \simeq \mathbb{R}^3$ as the momentum space of the system. Cotangent bundles are endowed with a canonical symplectic structure that, together with ordinary pointwise multiplication $\cdot$ on $C^\infty(T^*SO(3))$, uniquely determines the Poisson algebra $\mathcal{P}_{SO(3)}$.
Choosing to represent the algebra variables with right-invariant vector-fields, the Poisson brackets $\{\cdot,\cdot\}_{PB}$ on the phase space $T^*SO(3)$ are given by [36]:

$$\{F,G\}_{PB} = \frac{\partial F}{\partial P_i}\mathcal{L}_i g - \mathcal{L}_i f \frac{\partial G}{\partial P_i} + \epsilon_{ij}^k \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial P_j} P_k \tag{4.2}$$

for $F, G \in C^1(T^*SO(3))$, where $\mathcal{L}_i$ are Lie derivatives on $SO(3)$ with respect to an orthonormal basis of right-invariant vector fields and P_i are Euclidean coordinates in the momentum space $\mathfrak{so}(3) \simeq \mathbb{R}^3$. From here on, where not differently specified, right-invariant vector fields are considered.

4.2 Parametrization of the phase space $T^*SO(3)$

Now, we need a parametrization of the phase space $T^*SO(3) = SO(3) \times \mathfrak{so}(3)$. Regarding the cotangent space, we can use the coordinate P_i, which are globally well-defined due to the parallelazability of $T^*SO(3)$. Conversely, we cannot introduce a unique coordinate system $\zeta^i \in C^\infty(SO(3))$ on $SO(3)$, since $SO(3)$ is a compact group that cannot be covered with only a coordinate patch. However, we can choose a set of coordinates $\zeta^i : SO(3) \to \mathbb{R}$ by imposing $f \overset{!}{=} f_\zeta(\zeta^i)$, where $f_\zeta \circ \vec{\zeta} \equiv f$, for all $f \in C^\infty(SO(3))$ and we assume $\zeta(e) = 0$ and $\mathcal{L}_i\zeta^j(e) = \delta_i^j$. With these choices the Poisson brackets, induced by the right-invariant vector fields, read

$$\{\zeta^i,\zeta^j\}_{PB} = 0 \quad , \quad \{\zeta^i, P_j\}_{PB} = \mathcal{L}_{T_k}\zeta^k \quad , \quad \{P_i,P_j\}_{PB} = \epsilon_{ij}{}^k P_k. \tag{4.3}$$

Particularly, we can choose a coordinate system $\xi_h^i(g)$ in a neighbourhood of any element $h \in SO(3)$, such that $g \equiv \exp[i\xi_h^i(g)t_i]h$, where $t_i \equiv T_i(e) \in T_eSO(3) \cong \mathfrak{so}(3)$ is a Lie algebra basis. With this choice, the Poisson brackets (4.3) read:

$$\{\xi_h^i,\xi_h^j\}_{PB} = 0 \quad , \quad \{\xi_h^i, P_j\}_{PB}\Big|_h = \delta_j^i \quad , \quad \{P_i,P_j\}_{PB} = \epsilon_{ij}{}^k P_k \tag{4.4}$$

for all $h \in SO(3)$.

4.3 The quantization of the phase space $T^*SO(3)$

The Algebraic canonical quantization of cotangent bundles and the quantum representations for general (weakly exponential) Lie groups have been extensively studied in [36,37,88,89]. See also references therein. The result is that one can define a quantization map $\mathcal{Q}$ giving a canonical formulation of quantum mechanics corresponding to the classical phase space $T^*SO(3)$, by choosing for canonical variables the group elements $g \in SO(3)$ and the Lie algebra elements $P \in \mathfrak{so}(3) \simeq \mathbb{R}^3$. Actually, we define $\hat{g} = \mathcal{Q}(g)$ and $\hat{P}_i = \mathcal{Q}(P_i)$, such that the commutators of the operators $\hat{P}_i$ and the coordinate operators $\hat{\zeta}^i := \zeta^i(\hat{g})$ corresponding to the coordinate functions $\zeta^i : SO(3) \to \mathbb{R}^3$, induced by the right-invariant vector fields, read

$$[\hat{\zeta}^i,\hat{\zeta}^j] = 0 \quad , \quad [\hat{\zeta}^i,\hat{P}_j] = i\widehat{\mathcal{L}_{T_k}\zeta^k} \quad , \quad [\hat{P}_i,\hat{P}_j] = i\epsilon_{ij}{}^k\hat{P}_k \tag{4.5}$$

4.4 Group basis

We first consider the group representation on $L^2(SO(3))$ defined as the one diagonalizing all the operators $\hat{f} \equiv \mathcal{Q}(f) \in \mathfrak{A}_{SO(3)}$, where $f \in C^\infty(SO(3))$ and $\mathfrak{A}_{SO(3)}$ is an abstract *-algebra of operators. Particularly, we consider a complete set of orthonormal basis states $\{|g\rangle \mid g \in SO(3)\}$, labelled by group elements, which we will call the group basis, and satisfying

$$f(\hat{g})|g\rangle \equiv f(g)|g\rangle \tag{4.6}$$

and

$$\langle g|g'\rangle \equiv \delta(g^{-1}g') \quad \text{and} \quad \int_{SO(3)} \mathrm{d}g \ |g\rangle\langle g| \equiv \hat{\mathbb{I}}_{L^2(SO(3))} \ , \tag{4.7}$$

for any function $f \in C^\infty(SO(3))$ over $SO(3)$, where $\hat{\mathbb{I}}_{L^2(SO(3))}$ denotes the unit operator, dg the normalized Haar measure. As noted in [37], the property $f(\hat{g})|g\rangle \equiv f(g)|g\rangle$ guarantees that, for *any coordinate system* on $SO(3)$, $|g\rangle$ are the eigenstates of the coordinates. Then, we can define the Hilbert space $\mathcal{H}$ as the set of all states $|\psi\rangle$, whose decomposition in the $|g\rangle$ basis can be expressed in the form

$$|\psi\rangle = \int_{SO(3)} \mathrm{d}g \ \psi(g)|g\rangle \tag{4.8}$$

where $\psi \in L^2(SO(3), dg)$, and $\langle\psi|\psi\rangle = 1$, so that:

$$\langle g|f(\hat{g})|\psi\rangle = f(g)\psi(g) \tag{4.9}$$

For the Lie algebra operators $\hat{P}_i$, corresponding to momentum operators, in the group basis $\{|g\rangle|\ g \in SO(3)\}$ we define the self-adjoint operators in $L^2(SO(3), \mathrm{d}g)$:

$$\hat{P}_i \equiv -i\mathcal{L}_{T_i} \tag{4.10}$$

in the sense

$$\langle g|\hat{P}_i|\psi\rangle = -i\mathcal{L}_{T_i}\psi(g) \tag{4.11}$$

where $\mathcal{L}_{T_i}$ is the Lie derivative on $SO(3)$ with respect to the right-invariant basis vector field $T_i \in TSO(3)$, since these are the generators of translations on $SO(3)$.

With these choices, (4.5) are satisfied due to the properties of Lie derivatives, so that the commutators correctly reflect the classical Poisson structure (4.3) of the canonical variables. Finally, the quadratic Casimir operator is $\hat{P}^2 := \sum_i \hat{P}_i^2 \equiv -\Delta_{LB}$, where Δ_{LB} is the Laplace-Beltrami operator on $SO(3)$.

4.5 Spin basis and coherent states

In this section we review the harmonic analysis of functions on $SO(3)$ in terms of spin representations using Peter-Weyl theorem. We also introduce an over-completed basis of coherent states [37], related to the Perelomov coherent states on a 2-sphere S^2 [90], since they are a very useful tool in the new Spin Foam models [38].

From harmonic analysis of functions on compact Lie groups, it is known that every function on $SO(3)$ can be decomposed in terms of spin representations. Specifically, the Peter-Weyl theorem shows that $L^2(SO(3))$ decomposes into an orthogonal direct sum of all the irreducible unitary representations, (also called Wigner D-functions) $D^j_{mn}(g)$, $j \in \mathbb{N}_0$, $m, n = 0, \pm 1, \dots, \pm j$, in which the multiplicity of each irreducible representation is equal to its degree (that is, the dimension of the underlying space of the representation), so that

$$f(g) = \sum_{jmn} d_j f^j_{mn} D^j_{mn}(g) \tag{4.12}$$

where $d_j = 2j+1$ is the dimension of the representation j, and:

$$\int_{SO(3)} \mathrm{d}g\ \overline{D^{j'}_{m'n'}(g)} D^j_{mn}(g) = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'} \tag{4.13}$$

Therefore, we can define the spin network states $|j;n,m\rangle$ such that $\langle g|j;n,m\rangle = D^j_{mn}(g)$, which constitute a basis in the Hilbert space $\mathcal{H} \cong L^2(SO(3))$, and satisfy

$$\langle j';m',n'|j;m,n\rangle = \delta^{j'j}\delta_{m'm}\delta_{n'n} \quad \text{and} \quad \sum_{j,m,n} d_j |j;m,n\rangle\langle j;m,n| = \hat{\mathbb{I}}_{L^2(SO(3))}\ , \tag{4.14}$$

where $j \in \mathbb{N}_0$, $m,n = 0,\pm 1,\ldots,\pm j$.
Any state $|\psi\rangle \in \mathcal{H}$ can be expanded in the spin basis as

$$\begin{aligned} |\psi\rangle &= \sum_{j,m,n} d_j |j;n,m\rangle\langle j;n,m|\psi\rangle = \sum_{j,m,n} d_j|j;n,m\rangle \int_{SO(3)} \mathrm{d}g \langle j;n,m|g\rangle\, \langle g|\psi\rangle \\ &= \sum_{j,m,n} d_j |j;n,m\rangle \int_{SO(3)} \mathrm{d}g \overline{D^j_{mn}}(g)\psi(g) \end{aligned} \tag{4.15}$$

From the associativity of the group product, it follows that left- and right-invariant derivatives commute, so that the operators $\hat{P}^{L,R}_z := -i\mathcal{L}_{T^{L,R}_z}$, where $T^{L,R}_z \in TSO(3)$ are left- and right-invariant vector fields such that $T^L_z(e) = T^R_z(e) \equiv t_z \in \mathfrak{so}(3)$, together with the Casimir operator $\hat{P}^2$ form a maximal commuting set of operators on the Hilbert space $\mathcal{H}$. Thus, we have

$$\begin{aligned} \hat{P}^2|j;m,n\rangle &= j(j+1)|j;m,n\rangle \\ \hat{P}^L_z|j;m,n\rangle &= m|j;m,n\rangle \\ \hat{P}^R_z|j;m,n\rangle &= n|j;m,n\rangle \end{aligned} \tag{4.16}$$

We note that Casimir operators corresponding to left- and right-invariant Lie derivatives coincide, $\hat{P}^2 = (-i)^2\sum_i \mathcal{L}_{T_i}\mathcal{L}_{T_i}$.

As we mentioned above, we can define a set of coherent states $|j;\vec{m},\vec{n}\rangle$, $j \in \mathbb{N}_0$, $\vec{m},\vec{n} \in \mathbb{S}^2$ (the unit 2-sphere in $\mathbb{R}^3$), in terms of the representation matrices as

$$\langle g|j;\vec{m},\vec{n}\rangle := D^j_{jj}(g^{-1}_{\vec{n}} g g_{\vec{m}})\ , \tag{4.17}$$

so that

$$\sum_j d_j^2 \int \mathrm{d}^2\vec{n}\mathrm{d}^2\vec{m} |j;\vec{m},\vec{n}\rangle\langle j;\vec{m},\vec{n}| = \hat{\mathbb{I}}_{L^2(SO(3))} \tag{4.18}$$

where $g_{\vec{m}} \in SO(3)$ is the unique element, which rotates the unit vector in z-direction in $\mathbb{R}^3$ to $\vec{m}$ with the axis of rotation in the x-y-plane (similarly for $\vec{n}$). These states are related to Perelomov coherent states [90] $|j,\vec{n}\rangle_P$ on a 2-sphere S^2 via $\langle g|j;\vec{m},\vec{n}\rangle \equiv {}_P\langle j,\vec{m}|D^j(g)|j,\vec{n}\rangle_P$ and satisfy the following properties:

$$\hat{P}^2|j;\vec{m},\vec{n}\rangle = j(j+1)|j;\vec{m},\vec{n}\rangle \tag{4.19}$$

$$\begin{aligned} \vec{m}\cdot\hat{P}^L|j;\vec{m},\vec{n}\rangle = j|j;\vec{m},\vec{n}\rangle \quad &\text{and} \quad \vec{n}\cdot\hat{P}^R|j;\vec{m},\vec{n}\rangle = j|j;\vec{m},\vec{n}\rangle\ , \\ \langle j;\vec{m},\vec{n}|\hat{P}^L_i|j;\vec{m},\vec{n}\rangle = j\vec{m}_i \quad &\text{and} \quad \langle j;\vec{m},\vec{n}|\hat{P}^R_i|j;\vec{m},\vec{n}\rangle = j\vec{n}_i\ , \end{aligned} \tag{4.20}$$

4.6 The non-commutative momentum basis

Now, recall that the phase space decomposes as $T^*SO(3) = SO(3) \times \mathfrak{so}(3)$. In the previous section we have seen how to represent functions defined on the group $SO(3)$, the coordinate space. In this section, following [36, 37] we introduce a representation acting on functions of the classical dual space $\mathfrak{so}(3)$, the momentum space.
Since the operators $\hat{P}_i$ are non-commuting, we cannot find a basis of states diagonalizing them simultaneously. However, we can introduce a star-product $\star$ that deforms the theory, so that the commutation relations are satisfied. Thus, we define a set of states $\{|P\rangle \mid P \in \mathbb{R}^3_\star\}$ such that for $i = 1, 2, 3$:

$$\begin{aligned} \langle P|\hat{P}_i^L|\psi\rangle &= \psi(P) \star P_i \\ \langle P|\hat{P}_i^R|\psi\rangle &= P_i \star \psi(P) \\ \langle P|\hat{\zeta}^i|\psi\rangle &= -i\partial^i \psi(P) \end{aligned} \tag{4.21}$$

Since, as we have seen before, left- and right-invariant derivatives commute, the states $|P\rangle$ are simultaneously eigenstates of $\hat{P}_i^L = -i\mathcal{L}_{T_i^L}$ and $\hat{P}_i^R = -i\mathcal{L}_{T_i^R}$ corresponding to left- and right-invariant basis vector fields T_i^L and T_i^R.
Moreover, we have:

$$\langle P|P'\rangle = \delta_\star(P - P') \quad \text{and} \quad \int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} |P\rangle \star \langle P| = \hat{\mathbb{I}}_{L^2(SO(3))} \ , \tag{4.22}$$

So that, any state $|\psi\rangle \in \mathcal{H}$ has a representation in terms of the momentum basis as

$$|\psi\rangle = \int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} |P\rangle \star \langle P|\psi\rangle = \int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} |P\rangle \star \psi(P) \tag{4.23}$$

where $\psi(P) \equiv \langle P|\psi\rangle \in L^2(\mathbb{R}^3_\star)$.

Finally, we need to clarify the relation between the quantization map $\mathcal{Q}$, the $\star$-product, and the operator ordering. From the commutation relation (4.5), the $\star$-product needs to satisfy the condition:

$$\begin{aligned} \langle P|[\hat{P}_i, \hat{P}_i]|\psi\rangle &= [i\epsilon_{ij}^k P_k] \star \psi(P) \\ &= (P_i \star P_j - P_j \star P_i) \star \psi(P) \end{aligned} \tag{4.24}$$

giving a condition on the $\star$-product. In fact, we impose the stronger condition [36]:

$$\langle P|f(\hat{P})|\psi\rangle = f_\star(P) \star \psi(P) \tag{4.25}$$

for all $f_\star \in C^\infty(\mathfrak{so}(3))$ such that $f(\hat{P}) = \mathcal{Q}(f_\star)$.

Importantly, the choice of the quantization map $\mathcal{Q}$ determines uniquely the $\star$-product, since for all $f_\star, f'_\star \in C^\infty(\mathfrak{so}(3))$ we have [36]:

$$f_\star \star f'_\star = \mathcal{Q}^{-1}(\mathcal{Q}(f_\star)\mathcal{Q}(f'_\star)) \tag{4.26}$$

Moreover, the choice of the quantization map $\mathcal{Q}$ encodes the operator ordering coming from the non-commutativity of the elements $\hat{P}_i$. For example, one can choose the standard ordering $\mathcal{Q}(P_i^n P_j^m) = \hat{P}_i^n \hat{P}_j^m$, or the Weyl ordering $\mathcal{Q}(P_i^n P_j^m) = \mathcal{S}(P_i^n P_j^m)$, where $\mathcal{S}$ is the total symmetrization map, or the ordering coming from the Duflo map $\mathcal{D}$, $\mathcal{Q}(P_i^n P_j^m) = \mathcal{D}(P_i^n P_j^m)$, etc.

4.7 Non-commutative Fourier transform

In Section 4.5 we have seen that the group basis $|g\rangle$ and the spin basis $|j;m,n\rangle$ are related by the unitary transformation, the Wigner D matrix:

$$|j;m,n\rangle = \int_{SO(3)} \mathrm{d}g|g\rangle D^j_{nm}(g) \tag{4.27}$$

In this section we study the unitary transformation that relates the non-commutative momentum basis and the group basis, the non-commutative Fourier transform on $SO(3)$, first introduced in [91, 92], and subsequently extended to $SU(2)$ in [93] and based on the theory of quantum groups and Hopf algebras [94]. Recently, it has been utilized in the formulation of a non-commutative flux representation of Loop Quantum Gravity [54] and a metric representation of Group Field Theory [48, 95], which will be a key point of this thesis. Despite having a solid mathematical formulation, we are interested mainly in its pragmatical aspects, however the interested reader can fine more mathematical details in [91, 92].

The non-commutative Fourier transform on $SO(3)$ is an intertwiner $\mathcal{F} : L^2(SO(3)) \to L^2_\star(\mathfrak{so}(3))$ between the group and algebra representations, which can be expressed as an integral transform. Namely,

$$\begin{aligned}\psi(P) \equiv \langle P|\psi\rangle := \mathcal{F}(\psi)(P) &= \int_{SO(3)} \mathrm{d}g\, \langle P|g\rangle\, \langle g|\psi\rangle \\ &= \int_{SO(3)} \mathrm{d}g\, E(g,P)\, \psi(g) \in L^2_\star(\mathfrak{so}(3))\end{aligned}$$

where $\psi \in L^2(SO(3))$, and $E(g,P)$ are the non-commutative plane waves defined, in terms of the $\star$-product, as

$$E(g,P) = \eta(g) e^{i\zeta(g)\cdot P} = e_\star^{ik(g)\cdot P} \tag{4.28}$$

where the prefactor $\eta(g) := E(g,0)$ may be non-trivial, depending on the $\star$-product or, equivalently, the quantization map $\mathcal{Q}$ chosen. Moreover, $k(h) = -i\ln(h) \in \mathfrak{g}$ from any branch of the logarithm, and we introduced the $\star$-exponential notation

$$e_\star^{f(P)} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star \cdots \star f}_{n \text{ times}}(P)\,.$$

4.8 Properties of the non-commutative Fourier transform

Let us list some important properties of the non-commutative plane wave $E_g(P)$ [36], which we will use in the following:

$$E_g(P) = e_\star^{ik(g)\cdot P} \;=\; \eta(g) e^{i\zeta(g)\cdot P}\,, \tag{4.29}$$

$$E_e(P) = 1\,, \tag{4.30}$$

$$\mathcal{Q}(E_g(P)) = e^{ik(g)\cdot \hat{P}} \quad \in \mathfrak{A}_{\mathfrak{so}(3)}\,, \tag{4.31}$$

$$E_{g^{-1}}(P) = \overline{E_g(P)} = E_g(-P)\,, \tag{4.32}$$

$$E_{gh}(P) = E_g(P) \star E_h(P). \tag{4.33}$$

In addition, using $\eta(e) = E_e(0) \equiv 1$ and the properties of the ζ-coordinates, namely, $\zeta(e) = 0$ and $\mathcal{L}_i \zeta^j(e) = \delta_i^j$, it is possible to show that:

$$\int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} E_g(P) = \delta^3(\zeta(g)) = \delta(g) \,, \tag{4.34}$$

where the right-hand side is the Dirac delta distribution with respect to the right-invariant Haar measure on $SO(3)$.

Moreover, we have the important identity [37]:

$$\int_{SO(3)} \mathrm{d}g \; E_g(P) = (2\pi)^3 \delta_\star(P) \;, \tag{4.35}$$

where $\delta_\star$ is the delta function with respect to the $\star$-product in the sense

$$\int_{\mathbb{R}^3_\star} \mathrm{d}^3 P \; \delta_\star(P) \star f(P) = f(0) = \int_{\mathbb{R}^3_\star} \mathrm{d}^3 P \; f(P) \star \delta_\star(P) \tag{4.36}$$

We should note that $\delta_\star(P)$ is a regular function on $\mathbb{R}^3$, and in particular $\delta_\star(0) < \infty$. Furthermore, it has been shown [36, 37] that the non-commutative group Fourier transform is an isometry with respect to the L^2-norms:

$$\int_{SO(3)} \mathrm{d}g \overline{f(g)} f'(g) = \int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} \; \overline{f(P)} \star f'(P) \tag{4.37}$$

Finally, the inverse transform is given by:

$$f(g) \equiv \int_{\mathbb{R}^3_\star} \frac{\mathrm{d}^3 P}{(2\pi)^3} \; \overline{E_g(P)} \star f(P) \tag{4.38}$$

where $\mathrm{d}^3 P$ is the usual Lebesgue measure on $\mathbb{R}^3$.

4.9 An explicit example: the Freidel-Livine-Majid map

As we have seen, the choice of the quantization map determines the $\star$-product used in defining the algebra representation, the ordering of the operators, and the definition of the non-commutative Fourier transform, particularly the form of the plane waves.
Following [36], we now provide an example of our construction: the Freidel-Livine-Majid quantization map $\mathcal{Q}_{FLM}$, that we will use in the following to make explicit calculations.

Actually, we introduce $\mathcal{Q}_{FLM}$ by defining the non-commutative plane waves on $SO(3)$ [37]:

$$E_g(P) := \exp[Tr(gP)] \equiv \exp[i\chi^i(g) P_i] \tag{4.39}$$

where the trace $Tr(gP)$ is taken in the fundamental 2-dimensional representation of $SO(3)$, which is obtained from $g' \in SU(2)$ via the 2-to-1 map $g' \mapsto \mathrm{sgn}(Tr(g'))g' \equiv g \in SO(3)$.
The coordinates $\chi^i(g)$ in the exponential are given by:

$$\chi^i(g) \equiv Tr(g\sigma^i) \equiv \xi^i(g) \frac{\sin(|X(g)|)}{|X(g)|} \;, \tag{4.40}$$

where $g \equiv \exp[i\xi^i(g) t_i]$.

The non-commutative $\star$-product is then defined through the relation:

$$E_g(P) \star E_h(P) \equiv E_{gh}(P) \qquad \forall \; g, h \in SO(3), \; P \in \mathfrak{so}(3) \tag{4.41}$$

Moreover, by explicit calculation, it can be shown that [37]:

$$P_i \star P_j = P_i P_j + i\epsilon_{ij}{}^k P_k \ , \tag{4.42}$$

so the coordinates P_i satisfy Lie algebra type of commutation relations, which follow directly from the properties of the Lie derivative.

Furthermore, with this choice of coordinates, the transformation between the spin basis and the non-commutative momentum basis is mediated by the transformation kernel [37, 96, 97]:

$$\langle j; m, n|P\rangle \;=\; \int_{SO(3)} \mathrm{d}g \;\; \overline{D^j_{nm}(g)}\,\overline{E_g(P)} = \frac{2J_{2j+1}(|P|)}{i^{2j}|P|} D^j_{mn}\left(e^{i\frac{\pi}{2}\frac{P}{|P|}}\right)$$

In the semi-classical limit $j \to \infty$, the function $\frac{2J_{2j+1}(|P|)}{i^{2j}|P|}$ approximates a delta function $\delta(|P|-j)$ [37, 96].

Moreover, as also noted in [37, 54], for the coherent states $|j; \vec{n}; \vec{n}\rangle$ it can be shown that in the semi-classical limit, $j \to \infty$, the momentum variables P become commutative, their non-commutative Fourier transform

$$\langle P|j; \vec{n}; \vec{n}\rangle = \int_{SO(3)} \mathrm{d}g \; E_g(P) D^j_{jj}(g^{-1}_{\vec{n}} g g_{\vec{n}}) \tag{4.43}$$

peaks sharply at $P = j\vec{n}$, and therefore we find an identification of the non-commutative dual variables and the coherent state variables in this limit.

Chapter 5

Spin Foams and the path integral for gravity

In this section, following [20], we introduce the general idea behind Spin Foam models, a covariant approach to the construction of a quantum theory of gravity.
Particularly, the aim of Spin Foams is to represent a formulation of Quantum Gravity which is fully background independent and non-perturbative, in the sense that there is no fixed background geometry, and quantization follows the spirit of path integral formulation of Quantum Field Theories, as a sum-over-histories in a purely combinatorial-algebraic context [17].
Historically, covariant quantum gravity based on Feynman's path integral has been considered by Misner, Hawking, Hartle and others [98, 99]. Given a 4-dimensional globally hyperbolic space-time $\mathcal{M} = \Sigma \times \mathbb{R}$, where, for simplicity, Σ is usually chosen to be compact and simply connected, with the topology of the 3-sphere S^3, we consider all the possible space-time metrics G on $\mathcal{M}$, and we call geometries $[g] \in G/Diff(\mathcal{M})$ the space-time metrics up to 4-diffeomorphisms that are compatible with $\mathcal{M}$. Let Σ_1 and Σ_2 be boundaries of $\mathcal{M}$. Then, the transition amplitude between $|[q_{ab}]\rangle$ on Σ_1 and $|[q'_{ab}]\rangle$ on Σ_2 is formally

$$\langle [q_{ab}]\, |[\, q'_{ab}\,]\rangle = \int\limits_{[g]} \mathcal{D}[g] \quad e^{iS([g])}, \tag{5.1}$$

where the integration on the right is performed over all space-time geometries with fixed boundary values up to 3-diffeomorphisms $[q_{ab}]$, $[q'_{ab}]$, respectively. The expression above (5.1) is purely formal, for a variety of well-known reasons (see for example [20]).

Fortunately, Spin Foam models have several advantages with respect to this seminal idea, and many different approaches converged recently to this new framework: Loop Quantum Gravity (LQG), Topological Quantum Field Theory (TQFT), lattice gauge theory, path integral or sum-over-histories quantum gravity, Causal sets, Regge calculus, Group Field Theories (GFT) to name but a few [17].

Particularly, the introduction of spin-network states describing 3-geometry solves many of the problems of original covariant quantization, since they carry the diff-invariant information of the Riemannian structure of Σ. Furthermore, these states suggest the possibility of constructing a notion of Feynman 'path integral' in a combinatorial manner involving sums over spin network world sheets amplitudes. Heuristically, '4-geometries' are to be represented by 'histories' of quantum states of 3-geometries or spin network states. These 'histories' are Spin Foams [17, 20, 32], i.e. 2-complexes (collection of faces bounded by links joining at vertices) with representations of the Lorentz (or $SU(2)$) group attached to its faces, in such a way that any slice or any boundary of it, corresponding to a spatial hyper-surface, will be given by a spin network. Therefore, the 2-complex can be thought of as representing space-time while the boundary graphs as representing space.

Spin Foam models [17,20,32] are intended to give a path integral quantization of gravity based on these purely algebraic and combinatorial structures.

Mathematically, the more general definition of a Spin Foam is based on category theory [19,100]. For simplicity, here we give the following [19,20]:

Definition 5.0.1. A Spin Foam $\mathcal{F} : s \to s'$, representing a transition from the spin-network $s = (\gamma, \{j_\ell\}, \{\iota_n\})$ into $s' = (\gamma', \{j_{\ell'}\}, \{\iota_{n'}\})$, is defined by

- a 2-complex $\mathcal{J}$ bordered by the graphs of γ and γ' respectively,
- a collection of spins $\{j_f\}$ associated with faces $f \in \mathcal{J}$,
- a collection of intertwiners $\{\iota_e\}$ associated to edges $e \in \mathcal{J}$.

Both spins and intertwiners of exterior faces and edges match the boundary values defined by the spin networks s and s' respectively.

Spin Foams $\mathcal{F} : s \to s'$ and $\mathcal{F}' : s' \to s''$ can be composed into $\mathcal{F}\mathcal{F}' : s \to s''$ by gluing together the two corresponding 2-complexes at s'.

Finally,

Definition 5.0.2. A Spin Foam model is an assignment of amplitudes $A[\mathcal{F}]$ which is consistent with this composition rule in the sense that

$$A[\mathcal{F}\mathcal{F}'] = A[\mathcal{F}]A[\mathcal{F}']. \tag{5.2}$$

Transition amplitudes between spin network states are defined by

$$\langle s, s' \rangle_{phys} = \sum_{\mathcal{F}:s\to s'} A[\mathcal{F}], \tag{5.3}$$

where the notation anticipates the interpretation of such amplitudes as defining the physical scalar product.

We conclude this introductory section by noticing that [20]:

- These definition are formal, as we need to specify the set of allowed Spin Foams in the sum and define the corresponding amplitudes.
- The background-independent character of Spin Foams is related to the fact that geometry is encoded in the spin labellings which represent the degrees of freedom of the gravitational field. Thus, we have an implementation of a sum-over-histories for gravity in a purely combinatorial-algebraic context.

5.1 Spin Foams and the projection operator into $\mathcal{H}_{phys}$

As we have seen, the boundary states of Spin Foam models are represented by graphs coloured with representations and intertwiners of a certain group, i.e. spin networks states, which appear as the so-called kinematical states in the canonical loop quantization of first order gravity. Loop Quantum Gravity is characterized by three constraints: Gauss constraint, spatial diffeomorphisms constraint (both solved by generalized spin network states), and the Hamiltonian constraint, which implements the dynamics of the theory and whose solutions are still unknown. The crucial observation to relate Loop Quantum Gravity and Spin Foams is that in a generally covariant theory the path integral provides a device for constructing solutions to the quantum constraints. Thus, Spin Foams naturally arise in the formal definition of the exponentiation of the Hamiltonian constraint as studied in [101, 102]. Following [20, 83] we present the idea of Reisenberger and Rovelli, that is to construct the projection operator using the path integral representation of the δ-distribution,

$$P = \prod_{x\in\Sigma} \delta(\hat{\mathcal{S}}(x)) = \int \mathcal{D}[N]\; e^{i\hat{\mathcal{S}}[N]}, \tag{5.4}$$

where $N(x)$ is the lapse function:

$$\hat{\mathcal{S}}[N] = \int \mathrm{d}x^3 N(x)\hat{\mathcal{S}}(x), \tag{5.5}$$

We can think of this formula as a group averaging map which enables to define the physical inner product as

$$\langle s, s'\rangle_{phys} = \langle sP, s'\rangle\,, \tag{5.6}$$

where s and s' are two kinematical states.
Reisenberger and Rovelli considered a truncated version P_Λ (where Λ can be regarded as an infrared cut-off), and a diffeomorphism invariant measure $\mathcal{D}[N]$ generalizing the techniques of [103]. Then, using an expansion of the exponential in (5.4), this inner product becomes

$$\langle sP_\Lambda, s'\rangle = \int\limits_{|N(x)|\le\Lambda} \mathcal{D}[N]\left\langle s\sum_{n=0}^{\infty}\frac{i^n}{n!}(\mathcal{S}[N])^n, s'\right\rangle. \tag{5.7}$$

A key property of the Hamiltonian constraint is that it acts only at the nodes of a graph, creating new links and nodes, and thereby changing the geometry encoded in the state. Thus,

$$\langle sP_\Lambda, s'\rangle = \sum_{n=0}^{\infty}\frac{i^n\Lambda^n}{n!}\sum_{\mathcal{F}_n:s\to s'} A[\mathcal{F}_n] = \sum_{n=0}^{\infty}\frac{i^n\Lambda^n}{n!}\sum_{\mathcal{F}_n:s\to s'}\prod_v A_v(\rho_v,\iota_v),$$

where $\mathcal{F}_n : s \to s'$ are Spin Foams generated by n actions of the scalar constraint, i.e., spin foams with n vertices.

Remarkably, we see that the Spin Foam amplitude $A[\mathcal{F}_n]$ precisely factorizes in a product of vertex contributions $A_v(\rho_v, \iota_v)$ depending on the spins j_v and ι_v neighbouring faces and edges. Thus, we see that the properties of spin network states, together with a suitably regularized Hamiltonian constraint, should lead to a certain Spin Foam model [83].

Finally, we would like to discuss the relation between the matrix elements of the P operator and the discretization independence obtained by a refinement procedure [20].
Particularly, if discretization independence is obtained by a refinement procedure, then the number of simplexes N in the triangulation plays the role of cut-off and we define:

$$\langle s, s'\rangle_{phys} = \langle sP, s'\rangle = \lim_{N\to\infty}\langle sP_N, s'\rangle \tag{5.8}$$

where $\langle sP_N, s'\rangle$ are the transition amplitudes corresponding to a fixed triangulation with N simplexes.

5.2 Spin Foams from lattice gravity

In this section we consider the relation between Spin Foam models and lattice gauge theory, originally studied by Reisenberger [66].

In the case of Quantum Gravity, both lattice gauge theory and Spin Foam models result in a covariant quantization of four dimensional gravity, based on discrete structures, however the interpretation of these structures is fundamentally different. Indeed, the background-independent character of Spin Foams forbids the interpretation of the truncation simply as a UV regulator.

Following [20], we present a brief outline of the formulation, for details see [22, 66, 104, 105]. Starting from the formal path integral of the first order formulation of gravity

$$\mathcal{Z} = \int \mathcal{D}[e]\ \mathcal{D}[A]\ e^{iS(e,A)} = \int \mathcal{D}[A]\ e^{iS_{\text{eff}}(A)}, \tag{5.9}$$

where we formally integrated over the tetrad e obtaining an effective action S_{eff}.
Then we replace continuum space-time by a cellular decomposition, specifically a simplicial complex Δ. The triangulation specifies another decomposition of the manifold into cells called the 'dual complex'. There is a one-to-one correspondence between k-simplices in the triangulation of an n-manifold and $(n-k)$-cells in the dual complex, each k-simplex intersecting its corresponding dual $(n-k)$-cell in a single point. In four dimensional Quantum Gravity, we denote the dual 2-skeleton Δ^* constituted by edges $e \in \Delta^*$ and faces $f \in \Delta^*$, and we discretize the connection by assigning a group element g_e to edges

$$A \to \{g_e\}.$$

The connection integration is represent by the Haar measure on the group:

$$\mathcal{D}[A] \to \prod_{e\in\Delta^*} \mathrm{d}g_e.$$

Finally, upon discretization, we assign to the curvature $F(A)$ the holonomy around faces g_f corresponding to the product of the g_e's which we denote $g_f = g_e^{(1)} \cdots g_e^{(n)}$:

$$F(A) \to \{g_f\}.$$

Since the action of gravity depends on the connection A only through the curvature $F(A)$, we have $S_{eff}(A) \to S_{eff}(\{g_f\})$ and the lattice path integral becomes:

$$\mathcal{Z} = \int \prod_{e\in\Delta^*} \mathrm{d}g_e\ \exp\left[iS_{\text{eff}}(\{g_f\})\right]. \tag{5.10}$$

Following Reisenberger, we can assume that $S_{\text{eff}}(\{g_f\})$ is local by demanding that the amplitude of any piece of the 2-complex, obtained as its intersection with a ball, depends only on the value of the connection on the corresponding boundary. As a consequence there is a maximal splitting corresponding to the intersection of a dual vertex in Δ^* (corresponding to a 4-simplex in Δ) with a 3-sphere defining elementary building blocks, called *atoms*.

The atom amplitude, also called vertex amplitude, depends on the boundary data given by the value of the holonomies around the ten faces of the 4-simplex σ. This amplitude can be represented by a function $\mathcal{V}(\rho_v, \iota_v)$ depending on ten spins ρ_{ij} labelling the faces and five intertwiners ι_i labelling the edges. The total amplitude is given by gluing the atoms together, summing over common values of spin labels and intertwiners:

$$\mathcal{Z}[\Delta^*] = \sum_{\mathcal{C}_f:\{f\}\to\{\rho_f\}} \sum_{\mathcal{C}_e:\{e\}\to\{\iota_e\}} \prod_{f\in\Delta^*} A_f(\rho_f, \iota_f) \prod_{v\in\Delta^*} \mathcal{V}(\rho_v, \iota_v), \tag{5.11}$$

where $\mathcal{C}_e : \{e\} \to \{\iota_e\}$ denotes the assignment of intertwiners to edges, $\mathcal{C}_f : \{\rho_f\} \to \{f\}$ the assignment of spins ρ_f to faces, and $A_f(\rho_f, \iota_f)$ is the face amplitude arising in the integration over the lattice connection. Thus, as we have anticipated, the lattice definition of the path integral for gravity and covariant gauge theories becomes a discrete sum of Spin Foam amplitudes.

5.3 Spin Foams as Feynman diagrams

In the previous sections we have seen that we can compute the transition amplitude between two spin networks as a sum over Spin Foams going from the first spin network to the second. Each Spin Foam contributes to the amplitude an amount given by a product of amplitudes associated to its vertices, edges, and faces. Moreover, we have seen that we can define a vertex function by intersecting the Spin Foam with a small sphere centered at the vertex. As already pointed out in [19], Spin Foams can be interpreted in close analogy to Feynman diagrams. Standard Feynman graphs are generalized to 2-complexes where the edge and face amplitudes can be thought of as propagators, while Spin Foam vertices can be thought of as interactions, and the vertex amplitudes characterize the non-trivial dynamics of the theory [20].
This analogy has been motivated by the generalized matrix models of Boulatov and Ooguri [106,107] and recently has been realized in the so-called Group Field Theory/Spin Foam duality. The remarkable fact is that every Spin Foam model admit a dual formulation in terms of a field theory over a group manifold [108–110], in the sense that Spin Foam amplitudes correspond to Feynman diagrams of the GFT, which in turns correspond to simplicial path integral [32], as we will see in Section 5.7.

5.4 Discretization dependence

As we have seen, Spin Foam models are defined on a fixed cellular decomposition of $\mathcal{M}$, which reduces the infinite dimensional functional integral to a multiple integration over a finite number of variables.
However, transition amplitudes are discretization dependent, as the fixed cellular decomposition imposes restrictions on the allowed Spin Foams in the path integral (only those that can be obtained by all possible coloring of the underlying 2-complex) and on the possible 3-geometry spin network states on the boundary (only those that can be obtained by all possible coloring of the underlying boundary graph).
Therefore, a consistent definition of the path integral using Spin Foams should include a prescription to eliminate the discretization dependence [20].

Importantly, the nature of the truncation is different from the UV regulator of lattice gauge theories given by the lattice spacing. In fact, since in Spin Foam context geometry is encoded in the coloring (that can take any spin values) the configurations involve fluctuations all the way to Planck scale, and the truncation cannot be regarded as an UV cut-off as in the background dependent context of lattice gauge theories. Moreover, in the background independent context the triangulation contains only topological information and there is no geometrical meaning associated to its components.

A special case is that of topological theories, where there are no local excitations (no gravitons), and the result is independent of the chosen cellular decomposition.
In this case, we can define the sum over Spin Foams with the aid of a fixed cellular decomposition Δ of the manifold, which suffices to capture the degrees of freedom of the topological theory.

For the general case there are two main approaches [20]:

- *Refinement of the discretization:* According to this idea, topology is fixed by the simplicial decomposition. The truncation in the number of degrees of freedom should be removed by considering triangulations of increasing number of simplexes for that fixed topology. The flow in the space of possible triangulations is controlled by the Pachner moves. In the general case there is a great deal of ambiguity involved in the definition of refinement. The hope is that the nature of the transition amplitudes would be such that these ambiguities will not affect the final result.

- *Spin Foams as Feynman diagrams:* According to this idea the discretization independence is recover in the context of Group Field Theories, where the perturbative Feynman expansion of the GFT provides a definition of *sum over* discretizations which is fully combinatorial and hence independent of any manifold structure.
  Nevertheless, the convergence issues become more involved. Despite the fact the perturbative series are generically divergent, physical information can be obtained by the asymptotic limit, or using re-summation techniques.

5.5 A note on the signature: Riemannian gravity

Until now, we have considered the Lorentzian theory, since it is the physical one. However, the Lorentz group is non compact and this leads to some difficulties. These problems are in general technical difficulties rather than physical or conceptual. Specifically, the non-commutative Fourier transform has not been yet developed for non compact group. Therefore, from here on we will consider the Riemannian case, characterized by the gauge group $SO(4)$ or its double covering $Spin(4)$. As soon as new techniques are developed, we generalize the results of this thesis to the Lorenzian case.

5.6 Spin Foam model for BF theory with compact gauge group

In this section, following [14, 111], we consider the Spin Foam quantization of BF theory since it is the starting point for the definition of the models we will present in this thesis.
The reason is that General Relativity can be described by certain BF theory action plus Lagrange multiplier terms imposing certain algebraic constraints on the fields [112].
In the case of BF theory, the integration over the tetrad we formally performed in (5.9) is possible, thus BF theory can be quantized along the lines of the previous sections. Interestingly, due to the topological nature of BF theory, BF Spin Foam amplitudes are simply given by certain invariants in the representation theory of the gauge group.
Therefore, the standard strategy to define a Spin Foam model for gravity is to quantize the BF theory and then impose restrictions on the Spin Foams that enter in the partition function of the BF theory. These restrictions are essentially the solution of the quantum version of the simplicity constraints, i.e. the constraints that reduce BF theory to General Relativity.

Moreover, BF theory in three dimension is particularly studied as it describes $2+1$ dimensional quantum gravity [111].

Now, we begin the quantization of BF theory.
Let G be a compact group whose Lie algebra $\mathfrak{g}$ has an invariant inner product here denoted $\langle \cdot, \cdot \rangle$, and $\mathcal{M}$ a d-dimensional manifold.
Classical BF theory is defined by the action

$$S[B, \omega] = \int_{\mathcal{M}} \langle B \wedge F(\omega) \rangle \,, \tag{5.12}$$

where B is a $\mathfrak{g}$ valued $(d-2)$-form, ω is a connection on a G principal bundle over $\mathcal{M}$.
As mentioned before, BF theory is a topological theory, as there are only global or topological degrees of freedom. Actually, no local excitations are present, all solutions of the equations of

motion are locally related by gauge transformations, locally pure gauge:

$$\delta B = [B, \alpha] \quad \text{and} \quad \delta\omega = d_\omega \alpha \ , \tag{5.13}$$

where α is a $\mathfrak{g}$-valued 0-form, and the 'topological' gauge transformation:

$$\delta B = \mathrm{d}_\omega \quad \text{and} \quad \delta\omega = 0 \ , \tag{5.14}$$

where d_ω denotes the covariant exterior derivative and η is a $\mathfrak{g}$-valued 0-form.

For the moment we assume $\mathcal{M}$ to be compact and orientable. The partition function, $\mathcal{Z}$, is given by

$$\mathcal{Z} = \int \mathcal{D}[B]\mathcal{D}[\omega] e^{i\int S[B,\omega]} = \int \mathcal{D}[B]\mathcal{D}[\omega] \exp\left(i \int_{\mathcal{M}} \langle B \wedge F(\omega) \rangle \right) \tag{5.15}$$

Formally integrating over the B field in (5.15) we obtain that $\mathcal{Z}$ corresponds to the volume of the space of flat connections on $\mathcal{M}$:

$$\mathcal{Z} = \int \mathcal{D}[\omega] \delta(F(\omega)). \tag{5.16}$$

The next step is to give a physical meaning to the formal expressions above.
The standard procedure is to replace the d-dimensional manifold $\mathcal{M}$ with an arbitrary cellular decomposition Δ, that we assume for simplicity to be a triangulation.
Moreover, to each cellular decomposition Δ we can define the associated dual 2-complex Δ^*, i.e. a combinatorial object defined by a set of vertices $v \in \Delta^*$ (dual to d-cells in Δ) edges $e \in \Delta^*$ (dual to $(d-1)$-cells in Δ) and faces $f \in \Delta^*$ (dual to $(d-2)$-cells in Δ).

In the discretization process we smear continuous $(d-2)$-form B on the $(d-2)$-cells in Δ, dual to the faces $f \in \Delta^*$. Particularly, we define the Lie algebra elements B_f assigned to faces $f \in \Delta^*$:

$$B_f = \int_{(d-2)-cell} B \tag{5.17}$$

where we have used the one-to-one correspondence between faces $f \in \Delta^*$ and $(d-2)$-cells in Δ to label the discretization of the B field B_f.

Furthermore, we need to discretize also the connection ω, by the assignment of group elements $g_e \in G$ to edges $e \in \Delta^*$, interpreted as the holonomy of ω along $e \in \Delta^*$, namely

$$g_e = \mathcal{P}\exp\left(-\int_e \omega \right) \tag{5.18}$$

where the symbol "$\mathcal{P}\exp$" denotes the path-order-exponential.

At this point, the BF partition function on the simplicial decomposition Δ is:

$$\mathcal{Z}(\Delta) = \int \prod_{e\in\Delta^*} \mathrm{d}g_e \prod_{f\in\Delta^*} \mathrm{d}B_f e^{iB_f U_f} = \int \prod_{e\in\Delta^*} \mathrm{d}g_e \prod_{f\in\Delta^*} \delta(g_{e_1} \dots g_{e_n}), \tag{5.19}$$

where $U_f = g_{e_1} \dots g_{e_n}$ denotes the holonomy around faces, the integration measure $\mathrm{d}B_f$ is the standard Lebesgue measure and the integration measure $\mathrm{d}g_e$ is the invariant measure in G (which is the unique Haar measure when G is compact), satisfying

$$\int \mathrm{d}gF(g) = \int \mathrm{d}gF(g^{-1}) = \int \mathrm{d}gF(gh) = \int \mathrm{d}gF(hg) \tag{5.20}$$

where $h \in G$ and $F(g)$ is a test function.
Using the Peter-Weyl's theorem we can decompose the Dirac delta distribution appearing in (5.19) in terms of irreducible unitary representations of G, denoted by ρ

$$\delta(g) = \sum_\rho d_\rho Tr[\rho(g)] \tag{5.21}$$

Thus, we have:

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{\rho\}\to\{f\}} \int \prod_{e\in\Delta^*} \mathrm{d}g_e \prod_{f\in\Delta^*} d\rho_f Tr[\rho_f(g_{e_1}\dots g_{e_n})] \tag{5.22}$$

Integration over the connection can be performed as follows. In a triangulation Δ, the edges $e \in \Delta^*$ bound precisely d different faces; therefore, the g_e's in (5.22) appear in d different traces. The relevant formula is

$$P^e_{inv}(\rho_1\dots\rho_d) := \int \mathrm{d}g_e \rho_1(g_e) \otimes \rho_2(g_e) \otimes \dots \rho_d(g_e) = \sum \overline{C^\iota_{\rho_1\rho_2\cdots\rho_d}} C^\iota_{\rho_1\rho_2\cdots\rho_d} \tag{5.23}$$

For compact G it is easy to prove using the invariance (and normalization) of the the integration measure (5.20) that $P^e_{inv} = (P^e_{inv})^2$ is the projector onto $Inv[\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_d]$, and on the right hand side we have expressed the projector in terms of normalized intertwiners.
Finally, integration over the connection yields

$$\mathcal{Z}_{BF}(\Delta) = \sum_{\mathcal{C}_f:\{f\}\to\{\rho_f\}} \prod_{f\in\Delta^*} d_{\rho_f} \prod_{e\in\Delta^*} P^e_{inv}(\rho_1\dots\rho_d) \tag{5.24}$$

In the case $d = 4$ and $G = SO(4)$, we have

$$\mathcal{Z}_{BF}(\Delta) = \sum_{\mathcal{C}_f:\{f\}\to\{\rho_f\}} \prod_{f\in\Delta^*} d_{\rho_f^-} d_{\rho_f^+} \sum_{\mathcal{C}_e:\{e\}\to\{\iota_e\}} \prod_{v\in\Delta^*} \left\{15j\right\}^-_v \left\{15j\right\}^+_v \tag{5.25}$$

where we used the self-dual/anti-self-dual splitting of $SO(4)$ representations, and the symbol $\left\{15j\right\}_v$ associated to the vertex v is the well-known $15j$ symbol from the recoupling theory of angular momentum.

In other words, the BF amplitude associated to a two complex Δ^* is simply given by the sum over all possible assignments of irreducible representations of G to faces of the number obtained by the natural contraction of the network of projectors P^e_{inv} according to the pattern defined by the two-complex Δ^*.
The state sum (5.24) is topologically invariant, discretization independent, but generically divergent, nevertheless Crane and Yetter introduced in [113, 114] a regularized version based on quantum groups, for example if $G = SO(4) \supset SU(2) \times SU(2)$ then the regularized version is based on $SU_q(2) \times SU_q(2)$.

5.7 GFT models for BF theory with compact gauge group

As we have seen in Section 5.3 Spin Foams can be interpreted in close analogy to Feynman diagrams [19, 20]. This analogy is clarified in the context of Group Field Theories.

Group Field Theories, introduced by Boulatov [106] and Ooguri [107], can be seen as a generalization of the matrix models used to construct quantum gravity models in 2d [32], or as a second quantized formulation of canonical Loop Quantum Gravity [115, 116]. Moreover, Group Field Theories can be re-written in terms of Spin Foams or Simplicial path integrals used in simplicial quantum gravity approaches [32], like quantum Regge calculus and Causal Dynamical Triangulations [34].
Particularly, the duality Group Field Theory/Spin Foam model states that given a Spin Foam model defined on an arbitrary 2-complex Δ^* (dual to a triangulation Δ) and the partition function $\mathcal{Z}(\Delta^*)$, there exists a GFT such that the perturbative expansion of the field theory partition function generalizes to a sum over 2-complexes represented by Feynman diagrams of the field theory.

These diagrams look locally as dual to triangulations (vertices are 5-valent, edges are 4-valent) but they are no longer tied to any manifold structure [117]. Furthermore, recently, tools and ideas from non-commutative geometry have been introduced. As a result the GFT formalism is a very flexible and powerful tool, particularly because it is a generalization of well-known Quantum Field Theories (defined on Lie groups manifold or on the corresponding Lie algebras), with combinatorially non-local properties, as we will easily read in the GFT action.
In this section we follow [49, 50], starting by recalling the standard Ooguri GFT for BF theory and its non-commutative bivector representation. Other reviews of the GFT formalism that we may suggest to the reader are [32–34, 118].
In addition, we present an extension of the GFT formalism, where the usual field variables, associated to the four triangles of a tetrahedron, are supplemented by an S^3 vector playing the role of the normal to the tetrahedron. As we will see, it will allow us to implement the linear simplicity constraints (6.16) in a covariant way.

Our notations and conventions are the same as [50]: we identify functions on $SO(4)$ with functions on $SU(2)^- \times SU(2)^+/\mathbb{Z}_2$ and denote by $g = (g^-, g^+)$ the $SU(2)$ decomposition of the field variables. We also use the decomposition of $\mathfrak{so}(4)$ in anti-self dual and self-dual sectors $\mathfrak{so}(4) = \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$ and denote by $x = (x^-, x^+)$ the corresponding decomposition of its elements. From Section 5.7.2 on, based on the $SO(3)$ Fourier transform [91, 92], we further assume an invariance of group functions under $g \to -g$, so that they are effectively functions on $SO(3) \times SO(3)$. However, note that an extension of the non-commutative Fourier transform to the whole $SU(2)$ has been developed in [93], while different $SU(2)$ transform has been proposed and studied in [119]. As in [50], we do not use it in this thesis, because we do not expect the results to be very much modified by such extension.

5.7.1 Connection and spin formulations

The GFT for $SO(4)$ BF theory, also known as the Ooguri GFT model [107] for BF theory, is described in terms of a field $\varphi_{1234} := \varphi(g_1, \cdots, g_4)$ on four copies of the gauge group, $\varphi(g_1, \ldots, g_4) : SO(4)^{\times 4} \to \mathbb{R}$, symmetric under diagonal left action:

$$\forall h \in \mathrm{SO}(4), \quad \varphi(g_1, \cdots g_4) = \varphi(hg_1, \cdots hg_4) \tag{5.26}$$

The dynamics is governed by the GFT action:

$$S[\phi] = \frac{1}{2}\int [\mathrm{d}g_i]^4 \varphi_{1234}^2 + \frac{\lambda}{5!}\int [\mathrm{d}g_i]^{10} \varphi_{1234}\,\varphi_{4567}\,\varphi_{7389}\,\varphi_{962\,10}\,\varphi_{10\,851} \tag{5.27}$$

where $\mathrm{d}g$ is the normalized Haar measure and λ is a coupling constant.
The partition function is:

$$\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-S[\varphi]} \tag{5.28}$$

The perturbative expansion in λ generates 4-stranded Feynman diagrams are 2-complexes dual to 4d simplicial complexes:

$$\mathcal{Z} = \sum_{\Delta^*} \frac{\lambda^N}{sym[\Delta^*]} \mathcal{Z}(\Delta^*) \tag{5.29}$$

where N is the number of interaction vertices in the Feynman graph Δ^*, $sym[\Delta^*]$ is a symmetry factor for the graph and $\mathcal{Z}(\Delta^*)$ the corresponding Feynman amplitude.
Particularly, the combinatorics of the quintic interaction in the action represents the gluing of five tetrahedra (each corresponding to a field φ) pairwise along boundary triangles (each corresponding to one of the 4 arguments of φ) to form a 4-simplex. The kinetic term (quadratic in the field φ) dictates the gluing rules for 4-simplices along tetrahedra.

Labelling with indices $i, j = 1, \ldots, 5$ the tetrahedra in each 4-simplex, with $h \in SO(4)$ the group elements imposing left diagonal invariance, and by a pair (ij) the triangles shared by the tetrahedra i and j, the kinetic and vertex functions are:

$$\mathcal{K}(g_i, \tilde{g}_i) = \int \mathrm{d}h \prod_{i=1}^{4} \delta(g_i h \tilde{g}_i^{-1}) \qquad \mathcal{V}(g_{ij}) = \prod_{i,j=1}^{5} \int \mathrm{d}h_i \prod_{i \neq j} \delta(g_{ij} h_i h_j^{-1} g_{ji}^{-1}) \tag{5.30}$$

where g_{ij} and g_{ji} are associated to the same triangle in the tetrahedra i and j.
The piecewise-flat context in which the GFT models are best understood can be read out by the geometric content of the ten delta functions in the vertex term encoding the flatness of each "wedge", i.e. of the portion of each dual face inside a single 4-simplex [17, 20].

Upon Peter-Weyl decomposition, the gauge invariant field is expanded into four $SO(4)$ irreducible representations, labelled by pairs of $SU(2)$ spins $J = (j^-, j^+)$, and 4-valent intertwiners $\iota = (\iota^-, \iota^+)$ labelled by a pair of intermediate $SU(2)$ spins:

$$\begin{aligned} \varphi(g_1, \ldots, g_4) = \sum \varphi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+} \overline{(\iota_{k^-})^{j_i^-}_{b_i^- c_i^-}} (\iota_{k^-})^{j_i^-}_{c_i^- d_i^-} \overline{(\iota_{k^+})^{j_i^+}_{b_i^+ c_i^+}} (\iota_{k^+})^{j_i^+}_{c_i^+ d_i^+} \\ \prod_{i=1}^{4} d_{j_i^-} d_{j_i^+} D^{j_i^-}_{d_i^- a_i^-}(g_i^-) D^{j_i^+}_{d_i^+ a_i^+}(g_i^+) \end{aligned} \tag{5.31}$$

where $D^{j^{\pm}}(g^{\pm})$ are the $SU(2)$ Wigner matrices, $(\iota_k)^{j_i}_{c_i d_i}$ form a basis of four-valent $SO(3)$ intertwiners, labelled by an intermediate spin k, and summation, given by the recoupling theory, is on all indices.
Importantly, in the thesis we face summations over many indices, and we adopt Einstein convention, summing over repeated indices. However, we usually write the symbol $\sum$ in order to point out that a non-trivial sum is involved in the equation.

The partition function can be constructed from the action, re-written in the spin network basis. The interaction vertex, associated to each 4-simplex, is expressed in terms of two $SU(2)$ Wigner 15j-symbols [107], and the Feynman amplitudes takes the form of a state sum model on the simplicial complex dual Δ to the graph:

$$\mathcal{Z}(\Delta) = \sum_{\{J_t, \imath_\tau\}} \prod_t d_{j_t^-} d_{j_t^+} \prod_\sigma \{15j\}^-_\sigma \{15j\}^+_\sigma \tag{5.32}$$

The sum is over the $SO(4)$ representations J_t and intertwiners $\imath_\tau$ labelling the triangles and the tetrahedra; $d_j = 2j + 1$ is the dimension of the $SU(2)$ representation j. Remarkably, this is precisely the partition function of $SO(4)$ BF Spin Foam theory (5.24).
Despite the fact that the geometric interpretation may be less transparent, in this formulation we can work directly with quantum numbers labelling the states of the theory.

5.7.2 Non-commutative Fourier transform and bivector formulation

In this section we present one of the greatest property of the Group Field Theory formalism, the metric representation, encoding the simplicial geometry and, thus, bridging Spin Foams to simplicial path integral. As we have seen, the GFT field φ is a square-integrable function in the Hilbert space $\mathcal{H} = L^2(SO(4)^{\times 4})$, which admits three representation: group representation, spin representation and metric representation. The standard formulation of GFT starts from the field $|\varphi\rangle \in \mathcal{H}$ represented in the group basis, $\varphi(g_1, g_2, g_3, g_4) = \langle g_1| \otimes \langle g_2| \otimes \langle g_3| \otimes \langle g_4|\varphi\rangle$.
Moreover, we have seen that using Peter-Weyl decomposition, we can represent the GFT field

also in the spin network basis:

$$
\begin{aligned}
\varphi(g_1, g_2, g_3, g_4) &= \langle g_1| \otimes \langle g_2| \otimes \langle g_3| \otimes \langle g_4|\varphi\rangle \\
&= \sum d_{j_1} d_{j_2} d_{j_3} d_{j_4} \langle g_1|j_1; m_1, n_1\rangle \langle g_2|j_2; m_2, n_2\rangle \langle g_3|j_3; m_3, n_3\rangle \langle g_4|j_4; m_4, n_4\rangle \\
&\qquad \langle j_1; m_1, n_1| \otimes \langle j_2; m_2, n_2| \otimes \langle j_3; m_3, n_3| \otimes \langle j_4; m_4, n_4|\varphi\rangle \\
&= \sum d_{j_1} d_{j_2} d_{j_3} d_{j_4} D^{j_1}_{n_1 m_1}(g_1) D^{j_2}_{n_2 m_2}(g_2) D^{j_3}_{n_3 m_3}(g_3) D^{j_4}_{n_4 m_4}(g_4) \varphi^{j_i}_{n_i m_i}
\end{aligned}
\tag{5.33}
$$

where summation is intended over all indices.
Demanding the gauge invariance condition we obtain (5.31).
Finally, using the non-commutative Fourier transform, we can represent the GFT field in the metric representation as:

$$
\varphi_{1234} \equiv \varphi(x_1, x_2, x_3, x_4) = \langle x_1| \otimes \langle x_2| \otimes \langle x_3| \otimes \langle x_4|\varphi\rangle \tag{5.34}
$$

The variables x_i belong to the Lie algebra $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$. The relevant Fourier transform here is the obvious extension of the non-commutative $SO(3)$ Fourier transform [91–93] to the group $[SO(3) \times SO(3)]^4$:

$$
\varphi(x_1, \ldots, x_4) := \int [\mathrm{d}g_i]^4 \, \varphi(g_1, \ldots, g_4) \, e^{iTrx_1 g_1} \ldots e^{iTrx_4 g_4} \tag{5.35}
$$

The kernel of the Fourier transform is a product of 'plane waves':

$$
E_g(x) = e^{iTrxg} = e^{i\sum_{\pm} \epsilon_{g^\pm} \mathrm{tr}[x^\pm g^\pm]} = e^{i\epsilon_{g^-} \mathrm{tr} x^- g^-} e^{i\epsilon_{g^+} \mathrm{tr} x^+ g^+} := e_{g^-}(x^-) e_{g^+}(x^+) \tag{5.36}
$$

with $\epsilon_{g^\pm} = \mathrm{sign}(\mathrm{tr} g^\pm)$, and $e_{g^\pm}(x^\pm)$ are $SO(3)$ plane waves.
The plane waves satisfy the properties:

$$
\int \mathrm{d}^6 x \, E_g(x) = \delta(g), \quad E_{g^{-1}}(x) = E_g(-x) \tag{5.37}
$$

where $\mathrm{d}^6 x$ is the Lebesgue measure on $\mathfrak{so}(4) \sim \mathbb{R}^6$.

Now, we can write the GFT action (5.27) in the metric representation:

$$
\begin{aligned}
S = &\frac{1}{2} \int \left[\mathrm{d}^6 x_i\right]^4 \varphi_{1234} \star \varphi_{-1-2-3-4} \\
&+ \frac{\lambda}{5!} \int \left[\mathrm{d}^6 x_i\right]^{10} \varphi_{1234} \star \varphi_{-4567} \star \varphi_{-7-389} \star \varphi_{-9-6-2\,10} \star \varphi_{-10\,-8-5-1}
\end{aligned}
$$

where the $\star$-product is defined on $SO(3)$ plane waves as $e_g \star e_{g'}(x) = e_{gg'}$, extended to $E_g \star E_{g'}(x) = E_{gg'}(x)$ and by linearity to the algebra functions.

In the metric representation, gauge invariance has a clear geometric interpretation. To see this we observe that in the metric representation gauge invariance translates into the invariance of the dual field under $\star$-multiplication by a product of four plane waves $E_h(x_1) \ldots E_h(x_4) = E_h(x_1 + x_2 + x_3 + x_4)$ labelled by the same h:

$$
\forall h \in \mathrm{SO}(4), \quad \varphi(x_i) = E_h \ldots E_h \star \varphi(x_i) \tag{5.38}
$$

Integrating over h on both sides of this equality gives:

$$
\varphi(x_i) = \delta(x_1 + x_2 + x_3 + x_4) \star \varphi(x_i) = \hat{C} \triangleright \varphi(x_i) \tag{5.39}
$$

where the closure constraint is implemented by a projector, since the δ, defined as

$$\delta(x) := \int \mathrm{d}h E_h(x), \tag{5.40}$$

plays the role of non-commutative delta function on algebra functions $\delta \star \phi(x) = \phi(0)\delta(x)$, and satisfies $\delta \star \delta = \delta$.
Thus, we see that gauge invariance imposes a closure constraint on the metric variables of the dual fields $\varphi(x_i)$: $x_1 + x_2 + x_3 + x_4 = 0$.

Geometrically, the Lie algebra variables represent the four bivector variables $x_f = (x_f^+, x_f^-)$ associated to the four triangles of each tetrahedron, represented by the field $\varphi(x_1, \ldots, x_4)$ in a simplicial discretization of 4d BF theory, coming from the discretization of the continuum B field [120, 121].
At the same time, the group elements associated to links of the 2-complex (dual to the simplicial complex) are related to the discretization of the continuum 1-form connection, giving holonomies (discrete curvature) associated to each 2-cell to a triangle of the same.

The propagator and the vertex in this representation, are given by

$$P(x, x') = \prod_{i=1}^{4} \delta_{-x_i}(x_i'), \qquad V(x, x') = \int [\mathrm{d}h_\ell]^5 \prod_{i=1}^{10} (\delta_{-x_i^\ell} \star E_{h_\ell h_{\ell'}^{-1}})(x_i^{\ell'}) \tag{5.41}$$

where i labels the oriented strands (triangles) and ℓ the half lines (tetrahedra) of the graphs, $\delta_x(y) := \delta(x - y)$, with δ defined as in (5.40), and we have chosen to gauge-averaging only the vertex function. However, since gauge averaging is a projection $\hat{C}^2 = \hat{C}$, one could gauge-average only the propagator, or both without affecting the amplitudes.
Geometrically, the vertex function encodes the identification, up to parallel transport $h_\ell h_{\ell'}^{-1}$, of the two bivectors $x_i^\ell, x_i^{\ell'}$ associated to the same triangle in different tetrahedral frames ℓ, ℓ' sharing the triangle, modulo a sign related to a flip of the face orientation.

Finally, we consider Feynman amplitudes, corresponding to simplicial path integrals, by taking the $\star$-product of propagator and vertex functions, following the strands of the graph [48]. Particularly, for a given closed graph dual to a simplicial complex, it is possible to show that the result is an integration over one $SO(4)$ variable h_l for each link of the Feynman graph, equivalently for each tetrahedron of the dual simplicial complex, which is interpreted as parallel transport between the two 4-simplices sharing that tetrahedron, involving a single $\mathfrak{so}(4)$ variable x_t for each triangle t of the dual simplicial complex:

$$\mathcal{Z}(\Delta) = \int \prod_l \mathrm{d}h_l \prod_e [\mathrm{d}^6 x_t] \, e^{i \sum_t Tr \, x_t H_t} \tag{5.42}$$

where group element:

$$H_t = \overrightarrow{\prod_{l \in \partial f_t}} h_l \tag{5.43}$$

is the holonomy along the links on the boundary of the face (2-cell) f_t of the graph dual to the triangle t. Remarkably, we recognize the amplitudes of simplicial path integrals for BF theory, where field variables $x \in \mathfrak{so}(4)$ and group elements $h \in SO(4)$ arising from gauge invariance play the respective roles of discrete B field and discrete connection.

We conclude this section by summarizing the marvellous aspect of the GFT formalism. Indeed, the partition function of the Ooguri GFT model for BF theory, usually written in group representation, corresponds to Spin Foam model for BF theory in spin representation, and to simplicial path integrals for BF theory in metric representation:

$$
\begin{aligned}
\mathcal{Z}(\Delta) &= \prod_f \int_{\mathfrak{so}(4)} \mathrm{d}x_f \prod_L \int_{SO(4)} \mathrm{d}h_L e^{i\sum_f tr(x_f H_f)} \\
&= \prod_{L\in\Gamma} \int \mathrm{d}h_L \prod_f \delta\left(\prod_{L\in\partial f} h_L\right) \\
&= \sum_{\{j^- j^+\}} \prod_f d_{j^-} d_{j^+} \prod_v \{15j\}_v^- \{15j\}_v^+
\end{aligned} \tag{5.44}
$$

5.7.3 Introducing normals: extended GFT formalism

The metric representation of GFT suggests the identification of the bivectors of a tetrahedron with the Lie algebra variables of the GFT field. However, the linear simplicity constraints state that four triangles described by bivectors B_f, which belong to the same tetrahedron $\tau \supset f$, lie in the same hyperplane. This is equivalent to the condition that there exists a 4-vector k_τ, normal to the tetrahedron, such that

$$
(*B_f)^{IJ}(k_\tau)_J = 0 \tag{5.45}
$$

Therefore, the linear constraints involve an another geometrical variable: the normal to the tetrahedron. In this section, we review an extension of the usual GFT formalism introduced in [48, 122], which includes the normals as an additional field variable.

The extended GFT formalism [49,50] is based on the addition of a fifth variable $k \in SU(2) \sim S^3$, viewed as a unit vector in $\mathbb{R}^4$, to the Ooguri field. In geometrical models, k will be then interpreted as the normal to a tetrahedron. Thus, in this extended formalism, Group Field Theories are defined in terms of fields on $SO(4)^4 \times SU(2)$:

$$
\begin{aligned}
&\varphi : SO(4)^4 \times SU(2) \to \mathbb{R} \\
&\quad (g_1, g_2, g_3, g_4; k) \mapsto \varphi_k(g_1, g_2, g_3, g_4) := \varphi(g_1, g_2, g_3, g_4; k)
\end{aligned} \tag{5.46}
$$

Recall that the Ooguri field is gauge invariant in the sense (5.26), corresponding to the invariance under change of local frame in each tetrahedron. In the extended formalism, however, the local rotation should rotate simultaneously the bivectors and the normal vectors, thus we require gauge covariance of the $SO(4)$ arguments with respect to the normal k, with the overall 5-argument field being invariant:

$$
\forall h \in SO(4), \quad \varphi_k(g_1, \cdots g_4) = \varphi_{h\triangleright k}(hg_1, \cdots hg_4) \tag{5.47}
$$

where $h \triangleright k := h^+ k (h^-)^{-1}$ is the normal rotated by h.
Moreover, we note that gauge covariance (5.47) induces an invariance under the stabilizer subgroup $\mathrm{SO}(3)_k = \{h \in \mathrm{SO}(4), h^+ k (h^-)^{-1} = k\}$ of the normal k, affecting only the four group arguments of the field. Indeed, for for $h^+ k (h^-)^{-1} = k$ we have:

$$
\forall h, \quad \varphi_k(g_1, \cdots g_4) = \varphi_{h^+ k (h^-)^{-1}}(hg_1, \ldots, hg_4) = \varphi_k(hg_1, \ldots, hg_4) \tag{5.48}
$$

By using the harmonic analysis on $SO(4)$, gauge invariant fields are expanded into four irreducible SO(4) representations (given by pairs of SU(2) spins $J_i = (j_i^-, j_i^+)$, $i = 1 \cdots 4$), each of which can be further decomposed into $\mathrm{SO}(3)_k$ representations, and we have:

$$
\begin{aligned}
\phi_k(g_i) = \sum \phi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+}(k) \prod_{i=1}^4 \overline{\tilde{C}^{j_i^- j_i^+ w_i}_{a_i^- a_i^+ e_i}}(k) (\overline{\iota_g^y})^{w_i}_{e_i} (\iota_g^y)^{r_i}_{m_i} \tilde{C}^{l_i^- l_i^+ r_i}_{u^- u^+ m_i}(k) \\
d_{j_i^-} d_{j_i^+} D^{l_i^-}_{u_i^- v_i^-}(g_i^-) D^{l_i^+}_{u_i^+ v_i^+}(g_i^+) \overline{\tilde{C}^{l_i^- l_i^+ s_i}_{v_i^- v_i^+ n_i}}(k) (\overline{\iota_h^z})^{s_i}_{n_i} (\iota_h^z)^{x_i}_{f_i} \tilde{C}^{j_i^- j_i^+ x_i}_{b_i^- b_i^+ f_i}(k)
\end{aligned} \tag{5.49}
$$

where summation, given by the recoupling theory, is over all the indices and we have defined:

- the rotated Clebsch-Gordon coefficients:

$$\begin{aligned} \tilde{C}^{j^- j^+ j}_{m^- m^+ q}(k) &= \langle (j^-, j^+)\ j, q | j^-; m^- \rangle_k \otimes |j^+, m^+\rangle \\ \overline{\tilde{C}^{j^- j^+ l}_{n^- n^+ p}}(k) &= {}_k\langle j^-, n^- | \otimes \langle j^+, n^+ | (j^-, j^+)\ l, p\rangle \end{aligned} \tag{5.50}$$

- a basis of four-valent $SO(3)$ intertwiners, labelled by an intermediate spin j:

$$\begin{aligned} (\iota^j_g)^{r_i}_{m_i} &= \langle j, g | r_i, m_i\rangle = \langle j, g | r_1, m_1\rangle \otimes |r_2, m_2\rangle \otimes |r_3, m_3\rangle \otimes |r_4, m_4\rangle \\ (\overline{\iota^j_g})^{w_i}_{e_i} &= \langle w_i, e_i | j; g\rangle = \langle w_1, e_1 | \otimes \langle w_2, e_2 | \otimes \langle w_3, e_3 | \otimes \langle w_4, e_4 | j, g\rangle \end{aligned} \tag{5.51}$$

Proof. In order to show this result, first we expand gauge invariant fields are expanded into four irreducible SO(4) representations (given by pairs of SU(2) spins $J_i = (j_i^-, j_i^+)$, $i = 1 \cdots 4$):

$$\begin{aligned} \phi_k(g_i) &= \sum \phi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+}(k) \prod_{i=1}^4 d_{j_i^-} d_{j_i^+} D^{j_i^-}_{a_i^- b_i^-}(g_i^-) D^{j_i^+}_{a_i^+ b_i^+}(g_i^+) \\ &= \sum \phi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+}(k) \prod_{i=1}^4 d_{j_i^-} d_{j_i^+} \langle j_i^-, a_i^- | D^{j^-}(g_i^-) | j_i^-, b_i^-\rangle \langle j_i^+, a_i^+ | D^{j^+}(g_i^+) | j_i^+, b_i^+\rangle \end{aligned} \tag{5.52}$$

Then, we define the k-rotated states:

$$|j^-, m\rangle_k = D(k^{-1}) |j^-, m\rangle \tag{5.53}$$

which allows us to decompose the $SU(2) \times SU(2) \subset SO(4)$ representations into $\mathrm{SO}(3)_k$ representations. Indeed, introducing several resolutions of identity of the space $\mathcal{H}_{[SU(2)^- \times SU(2)^+]^{\times 4}}$, we have:

$$\begin{aligned} \phi_k(g_i) = &\sum \phi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+}(k) \prod_{i=1}^4 d_{j_i^-} d_{j_i^+} \langle j_i^-, a_i^- | j_i^-, c_i^-\rangle_k \langle j_i^+, a_i^+ | j_i^+, c_i^+\rangle_k \langle j_i^-, c_i^- | \otimes \langle j_i^+, c_i^+ | \\ & |(j_i^-, j_i^+)\ w_i, e_i\rangle \langle (j_i^-, j_i^+)\ w_i, e_i | (w_i, e_i)\ y, g\rangle \langle (w_i, e_i)\ y, g | \\ & |(l_i^-, l_i^+)\ r_i, m_i\rangle \langle (l_i^-, l_i^+)\ r_i, m_i | l_i^-, p_i^-\rangle_k \otimes |l_i^+, p_i^+\rangle_k \langle l_i^-, p_i^- | l_i^-, u_i^-\rangle \langle l_i^+, p_i^+ | l_i^+, u_i^+\rangle \\ & \langle l_i^-, u_i^- | D^{j^-}(g_i^-) | l_i^-, v_i^-\rangle \langle l_i^+, u_i^+ | D^{j^+}(g_i^+) | l_i^+, v_i^+\rangle \\ & \langle l_i^-, v_i^- | l_i^-, q_i^-\rangle_k \langle l_i^+, v_i^+ | l_i^+, q_i^+\rangle_k \langle l_i^-, q_i^- | \otimes \langle l_i^+, q_i^+ | (l_i^-, l_i^+)\ s_i, n_i\rangle \langle (l_i^-, l_i^+) s_i, n_i | \\ & |(x_i, f_i)\ z, h\rangle \langle (x_i, f_i)\ z, h | (j_i^-, j_i^+)\ x_i, f_i\rangle \langle (j_i^-, j_i^+)\ x_i, f_i | \\ & |j_i^-, d_i^-\rangle_k \otimes |j_i^+, d_i^+\rangle_k \langle j_i^-, d_i^- | j_i^-, b_i^-\rangle \langle j_i^+, d_i^+ | j_i^+, b_i^+\rangle \\ = &\sum \phi^{j_i^- j_i^+}_{a_i^- b_i^- a_i^+ b_i^+}(k) \prod_{i=1}^4 \overline{\tilde{C}^{j_i^- j_i^+ w_i}_{a_i^- a_i^+ e_i}}(k) (\overline{\iota^y_g})^{w_i}_{e_i} (\iota^y_g)^{r_i}_{m_i} \tilde{C}^{l_i^- l_i^+ r_i}_{u^- u^+ m_i}(k) \\ & d_{j_i^-} d_{j_i^+} D^{l_i^-}_{u_i^- v_i^-}(g_i^-) D^{l_i^+}_{u_i^+ v_i^+}(g_i^+) \overline{\tilde{C}^{l_i^- l_i^+ s_i}_{v_i^- v_i^+ n_i}}(k) (\overline{\iota^z_h})^{s_i}_{n_i} (\iota^z_h)^{x_i}_{f_i} \tilde{C}^{j_i^- j_i^+ x_i}_{b_i^- b_i^+ f_i}(k) \end{aligned} \tag{5.54}$$

□

As noted in [49,50], a set of basis functions is given by:

$$\Psi^{(J_i, k_i, j)}_{m_i^-, m_i^+}(g_i; k) = \left(\prod_{i=1}^4 D^{j_i^-}_{n_i^- m_i^-}(g_i^-) D^{j_i^+}_{n_i^+ m_i^+}(g_i^+) \tilde{C}^{j_i^- j_i^+ k_i}_{n_i^- n_i^+ p_i}(k) \right) (\iota_j)^{k_i}_{p_i} \tag{5.55}$$

where repeated lower indices are summed over. The k-dependent coefficients, defined in terms of the $SO(3)$ Clebsch-Gordan coefficients $C^{j^- j^+ k}_{mnp}$ as

$$\tilde{C}^{j^- j^+ k}_{m^- m^+ p}(k) = C^{j^- j^+ k}_{m m^+ p} D^{j^-}_{m m^-}(k) \tag{5.56}$$

and

$$\overline{\tilde{C}^{j^-j^+k}_{m^-m^+p}}(k) = \overline{C^{j^-j^+k}_{mm^+p}D^{j^-}_{mm^-}}(k) = D^{j^-}_{m^-m}(k^{-1})\overline{C^{j^-j^+k}_{mm^+p}} \tag{5.57}$$

define a tensor that intertwines the action of $SO(3)_k$ in the representation $j^- \otimes j^+$ and the action of $SO(3)$ in the representation k. Namely, given $\mathbf{u}_k = (k^{-1}uk, u) \in SO(3)_k$, we have:

$$\tilde{C}^{j^-j^+k}_{m^-m^+p}(k)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{m^+n^+}(u) = \tilde{C}^{j^-j^+k}_{n^-n^+q}(k)D^k_{pq}(u) \tag{5.58}$$

The proof is straightforward.

Proof.

$$\begin{aligned}
&\tilde{C}^{j^-j^+k}_{m^-m^+p}(k)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{m^+n^+}(u) = C^{j^-j^+k}_{mm^+p}D^{j^-}_{mm^-}(k)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{m^+n^+}(u)\\
&= C^{j^-j^+k}_{mm^+p}D^{j^-}_{mn^-}(uk)D^{j^+}_{m^+n^+}(u) = C^{j^-j^+k}_{mm^+p}D^{j^-}_{mr^-}(u)D^{j^+}_{m^+n^+}(u)D^{j^-}_{r^-n^-}(k)\\
&= C^{j^-j^+k}_{mm^+p}\overline{C^{j^-j^+k}_{mm^+x}}C^{j^-j^+k}_{r^-n^+y}D^k_{xy}(u)D^{j^-}_{r^-n^-}(k) = \delta^k_{px}C^{j^-j^+k}_{r^-n^+y}D^k_{xy}(u)D^{j^-}_{r^-n^-}(k)\\
&= \tilde{C}^{j^-j^+k}_{n^-n^+q}(k)D^k_{pq}(u)
\end{aligned} \tag{5.59}$$

□

Remarkably, as observed in [49,50], (5.55) corresponds to the vertex structure of the so-called projected spin networks of the covariant approach to Loop Quantum Gravity [87], which thus label the polynomial gauge invariant operators in the extended formalism, i.e. the observables.

Now we consider the metric representation of the extended GFT field, obtained by means of the non-commutative Fourier transform:

$$\begin{aligned}
\varphi_k(x_1,\ldots,x_4) :&= \varphi(x_1,\ldots,x_4;k)\\
&= \int [\mathrm{d}g]^4 \varphi(g_1,\ldots,g_4;k)E_{g_1}(x_1)E_{g_2}(x_2)E_{g_3}(x_3)E_{g_4}(x_4)
\end{aligned} \tag{5.60}$$

Gauge covariance translates into:

$$\forall h \in SO(4), \qquad \varphi_k = E_h \cdots E_h \star \varphi_{h^{-1}\triangleright k} \tag{5.61}$$

where $h^{-1} \triangleright k = (h^+)^{-1}kh^-$.
Interestingly, gauge covariance can be imposed by means of the gauge invariance projector, acting on extended fields as

$$(\hat{C} \triangleright \varphi)_k = \int \mathrm{d}h\, E_h \cdots E_h \star \varphi_{h^{-1}\triangleright k} \tag{5.62}$$

Note that gauge invariance (5.39), and thus the closure of the four bivector variables, can be recovered upon integration over the normal.
In fact, if $\psi(x_i) := \int_{SU(2)} \mathrm{d}k\varphi_k(x_i)$, then:

$$\begin{aligned}
\psi(x_i) &= \int \mathrm{d}h\mathrm{d}kE_h\ldots E_h \star \varphi_{h^{-1}\triangleright k}(x_i) = \int \mathrm{d}h\mathrm{d}kE_h\ldots E_h \star \varphi_k(x_i)\\
&= \int \mathrm{d}hE_h\ldots E_h \star \psi(x_i) = \delta(x_1+x_2+x_3+x_4) \star \psi(x_i)
\end{aligned} \tag{5.63}$$

where we used the gauge covariance of the extended field (5.61) and the properties of the Haar measure.

We conclude this section by considering the dynamics of the theory, that, just like in the standard formulation, generates simplicial BF path integrals as Feynman amplitudes, thus still corresponds, at the perturbative level, to quantum simplicial BF theory.
Specifically, in group representation, we can choose the action:

$$S[\varphi] = \frac{1}{2}\int [\mathrm{d}g_i]^4 dk\, \varphi^2_{k1234} + \frac{\lambda}{5!}\int [\mathrm{d}g_i]^{10}[\mathrm{d}k_i]^5\, \varphi_{k_1 1234}\varphi_{k_2 4567}\varphi_{k_3 7389}\varphi_{k_4 962\,10}\varphi_{k_5 10\,851} \tag{5.64}$$

where φ_{k1234} is a shorthand notation for $\varphi_k(g_1, \cdots g_4)$, $\mathrm{d}g$ and $\mathrm{d}k$ are the Haar measures on $SO(4)$ and $SU(2)$.
In the action, the kinetic term, encoding the glueing rule of 4-simplices along a tetrahedron, identifies both group elements and normals, whereas the interaction does not couple the normals.
Using the non-commutative Fourier transform, we can write the action in metric representation:

$$\begin{aligned} S = &\frac{1}{2}\int [\mathrm{d}^6 x_i]^4 [\mathrm{d}k]\, \varphi_{k1234} \star \varphi_{k-1-2-3-4} \\ &+ \frac{\lambda}{5!}\int [\mathrm{d}^6 x_i]^{10}\, \psi_{1234} \star \psi_{-4567} \star \psi_{-7-389} \star \psi_{-9-6-2\,10} \star \psi_{-10\,-8-5-1} \end{aligned}$$

where we note that the interaction term (thus the vertex amplitude), written in terms of $\psi(x_i) = \int_{SU(2)} \mathrm{d}k \varphi_k(x_i)$, corresponds to the standard GFT action.

However [50], the propagator of the extended GFT formalism is different as it is supplemented with an additional strand which identifies the normals k up to parallel transport arising from gauge invariance, which can be imposed, as in the non-extended case, in the propagator only, in the vertex only, or in both. Particularly, in the metric representation, choosing to implement gauge invariance in the vertex only, the propagator reads:

$$P(x, k; x', k') = \prod_{i=1}^{4} \delta_{-x_i}(x'_i)\, \delta(k' k^{-1}) \tag{5.65}$$

where the additional delta reduces the number of SU(2) variables to one per link, hence to one k_τ per tetrahedron (each tetrahedron τ has its normal vector k_τ).
As we have mentioned in Section 3.6, the boundary states appearing in the amplitudes for GFT n-point functions are projected spin network states, thus different from the standard $SU(2)$ spin network states of the standard Ooguri model for BF theory, that can be recovered by averaging out the normal variables independently at each tetrahedron in the boundary. Actually, the extended formulation only modifies the structure of boundary states.
In fact, as the integrals over the normals on the internal links (bulk tetrahedra) drop from the amplitudes, the amplitudes of closed diagrams will not depend on the normals [50].

Chapter 6

Gravity as a constrained BF theory

In this chapter we present the first order formulation of General Relativity, and we see how it can be written as a constrained topological BF field theory. This formulation is very important in Quantum Gravity since it is the starting point of most Spin Foam and GFT models for quantum gravity.

6.1 The Plebanski formulation of General Relativity

In this section, following [83], we relate the Palatini formulation of General Relativity to a constrained topological BF theory.

Let ω be a 1-form connection with values in the Lie algebra $\mathfrak{so}(4)$. Consider a 2-form B also with values in the Lie algebra $\mathfrak{so}(4)$. The Plebanski formulation of General Relativity is defined by the classical continuum action:

$$S_{Pl}[\omega, B, \mu] = \int_{\mathcal{M}} \left(B^{IJ} \wedge F_{IJ}(\omega) + \frac{\Lambda}{2} (*B)^{IJ} \wedge B_{IJ} + \frac{1}{4} \mu_{IJKL} B^{IJ} \wedge B^{KL} \right) \tag{6.1}$$

where the Lagrange multiplier μ_{IJKL} is symmetric under the exchange of the pairs $_{[IJ]}$ and $_{[KL]}$, $\mu_{[IJ][KL]} = \mu_{[KJ][IJ]}$, and satisfies the tracelessness condition $\epsilon^{IJKL}\mu_{IJKL} = 0$. It generates the 20 simplicity constraints ensuring that the 36 components of $B^{IJ}_{\mu\nu}$ reduce to 16 components associated to a tetrad field e^I_μ.

The equations of motion derived from (6.1) are given by:

$$\begin{aligned} &DB = dB + [\omega, B] = 0, \\ &F^{IJ} + \frac{1}{2}(\Lambda \epsilon^{IJKL} + \mu^{IJKL}) B_{KL} = 0 \\ &B^{IJ} \wedge B^{KL} = \frac{1}{4!} \epsilon_{IJKL} B^{IJ} \wedge B^{KL} \epsilon^{IJKL} \equiv \mathcal{V} \epsilon^{IJKL} \end{aligned} \tag{6.2}$$

From here on, for simplicity we consider $\Lambda = 0$, as models with non-zero cosmological constant involve quantum groups.
The last equation of motion is known as the simplicity constraint and it implies that the B field can be written as:

- topological sector ($I\pm$): $B^{IJ} = \pm e^I \wedge e^J$
- gravitational sector ($II\pm$): $B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L$

We can recover the Palatini formulation by plugging the solution for the gravitational sector ($B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L$) back into the Plebanski action.

These set of continuum simplicity constraints are called quadratic simplicity constraints, and in the non-degenerate case $\mathcal{V} \neq 0$, can be equivalently rewritten as

$$B^{IJ} \wedge B^{KL} = \mathcal{V}\epsilon^{IJKL} \Longleftrightarrow \epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = \mathcal{V}\epsilon_{\mu\nu\rho\sigma} \tag{6.3}$$

In summary, at the classical level, the gravitational sector of the Plebanki theory is equivalent to the Palatini formalism of General Relativity.

Furthermore, in the classical theory, initial gravitation solutions remain within the same branch of solutions [104]. However, problems arise in the quantum theory, where one has necessarily contributions from all branches. Specifically, the topological sector may "interfere" with the physical one at the quantum level.

Historically, the first Spin Foam model of four-dimensional quantum gravity was the Barrett-Crane model, based on the discretization of the quadratic simplicity constraints. Nevertheless, the Barrett-Crane model does not distinguish between the two sectors and therefore one cannot be sure that one describes pure gravity.

One can simply resolve this problem, selecting a single (gravitational) sector, by reformulating the simplicity constraints in such a way that these two sectors are distinguished. This can be done by imposing a stronger version of the simplicity constraints: the linear version of the simplicity constraints, which in turn have a clear geometric meaning.

Linear simplicity constraints for the classical continuum theory were first introduced in [39], motivated by the fact that a linear version of the discrete simplicity constraints had been introduced in [38, 43, 123, 124] to cure several problematic features of the Barrett-Crane model. The conclusion of [39] is that the replacement of the quadratic simplicity constraints with linear constraints at the continuum level is relatively straightforward, after the introduction of a new variable n_A forming a basis of three-forms at each point. Moreover, linear simplicity constraints do not require an additional non-degeneracy assumption on B^{IJ}, but a non-degeneracy assumption is still necessary on the three-forms n_A.

6.2 Discrete Quadratic Constraints

In discrete approaches to Quantum Gravity continuum space-time is replaced by a cellular complex. Therefore, in order to implement the simplicity constraints in the Spin Foam / GFT framework, we need to discretize the Plebanski constraint on an arbitrary simplicial complex [38, 43, 47, 87, 120, 121, 123–125].

Particularly, we introduce a simplicial decomposition Δ of space-time, consisting of 4-simplices, tetrahedra, and triangles, dual respectively to vertices v, edges e, and faces f in the dual 2-complex.

We assume that the curvature is concentrated on the "bones" f, and is coded in the holonomy around the "link" of each f, while on each 4-simplex geometry is flat.

This geometry can be described by choosing the tetrad one-form $e(v)^I$ in a cartesian coordinate patch covering the simplex v; $e(t)^I$ a is a tetrad one-form in a cartesian coordinate patch covering the tetrahedron t.

The matrix $(V_{vt})^I_J \in SO(4)$ is defined by $(V_{vt})^I_J e(v)^J_a = e(t)^I_a$ when t bounds v and in a common coordinate patch.

Moreover, for each triangle f in t, we define [43]:

$$B_f(t)^{IJ} := \int_f *(e(t)^I \wedge e(t)^J) \quad \in \quad \mathfrak{so}(4). \tag{6.4}$$

where $*$ stand for the Hodge dual in the internal indices I, J. Finally, for each triangle f and each pair of tetrahedra t, t' in the link of f, we define:

$$U_f(t, t') := V_{tv_1} V_{v_1 t_1} V_{t_1 v_2} \dots V_{v_n t'}, \tag{6.5}$$

where the product is around the link in the clock-wise direction from t' to t.
In the discrete setting, the quadratic simplicity constraints on $B_f(t)$, considered as an independent variable instead of the tetrads, can be stated as follows [43]:

1. $\forall f$ and $t, t' \in Link(f)$,
$$U_f(t,t')B_f(t') = B_f(t)U_f(t,t'); \tag{6.6}$$

2. Closure, $\forall t$
$$\sum_{f\in t} B_f(t) = 0; \tag{6.7}$$

3. Diagonal simplicity constraint, $\forall f$
$$C_{ff} := *B_f(t) \cdot B_f(t) \approx 0; \tag{6.8}$$

4. Off-diagonal simplicity constraint, $\forall f, f' \in t$
$$C_{ff'} := *B_f(t) \cdot B_{f'}(t) \approx 0; \tag{6.9}$$

5. Dynamical (or volume) simplicity constraint, $f, f' \in v$ not in the same tetrahedron t
$$*B_f(v) \cdot B_{f'}(v) \approx \pm 12\mathcal{V}(v). \tag{6.10}$$
where $\mathcal{V}(v)$ is the four volume of the 4-simplex and the sign depends on relative face orientation [126] and the dot stands for the scalar product in the algebra.

Now, (6.6) is assumed to hold prior to varying the action [43].
The closure constraint (6.7) generates the internal gauge transformation. In the context of Group Field Theories the closure constraint is automatically satisfied by the invariance under the diagonal (left) action of the group $SO(4)$ on the four group arguments of the GFT field. In the context of Spin Foams model the vertex amplitude (the dynamics in quantum theory) turns out to project on the gauge invariant subspace. Therefore the closure constraint is automatically implemented by the dynamics in quantum theory [43].
Diagonal (6.8) and Off-diagonal (6.9) simplicity constraints are second class constraints, and must be separately imposed on the state space.
Finally, as noted in [46, 47, 124], the volume constraint (6.10) is automatically satisfied when the rest of the constraints are satisfied.

6.3 Discrete Linear Constraints

As we have mentioned, quadratic simplicity constraints can be replaced with the lightly stronger linear simplicity constraints, used in the new Spin Foam models presented in this thesis.

Indeed, in the absence of Immirzi parameter, the simplicity constraints state that the hodge dual $*B_j^{IJ}$ are the area bivectors of a geometric (metric) tetrahedron: this is the discrete equivalent of $B = *e \wedge e$. Following [38, 44, 87], a discrete tetrad can be reconstructed for the whole simplicial complex, and it determines the discrete bivectors B_f associated to triangles of the same, if one requires that:

$$\forall\ tetrahedra\ t \in \Delta\ \exists\ k_t \in \mathcal{S}^3\ /\ (*B_f)^{IJ} k_{tJ} = 0,\ \forall B_f\ f \subset t \tag{6.11}$$

In words, for each tetrahedron, the four $*B_j^{IJ}$, with $j = 1, \ldots, 4$, lie in the same hypersurface normal to a given unit vector k_{tJ} in $\mathbb{R}^4 \sim \mathcal{S}^3$.

Therefore, we can consider the unit vector k_{tJ} as the normal vector to the tetrahedron t.
We should supplemented this condition by demanding the closure condition of all the tetrahedra in Δ, plus an orientation and a non-degeneracy condition [38, 43, 47, 123–125].

The non-degeneracy condition is studied in [39], where a complete analysis of the linear simplicity constraints is performed. Specifically, Oriti and Gielen introduced the linearised version of the diagonal and cross-diagonal simplicity constraints, as well as the linear volume constraint. The result is that if linear simplicity and cross-simplicity constraints and closure constraints for both bivector variables and normals are imposed everywhere, a sufficient set of linear volume constraints follows. Thus, we can either impose linear simplicity constraints, standard closure constraints (the requirement for the triangles specified by bivectors close up to form a tetrahedron) and linear volume constraints or we can impose linear simplicity constraints, standard closure constraints (the requirement for the triangles specified by bivectors close up to form a tetrahedron) and a four-dimensional closure constraint that can be interpreted as demanding that the five tetrahedra, specified by the normals, close up to form a 4-simplex [40]. As a consequence [39], non-geometric bivector configurations cannot appear if the normals are non-degenerate and the additional closure constrain on the normals holds.

Thus, if ten bivectors associated to the faces of a 4-simplex satisfy simplicity and closure constraints for each tetrahedron of the 4-simplex (there are five tetrahedra in a 4-simplex), then they define a geometric 4-simplex (for non-degenerate configurations).

Furthermore, if the (constrained) bivectors associated to a given tetrahedron are also correctly identified across the two 4-simplices sharing it, then the reconstruction of a discrete tetrad can be carried out for the whole simplicial complex, (again excluding degenerate configurations).

Finally, we can write the same constraint, in the self-dual/anti-self-dual splitting,

$$\forall \; tetrahedra \; t \in \Delta \; \exists \; k_t \in S^3 \simeq SU(2) \; / \; b^i_+ + (k \cdot b_- \cdot k^{-1})^i = 0, \; \forall B_f = (b^i_{f+}, b^i_{f-}) \tag{6.12}$$

where we use the natural action of $SU(2)$ in the fundamental representation.

In the above procedure we haven't considered the Immirzi parameter. Fortunately, all this can be generalized to include the Immirzi parameter γ (which plays a crucial role in LQG), by adding a topological term to the Plebanski action:

$$S(\omega, B, \mu) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) + \frac{1}{\gamma} (*B)^{IJ} \wedge F_{IJ} + \frac{1}{4} \mu_{IJKL} B^{IJ} \wedge B^{KL} \right] \tag{6.13}$$

where the the additional term vanishes on-shell (when imposing metricity and torsion freeness of the connection).
After inserting the solution to the constraints, this becomes the Holst-Plebanski action for gravity:

$$S(\omega, e, \mu) = \int_{\mathcal{M}} \left[\frac{1}{2} \epsilon^{IJ}{}_{KL} e^K \wedge e^L \wedge F_{IJ}(\omega) + \frac{1}{2\gamma} e^I \wedge e^J \wedge F_{IJ}(\omega) \right] \tag{6.14}$$

which is the classical starting point of Loop Quantum Gravity.

The linear discrete simplicity constraints with finite Immirzi parameter, that we impose on the BF action, are:

$$\forall \; tetrahedra \; t \in \Delta \; \exists \; k_t \in S^3 \; / \; (B_f - \gamma * B_f)^{IJ} k_{tJ} = 0, \; \forall B_f \; f \subset t \tag{6.15}$$

which corresponds, in the self-dual/anti-self-dual splitting, to

$$\forall \; tetrahedra \; t \in \Delta \; \exists \; k_t \in SU(2) \; / \; \beta b^i_+ + (k \cdot b_- \cdot k^{-1})^i = 0, \; \forall B_f = (b^i_{f+}, b^i_{f-}) \tag{6.16}$$

where the parameter β is related to the Immirzi parameter as:

$$\beta = \frac{\gamma - 1}{\gamma + 1} \tag{6.17}$$

Chapter 7

GFT models for four dimensional Riemannian Quantum Gravity

Most of the approaches to covariant quantum gravity start directly in the Spin Foam formalism, however in this thesis we work in a GFT context because it is the most complete setting to study the dynamics of spin networks and simplicial structures. However, we would like to point out that, due to the Group Field Theory / Spin Foams duality, the entire construction could be carried through directly at the level of simplicial path integrals (as done in [127–129]) or Spin Foam amplitudes.

Nevertheless, we start this section by recalling the standard Spin Foam quantization.
Given the knowledge we have on BF models and their relation with gravity, the usual procedure for defining a Spin Foam model for four dimensional quantum gravity stems from a discretization of the classical theory, by choosing a triangulation Δ on $\mathcal{M}$. Clearly, the most direct route to quantization would be to include a discrete analogue of the constraints $\mathcal{C}(B)$ into the definition of the measure of the discretized path integral [105, 130]:

$$\mathcal{Z}(\Delta) = \int \mathcal{D}[\omega_\Delta, B_\Delta]\delta(\mathcal{C}(B_\Delta))\, e^{iTr\, B_\Delta F_\Delta}, \tag{7.1}$$

However, this has proven very difficult up to now.
Since we know how to discretize and quantize of BF theories in any dimension [107, 130], the standard Spin Foam strategy consists of quantizing first the topological BF part of the discretized theory, and then to implement a quantum version of the constraints.
Alternatively, GFT models for quantum gravity are based on modifications of the Ooguri model for 4d BF theory, or from the corresponding simplicial path integral or Spin Foam expression, where suitable restriction are imposed on the dynamical variables representing the discrete (classical or quantum) B field variables. These processes would break the topological invariance, thus recovering the gravity degrees of freedom, hoping to obtain the correct result, despite the ambiguities involved in this way of proceeding. For example, there are several ways to impose the quantum simplicity constraints.
The first proposal for the implementation of the constraints (without Immirzi parameter), motivated by the geometric quantization of simplicial structures [42, 131], led to the famous Barrett-Crane model [125], characterized by strong imposition of the constraints. However, some features of which have raised criticisms over the years [123, 124], particularly it seemed that the imposition of the constraints was too strong, leading to the so-called ultra-locality problem, which originates in the uniqueness of the Barrett-Crane intertwiner and results in the fact that the simplices do not talk to each other.
Then, the EPRL model have been developed imposing the linear simplicity constraints weakly, using a Gupta-Bleuler procedure or Master constraint techniques [43], which include the Immirzi

parameter and reduce to Barrett-Crane when $\gamma = \infty$.
The Freidel Krasnov model shares with the EPRL model the linearisation of the simplicity constraints, the inclusion of the Immirzi parameter and the weak imposition of the constraints, and they coincide when $\gamma < 1$, but they differ for $\gamma > 1$. Actually, the difference is that in the FK model constraints are imposed in average on coherent states.
The strategy of these models is the following.
First, we find a quantum version of the simplicity constraints as an operator $\hat{D}$ acting on BF spin network states, $|j^-; m^-, n^-\rangle \otimes |j^+; m^+, n^+\rangle$, or BF coherent states, $|j^-; \vec{u}^-, \vec{v}^-\rangle \otimes |j^+; \vec{u}^+, \vec{v}^+\rangle$.
As we will see that the classical B variables, at the quantum level, are identified with generators $J^{IJ} = (J_i^+, J_i^-)$ of the $\mathfrak{so}(4)$ Lie algebra, associated to the operators $(\hat{X}_i^{+R}, \hat{X}_i^{-R})$ acting as operators on the $SO(4)$ spin network states. Therefore, classical constraints, functions of the B's, are then turned into operators and imposed at the quantum level on such BF spin networks, to give restrictions on both the representations attached to their links and on the intertwiners associated to their vertices [83].
For example, simplicity operator $\hat{D} = \beta \hat{X}^+ + k\hat{X}^- k^{-1}$ can be implemented:

1. via Master constraint criterion

$$0 \approx \hat{D}^2|\psi\rangle = \sum_{j^\pm, m^\pm, n^\pm} \hat{D}^2 |j^-; m^-, n^-\rangle \otimes |j^+; m^+, n^+\rangle d_{j^-} d_{j^+} \psi^{j^- j^+}_{m^- n^- m^+ n^+} \tag{7.2}$$

2. via Gupta-Bleuler criterion

$$\begin{aligned} 0 = \langle\phi|\hat{D}|\psi\rangle = &\sum_{j^\pm, m^\pm, n^\pm} \sum_{j'^\pm, m'^\pm, n'^\pm} d_{j'^-} d_{j'^+} \overline{\phi^{j'^- j'^+}_{m'^- n'^- m'^+ n'^+}} d_{j^-} d_{j^+} \psi^{j^- j^+}_{m^- n^- m^+ n^+} \\ &\langle j'^-; m'^-, n'^-| \otimes \langle j'^+; m'^+, n'^+|\hat{D}|j^-; m^-, n^-\rangle \otimes |j^+; m^+, n^+\rangle \end{aligned} \tag{7.3}$$

3. in average, using coherent states

$$0 = \langle j'^-; \vec{u}^-, \vec{v}^-| \otimes \langle j^+; \vec{u}^+, \vec{v}^+|\hat{D}|j^-; \vec{u}^-, \vec{v}^-\rangle \otimes |j^+; \vec{u}^+, \vec{v}^+\rangle \tag{7.4}$$

The states $|\phi\rangle_{phys}$, satisfying these equations, are the solutions of the simplicity constraints for the specific model, and they are given by:

$$|\phi\rangle_{phys} = \hat{S}|\phi\rangle \tag{7.5}$$

where $\hat{S}$ is the operator that imposes the simplicity constraints for the considered model.
Then, the Group Field Theory corresponding to the specific model is obtained by substituting in the BF action the constrained field $\phi_{phys}(x) = \langle x|\phi\rangle_{phys}$. The expansion in Feynman diagrams results in a sum of 2-complexes dual to simplicial complexes satisfying the simplicity constraints.
However, in the GFT context, we can use the duality between simplicial gravity path integrals and Spin Foam models and follow an other strategy, characteristic of the Baratin Oriti model. Starting from the BF GFT model in terms of Lie algebra variables, or from the resulting (non-commutative) simplicial path integral, we can impose the geometric constraints directly on the bivectors. In this way, we obtain a model whose Feynman amplitudes are manifestly simplicial path integrals for BF theory with constraints; then, we can also re-write the same amplitudes as Spin Foam models and identify the corresponding space of boundary states in a spin network basis.
Although these two strategies are different, since in the state sum strategy constraints are imposed on BF spin network states, while in the (non-commutative) geometric strategy they are imposed on the Lie algebra variables (bivectors) of the GFT field, we can use the Group

Field Theory/Spin Foam duality to set the problem in the same framework.
For example, given the generic state $|\psi\rangle \in \mathcal{H} = L^2(SO(4))$, we can define the operator imposing the simplicity constraints $\hat{S}$ such that:

$$
\begin{aligned}
|\phi\rangle_{phys} &= \int \mathrm{d}g^- \mathrm{d}g^+ \hat{S}|g\rangle \otimes |g^+\rangle \psi(g^-, g^+) \\
&= \int_{\mathbb{R}^6_\star} \frac{\mathrm{d}^3 x^-}{(2\pi)^3} \frac{\mathrm{d}^3 x^+}{(2\pi)^3} \hat{S}|x^-\rangle \otimes |x^+\rangle \star \psi(x^-, x^+) \\
&= \sum_{j^\pm, m^\pm, n^\pm} d_{j^-} d_{j^+} \hat{S}|j^-; m^-, n^-\rangle \otimes |j^-; m^-, n^-\rangle \psi^{j^- j^+}_{m^- n^- m^+ n^+} \\
&= \sum_{j^\pm} d^2_{j^-} d^2_{j^+} \int_{S^2} \mathrm{d}^2 \vec{u}^\pm \mathrm{d}^2 \vec{v}^\pm \hat{S}|j^-; \vec{u}^-, \vec{v}^-\rangle \otimes |j^+; \vec{u}^+, \vec{v}^+\rangle \psi^{j^- j^+}_{(\vec{u}^-, \vec{v}^-, \vec{u}^+, \vec{v}^+)}
\end{aligned}
\tag{7.6}
$$

Therefore, we see that the only difference between the above strategies is that in some model (EPRL, Alexandrov, Ding-Han-Rovelli) constraints are diagonal in the spin network basis, in other models (Freidel Krasnov) in coherent states basis, and in other (Baratin Oriti) in non-commutative metric basis.
All these strategies present quantization ambiguities, which should be clarified, have several advantages and disadvantages, and may lead to different amplitudes.
However, the correct strategy should encode simplicial geometry and simplicial gravity dynamics, reproduce the correct asymptotic limit and should describe, in some regime, the effective (continuum) gravitational physics.
In this thesis we focus only on the implementation of the simplicity constraints on the GFT field for these models, and we will not study the resulting Feynman diagrams, nor their asymptotic limit.
Moreover, for simplicity we will consider only a face of the tetrahedron. However, we don't loose generality since if the simplicity constraints are imposed on four faces of the same tetrahedron, with respect to the same normal, then one recovers the general solution.
In this case the extended GFT field decomposition is:

$$
\begin{aligned}
\phi_k(g) = \sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) \overline{\tilde{C}^{j^- j^+ k}_{m^- m^+ q}}(k) \tilde{C}^{j^- j^+ k}_{r^- r^+ q}(k) \\
D^{j^-}_{r^- s^-}(g^-) D^{j^+}_{r^+ s^+}(g^+) \overline{\tilde{C}^{j^- j^+ l}_{s^- s^+ p}}(k) \tilde{C}^{j^- j^+ l}_{n^- n^+ p}(k)
\end{aligned}
\tag{7.7}
$$

7.1 Quantum Bivectors

In this section, we show that the classical B variables, the bivectors, at the quantum level, are identified with generators $J^{IJ} = (J_i^+, J_i)$ of the $\mathfrak{so}(4)$ Lie algebra, which then act as operators on the $SO(4)$ spin network states. We start recalling the mathematical definition of classical bivectors, and we use tools from Kirillov's orbit method to find the associated Kirillov-Kostant Poisson structure. Then, using Geometric Quantization, particularly Käler quantization, we, briefly, describe the quantization of the Poisson structure, following [42].
By definition, a *bivector* in n dimensions is an element of $\Lambda^2\mathbb{R}^n$. In words, it can be imagined as a generalization of a triangle, with edge vectors, e, f, g in $\mathbb{R}^n$, in fact a bivector determines a 2-dimensional plane in $\mathbb{R}^n$ with an orientation, and norm corresponding to twice the area of an associated triangle with the same orientation, and laying in the same 2-dimensional plane. Particularly, the bivector for the triangle is

$$E = e \wedge f = f \wedge g = g \wedge e. \tag{7.8}$$

However, no further details of the geometry of the triangle are recorded.
The first step to quantize the geometry of a bivector is to associate to each bivector in $\mathbb{R}^n$ an element in $\mathfrak{so}(n)^*$.
Particularly, the isomorphism $\beta : \Lambda^2\mathbb{R}^n \to \mathfrak{so}(n)^*$ is given by

$$\beta(e \wedge f)(l) = \eta(le, f) \tag{7.9}$$

for any bivector $e \wedge f$, $l \in \mathfrak{so}(n)$ and η is the Euclidean metric on $\mathbb{R}^n$.
The relevant case for our purpose is the 4-dimensional case. This case is not particularly complicated, since using the isomorphism $\mathfrak{so}(4)^* \cong \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)^*$, which corresponds to the splitting of a bivector into its self-dual and anti-self-dual parts, we can reduce the problem to simpler 3-dimensional case, characterized by $\mathfrak{so}(3)^*$.
The second step is based on the geometric quantization of the Kirillov-Kostant Poisson structure of the Poisson manifold $\mathfrak{so}(3)^*$. Actually, we note that in any dimension the dual of any Lie algebra has a natural Poisson structure, (the Kirillov-Kostant Poisson structure on the dual of $\mathfrak{so}(n)$), so that we can identify the space of geometries of a bivector as a classical phase space. Using geometric quantization, we can quantize this phase space and construct a Hilbert space called the space of states of a 'quantum bivector'. Therefore, we first focus on the quantization of a bivector in three dimensions, which is isomorphic to an element in $\mathfrak{so}(3)^*$.
As mentioned, $\mathfrak{so}(4)^*$ with its Kirillov-Kostant Poisson structure is the product of two copies of the Poisson manifold $\mathfrak{so}(3)^*$, thus the generalization from $\mathfrak{so}(3)^*$ to $\mathfrak{so}(4)^*$ is not an hard task. However, we need to be careful since, as we will see, for $\mathfrak{so}(4)^*$ there are two possible Poisson structures: the flipped and the non-flipped one, for more details [42, 132].

7.2 Geometric Quantization of the Kirillov-Kostant Poisson structure

As we mentioned above, Kirillov and Kostant showed in [133, 134] that the dual of any Lie algebra $\mathfrak{g}$, and in our case $\mathfrak{g} = \mathfrak{so}(3)$, is a vector space with an additional structure that makes it a Poisson manifold, i.e. a manifold with a Poisson bracket on its algebra of functions. Thus, we can define a 2-form ω compatible with the Poisson structure, called a symplectic leaf on the Poisson manifold. Interestingly, if $\mathfrak{g} = \mathfrak{so}(3)$, the symplectic leaves are spheres centered at the origin. The quantization of the Poisson manifold is performed by constructing a Hilbert space (via Kähler quantization [135–137]) for each symplectic leaf and then by considering the direct sum of all these Hilbert spaces.
The first step is to choose a complex structure J on the leaf that preserves the symplectic form ω, making the leaf into a Kähler manifold, see [42].
Secondly, we choose a holomorphic complex line bundle L over the leaf, called the *prequantum line bundle* equipped with a connection whose curvature equals ω. If the symplectic leaf is integral, we define the Hilbert space for the leaf to be the space of square-integrable holomorphic sections of L. For nonintegral leaves, we define the Hilbert space to be 0-dimensional. Particularly, in the case of interest, $\mathfrak{so}(3)^*$, it is possible to show [42] that the integral symplectic leaves are the spheres, S_j, centered at the origin with radii given by non-negative half-integers j. When $j = 0$, S_j is trivially a Kähler manifold. In this case the prequantum line bundle is trivial and the Hilbert space of holomorphic sections is 1-dimensional.

The direct sum of all these spaces is the Hilbert space of a quantum bivector in three dimensions, denoted by

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j. \tag{7.10}$$

Following [42], we can define self-adjoint operators $\hat{J}^i$, the observables associated to the three components of the quantum bivector, acting on $\mathcal{H}$ and satisfying the usual angular momentum commutation relations:

$$[\hat{J}^1, \hat{J}^2] = i\hat{J}^3, \qquad [\hat{J}^2, \hat{J}^3] = i\hat{J}^1, \qquad [\hat{J}^3, \hat{J}^1] = i\hat{J}^2. \tag{7.11}$$

We note that they don't commute so that we cannot measure simultaneously the three component of a quantum bivector with complete precision.
We can consider operators $\hat{J}^i = \mathcal{Q}(J^i)$ as the quantized coordinate functions (via the quantization map $\mathcal{Q}$) on $\mathfrak{so}(3)^*$, where J^i are a basis of $\mathfrak{so}(3)$, satisfying the Poisson bracket relations:

$$\{J^1, J^2\} = J^3, \qquad \{J^2, J^3\} = J^1, \qquad \{J^3, J^1\} = J^2. \tag{7.12}$$

In other words a natural way to quantize a bivector split in the self-dual and anti-self-dual part (representing two 3-vectors in three dimensions) is to associate to each part an element of the canonical basis of the $SO(3)$ Lie algebra, J^i, in some representation j. Then we choose a quantization map $\mathcal{Q}$, mapping the classical generators in the representation j to quantum operators acting on the corresponding Hilbert space $\mathcal{H}_j$. Doing this, the operator corresponding to the square of the 3-vector length is given by the $SU(2)$ Casimir $C = L^2 = J \cdot J$, which is diagonal on the representation space $\mathcal{H}_j$ with eigenvalue $L^2 = j(j+1)$ or $L^2 = (j+1/2)^2$, etc. according on the ordering of the operators entering in the definition of the length, i.e. on the choice of the quantization map $\mathcal{Q}$. Therefore we see that the representation label j gives the quantum length of the 3-vector to which the corresponding representation is assigned, thus the Hilbert space $\mathcal{H}_j$ can be interpreted as the Hilbert space for a 3-vector with squared length $j(j+1)$. In general, we define the Hilbert space describing the geometry of a quantum 3-vector, $\mathcal{H} = \oplus_j \mathcal{H}_j$, i.e. as a direct sum of all the Hilbert spaces corresponding to a possible fixed length.

7.3 Quantum bivectors in four dimensions

From the observation of the previous sections, in the four dimensional case, where the symplectic manifold is $\mathfrak{so}(4)^*$, any symplectic leaf is of the form $S_j \times S_k$, that is, the product of a sphere of radius j and a sphere of radius k, where j, k are independent arbitrary spins. This symplectic leaf can be turned into a Käler manifold, that can be equipped with a line bundle given by the tensor product of the line bundles over S_j and S_k. Finally, after geometric quantization, we obtain a Hilbert space equal to the tensor product of the spin-j representation of the self-dual copy of $\mathfrak{so}(3)$ and the spin-k representation of the anti-self-dual copy.
Taking the direct sum over all leaves, we thus obtain the Hilbert space, characterized by:

- the non-flipped Poisson structure

$$\mathcal{H} = \bigoplus_{j,k} \mathcal{H}_j^+ \otimes \mathcal{H}_k^-, \tag{7.13}$$

corresponding to the Poisson structure

$$\{J^{+i}, J^{+j}\} = \epsilon^{ij}{}_k J^{+k}, \qquad \{J^{-i}, J^{-j}\} = \epsilon^{ij}{}_k J^{-k}, \qquad \{J^{+i}, J^{-j}\} = 0. \tag{7.14}$$

where $\mathcal{H}_j^{\pm}$ are the Hilbert space describing the self-dual and anti-self-dual part, respectively.

- the flipped Poisson structure

$$\mathcal{H} = \bigoplus_{j,k} \mathcal{H}_j^+ \otimes \tilde{\mathcal{H}}_k^- \tag{7.15}$$

obtained by reversing the sign of the Poisson structure on the anti-self-dual summand in $\mathfrak{so}(3)^* \otimes \mathfrak{so}(3)^*$, which is the one that determines the chirality of the tetrahedron (described by four flipped bivectors) correctly:

$$\{J^{+i}, J^{+j}\} = \epsilon^{ij}{}_k J^{+k}, \qquad \{J^{-i}, J^{-j}\} = -\epsilon^{ij}{}_k J^{-k}, \qquad \{J^{+i}, J^{-j}\} = 0. \tag{7.16}$$

The choice of the flipped Poisson structure on $\Lambda^2\mathbb{R}^4$, identified with $\mathfrak{so}(4)^*$ is related to the isomorphism β (7.9). Actually, we could achieve the same Poisson structure on $\Lambda^2\mathbb{R}^4$ by starting with the standard Poisson structure on $\mathfrak{so}(4)^*$ and instead using the isomorphism $\beta \circ *$, which differs by the Hodge dual $*$ acting on the bivectors.

7.4 Relation between Spin network states and Angular momentum states

We have seen that the geometries of quantized bivectors are described by states in the Hilbert space:

$$\mathcal{H} = \bigoplus_{j,k} \mathcal{H}_j^+ \otimes \tilde{\mathcal{H}}_k^- \tag{7.17}$$

where each $\mathcal{H}_j^\pm$ is the usual space of states of ordinary quantum angular momentum, $|j, m\rangle$. However, for our purpose, we would like quantum bivectors acting as operators on the $SO(4)$ spin network states $|j^-; m^-, n^-\rangle \otimes |j^+; m^+, n^+\rangle \in L^2(SO(4))$, and not on ordinary angular momentum states.
In this section we consider the relation between angular momentum states of quantum mechanics and spin network states, particularly we see how to embed angular momentum states in spin network states.

The angular momentum is the quantity that is conserved if a physical system is symmetric under rotations. Rotations are described by a matrix representing elements $g \in SO(3)$, or its double covering $SU(2)$. The generators of $SO(3)$ are orthonormal vectors $J^i \in \mathfrak{so}(3)$, the Lie algebra $\mathfrak{so}(3)$ is related to the Lie algebra of invariant vector fields on the group manifold, and we have:

$$\{J^i, J^j\} = \epsilon^{ij}{}_k J^k \tag{7.18}$$

since the generators do not commute we can simultaneously diagonalize only the Casimir operator J^2 and a component, for example, J^z. The $(2j+1)$-dimensional representation space $|j, m\rangle$ is labelled by j and m eigenvalues respectively of J^2 and J^z.

In quantum mechanics symmetries are implemented by unitary operators, and in the case of rotations we have that rotation symmetries are implemented by unitary representations $D^j(g)$, acting on a $(2j+1)$-dimensional Hilbert space, due to the fact that unitary representations of compact groups are finite-dimensional. A basis of the Hilbert space is given by the states $|j, m\rangle$ with $-j \leq m \leq j$.

The Stone's theorem establishes a one-to-one correspondence between self-adjoint operators on a Hilbert space and one-parameter families of unitary representations.

The commutation relations among the self-adjoint operators $\hat{J}^i$ are

$$[\hat{J}^i, \hat{J}^j] = i\hbar \epsilon^{ij}{}_k \hat{J}^k \tag{7.19}$$

The quantum generators can be obtained by canonical quantization of the Lie algebra by means of a quantization map $\mathcal{Q}$, such that $\hat{J}^i = \mathcal{Q}(J^i)$, and we have

$$\hat{J}^z|j,m\rangle = \hbar m|j,m\rangle \quad \text{and} \quad \hat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \ , \tag{7.20}$$

where we have restored the Planck's constant $\hbar$. Finally, the relation between angular momentum states and spin network states is given by the definition of the Wigner D matrix:

$$D^j_{mn}(g) = \langle j,m|D^j(g)|j,n\rangle \tag{7.21}$$

As we have seen, spin network states $|j;m,n\rangle$ form a basis of the Hilbert Space $\mathcal{H} = L^2(SO(3))$. Indeed, a decomposition of functions on $SO(3)$ in terms of spin representations follows from the Peter-Weyl theorem, which shows the orthogonality of the elements of representation matrices $D^j_{mn}(g)$. Accordingly, the states $|j;m,n\rangle$ defined via $\langle g|j;m,n\rangle = D^j_{nm}(g)$ constitute a basis in the Hilbert space $\mathcal{H} = L^2(SO(3))$, the spin basis.
Recall that the spin label j of the state refers to the eigenvalues of the quadratic Casimir operator $\hat{X}^2 = (-i\hbar)^2\mathcal{L}_{T_i}\mathcal{L}_{T_i}$,

$$\hat{X}^2|j;m,n\rangle = \hbar^2 j(j+1)|j;m,n\rangle \tag{7.22}$$

which label the irreducible unitary representations of $SO(3)$, while the indices m,n refer to eigenvalues of the Lie derivative operators $\hat{X}^{L,R}_z = -i\hbar\mathcal{L}_{T^{L,R}_z}$ with respect to left- and right-invariant vector fields $T^{L,R}_z \in TSO(3)$, such that $T^L_z(e) = T^R_z(e) = t_z \in \mathfrak{so}(3)$,

$$\hat{X}^L_z|j;m,n\rangle = \hbar m|j;m,n\rangle \quad \text{and} \quad \hat{X}^R_z|j;m,n\rangle = \hbar n|j;m,n\rangle \ , \tag{7.23}$$

Now, we point out that the states $|j;m,n\rangle$ are not the usual angular momentum $|j,m\rangle$, even though they are related through the definition of representation matrix elements $D^j_{mn}(g) = \langle j,n|D^j(g)|j,m\rangle$. Specifically, the space of angular momentum states, $|j,m\rangle$, is a $(2j+1)$-dimensional representation space of the symmetry group of rotations around a point ($SO(3)$) and its double-cover $SU(2)$, $L^2(S^2) = \sum_j \mathcal{H}^j_{SO(3)}$, where $\mathcal{H}^j_{SO(3)}$ is an irreducible representation space of $SO(3)$.
The space of Spin network states, $|j;m,n\rangle$, is a $(2j+1)^2$-dimensional representation space corresponding to the Hilbert space $\mathcal{H} = L^2(SO(3)) = \sum_j [H^j_{SO(3)}]^L \times [\mathcal{H}^j_{SO(3)}]^R$, giving a full representation of both left and right multiplication.
However, we can represent angular momentum states $|j,m\rangle$ in the Hilbert space $L^2(SO(3))$ by the identification $\mathcal{H}^j_{SO(3)} \equiv [\mathcal{H}^j_{SO(3)}]^{R/L}$. In the thesis, we have chosen to use right-invariant vector fields, thus the embedding is $\mathcal{H}^j_{SO(3)} \equiv [\mathcal{H}^j_{SO(3)}]^R \subset [\mathcal{H}^j_{SO(3)}]^L \times [\mathcal{H}^j_{SO(3)}]^R$, and:

$$\hat{J}^z|j,n\rangle = n|j,n\rangle \equiv \hat{X}^R_z|j;m,n\rangle = n|j;m,n\rangle \tag{7.24}$$

In conclusion, we have shown that the classical B variables, the bivectors, at the quantum level, are identified, in the splitting self-dual anti-self-dual with momentum operators $[\hat{X}^R_i]^\pm$, acting on the Hilbert space $L^2(SO(3)^\pm)$, whose states are spin network states $|j^\pm;m^\pm,n^\pm\rangle$, or non-commutative metric states $|x^\pm\rangle$, described in Chapter 4.

7.5 Quantum simplicity constraints

Now, we can finally give the quantum version of Plebanski constraints as operator equations. Recall that the classical simplicity constraint reads, in the self-dual/anti-self-dual splitting,

$$\forall \ tetrahedra \ t \in \Delta \ \exists \ k_t \in SU(2) \ / \ \beta x^+_i + k \cdot x^-_i \cdot k^{-1} = 0, \ \forall X_f = (x^+_f, x^-_f) \tag{7.25}$$

Thus, at the quantum level we can define the quantum operator imposing the simplicity constraints:

$$\hat{D}_i = \beta \hat{X}_i^+ + k\hat{X}_i^- k^{-1} \tag{7.26}$$

where $\hat{X}_i^{\pm} \equiv [\hat{b}_i^L]^{\pm}$ are quantum operators associated to the generators of $SU(2)^{\pm}$ related to left-invariant vector fields. For example, they act on spin network states as:

$$\begin{aligned} \hat{X}_z^{\pm}|j^{\pm}; m^{\pm}, n^{\pm}\rangle &= m^{\pm}|j^{\pm}; m^{\pm}, n^{\pm}\rangle \\ \hat{X}^{\pm} \cdot \hat{X}^{\pm}|j^{\pm}; m^{\pm}, n^{\pm}\rangle &= j^{\pm}(j^{\pm}+1)|j^{\pm}; m^{\pm}, n^{\pm}\rangle \end{aligned} \tag{7.27}$$

While on non-commutative metric states as [37]:

$$\begin{aligned} \hat{X}^{\pm}|x^{\pm}\rangle &= x^{\pm} \star |x^{\pm}\rangle \\ \hat{X}^{\pm} \cdot \hat{X}^{\pm}|x^{\pm}\rangle &= (x^{\pm})^2 \star |x^{\pm}\rangle \end{aligned} \tag{7.28}$$

Chapter 8

The Riemannian EPRL model

In this section we present the Riemmanian EPRL (Engle, Pereira, Rovelli and Livine) model, developed firstly by Engle, Pereira and Rovelli in [123,124] without the Immirzi parameter, and subsequently generalized in [43] to the case of finite Immirzi parameter. Remarkably, the EPRL model is the first "new" model proposed, implementing several feature mentioned in previous chapters as the Immirzi parameter, the linear simplicity constraints and the weak imposition of the constraints.
Historically, the first Spin Foam model proposed was the Barret-Crane model. We will not review this model in the thesis because it corresponds to the limit $\gamma \to \infty$ of the EPRL model, that we are going to present in the next sections.
Nevertheless, in order to understand the motivations and the importance of the EPRL model it is important to describe briefly the Barrett-Crane features, as the new models have been proposed to cure some of the Barrett-Crane problems.

8.1 The problems of the Barrett-Crane model

As mentioned, the Barrett-Crane model, as was originally defined, is based on the discretization of the quadratic simplicity constraints. At the quantum level, using geometric quantization we can identify the classical bivectors B_f^{IJ} with the generators J^{IJ} of the Lie algebra of the gauge group $G = SO(4)$. Thus, in the boundary Hilbert space bivectors are represented by a derivative operator whose action on the spin networks is equivalent to an insertion of the generator of the Lie algebra in the representation $\lambda = (j^-, j^+)$ associated with the given face. This identification is justified by the observation that in BF theory the B field is canonically conjugated to the spin connection [83].
Once this identification has been made, the simplicity constraints are imposed strongly on spin network states, the boundary states of the BF theory, which have the effect of fixing the representations and the intertwiners coloring the graphs.
The geometric interpretation is that four quantum bivectors, satisfying the quantum simplicity constraints, form a quantum tetrahedron, represented by an intertwining operator.
Particularly, in absence of the Immirzi parameter, the diagonal simplicity constraint (which involves only the bivectors referring to one face f) restricts the allowed representations $\lambda_f = (j_f^-, j_f^+)$ to be simple representations $\lambda_f^s = (j_f^-, j_f^+) = (j_f, j_f)$, thus imposing the relation $j_f^- = j_f^+$ [83].
Regarding the cross simplicity constraint, we note that it involves different faces dual to triangles sharing an edge.
These triangles are in the boundary of a tetrahedron τ, characterized by an intertwining operator, which can be always decomposed into the sum of three-valent intertwiners.
Actually, three-valent intertwiners are characterized by intermediate representations.

Then, the imposition of the cross simplicity constraint requires that we look for intertwiners that are compatible with the coupling of simple representations, i.e. the intermediate representations of the decomposition of the intertwniner are simple [83].
The intertwiner satisfying this condition, called BC intertwiner was proposed in [125] and was shown to be unique in [138].
Despite the great success of the Barrett-Crane model, it has been argued that some of its features that may be considered as problems, in the spirit of the Spin Foam approach described in Section 5.1.
Specifically, on one hand a lot of criticism has been raised concerning the ability of the BC model to reproduce correctly quantum simplicial geometry, as the asymptotic (i.e. the large spin limit) behaviour of the vertex amplitude [139–141] does not lead to the exponential of the Regge action, but rather to crippled geometric configurations.
On the other hand, the original version of the Barrett-Crane model has strange geometric properties, specifically [49]:

- 4-simplices speak only through face representations, i.e. triangle areas [142];
- bivectors associated to same triangle in different simplices are not identified [38, 43, 47, 123, 124, 127–129];
- normal vectors to the same tetrahedron seen in different 4-simplices are uncorrelated [48, 122];
- the simplicity constraints are imposed in a non-covariant fashion [48];
- because of this non-covariance, there are missing constraints over the connection variables [48].

These features are a consequence of the uniqueness of the BC intertwiner, but they have been reconsidered in the Barrett-Crane version ($\gamma = \infty$) of the Baratin Oriti model, where the last three issues are solved if constraints are imposed covariantly (see the conclusions of [49]).

Finally, an other important issue comes from the analysis of the constraints. Indeed, in the case of quadratic simplicity constraints or in presence of finite Immirzi parameter (relating Spin Foam models and Loop Quantum Gravity), constraints to be imposed are second class, and shouldn't be imposed strongly.

Remarkably, the EPRL model was the first model implementing three new inputs [83]:

- linearisation of the simplicity constraints,
- inclusion of the Immirzi parameter,
- imposition of the simplicity constraints in a weak sense.

Let us present these new ideas and their effects on the Spin Foam quantization.

From the classical analysis, the Plebanski formulation is equivalent to four- dimensional gravity only in the so-called gravitational sector (where $B^{IJ} = \pm\frac{1}{2}\epsilon_{IJKL}e^K \wedge e^L$) of the simplicity constraints. However, the topological sector (where $B^{IJ} = \pm e^I \wedge e^J$) may "interfere" with the physical one at the quantum level. The original quadratic version of the simplicity constraints does not distinguish between the two sectors and therefore one cannot be sure that one describes pure gravity.
The linearised version of the simplicity constraints turns out to be stronger than the original one, allowing the distinction of the two sectors, and has a direct geometric interpretation: indeed, linear simplicity constraints simply mean that the four triangles described by bivectors B_f, which belong to the same tetrahedron $\tau \supset f$, lie in the same hyperplane.

The second point is to incorporate the Immirzi parameter into the quantization scheme. This is, of course, motivated by the hope to find a Spin Foam model consistent with LQG, whose results depend on this (classically irrelevant) parameter in a non-trivial way.
The third point is related to the way the simplicity constraints have to be imposed. We have seen that strong imposition leads to the Barrett-Crane model, whereas the EPRL model imposes the constraints weakly.
However, we would like to point out the remarkable fact that an analysis of the constraints [143] has shown that in the pure gravitational of Holst gravity, or in the Plabanski theory with linear constraints, simplicity constraints close, thus they are first class and should be imposed strongly.

8.2 The EPRL model and the Master Constraint criterion

There are several ways for weak imposition of the constraints. The first proposed is the Master constraint criterion, that can be developed by studying the spectrum of the Master constraint operator $\hat{M} = \hat{D} \cdot \hat{D}$. Strong imposition of the constraint $\hat{D}$ would amount to looking for the kernel of the master constraint $\hat{M}$. However, generically the positive operator associated with the master constraint does not contain the zero eigenvalue in the spectrum. However, the proposal of [143] is to look for the minimum eigenvalue among spaces $\mathcal{H}_j \in \mathcal{H}_{j^- j^+}$. Explicitly, the Master constraints is:

$$\begin{aligned}\hat{M}|\psi\rangle =&\Big\{k(\hat{X}_i^-)^2k^{-1} + \beta^2(\hat{X}_i^+)^2 + \\ &+ \beta[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2]\Big\}|\psi\rangle \\ =&\frac{1}{(\gamma+1)^2}\Big\{(\gamma+1)^2k(\hat{X}_i^-)^2k^{-1} + (\gamma-1)^2(\hat{X}_i^+)^2 + \\ &+ (\gamma^2-1)[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2]\Big\}|\psi\rangle\end{aligned} \tag{8.1}$$

The derivation is straightforward.

Proof.

$$\begin{aligned}&(k\hat{X}_i^-k^{-1} + \beta\hat{X}_i^+)^2|\psi\rangle = \\ &= \{k\hat{X}_i^-k^{-1}k\hat{X}_i^-k^{-1} + k\hat{X}_i^-k^{-1}\beta\hat{X}_i^+ + \beta\hat{X}_i^+k\hat{X}_i^-k^{-1} + \beta\hat{X}_i^+\beta\hat{X}_i^+\}|\psi\rangle \\ &= \{k(\hat{X}_i^-)^2k^{-1} + \beta^2(\hat{X}_i^+)^2 + \beta[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2]\}|\psi\rangle \\ &= \Big\{\frac{(\gamma+1)^2}{(\gamma+1)^2}k(\hat{X}_i^-)^2k^{-1} + \frac{(\gamma-1)^2}{(\gamma+1)^2}(\hat{X}_i^+)^2 + \\ &+ \frac{(\gamma+1)^2}{(\gamma+1)^2}\frac{\gamma-1}{\gamma+1}\Big[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2\Big]\Big\}|\psi\rangle \\ &= \frac{1}{(\gamma+1)^2}\Big\{(\gamma+1)^2k(\hat{X}_i^-)^2k^{-1} + (\gamma-1)^2(\hat{X}_i^+)^2 + \\ &+ (\gamma-1)(\gamma+1)[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2]\Big\}|\psi\rangle \\ &= \frac{1}{(\gamma+1)^2}\Big\{(\gamma+1)^2k(\hat{X}_i^-)^2k^{-1} + (\gamma-1)^2(\hat{X}_i^+)^2 + \\ &+ (\gamma^2-1)[(k\hat{X}_i^-k^{-1} + \hat{X}_i^+)^2 - k(\hat{X}_i^-)^2k^{-1} - (\hat{X}_i^+)^2]\Big\}|\psi\rangle\end{aligned} \tag{8.2}$$

□

If we consider the standard eigenvalues of the quadratic operator:

$$\begin{aligned} [k(\hat{X}_i^-)^2k^{-1}]|\psi\rangle &= j^-(j^-+1)|\psi\rangle \\ (\hat{X}_i^+)^2|\psi\rangle &= j^+(j^++1)|\psi\rangle \\ (k\hat{X}_i^-k^{-1}+\hat{X}_i^+)^2|\psi\rangle &= k(k+1)|\psi\rangle \end{aligned} \tag{8.3}$$

We have the following equation:

$$\begin{aligned} \hat{M}|\psi\rangle =& \frac{1}{(\gamma+1)^2}\Big\{(\gamma+1)^2j^-(j^-+1)+(\gamma-1)^2j^+(j^++1)+ \\ &+(\gamma^2-1)[j(j+1)-j^-(j^-+1)-(j^++1)]\Big\}|\psi\rangle \\ =& m_{j^\pm j}|\psi\rangle \end{aligned} \tag{8.4}$$

There are two cases:

- Case $\gamma < 1$: In this case the minimum eigenvalue is obtained for

$$\begin{cases} j^- = \frac{1-\gamma}{2}j = |\beta|j^+ \\ j^+ = \frac{1+\gamma}{2}j \\ j = j^+ + j^- = j^+(1+|\beta|) \end{cases} \tag{8.5}$$

  i.e., one has

$$m_{j^\pm j} \geq m_j \tag{8.6}$$

  where $m_j = \frac{\hbar^2(1-\gamma^2)j}{(1+\gamma)^2}$ where we have restored the explicit dependence on $\hbar$ so that it is apparent that selected eigenvalue vanishes in the semi-classical limit $\hbar \to 0$ and $j \to \infty$ with $j\hbar =$ constant.

  Proof.

$$\begin{aligned} \hat{M}|\psi\rangle &= \{j^-(j^-+1)+\beta^2j^+(j^++1)+ \\ &+\beta[k(k+1)-j^-(j^-+1)-j^+(j^++1)]\}|\psi\rangle \\ &= \{|\beta|j^+(|\beta|j^++1)+\beta^2j^+(j^++1)+ \\ &+\beta[j^+(1+|\beta|)(j^+(1+|\beta|)+1)-|\beta|j^+(|\beta|j^++1)-j^+(j^++1)]\}|\psi\rangle \\ &= \{|\beta|^2(j^+)^2+|\beta|j^++\beta^2(j^+)^2+\beta^2j^++ \\ &+\beta[(j^+(1+|\beta|))^2+j^++|\beta|j^+-|\beta|^2(j^+)^2-|\beta|j^+-(j^+)^2-j^+]\}|\psi\rangle \\ &= \{2\beta^2(j^+)^2+|\beta|j^++\beta^2j^++ \\ &+\beta[(j^+)^2(1+2|\beta|+|\beta|^2)-|\beta|^2(j^+)^2-(j^+)^2]\}|\psi\rangle \\ &= \{2\beta^2(j^+)^2+|\beta|j^++\beta^2j^++2\beta|\beta|(j^+)^2\}|\psi\rangle \\ &= \{2\beta^2(j^+)^2+|\beta|j^++\beta^2j^+-2|\beta|^2(j^+)^2\}|\psi\rangle \\ &= |\beta|j^+(1+|\beta|)|\psi\rangle = |\beta|j|\psi\rangle = \frac{1-\gamma}{\gamma+1}j|\psi\rangle = \frac{1-\gamma^2}{(1+\gamma)^2}j|\psi\rangle \end{aligned} \tag{8.7}$$

  restoring $\hbar$ we complete the proof. □

- Case $\gamma > 1$: In this case the minimum eigenvalue is obtained for

$$\begin{cases} j^- = \frac{\gamma-1}{2}j = |\beta|j^+ \\ j^+ = \frac{\gamma+1}{2}j \\ j = j^+ - j^- = j^+(1-|\beta|) \end{cases} \tag{8.8}$$

i.e., one has

$$m_{j^\pm j} \geq m_j \tag{8.9}$$

where $m_j = \frac{\hbar^2(\gamma^2-1)j}{(1+\gamma)^2}$ where we have restored the explicit dependence on $\hbar$ so that it is apparent that selected eigenvalue vanishes in the semi-classical limit $\hbar \to 0$ and $j \to \infty$ with $j\hbar =$ constant.

Proof.

$$\begin{aligned}
\hat{M}|\psi\rangle &= \{j^-(j^-+1) + \beta^2 j^+(j^++1)+ \\
&+ \beta[k(k+1) - j^-(j^-+1) - j^+(j^++1)]\}|\psi\rangle \\
&= \{|\beta|j^+(|\beta|j^++1) + \beta^2 j^+(j^++1)+ \\
&+ \beta[j^+(1-|\beta|)(j^+(1-|\beta|)+1) - |\beta|j^+(|\beta|j^++1) - j^+(j^++1)]\}|\psi\rangle \\
&= \{|\beta|^2(j^+)^2 + |\beta|j^+ + \beta^2(j^+)^2 + \beta^2 j^+ + \\
&+ \beta[(j^+(1-|\beta|))^2 + j^+ - |\beta|j^+ - |\beta|^2(j^+)^2 - |\beta|j^+ - (j^+)^2 - j^+]\}|\psi\rangle \\
&= \{2\beta^2(j^+)^2 + |\beta|j^+ + \beta^2 j^+ + \\
&+ \beta[(j^+)^2(1-2|\beta|+|\beta|^2) - 2|\beta|j^+ - |\beta|^2(j^+)^2 - (j^+)^2]\}|\psi\rangle \\
&= \{2\beta^2(j^+)^2 + |\beta|j^+ + \beta^2 j^+ + \\
&+ \beta[(j^+)^2 - 2|\beta|(j^+)^2 + |\beta|^2(j^+)^2 - 2|\beta|j^+ - |\beta|^2(j^+)^2 - (j^+)^2]\}|\psi\rangle \\
&= \{2\beta^2(j^+)^2 + |\beta|j^+ + \beta^2 j^+ - 2\beta^2(j^+)^2 - 2\beta^2 j^+\}|\psi\rangle \\
&= \beta j^+(1-\beta)|\psi\rangle = \beta j|\psi\rangle = \frac{\gamma-1}{\gamma+1}j|\psi\rangle = \frac{\gamma^2-1}{(\gamma+1)^2}j|\psi\rangle
\end{aligned} \tag{8.10}$$

restoring $\hbar$ we complete the proof. □

However, we note that this implies that the Immirzi parameter γ is quantized to be a rational number [14]. Remarkably, this is a common feature of the EPRL model, the Alexandrov's proposal, the Ding-Han-Rovelli model and the Freidel Krasnov model.

8.3 The Master constraint criterion and the Casimir operators

The Master constraint criterion can be equivalent re-written in an equivalent way involving only $Spin(4)$ and $SU(2)$ Casimir operators, as noted in [14] [83].
The criterion is encoded in the following two equations:

- First constraint equation

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle \approx 0 \tag{8.11}$$

- Second constraint equation

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle \approx 0 \tag{8.12}$$

where:

$$\begin{aligned}
\hat{C}_G^{(1)} &= (k\hat{X}_i^- k^{-1})^2 + (\hat{X}_i^+)^2 \\
\hat{C}_G^{(2)} &= (\hat{X}_i^+)^2 - (k\hat{X}_i^- k^{-1})^2 \\
\hat{C}_H &= (k\hat{X}_i^- k^{-1} + \hat{X}_i^+)^2
\end{aligned} \tag{8.13}$$

Proof. Classically, for the diagonal simplicity constraint, we have:

$$\begin{aligned}
0 &= kx_i^- k^{-1} + \beta x_i^+ = (kx_i^- k^{-1})^2 - \beta^2 (x_i^+)^2 = (kx_i^- k^{-1})^2 - \left(\frac{\gamma - 1}{\gamma + 1}\right)^2 (x_i^+)^2 \\
&= \frac{(\gamma+1)^2 (kx_i^- k^{-1})^2 - (\gamma - 1)^2 (x_i^+)^2}{(\gamma+1)^2} \\
&= \gamma^2 (kx_i^- k^{-1})^2 + 2\gamma (kx_i^- k^{-1})^2 + (kx_i^- k^{-1})^2 - \gamma^2 (x_i^+)^2 + 2\gamma (x_i^+)^2 - (x_i^+)^2 \\
&= (\gamma^2 + 1)[(x_i^+)^2 - (kx_i^- k^{-1})^2] - 2\gamma [(x_i^+)^2 + (kx_i^- k^{-1})^2] \\
&= (\gamma^2 + 1) C_G^{(2)} - 2\gamma C_G^{(1)}
\end{aligned} \tag{8.14}$$

Classically, for the cross simplicity constraint, we have:

$$\begin{aligned}
0 &= kx_i^- k^{-1} + \beta x_i^+ = kx_i^- k^{-1} + \frac{\gamma - 1}{\gamma + 1} x_i^+ = \frac{(\gamma + 1) kx_i^- k^{-1} + (\gamma - 1) x_i^+}{\gamma + 1} = \\
&= \gamma (kx_i^- k^{-1} + x_i^+) + kx_i^- k^{-1} - x_i^+ = \gamma (kx_i^- k^{-1} + x_i^+)^2 - [(x_i^+)^2 - (kx_i^- k^{-1})^2] \\
&= \gamma C_H - C_G^{(2)}
\end{aligned} \tag{8.15}$$

At the quantum level we have to define a quantization map such that $\hat{X}_i^- = \mathcal{Q}(x_i^-)$ and $\hat{X}_i^+ = \mathcal{Q}(x_i^+)$, as a result we have ambiguities in the spectrum of the quantum Casimir operators $\hat{C}_G^{(1)}$, $\hat{C}_G^{(2)}$ and $\hat{C}_H$ due to the operator ordering problem. □

Several comments concerning this result are in order [83]:

- The first equation derives from the diagonal simplicity constraint, as it involves only a relation between j^- and j^+ representations, precisely as the diagonal simplicity constraint. Moreover, it can be imposed strongly. The distinguishing role of this constraint is justified by the fact that it lies in the center of the algebra formed by the quantum constraint operators, i.e. it commutes with all the other constraints, and can therefore be considered as first class [83].

- The second equation is a consequence of the cross simplicity, as it impose a relation between the $SU(2)^-$, $SU(2)^+$ representations and the intermediate representation of the $SU(2) \subset SU(2)^- \times SU(2)^+ \subset SO(4)$ subgroup, which is the stabilizer subgroup of the normal of the tetrahedron.

- The presence of the $SU(2)$ subgroup and the normal immediately implies that the solution to the quantum simplicity constraints will lie in the class of projected spin networks. Consequently, we will impose the EPRL constraints on the extended GFT field.

- Finally, the two equations do not have common exact solutions with the standard choice of the spectrum of the Casimir operators. However, it is known that they may acquire quantum corrections due to ordering ambiguities related to the quantization map chosen. Moreover, these corrections are assumed to be such that the equations do have non-trivial solutions. More details in the following. Specifically, the EPRL solutions can be exact solutions with a particular choice of such corrections, but it should be mentioned that there are also other interesting possibilities [44, 144].

8.4 The EPRL solutions and the Casimir operators

In the light of the previous section, we now study the dependence of the two constraint equation on the choice of the quantum corrections due to ordering ambiguities.

Particularly, if we consider the prescriptions (8.5) and (8.8), we observe that for the standard spectrum of the Casimir operators:

$$\begin{aligned}\hat{C}_G^{(1)}|\psi\rangle =& j^+(j^+ + 1) + j^-(j^- + 1)|\psi\rangle \\ \hat{C}_G^{(2)}|\psi\rangle =& j^+(j^+ + 1) - j^-(j^- + 1)|\psi\rangle \\ \hat{C}_H|\psi\rangle =& j(j+1)|\psi\rangle\end{aligned} \tag{8.16}$$

we have that:

- In the case $\beta < 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \gamma j(\gamma^2 - 1) \tag{8.17}$$

  while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{8.18}$$

- In the case $\beta > 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = j(1-\gamma^2)|\psi\rangle \tag{8.19}$$

  while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [j(\gamma - 1)]|\psi\rangle \tag{8.20}$$

Proof. In the case $\beta < 0$, we have for the first simplicity equation:

$$\begin{aligned}&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\ &= \{(\gamma^2+1)\left[j^+(j^+ + 1) - j^-(j^- + 1)\right] - 2\gamma[j^+(j^+ + 1) + j^-(j^- + 1)]\}|\psi\rangle \\ &= \Bigg\{(\gamma^2+1)\left[\frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) - \frac{1-\gamma}{2}j\left(\frac{1-\gamma}{2}j+1\right)\right] + \\ &- 2\gamma\left[\frac{1+\gamma}{2}j(\frac{1+\gamma}{2}j+1) + \frac{1-\gamma}{2}j(\frac{1-\gamma}{2}j+1)\right]\Bigg\}|\psi\rangle \\ &= \Bigg\{\frac{(\gamma^2+1)}{4}\left[j^2 + 2\gamma j^2 + \gamma^2 j^2 + 2\gamma j + 2j - j^2 + 2\gamma j^2 - \gamma^2 j^2 - 2j + 2\gamma j\right] + \\ &- \frac{2\gamma}{4}\left[j^2 + 2\gamma j^2 + \gamma^2 j^2 + 2\gamma j + 2j + j^2 - 2\gamma j^2 + \gamma^2 j^2 + 2j - 2\gamma j\right]\Bigg\}|\psi\rangle \\ &= [\gamma^3 j^2 + \gamma^3 j + \gamma j^2 + \gamma j - \gamma j^2 - \gamma^3 j^2 - 2\gamma j]|\psi\rangle = \gamma j(\gamma^2 - 1)|\psi\rangle\end{aligned} \tag{8.21}$$

In the case $\beta < 0$, we have for the second simplicity equation:

$$\begin{aligned}&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j(j+1) - j^+(j^+ + 1) + j^-(j^- + 1)]|\psi\rangle \\ &= \left[\gamma j(j+1) - \frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) + \frac{1-\gamma}{2}j\left(\frac{1-\gamma}{2}j+1\right)\right]|\psi\rangle \\ &= \left[\gamma j(j+1) - \frac{j^2 + 2\gamma j^2 + \gamma^2 j^2 + 2\gamma j + 2j}{4} + \frac{j^2 - 2\gamma j^2 + \gamma^2 j^2 + 2j - 2\gamma j}{4}\right]|\psi\rangle \\ &= \left[\gamma j(j+1) - \frac{4\gamma j^2 + 4\gamma j}{4}\right]|\psi\rangle = [\gamma j(j+1) - \gamma j(j+1)]|\psi\rangle = 0\end{aligned} \tag{8.22}$$

In the case $\beta > 0$, we have for the first simplicity equation:

$$
\begin{aligned}
&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\
&= \{[(\gamma^2+1)[j^+(j^++1) - j^-(j^-+1)] - 2\gamma[j^+(j^++1) + j^-(j^-+1)]\}|\psi\rangle \\
&= \left\{(\gamma^2+1)\left[\frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) - \frac{\gamma-1}{2}j\left(\frac{\gamma-1}{2}j+1\right)\right] + \right. \\
&\left. - 2\gamma\left[\frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) + \frac{\gamma-1}{2}j\left(\frac{\gamma-1}{2}j+1\right)\right]\right\}|\psi\rangle \\
&= \left\{\frac{(\gamma^2+1)}{4}\left[j^2+2\gamma j^2+\gamma^2j^2+2\gamma j+2j-j^2+2\gamma j^2-\gamma^2j^2+2j-2\gamma j\right] + \right. \\
&\left. -\frac{2\gamma}{4}\left[j^2+2\gamma j^2+\gamma^2j^2+2\gamma j+2j+j^2-2\gamma j^2+\gamma^2j^2-2j+2\gamma j\right]\right\}|\psi\rangle \\
&= [\gamma^3j^2+\gamma^2j+\gamma j^2+j-\gamma j^2-\gamma^3j^2-2\gamma^2j]|\psi\rangle = j(1-\gamma^2)|\psi\rangle
\end{aligned}
\tag{8.23}
$$

In the case $\beta > 0$, we have for the second simplicity equation:

$$
\begin{aligned}
&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j(j+1) - j^+(j^++1) + j^-(j^-+1)]|\psi\rangle \\
&= \left[\gamma j(j+1) - \frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) + \frac{\gamma-1}{2}j\left(\frac{\gamma-1}{2}j+1\right)\right]|\psi\rangle \\
&= \left[\gamma j(j+1) - \frac{j^2+2\gamma j^2+\gamma^2j^2+2\gamma j+2j}{4} + \frac{j^2-2\gamma j^2+\gamma^2j^2+2\gamma j-2j}{4}\right]|\psi\rangle \\
&= \left[\gamma j(j+1) - \frac{4\gamma j^2+4j}{4}\right]|\psi\rangle = j(\gamma-1)|\psi\rangle
\end{aligned}
\tag{8.24}
$$

□

In order to summarize, we have seen that the standard choice of the Casimir does not solve exactply the Master constraint criterion for both $\gamma > 1$ and $\gamma < 1$. Indeed, generically the positive operator associated with the Master constraint does not contain the zero eigenvalue in the spectrum and weak imposition of the constraint corresponds to look for the minimum eigenvalue among spaces $\mathcal{H}_j \in \mathcal{H}_{j^-j^+}$.

Moreover, the Master constraint criterion can be re-written in terms of two equations involving $SO(4)$ and $SU(2)$ Casimir operators. As expected, the standard choice of the Casimir does not solve both solutions exactly for both the cases $\gamma > 1$ and $\gamma < 1$. Interestingly, we note that, with the standard choice of the Casimir, the second constraint equation is solved for $\gamma < 1$, we will use this result in Section 9.

However, as we have seen, there is an other possibility, related to the quantum corrections due to ordering ambiguities.
In fact, since the Master constraint involves the Casimir operators, the solution depends on the operator ordering of the Casimir, which depends on the choice of the quantization map $\mathcal{Q}$. Actually, an ordering ambiguity is always present in quantum theory, and can be related, in the Spin Foam context to ambiguities in the path integral measure [43]. For instance, different ordering of the Casimir can yields spectra $j(j+1)$ or j^2 or $(j+\frac{1}{2})^2$, or anything differing from these by a linear or constant shift [43]. In the EPRL context, the natural ordering in order to find solutions to the simplicity constraints seems to favour the spectrum j^2 for the $SU(2)$ Casimir operator instead of the usual $j(j+1)$, leading to an area spectrum with a constant spacing of j.

Particularly, if we consider the prescription (8.5) and (8.8), we observe that for the j^2 spectrum of the Casimir operators:

$$\begin{aligned}\hat{C}_G^{(1)}|\psi\rangle =&(j^+)^2 + (j^-)^2|\psi\rangle \\ \hat{C}_G^{(2)}|\psi\rangle =&(j^+)^2 - (j^-)^2|\psi\rangle \\ \hat{C}_H|\psi\rangle =&j^2|\psi\rangle\end{aligned} \tag{8.25}$$

we have that:

- In the case $\beta < 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = 0 \tag{8.26}$$

while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{8.27}$$

- In the case $\beta > 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = 0 \tag{8.28}$$

while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{8.29}$$

Proof. In the case $\beta < 0$, we have for the first simplicity equation:

$$\begin{aligned}&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\ &= \left\{(\gamma^2+1)\left[\frac{(1+\gamma)^2j^2}{4} - \frac{(1-\gamma)^2j^2}{4}\right] - 2\gamma\left[\frac{(1+\gamma)^2j^2}{4} + \frac{(1-\gamma)^2j^2}{4}\right]\right\}|\psi\rangle \\ &= \left\{(\gamma^2+1)\left[\frac{2\gamma j^2+2\gamma j^2}{4}\right] - 2\gamma\left[\frac{j^2+\gamma^2j^2+j^2+\gamma^2j^2}{4}\right]\right\}|\psi\rangle \\ &= \left\{(\gamma^2+1)[\gamma j^2] - \gamma j^2[1+\gamma^2]\right\}|\psi\rangle = 0\end{aligned} \tag{8.30}$$

In the case $\beta < 0$, we have for the second simplicity equation:

$$\begin{aligned}[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle &= [\gamma j^2 - (j^+)^2 + (j^-)^2]|\psi\rangle \\ &= \left[\gamma j^2 - \frac{(1+\gamma)^2}{4}j^2 + \frac{(1-\gamma)^2}{4}j^2\right]|\psi\rangle \\ &= \left[\gamma j^2 - \frac{2\gamma j^2+2\gamma j^2}{4}\right]|\psi\rangle = \left[\gamma j^2 - \gamma j^2\right]|\psi\rangle = 0\end{aligned} \tag{8.31}$$

In the case $\beta > 0$, we have for the first simplicity equation:

$$\begin{aligned}&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\ &= \left\{(\gamma^2+1)\left[\frac{(1+\gamma)^2j^2}{4} - \frac{(\gamma-1)^2j^2}{4}\right] - 2\gamma\left[\frac{(1+\gamma)^2j^2}{4} + \frac{(\gamma-1)^2j^2}{4}\right]\right\}|\psi\rangle \\ &= \left\{(\gamma^2+1)\left[\frac{2\gamma j^2+2\gamma j^2}{4}\right] - 2\gamma\left[\frac{j^2+\gamma^2j^2+j^2+\gamma^2j^2}{4}\right]\right\}|\psi\rangle \\ &= \left\{(\gamma^2+1)[\gamma j^2] - \gamma j^2[1+\gamma^2]\right\}|\psi\rangle = 0\end{aligned} \tag{8.32}$$

In the case $\beta > 0$, we have for the second simplicity equation:

$$\begin{aligned}
&[\gamma \hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j^2 - (j^+)^2 + (j^-)^2]|\psi\rangle \\
&= \left[\gamma j^2 - \frac{(1+\gamma)^2}{4} j^2 + \frac{(\gamma-1)^2}{4} j^2\right]|\psi\rangle \\
&= \left[\gamma j^2 - \frac{2\gamma j^2 + 2\gamma j^2}{4}\right]|\psi\rangle = [\gamma j^2 - \gamma j^2]|\psi\rangle = 0
\end{aligned} \tag{8.33}$$

□

Therefore, we see that there is a quantization procedure for which the simplicity constraints are satisfied exactly.

8.5 A note on the quantization map

As we have seen one can find a particular spectrum of the Casimir operators such that the EPRL prescription satisfies the Master constraint criterion exactly for both $\gamma > 1$ and $\gamma < 1$. However, the quantization procedure leading to the spectrum j^2 has not been studied in detail, specifically it is not known the expression of the plane waves appearing in the non-commutative Fourier transform. Actually, the non-commutative Fourier transform has been defined for the Freidel-Livine-Majid (or the Duflo or the Symmetric) quantization map, characterized by the Casimir spectrum $j(j+1)$ (or $(j+\frac{1}{2})^2$). Thus, explicit calculation in the metric representation will be performed with the Freidel-Livine-Majid quantization map for which the EPRL constraints are not satisfied exactly.
Nevertheless, we would like to point out that probably exact solutions of the Master constraint criterion are related to how the constraints are imposed, rather than the particular choice of quantization map. Indeed, we will show that the Baratin Oriti model satisfies the Master constraint criterion and the Gupta-Bleuler criterion independently by the choice of the quantization map (the only requirement is that the quantization procedure is well defined). So that we expect that the fundamental problem is not what is the proper choice of the operator ordering, but rather what is the proper way to implement the constraints (and the Baratin Oriti model seems the more general). Finally, in the last sections of the thesis, we will compare the new models in the large-j limit. These results will not depend on the choice of the choice of the quantization map. In conclusion, the issue of the operator ordering, solving exactly the Master constraint criterion, may be not a fundamental problem of this thesis. However, it may play a role with respect to the relation between the Ashtekar-Barbero connection and the spin connection, as we will see in Section 9.

8.6 The EPRL solutions in the extended GFT formalism

Finally, in this section we consider the extended GFT field solution of the EPRL model, in different representations.
We start noticing that the EPRL model imposes a restriction on the possible representations. In other words, it is diagonal in the spin basis, where it acts as two Kronecker deltas. The first impose the relation $j^- = |\beta| j^+$ while the other the relation $j = j^+ \pm j^-$ according of $\gamma \lessgtr 1$.

Therefore, in the spin representation we have:

- If $\beta < 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\phi\rangle = \\
&= \sum \phi^{j^-j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{j^-j^+j}_{m^-m^+q}}(k)\tilde{C}^{j^-j^+j}_{r^-r^+q}(k)\overline{\tilde{C}^{j^-j^+l}_{s^-s^+p}}(k)\tilde{C}^{j^-j^+l}_{n^-n^+p}(k)\delta^{j^-|\beta|j^+}\delta^{jj^+(1+|\beta|)} \\
&= \sum \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k)\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \\
&= [\phi^{\beta<0}_{EPRL}]^{|\beta|j^+\ j^+}_{r^-s^-r^+s^+}(k)
\end{aligned} \tag{8.34}$$

- If $\beta > 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\phi\rangle = \\
&= \sum \phi^{j^-j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{j^-j^+j}_{m^-m^+q}}(k)\tilde{C}^{j^-j^+j}_{r^-r^+q}(k)\overline{\tilde{C}^{j^-j^+l}_{s^-s^+p}}(k)\tilde{C}^{j^-j^+l}_{n^-n^+p}(k)\delta^{j^-|\beta|j^+}\delta^{jj^+(1-|\beta|)} \\
&= \sum \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{m^-m^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k)\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \\
&= [\phi^{\beta>0}_{EPRL}]^{|\beta|j^+\ j^+}_{r^-s^-r^+s^+}(k)
\end{aligned} \tag{8.35}$$

Moreover, in a basis of coherent states these solutions correspond to:

- If $\beta < 0$

$$\begin{aligned}
&\langle j^-; \vec{n}^-, \vec{m}^-| \otimes \langle j^+; \vec{n}^+, \vec{m}^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\psi\rangle = \\
&= \sum d_{j^-}d_{j^+}\langle j^-; \vec{n}^-, \vec{m}^-|j^-; s^-, r^-\rangle\langle j^+; \vec{n}^+, \vec{m}^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\psi\rangle \\
&= \sum d_{|\beta|j^+}d_{j^+}\phi^{|\beta|j^+j^+}_{x^-y^-x^+y^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{x^-x^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \\
&\qquad D^{|\beta|j^+}_{r^-|\beta|j^+}(g_{\vec{n}^-})\overline{D^{|\beta|j^+}_{s^-|\beta|j^+}}(g_{\vec{m}^-})D^{j^+}_{r^+j^+}(g_{\vec{n}^+})\overline{D^{j^+}_{s^+j^+}}(g_{\vec{m}^+})\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{y^-y^+p}(k) \\
&= [\phi^{\beta<0}_{EPRL}]^{|\beta|j^+\ j^+}_{(\vec{m}^-,\vec{n}^-,\vec{m}^+,\vec{n}^+)}(k)
\end{aligned} \tag{8.36}$$

- If $\beta > 0$

$$\begin{aligned}
&\langle j^-; \vec{n}^-, \vec{m}^-| \otimes \langle j^+; \vec{n}^+, \vec{m}^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\psi\rangle = \\
&= \sum d_{j^-}d_{j^+}\langle j^-; \vec{n}^-, \vec{m}^-|j^-; s^-, r^-\rangle\langle j^+; \vec{n}^+, \vec{m}^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\psi\rangle \\
&= \sum d_{|\beta|j^+}d_{j^+}\phi^{|\beta|j^+j^+}_{x^-y^-x^+y^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{x^-x^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k) \\
&\qquad D^{|\beta|j^+}_{r^-|\beta|j^+}(g_{\vec{n}^-})\overline{D^{|\beta|j^+}_{s^-|\beta|j^+}}(g_{\vec{m}^-})D^{j^+}_{r^+j^+}(g_{\vec{n}^+})\overline{D^{j^+}_{s^+j^+}}(g_{\vec{m}^+})\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{y^-y^+p}(k) \\
&= [\phi^{\beta>0}_{EPRL}]^{|\beta|j^+\ j^+}_{(\vec{m}^-,\vec{n}^-,\vec{m}^+,\vec{n}^+)}(k)
\end{aligned} \tag{8.37}$$

In the group representation we have:

- If $\beta < 0$

$$
\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle g^-|j^-; s^-, r^-\rangle \otimes \langle g^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\phi\rangle \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \\
&\qquad D^{|\beta|j^+}_{r^-s^-}(g^-) D^{j^+}_{r^+s^+}(g^+) \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \\
&= \phi(g^-, g^+)^{\beta<0}_{EPRL}(k)
\end{aligned}
\tag{8.38}
$$

- If $\beta > 0$

$$
\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle g^-|j^-; s^-, r^-\rangle \otimes \langle g^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\phi\rangle \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k) \\
&\qquad D^{|\beta|j^+}_{r^-s^-}(g^-) D^{j^+}_{r^+s^+}(g^+) \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \\
&= \phi(g^-, g^+)^{\beta>0}_{EPRL}(k)
\end{aligned}
\tag{8.39}
$$

Now, performing the non-commutative Fourier transform we can write the solutions in the metric representation. As mentioned before we choose for explicit calculation the Freidel-Livine-Majid quantization map, described in Section 4.9.
We have:

- If $\beta < 0$

$$
\begin{aligned}
&\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}^{\beta<0}_{EPRL}|\psi\rangle = \int \mathrm{d}g^-\mathrm{d}g^+ \langle x^-|g^+\rangle\langle x^+|g^+\rangle\langle g^-| \otimes \langle g^+|\hat{S}^{\beta<0}_{EPRL}|\psi\rangle \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \\
&\quad \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \int \mathrm{d}g^- E_{g^-}(x^-) D^{|\beta|j^+}_{r^-s^-}(g^-) \int \mathrm{d}g^- E_{g^+}(x^+) D^{j^+}_{r^+s^+}(g^+) \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \\
&\quad \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \int \mathrm{d}g^- \overline{E}_{g^-}(x^-) \overline{D^{|\beta|j^+}_{s^-r^-}}(g^-) \int \mathrm{d}g^+ \overline{E}_{g^+}(x^+) \overline{D^{j^+}_{s^+r^+}}(g^+) \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \\
&\quad \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \frac{2J_{[2|\beta|j^++1]}(|x^-|)}{i^{2|\beta|j^+}|x^-|} D^{|\beta|j^+}_{r^-s^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|} D^{j^+}_{r^+s^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&= \phi(x^-, x^+)^{\beta<0}_{EPRL}(k)
\end{aligned}
\tag{8.40}
$$

- If $\beta > 0$

$$\begin{aligned}
&\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}^{\beta>0}_{EPRL}|\psi\rangle = \int dg^- dg^+ \langle x^-|g^+\rangle\langle x^+|g^+\rangle\langle g^-| \otimes \langle g^+|\hat{S}^{\beta>0}_{EPRL}|\psi\rangle \\
=&\sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{m^-m^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k) \\
&\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k)\int \mathrm{d}g^- E_{g^-}(x^-)D^{|\beta|j^+}_{r^-s^-}(g^-)\int \mathrm{d}g^- E_{g^+}(x^+)D^{j^+}_{r^+s^+}(g^+) \\
=&\sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{m^-m^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k) \\
&\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k)\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k)\int \mathrm{d}g^- \overline{E}_{g^-}(x^-)\overline{D^{|\beta|j^+}_{s^-r^-}}(g^-)\int \mathrm{d}g^+ \overline{E}_{g^+}(x^+)\overline{D^{j^+}_{s^+r^+}}(g^+) \\
=&\sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k)\overline{\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{m^-m^+q}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1-|\beta|)}_{r^-r^+q}(k)\overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \\
&\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k)\frac{2J_{[2|\beta|j^++1]}(|x^-|)}{i^{2|\beta|j^+}|x^-|}D^{|\beta|j^+}_{r^-s^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|}D^{j^+}_{r^+s^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
=&\phi(x^-,x^+)^{\beta>0}_{EPRL}(k)
\end{aligned} \tag{8.41}$$

Chapter 9

The Alexandrov's proposal

In [44] Sergei Alexandrov studied the relation between the Ashtekar-Barbero (AB) connection A^a, necessary in the canonical Loop Quantum Gravity, and the spin connection ω^{IJ}, characterizing the Spin Foam quantization. As a result, he realized that the spin connection projected to the subspace defined by the EPRL model with a slightly modified restriction on representations naturally gives rise to the AB connection. Particularly, recall the two Casimir equations of the simplicity constraints:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = 0 \tag{9.1}$$

and,

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{9.2}$$

where:

$$\begin{aligned} \hat{C}_G^{(1)} =& (k\hat{X}_i^- k^{-1})^2 + (\hat{X}_i^+)^2 \\ \hat{C}_G^{(2)} =& (\hat{X}_i^+)^2 - (k\hat{X}_i^- k^{-1})^2 \\ \hat{C}_H =& (k\hat{X}_i^- k^{-1} + \hat{X}_i^+)^2 \end{aligned} \tag{9.3}$$

We know that a concrete solution of these conditions on the Casimir operators depends the value of the Immirzi parameter γ, which we have assumed to be positive.
However, as we have seen for the EPRL model the general feature is that these conditions do not have any solutions in terms of unitary representations, and the usual strategy [43] is to adjust the values of the Casimir operators by linear and constant terms in representation labels in such a way that solutions do exist [44]. These adjustment is a consequence of the quantization process, particularly related to the ordering ambiguity at quantum level.
Moreover, we know that for the standard choice of the Casimir the solutions are valid in a semi-classical limit, for large j, and correspond to the minimum of the Master constraint criterion.

In [44] Alexandrov investigated the relation between the Ashtekar-Barbero connection and the spin connection. The great result is that if (9.2) is solved exactly then one does have a possibility to extract the LQG connection from the EPRL constraints. But this requires the exact solution of one of the constraints.

9.1 The Alexandrov's prescription

The Alexandrov's proposal is a modification of the original EPRL solutions, in order to solve the second Casimir simplicity equation (9.2) exactly with the standard choice of the Casimir. Particularly, the Alexandrov's prescription is:

- If $\beta < 0$

$$\begin{cases} j^- = \frac{1-\gamma}{2} j = |\beta| j^+ \\ j^+ = \frac{1+\gamma}{2} j \\ j = j^+ + j^- = j^+(1+|\beta|) \end{cases} \tag{9.4}$$

- If $\beta > 0$

$$\begin{cases} j^- = \frac{\gamma-1}{2}(j+1) = |\beta|(j^+ + 1) \\ j^+ = \frac{\gamma+1}{2}(j+1) - 1 \\ j = j^+ - j^- = j^+(1-|\beta|) - |\beta| \end{cases} \tag{9.5}$$

We observe that with this choice the second simplicity constraint equation (9.1) is solved with the standard choice of the Casimir ($j(j+1)$), for both $\gamma > 1$ and $\gamma < 1$. Specifically, if

$$\begin{aligned} \hat{C}_G^{(1)}|\psi\rangle =& j^+(j^+ + 1) + j^-(j^- + 1)|\psi\rangle \\ \hat{C}_G^{(2)}|\psi\rangle =& j^+(j^+ + 1) - j^-(j^- + 1)|\psi\rangle \\ \hat{C}_H|\psi\rangle =& j(j+1)|\psi\rangle \end{aligned} \tag{9.6}$$

then:

- In the case $\beta < 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \gamma j(\gamma^2 - 1) \tag{9.7}$$

  and the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{9.8}$$

- In the case $\beta > 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = -(j+1)(\gamma-1)^2 \tag{9.9}$$

  and the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{9.10}$$

Proof. In the case $\beta < 0$, we have for the first simplicity equation:

$$\begin{aligned} &[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\ &= \{[(\gamma^2+1)[j^+(j^+ + 1) - j^-(j^- + 1)] - 2\gamma[j^+(j^+ + 1) + j^-(j^- + 1)]\}|\psi\rangle \\ &= \Bigg\{(\gamma^2+1)\left[\frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) - \frac{1-\gamma}{2}j\left(\frac{1-\gamma}{2}j+1\right)\right] + \\ &- 2\gamma\left[\frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) + \frac{1-\gamma}{2}j\left(\frac{1-\gamma}{2}j+1\right)\right]\Bigg\}|\psi\rangle \\ &= \Bigg\{\frac{(\gamma^2+1)}{4}\left[j^2 + 2\gamma j^2 + \gamma^2 j^2 + 2\gamma j + 2j - j^2 + 2\gamma j^2 - \gamma^2 j^2 - 2j + 2\gamma j\right] + \\ &- \frac{2\gamma}{4}\left[j^2 + 2\gamma j^2 + \gamma^2 j^2 + 2\gamma j + 2j + j^2 - 2\gamma j^2 + \gamma^2 j^2 + 2j - 2\gamma j\right]\Bigg\}|\psi\rangle \\ &= [\gamma^3 j^2 + \gamma^3 j + \gamma j^2 + \gamma j - \gamma j^2 - \gamma^3 j^2 - 2\gamma j]|\psi\rangle = \gamma j(\gamma^2 - 1)|\psi\rangle \end{aligned} \tag{9.11}$$

In the case $\beta < 0$, we have for the second simplicity equation:

$$\begin{aligned}
&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j(j+1) - j^+(j^+ + 1) + j^-(j^- + 1)]|\psi\rangle \\
&= \left[\gamma j(j+1) - \frac{1+\gamma}{2}j\left(\frac{1+\gamma}{2}j+1\right) + \frac{1-\gamma}{2}j\left(\frac{1-\gamma}{2}j+1\right)\right]|\psi\rangle \\
&= [\gamma j(j+1) - \frac{j^2+2\gamma j^2+\gamma^2 j^2+2\gamma j+2j}{4} + \frac{j^2-2\gamma j^2+\gamma^2 j^2+2j-2\gamma j}{4}]|\psi\rangle \\
&= [\gamma j(j+1) - \frac{4\gamma j^2+4\gamma j}{4}]|\psi\rangle = [\gamma j(j+1) - \gamma j(j+1)]|\psi\rangle = 0
\end{aligned} \tag{9.12}$$

In the case $\beta > 0$, we have for the first simplicity equation:

$$\begin{aligned}
&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\
&= \{(\gamma^2+1)[j^+(j^++1) - j^-(j^-+1)] - 2\gamma[j^+(j^++1) + j^-(j^-+1)]\}|\psi\rangle \\
&= \left\{(\gamma^2+1)(j+1)\left[\left(\frac{\gamma+1}{2}(j+1)-1\right)\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\left(\frac{\gamma-1}{2}(j+1)+1\right)\right] + \right. \\
&\left. - 2\gamma(j+1)\left[\left(\frac{\gamma+1}{2}(j+1)-1\right)\frac{\gamma+1}{2} + \frac{\gamma-1}{2}\left(\frac{\gamma-1}{2}(j+1)+1\right)\right]\right\}|\psi\rangle \\
&= \{(\gamma^2+1)(j+1)[(j+1)\gamma-1] - \gamma(j+1)[(j+1)(\gamma^2+1)-2]\}|\psi\rangle \\
&= \{(\gamma^2+1)(j+1)^2(\gamma-\gamma) - (j+1)(\gamma-1)^2\}|\psi\rangle = -(j+1)(\gamma-1)^2|\psi\rangle
\end{aligned} \tag{9.13}$$

In the case $\beta > 0$, we have for the second simplicity equation:

$$\begin{aligned}
&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j(j+1) - j^+(j^++1) + j^-(j^-+1)]|\psi\rangle \\
&= \left[\gamma j(j+1) - \left(\frac{1+\gamma}{2}(j+1)-1\right)\frac{1+\gamma}{2}(j+1) + \right. \\
&\left. - \frac{\gamma-1}{2}(j+1)\left(\frac{\gamma-1}{2}(j+1)+1\right)\right]|\psi\rangle \\
&= (j+1)\left[\gamma j - \frac{(j+1)(\gamma^2+2\gamma+1-\gamma^2+2\gamma-1) - 2(\gamma+1+\gamma-1)}{4}\right]|\psi\rangle \\
&= (j+1)[\gamma j - \gamma j]|\psi\rangle = 0
\end{aligned} \tag{9.14}$$

□

Moreover, Alexandrov has found a particular spectra of the Casimir such that the restrictions on the representation (9.4) and (9.5) solve exactly both the constraints (9.1) and (9.2). Indeed, with the expression for the Casimir operators given by:

$$\begin{aligned}
\hat{C}_G^{(1)}|\psi\rangle &= (j^++1)^2 + (j^-)^2|\psi\rangle \\
\hat{C}_G^{(2)}|\psi\rangle &= (j^++1)^2 - (j^-)^2|\psi\rangle \\
\hat{C}_H|\psi\rangle &= (k+1)^2|\psi\rangle
\end{aligned} \tag{9.15}$$

and the prescriptions (9.4), and (9.5), we have:

- In the case $\beta < 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = 0 \tag{9.16}$$

  while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{9.17}$$

- In the case $\beta > 0$, the first simplicity equation is:

$$[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = 0 \tag{9.18}$$

while the second simplicity equation reads:

$$[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = 0 \tag{9.19}$$

Proof. In the case $\beta < 0$, we have for the first simplicity equation:

$$\begin{aligned}
&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\
&= \left\{(\gamma^2+1)\left[\frac{(1+\gamma)^2j^2}{4} - \frac{(1-\gamma)^2j^2}{4}\right] - 2\gamma\left[\frac{(1+\gamma)^2j^2}{4} + \frac{(1-\gamma)^2j^2}{4}\right]\right\}|\psi\rangle \\
&= \left\{(\gamma^2+1)\left[\frac{2\gamma j^2+2\gamma j^2}{4}\right] - 2\gamma\left[\frac{j^2+\gamma^2j^2+j^2+\gamma^2j^2}{4}\right]\right\}|\psi\rangle \\
&= \left\{(\gamma^2+1)[\gamma j^2] - \gamma j^2[1+\gamma^2]\right\}|\psi\rangle = 0
\end{aligned} \tag{9.20}$$

In the case $\beta < 0$, we have for the second simplicity equation:

$$\begin{aligned}
&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma j^2 - (j^+)^2 + (j^-)^2]|\psi\rangle \\
&= \left[\gamma j^2 - \frac{(1+\gamma)^2}{4}j^2 + \frac{(1-\gamma)^2}{4}j^2\right]|\psi\rangle \\
&= \left[\gamma j^2 - \frac{2\gamma j^2+2\gamma j^2}{4}\right]|\psi\rangle = [\gamma j^2 - \gamma j^2]|\psi\rangle = 0
\end{aligned} \tag{9.21}$$

In the case $\beta > 0$, we have for the first simplicity equation:

$$\begin{aligned}
&[(\gamma^2+1)\hat{C}_G^{(2)} - 2\gamma\hat{C}_G^{(1)}]|\psi\rangle = \\
&= \{(\gamma^2+1)[(j^++1)^2 - (j^-)^2] - 2\gamma[(j^++1)^2 + (j^-)^2]\}|\psi\rangle \\
&= \left\{(\gamma^2+1)\left[\frac{(\gamma+1)^2}{4}(j+1)^2 - \frac{(\gamma-1)^2}{4}(j+1)^2\right] + \right. \\
&\left. - 2\gamma\left[\frac{(\gamma+1)^2}{4}(j+1)^2 + \frac{(\gamma-1)^2}{4}(j+1)^2\right]\right\}|\psi\rangle \\
&= \left\{(\gamma^2+1)\frac{(j+1)^2}{4}(\gamma^2+2\gamma+1-\gamma^2+2\gamma-1)\right. \\
&\left. - \frac{\gamma(j+1)^2}{2}(\gamma^2+2\gamma+1+\gamma^2-2\gamma+1)\right\}|\psi\rangle \\
&= \left\{(\gamma^2+1)(j+1)^2\gamma - \gamma(\gamma^2+1)(j+1)^2\right\}|\psi\rangle = 0
\end{aligned} \tag{9.22}$$

In the case $\beta > 0$, we have for the second simplicity equation:

$$\begin{aligned}
&[\gamma\hat{C}_H - \hat{C}_G^{(2)}]|\psi\rangle = [\gamma(j+1)^2 - (j^++1)^2 + (j^-)^2]|\psi\rangle \\
&= \left[\gamma(j+1)^2 - \frac{(1+\gamma)^2}{4}(j+1)^2) + \frac{(\gamma-1)^2}{4}(j+1)^2\right]|\psi\rangle \\
&= (j+1)^2\left[\gamma + \frac{-1-2\gamma-\gamma^2+1-2\gamma+\gamma^2}{4}\right]|\psi\rangle \\
&= (j+1)^2\left[\gamma-\gamma\right]|\psi\rangle = 0
\end{aligned} \tag{9.23}$$

□

9.2 The Alexandrov's solutions in the extended GFT formalism

In this section, we consider the Alexandrov's modification to the extended GFT field, in the different representations. As the EPRL model, the Alexandrov's proposal imposes a restriction on the possible representations, (9.4) and (9.5) In other words, it is diagonal in the spin basis, where it acts as two Kronecker deltas. However, for $\beta < 0$ the Alexandrov's prescription corresponds to the EPRL model: $\delta^{j^- \ |\beta| j^+}$ and $\delta^{j \ j^+(1+|\beta|)}$, while for $\beta > 0$ we have $\delta^{j^- \ |\beta|(j^+ +1)}$ and $\delta^{j \ j^+(1-|\beta|)-|\beta|}$.

Therefore, in the spin representation we have:

- If $\beta < 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_A^{\beta<0}|\phi\rangle = \\
&= \sum \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{j^- j^+ j}_{m^- m^+ q}}(k)\tilde{C}^{j^- j^+ j}_{r^- r^+ q}(k)\overline{\tilde{C}^{j^- j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{j^- j^+ l}_{n^- n^+ p}(k)\delta^{j^- |\beta| j^+}\delta^{j j^+(1+|\beta|)} \\
&= \sum \phi^{|\beta| j^+ j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{m^- m^+ q}}(k)\tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{r^- r^+ q}(k)\overline{\tilde{C}^{|\beta| j^+ j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{|\beta| j^+ j^+ l}_{n^- n^+ p}(k) \\
&= [\phi_A^{\beta<0}]^{|\beta| j^+ \ j^+}_{r^- s^- r^+ s^+}(k)
\end{aligned} \tag{9.24}$$

- If $\beta > 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_A^{\beta>0}|\phi\rangle = \\
&= \sum \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{j^- j^+ j}_{m^- m^+ q}}(k)\tilde{C}^{j^- j^+ j}_{r^- r^+ q}(k)\overline{\tilde{C}^{j^- j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{j^- j^+ l}_{n^- n^+ p}(k) \\
&\qquad \delta^{j^- |\beta|(j^+ +1)}\delta^{j [j^+(1-|\beta|)-|\beta|]} \\
&= \sum \phi^{|\beta|(j^+ +1) j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{|\beta|(j^+ +1) j^+ [j^+(1-|\beta|)-|\beta|]}_{m^- m^+ q}}(k) \\
&\qquad \tilde{C}^{|\beta|(j^+ +1) j^+ [j^+(1-|\beta|)-|\beta|]}_{r^- r^+ q}(k)\overline{\tilde{C}^{|\beta|(j^+ +1) j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{|\beta|(j^+ +1) j^+ l}_{n^- n^+ p}(k) \\
&= [\phi_A^{\beta>0}]^{|\beta|(j^+ +1) \ j^+}_{r^- s^- r^+ s^+}(k)
\end{aligned} \tag{9.25}$$

Moreover, in a basis of coherent states these solutions correspond to:

- If $\beta < 0$

$$\begin{aligned}
&\langle j^-; \vec{n}^-, \vec{m}^-| \otimes \langle j^+; \vec{n}^+, \vec{m}^+| \otimes \langle k|\hat{S}_A^{\beta<0}|\psi\rangle = \\
&= \sum d_{|\beta| j^+} d_{j^+} \phi^{|\beta| j^+ j^+}_{x^- y^- x^+ y^+}(k)\overline{\tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{x^- x^+ q}}(k)\tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{r^- r^+ q}(k) \\
&\qquad D^{|\beta| j^+}_{r^- |\beta| j^+}(g_{\vec{n}^-})\overline{D^{|\beta| j^+}_{s^- |\beta| j^+}}(g_{\vec{m}^-})D^{j^+}_{r^+ j^+}(g_{\vec{n}^+})\overline{D^{j^+}_{s^+ j^+}}(g_{\vec{m}^+})\overline{\tilde{C}^{|\beta| j^+ j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{|\beta| j^+ j^+ l}_{y^- y^+ p}(k) \\
&= [\phi_A^{\beta<0}]^{|\beta| j^+ \ j^+}_{(\vec{m}^-, \vec{n}^-, \vec{m}^+, \vec{n}^+)}(k)
\end{aligned} \tag{9.26}$$

- If $\beta > 0$

$$\begin{aligned}
&\langle j^-; \vec{n}^-, \vec{m}^-| \otimes \langle j^+; \vec{n}^+, \vec{m}^+| \otimes \langle k|\hat{S}_A^{\beta>0}|\psi\rangle = \\
&= \sum d_{|\beta|(j^+ +1)} d_{j^+} \phi^{|\beta|(j^+ +1) j^+}_{x^- y^- x^+ y^+}(k)\overline{\tilde{C}^{|\beta|(j^+ +1) j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{|\beta|(j^+ +1) j^+ l}_{y^- y^+ p}(k) \\
&\qquad D^{|\beta|(j^+ +1)}_{r^- |\beta|(j^+ +1)}(g_{\vec{n}^-})\overline{D^{|\beta|(j^+ +1)}_{s^- |\beta|(j^+ +1)}}(g_{\vec{m}^-})D^{j^+}_{r^+ j^+}(g_{\vec{n}^+})\overline{D^{j^+}_{s^+ j^+}}(g_{\vec{m}^+}) \\
&\qquad \overline{\tilde{C}^{|\beta|(j^+ +1) j^+ [j^+(1-|\beta|)-|\beta|]}_{x^- x^+ q}}(k)\tilde{C}^{|\beta|(j^+ +1)+j^+ [j^+(1-|\beta|)-|\beta|]}_{r^- r^+ q}(k) \\
&= [\phi_A^{\beta>0}]^{|\beta|(j^+ +1) \ j^+}_{(\vec{m}^-, \vec{n}^-, \vec{m}^+, \vec{n}^+)}(k)
\end{aligned} \tag{9.27}$$

In the group representation we have:

- If $\beta < 0$

$$
\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}_A^{\beta<0}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle g^-|j^-; s^-, r^-\rangle \otimes \langle g^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_A^{\beta<0}|\phi\rangle \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \\
&\qquad D^{|\beta|j^+}_{r^-s^-}(g^-) D^{j^+}_{r^+s^+}(g^+) \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \\
&= \phi(g^-, g^+)^{\beta<0}_A(k)
\end{aligned}
\tag{9.28}
$$

- If $\beta > 0$

$$
\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}_A^{\beta>0}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle g^-|j^-; s^-, r^-\rangle \otimes \langle g^+|j^+; s^+, r^+\rangle \\
&\qquad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_A^{\beta>0}|\phi\rangle \\
&= \sum d_{|\beta|(j^++1)} d_{j^+} \phi^{|\beta|(j^++1)j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|(j^++1)j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|(j^++1)j^+l}_{n^-n^+p}(k) \\
&\quad D^{|\beta|(j^++1)}_{r^-s^-}(g^-) D^{j^+}_{r^+s^+}(g^+) \overline{\tilde{C}^{|\beta|(j^++1)j^+[j^+(1-|\beta|)-|\beta|]}_{m^-m^+q}}(k) \tilde{C}^{|\beta|(j^++1)j^+[j^+(1-|\beta|)-|\beta|]}_{r^-r^+q}(k) \\
&= \phi(g^-, g^+)^{\beta>0}_A(k)
\end{aligned}
\tag{9.29}
$$

Now, performing the non-commutative Fourier transform we can write the solutions in the metric representation. As mentioned before we choose for explicit calculation the Freidel-Livine-Majid quantization map, described in Section 4.9. We have:

- If $\beta < 0$

$$
\begin{aligned}
&\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_A^{\beta<0}|\psi\rangle = \int \mathrm{d}g^- dg^+ \langle x^-|g^+\rangle \langle x^+|g^+\rangle \langle g^-| \otimes \langle g^+|\hat{S}_A^{\beta<0}|\psi\rangle \\
&= \sum d_{|\beta|j^+} d_{j^+} \phi^{|\beta|j^+j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}(k) \overline{\tilde{C}^{|\beta|j^+j^+l}_{s^-s^+p}}(k) \\
&\tilde{C}^{|\beta|j^+j^+l}_{n^-n^+p}(k) \frac{2J_{[2|\beta|j^++1]}(|x^-|)}{i^{2|\beta|j^+}|x^-|} D^{|\beta|j^+}_{r^-s^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|} D^{j^+}_{r^+s^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&= \phi(x^-, x^+)^{\beta<0}_A(k)
\end{aligned}
\tag{9.30}
$$

- If $\beta > 0$

$$
\begin{aligned}
&\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_A^{\beta>0}|\psi\rangle = \int \mathrm{d}g^- \mathrm{d}g^+ \langle x^-|g^+\rangle \langle x^+|g^+\rangle \langle g^-| \otimes \langle g^+|\hat{S}_A^{\beta>0}|\psi\rangle \\
&= \sum d_{|\beta|(j^++1)} d_{j^+} \phi^{|\beta|(j^++1)j^+}_{m^-n^-m^+n^+}(k) \overline{\tilde{C}^{|\beta|(j^++1)j^+l}_{s^-s^+p}}(k) \tilde{C}^{|\beta|(j^++1)j^+l}_{n^-n^+p}(k) \\
&\overline{\tilde{C}^{|\beta|(j^++1)j^+[j^+(1-|\beta|)-|\beta|]}_{m^-m^+q}}(k) \tilde{C}^{|\beta|(j^++1)j^+[j^+(1-|\beta|)-|\beta|]}_{r^-r^+q}(k) \\
&\frac{2J_{[2|\beta|(j^++1)+1]}(|x^-|)}{i^{2|\beta|(j^++1)}|x^-|} D^{|\beta|(j^++1)}_{r^-s^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|} D^{j^+}_{r^+s^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&= \phi(x^-, x^+)^{\beta>0}_A(k)
\end{aligned}
\tag{9.31}
$$

Chapter 10

The Ding-Han-Rovelli solutions of the Gupta-Bleuler criterion

In the previous chapters we have seen that the problems of the BC model may be related to an over-imposition of the simplicity constraints. Mathematically the problem can be traced back to the fact that after quantization the bivectors become non-commuting operators, thus representing a second class system of constraint. Then strong imposition corresponds to the requirement that all their commutators be vanishing as well. To avoid this problem, an other criterion for weak imposition of the constraint, suggested in [123], is to require the vanishing of the matrix elements:

$$\langle\psi|\hat{D}|\phi\rangle = \langle\psi|[k\hat{X}^{-}k^{-1} + \beta\hat{X}^{+}]|\phi\rangle = 0 \tag{10.1}$$

In other words, weak imposition of the simplicity constraints can be achieved via a Gupta-Bleuler criterion, i.e. selecting a suitable class of states for which (10.1) is satisfied.

In [144] it was shown that the EPRL solutions for both $\beta < 0$ and $\beta > 0$ satisfy the Gupta-Bleuler criterion.

Moreover, in [45] it has been proposed an extension of the EPRL solutions satisfying the Gupta-Bleuler criterion:

$$\begin{cases} j^{-} = \frac{1-\gamma}{2}j + r \\ j^{+} = \frac{1+\gamma}{2}j \\ j = j^{+} + j^{-} - r \end{cases} \tag{10.2}$$

where the $Spin(4)$ irreps for a given Barbero-Immirzi parameter γ, should be such that

$$r = \frac{(1+\gamma)j^{-} - (1-\gamma)j^{+}}{1+\gamma} \tag{10.3}$$

is a non-negative integer, and satisfies

$$0 \leq r \leq j^{+} + j^{-} - |j^{+} - j^{-}| \tag{10.4}$$

implying

$$(\gamma - 1)j \leq r \leq \gamma j \Longleftrightarrow \frac{|1-\gamma|}{1+\gamma} \leq j^{-} \leq j^{+} \tag{10.5}$$

or

$$\gamma j \leq r \leq (1+\gamma)j \Longleftrightarrow j^{+} \leq j^{-} \leq \frac{3+\gamma}{1+\gamma}j^{+} \tag{10.6}$$

As for the EPRL model, we note that the Immirzi parameter γ is quantized to be a rational number.

First, we point out that the proof relies on a representation of the field in terms of spinors,

which we don't consider in this thesis. Recently, several models have been proposed using the holomorphic representation. The reason is that spinors can be related to biverctor, and thus have a clear geometric content (as the metric variables of the GFT formalism). However, we will not consider these models in the thesis, and we leave their analysis for future works. Moreover, let us mention that due to the geometric interpretation of the holomorphic representation as well as the metric representation, a comparison between the Baratin Oriti model and the holomorphic models would be really interesting.
Secondly, we observe that the EPRL solutions for both $\beta > 0$ and $\beta < 0$ are indeed solutions of the Gupta-Bleuler criterion with $r = 0$ (for $\gamma < 1$) and $r = (\gamma - 1)j$ (for $\gamma > 1$). Also Alexandrov's solutions for $\beta < 0$ satisfy the Gupta-Bleuler criterion with $r = 0$, while for $\beta > 0$ we have to choose $r = \beta(2j^+ + 1)$, and this choice is compatible with (10.5) or (10.6) if $\frac{\gamma-1}{2} \leq j^+$ or $\frac{\gamma-1}{4} \leq j^+$, respectively. Thus in the asymptotic limit $j^\pm \to \infty$, Alexandrov solutions are solutions of the Ding-Han-Rovelli model.

Secondly, we observe that we have a new degree of freedom r parametrizing the solutions. In [45], the authors noted that in the classical theory we have the well known relation

$$|x^-|^2 = \beta^2 |x^+|^2 \tag{10.7}$$

which implies, that in the large-j asymptotic regime

$$j^- = |\beta| j^+ \tag{10.8}$$

We can still obtain states compatible with General Relativity in the classical limit by demanding that

$$\lim_{j^\pm \to \infty} \frac{r}{j^-} = 0 \; for \; 0 < \gamma < 1 \tag{10.9}$$

or

$$\lim_{j^\pm \to \infty} \frac{r}{j^-} = 2 \; for \; \gamma > 1 \tag{10.10}$$

The conclusions of the Ding-Han-Rovelli article [45] suggest that it may be reasonable to suspect that the weak imposition of the simplicity constraints may be too weak to properly define quantum General Relativity, in the same sense in which the strong imposition of these constraints in the old Barrett- Crane model was too strong.
However, according to Ding, Han and Rovelli there is a simple way out, which is to impose the (non-commuting) simplicity constraints weakly, and the diagonal simplicity constraint strongly. With this choice of constraints, properly ordered, one obtains $r_f = 0$, precisely the LQG state space in the boundary.
An other possibility suggested in [45] is that one could also take the point of view that the quantum numbers r_f label different possible definitions of the spin-foam models. In each of these spin-foam models, the boundary Hilbert space solves the simplicity constraints weakly. And for different choices of r_f the boundary Hilbert spaces are isometric to each other.

10.1 The Ding-Han-Rovelli solutions in the extended GFT formalism

Finally, in this section we consider the extended GFT field solution of the EPRL model, in the different representations.
We start noticing that the EPRL model imposes a restriction on the possible representations. In other words, it is diagonal in the spin basis, where it acts as two Kronecker deltas. The first impose the relation $j^- = -\beta j^+ + r$ while the other the relation $j = j^+(1-\beta)$. Therefore, in the spin representation we have:

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{DHR}|\phi\rangle = \\
&= \sum \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{j^- j^+ j}_{m^- m^+ q}}(k)\tilde{C}^{j^- j^+ j}_{r^- r^+ q}(k)\overline{\tilde{C}^{j^- j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{j^- j^+ l}_{n^- n^+ p}(k)\delta^{j^- -\beta j^+ + r}\delta^{j j^+(1-\beta)} \\
&= \sum \phi^{-\beta j^+ + r j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{m^- m^+ q}}(k)\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{r^- r^+ q}(k) \\
&\quad \overline{\tilde{C}^{-\beta j^+ + r j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{-\beta j^+ + r j^+ l}_{n^- n^+ p}(k) \\
&= [\phi_{DHR}]^{-\beta j^+ + r\ j^+}_{r^- s^- r^+ s^+}(k)
\end{aligned} \tag{10.11}$$

Moreover, in a basis of coherent states these solutions correspond to:

$$\begin{aligned}
&\langle j^-; \vec{n}^-, \vec{m}^-| \otimes \langle j^+; \vec{n}^+, \vec{m}^+| \otimes \langle k|\hat{S}_{DHR}|\psi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle j^-; \vec{n}^-, \vec{m}^-|j^-; s^-, r^-\rangle\langle j^+; \vec{n}^+, \vec{m}^+|j^+; s^+, r^+\rangle \\
&\quad \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{DHR}|\psi\rangle \\
&= \sum d_{-\beta j^+ + r} d_{j^+} \phi^{-\beta j^+ + r j^+}_{x^- y^- x^+ y^+}(k)\overline{\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{x^- x^+ q}}(k)\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{r^- r^+ q}(k) \\
&\quad D^{-\beta j^+ + r}_{r^- -\beta j^+ + r}(g_{\vec{n}^-})\overline{D^{-\beta j^+ + r}_{s^- |\beta| j^+}}(g_{\vec{m}^-})D^{j^+}_{r^+ j^+}(g_{\vec{n}^+})\overline{D^{j^+}_{s^+ j^+}}(g_{\vec{m}^+}) \\
&\quad \overline{\tilde{C}^{-\beta j^+ + r j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{-\beta j^+ + r j^+ l}_{y^- y^+ p}(k) \\
&= [\phi_{DHR}]^{-\beta j^+ + r\ j^+}_{(\vec{m}^-, \vec{n}^-, \vec{m}^+, \vec{n}^+)}(k)
\end{aligned} \tag{10.12}$$

In the group representation we have:

$$\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}_{DHR}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \langle g^-|j^-; s^-, r^-\rangle \otimes \langle g^+|j^+; s^+, r^+\rangle\ \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{DHR}|\phi\rangle \\
&= \sum d_{-\beta j^+ + r} d_{j^+} \phi^{-\beta j^+ + r j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{m^- m^+ q}}(k)\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{r^- r^+ q}(k) \\
&\quad D^{-\beta j^+ + r}_{r^- s^-}(g^-)D^{j^+}_{r^+ s^+}(g^+)\overline{\tilde{C}^{-\beta j^+ + r j^+ l}_{s^- s^+ p}}(k)\tilde{C}^{-\beta j^+ + r j^+ l}_{n^- n^+ p}(k) \\
&= \phi(g^-, g^+)_{DHR}(k)
\end{aligned} \tag{10.13}$$

Now, performing the non-commutative Fourier transform we can write the solutions in the metric representation. As mentioned before we choose for explicit calculation the Freidel-Livine-Majid quantization map. We have:

$$\begin{aligned}
&\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_{DHR}|\psi\rangle = \int \mathrm{d}g^-\mathrm{d}g^+ \langle x^-|g^+\rangle\langle x^+|g^+\rangle\langle g^-| \otimes \langle g^+|\hat{S}^{\beta>0}_{DHR}|\psi\rangle \\
&= \sum d_{-\beta j^+ + r} d_{j^+} \phi^{-\beta j^+ + r j^+}_{m^- n^- m^+ n^+}(k)\overline{\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{m^- m^+ q}}(k)\tilde{C}^{-\beta j^+ + r j^+ j^+(1-\beta)}_{r^- r^+ q}(k)\overline{\tilde{C}^{-\beta j^+ + r j^+ l}_{s^- s^+ p}}(k) \\
&\quad \tilde{C}^{-\beta j^+ + r j^+ l}_{n^- n^+ p}(k)\frac{2J_{[2(-\beta j^+ + r)+1]}(|x^-|)}{i^{2(-\beta j^+ + r)}|x^-|}D^{-\beta j^+ + r}_{r^- s^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\ \frac{2J_{[2j^+ + 1]}(|x^+|)}{i^{2j^+}|x^+|}D^{j^+}_{r^+ s^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&= \phi(x^-, x^+)_{DHR}(k)
\end{aligned} \tag{10.14}$$

Chapter 11

The Freidel Krasnov model

In this chapter we consider the model introduced by Freidel and Krasnov in [38], based on the imposition of the simplicity constraints using the Livine-Speziale coherent states [46]. The FK model is based on the same ideas that characterize the EPRL model, namely, the linearisation of the simplicity constraints, the inclusion of the Immirzi parameter and the imposition of the constraints in a weak sense, but this last step is realized in a different way. Specifically, the simplicity constraints are imposed in average on coherent states, so that semi-classical bivectors satisfy the geometricity conditions.

Although in Section 4.5 we defined a coherent states in the Hilbert space $L^2(SO(3))$, that can be generalized to $L^2(SO(4))$, the FK model was first introduced in terms of $SO(4)$ coherent states [46], which we are going to define. Remarkably, they can be embedded in the $L^2(SO(4))$ coherent states, precisely as the $SU(2)$ angular momentum states in $L^2(SU(2))$ spin network states.

11.1 Coherent states

In order to define $SO(4)$ coherent states, we first describe the coherent states for the group $SU(2)$, which can be trivially generalized to $SO(4)$.
For the $SU(2)$ group, these states are defined by acting with an $SU(2)$ element on the highest weight state $|j,j\rangle$ in a representation of spin j:

$$|j,h\rangle = D^j(h)|j,j\rangle \qquad h \in SU(2), \tag{11.1}$$

and they form an over-completed basis that can be used to write a decomposition of the identity operator in the representation space V^j of dimension $d_j = 2j+1$, namely:

$$\mathbb{I}_j = \sum_{mm'} |j,m\rangle\langle j,m'| d_j \int_{SU(2)} \mathrm{d}g D^j_{mj}(g)\overline{D^j_{m'j}}(g) = d_j \int_{SU(2)} \mathrm{d}g |j,g\rangle\langle j,g| \tag{11.2}$$

where $|j,m\rangle$, $m \in [-j,j]$ is the usual orthonormal basis in V^j.

Proof. This results simply follows from the fact that:

$$\delta_{mm'} = d_j \int_{SU(2)} \mathrm{d}g D^j_{mj}(g)\overline{D^j_{m'j}}(g) \tag{11.3}$$

Here $D^j_{mj}(g) = \langle j,m|D^j(d)|j,j\rangle$ is the matrix element of the group element g in the representation j computed between the states $\langle j,m|$ and $|j,j\rangle$, the latter being the highest weight state. □

Interestingly, it is possible to reduce the integral in the decomposition of the identity (11.2) by noticing that the matrix elements $D^j_{mj}(g)$ and $D^j_{mj}(gh)$ differ only by a phase for any group element h from the $U(1)$ subgroup of $SU(2)$. Therefore the integral in (11.2) can be taken over the coset $SU(2)/U(1) = S^2$. Thus, we have:

$$\mathbb{I}_j = d_j \int_{S^2} \mathrm{d}\vec{n} |j, \vec{n}\rangle\langle j, \vec{n}| \tag{11.4}$$

where $\vec{n} \in S^2$ is integrated with the invariant measure on the sphere.
We will suppress the domain of integration in what follows.

As mentioned before, the coherent states have a clear geometrical interpretation, since they represent a semi-classical vector. Indeed, if $\hat{J}^i$ are the generators of $SU(2)$ Lie algebra $\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}_k$, then the expectation value of these operators in a given coherent state is:

$$\langle j, \vec{n}|\hat{J}^i|j, \vec{n}\rangle = jn^i \tag{11.5}$$

Thus we see that $|j, \vec{n}\rangle$ describes a vector in $\mathbb{R}^3$ of length j and of direction $\vec{n}$.

Finally, as noted in [38], the coherent states minimize the dispersion of the quadratic operator $\hat{J}^2$. Indeed, in principle we can define the coherent states as $g|j, m\rangle$, and the dispersion of $\hat{J}^2$ results:

$$\Delta j^2 = j + j^2 - m^2 \tag{11.6}$$

Thus, we see that only the highest and lowest states $m = \pm j$ lead to the coherent states that minimize the uncertainty relation.
The geometrical interpretation, is that while $g|j, j\rangle$ represent a vector lying on the sphere on radius j, states $g|j, m\rangle$ correspond to a circle on this sphere of radius $\sqrt{j^2 - m^2}$.

11.2 Imposing the simplicity constraints

Now, we can use the geometric properties of coherent states to implement the simplicity constraints. The idea relates to the fact that we can generalize the $SU(2)$ coherent states to $SO(4) \subset SU(2) \otimes SU(2)$ coherent states, describing bivectors. Specifically, we can associate to each state $|j^-, n^-\rangle \otimes |j^+, n^+\rangle$ a bivector

$$x^{IJ} = (x^-, x^+) = \langle j^-, \vec{n}^-| \otimes \langle j^+, \vec{n}^+|\hat{J}^{IJ}|j^-, \vec{n}^-\rangle \otimes |j^+, \vec{n}^+\rangle = (j^-\vec{n}^-, j^+\vec{n}^+) \tag{11.7}$$

where $\hat{J}^{IJ}$ are the operators corresponding to the $SO(4)$ generators, that we identify with $\hat{X}^\pm$, in the self-dual / anti-self-dual split.

Now, recall that, classically, the discrete simplicity constraints in terms of the self-dual part and anti-self-dual part of the bivector read:

$$kx^-k^{-1} + \beta x^+ = 0 \tag{11.8}$$

At the quantum level $\hat{D} = k\hat{X}^-k^{-1} + \beta\hat{X}^+$, we require that simplicity constraints are satisfied in average, on coherent states. Particularly, on a coherent states basis, we have:

$$\begin{aligned} 0 =& \langle j^-, \vec{u}^-| \otimes \langle j^-, \vec{u}^-|[\hat{D}]|j^i, \vec{u}^-\rangle \otimes |j^-, \vec{u}^-\rangle \\ =& k(j^-\vec{u}^-)k^{-1} + sign(\beta)|\beta|j^+\vec{u}^+ \end{aligned} \tag{11.9}$$

with solution:

$$\begin{cases} j^- = |\beta|j^+ \\ \vec{u}^- = -sign(\beta)k^{-1}\vec{u}^+k \end{cases} \tag{11.10}$$

Then, the FK model amounts to consider the GFT field, decomposed in terms of coherent states:

$$\begin{aligned}
\phi_k(g) =& \sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) \delta^{j^-}_{m^- r^-} \delta^{j^+}_{m^+ r^+} D^{j^-}_{r^- n^-}(g^-) D^{j^+}_{r^+ n^+}(g^+) \\
=& \sum d^2_{j^-} d^2_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) D^{j^-}_{r^- n^-}(g^-) D^{j^+}_{r^+ n^+}(g^+) \\
& \int \mathrm{d}\vec{u}^- \langle j^-, m^- | j^-, \vec{u}^- \rangle \langle j^-, \vec{u}^- | j^-, r^- \rangle \\
& \int \mathrm{d}\vec{u}^+ \langle j^+, m^+ | j^+, \vec{u}^+ \rangle \langle j^+, \vec{u}^+ | j^+, r^+ \rangle
\end{aligned} \tag{11.11}$$

and restrict the coherent states in the decomposition, corresponding to the representation of the right invariant vector fields, to be

$$||\beta|j^+, -sign(\beta) k^{-1} \vec{u}^+ k\rangle \otimes |j^+, \vec{u}^+\rangle \tag{11.12}$$

11.3 The FK solutions in the extended GFT formalism

Equivalently, we can write the FK model in terms of the extended GFT field embedding $SO(4)$ coherent states in the $L^2(SO(4))$ coherent states. Specifically, in a coherent states basis we have:

$$\begin{aligned}
[\phi_{FK}]^{j^- \; j^+}_{(\vec{m}^-, \vec{n}^-, \vec{m}^+, \vec{n}^+)} =& \delta^{|\beta| j^+ j^-} \delta(k \vec{m}^- k^{-1} + sign[\beta] \vec{m}^+) \\
& \langle j^-; \vec{n}^-, \vec{m}^- | \otimes \langle j^+; \vec{n}^+, \vec{m}^+ | \phi \rangle
\end{aligned} \tag{11.13}$$

and the imposition of the FK constraint on these coherent states states reads:

$$\begin{aligned}
0 = \langle |\beta| j^+; \vec{n}^-, -sign(\beta) k^{-1} \vec{m}^+ k | \otimes \langle j^+; \vec{n}^+, \vec{m}^+ | \left[k \hat{X}^{R-}_i k^{-1} + \beta \hat{X}^{R+}_i \right] \\
||\beta| j^+; \vec{n}^-, -sign(\beta) k^{-1} \vec{m}^+ k \rangle \otimes |j^+; \vec{n}^+, \vec{m}^+\rangle
\end{aligned} \tag{11.14}$$

Now, in order to calculate the FK constrained GFT field in the spin representation, we show that:

- For $\beta < 0$ we have:

$$\begin{aligned}
& \langle j^-, m^- | \otimes \langle j^+, m^+ | \hat{S}_{FK} | j^-, r^- \rangle \otimes |j^+, r^+\rangle = \\
&= \sum \frac{d_{|\beta| j^+} d_{j^+}}{d_{j^+(1+|\beta|)}} \overline{\tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{m^- m^+ p}}(k) \tilde{C}^{|\beta| j^+ j^+ j^+(1+|\beta|)}_{r^- r^+ p}(k)
\end{aligned} \tag{11.15}$$

- For $\beta > 0$ we have:

$$\begin{aligned}
& \langle j^-, m^- | \otimes \langle j^+, m^+ | \hat{S}_{FK} | j^-, r^- \rangle \otimes |j^+, r^+\rangle = \\
&= \sum \frac{d_{|\beta| j^+} d_{j^+}}{d_k} \left| C^{|\beta| j^+ j^+ k}_{-|\beta| j^+ j^+ q} \right|^2 \overline{\tilde{C}^{|\beta| j^+ j^+ k}_{m^- m^+ p}}(k) \tilde{C}^{|\beta| j^+ j^+ k}_{r^- r^+ p}(k)
\end{aligned} \tag{11.16}$$

Proof. Case: $\beta < 0$

$$\begin{aligned}
&\langle j^-, m^-| \otimes \langle j^+, m^+|\hat{S}_{FK}|j^-, r^-\rangle \otimes |j^+, r^+\rangle = \\
&= \sum d_{j^-} d_{j^+} \int \mathrm{d}\vec{u} \langle j^-, m^-||\beta|j^+, k^{-1}\vec{u}k\rangle\langle|\beta|j^+, k^{-1}\vec{u}k|j^-, r^-\rangle \\
&\langle j^+, m^+|j^+, \vec{u}\rangle\langle j^+, \vec{u}|j^+, r^+\rangle \\
&= \sum d_{j^-} d_{j^+} \langle j^-, m^-||\beta|j^+, x^-\rangle\langle j^+, m^+|j^+, x^+\rangle \\
&\int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{x^-|\beta|j^+}(k^{-1}g_{\vec{u}}k)\overline{D^{|\beta|j^+}_{y^-|\beta|j^+}}(k^{-1}g_{\vec{u}}k) D^{j^+}_{x^+j^+}(g_{\vec{u}})\overline{D^{j^+}_{y^+j^+}}(g_{\vec{u}}) \\
&\langle|\beta|j^+, y^-|j^-, r^-\rangle\langle j^+, y^+|j^+, r^+\rangle \\
&= \sum d_{j^-} d_{j^+} \delta^{j^-|\beta|j^+} \delta_{m^-x^-}\delta_{m^+x^+}\delta_{y^-r^-}\delta_{y^+r^+} \\
&\int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{x^-|\beta|j^+}(k^{-1}g_{\vec{u}}k)\overline{D^{|\beta|j^+}_{y^-|\beta|j^+}}(k^{-1}g_{\vec{u}}k) D^{j^+}_{x^+j^+}(g_{\vec{u}})\overline{D^{j^+}_{y^+j^+}}(g_{\vec{u}}) \\
&= \sum d_{|\beta|j^+} d_{j^+} D^{|\beta|j^+}_{m^-a^-}(k^{-1}) D^{|\beta|j^+}_{b^-|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^-c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^-|\beta|j^+}}(k) \\
&\int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{a^-b^-}(g_{\vec{u}})\overline{D^{|\beta|j^+}_{c^-d^-}}(g_{\vec{u}}) D^{j^+}_{m^+j^+}(g_{\vec{u}})\overline{D^{j^+}_{r^+j^+}}(g_{\vec{u}}) \\
&= \sum d_{|\beta|j^+} d_{j^+} D^{|\beta|j^+}_{m^-a^-}(k^{-1}) D^{|\beta|j^+}_{b^-|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^-c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^-|\beta|j^+}}(k) \\
&\overline{C^{|\beta|j^+j^+k}_{a^-m^+p}} C^{|\beta|j^+j^+k}_{b^-j^+q} C^{|\beta|j^+j^+l}_{c^-r^+e}\overline{C^{|\beta|j^+j^+l}_{d^-j^+f}} \int \mathrm{d}g_{\vec{u}} D^k_{pq}(g_{\vec{u}})\overline{D^l_{ef}}(g_{\vec{u}}) \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} D^{|\beta|j^+}_{m^-a^-}(k^{-1}) D^{|\beta|j^+}_{b^-|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^-c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^-|\beta|j^+}}(k) \\
&\overline{C^{|\beta|j^+j^+k}_{a^-m^+p}} C^{|\beta|j^+j^+k}_{b^-j^+q} C^{|\beta|j^+j^+l}_{c^-r^+e}\overline{C^{|\beta|j^+j^+l}_{d^-j^+f}} \delta^{lk}\delta_{pe}\delta_{qf} \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} D^{|\beta|j^+}_{|\beta|j^+d^-}(k^{-1})\overline{C^{|\beta|j^+j^+k}_{d^-j^+q}} C^{|\beta|j^+j^+k}_{b^-j^+q} D^{|\beta|j^+}_{b^-|\beta|j^+}(k) \\
&D^{|\beta|j^+}_{m^-a^-}(k^{-1})\overline{C^{|\beta|j^+j^+k}_{a^-m^+p}} C^{|\beta|j^+j^+k}_{c^-r^+p} D^{|\beta|j^+}_{c^-r^-}(k) \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}} \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+p}}(k)\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+p}(k)
\end{aligned} \tag{11.17}$$

Where we have used:

$$\begin{aligned}
&\sum_{qd^-b^-} \frac{1}{d_k} D^{|\beta|j^+}_{|\beta|j^+d^-}(k^{-1})\overline{C^{|\beta|j^+j^+k}_{d^-j^+q}} C^{|\beta|j^+j^+k}_{b^-j^+q} D^{|\beta|j^+}_{b^-|\beta|j^+}(k) \\
&= \sum_{qd^-b^-} \frac{1}{d_k} \langle|\beta|j^+, |\beta|j^+|D(k^{-1})||\beta|j^+, d^-\rangle_k \langle|\beta|j^+, d^-| \otimes \langle j^+, j^+|(|\beta|j^+, j^+)\ k, q\rangle \\
&\quad \langle(|\beta|j^+, j^+)\ k, q||\beta|j^+, b^-\rangle_k \otimes |j^+, j^+\rangle\langle|\beta|j^+, b^-|D(k)||\beta|j^+, |\beta|j^+\rangle \\
&= \sum_{qd^-b^-} \frac{1}{d_k} \langle|\beta|j^+, |\beta|j^+|D(k^{-1})||\beta|j^+, d^-\rangle\langle|\beta|j^+, d^-|D(k) \otimes \langle j^+, j^+|(|\beta|j^+, j^+)\ k, q\rangle \\
&\quad \langle(|\beta|j^+, j^+)\ k, q|D(k^{-1})||\beta|j^+, b^-\rangle \otimes |j^+, j^+\rangle\langle|\beta|j^+, b^-|D(k)||\beta|j^+, |\beta|j^+\rangle \\
&= \sum_q \frac{1}{d_k} \langle|\beta|j^+, |\beta|j^+| \otimes \langle j^+, j^+|(|\beta|j^+, j^+)\ k, q\rangle \\
&\quad \langle(|\beta|j^+, j^+)\ k, q||\beta|j^+, |\beta|j^+\rangle \otimes |j^+, j^+\rangle \\
&= \sum_q \frac{1}{d_k} \overline{C^{|\beta|j^+j^+k}_{|\beta|j^+j^+q}} C^{|\beta|j^+j^+k}_{|\beta|j^+j^+q} = \frac{1}{d_{j^+(1+|\beta|)}}
\end{aligned} \tag{11.18}$$

Case: $\beta > 0$

$$
\begin{aligned}
&\langle j^-, m^-| \otimes \langle j^+, m^+|\hat{S}_{FK}|j^-, r^-\rangle \otimes |j^+, r^+\rangle = \\
&= \sum d_{j^-} d_{j^+} \int \mathrm{d}^2\vec{u} \langle j^-, m^-||\beta|j^+, -k^{-1}\vec{u}k\rangle\langle|\beta|j^+, -k^{-1}\vec{u}k|j^-, r^-\rangle \\
&\langle j^+, m^+|j^+, \vec{u}\rangle\langle j^+, \vec{u}|j^+, r^+\rangle \\
&= \sum d_{j^-} d_{j^+} \langle j^-, m^-||\beta|j^+, x^-\rangle\langle j^+, m^+|j^+, x^+\rangle\langle|\beta|j^+, y^-|j^-, r^-\rangle\langle j^+, y^+|j^+, r^+\rangle \\
&\int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{x^- -|\beta|j^+}(k^{-1}g_{\vec{u}}k)\overline{D^{|\beta|j^+}_{y^- -|\beta|j^+}}(k^{-1}g_{\vec{u}}k) D^{j^+}_{x^+ j^+}(g_{\vec{u}})\overline{D^{j^+}_{y^+ j^+}}(g_{\vec{u}}) \\
=&\sum d_{j^-} d_{j^+} \delta^{j^- |\beta|j^+}\delta_{m^- x^-}\delta_{m^+ x^+}\delta_{y^- r^-}\delta_{y^+ r^+} D^{|\beta|j^+}_{x^- a^-}(k^{-1}) D^{|\beta|j^+}_{b^- -|\beta|j^+}(k) \\
&\overline{D^{|\beta|j^+}_{y^- c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^- -|\beta|j^+}}(k) \int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{a^- b^-}(g_{\vec{u}})\overline{D^{|\beta|j^+}_{c^- d^-}}(g_{\vec{u}}) D^{j^+}_{x^+ j^+}(g_{\vec{u}})\overline{D^{j^+}_{y^+ j^+}}(g_{\vec{u}}) \\
=&\sum d_{|\beta|j^+} d_{j^+} D^{|\beta|j^+}_{m^- a^-}(k^{-1}) D^{|\beta|j^+}_{b^- -|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^- c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^- -|\beta|j^+}}(k) \\
&\int \mathrm{d}g_{\vec{u}} D^{|\beta|j^+}_{a^- b^-}(g_{\vec{u}})\overline{D^{|\beta|j^+}_{c^- d^-}}(g_{\vec{u}}) D^{j^+}_{m^+ j^+}(g_{\vec{u}})\overline{D^{j^+}_{r^+ j^+}}(g_{\vec{u}}) \\
=&\sum d_{|\beta|j^+} d_{j^+} D^{|\beta|j^+}_{m^- a^-}(k^{-1}) D^{|\beta|j^+}_{b^- -|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^- c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^- -|\beta|j^+}}(k) \\
&\overline{C^{|\beta|j^+ j^+ k}_{a^- m^+ p}} C^{|\beta|j^+ j^+ k}_{b^- j^+ q} C^{|\beta|j^+ j^+ l}_{c^- r^+ e}\overline{C^{|\beta|j^+ j^+ l}_{d^- j^+ f}} \int \mathrm{d}g_{\vec{u}} D^k_{pq}(\mathrm{d}g_{\vec{u}})\overline{D^l_{ef}}(\mathrm{d}g_{\vec{u}}) \\
=&\sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} D^{|\beta|j^+}_{m^- a^-}(k^{-1}) D^{|\beta|j^+}_{b^- -|\beta|j^+}(k)\overline{D^{|\beta|j^+}_{r^- c^-}}(k^{-1})\overline{D^{|\beta|j^+}_{d^- -|\beta|j^+}}(k) \\
&\overline{C^{|\beta|j^+ j^+ k}_{a^- m^+ p}} C^{|\beta|j^+ j^+ k}_{b^- j^+ q} C^{|\beta|j^+ j^+ l}_{c^- r^+ e}\overline{C^{|\beta|j^+ j^+ l}_{d^- j^+ f}} \delta^{lk}\delta_{pe}\delta_{qf} \\
=&\sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} D^{|\beta|j^+}_{-|\beta|j^+ d^-}(k^{-1})\overline{C^{|\beta|j^+ j^+ k}_{d^- j^+ q}} C^{|\beta|j^+ j^+ k}_{b^- j^+ q} D^{|\beta|j^+}_{b^- -|\beta|j^+}(k) \\
&D^{|\beta|j^+}_{m^- a^-}(k^{-1})\overline{C^{|\beta|j^+ j^+ k}_{a^- m^+ p}} C^{|\beta|j^+ j^+ k}_{c^- r^+ p} D^{|\beta|j^+}_{c^- r^-}(k) \\
=&\sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} \left| C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ q}\right|^2 \overline{\tilde{C}^{|\beta|j^+ j^+ k}_{m^- m^+ p}}(k)\tilde{C}^{|\beta|j^+ j^+ k}_{r^- r^+ p}(k)
\end{aligned}
\tag{11.19}
$$

Where we have used:

$$
\begin{aligned}
&\sum_{q b^- d^-} D^{|\beta|j^+}_{-|\beta|j^+ d^-}(k^{-1})\overline{C^{|\beta|j^+ j^+ k}_{d^- j^+ q}} C^{|\beta|j^+ j^+ k}_{b^- j^+ q} D^{|\beta|j^+}_{b^- -|\beta|j^+}(k) \\
=&\sum_{q b^- d^-} \langle|\beta|j^+, -|\beta|j^+|D(k^{-1})||\beta|j^+, d^-\rangle_k\langle|\beta|j^+, d^-| \otimes \langle j^+, j^+|(|\beta|j^+, j^+)\ k, q\rangle \\
&\langle(|\beta|j^+, j^+)\ k, q||\beta|j^+, b^-\rangle_k \otimes \langle j^+, j^+|\langle|\beta|j^+, b^-|D(k)||\beta|j^+, -|\beta|j^+\rangle \\
=&\sum_{q b^- d^-} \langle|\beta|j^+, -|\beta|j^+|D(k^{-1})||\beta|j^+, d^-\rangle\langle|\beta|j^+, d^-|D(k) \otimes \langle j^+, j^+|(|\beta|j^+, j^+)k, q\rangle \\
&\langle(|\beta|j^+, j^+)\ k, q|D(k^{-1})||\beta|j^+, b^-\rangle \otimes |j^+, j^+\rangle\langle|\beta|j^+, b^-|D(k)||\beta|j^+, -|\beta|j^+\rangle \\
=&\sum_q \langle|\beta|j^+, -|\beta|j^+| \otimes \langle j^+, j^+|(|\beta|j^+, j^+)\ k, q\rangle \\
&\langle(|\beta|j^+, j^+)\ k, q||\beta|j^+, -|\beta|j^+\rangle \otimes |j^+, j^+\rangle \\
=&\overline{C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ q}} C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ q} = \left| C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ q}\right|^2
\end{aligned}
\tag{11.20}
$$

□

Interestingly, these results show clearly that the FK operator $\hat{S}_{FK}$ for $\beta < 0$ is proportional to a projector, while for $\beta > 0$ it is not a projector.

Proof.

- For $\beta < 0$, we have:

$$\begin{aligned}
&\langle j^-, m^-| \otimes \langle j^+, m^+|\hat{S}^2_{FK}|j^-, s^-\rangle \otimes |j^+, s^+\rangle = \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}} \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+p}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+p}(k) \\
&\frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}} \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{r^-r^+q}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{s^-s^+q}(k) \\
&= \sum \left[\frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}}\right]^2 \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+p}}(k) \delta_{pq} \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{s^-s^+q}(k) \\
&= \sum \left[\frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}}\right]^2 \overline{\tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{m^-m^+p}}(k) \tilde{C}^{|\beta|j^+j^+j^+(1+|\beta|)}_{s^-s^+p}(k) \\
&= \left[\frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}}\right] \langle j^-, m^-| \otimes \langle j^+, m^+|\hat{S}_{FK}|j^-, s^-\rangle \otimes |j^+, s^+\rangle
\end{aligned} \tag{11.21}$$

- For $\beta > 0$, we have:

$$\begin{aligned}
&\langle j^-, m^-| \otimes \langle j^+, m^+|\hat{S}^2_{FK}|j^-, s^-\rangle \otimes |j^+, s^+\rangle = \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} \left|C^{|\beta|j^+j^+k}_{-|\beta|j^+j^+q}\right|^2 \overline{\tilde{C}^{|\beta|j^+j^+k}_{m^-m^+p}}(k) \tilde{C}^{|\beta|j^+j^+k}_{r^-r^+p}(k) \\
&\qquad \frac{d_{|\beta|j^+} d_{j^+}}{d_z} \left|C^{|\beta|j^+j^+z}_{-|\beta|j^+j^+x}\right|^2 \overline{\tilde{C}^{|\beta|j^+j^+z}_{r^-r^+q}}(k) \tilde{C}^{|\beta|j^+j^+z}_{s^-s^+q}(k) \\
&= \sum \left[\frac{d_{|\beta|j^+} d_{j^+}}{d_k}\right] \left|C^{|\beta|j^+j^+k}_{-|\beta|j^+j^+q}\right|^2 \left[\frac{d_{|\beta|j^+} d_{j^+}}{d_z}\right] \left|C^{|\beta|j^+j^+z}_{-|\beta|j^+j^+x}\right|^2 \\
&\qquad \overline{\tilde{C}^{|\beta|j^+j^+k}_{m^-m^+p}}(k) \tilde{C}^{|\beta|j^+j^+k}_{s^-s^+p}(k)
\end{aligned} \tag{11.22}$$

□

With these results, we can finally write the FK model in spin representation:

- Case $\beta < 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{FK}^{\beta<0}|\phi\rangle = \\
&= d^2_{|\beta|j^+} d^2_{j^+} \int \mathrm{d}^2\vec{m}^- \mathrm{d}^2\vec{m}^+ \mathrm{d}^2\vec{n}^+ \langle j^-; s^-, r^- || \beta| j^+; \vec{m}^-, k^{-1}\vec{n}^+ k\rangle \\
&\langle j^+; s^+, r^+ | j^+; \vec{m}^+, \vec{n}^+\rangle \langle |\beta| j^+; \vec{m}^-, k^{-1}\vec{n}^+ k| \otimes \langle j^+; \vec{m}^+, \vec{n}^+| \otimes \langle k|\phi\rangle \\
&= \sum (d_{|\beta|j^+} d_{j^+})^2 \phi^{|\beta|j^+ j^+}_{w^- x^- w^+ x^+}(k) \\
&\int \mathrm{d}g_{\vec{m}^-} D^{|\beta|j^+}_{s^- |\beta|j^+}(g_{\vec{m}^-}) \overline{D^{|\beta|j^+}_{x^- |\beta|j^+}(g_{\vec{m}^-})} \int \mathrm{d}g_{\vec{m}^+} D^{j^+}_{s^+ j^+}(g_{\vec{m}^+}) \overline{D^{j^+}_{x^+ j^+}(g_{\vec{m}^+})} \\
&\int \mathrm{d}g_{\vec{n}^+} \overline{D^{|\beta|j^+}_{r^- |\beta|j^+}(k^{-1}g_{\vec{n}^+}k)} D^{|\beta|j^+}_{w^- |\beta|j^+}(k^{-1}g_{\vec{n}^+}k) D^{j^+}_{w^+ j^+}(g_{\vec{n}^+}) \overline{D^{j^+}_{r^+ j^+}(g_{\vec{n}^+})} \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_{j^+(1+|\beta|)}} \phi^{|\beta|j^+ j^+}_{w^- s^- w^+ s^+}(k) \overline{\tilde{C}^{|\beta|j^+ j^+ j^+(1+|\beta|)}_{w^- w^+ p}(k)} \tilde{C}^{|\beta|j^+ j^+ j^+(1+|\beta|)}_{r^- r^+ p}(k)
\end{aligned} \tag{11.23}$$

- Case $\beta > 0$

$$\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{FK}^{\beta>0}|\phi\rangle = \\
&= d^2_{|\beta|j^+} d^2_{j^+} \int \mathrm{d}^2\vec{m}^- \mathrm{d}^2\vec{m}^+ \mathrm{d}^2\vec{n}^+ \langle j^-; s^-, r^- || \beta| j^+; \vec{m}^-, -k^{-1}\vec{n}^+ k\rangle \\
&\langle j^+; s^+, r^+ | j^+; \vec{m}^+, \vec{n}^+\rangle \langle |\beta| j^+; \vec{m}^-, -k^{-1}\vec{n}^+ k| \otimes \langle j^+; \vec{m}^+, \vec{n}^+| \otimes \langle k|\phi\rangle \\
&= \sum \frac{d_{|\beta|j^+} d_{j^+}}{d_k} \phi^{|\beta|j^+ j^+}_{w^- s^- w^+ s^+}(k) \left| C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ x} \right|^2 \overline{\tilde{C}^{|\beta|j^+ j^+ k}_{w^- w^+ p}(k)} \tilde{C}^{|\beta|j^+ j^+ k}_{r^- r^+ p}(k)
\end{aligned} \tag{11.24}$$

And, thus, in group representation we have:

- If $\beta < 0$

$$\begin{aligned}
[\phi(g^-, g^+)_k]^{\beta>0}_{FK} = \sum & \frac{d^2_{|\beta|j^+} d^2_{j^+}}{d_{j^+(1+|\beta|)}} \psi^{|\beta|j^+ j^+}_{r^- t^- r^+ t^+}(k) \overline{\tilde{C}^{|\beta|j^+ j^+ j^+(1+|\beta|)}_{r^- r^+ q}(k)} \tilde{C}^{|\beta|j^+ j^+ j^+(1+|\beta|)}_{s^- s^+ q}(k) \\
& D^{|\beta|j^+}_{s^- t^-}(g^-) D^{j^+}_{s^+ t^+}(g^+)
\end{aligned} \tag{11.25}$$

- If $\beta > 0$

$$\begin{aligned}
[\phi(g^-, g^+)_k]^{\beta>0}_{FK} = \sum & \frac{d^2_{|\beta|j^+} d^2_{j^+}}{d_k} \phi^{|\beta|j^+ j^+}_{r^- t^- r^+ t^+}(k) \left| C^{|\beta|j^+ j^+ k}_{-|\beta|j^+ j^+ x} \right|^2 \overline{\tilde{C}^{|\beta|j^+ j^+ k}_{r^- r^+ q}}(k) \tilde{C}^{|\beta|j^+ j^+ k}_{s^- s^+ q}(k) \\
& D^{|\beta|j^+}_{s^- t^-}(g^-) D^{j^+}_{s^+ t^+}(g^+)
\end{aligned} \tag{11.26}$$

Remarkably, we note that the FK model for $\beta < 0$ is proportional to the EPRL model, and it does not require a separate discussion. However, for $\beta > 0$ the FK model differs from the EPRL one, suggesting an alternative Spin Foam model. Moreover, while in the limit $\gamma \to \infty$ (equivalently $\beta = 1$) the EPRL model reduces exactly to the original Barrett-Crane model, the FK model is different. Finally, we conclude this section, by performing the non-commutative group Fourier anti-transformation, and writing the FK model in metric representation.

$$\begin{aligned}
&[\phi(x^-, x^+)_k]_{FK} = \\
&= (d_{|\beta|j^+} d_{j^+})^2 \int \mathrm{d}^2\vec{m}^- \mathrm{d}^2\vec{m}^+ \mathrm{d}^2\vec{n}^+ E_{([-sign(\beta)k^{-1}g_{\vec{n}^-}k][g_{\vec{m}^-}]^{-1})}(x^-) \star \hat{D}^{|\beta|j^+}_{[\vec{m}^-][\vec{m}^-]}(x^-) \\
&\quad E_{(g_{\vec{n}^+} g^{-1}_{\vec{m}^+})}(x^+) \star \hat{D}^{|\beta|j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \phi^{|\beta|j^+ j^+}_{(\vec{m}^-, -sign(\beta)k^{-1}\vec{n}^+ k, \vec{m}^+, \vec{n}^+)}(k)
\end{aligned} \tag{11.27}$$

Proof.

$$
\begin{aligned}
&[\phi(x^-,x^+)_k]_{FK} = \\
&= (d_{j^-}d_{j^+})^2 \int \mathrm{d}^2\vec{m}^-\mathrm{d}^2\vec{m}^+\mathrm{d}^2\vec{n}^-\mathrm{d}^2\vec{n}^+ \langle x^-|j^-;\vec{m}^-,\vec{n}^-\rangle\langle x^+|j^+;\vec{m}^+,\vec{n}^+\rangle \\
&\langle j^-;\vec{m}^-,\vec{n}^-| \otimes \langle j^+;\vec{m}^+,\vec{n}^+| \otimes \langle k|S_{FK}|\phi\rangle \\
&= (d_{|\beta|j^+}d_{j^+})^2 \int \mathrm{d}^2\vec{m}^-\mathrm{d}^2\vec{m}^+\mathrm{d}"\vec{n}^+ \langle x^-||\beta|j^+;\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k\rangle\langle x^+|j^+;\vec{m}^+,\vec{n}^+\rangle \\
&\langle|\beta|j^+;m^-,-sign(\beta)k^{-1}\vec{n}^+k| \otimes \langle j^+;\vec{m}^+,\vec{n}^+| \otimes \langle k|\phi\rangle \\
&= \int \mathrm{d}^2\vec{m}^-\mathrm{d}^2\vec{m}^+\mathrm{d}^2\vec{n}^+\mathrm{d}g^-\mathrm{d}g^+ \langle x^-|g^-\rangle\langle x^+|g^+\rangle\langle g^-||\beta|j^+;m^-,-sign(\beta)k^{-1}\vec{n}^+k\rangle \\
&\langle g^+|j^+;\vec{m}^+,\vec{n}^+\rangle(d_{|\beta|j^+}d_{j^+})^2\psi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k) \\
&= \int \mathrm{d}g_{\vec{m}^-}\mathrm{d}g_{\vec{m}^+}\mathrm{d}g_{\vec{n}^+}\mathrm{d}g^-\mathrm{d}g^+ E_{g^-}(x^-)D^{|\beta|j^+}_{|\beta|j^+|\beta|j^+}([-sign(\beta)k^{-1}g_{\vec{n}^-}k]^{-1}g^-[g_{\vec{m}^-}]) \\
&E_{g^+}(x^+)d_{j^+}D^{j^+}_{j^+j^+}(g^{-1}_{\vec{n}^+}g^+g_{\vec{m}^+})(d_{|\beta|j^+}d_{j^+})^2\phi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k) \\
&= \int \mathrm{d}g_{\vec{m}^-}\mathrm{d}g_{\vec{m}^+}\mathrm{d}g_{\vec{n}^+}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+ E_{([-sign(\beta)k^{-1}g_{\vec{n}^-}k]\tilde{g}^-[g_{\vec{m}^-}]^{-1})}(x^-)D^{|\beta|j^+}_{|\beta|j^+|\beta|j^+}(\tilde{g}^-) \\
&E_{g_{\vec{n}^+}\tilde{g}^+g^{-1}_{\vec{m}^+}}(x^+)D^{j^+}_{j^+j^+}(\tilde{g}^+)(d_{|\beta|j^+}d_{j^+})^2\phi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k) \\
&= \int \mathrm{d}g_{\vec{m}^-}\mathrm{d}g_{\vec{m}^+}\mathrm{d}g_{\vec{n}^+}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+ E_{([-sign(\beta)k^{-1}g_{\vec{n}^-}k][g_{\vec{m}^-}]^{-1})}(x^-)\star E_{([g_{\vec{m}^-}]\tilde{g}^-[g_{\vec{m}^-}]^{-1})}(x^-) \\
&D^{|\beta|j^+}_{|\beta|j^+|\beta|j^+}(\tilde{g}^-)E_{(g_{\vec{n}^+}g^{-1}_{\vec{m}^+})}(x^+)\star E_{(g_{\vec{m}^+}\tilde{g}^+g^{-1}_{\vec{m}^+})}(x^+)D^{j^+}_{j^+j^+}(\tilde{g}^+) \\
&(d_{|\beta|j^+}d_{j^+})^2\phi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k) \\
&= \int \mathrm{d}g_{\vec{m}^-}\mathrm{d}g_{\vec{m}^+}\mathrm{d}g_{\vec{n}^+}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+ E_{([-sign(\beta)k^{-1}g_{\vec{n}^-}k][g_{\vec{m}^-}]^{-1})}(x^-)\star E_{(\tilde{g}^-)}([g_{\vec{m}^-}]^{-1}x^-[g_{\vec{m}^-}]) \\
&D^{|\beta|j^+}_{|\beta|j^+|\beta|j^+}(\tilde{g}^-)E_{(g_{\vec{n}^+}g^{-1}_{\vec{m}^+})}(x^+)\star E_{(\tilde{g}^+)}(g^{-1}_{\vec{m}^+}x^+g_{\vec{m}^+})D^{j^+}_{j^+j^+}(\tilde{g}^+) \\
&(d_{|\beta|j^+}d_{j^+})^2\phi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k) \\
&= \int \mathrm{d}g_{\vec{m}^-}\mathrm{d}g_{\vec{m}^+}\mathrm{d}g_{\vec{n}^+} E_{([-sign(\beta)k^{-1}g_{\vec{n}^-}k][g_{\vec{m}^-}]^{-1})}(x^-)\star \hat{D}^{|\beta|j^+}_{[\vec{m}^-][\vec{m}^-]}(x^-) \\
&E_{(g_{\vec{n}^+}g^{-1}_{\vec{m}^+})}(x^+)\star \hat{D}^{|\beta|j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+)(d_{|\beta|j^+}d_{j^+})^2\phi^{|\beta|j^+j^+}_{(\vec{m}^-,-sign(\beta)k^{-1}\vec{n}^+k,\vec{m}^+,\vec{n}^+)}(k)
\end{aligned}
\tag{11.28}
$$

□

Chapter 12

The Baratin Oriti model

In this chapter we present the Baratin Oriti model firstly proposed for the Barrett-Crane model in [49] and then generalized in [50] to the case with finite Immirzi parameter.
While the other models presented in this thesis were originally formulated in a Spin Foam context, the Baratin Oriti model was introduced in a Group Field Theory framework as it deeply relies on the metric representation of the GFT field.

Indeed, the Baratin Oriti model encodes the simplicity condition (6.16) of the bivector variables x_j as a constraint on the extended GFT field ϕ_k. The geometrical interpretation is clear since the extended GFT field describes the geometry of a tetrahedron and its normal vector k^I identifies by the extra variable $k \in SU(2)$.

The idea is to impose the simplicity condition using non-commutative delta functions directly on the GFT field. Actually, since the non-commutative δ functions act as Dirac distributions for the $\star$-product, the imposition of the simplicity conditions via non-commutative δ functions on the field will effectively amount to constrain the measure on the bivectors. Moreover, both the delta function and the $\star$-product reflect the non-commutativity of the metric representation.
Particularly, the Baratin Oriti model introduces the following function of $x = (x^-, x^+) \in \mathfrak{so}(4)$:

$$S_{BO}(x) = \delta_{-kx^-k^{-1}}(\beta x^+) = \int_{SU(2)} \mathrm{d}u e^{iTr[k^{-1}ukx^-} e^{i\beta Tr[ux^+]} \tag{12.1}$$

where $\delta_{-a}(b) := \delta(a+b)$ and δ is the $\mathfrak{su}(2)$ non-commutative delta function.

Thus, the geometrical GFT model is defined by constraining the field $\phi_k(x)$ by an operator $\hat{S}_{BO} = \mathcal{Q}(S_{BO})$ acting on it by $\star$-multiplication. Specifically, in the case of the field describing only a face of the tetrahedron, we have:

$$\langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle = S_{BO}(x) \star_{\mathcal{Q}} \phi_k(x) = \delta_{-kx^-k^{-1}}(\beta x^+) \star_{\mathcal{Q}} \phi_k(x) \tag{12.2}$$

However, the generalization to the complete tetrahedron is simple and is given by:

$$(\hat{S}_{BO} \triangleright \phi)_k(x_1, \dots, x_4) = \prod_{j=1}^{4} \hat{S}_{BO}(x_j) \star_{\mathcal{Q}} \phi_k(x_1, \dots, x_4) \tag{12.3}$$

In [50], with the choice of the Freidel-Livine-Majid quantization map, it has been shown that the simplicity function (6.16) is well-defined. Particularly, in order to take the $\star$-product of the S_k^β with the field, it is necessary that the function (12.1) is in the image of the non-commutative Fourier transform. In [50] it has been shown that this is indeed the case. In fact, since $u = e^{\theta n^j \tau_j} \in SU(2)$ is parametrized by the angle $\theta \in [0, \pi]$ and the unit $\mathbb{R}^3$-vector $\vec{n}$, where the τ_j are i times the Pauli matrices, we can define $u_\beta = e^{\theta_\beta n_\beta^j \tau_j}$, where the parameters θ_β and $\vec{n}_\beta$ are:

$$\sin\theta_\beta = |\beta| \sin\theta, \ \mathrm{sign}(\cos\theta_\beta) = \mathrm{sign}(\cos\theta); \quad \vec{n}_\beta = \mathrm{sign}(\beta)\vec{n} \tag{12.4}$$

and the simplicity function can be written as a superposition of plane waves $E_g(x)=e^{iTrgx}$:

$$\delta_{-kx^-k^{-1}}(\beta x^+) = \int_{\mathrm{SU}(2)} \mathrm{d}u \, E_{\mathbf{u}^k_\beta}(x) \tag{12.5}$$

where we introduced $\mathbf{u}^k_\beta=(k^{-1}uk, u_\beta)\in\mathrm{SU}(2)\times\mathrm{SU}(2)$.
Moreover, we note that the operator $\hat{S}$ is not a projector for generic values of β, unless $\beta=0,1$ (which corresponds to $\gamma=1,\infty$).
The reason can be traced back to the non-linearity of the scaling by β in the definition (12.4), specifically we have that $(uv)_\beta \neq u_\beta v_\beta$, and thus $S^\beta_k \star S^\beta_k \neq S^\beta_k$.

Now, the issue concerning the projector property of the simplicity constraint operators in the different models is far from being trivial. Indeed, in Spin Foam models one needs to impose simplicity constraints and closure constraints. For example, in the standard (not extended) EPRL and FK (with $\gamma < 1$) models simplicity constraints are implemented by a projector operator (two Kronecker deltas) which does not commute with the closure constraint, and the result of the composition of the two operators is not a projector. In the standard FK model $\gamma > 1$ the operator implementing the simplicity constraints is not a projector, as well as in the Baratin Oriti model.
As we have seen, in the GFT context, we can choose to impose gauge invariance and simplicity in the propagator only, in the vertex only, or in both. Unfortunately, if the full geometric operator is not a propagator then the resulting amplitude depends on this choice.
One could ask whether the fact that the operator imposing the simplicity constraints is a projector or not depends on the choice of the quantization map. However, preliminary results for the Baratin Oriti model seems to indicate that also for the Duflo map $\hat{S}_{BO}$ is not a projector [145]. However, further investigation is needed for all these models in the extended GFT formalism, as relaxed closure might commute with the EPRL/FK simplicity operator.

Despite this fact, we note the remarkable fact that the action of $\hat{S}$ is well-defined on gauge invariant fields, as it commutes with the gauge transformations (5.47):

$$\hat{S}^\beta \triangleright [E_h \cdots E_h \star \varphi_{h^{-1}\triangleright k}] = E_h \cdots E_h \star (\hat{S}^\beta \triangleright \varphi)_{h^{-1}\triangleright k} \tag{12.6}$$

thanks to the commutation relations between plane waves and simplicity functions:

$$E_h \star S^\beta_k = S^\beta_{h\triangleright k} \star E_h \tag{12.7}$$

where $h \triangleright k := h^+ k (h^-)^{-1}$.

Now, we prove this equation by starting from enunciating the following lemma:

Lemma 12.0.1. *The following relation holds:*

$$(u^{-1}vu)_\beta = u^{-1}v_\beta u \tag{12.8}$$

Proof. We can prove the lemma by noticing the fact that by definition we have:

$$e^{i\beta tr[av]} = e^{itr[av_\beta]}, \qquad a \in \mathfrak{su}(2), v, v_\beta \in SU(2) \tag{12.9}$$

So that:

$$e^{itr[x(u^{-1}vu)_\beta]} = e^{i\beta tr[x(u^{-1}vu)]} = e^{i\beta tr[(uxu^{-1})v]} = e^{itr[(uxu^{-1})v_\beta]} = e^{itr[x(u^{-1}v_\beta u)]} \tag{12.10}$$

□

Now, using the invariant properties of the Haar measure, we have:

$$\begin{aligned}
E_h \star S_k^\beta &= \int_{SU(2)} \mathrm{d}u \left(E_{(h^-,h^+)} \star E_{(k^{-1}uk,u_\beta)} \right)(x^-,x^+) \\
&= \int_{SU(2)} \mathrm{d}u E_{(h^-k^{-1}uk,h^+u_\beta)}(x^-,x^+) \\
&= \int_{SU(2)} \mathrm{d}u E_{(h^-k^{-1}(h^+)^{-1}uh^+k(h^-)^{-1}h^-,h^+(h^+)^{-1}u_\beta h^+)}(x^-,x^+) \\
&= \int_{SU(2)} \mathrm{d}u E_{((h^+k(h^-)^{-1})^{-1}u(h^+k(h^-)^{-1})h^-,u_\beta h^+)}(x^-,x^+) \\
&= \int_{SU(2)} \mathrm{d}u \left(E_{((h^+k(h^-)^{-1})^{-1}u(h^+k(h^-)^{-1}),u_\beta)} \star E_{(h^-,h^+)} \right)(x^-,x^+) = S^\beta_{h \triangleright k} \star E_h
\end{aligned} \tag{12.11}$$

These properties have a geometric interpretation, as they expresses the fact that rotating a bivector which is simple with respect to a normal k gives a bivector which is simple with respect to the rotated normal $h \triangleright k := h^+k(h^-)^{-1}$ [50].

Remarkably, in the extended GFT formalism, the linear simplicity constraints on the bivectors can be imposed in a covariant way, whereas in the standard formulation of the Barrett-Crane model, and the EPRL/FK model, simplicity and gauge invariance are implemented by means of two non-commuting projectors.

Particularly, one could ask whether after the imposition of the Baratin Oriti simplicity constraints one needs also to impose the closure constraint. However, due to the fact that the constraints are imposed using non-commutative delta function and $\star$-product, because of gauge invariance, the closure constraint holds after integration over the normal: let $\psi := \int \mathrm{d}k \hat{S}^\beta \triangleright \varphi_k$, then $\psi = \delta(x_1 + x_2 + x_3 + x_4) \star \psi$.

12.1 The Baratin Oriti model and in the extended GFT formalism

In this section we describe the Baratin Oriti solutions in different representations. Particularly, we start by noticing that the Baratin Oriti operator is diagonal in the metric representation, namely:

$$\langle x^- | \otimes \langle x^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle = \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x) = \frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \phi_k(x) \tag{12.12}$$

Now, let us consider the action of the Baratin Oriti operator on the extended GFT field in the group representation. Using the non-commutative Fourier anti-transformation, we have:

$$\begin{aligned}
&\langle g^- | \otimes \langle g^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle = \\
&= \int \frac{\mathrm{d}^3x^-}{(2\pi)^3} \frac{\mathrm{d}^3x^+}{(2\pi)^3} \overline{E}_{(g^-,g^+)}(x^-,x^+) \star S_k^\beta(x^-,x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-}{(2\pi)^3} \frac{\mathrm{d}^3x^+}{(2\pi)^3} \mathrm{d}u E_{((g^-)^{-1}k^{-1}uk,(g^+)^{-1}u_\beta)}(x^-,x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-}{(2\pi)^3} \frac{\mathrm{d}^3x^+}{(2\pi)^3} \mathrm{d}u \overline{E}_{([(g^-)^{-1}k^{-1}uk]^{-1},[(g^+)^{-1}u_\beta]^{-1})}(x^-,x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-}{(2\pi)^3} \frac{\mathrm{d}^3x^+}{(2\pi)^3} \mathrm{d}u \overline{E}_{(k^{-1}u^{-1}kg^-,(u_\beta)^{-1}g^+)}(x^-,x^+) \star \phi_k(x^-,x^+) \\
&= \int \mathrm{d}u \phi_k(k^{-1}u^{-1}kg^-, u_\beta^{-1}g^+) = \int \mathrm{d}u \phi_k(k^{-1}ukg^-, u_\beta g^+) = \int \mathrm{d}u \phi_k(\mathbf{u}^k_\beta g)
\end{aligned} \tag{12.13}$$

where $\mathbf{u}_\beta^k = (k^{-1}uk, u_\beta) \in \mathrm{SU}(2)\times\mathrm{SU}(2)$ and u_β is defined as in (12.4).
As we have seen, in the Barrett-Crane case, corresponding to the limit $\gamma \to \infty$, thus $\beta = 1$, the Baratin Oriti operator becomes a projector. Specifically, it reduces to the projector onto field on the homogeneous space $\mathrm{SO}(4)/\mathrm{SO}(3)_k$. However, in order to obtain the standard GFT model dual to the Riemannian Barrett-Crane model [49, 125], we need to gauge fix the normal to the value $k = 1$ (time gauge), by means of the invariance (5.47). However, as noted in [50], the difference is that the gauge fixed extended fields are not gauge invariant under the full SO(4), but only under the diagonal SU(2) subgroup.

Finally, let us consider the action of the Baratin Oriti operator on the extended GFT field in the spin representation. Using Peter-Weyl decomposition of the constrained field $(\hat{S}^\beta \triangleright \varphi)_k$:

$$\begin{aligned}
&\langle g^-| \otimes \langle g^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle = \\
&= \sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) D^{j^-}_{m^- r^-}(k^{-1}) C^{j^- j^+ k}_{r^- m^+ q} C^{j^- j^+ k}_{s^- s^+ q} D^{j^-}_{s^- t^-}(k) \\
&\int \mathrm{d}u D^{j^-}_{t^- c^-}(k^{-1}ukg^-) D^{j^+}_{s^+ b^+}(u_\beta g^+) D^{j^-}_{c^- b^-}(k^{-1}) C^{j^- j^+ k}_{b^- b^+ p} C^{j^- j^+ k}_{a^- n^+ p} D^{j^-}_{a^- n^-}(k) \\
=&\sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) D^{j^-}_{m^- r^-}(k^{-1}) C^{j^- j^+ k}_{r^- m^+ q} D^{j^-}_{c^- b^-}(k^{-1}) C^{j^- j^+ k}_{b^- b^+ p} C^{j^- j^+ k}_{a^- n^+ p} D^{j^-}_{a^- n^-}(k) \\
&\int \mathrm{d}u D^{j^-}_{t^- d^-}(k^{-1}uk) C^{j^- j^+ k}_{s^- s^+ q} D^{j^-}_{s^- t^-}(k) D^{j^+}_{s^+ d^+}(u_\beta) D^{j^-}_{d^- c^-}(g^-) D^{j^-}_{d^+ b^+}(g^-) \\
=&\sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) D^{j^-}_{m^- m}(k^{-1}) C^{j^- j^+ k}_{m m^+ q} F^{j^- j^+ k}_{r^- r^+ q}(k) \\
&D^{j^-}_{r^- s^-}(g^-) D^{j^-}_{r^+ s^+}(g^-) D^{j^-}_{s^- s}(k^{-1}) C^{j^- j^+ k}_{s s^+ p} C^{j^- j^+ k}_{n n^+ p} D^{j^-}_{n n^-}(k) \\
=&\sum d_{j^-} d_{j^+} \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) \overline{\tilde{C}^{j^- j^+ k}_{m^- m^+ q}}(k) F^{j^- j^+ k}_{r^- r^+ q}(k) D^{j^-}_{r^- s^-}(g^-) D^{j^-}_{r^+ s^+}(g^-)
\end{aligned} \tag{12.14}$$

where we have defined:

$$F^{j^- j^+ k}_{x^- x^+ q}(k) = \int \mathrm{d}u D^{j^-}_{t^- x^-}(k^{-1}uk) D^{j^+}_{t^+ x^+}(u_\beta) \tilde{C}^{j^- j^+ k}_{t^- t^+ q}(k) \tag{12.15}$$

with u_β given as in (12.4).

Thus, the Baratin Oriti model in the spin representation is:

$$\begin{aligned}
[\phi_{BO}]^{j^-\; j^+}_{r^- s^- r^+ s^+}(k) &= \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle \\
&= \sum \phi^{j^- j^+}_{m^- n^- m^+ n^+}(k) \overline{\tilde{C}^{j^- j^+ k}_{m^- m^+ q}}(k) F^{j^- j^+ k}_{r^- r^+ q}(k) \overline{\tilde{C}^{j^- j^+ k}_{s^- s^+ p}}(k)
\end{aligned} \tag{12.16}$$

More generally, a set of basis functions describing the complete tetrahedron is given by the action of the $\hat{S}$ on the functions (5.55):

$$\hat{S}^\beta \triangleright \Psi^{(J_i, k_i, j)}_{m_i^-, m_i^+}(g_i; k) = \left(\prod_{i=1}^{4} D^{j_i^-}_{n_i^- m_i^-}(g_i^-) D^{j_i^+}_{n_i^+ m_i^+}(g_i^+) F^{j_i^- j_i^+ k_i}_{n_i^- n_i^+ p_i}(k) \right) (\iota_j)^{k_i}_{p_i} \tag{12.17}$$

where repeated lower indices are summed over. This expression is obtained from (5.55) by replacing the k-dependent coefficients $\tilde{C}^{j^- j^+ k}_{m^- m^+ p}(k) = C^{j^- j^+ k}_{m m^+ p} D^{j^-}_{m m^-}(k)$ by new ones $F^{j^- j^+ k}_{x^- x^+ q}(k)$. As noted in [50], these new coefficients share with the k-dependent coefficients $\tilde{C}^{j^- j^+ k}_{m^- m^+ p}(k)$ the property to intertwine the action of stabilizer subgroup $\mathrm{SO}(3)_k$ in the representation $j^- \otimes j^+$ and the action of SO(3) in the representation k. Namely, given $\mathbf{u}_k = (k^{-1}uk, u) \in \mathrm{SO}(3)_k$, we have:

$$F^{j_i^- j_i^+ k_i}_{m_i^- m_i^+ p_i}(k) D^{j_i^-}_{m_i^- n_i^-}(\mathbf{u}_k^-) D^{j_i^+}_{m^+ n_i^+}(\mathbf{u}_k^+) = F^{j_i^- j_i^+ k_i}_{n_i^- n_i^+ q_i}(k) D^{k_i}_{q_i p_i}(u). \tag{12.18}$$

It is easy to show that this property holds.

Proof. Actually, the l.h.s. of (12.18) is:

$$
\begin{aligned}
&F^{j^-j^+k}_{m^-m^+p}(k)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{m^+n^+}(u) = \\
&= \int \mathrm{d}v D^{j^-}_{t^-m^-}(k^{-1}vk)D^{j^+}_{t^+m^+}(v_\beta)\tilde{C}^{j^-j^+k}_{t^-t^+q}(k)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{m^+n^+}(u) \\
&= \int \mathrm{d}v D^{j^-}_{t^-m^-}(k^{-1}vk)D^{j^-}_{m^-n^-}(k^{-1}uk)D^{j^+}_{t^+m^+}(v_\beta)D^{j^+}_{m^+n^+}(u)\tilde{C}^{j^-j^+k}_{t^-t^+q}(k) \\
&= \int \mathrm{d}v D^{j^-}_{t^-n^-}(k^{-1}vuk)D^{j^+}_{t^+n^+}(v_\beta u)\tilde{C}^{j^-j^+k}_{t^-t^+q}(k) \\
&= \int \mathrm{d}v D^{j^-}_{t^-n^-}(k^{-1}vuk)D^{j^+}_{t^+n^+}(v_\beta u)C^{j^-j^+k}_{tt^+q}D^{j^-}_{tt^-}(k) \\
&= \int \mathrm{d}v D^{j^-}_{s^-n^-}(vuk)D^{j^+}_{s^+n^+}(v_\beta u)C^{j^-j^+k}_{s^-s^+q} \\
&= \int \mathrm{d}\tilde{v} D^{j^-}_{s^-n^-}(u\tilde{v}k)D^{j^+}_{s^+n^+}((u\tilde{v}u^{-1})_\beta u)C^{j^-j^+k}_{s^-s^+q} \\
&= \int \mathrm{d}\tilde{v} D^{j^-}_{s^-n^-}(u\tilde{v}k)D^{j^+}_{s^+n^+}(u\tilde{v}_\beta)C^{j^-j^+k}_{s^-s^+q}
\end{aligned}
\tag{12.19}
$$

Where we have used the relation:

$$
(u^{-1}vu)_\beta = u^{-1}v_\beta u \tag{12.20}
$$

which follows from Lemma (12.8).

Accordingly, the r.h.s. of (12.18) is:

$$
\begin{aligned}
&F^{j^-j^+k}_{n^-n^+q}(k)D^{k}_{pq}(u) = \int \mathrm{d}w D^{j^-}_{t^-n^-}(k^{-1}wk)D^{j^+}_{t^+n^+}(w_\beta)\tilde{C}^{j^-j^+k}_{t^-t^+q}(k)D^{k}_{pq}(u) \\
&= \int \mathrm{d}w D^{j^-}_{t^-n^-}(k^{-1}wk)D^{j^+}_{t^+n^+}(w_\beta)C^{j^-j^+k}_{tt^+q}D^{j^-}_{tt^-}(k)D^{k}_{pq}(u) \\
&= \int \mathrm{d}w D^{j^-}_{tt^-}(k)D^{j^-}_{t^-n^-}(k^{-1}wk)D^{j^+}_{t^+n^+}(w_\beta)C^{j^-j^+k}_{tt^+q}D^{k}_{pq}(u) \\
&= \int \mathrm{d}w D^{j^-}_{tn^-}(wk)D^{j^+}_{t^+n^+}(w_\beta)C^{j^-j^+k}_{tt^+q}D^{k}_{pq}(u) \\
&= \int \mathrm{d}w D^{j^-}_{tn^-}(wk)D^{j^+}_{t^+n^+}(w_\beta)C^{j^-j^+k}_{s^-s^+p}D^{j^-}_{s^-t}(u)D^{j^+}_{s^+t^+}(u) \\
&= \int \mathrm{d}w D^{j^-}_{s^-t}(u)D^{j^-}_{tn^-}(wk)D^{j^+}_{s^+t^+}(u)D^{j^+}_{t^+n^+}(w_\beta)C^{j^-j^+k}_{s^-s^+p} \\
&= \int \mathrm{d}w D^{j^-}_{s^-n^-}(uwk)D^{j^+}_{s^+n^+}(uw_\beta)C^{j^-j^+k}_{s^-s^+p}
\end{aligned}
\tag{12.21}
$$

□

However, there is an other possibility to write the Baratin Oriti field in spin representation, which relies on the fact that , although behaving like a proper delta distribution with respect to the star product, under integration, the non-commutative delta function is a regular function when seen as a function on $\mathbb{R}^3$ [91, 92]. In fact, seen as a function of $\mathbb{R}^3$:

$$
\delta_{-kx^-k^{-1}}(\beta x^+) = \frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \tag{12.22}
$$

Thus, we can write:

$$
\begin{aligned}
&\langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle = \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \langle j^-; s^-, r^-|x^-\rangle\langle j^+; s^+, r^+|x^+\rangle \star \langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} dg^- dg^+ \langle j^-; s^-, r^-|g^-\rangle\langle g^-|x^-\rangle\langle j^+; s^+, r^+|g^+\rangle\langle g^+|x^+\rangle\star \\
&\star\, \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \mathrm{d}g^-\mathrm{d}g^+ \overline{D^{j^-}_{r^-s^-}}(g^-)\overline{E}_{g^-}(x^-)\overline{D^{j^+}_{r^+s^+}}(g^+)\overline{E}_{g^+}(x^+)\star \\
&\star\, \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \frac{2J_{[2j^-+1]}(|x^-|)}{i^{2j^-}|x^-|} D^{j^-}_{s^-r^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|} D^{j^+}_{s^+r^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&\star\, \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \frac{2J_{[2j^-+1]}(|x^-|)}{i^{2j^-}|x^-|} D^{j^-}_{s^-r^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|} D^{j^+}_{s^+r^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&\frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+)
\end{aligned}
\tag{12.23}
$$

Remarkably, both the coefficients $F^{j^-j^+k}_{x^-x^+q}(k)$ and the Bessel functions (12.22) convoluted with the unconstrained field, contain all the information about the simplicity constraints and the specific form of the operator implementing them.

Indeed, as noted in [50], in the new coefficients $F^{j^-j^+k}_{x^-x^+q}(k)$ the integral of the two Wigner matrices encodes a relation between the spins (j^-, j^+), which depends on the Immirzi parameter; for example $j^- = j^+$ when $\beta \in \{-1, 1\}$, namely when $\gamma \in \{0, \infty\}$. However for generic values of β, it does not enforce the spin relations $j^- = |\beta|j^+$ characteristic of the EPRL/FK models.

However, the relation between the EPRL/FK models and the Baratin Oriti model is more clear in the Bessel formulation (12.22), as the asymptotic behaviour for large spins is easier to study. As we will see in Section 12.5 and 12.8, although the Baratin Oriti model does not imply the EPRL/FK condition, the Bessel function in the large-j limit peaks on $j^- = |\beta|j^+$.

An other remarkable feature of the Baratin Oriti model is that there aren't any rationality conditions imposed on the Immirzi parameter γ.

Finally, we can write the Baratin Oriti model in terms of coherent states.

$$
\begin{aligned}
&\langle j^-; \vec{m}^-, \vec{n}^-| \otimes \langle j^-; \vec{m}^+, \vec{n}^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle = \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-) \star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \\
&\frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+)
\end{aligned}
\tag{12.24}
$$

Before proving this equation, we note that in the formalism we have used in the definition of the constrained GFT field is dual to the geometric quantization of a tetrahedron, characterized by its four constrained bivectors, and its normal. Indeed, the GFT field corresponds to a state in the Hilbert space $\mathcal{H} = L^2(SO(4)^4) \otimes L^2(SU(2)) \simeq \bigotimes_{i=1}^4 L^2_\star(\mathbb{R}^6) \otimes L^2(\mathrm{SU}(2))$, where the $L^2_\star$ spaces, which also appear as state spaces in the flux representation of loop quantum gravity [54], are spaces of functions on $\mathfrak{so}(4) \sim \mathbb{R}^6$ endowed with the scalar product $\int \mathrm{d}^6x(\bar{f} \star g)(x)$

$$
\langle f, g\rangle = \int \mathrm{d}^6x(\bar{f} \star g)(x)
$$

where $\bar{f}(x) = f(-x)$.

This Hilbert space is the result of geometric quantization of the classical configurations $\{x_j\} \in \mathfrak{so}(4)$, $k \in \mathrm{SU}(2) \sim \mathcal{S}^3$ of bivectors and normal. Simplicity constraints are then implemented by using two commuting operators: the simplicity operator $\hat{S}^\beta$ and the gauge projector $\hat{C}$ defined by (5.39), allowing the unambiguous definition of a 'geometricity operator' $\hat{G} = \hat{S}^\beta \hat{C} = \hat{C}\hat{S}^\beta$, acting by $\star$-multiplication on the states $|x^-\rangle \otimes |x^+\rangle \otimes |k\rangle \in L^2_\star(\mathbb{R}^6) \otimes L^2(\mathrm{SU}(2))$. The duality comes from the fact that all constraints are imposed by means of non-commutative delta-functions, acting as Dirac distributions for the star product, so that it effectively amounts to constrain the measures $\mathrm{d}^6 x_j$ on the classical field variables [50].

We conclude this section with the proof of (12.24):

Proof.

$$
\begin{aligned}
&\langle j^-; \vec{m}^-, \vec{n}^- | \otimes \langle j^-; \vec{m}^+, \vec{n}^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle = \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \langle j^-; \vec{m}^-, \vec{n}^- | x^- \rangle \langle j^+; \vec{m}^+, \vec{n}^+ | x^+ \rangle \star \langle x^- | \otimes \langle x^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}g^- \mathrm{d}g^+ \langle j^-; \vec{m}^-, \vec{n}^- | g^- \rangle \langle g^- | x^- \rangle \langle j^+; \vec{m}^+, \vec{n}^+ | g^+ \rangle \langle g^+ | x^+ \rangle \\
&\star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}g^- \mathrm{d}g^+ \overline{D^{j^-}_{j^- j^-}}(g^{-1}_{\vec{n}^-} g^- g_{\vec{m}^-}) \overline{E}_{g^-}(x^-) \overline{D^{j^+}_{j^+ j^+}}(g^{-1}_{\vec{n}^+} g^+ g_{\vec{m}^+}) \overline{E}_{g^+}(x^+) \\
&\star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}g^- \mathrm{d}g^+ D^{j^-}_{j^- j^-}(g^{-1}_{\vec{m}^-} g^- g_{\vec{n}^-}) E_{g^-}(x^-) D^{j^+}_{j^+ j^+}(g^{-1}_{\vec{m}^+} g^+ g_{\vec{n}^+}) E_{g^+}(x^+) \\
&\star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}\tilde{g}^- \mathrm{d}\tilde{g}^+ D^{j^-}_{j^- j^-}(\tilde{g}^-) E_{g_{\vec{m}^-} \tilde{g}^- g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{j^+ j^+}(\tilde{g}^+) E_{g_{\vec{m}^+} \tilde{g}^+ g^{-1}_{\vec{n}^+}}(x^+) \\
&\star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}\tilde{g}^- \mathrm{d}\tilde{g}^+ D^{j^-}_{j^- j^-}(\tilde{g}^-) E_{g_{\vec{m}^-} \tilde{g}^- g^{-1}_{\vec{m}^-}}(x^-) \star E_{g_{\vec{m}^-} g^{-1}_{\vec{n}^-}}(x^-) \\
&D^{j^+}_{j^+ j^+}(\tilde{g}^+) E_{g_{\vec{m}^+} \tilde{g}^+ g^{-1}_{\vec{m}^+}}(x^+) \star E_{g_{\vec{m}^+} g^{-1}_{\vec{n}^+}}(x^+) \star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}\tilde{g}^- \mathrm{d}\tilde{g}^+ D^{j^-}_{j^- j^-}(\tilde{g}^-) E_{\tilde{g}^-}(g^{-1}_{\vec{m}^-} x^- g_{\vec{m}^-}) \star E_{g_{\vec{m}^-} g^{-1}_{\vec{n}^-}}(x^-) \\
&D^{j^+}_{j^+ j^+}(\tilde{g}^+) E_{\tilde{g}^+}(g^{-1}_{\vec{m}^+} x^+ g_{\vec{m}^+}) \star E_{g_{\vec{m}^+} g^{-1}_{\vec{n}^+}}(x^+) \star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-) \star E_{g_{\vec{m}^-} g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \star E_{g_{\vec{m}^+} g^{-1}_{\vec{n}^+}}(x^+) \\
&\star \delta_{-kx^-k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-) \star E_{g_{\vec{m}^-} g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \star E_{g_{\vec{m}^+} g^{-1}_{\vec{n}^+}}(x^+) \\
&\frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+)
\end{aligned}
\tag{12.25}
$$

□

12.2 The Baratin Oriti model and the Master Constraint criterion

In this section we show the first important result of this thesis. As we have seen, both the EPRL solutions and the Alexandrov solutions satisfy the Master constraint criterion approximately with the standard choice of the Casimir operator. However, with a particular choice of the operator ordering, i.e. with a specific quantization map, they can satisfy the Master constraint criterion exactly.

Obviously, it would be better if we find a particular way to impose the simplicity constraints such that the Master constraint criterion is satisfied exactly for all the admissible quantization maps. Interestingly, the Baratin Oriti solutions precisely satisfy the Master constraint criterion exactly, independently from the choice of (well-defined) quantization map.

In order to show this important result, the first step is to observe that we can write the Master constraint criterion in metric representation as:

$$\begin{aligned} 0 = \hat{D}\cdot\hat{D}|\phi\rangle &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\hat{D}|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} \langle x^-|\otimes\langle x^+|\otimes\langle k|\phi\rangle \\ &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}[k\hat{X}^-k^{-1}+\beta\hat{X}^+]^2|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} \phi_k(x^-,x^+) \\ &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} [kx^-k^{-1}+\beta x^+]^{\star 2} \star_{\mathcal{Q}} \phi_k(x^-,x^+) \end{aligned} \tag{12.26}$$

where

$$[kx^-k^{-1}+\beta x^+]^{\star 2} = [kx^-k^{-1}+\beta x^+]\star_{\mathcal{Q}}[kx^-k^{-1}+\beta x^+] \tag{12.27}$$

Now, we consider the action of the Master constraint operator on states satisfying the Baratin Oriti constraints. Namely,

$$\begin{aligned} \hat{D}\cdot\hat{D}\hat{S}_{BO}|\phi\rangle &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} \langle x^-|\otimes\langle x^+|\otimes\langle k|\hat{S}_{BO}|\phi\rangle \\ &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}[k\hat{X}^-k^{-1}+\beta\hat{X}^+]^2|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} \delta_{-kx^-k^{-1}}(\beta x^+)\star_{\mathcal{Q}} \\ &\quad \star_{\mathcal{Q}} \phi_k(x^-,x^+) \\ &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}|x^-\rangle\otimes|x^+\rangle \star_{\mathcal{Q}} [kx^-k^{-1}+\beta x^+]^{\star 2} \star_{\mathcal{Q}} \delta_{-kx^-k^{-1}}(\beta x^+)\star_{\mathcal{Q}} \\ &\quad \star_{\mathcal{Q}} \phi_k(x^-,x^+) \\ &= 0 \end{aligned} \tag{12.28}$$

Therefore, the Baratin Oriti solutions are annihilated by the Master constraint operator, and thus they satisfy the Master constraint criterion exactly. Interestingly, the above procedure is general, as we have not specified any preferred quantization scheme.

The remarkable fact is that, as we have seen, the Baratin Oriti model allows more solutions than the EPRL model, one could ask whether the Baratin Oriti solutions are proper solutions and their contribution to the field amplitude have to be taken into account, or rather they should be overlooked. This simple result shows that they are as good as the EPRL one, since they satisfy the same criterion. Moreover, the Baratin Oriti model seems more general since it solves the Master constraint criterion exactly, for every (well-defined) quantization map, and at the same time it allows more solutions than the EPRL one, without imposing any rationality condition on the Immirzi parameter.

12.3 The Baratin Oriti model and the Gupta-Bleuler criterion

In this section we show the second important result of this thesis. Indeed, we have seen that the Baratin Oriti solutions satisfy the Master constraint criterion exactly, independently on the choice of the quantization map. In this section, we see that the Baratin Oriti solutions satisfy also the Gupta-Bleuler criterion, for every (well-defined) quantization map.

In order to show this important result, the first step is to note that we can write the Gupta-Bleuler criterion in metric representation as:

$$\begin{aligned}
0 =&\langle\psi|\hat{D}|\phi\rangle \\
=&\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\langle\psi|\hat{D}|k\rangle\otimes|x^-\rangle\otimes|x^+\rangle\star_{\mathcal{Q}}\langle x^-|\otimes\langle x^+|\otimes\langle k|\phi\rangle \\
=&\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\langle\psi|[k\hat{X}^-k^{-1}+\beta\hat{X}^+]|k\rangle\otimes|x^-\rangle\otimes|x^+\rangle\star_{\mathcal{Q}}\phi_k(x'^-,x'^+) \\
=&\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}[kx^-k^{-1}+\beta x^+]\star_{\mathcal{Q}}\bar{\psi}_k(x^-,x^+)\star_{\mathcal{Q}}\phi_k(x^-,x^+)
\end{aligned} \tag{12.29}$$

Now, we observe that the states that satisfy the Baratin Oriti simplicity constraint satisfy also the simplicity constraints imposed via a Gupta-Bleuler method.
Indeed, we have:

$$\begin{aligned}
&\langle\psi|\hat{D}\hat{S}_{BO}|\phi\rangle = \\
&=\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\langle\psi|\hat{D}|k\rangle\otimes|x^-\rangle\otimes|x^+\rangle\star_{\mathcal{Q}}\langle x^-|\otimes\langle x^+|\otimes\langle k|\hat{S}_{BO}|\phi\rangle \\
&=\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\bar{\psi}_k(x^-,x^+)\star_{\mathcal{Q}}[kx^-k^{-1}+\beta x^+]\star_{\mathcal{Q}}\delta_{-kx^-k^{-1}}(\beta x^+)\star_{\mathcal{Q}}\phi_k(x^-,x^+) \\
&=\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\bar{\psi}_k(x^-,x^+)\star_{\mathcal{Q}}[kx^-k^{-1}+\beta x^+]\star_{\mathcal{Q}}\delta_{-kx^-k^{-1}}(\beta x^+)\star_{\mathcal{Q}}\phi_k(x^-,x^+) \\
&=0
\end{aligned} \tag{12.30}$$

Therefore, the Baratin Oriti solutions satisfy the Gupta-Bleuler criterion exactly. Interestingly, the above procedure is general, as we have not specified any preferred quantization scheme.

This result has important consequences. Firstly, because it is an other confirmation that all the solutions of the Baratin Oriti model are good solutions, since they satisfy both the Master constraint and Gupta-Bleuler criterion, independently from the quantization map. The second important result is that also the Ding-Han-Rovelli model allows a finite number of extra solutions with respect to the EPRL model (but significantly less than the Batain Oriti model). However, the Ding-Han-Rovelli extra solutions, satisfying the Gupta-Bleuler criterion are generally overlooked as they seem to have a strange large-j asymptotic behaviour. In this thesis, we present the idea that both the Baratin Oriti and Ding-Han-Rovelli extra solutions are good solutions, and have a correct large-j asymptotic behaviour, due to the fact that they are weighted by particular functions that suppress solutions with suspicious semi-classical limit. For more details, in Section 12.7 we compare (a specific subset of) solutions of the two models.

12.4 Discussion

As we have seen, the Baratin Oriti solutions satisfy both the Master constraint criterion and the Gupta-Bleuler criterion. The reason is that the Baratin Oriti model imposes the simplicity constraints by means of non-commutative delta functions, making possible to follow a more standard path integral formulation. Particularly, for $\beta = 1$, i.e. for the Barrett-Crane model, this imposition is consistent as it satisfies:

$$\begin{aligned}
\hat{D}\hat{S}_{BO}|\phi\rangle &= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}\hat{D}|x^-\rangle \otimes |x^+\rangle \star_{\mathcal{Q}} \langle x^-| \otimes \langle x^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}[k\hat{X}^-k^{-1} + \hat{X}^+]|x^-\rangle \otimes |x^+\rangle \star_{\mathcal{Q}} \delta_{-kx^-k^{-1}}(x^+)\star_{\mathcal{Q}} \\
&\star_{\mathcal{Q}} \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+\mathrm{d}k}{(2\pi)^6}|x^-\rangle \otimes |x^+\rangle \star_{\mathcal{Q}} [kx^-k^{-1} + x^+] \star_{\mathcal{Q}} \delta_{-kx^-k^{-1}}(x^+)\star_{\mathcal{Q}} \\
&\star_{\mathcal{Q}} \phi_k(x^-, x^+) \\
&= 0
\end{aligned} \tag{12.31}$$

Following [49], we would like to conclude this section reconsidering the issues concerning the quantum simplicial geometry in the Barrett-Crane model, and how they are treated in the Baratin Oriti model.

Actually, we have seen that in the Baratin Oriti model, implemented in the extended GFT formalism, normal vectors to the same tetrahedron, seen in different 4-simplices are correlated, as simplicity and closure constraints are imposed covariantly. Moreover, all the linear simplicity constraints characterizing the classical theory are correctly implemented, so that in the Baratin Oriti model there are no missing constraints over the connection variables.

Regarding the fact that in the Barrett-Crane model 4-simplices speak only through face representations, i.e. triangle areas, we would like to point out that in the metric representation of the Baratin Oriti model, 4-simplices are described by ten bivector variables, correctly identified across 4-simplices sharing the same tetrahedron (and thus the same triangles). Indeed, the correct identifications are dictated by the combinatorial non-local structure characterizing the the GFT action.

Moreover, as noted in [49], in the Baratin Oriti model the properties of the non-commutative $\star$-product result in a relaxation of the parallel transport, given by the interplay between simplicity constraints and gauge covariance. Indeed, using the decomposition of the simplicity constraints into plane-waves and the properties of the $\star$-product, the unconstraint BF action and the simplicity constraints can be recombined into a single an effective action [49, 127–129], whose equations of motion do not force the identification (up to parallel transport) of bivectors across neighbouring tetrahedra.

12.5 The Baratin Oriti model and the EPRL model

In this section we consider the relation between the Baratin Oriti model and the EPRL model. Particularly, we consider the Baratin Oriti GFT field in spin representation, and we study the function imposing the simplicity constraints in the large-j limit. We will see that the function peaks on the first EPRL condition $j^- = |\beta| j^+$. Moreover, we formulate some hypothesis regarding the second EPRL condition $j = j^+ \pm j^-$ if $\beta \lessgtr 0$, from the Baratin Oriti prospective.

The first step is to write the Baratin Oriti GFT field in the spin representation, using the Bessel formulation (12.22)

$$\begin{aligned}
&\langle j^-; s^-, r^- | \otimes \langle j^+; s^+, r^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle = \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \langle j^-; s^-, r^- | x^- \rangle \langle j^+; s^+, r^+ | x^+ \rangle \star \langle x^- | \otimes \langle x^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}g^- \mathrm{d}g^+ \langle j^-; s^-, r^- | g^- \rangle \langle g^- | x^- \rangle \langle j^+; s^+, r^+ | g^+ \rangle \langle g^+ | x^+ \rangle \\
&\star \delta_{-kx^- k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \mathrm{d}g^- \mathrm{d}g^+ \overline{D^{j^-}_{r^- s^-}}(g^-) \overline{E}_{g^-}(x^-) \overline{D^{j^+}_{r^+ s^+}}(g^+) \overline{E}_{g^+}(x^+) \\
&\star \delta_{-kx^- k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \frac{2 J_{[2j^-+1]}(|x^-|)}{i^{2j^-} |x^-|} D^{j^-}_{s^- r^-} \left(e^{i\pi \frac{x^-}{|x^-|}} \right) \frac{2 J_{[2j^++1]}(|x^+|)}{i^{2j^+} |x^+|} D^{j^+}_{s^+ r^+} \left(e^{i\pi \frac{x^+}{|x^+|}} \right) \\
&\star \delta_{-kx^- k^{-1}}(\beta x^+) \star \phi_k(x^-, x^+) \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \frac{2 J_{[2j^-+1]}(|x^-|)}{i^{2j^-} |x^-|} D^{j^-}_{s^- r^-} \left(e^{i\pi \frac{x^-}{|x^-|}} \right) \frac{2 J_{[2j^++1]}(|x^+|)}{i^{2j^+} |x^+|} D^{j^+}_{s^+ r^+} \left(e^{i\pi \frac{x^+}{|x^+|}} \right) \\
&\frac{2 J_1(|kx^- k^{-1} + \beta x^+|)}{|kx^- k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+)
\end{aligned} \tag{12.32}$$

where we used the fact that, although behaving like a proper delta distribution with respect to the star product, under integration, the delta function is a regular function when seen as a function on $\mathbb{R}^3$ [91, 92]. In fact, recall that:

$$\delta_{-kx^- k^{-1}}(\beta x^+) = \frac{2 J_1(|kx^- k^{-1} + \beta x^+|)}{|kx^- k^{-1} + \beta x^+|} \tag{12.33}$$

In order to make easier the comparison with the other models we write:

$$\begin{aligned}
&\langle j^-; s^-, r^- | \otimes \langle j^+; s^+, r^+ | \otimes \langle k | \hat{S}_{BO} | \phi \rangle = \\
&= \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \frac{2 J_{[2j^-+1]}(|x^-|)}{i^{2j^-} |x^-|} D^{j^-}_{s^- r^-} \left(e^{i\pi \frac{x^-}{|x^-|}} \right) \frac{2 J_{[2j^++1]}(|x^+|)}{i^{2j^+} |x^+|} D^{j^+}_{s^+ r^+} \left(e^{i\pi \frac{x^+}{|x^+|}} \right) \\
&\frac{2 J_1(|kx^- k^{-1} + \beta x^+|)}{|kx^- k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+) \\
&= \sum \int \frac{\mathrm{d}^3 x^- \mathrm{d}^3 x^+}{(2\pi)^6} \frac{2 J_{[2j^-+1]}(|x^-|)}{i^{2j^-} |x^-|} D^{j^-}_{n^- m^-} \left(e^{i\pi \frac{x^-}{|x^-|}} \right) \frac{2 J_{[2j^++1]}(|x^+|)}{i^{2j^+} |x^+|} D^{j^+}_{n^+ m^+} \left(e^{i\pi \frac{x^+}{|x^+|}} \right) \\
&\delta_{m^- r^-} \delta_{m^+ r^+} \delta_{s^- n^-} \delta_{s^+ n^+} \frac{2 J_1(|kx^- k^{-1} + \beta x^+|)}{|kx^- k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+)
\end{aligned} \tag{12.34}$$

Now, observe that $\frac{2J_{[2j+1]}(|x|)}{i^{2j}|x|}$ is peaked about $|x| \sim j$ [96]. Particularly, in the limit $j \to \infty$, it approximates a delta function $\delta(|x| - j)$ [37], where the δ function is a delta distribution with respect to the point-wise product. So that in the limit $j^\pm \to \infty$, we have:

$$\begin{aligned}
&\lim_{j^\pm\to\infty} \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{BO}|\phi\rangle = \\
&= \sum \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \delta(|x^-| - j^-) D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \delta(|x^+| - j^+) D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&\delta_{m^-r^-}\delta_{m^+r^+}\delta_{s^-n^-}\delta_{s^+n^+} \frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|} \star \phi_k(x^-, x^+) \\
&= \sum \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} \delta(|x^-| - j^-) D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right) \delta(|x^+| - j^+) D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right) \\
&\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm n^\pm s^\pm}(x^-, x^+) \star \phi_k(x^-, x^+)
\end{aligned} \tag{12.35}$$

where we have defined the function imposing the Baratin Oriti constraints:

$$\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm n^\pm s^\pm}(x^-, x^+) = \delta_{m^-r^-}\delta_{m^+r^+}\delta_{s^-n^-}\delta_{s^+n^+} \frac{2J_1(|k\frac{x^-}{|x^-|}k^{-1}j^- + \beta\frac{x^+}{|x^+|}j^+|)}{\left|k\frac{x^-}{|x^-|}k^{-1}j^- + \beta\frac{x^+}{|x^+|}j^+\right|} \tag{12.36}$$

In the definition of $\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm s^\pm n^\pm}(x^-, x^+)$ we have implicitly considered that $|x^\pm| = j^\pm$, due to the Dirac deltas $\delta(|x^\pm| - j^\pm)$. Moreover, we will consider only the case $m^\pm = r^\pm$ and $n^\pm = s^\pm$, since for other values the function is trivially zero.

First of all, we note that since $\frac{2J_{[2j+1]}(|x|)}{i^{2j}|x|}$ is peaked about $|x| \sim j$, we have that $\frac{2J_{[1]}(|x|)}{|x|}$ is peaked about $|x| \sim 0$. Specifically, the function $\frac{2J_{[1]}(|x|)}{|x|}$ is plotted in Figure 12.1.

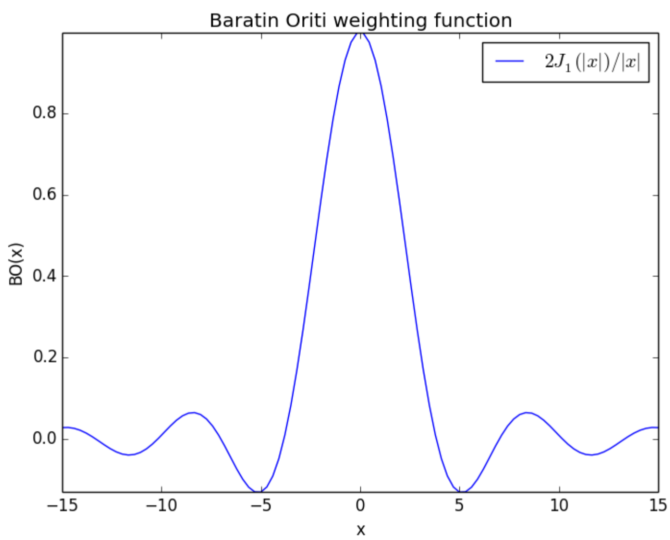

Figure 12.1: Baratin Oriti weighting function

Thus, the peak corresponds to:

$$\begin{cases} j^- = |\beta| j^+ \\ k\frac{x^-}{|x^-|}k^{-1} + sign(\beta)\frac{x^+}{|x^+|} = 0 \end{cases} \tag{12.37}$$

where we recognize the first EPRL condition $j^- = |\beta| j^+$.

Indeed, if $j^- = |\beta| j^+$ and $k\frac{x^-}{|x^-|}k^{-1} + sign(\beta)\frac{x^+}{|x^+|} = 0$ we have

$$\begin{aligned} \frac{2J_1(|k\frac{x^-}{|x^-|}k^{-1}|x|^- + \beta\frac{x^+}{|x^+|}|x^+||)}{\left|k\frac{x^-}{|x^-|}k^{-1}|x|^- + \beta\frac{x^+}{|x^+|}|x^+|\right|} &= \frac{2J_1(|k\frac{x^-}{|x^-|}k^{-1}|\beta|j^+ + \beta\frac{x^+}{|x^+|}j^+|)}{\left|k\frac{x^-}{|x^-|}k^{-1}|\beta|j^+ + \beta\frac{x^+}{|x^+|}j^+\right|} \\ = \frac{2J_1(|\beta|j^+(|k\frac{x^-}{|x^-|}k^{-1} + sign(\beta)\frac{x^+}{|x^+|}|))}{|\beta|j^+\left|k\frac{x^-}{|x^-|}k^{-1} + sign(\beta)\frac{x^+}{|x^+|}\right|} &= \left.\frac{2J_1(|x|)}{|x|}\right|_{x=0} \end{aligned} \tag{12.38}$$

Thus, the first result of this section is that in the large-j limit, the Baratin Oriti weighting function peaks on the first EPRL condition.

Secondly, we have that the condition $k\frac{\vec{x}^-}{|x^-|}k^{-1} + sign(\beta)\frac{\vec{x}^+}{|x^+|} = 0$ distinguishes the cases $\beta \lesseqgtr 0$ and it is probably related to the second EPRL condition $j = j^+ \pm j^-$. The idea is that, while the Freidel Krasnov model and the Baratin Oriti model have a clear geometric content, the EPRL model imposes the constraints only on the norms $j^\pm$ of the bivectors, expressed by the first EPRL condition $j^- = |\beta| j^+$ (which characterizes also the Freidel Krasov model. However, there is a reminiscence of the directional constraints in the second EPRL condition, selecting the $SU(2) \subset SU(2)^- \otimes SU(2)^+ \subset SO(4)$ subgroup representation.
More specifically, we expect that for $\beta < 0$, where the EPRL model and the FK model coincide, $\lim_{j^\pm\to\infty} F^{j^- j^+ k}_{x^- x^+ q}(k)$ peaks on $C^{j^- j^+ j^+ + j^-}_{x^- x^+ q}$. For $\beta > 0$ we would expect a different condition, as the Baratin Oriti model is expected to be more similar to the FK model (due to the geometric character of the imposition of the constraints), which for $\beta > 0$ differs from the EPRL model.

However, this calculation is really difficult. A strategy would be to consider the coefficients in coherent states, in fact, this is the standard procedure in asymptotic calculations [146–152], though there are different proposes, for instance see [153]. However, we leave this for future works.

Finally, we note that in the Baratin Oriti model:

$$\lim_{x\to\infty} \frac{2J_{[1]}(|x|)}{|x|} = 0 \tag{12.39}$$

so that solutions far from the EPRL condition are suppressed. In accordance with the classical theory.
Interestingly, the Baratin Oriti model does not select a particular representation of the $SU(2) \subset SU(2)^- \otimes SU(2)^+ \subset SO(4)$ subgroup.

In conclusion, we have seen that all the Baratin Oriti solutions are good solutions, since they satisfy both the Master constraint criterion and the Gupta-Bleuler criterion exactly. Interestingly, the Baratin Oriti model allows a continuous set of solutions (whereas the EPRL model implies only the solution corresponding to $j^- = |\beta| j^+$ and $j = j^+ \pm j^-$) with a non-trivial relation between j^- and j^+, and without imposing any condition on the representation of the $SU(2) \subset SU(2)^- \otimes SU(2)^+ \subset SO(4)$ subgroup. Results are in accordance with the classical theory, since the contribution of the solutions far from the EPRL conditions (which do not reproduce the correct semi-classical behaviour) is close to zero. Finally, the function implementing the Baratin Oriti constraints peaks on the first EPRL condition $j^- = |\beta| j^+$.

12.6 The Baratin Oriti model and the Alexandrov's proposal

In this section we consider the relation between the Baratin Oriti model and the solutions proposed by Alexandrov. Particularly, we study the function implementing the Baratin Oriti constraints in the spin representation and in the large-j limit. As we have seen, in the large-j limit the function $\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm n^\pm s^\pm}(x^-,x^+)$ has a non trivial dependence on the norms of the self-dual and anti-self-dual parts of the bivectors $|x^-|$ and $|x^+|$ (corresponding to j^- and j^+) and on the directions $\frac{x^-}{|x^-|}$ and $\frac{x^+}{|x^+|}$.

Now, in order to continue our analysis we need to simplify the problem. Although we loose some information about the behaviour of the complete set of solutions, we are still able to compare particular solutions of the Baratin Oriti model satisfying the Alexadrov's conditions in a specific regime. Particularly, from here on we consider only the solutions that satisfy the directional condition:

$$k\frac{x^-}{|x^-|}k^{-1} = -sign(\beta)\frac{x^+}{|x^+|} = sign\left(\frac{1-\gamma}{1+\gamma}\right)\frac{x^+}{|x^+|} \tag{12.40}$$

For simplicity, we can assume $k=\mathbb{I}$.
This corresponds to look at the maximum values of (12.36) as a function of only $|x^-|$ and $|x^+|$ (and so j^- and j^+). Indeed, we are neglecting the dependence of $\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm n^p m s^\pm}(x^-,x^+)$ on the directional parts, by fixing this dependence to be constant and corresponding to the case in which the effects are maximized. For other directions the function $\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm n^p m s^\pm}(x^-,x^+)$ (and thus the integrand) assumes values closer to zero, so that we are maximizing the integrand in order to maximize the integral.

Therefore, we consider the function imposing the Baratin Oriti constraints, satisfying the directional condition, which does not depend explicitly on $x^\pm$:

$$\mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-,j^+) = \delta_{m^-r^-}\delta_{m^+r^+}\delta_{s^-n^-}\delta_{s^+n^+}\frac{2J_1(|j^-+|\beta|j^+|)}{|j^-+|\beta|j^+|} \tag{12.41}$$

for which we have:

$$\left|\mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-,j^+)\right| \geq \left|\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm s^\pm n^\pm}(x^-,x^+)\right| \tag{12.42}$$

so that

$$\begin{aligned}
&\Bigg|\sum \mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-,j^+)\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)\delta(|x^+|-j^+)\\
&D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+)\Bigg|\\
&\geq\\
&\Bigg|\sum\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)\delta(|x^+|-j^+)\\
&D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\mathcal{BO}^{j^\pm=|x^\pm|}_{m^\pm r^\pm s^\pm n^\pm}(x^-,x^+)\star\phi_k(x^-,x^+)\Bigg|
\end{aligned} \tag{12.43}$$

In Chapter 9 we have seen that if $\beta<0$ the Alexandrov's proposal corresponds to the EPRL model, and the relation between the two models is studied in Section 12.5.
For $\beta>0$ the condition we need to impose are:

$$\begin{cases} j^- = \beta(1+j^+)\\ j = j^+(1-\beta)-\beta \end{cases} \tag{12.44}$$

The complete analysis would amount to restrict the representation labels appearing in the coefficients $F^{j^- j^+ j}_{m^- m^+ p}(k)$ appearing in (12.15), using Kronecker deltas imposing (12.44). However, since the asymptotic behaviour of the $F^{j^- j^+ j}_{m^- m^+ p}(k)$ has not been studied yet, we limit our analysis to the first Alexandrov's condition $j^- = \beta(1 + j^+)$.

Particularly, we ask what is the value of the function imposing the Baratin Oriti constraints corresponding to the choice $j^- = \beta(1 + j^+)$. Specifically, does the solution $j^- = \beta(1 + j^+)$ contribute to the Baratin Oriti GFT field? Or is it suppressed? In order to answer these questions, we can calculate the value of $\mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-, j^+)$ for $j^- = \beta(1 + j^+)$ and $\beta > 0$. Actually, with directional condition satisfied, we have:

$$\begin{aligned}\mathcal{BO}^{DIR}[\beta(1+j^+), j^+] =& \frac{2J_1(|j^- - |\beta| j^+|)}{j^- - |\beta| j^+|} = \frac{2J_1(|\beta(1+j^+) - \beta j^+|)}{\beta(1+j^+) - \beta j^+|} \\ &= \frac{2J_1(|\beta + \beta j^+ - \beta j^+|)}{(|\beta + \beta j^+ - \beta j^+|)} = \frac{2J_1(|\beta|)}{|\beta|}\end{aligned} \tag{12.45}$$

and for $0 \leq \beta \leq 1$ we have that $0.9 \leq \frac{2J_1(|\beta|)}{|\beta|} \leq 1$. Thus, the solutions of Baratin Oriti model satisfying the first Alexandrov's condition (with the directional condition satisfied) are good solutions, and they contribute significantly to the amplitude.

12.7 The Baratin Oriti model and the Ding-Han-Rovelli solutions

In this section we investigate the relation between the solutions of the Baratin Oriti model and the Ding-Han-Rovelli solutions. This comparison is particularly interesting since they both satisfy the Gupta-Bleuler criterion.
Particularly, we consider the solutions of the Baratin Oriti model, in the spin network basis, and we investigate the behaviour of the function $\mathcal{BO}^{DIR}$, satisfying the directional condition $k\frac{x^-}{|x^-|}k^{-1} = -sign(\beta)\frac{x^+}{|x^+|}$, in the large-$j$ limit. Although we loose some information about the complete behaviour of the function, this makes easier the comparison with the Ding-Han-Rovelli model.
Moreover, we give a numerical example for both $\gamma < 1$ and $\gamma > 1$. We will see that, in both cases, the solutions are weighted by the function implementing the simplicity constraints, $\mathcal{BO}^{DIR}$, which, as we have seen in Section 12.5, peaks on the EPRL solutions.
However, we know that the Baratin Oriti model allows more solutions with a non-trivial relation between j^- and j^+, without imposing any condition on the representation of the $SU(2) \subset SU(2)^- \otimes SU(2)^+ \subset SO(4)$ subgroup, and without forcing the Immirzi parameter to be a rational number. Furthermore, results are in accordance with the classical theory, since the contribution of the solutions far from the EPRL, leading to a strange large-j asymptotic limit, is close to zero.

In order to compare the two model, we also present the solutions found by Ding, Han and Rovelli, in the spin network basis, and in the large-j limit.
The function implementing the simplicity constraints is a Clebsch-Gordan function which imposes a specific relation between representations of $SU(2)^-, SU(2)^+$ and $SU(2) \subset SU(2)^- \otimes SU(2)^+ \subset SO(4)$. However, as we have seen in Chapter 10, it allows more solutions than the EPRL model, with a non-trivial relation between j^- and j^+ parametrized by an extra degree of freedom r, which has two different domains of definition, although they are not disjoint sets, $(\gamma - 1)j \leq r \leq \gamma j$ and $\gamma j \leq r \leq (1 + \gamma)j$.
Remarkably, we note that if $r = (\gamma - 1)j$ (for $\gamma > 1$) or if $r = 0$ (for $\gamma < 1$) we get the EPRL solutions.

Finally, we give a numerical example for both $\gamma < 1$ and $\gamma > 1$, considering the different domains of the parameter r.
In order to make a comparison with the Baratin Oriti model, we study the function implementing the Ding-Han-Rovelli constraints, the Clebsch-Gordan function denoted by $\mathcal{DHR}$, in the large-j limit and for a specific choice of the labels in the Clebsh-Gordon coefficients, that maximize $\mathcal{DHR}$. Despite the fact we loose some information about the complete behaviour of the function, we still be able to get some important informations.

Indeed, if we consider the domain $(\gamma - 1)j \leq r \leq \gamma j$, the function $\mathcal{DHR}$ peaks on the EPRL solutions, while the contribution of the solutions far from the EPRL conditions is close to zero, in accordance with the classical theory.

If we consider the domain $(\gamma-1)j \leq r \leq \gamma j$, the function $\mathcal{DHR}$ is significantly suppressed, indicating that the Ding-Han-Rovelli solutions in this case are good solutions of the constraints, but in facts they don't contribute to the field amplitude. This is in accordance with the classical limit.

An other important aspect we would like to point out regards the extra solutions. Actually, the Ding-Han-Rovelli solutions are labelled by the extra degree of freedom r which can assume only non-negative integer values and has a finite domain of definition: $(\gamma-1)j \leq r \leq (1+\gamma)j$ or $0 \leq r \leq (1+\gamma)j$, according if $\gamma \lesseqgtr 1$. At the same time, the Baratin Oriti model is more general, since extra solutions can be rewritten in terms of a continuum parameter $r \in \mathbb{R}$. Therefore, the Ding-Han-Rovelli solutions are a subset of the Baratin Oriti solutions.
Furthermore, in the regime we have considered (large-j limit, with the Baratin Oriti solutions satisfying the directional condition and the specific choice of the Clebsh-Goradan labels) the Ding-Han-Rovelli amplitudes are very similar to the Baratin Oriti amplitudes, although they are not the just discretized counterpart. In this thesis we cannot argue whether they do not coincide due to the approximations we have considered or rather if the two models are just different (for example they could assign different amplitudes to the same solution).

Particularly, we note a general difference in the values of the peaks for the two models. In the Baratin Oriti model the maximum of the weighting function, $\mathcal{BO}^{DIR}$, is 1 for both $\gamma < 1$ and $\gamma > 1$ (the peak corresponds to the solution with $j^- = |\beta|j^+$).
Interestingly, the peak of the weighting function of the Ding-Han-Rovelli model, $\mathcal{DHR}$, for the domain $(\gamma - 1)j \leq r \leq \gamma j$ corresponds to:

$$\begin{cases} r = (\gamma - 1)j & for \;\; \gamma > 1 \\ r = 0 & for \;\; \gamma < 1 \end{cases} \tag{12.46}$$

i.e. the first EPRL condition, $j^- = |\beta|j^+$.
However, the maximum is 1 when $\gamma < 1$, while for $\gamma > 1$ the maximum is always less than 1. This may be related to the fact that the EPRL model and the FK model coincide when $\gamma < 1$, while they differ when $\gamma > 1$. Further investigation is needed.

Lets start our analysis.
First we want to find some particular Baratin Oriti solutions that we can easily study. For this purpose, we can consider the solutions satisfying the directional condition (12.40), $\mathcal{BO}^{DIR}$. As remarked above, although we loose some information about the behaviour of the complete set of solutions, we are still able to compare particular solutions of the Baratin Oriti model to some specific solutions of the Ding-Han-Rovelli model.

Recall that the function imposing the Baratin Oriti constraints, satisfying the directional condition is:

$$\mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-, j^+) = \delta_{m^-r^-}\delta_{m^+r^+}\delta_{s^-n^-}\delta_{s^+n^+}\frac{2J_1(|j^- + |\beta|j^+|)}{|j^- + |\beta|j^+|} \tag{12.47}$$

As have have noted before, it does not depend explicitly on $x^\pm$, and we assume $m^\pm = r^\pm$ and

$n^\pm = s^\pm$, so that:

$$\lim_{j^\pm\to\infty}\langle j^-;s^-,r^-|\otimes\langle j^+;s^+,r^+|\otimes\langle k|\hat{S}^{DIR}_{BO}|\phi\rangle =$$
$$= \sum\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\delta(|x^+|-j^+)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)$$
$$\delta_{m^-r^-}\delta_{m^+r^+}\delta_{s^-n^-}\delta_{s^+n^+}\frac{2J_1(|j^-+|\beta|j^+|)}{|j^-+|\beta|j^+|}\star\phi_k(x^-,x^+) \tag{12.48}$$
$$= \sum\mathcal{BO}^{DIR}_{m^\pm r^\pm n^\pm s^\pm}(j^-,j^+)\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)\delta(|x^+|-j^+)$$
$$D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+)$$

Particularly, we will study the behaviour of (12.47), in the particular regime described above, as a function of j^-, having fixed the value of j^+, and we will compare it to some solutions of the Ding-Han-Rovelli model that we are going to define. However, before doing so, in order to make more evident the analogies between the Baratin Oriti particular solutions and the Ding-Han-Rovelli solutions, we define the continuous positive parameters r, and j such that:

$$\begin{cases} j^- = \frac{1-\gamma}{2}j + r \\ j^+ = \frac{1+\gamma}{2}j \end{cases} \tag{12.49}$$

so that when $\gamma < 1$ ($\gamma > 1$) if $r = 0$ ($r = (\gamma-1)j$) we have the EPRL solutions, while different values of r parametrize other Baratin Oriti solutions.
Finally, to simplify further the comparison we choose $m^- = r^- = \frac{1-\gamma}{2}j$, $m^+ = r^+ = \frac{1+\gamma}{2}j$ and $n^\pm = s^\pm$.
Thus for $\gamma < 1$ we study:

$$\mathcal{BO}^{DIR}_{\gamma<1}(r) = \frac{2J_1(|j^-+\beta j^+|)}{|j^-+\beta j^+|} = \frac{2J_1(|\frac{1-\gamma}{2}j+r+\frac{\gamma-1}{2}j|)}{|\frac{1-\gamma}{2}j+r+\frac{\gamma-1}{2}j|} = \frac{2J_1(|r|)}{|r|} \tag{12.50}$$

For $\gamma > 1$ we have:

$$\mathcal{BO}^{DIR}_{\gamma>1}(r) = \frac{2J_1(|j^- - \beta j^+|)}{|j^- - \beta j^+|} = \frac{2J_1(|\frac{1-\gamma}{2}j+r-\frac{\gamma-1}{2}j|)}{|\frac{1-\gamma}{2}j+r-\frac{\gamma-1}{2}j|} = \frac{2J_1(|(1-\gamma)j+r|)}{|(1-\gamma)j+r|} \tag{12.51}$$

Now, lets consider the Ding-Han-Rovelli solutions in spin representation, which can re-written, by inserting a resolution of the identity in the non-commutative metric variables, as:

$$\langle j^-;m^-,n^-|\otimes\langle j^-;m^+,n^+|\otimes\langle k|\hat{S}_{DHR}|\phi\rangle =$$
$$= \sum\overline{C^{j^-j^+j}_{m^-m^+q}}(k)C^{j^-j^+j}_{r^-r^+q}(k)\overline{C^{j^-j^+l}_{s^-s^+p}}(k)C^{j^-j^+l}_{n^-n^+p}(k)$$
$$\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\frac{2J_{[2j^-+1]}(|x^-|)}{i^{2j^-}|x^-|}D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|}D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)$$
$$\left[\delta^{j^-[\frac{1-\gamma}{2}j+r]}\delta^{j^+[\frac{1+\gamma}{2}j]}\right]\star\phi_k(x^-,x^+) \tag{12.52}$$
$$= \sum\delta_{n^-s^-}\delta_{n^+s^+}\overline{C^{j^-j^+j}_{m^-m^+p}}(k)C^{j^-j^+j}_{r^-r^+p}(k)$$
$$\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\frac{2J_{[2j^-+1]}(|x^-|)}{i^{2j^-}|x^-|}D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\frac{2J_{[2j^++1]}(|x^+|)}{i^{2j^+}|x^+|}D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)$$
$$\left[\delta^{j^-[\frac{1-\gamma}{2}j+r]}\delta^{j^+[\frac{1+\gamma}{2}j]}\right]\star\phi_k(x^-,x^+)$$

where we have the two domains:

$$\begin{cases} Domain & \text{I: } (\gamma-1)j \le r \le \gamma j \quad (12.53) \\ Domain & \text{II: } \gamma j \le r \le (1+\gamma)j \quad (12.54) \end{cases}$$

As we have seen in the previous sections, in the limit $j \to \infty$, the function $\frac{2J_{[2j+1]}(|x|)}{i^{2j}|x|}$ approximates a delta function $\delta(|x|-j)$ [37], where the δ function is a delta distribution with respect to the point-wise product. Thus, we have:

$$\begin{aligned}
&\lim_{j^\pm \to \infty} \langle j^-; s^-, r^-| \otimes \langle j^+; s^+, r^+| \otimes \langle k|\hat{S}_{DHR}|\phi\rangle = \\
&= \sum \delta_{n^-s^-}\delta_{n^+s^+}\overline{C^{j^-j^+j}_{m^-m^+p}}(k)C^{j^-j^+j}_{r^-r^+p}(k)\left[\delta^{j^-[\frac{1-\gamma}{2}j+r]}\delta^{j^+[\frac{1+\gamma}{2}j]}\right] \\
&\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)\delta(|x^+|-j^+)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+) \qquad (12.55)\\
&= \sum \mathcal{DHR}^{j^-j^+}_{m^\pm r^\pm n^\pm s^\pm}\int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(|x^-|-j^-)\delta(|x^+|-j^+) \\
&D^{j^-}_{n^-m^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{j^+}_{n^+m^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+)
\end{aligned}$$

where we have defined the function imposing the Ding-Han-Rovelli constraints:

$$\mathcal{DHR}^{j^-j^+}_{m^\pm r^\pm n^\pm s^\pm} = \delta_{n^-s^-}\delta_{n^+s^+}\overline{C^{j^-j^+j}_{m^-m^+p}}(k)C^{j^-j^+j}_{r^-r^+p}(k)\left[\delta^{j^-[\frac{1-\gamma}{2}j+r]}\delta^{j^+[\frac{1+\gamma}{2}j]}\right] \qquad (12.56)$$

Now, recall that, regarding the Baratin Oriti model, we have not considered the whole set of solutions but rather only the solutions satisfying the directional condition (12.40), in order to simplify the problem. Analogously, in order to study the Ding-Han-Rovelli solutions, we focus on the particular case $m^+ = r^+ = j^+ = \frac{1+\gamma}{2}j$, $m^- = r^- = \frac{1-\gamma}{2}j$, $n^\pm = s^\pm$ and $p = j$. These choices correspond to looking at the peaks of the Clebsch-Gordan function (12.56).
Moreover, we observe that if we fix j^+ then j is given by:

$$j = \frac{2}{1+\gamma}j^+ \qquad (12.57)$$

Therefore, fixing the values of j and γ, we can study these functions as functions of r (and so as a function of j^-), distinguishing the cases $\gamma < 1$ and $\gamma > 1$.

In conclusion, the function imposing the Ding-Han-Rovelli that we consider is:

$$\mathcal{DHR}(r) = \overline{C^{[\frac{1-\gamma}{2}j+r]\frac{1+\gamma}{2}jj}_{\frac{1-\gamma}{2}j\frac{1+\gamma}{2}jj}}C^{[\frac{1-\gamma}{2}j+r]\frac{1+\gamma}{2}jj}_{\frac{1-\gamma}{2}j\frac{1+\gamma}{2}jj} \qquad (12.58)$$

where r satisfies (12.53) or (12.54).

Thus, we have now arrived to two expressions that are easily comparable. Indeed, we consider a particular subset of the solutions of the Ding-Han-Rovelli model, given by the following expression:

$$\begin{aligned}
&\lim_{j\to\infty}\langle(1-\gamma)j/2+r;(1-\gamma)j/2,n^-|\otimes\langle(1+\gamma)j/2;(1+\gamma)j/2,n^+|\otimes\langle k|\hat{S}_{DHR}|\phi\rangle = \\
&= \sum \mathcal{DHR}(r)\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta\left(|x^-|-\frac{1-\gamma}{2}j-r\right)\delta\left(|x^+|-\frac{1+\gamma}{2}j\right) \\
&D^{\frac{1-\gamma}{2}j+r}_{\frac{1-\gamma}{2}jm^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{\frac{1+\gamma}{2}j}_{\frac{1+\gamma}{2}jm^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+)
\end{aligned}$$

and we compare them to an other subset of solution of the Baratin Oriti model, given by:

$$\lim_{j\to\infty}\langle(1-\gamma)j/2+r;(1-\gamma)j/2,n^-|\otimes\langle(1+\gamma)j/2;(1+\gamma)j/2,n^+|\otimes\langle k|\hat{S}_{BO}|\phi\rangle =$$
$$=\sum \mathcal{BO}^{DIR}_{\gamma\lessgtr 1}(r)\int\frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta\left(|x^-|-\frac{1-\gamma}{2}j-r\right)\delta\left(|x^+|-\frac{1+\gamma}{2}j\right)$$
$$D^{\frac{1-\gamma}{2}j+r}_{\frac{1-\gamma}{2}jm^-}\left(e^{i\pi\frac{x^-}{|x^-|}}\right)D^{\frac{1+\gamma}{2}j}_{\frac{1+\gamma}{2}jm^+}\left(e^{i\pi\frac{x^+}{|x^+|}}\right)\star\phi_k(x^-,x^+)$$

We note that the differences are only in the functions implementing the constraints, thus in order to compare the two set of solutions we need to study $\mathcal{DHR}(r)$ and $\mathcal{BO}^{DIR}_{\gamma\lessgtr 1}(r)$. Particularly, fixing the value of j (and correspondingly j^+) and γ, we can study these functions as functions of r (and so as functions of j^-). Importantly, we have to distinguish the cases $\gamma<1$ or $\gamma>1$, and the domains (12.53) or (12.54).

12.7.1 A numerical example

In this section we give a numerical example of the functions $\mathcal{BO}^{DIR}_{\gamma\lessgtr 1}(r)$ and $\mathcal{DHR}(r)$.
First, we consider Ding-Han-Rovelli solutions, characterized by $\mathcal{DHR}(r)$, distinguishing the cases $\gamma<1$ and $\gamma>1$ and the *Domain I* (12.53) and *Domain II* (12.54).
For example, for the *Domain I*, (12.53), if $j=10$ and $\gamma=0.5<1$ the result is plotted in Figure 12.2, while the result in the case $j=10$ and $\gamma=2>1$ is plotted in Figure 12.3.

We can see that in the case $\gamma<1$ the function peaks when $r=0$, corresponding to $j^-=\frac{1-\gamma}{2}j=\frac{1-\gamma}{1+\gamma}j^+=|\beta|j^+$, i.e. the first EPRL condition. Particularly, the value of the peak is 1. As r increases the function rapidly goes to zero.

For $\gamma>1$ the function peaks when $r=(\gamma-1)j=10$ corresponding to $j^-=\frac{\gamma-1}{2}j=\frac{\gamma-1}{1+\gamma}j^+=|\beta|j^+$, i.e. the first EPRL condition. Particularly, the value of the peak is 0.681. As r increases the function rapidly goes to zero.

We note that a general feature of the Ding-Han-Rovelli model is that the maximum of the weighting function for the domain $0\le r\le\gamma j$, with $\gamma<1$, is 1, while for $(\gamma-1)j\le r\le\gamma j$, with $\gamma>1$, the maximum is always less than 1.

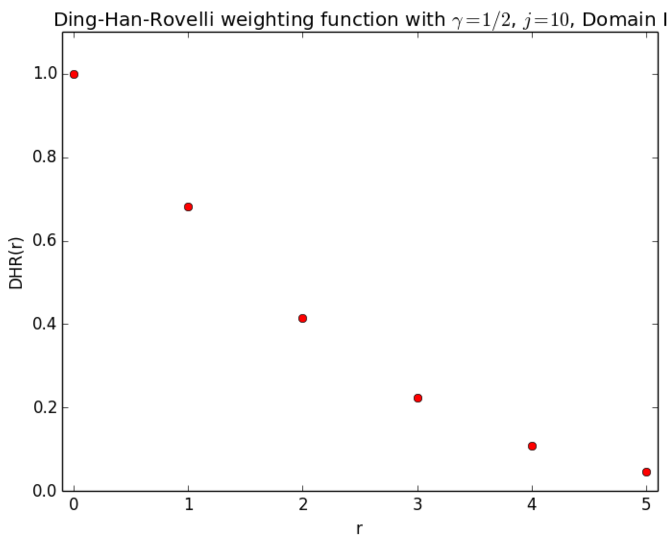

Figure 12.2: Ding-Han-Rovelli weighting function with $\gamma=\frac{1}{2}$, $j=10$, Domain I

For the *Domain II*, (12.54), if $j=10$ and $\gamma=0.5<1$ the result is plotted in Figure 12.4, while the result in the case $j=10$ and $\gamma=2>1$ is plotted in Figure 12.5, where we have

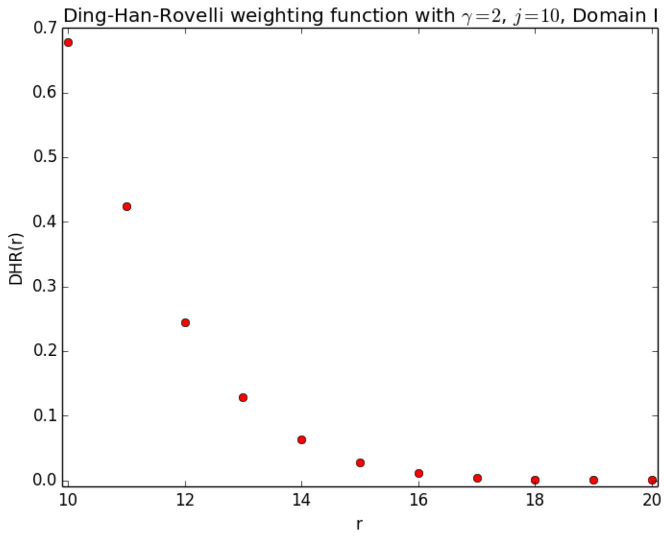

Figure 12.3: Ding-Han-Rovelli weighting function with $\gamma = 2$, $j = 10$, Domain I

considered a logarithmic scale in the ordinate axis, so that we can easily note the exponential decay characterizing the Ding-Han-Rovelli weighting function as r increases.
The peaks are 0.045652 (when $r = \gamma j = 5$) for $\gamma = 0.5 < 1$ and 0.000111 (when $r = \gamma j = 20$) for $\gamma = 2 > 1$.
Remarkably, we note that the Ding-Han-Rovelli function that imposes the simplicity constraints for the *Domain II*, (12.54), is significantly suppressed.
The reason is that for the *Domain II*, (12.54), we have $j^+ \leq j^- \leq \frac{3+\gamma}{1+\gamma} j^+$, and this condition is really far from the first EPRL condition $j^- = \left|\frac{\gamma-1}{\gamma+1}\right| j^+$, which gives the correct classical relation.

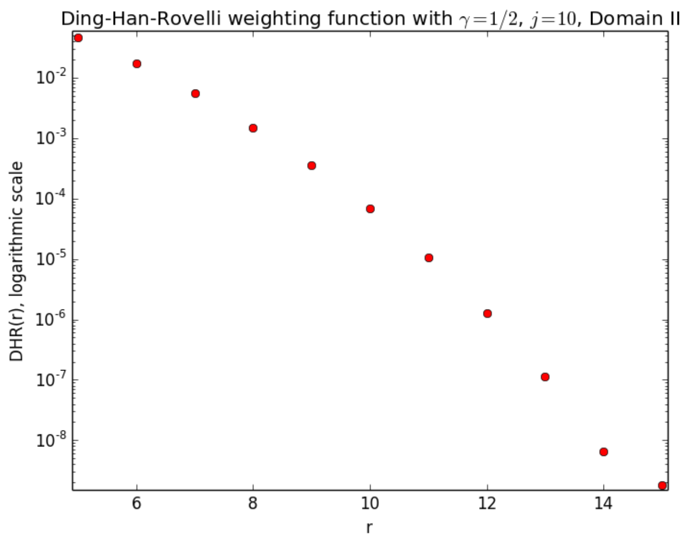

Figure 12.4: Ding-Han-Rovelli weighting function with $\gamma = \frac{1}{2}$, $j = 10$, Domain II

Therefore, the Ding-Han-Rovelli model allows solutions with a non trivial relation between j^- and j^+ ($j^- = -\beta j^+ + r$, with r such that (12.53) or (12.54)), but it assigns a different weight to these solutions given by the function $\mathcal{DHR}$, (12.56). Particularly, for the *Domain I* of the parameter r, given in (12.53), the peak of the function $\mathcal{DHR}$ corresponds to the first EPRL

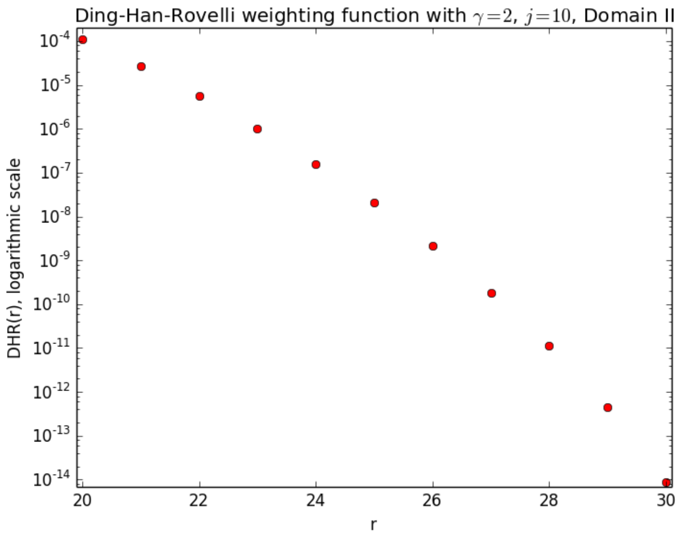

Figure 12.5: Ding-Han-Rovelli weighting function with $\gamma = 2$, $j = 10$, Domain II

condition, while the contribution of the solutions far from the first EPRL condition is close to zero.

Now, we consider the Baratin Oriti solutions, characterized by $\mathcal{BO}^{DIR}_{\gamma \lesseqgtr 1}(r)$, distinguishing the cases $\gamma < 1$ and $\gamma > 1$, in the same range of parameters of the previous analysis. Specifically, if $j = 10$, $\gamma = 0.5 < 1$ and we choose the *Domain I*, (12.53), the result is plotted in Figure 12.6, while the result in the case $j^+ = 10$ and $\gamma = 2 > 1$, for the same domain is plotted in Figure 12.7.
We can see that in the case $\gamma < 1$ the function peaks when $r = 0$, corresponding to $j^- = \frac{1-\gamma}{2} j = \frac{1-\gamma}{1+\gamma} j^+ = |\beta| j^+$, i.e. the first EPRL condition. Particularly, the value of the peak is 1. As r increases the function rapidly goes to zero.
For $\gamma > 1$ the function peaks when $r = (\gamma - 1)j = 10$, corresponding to $j^- = \frac{\gamma-1}{2} j = \frac{\gamma-1}{1+\gamma} j^+ = |\beta| j^+$, i.e. the first EPRL condition. Again, the value of the peak is 1 and the function rapidly goes to zero as r increases.

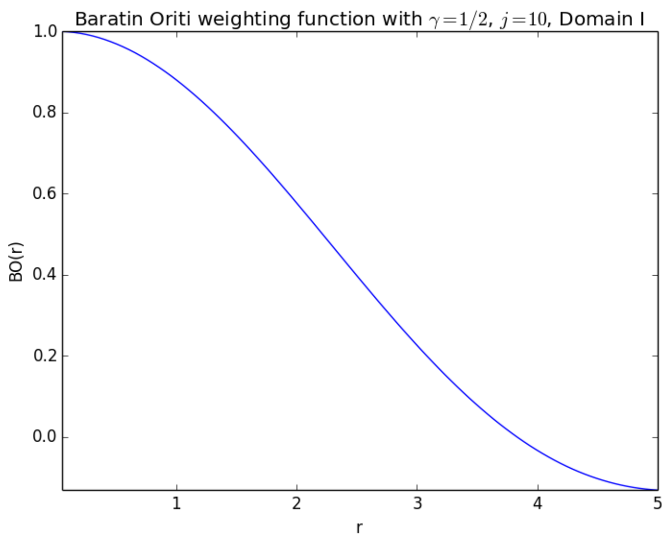

Figure 12.6: Baratin Oriti weighting function with $\gamma = \frac{1}{2}$, $j = 10$, Domain I

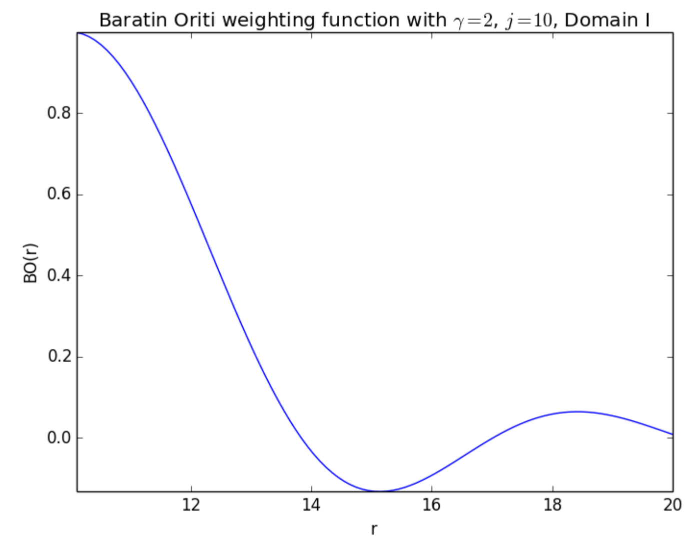

Figure 12.7: Baratin Oriti weighting function with $\gamma = 2$, $j = 10$, Domain I

Therefore, the Baratin Oriti model allows solutions with a non trivial relation between j^- and j^+, but it assigns a different weight to these solutions given by the function $\mathcal{BO}$, (12.36). Particularly, also solutions which would lead to a strange semi-classical behaviour are good solutions of the constraints, as they are suppressed in the large-j limit by the function (12.36).

Actually, we can consider the Baratin Oriti solutions in the *Domain II* and compare the results with the Ding-Han-Rovelli model, although in the Baratin Oriti model there is no upper bound to the domain of the continuum parameter, since $r \in [0, +\infty)$. Specifically, if $j = 10$, $\gamma = 0.5 < 1$ and we choose the second domain (12.54) the result is plotted in Figure 12.8, while the result in the case $j^+ = 10$ and $\gamma = 2 > 1$, for the same domain is plotted in Figure 12.9. In both the cases we observe that as r increases the Baratin Oriti weighting function oscillates closer to zero, in accordance with the semi-classical theory.

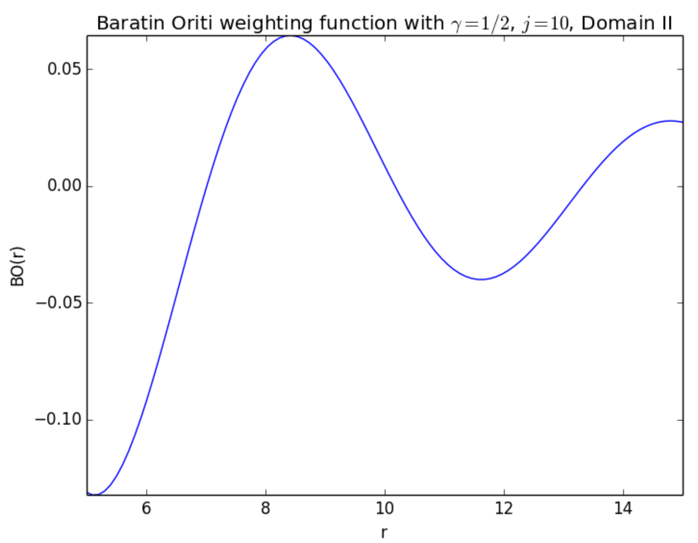

Figure 12.8: Baratin Oriti weighting function with $\gamma = \frac{1}{2}$, $j = 10$, Domain II

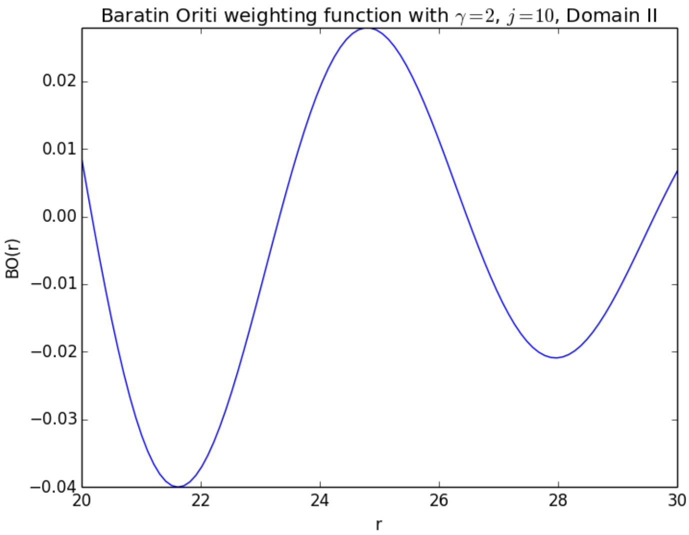

Figure 12.9: Baratin Oriti weighting function with $\gamma = 2$, $j = 10$, Domain II

12.8 The Baratin Oriti model and the Freidel Krasnov model

In this section we investigate the relation between the Baratin Oriti model and the Freidel Krasnov model in the large-j limit. Particularly, the comparison is interesting since both the models impose geometricity conditions directly on bivectors. Particularly, the Baratin Oriti model demands that quantum bivectors satisfy the simplicity constraints, whereas the Freidel Krsnov model imposes simplicity conditions on semi-classical bivectors.

Therefore in asymptotic large-j limit, we expect that they somehow coincide. Indeed, we will show that, in this limit, the Baratin Oriti weighting function peaks on the Freidel Krasnov conditions. However, the Baratin Oriti model is more general as it allows more solutions, and it does not imply any rationality condition on the Immirzi parameter.

In order to show that this is really the case, the strategy is to write both the models in coherent states.

Particularly, in Section 12.1, we have shown that the Baratin Oriti solutions in coherent states read (12.24):

$$\begin{aligned}
&\langle j^-;\vec{m}^-,\vec{n}^-|\otimes\langle j^-;\vec{m}^+,\vec{n}^+|\otimes\langle k|\hat{S}_{BO}|\phi\rangle = \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-)\star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+)\star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \\
&\frac{2J_1(|kx^-k^{-1}+\beta x^+|)}{|kx^-k^{-1}+\beta x^+|}\star\phi_k(x^-,x^+)
\end{aligned} \tag{12.59}$$

Moreover, we can write the Freidel Krasnov solutions as:

$$\begin{aligned}
&\langle j^-;\vec{m}^-,\vec{n}^-|\otimes\langle j^+;\vec{m}^+,\vec{n}^+|\otimes\langle k|\hat{S}_{FK}|\phi\rangle = \\
&= \int \frac{d^3x^-d^3x^+}{(2\pi)^6} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-)\star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-) D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+)\star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \\
&\delta(k\vec{m}^-k^{-1}+sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+}\star\phi_k(x^-,x^+)
\end{aligned} \tag{12.60}$$

Proof.

$$
\begin{aligned}
&\langle j^-;\vec{m}^-,\vec{n}^-| \otimes \langle j^+;\vec{m}^+,\vec{n}^+| \otimes \langle k|\hat{S}_{FK}|\phi\rangle = \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}g^-\mathrm{d}g^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&\langle j^-;\vec{m}^-,\vec{n}^-|g^-\rangle\langle g^-|x^-\rangle\langle j^+;\vec{m}^+,\vec{n}^+|g^+\rangle\langle g^+|x^+\rangle \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}g^-\mathrm{d}g^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&\overline{D^{j^-}_{j^-j^-}}(g^{-1}_{\vec{n}^-}g^-g_{\vec{m}^-})\overline{E}_{g^-}(x^-)\overline{D^{j^+}_{j^+j^+}}(g^{-1}_{\vec{n}^+}g^+g_{\vec{m}^+})\overline{E}_{g^+}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}g^-\mathrm{d}g^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&D^{j^-}_{j^-j^-}(g^{-1}_{\vec{m}^-}g^-g_{\vec{n}^-})E_{g^-}(x^-)D^{j^+}_{j^+j^+}(g^{-1}_{\vec{m}^+}g^+g_{\vec{n}^+})E_{g^+}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&D^{j^-}_{j^-j^-}(\tilde{g}^-)E_{g_{\vec{m}^-}\tilde{g}^-g^{-1}_{\vec{n}^-}}(x^-)D^{j^+}_{j^+j^+}(\tilde{g}^+)E_{g_{\vec{m}^+}\tilde{g}^+g^{-1}_{\vec{n}^+}}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&D^{j^-}_{j^-j^-}(\tilde{g}^-)E_{g_{\vec{m}^-}\tilde{g}^-g^{-1}_{\vec{m}^-}}(x^-) \star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-) \\
&D^{j^+}_{j^+j^+}(\tilde{g}^+)E_{g_{\vec{m}^+}\tilde{g}^+g^{-1}_{\vec{m}^+}}(x^+) \star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\mathrm{d}\tilde{g}^-\mathrm{d}\tilde{g}^+\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \\
&D^{j^-}_{j^-j^-}(\tilde{g}^-)E_{\tilde{g}^-}(g^{-1}_{\vec{m}^-}x^-g_{\vec{m}^-}) \star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-) \\
&D^{j^+}_{j^+j^+}(\tilde{g}^+)E_{\tilde{g}^+}(g^{-1}_{\vec{m}^+}x^+g_{\vec{m}^+}) \star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}\delta(km^-k^{-1} + sign[\beta]m^+)\delta^{j^-|\beta|j^+} \\
&D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-) \star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-)D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \star \phi_k(x^-,x^+) \\
&= \int \frac{\mathrm{d}^3x^-\mathrm{d}^3x^+}{(2\pi)^6}D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-) \star E_{g_{\vec{m}^-}g^{-1}_{\vec{n}^-}}(x^-)D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \star E_{g_{\vec{m}^+}g^{-1}_{\vec{n}^+}}(x^+) \\
&\delta(k\vec{m}^-k^{-1} + sign[\beta]\vec{m}^+)\delta^{j^-|\beta|j^+} \star \phi_k(x^-,x^+)
\end{aligned} \tag{12.61}
$$

□

Now, we use the property that in the limit $j \to \infty$ the function:

$$\langle x|j;\vec{m},\vec{m}\rangle = \int \mathrm{d}gE_g(x)D^j_{jj}(g^{-1}_{\vec{m}}gg_{\vec{m}}) = D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+) \tag{12.62}$$

peaks sharply at $x = j\vec{m}$.
Thus, in the limit $j^\pm \to \infty$, the function implementing the Baratin Oriti constraints reads:

$$
\begin{aligned}
&\lim_{j^\pm \to \infty} D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(x^-)D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(x^+)\frac{2J_1(|kx^-k^{-1} + \beta x^+|)}{|kx^-k^{-1} + \beta x^+|}) \\
&= D^{j^-}_{[\vec{m}^-][\vec{m}^-]}(j^-\vec{m}^-)D^{j^+}_{[\vec{m}^+][\vec{m}^+]}(j^+\vec{m}^+)\frac{2J_1(|kj^-\vec{m}^-k^{-1} + sign[\beta]|\beta|j^+\vec{m}^+|)}{(|kj^-\vec{m}^-k^{-1} + sign[\beta]|\beta|j^+\vec{m}^+|)}
\end{aligned} \tag{12.63}
$$

Finally, recall that since $\frac{2J_{[2j+1]}(|x|)}{i^{2j}|x|}$ is peaked about $|x| \sim j$, we have that $\frac{2J_{[1]}(|x|)}{|x|}$ is peaked about $|x| \sim 0$, i.e. when:

$$\begin{cases} j^- = |\beta| j^+ \\ k\vec{m}^- k^{-1} + sign[\beta]\vec{m}^+ = 0 \end{cases} \tag{12.64}$$

i.e. precisely the Freidel Krasnov constraints.
Finally, we recall the fact that in the Baratin Oriti model:

$$\lim_{x \to \infty} \frac{2J_{[1]}(|x|)}{|x|} = 0 \tag{12.65}$$

so that solutions far from the Freidel Krasnov conditions are suppressed. In accordance with the classical theory.

In order to summarize our analysis, we have seen that the Freidel Krasnov implements the simplicity constraints by means of a Kronecker delta imposing the condition $j^- = |\beta| j^+$, and a Dirac delta $\delta(k\vec{m}^- k^{-1} + sign[\beta]\vec{m}^+)$.

The Kronecker delta in the large-j limit imposes a condition on the norms of the self-dual $|x^+|$ and anti-self-dual $|x^-|$ parts of the bivector, while the Dirac delta imposes a directional condition on $\vec{m}^-$ and $\vec{m}^+$ describing, in the large-j limit the directions of the self-dual $\frac{x^+}{|x^+|}$ and anti-self-dual $\frac{x^-}{|x^-|}$ parts of the bivector.

Regarding the Baratin Oriti model, we have seen that the $\star$-delta implementing the simplicity constraints can be seen (with the non-commutative pointwise product) as a particular Bessel function that peaks on the Freidel Krasnov condition. The difference is that the Freidel Krasnov model implies the constraints by means of Delta functions, which are 1 when the conditions are satisfies and zero otherwise. The Baratin Oriti model is then more general, since it implements the constraints by means of a particular function that peaks when the Freidel Krasnov conditions are satisfied, but, at the same time, is broadened with respect to Dirac and Kronecker deltas, allowing a continuous set of solutions, that contribute non-trivially to the field amplitude. However, in accordance to the classical limit, solutions far from the peak are significantly suppressed.

Chapter 13

Conclusions

In the thesis we have considered several Spin Foam models for four dimensions Riemannian Quantum Gravity, particularly in the extended GFT formalism.

The reason we have chosen to focus on the Euclidean signature can be traced back to the difficulties dealing with non-compact groups. Specifically, the non-commutative Fourier transform, which plays a crucial role in the thesis, has not been yet defined for the Lorentz group. However, we precise that Euclidean gravity is deeply studied in Quantum Gravity literature in parallel with the Lorentzian case, as it is a testing ground for new ideas.

The standard strategy in order to formulate a GFT model for gravity is to start from the Holst-Plebanski formulation of gravity, i.e. a constrained BF theory. As we have seen in Chapter 5, BF theory, being a topological field theory, can be easily quantized in the Spin Foam and Group Field Theory framework. Moreover, results are independent on the discretization, and the theory is characterized by topological invariants.

However, the non-trivial aspect is the imposition of the simplicity constraints, which break the topological invariance and restore the local degrees of freedom, typical of General Relativity.

As we have described in Chapter 6 and 7, in order to implement the simplicity constraints in Spin Foam models, we need, firstly, to discretize the classical constraints of the Plebanski formulation (or a linearised version), and, secondly, we quantize them using geometric quantization tools, in order to obtain an operatorial equation. Several models have been proposed, imposing the quantum simplicity constraints differently. Actually, different criteria for the implementation of the constraints have been considered in the literature, for example strong imposition on quantum states or weak on matrix elements.

In Chapter 8, we have studied the Riemannian EPRL model, whose solutions satisfy the Master constraint criterion. Specifically, we have seen that with the standard choice of the spectrum of the Casimir operators, the solutions of the EPRL model minimize the Master constraint operator, without solving it exactly. Remarkably, the spectrum of the Casimir operators depends on the choice of the operator ordering, and thus on the choice of the quantization map.

However, modifications of the operator ordering allow to satisfy the criterion exactly, but the spectrum of the area operators results different from the standard one proposed in Loop Quantum Gravity. In the thesis, we have investigated the EPRL solutions of the simplicity constraints with respect to different choices of the spectrum of the Casimir operators.

Moreover, we have considered the EPRL solutions in the extended GFT formalism, implementing the EPRL constraints directly on the GFT field, by restricting the $SU(2)^-$ (anti-self-dual subgroup), $SU(2)^+$ (self-dual subgroup) and $SU(2) \subset SU(2)^- \times SU(2)^+ \subset SO(4)$ representations appearing in the Peter-Weyl decomposition of the GFT field, using Kronecker deltas.

In the thesis, we have limited our analysis to the GFT field because the amplitude can be obtained by considering the Feynman expansion of the GFT partition function, constructed with the constrained GFT fields. The expansion of the partition function in terms of Feynman diagrams generates simplicial complexes and Spin Foams corresponding, exactly, to the EPRL Spin Foam model. To be more precise, we have imposed the simplicity constraints only on a face of the tetrahedron (corresponding to a bivector), but the results are easily generalizable to the whole tetrahedron, if we consider the four faces of the tetrahedron be orthogonal to the same normal vector. Technically speaking, we can glue the four constrained faces, by contracting the four face amplitudes with an appropriate four-valent $SO(3)_k$ intertwiner.

Modifications to the EPRL model have been proposed in order to relate the spin connection, characteristic of the Spin Foam models (path integral quantization), and the Ashtekar-Barbero connection, appearing in the standard formulation of Loop Quantum Gravity (canonical quantization).

The strategy proposed by Alexandrov, that has been studied in Chapter 9 of this thesis, amounts to a modification in the spectrum of the Casimir operators, appearing in the operatorial equation implementing the simplicity constraints. The result is a modified relation between the representations of the $SU(2)$ subgroups. As for the EPRL model, we have considered the Alexandrov model in the extended GFT formalism, in different representations, implementing the constraints directly on the GFT field.

However, we mention that an other possibility to relate canonical and path integral quantization of gravity is to generalize the Ashtekar-Barbero connection to a Lorentz (or $SO(4)$) connection, and generalize $SU(2)$ spin networks to projected spin networks (which appear also in the extended GFT formalism described in the thesis). We hope to clarify the relation between extended GFT models, Covariant Loop Quantum Gravity and Spin Foam models in future works.

An other extension of the EPRL model, that we considered in Chapter 10 of this thesis, has been proposed by Ding, Han and Rovelli, based on weak imposition of the constraints using a Gupta-Bleuler criterion. Actually, in [45] it has been shown that the Ding-Han-Rovelli model allows more solutions than the EPRL model. Specifically, solutions are parametrized by an extra degree of freedom r, and the EPRL solutions are recovered for a particular choice of the r parameter.

However, these extra solutions have not been studied in the literature, rather they are usually overlooked as they are generally supposed to have a strange asymptotic behaviour. In this regard, we think that extra solutions are good solutions and their contribution to the field amplitude should be considered. The reason is that extra solutions are weighted by the function implementing the constraints (related to Clebsch-Gordan coefficients), and those with suspicious large-j behaviour are suppressed. As for the other models, we have considered the Ding-Han-Rovelli solutions in the extended GFT formalism, in different representations, implementing the constraints directly on the field, using Kronecker deltas.

Soon after the EPRL model was proposed, Freidel and Krasnov suggested an other model based on Livine-Speziale coherent states. The model is presented in Chapter 11, and it amounts to demand that simplicity constraints are satisfied in mean value, on coherent states. Due to the properties of the coherent states, the Freidel Krasnov model has a clear geometric interpretation, as it requires that semi-classical bivectors satisfy the simplicity constraints.

The Freidel Krasnov model differs from the other models as it is not defined in terms of Kronecker deltas restricting the representations of the sub-groups. Rather, it amounts to decompose the GFT field on an over-completed basis of coherent states and then restrict such states to the ones satisfying the simplicity constraints. Nevertheless, it turns out that for $\beta < 0$ the FK model coincides with the EPRL model, while for $\beta > 0$ the two models differ. In the thesis we have studied the Freidel Krasnov interwiner in the extended GFT formalism, in different representations.

In Chapter 12, we have considered the Baratin Oriti model, based on the metric representation of the extended GFT field. Similarly to the Freidel Krasnov model, the Baratin Oriti model has a clear geometric interpretation as it implements, using non-commutative delta functions, the simplicity constraints directly on the metric variable of the field, describing quantum bivectors. As pointed out in [50], the Baratin Oriti model may be the most general, as it does not imply any restriction on the Immirzi parameter, whereas all the other models impose a rationality condition.

In this thesis, we have deepened this idea. Particularly, our first great result is that the Baratin Oriti solutions satisfy both the Master constraint criterion and the Gupta-Bleuler criterion exactly, for every choice of well-defined quantization map.

The second great result comes from the introduction of a new formulation of the Baratin Oriti model in terms of Bessel functions, which allows an easy comparison with the other models in the large-j limit. Particularly, we have shown that, in this limit, the function weighting the Baratin Oriti solutions peaks exactly on the first EPRL condition (relating the representation of the self-dual and anti-self-dual subgroups).

Moreover, we have argued a way to recover also the second EPRL condition (relating the $SU(2) \subset SU(2)^- \times SU(2)^+ \subset SO(4)$ subgroup with the self-dual and anti-self-dual subgroups), although the calculation is left for future work.

Finally, last issue was to check whether the Baratin Oriti extra solutions lead to strange semi-classical limit.

Our result is that the weighting function implementing the Baratin Oriti constraints suppresses exactly the solutions that would lead to a suspicious asymptotic limit. Thus, Baratin Oriti extra solutions are good solutions and contribute to the field amplitude, but the more they are far from the EPRL conditions, the more their contribution to the field amplitude decreases.

The comparison between the Baratin Oriti solutions and the Alexandrov solutions is more involved, and thus we have considered only a subset of the Baratin Oriti solutions. Particularly, we have restricted our attention to the solutions satisfying a directional condition. This condition removes the dependence on the directions of the self-dual and anti-self dual parts of the bivector, and maximizes the weighting function with respect to the norms of the self-dual and anti-self dual parts, corresponding, in the large-j limit, to the representations of the $SU(2)^-$ and $SU(2)^+$ subgroups. Our result is that the Baratin Oriti solutions satisfying the directional condition and the first Alexandrov's condition ($j^- = \beta(1 + j^+)$, when $\beta > 0$) contribute significantly to the amplitude. For $\beta < 0$, the Alexandrov's proposal coincides with the EPRL model.

Also the comparison between the Baratin Oriti solutions and the Ding-Han-Rovelli solutions is generally non-trivial due to the complexity of the weighting functions implementing the constraints in the two models. Thus, we have performed several approximations. On one hand we have considered the Baratin Oriti solutions satisfying the directional condition. On the other hand, we have considered only a subset of the Ding-Han-Rovelli solutions, precisely those with the indices, appearing in the Clebsch-Gordan weighting function, that maximize the field amplitude.

Our result is the following.

Both the $\mathcal{BO}$ and $\mathcal{DHR}$ functions implementing the constraints peak on the first EPRL condition, and they reproduce the correct asymptotic limit. Also the field amplitudes are similar, although, on one hand the Baratin Oriti model admits a continuum spectrum of solutions, whereas the Ding-Han-Rovelli model admits only a finite set of solutions. On the other hand the field amplitudes of the Baratin Oriti solutions, in the regime we have considered, are not precisely the discretized counterpart of the Ding-Han-Rovelli solutions. We don't know if this discrepancy is due to the approximations and simplifications we have performed during the calculations, or rather the two models just weight differently the field amplitudes. A more deep analysis of the asymptotic of the two models is needed and left for future works.

Finally, we have compared the Baratin Oriti model and the Freidel Krasnov model in the large-j limit. In this case the relation between the two model is more evident because in the asymptotic limit quantum bivectors become semi-classical bivectors, described in terms of coherent states. Our analysis shows that the Baratin Oriti weighting function peaks exactly on the solution of the delta function, implementing the Freidel Krasnov constraints, but it is more widened, thus admitting more solutions.

In conclusion we think that the extra solutions characterizing the Ding-Han-Rovelli model and the Baratin Oriti model are good solutions, and should be considered, as they contribute to the GFT field amplitude. Particularly, the Baratin Oriti solutions seem the most general, as they satisfy both the Master constraint criterion and the Gupta-Bleuler criterion exactly, for every well-defined quantization map, and without imposing any rationality condition on the Immirzi parameter. Moreover, the asymptotic limit is generally in accordance with the other models, but at the same time the Baratin Oriti solutions seem the natural generalization of the EPRL/FK solutions.

13.1 Perspectives

As mentioned in the thesis several aspects should be deepened, more calculations should be done, more strategies need to be explored. First, a complete analysis of the asymptotic limit of the models presented in the thesis is needed. Actually, while the large-j limit of the EPRL and FK models have been deeply studied in the literature [146–152], the asymptotic limit of the Ding-Han-Rovelli solutions nor the Baratin Oriti model haven't been considered.

Particularly, it would be interesting to check whether the Baratin Oriti model implies, in the large-j limit, the second EPRL condition or the second Alexandrov condition. Moreover, we have shown that the Baratin Oriti solutions are more general with respect to the other models presented, but are they the most general solutions satisfying the Master constraint criterion or the Gupta-Bleuler criterion? We would like to answer this question in future works.

An other open issue regards the projector property of the operator implementing the simplicity constraints. Particularly, it would be interesting to investigate whether the EPRL/FK projector, in the extended GFT formalism, commutes with the relaxed closure constraint. In addition, we would like to know what are the properties of the Baratin Oriti operator implementing the simplicity constraints under integration, particularly when acting on observables. Again these issues are left for future studies.

Other possibles developments of the thesis are the comparisons between the Baratin Oriti model and other models that have been recently introduced. For example, the Warsaw proposal [154–156], an other modification of the EPRL model, and the holomorphic Spin Foam models [157–159], based on the imposition of the simplicity constraints in a new spinorial representation.

Finally, we have set all the discussion in the Euclidean setting, obviously a generalization of the results of this thesis to the Lorentzian case, possibly with non-zero cosmological constant and matter, is needed.

13.2 About the role of simplicity constraints...

In this thesis we have discussed the implementation of the simplicity constraints in recent Spin Foam and GFT models. As we have seen, at the classical level they are related to the Plebanski formulation of General Relativity. Thus, a natural interpretation of the simplicity constraints at the quantum level is to consider them as the constraints that one has to impose on quantum BF states to break the topological invariance of the quantum BF theory and recover a quantum version of the Holst-Plebanski formulation of General Relativity.

However, we would like to suggest an other interpretation of the constraints. Actually, General Relativity might be an effective theory, a low-energy approximation of a more fundamental theory. An indication of this might come from the perturbative non-renormalizability of the theory. More generally, one could expect quantum corrections to the Palatini action.

Actually, the relevant feature of General Relativity is the background independence and the geometric content of the theory, rather than the particular form of the action (many generalizations of the Einstein-Hilbert action have been proposed, see for instance $f(R)$ gravity, massive gravity, etc.).

Therefore, the idea we would like to present is that simplicity constraints might be interpret as a geometric condition that the fundamental degrees of freedom of the theory (the quanta of the Group Field Theory) have to satisfy. Specifically, the simplicity constraints imply that the GFT field describes a geometric tetrahedron.

However, other possible criteria to break the topological invariance may be related to the renormalizability of the GFT theory, rather than the geometricity condition. Regarding this, GFT renormalization has been currently studied [6, 160–162]. For example, preliminary results [162] indicate that the dynamics of the renormalized Lorentzian EPRL/FK model would result modified with respect to the tree level one. Future studies in GFT renormalization will clarify these issues.

13.3 About the imposition of simplicity constraints...

An other issue that we would like to point out is that the strategy of the Spin Foam program (summarized as first quantize the BF theory and then impose quantum simplicity constraints) is debated. Actually, in [87] Alexandrov concluded that the Spin Foam strategy to quantization in four dimensions with finite Immirzi parameter is not viable, as it contradicts the quantization rules for systems with second class constraints and leads to various inconsistencies at the quantum level. Alexandrov specifically considered the EPRL model and the Freidel Krasnov model and identified in the rationality condition imposed on the Immirzi parameter a manifestation of these problems. However, he mentioned that there is an exception to which his analysis cannot be applied, the FK model for $\beta = 1$.

Remarkably, in the Baratin Oriti model there is no rationality condition on the Immirzi parameter, and thus it might be an other exception to the Alexandrov's analysis, although further investigation is needed. However, this hypothesis is supported by the fact that in the Baratin Oriti model the simplicity conditions can be implemented in the path-integral by an effective gauge covariant measure on the space of discrete connections, as shown in [49, 50].

In conclusion, we would like to stress that the standard strategy "first quantize, then constrain" is really discussed (see for instance [41, 83]). Particularly, in [41] Geiller and Noui concluded that "in the end, the only way to discriminate between the various proposals is either to extract physical predictions to compare with experiments, or at least to test the strategies on toy models which bear a close analogy with gravity".

13.4 About the relation between simplicity constraints and physical observables...

In [8] Peres stressed that "Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory". Therefore, supposing simplicity constraints are correctly implemented, we may ask whether they are related to physical observables. Generally, extracting physics from Spin Foams is problematic, as, from the experimental point of view, we have no direct evidence of quantum gravity phenomena, although we can consider semi-classical quantum gravity effects near singularities (Black holes and the Big Bang).

Moreover, from the theoretical point of view the situation may be worse as both the classical limit and the continuum limit of Spin Foam/GFT models have not been fully understood. However, the general idea [32] is that the fundamental degrees of freedom of the GFT formalism are discrete, pre-geometric building blocks of combinatorial and algebraic nature. Particularly, the spatial manifold is, at very short distances and very high energy (quantum domain), a collection of (glued) building blocks and the field theory is defined on the space of possible geometries of each such building block. In this picture, the space-time as a continuum manifold is but an emergent concept corresponding to a collective configuration of a large number of quantum gravity building blocks, possibly after a phase transition [163, 164]. Thus, space and time disappear at microscopic scales. The main hypothesis is that what we call continuum space-time is but a phase of an underlying GFT system.

Recently, a new class of GFT states, the GFT condensates, has been proposed [165–169]. The hope is that these states could describe continuum macroscopic homogeneous (but anisotropic) geometries. Particularly, the appearance of macroscopic geometries is captured by a process similar to Bose-Einstein condensation of the microscopic GFT building blocks.

The effective dynamics can be extracted for a homogeneous quantum space, i.e. for GFT condensate, from a generic GFT model for quantum gravity, and it has been shown to have the form of a non-linear and non-local extension of quantum cosmology, similar to the one suggested in [170].

The (Gross-Pitaevskii-like) hydrodynamics of the GFT condensate, in a further classical approximation following the continuum (thermodynamic) limit of the quantum GFT system, would then correspond to General Relativity.

One can put forward a further hypothesis [171] and identify the process (phase transition) of quantum space-time condensation with the Big Bang singularity.

Therefore, the Big Bang would correspond to a geometrogenesis process, in other words before the condensation there was no space nor time.

In this picture, what is the physical relevance of simplicity constraints in the Big Bang phase transition? And can we test the correct imposition of the simplicity constraints looking at physical observables, for example the spectrum of the Cosmic Microwave Background Radiation? We hope to answer these questions in the next few years. Stay tuned and enjoy this stimulating quest!

Appendix A

Differential geometry

In this chapter we define several concepts of differential geometry used in the thesis. Particularly, following [89] and [172], we start from the definition of Banach spaces, then we introduce the notion of manifolds, tangent vectors and (co-)tangent bundles. After that we define exterior and Lie derivative, and finally the notion of flow on a manifold.

Definition A.0.1. A Banach space E is a vector space X over the field $\mathbb{R}$ of real numbers, or over the field $\mathbb{C}$ of complex numbers, equipped with a norm with respect to which they are (Cauchy) complete.

Particularly, any Hilbert space $\mathcal{H}$ serves as an example of a Banach space.

Definition A.0.2. A Hilbert space $\mathcal{H}$ on $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a Banach space complete for a norm of the form

$$\|x\|_{\mathcal{H}} = \sqrt{\langle x, x\rangle},$$

where

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$$

is the inner product, linear in its first argument that satisfies the following:

$$\forall x, y \in \mathcal{H} : \quad \langle y, x\rangle = \overline{\langle x, y\rangle}, \tag{A.1}$$

$$\forall x \in \mathcal{H} : \quad \langle x, x\rangle \geq 0, \tag{A.2}$$

$$\langle x, x\rangle = 0 \Leftrightarrow x = 0. \tag{A.3}$$

Now, we can introduce a fundamental concept in mathematical physics, the notion of manifold. For example, in General Relativity space-time is a four dimensional Lorentzian manifold, the classical phase space is described by a symplectic manifold with a Poisson structure, Calabi-Yau manifolds play a fundamental role in recent superstring theories, etc.

Let E be a real Banach space.

Definition A.0.3. A topological manifold M is a is a topological space, specifically a second countable Hausdorff space, such that around any of its points $m \in M$ there is an open set $U \subset M$ and a map $\phi : U \to \phi(U) \subseteq E$ that is continuous with continuous inverse — homomorphism. The couple (U, ϕ) is called local chart. Moreover, we define an atlas the collection of such local charts $\{(U_i, \phi_i)\}_i$ such that

- $\bigcup_i U_i = M_i$
- on a (non-empty) overlap $U_i \cap U_j$ the transition maps the transition maps

  $$\phi_j \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

  are homomorphisms for any pair of local charts (U_i, ϕ_i), (U_j, ϕ_j)

Generally, we would like to introduce concepts such as differentiable functions, tangent vectors, vector fields and differential forms on a manifold M. Accordingly, we need differentiable or smooth structures.

Definition A.0.4. A differentiable manifold is a topological manifold equipped with an equivalence class of atlases whose transition maps are all differentiable. More generally, a C^k-manifold is a topological manifold with an atlas whose transition maps are all k-times continuously differentiable.

Finally, we can define a C^k-submanifold of a C^k-submanifold.

Definition A.0.5. Given a C^k-manifold N of dimension n, then we say that a subset $M \subset N$ is a C^k-submanifold of dimension m if M can be covered by charts (U, ϕ) of N with the property that $M \cap U = \phi^{-1}(\mathbb{R}^m \times \{0\})$. (If $k = \infty$ we will also call this a smooth submanifold). Hence we must have $m \leq n$ and $\mathbb{R}^n$ is written as $\mathbb{R}^m \times \mathbb{R}^{n-m}$.

Generally, we would like to define real-valued smooth functions on a manifold. Particularly, we can define differentiable a function f on a differentiable manifold M by means of the differentiable structure of the atlas and the definition of differentiable functions on a Banach space. Actually,

Definition A.0.6. A real-valued function $f : M \to \mathbb{R}$ on M is said to be differentiable at $m \in M$ if there exists a chart (U, ϕ) with $m \in U$ such that the local representative $g := f \circ \phi^{-1} : V \subset E \to \mathbb{R}$ is differentiable at $u = \phi(m)$ as a function on E (V open in E).

Importantly, this definition is independent from the choice of the local chart due to the compatibility of the transition maps.

We note also that the space of all real-valued smooth functions on M, denoted $C^\infty(M, \mathbb{R})$, is an associative commutative (real) algebra under addition and pointwise multiplication of functions. Particularly, pointwise multiplication "$\cdot$" is a map $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ defined as

$$(f \cdot g)(x) := f(x)g(x) \qquad f, g \in C^\infty(M, \mathbb{R})$$

which satisfies the following conditions for all $f, g, h \in C^\infty(M, \mathbb{R})$ and $\lambda \in \mathbb{R}$

- Commutativity: $f \cdot g = g \cdot f$
- Linearity: $f \cdot (g + \lambda h) = f \cdot g + \lambda f \cdot h$
- Associativity: $f \cdot (g \cdot h) = (f \cdot g) \cdot h$

Now, we arrive to an essential definition, which will play a crucial role in the following: the notion of vector bundles.

Definition A.0.7. A vector bundle over the manifold M (called the base manifold) is a real manifold X (called the total space) together with a smooth map $\pi : X \to M$ (called the projection) such that

- For each $m \in M, X_m = \pi^{-1}(m)$ is a vector space over $\mathbb{K}$ of constant dimension r (X_m is called the fibre over $\{m\}$ and r is called the fibre dimension),
- For each $m_0 \in M$, there is a neighbourhood $U \ni m_0$ in M together with a diffeomorphism $\psi : U \times \mathbb{K}^r \to \pi^{-1}(U)$ such that $\psi|_m : \{m\} \times \mathbb{K}^r \to X_m$ is a linear isomorphism for each $m \in U$ (the pair (U, ψ) is called a local trivialization).

In the definition above M is a n-dimensional real manifold and $\mathbb{K}$ denotes the field of either the real or complex numbers. A map $s : U \subset M \to X$ is called a section of a vector bundle X if $\pi \circ s = Id$.

We will see different example of vector bundles, the first is the tangent vector bundle, which we are going to define. First of all, we start by introducing the fundamental concept of tangent vector v_m at a point $m \in U \subset M$.

Definition A.0.8. There are three standard definitions:

- Let $c : \mathbb{R} \to M$ be a smooth curve on M such that $c(0) = m$. On the set of germs (local representatives of smooth functions) at 0 of smooth curves we define the following equivalence relation: $c \sim \bar{c}$ if $c(0) = \bar{c}(0)$ and in a local chart (U, ϕ) we have $\frac{d}{dt}|_{t=0}(\phi \circ c)(t) = \frac{d}{dt}|_{t=0}(\phi \circ \bar{c})(t)$. A kinematic tangent vector is then such an equivalence class of (germs of) smooth curves $[c]_m$.
- An operational tangent vector v_m is a derivation of the algebra of smooth functions $f \in C^\infty(U, \mathbb{R})$:
$$v_m(f) := Df_m(v_m) := D(f \circ \phi^{-1})_{\phi(m)}(v_m).$$
That is, for each open $U \subset M$, v_m is a continuous (or bounded) linear map $v_m : C^\infty(U, \mathbb{R}) \to \mathbb{R}$, and $v_m(f\bar{f}) = v_m(f)\bar{f}(m) + f(m)v_m(\bar{f})$ for all $f, \bar{f} \in C^\infty(U, \mathbb{R})$. Note that the value of $v_m(f)$ depends only on the germ of f at m.
- A tangent vector v_m at $m \in M$ is an equivalence class of triples (U, ϕ, v_m), where two triples (U, ϕ, v_m), (U', ϕ', v'_m) are equivalent if $v'_m = D(\phi' \circ \phi^{-1})_{\phi(m)}(v_m)$, for (U, ϕ), (U', ϕ') local charts around m.

It is possible to show that these definitions may be equivalent in finite dimensions, however not generally in infinite dimensions.

Definition A.0.9. In finite dimensions, we may define a single notion of tangent space of M at m as the set of all tangent vectors at m, denoted as T_mM.

We observe that any tangent vector $a \in T_mM$ can be written as $a = a^k\partial_k$ where $a^k = a^k(x)$ are numerical coefficients and $\partial_k \equiv \frac{\partial}{\partial x^k}$ form a basis in the tangent space.

With the above definitions, we can define the differential of a smooth function on manifold:

Definition A.0.10. Given smooth manifolds M, N and a smooth mapping $\psi : M \to N$ we define the differential $d\psi_m$ at a point $m \in M$ as the following linear mapping:

$$\begin{aligned} d\psi_m : T_mM &\to T_{\psi(m)}N \\ v_m &\mapsto d\psi_m(v) \end{aligned} \tag{A.4}$$

with $d\psi_m(v)g := v_m(g \circ \psi)$ for germs g on $\psi(m)$.

Finally, the tangent bundle TM is the collection (in fact, disjoint union) of all tangent spaces

$$TM := \bigcup_{m \in M} T_mM,$$

and carries a natural manifold structure modelled on $E \times E$, with local charts (TU_i, ψ_i) where:

$$\psi_i : TU_i := \bigcup_{m \in U_i} T_mM \to \phi_i(U_i) \times E,$$

for (U_i, ϕ_i) local charts of M.

Definition A.0.11. A (tangent) vector field X on M is a smooth section of the tangent bundle TM, that is, a smooth map $X : M \to TM$ such that $\pi_M \circ X = id_M$, where $\pi_M : TM \to M$ is the tangent bundle projection.

We denote the space of all smooth vector fields on M by $Vect(M)$.
Now, after having defined the tangent space, we can define the cotangent space, by considering the dual:

Definition A.0.12. The cotangent space $T^*_m M$ is defined as the topological dual of $T_m M$, that is the set of all continuous linear maps $T_m M \to \mathbb{R}$.

The collection of all cotangent spaces defines the cotangent bundle

$$T^*M := \bigcup_{m\in M} T^*_m M$$

which may be shown to carry a manifold structure, with local charts (T^*U_i, ψ_i)

$$\psi_i : T^*U_i := \bigcup_{m\in U_i} T^*_m M \to \phi_i(U_i) \times E^*,$$

with (U_i, ϕ_i) local charts for M.

Recalling that a differential vector field on M is a section of its tangent bundle, we define:

Definition A.0.13. A differential 1-form on M is a section of its cotangent bundle.

We denote the space of all smooth differential 1-forms on M by $\Lambda^1(M)$, which is dual to $Vect(M)$.
Locally, we can write a differential 1-form as the expression $\omega = \omega_i dx^i$ for some chart U, where $\{dx^i\}_{1\leq i\leq n}$ forms the basis of $\Lambda^1(U)$ dual to the basis $\{\partial_i\}_{1\leq i\leq n}$ of $Vect(U)$.
Moreover, the space $\Lambda^k(M)$ of smooth differential k-forms on M can be defined as the k-th exterior power of $\Lambda^1(M)$. In a local coordinate system a k-form ω looks like

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1,\ldots,i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where all indices is, $1 \leq s \leq k$, run from 1 to n and the wedge product is bilinear, associative, and anti-symmetric.

On the space $\Lambda^k(M)$ of smooth differential k-forms on M, we can define the notion of exterior derivative:

Definition A.0.14. The exterior derivative d is the unique $\mathbb{R}$-linear mapping from k-forms to $(k + 1)$-forms satisfying the following properties:

- df is the differential 1-form of f for smooth functions f.
- flatness: $d \circ d = 0$.
- Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (1)^p(\alpha \wedge d\beta)$ where α is a differential p-form.

Finally, we end this section with a very important object in differential geometry, which is the Lie derivative $\mathcal{L}$ on a manifold M. For our purpose, the Lie derivative plays a crucial role when the classical phase space of a physical system is the cotangent bundle of a Lie group, since momenta are precisely Lie derivatives.

The general definition involves the tensor bundle of rank (p, q) of a manifold, i.e. the direct sum of all tensor products of the p tangent bundle and the q cotangent bundle of that manifold. Then, in analogy with vector fields, we define the tensor field as a section of some tensor bundle.

Now, we can give a general definition for the Lie derivative of a tensor field:

Definition A.0.15. Let T be a differentiable tensor field of rank (p, q) and Y be a differentiable vector field on a differential manifold M. Then an algebraic definition for the Lie derivative of a tensor field follows from the following four axioms:

- The Lie derivative of a function is the directional derivative of the function. So if f is a real valued function on M, then
$$\mathcal{L}_Y f = Y(f) \tag{A.5}$$
- The Lie derivative obeys the Leibniz rule. For any tensor fields S and T, we have
$$\mathcal{L}_Y(S \otimes T) = (\mathcal{L}_Y S) \otimes T + S \otimes (\mathcal{L}_Y T). \tag{A.6}$$
- The Lie derivative obeys the Leibniz rule with respect to contraction
$$\mathcal{L}_X(T(Y_1, \ldots, Y_n)) = (\mathcal{L}_X T)(Y_1, \ldots, Y_n) + T((\mathcal{L}_X Y_1), \ldots, Y_n) + \cdots + T(Y_1, \ldots, (\mathcal{L}_X Y_n)) \tag{A.7}$$
- The Lie derivative commutes with exterior derivative on functions
$$[\mathcal{L}_X, d] = 0 \tag{A.8}$$

In order to give an other definition of the Lie derivative, we first need to define an auxiliary concept: the flow on a manifold M.

Definition A.0.16. For any (kinematic) vector field $X \in Vect(M)$ there exists an associated local flow $Fl^X : J \subseteq \mathbb{R} \times M \to M$, as the unique solution to the differential equation
$$\begin{aligned} \frac{d}{dt} Fl^X_t(m) &= X_{Fl^X_t}(m), \\ Fl^X_0(m) &= m \end{aligned} \tag{A.9}$$
for all $(t, m) \in J \subseteq \mathbb{R} \times M$. Moreover, if $J = \mathbb{R} \times M$, the flow is called complete or global flow, and the associated vector field X a complete vector field.

Remarkably, the flow Fl^X satisfies the group law $Fl^X_s \circ Fl^X_t = Fl^X_{s+t}$, for all $s, t \in \mathbb{R}$, whenever Fl^X_s, Fl^X_t , Fl^X_{s+t} are defined.

Definition A.0.17. Let Fl^X be a complete flow on a manifold M associated to the differential vector field $X \in Vect(M)$, then the set $\{Fl^X_t : t \in \mathbb{R}\}$ is called a one-parameter group of transformations of M, otherwise a one-parameter semi-group of local diffeomorphisms.

Now, intuitively, if we consider a tensor field T and a vector field Y, then $\mathcal{L}_Y T$ is the infinitesimal change we would see when we flow T using the vector field $-Y$, which is the same thing as the infinitesimal change we would see in T if we flowed along the vector field Y.

More precisely,

Definition A.0.18. If we have a differentiable tensor field T of rank (p, q) and a differentiable vector field Y, then we can define the Lie derivative of T along Y. Let $\phi : M \times \mathbb{R} \to M$ be the one-parameter semigroup of local diffeomorphisms of M induced by the vector flow of Y and denote $\phi_t(p) := \phi(p, t)$. For each sufficiently small t, ϕ_t is a diffeomorphism from a neighbourhood in M to another neighbourhood in M, and ϕ_0 is the identity diffeomorphism. The Lie derivative of T is defined at a point p by
$$(\mathcal{L}_Y T)_p = \left.\frac{d}{dt}\right|_{t=0} \left((\varphi_{-t})_* T_{\varphi_t(p)}\right) = \left.\frac{d}{dt}\right|_{t=0} \left((\varphi_t)^* T\right)_p. \tag{A.10}$$
where $(\varphi_t)_*$ is the pushforward along the diffeomorphism and $(\varphi_t)^*$ is the pullback along the diffeomorphism.

A.1 Lie groups

In this section we introduce an other fundamental concept of modern physics: Lie groups. Their importance is related to continuous symmetries, which are a guiding principle when we formulate physical theories.

Definition A.1.1. A Lie group is a smooth (C^∞)-manifold G equipped with a group structure so that the maps $\mu : G \times G \to G$ mapping $(x, y) \mapsto xy$ and $\iota : G \to G$ mapping $x \mapsto x^{-1}$ are smooth.

An example of a Lie group, playing a very important role in representation theory, is the general linear group of degree n, $GL(n, \mathbb{R}) = \{A \in Mat\{n, \mathbb{R}\}, \ \ | \ \ det[A] \neq 0\}$, i.e. the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication. This forms a group, because the product of two invertible matrices is again invertible, and the inverse of an invertible matrix is invertible.

Consequently, we can define a subgroup H of a Lie group G:

Definition A.1.2. Let G be a Lie group, and let $H \subset G$ be both a subgroup and a smooth submanifold. Then H is a Lie group.

Next we come to the concept of isomorphic Lie groups.

Definition A.1.3. Let G and H be Lie groups.

- A Lie group homomorphism from G to H is a smooth map $\phi : G \to H$ that is a homomorphism of groups.
- A Lie group isomorphism from G to H is a bijective Lie group homomorphism $\phi : G \to H$ whose inverse is also a Lie group homomorphism. Remarkably, a Lie group isomorphism is also a diffeomorphism.
- A Lie group automorphism of G is a Lie group isomorphism of G onto itself.

Among the diffeomorphisms on Lie groups, we are particularly interested in the left and right translation diffeomorphisms:

Definition A.1.4. Let G be a Lie group and $g \in G$. Then the translation maps $L_g : G \to G$, mapping $h \mapsto gh$ and $R_g : G \to G$, mapping $h \mapsto hg$ are diffeomorphisms from G onto itself, hence they are automorphisms (the notions of diffeomorphism and isomorphism are identical for Lie groups). As a consequence the conjugation map $C_g = L_g \circ R_g^{-1} : G \to G$, mapping $h \mapsto ghg^{-1}$ is an automorphism too, also called an inner automorphism.

Having defined the left and right translation, we can define left and right invariant vector fields by:

Definition A.1.5. A vector field $v \in Vect(G)$ is called left invariant if $(dL_g)v = v$ for all $g \in G$.

From the above equation with $h = e$ we see that a left invariant vector field is completely determined by its value $v(e) \in T_eG$.

When we study the phase space associated to the cotangent bundle of a Lie group, we use the following:

Theorem A.1.1. *If G is a Lie group, then the tangent bundle TG is always trivializable, i.e. $TG \simeq G \times \mathbb{R}^n$ with $n = dim(G)$*

Now, we are interested in the action α of a Lie group G on a smooth manifold M:

Definition A.1.6. Let M be a smooth manifold and G a Lie group. A (left) action of G on M is a differentiable map $\alpha : G \times M \to M$ such that

- $\alpha(g_1, \alpha(g_2, m)) = \alpha(g_1 g_2, m)(m \in M, g_1, g_2 \in G)$;
- $\alpha(e, m) = m(m \in M)$.

Usually we use the notation $g \cdot m$ or gm for $\alpha(g, m)$.

If $g \in G$, then we sometimes use the notation α_g for the map $m \mapsto \alpha(g, m) = gm$, which is a bijection with inverse map equal to $\alpha_{g^{-1}}$.

Sets of the form $mG(m \in M)$ are called orbits of the action α. Moreover, the orbits constitute a partition of M, and the set of all orbits, called the orbit space, is denoted by M/G. The action of G on M is called transitive if it has only one orbit, the full manifold M. In this case the G-space M is said to be a homogeneous space for G.

We are now ready to define the notion of representation of a Lie group, playing a fundamental role in the thesis:

Definition A.1.7. Let V be a locally convex space. A representation $\pi = (\pi, V)$ of G in V is a left action $\pi : G \times V \to V$, such that $\pi(g) : v \mapsto \pi(g)v = \pi(g, v)$ is a linear endomorphism of V, for every $g \in G$. The representation is called finite dimensional if $dim(V) < \infty$.

Furthermore, it would be useful have a notion of invariant subspace, therefore:

Definition A.1.8. Let π be a representation of G in a linear space V. By an invariant subspace we mean a linear subspace $W \subset V$ such that $\pi(g)W \subset W$ for every $g \in G$. A continuous representation π of G in a complete locally convex space V is called irreducible, if 0 and V are the only closed invariant subspaces of V.

A.2 Lie algebras

In this section we define the notion of Lie algebra and we study the relation between Lie group and the associated Lie algebra. Finally, we define the adjoint representation, that we will use to define Coadjoint orbits.

Definition A.2.1. A real Lie algebra is a real linear space $\mathfrak{a}$ equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ such that for all $X, Y, Z \in \mathfrak{a}$ we have

- Anti-Symmetry: $[X, Y] = [Y, X]$
- Jacobi-identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Consequently, we can define Lie algebra homomorphisms:

Definition A.2.2. Let $\mathfrak{a}, \mathfrak{b}$ be Lie algebras. A Lie algebra homomorphism from $\mathfrak{a}$ to $\mathfrak{b}$ is a linear map $\phi : \mathfrak{a} \to \mathfrak{b}$ such that

$$\phi([X, Y]_{\mathfrak{a}}) = [\phi(X), \phi(Y)]_{\mathfrak{b}}, \quad \forall X, Y \in \mathfrak{a}. \tag{A.11}$$

Now, we can introduce the representation of a Lie algebra

Definition A.2.3. Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ in a complex linear space V is a bilinear map $\iota \times V \to V$, mapping $(X, v) \mapsto Xv$, such that

$$[X,Y]v = (XY)v - (YX)v, \quad \forall X, Y \in \mathfrak{g},\ v \in V. \tag{A.12}$$

In other words, the map $X \mapsto X\cdot$ is a Lie algebra homomorphism from $\mathfrak{g}$ into $End(V)$. Where $End(V)$ denoted the space of endomorphisms of a vector space V, i.e. linear maps $f : V \to V$.

Particularly, we can define the adjoint representation:

Definition A.2.4. If $g \in G$ we define $Ad(x) \in GL(T_eG)$ by $Ad(x) \equiv T_eC_g$. The map $Ad : G \to GL(T_eG)$ is called the adjoint representation of G in T_eG. If G is a matrix group the Lie algebra $\mathfrak{g}$ is a subspace of $Mat(n, \mathbb{R})$. Hence the adjoint representation is simply matrix conjugation:

$$Ad(g)X = g \cdot X \cdot g^{-1}, X \in \mathfrak{g}, g \in G. \tag{A.13}$$

This section shall be concluded with the justification of the identification of T_eG with $\mathfrak{g}$, the Lie algebra of G. Particularly, one way to derive the Lie algebra associated to a Lie group, using left invariant vector fields, follows by considering that:

- Firstly, we note that for (operational) vector fields $X, Y \in Vect(M)$, there exists a vector field $[X,Y] \in Vect(M)$ which is uniquely determined by

  $$[X,Y](f) := X(Y(f)) - Y(X(f))$$

  for each $f \in C^\infty(U, \mathbb{R})$ on each open set $U \subset M$ (as a consequence of the chain rule). The $\mathbb{R}$ - bilinear mapping $[\cdot,\cdot] : Vect(M) \times Vect(M) \to Vect(M)$ is called the Lie bracket on vector fields, and $(Vect(M), [\cdot,\cdot])$ constitutes a Lie algebra.

- If G is any group acting smoothly on the manifold M, then it acts on the vector fields, and the vector space of vector fields fixed by the group is closed under the Lie bracket and therefore also forms a Lie algebra.

- We apply this construction to the case when the manifold M is the underlying space of a Lie group G, with G acting on $G = M$ by left translations $L_g(h) = gh$. This shows that the space of left invariant vector fields (vector fields satisfying $dL_gX_h = X_{gh}$ for every $h \in G$, where dL_g denotes the differential of L_g) on a Lie group is a Lie algebra under the Lie bracket of vector fields.

- Any tangent vector at the identity of a Lie group can be extended to a left invariant vector field by left translating the tangent vector to other points of the manifold. Specifically, the left invariant extension of an element $v \in T_eG$ of the tangent space at the identity is the vector field defined by $\hat{v}_g = dL_gv$. This identifies the tangent space T_eG at the identity with the space of left invariant vector fields, and therefore makes the tangent space at the identity into a Lie algebra, called the Lie algebra of G, denoted by $\mathfrak{g}$. Thus the Lie bracket on $\mathfrak{g}$ is given explicitly by $[v, w] = [\hat{v}, \hat{w}]_e$. This Lie algebra $\mathfrak{g}$ is finite-dimensional and it has the same dimension as the manifold G

Finally, a Lie group G and the corresponding Lie algebra $\mathfrak{g} \simeq T_eG$ are related by the exponential map defined as:

Definition A.2.5. The exponential map $\exp = \exp_G : T_eG \to G$ is defined by $\exp(X) = \alpha_X(1)$ where α_X is the maximal integral curve with initial point e of the left invariant vector field v_X on G determined by $v_X(e) = X$, i.e.

$$\frac{d}{dt}\alpha_X(t) = v_X(\alpha_X(t)).$$

A.3 Symplectic structure

In this section, we consider a particular class of manifolds, symplectic manifolds, which are the mathematical structure describing the phase space of classical physics.

Definition A.3.1. A symplectic manifold is a pair (M, ω), where M is a $2n$-dimensional manifold (with $n \in \mathbb{N}$) modelled on a Banach space E, and ω is a strong symplectic structure on M, that is, a two-form on M such that

- ω is closed, i.e. $d\omega = 0$,
- for each $m \in M$, $\omega_m : T_mM \times T_mM \to \mathbb{R}$ is strongly non-degenerate.

Now, we introduce the Hamiltonian vector field that, together with the Poisson structure on the symplectic manifold, characterizes the physical system and determines the evolution in time.

Definition A.3.2. Let (M, ω) be a symplectic manifold. A vector field $X \in Vect(M)$ is called locally Hamiltonian if $\mathcal{L}_X\omega = 0$, f or equivalently if $i_X\omega$ is closed. If $i_X\omega$ is, additionally, exact, then X is called an Hamiltonian vector field.

The Poisson structure is given by the Poisson bracket:

Definition A.3.3. The Poisson bracket between two smooth functions $f, g : U \subset M \to \mathbb{R}$ on a symplectic manifold (M, ω) is defined as

$$\{f, g\} := \omega(X_f, X_g) \tag{A.14}$$

where X_f , X_g are the corresponding Hamiltonian vector fields.

The Poisson bracket satisfies the following properties:

- Linearity: $\{f, g + \lambda h\} = \{f, g\} + \lambda\{f, h\}$
- Anti-symmetry: $\{f, g\} = -\{g, f\}$
- Jacobi identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$

for all $f, g, h \in C^\infty(M, \mathbb{R})$.

That is, $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ is a Lie algebra under the Poisson bracket.
Moreover, the Lie algebra structure $\{\cdot, \cdot\}$ on $C^\infty(M, \mathbb{R})$ is compatible with the associative structure,

$$\{f, gh\} := -X_f(gh) = -X_f(g)h - gX_f(h) = \{f, g\}h + g\{f, h\} \tag{A.15}$$

making thus $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\}, \cdot)$ a Poisson algebra.

Finally, we consider certain submanifolds of symplectic manifolds, which will become important in the study of polarizations later in this thesis. Before introducing these submanifolds, let us begin with the following definition:

Definition A.3.4. A symplectic vector space is a pair (V, ω) in which V is a $2n$, $(n \in \mathbb{N})$, dimensional real vector space and $\omega : V \times V \to \mathbb{C}$ is a bilinear form on V satisfying

- Anti-symmetry: $\omega(X, Y) = \omega(Y, X) \qquad X, Y \in V$,
- Non-degeneracy: $\omega(X, \cdot) = 0 \iff X = 0 \qquad X \in V$.

Finally, we end this section with the definition of the Lagrangian subspace:

Definition A.3.5. A Lagrangian subspace $H \subset V$ is a subspace H of a symplectic vector space V with the properties

- $dim(H) = \frac{1}{2}dim(V) = n$,
- $\omega(X, Y) = 0$ $X, Y \in H$.

A.4 Coadjoint orbits

The notion of coadjoint orbits is the main ingredient of the orbit method.
Let G be a Lie group, $\mathfrak{g} = Lie(G)$ the associated Lie algebra, and $g \in G$.
First of all, consider $\mathfrak{g}^* \equiv \{f : \mathfrak{g} \to \mathbb{R} \mid f\ linear\}$, the vector space dual to $\mathfrak{g}$.
Then, for any linear representation (π, V) of a group G we can define a dual representation (π^*, V^*) in the dual space V^*:

$$\pi^*(g) : V^* \to V^* \pi^*(g) \equiv \pi(g^{-1})^* \tag{A.16}$$

where on the right-hand side we have denoted the dual operator in V^* by the asterisk, defined by:

$$\langle \pi(g^{-1})^* f, v\rangle \equiv \langle f, \pi(g^{-1})v\rangle \quad for \quad any \quad v \quad \in V, \quad f \in V^* \tag{A.17}$$

and by $\langle f, v\rangle$ we denote the value of the linear functional f on a vector v.

Now, given the adjoint representation, we can define the coadjoint representation as follows:

Definition A.4.1. Let $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$.
Then the coadjoint representation $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ of G in $\mathfrak{g}^*$ is defined by $Ad^*(g) : \mathfrak{g}^* \to \mathfrak{g}^*$ and $Ad^*(g) \equiv Ad(g^{-1})^*$.

And finally, we arrive to the notion of coadjoint orbits, that as we will see, are related to symplectic manifolds:

Definition A.4.2. Given $F \in \mathfrak{g}^*$. The coadjoint orbit $\mathfrak{O}(F)$ is the image of the map $\kappa : G \to \mathfrak{g}^*$, mapping $g \mapsto Ad^*(g)F$.

The important and beautiful fact that the coadjoint orbits of Lie groups are symplectic manifolds is encoded in the following theorem:

Theorem A.4.1. *On every coadjoint orbit $\mathfrak{O}$ of a group G, there exists a non-degenerate closed G-invariant differential 2-form B_Ω (called the Kirillov form) defined by*

$$B_{\mathfrak{O}}(F)(X, Y) \equiv \langle F, [X, Y]\rangle \quad X, Y \in \mathfrak{g}, \ F \in \mathfrak{g}^*.$$

This result is the most important new mathematical object that has been brought into consideration in connection with the orbit method, as we will see in the quantization of classical bivectors

Appendix B

Quantization

In this section we define some general concepts of what is called quantization of symplectic manifolds, while in next sections we will give more technical details. However, our description is intended to be a general overview, or an introduction to one of the most debated problems of modern mathematical physics. We refer to the literature for more specific topics [36,89,173,174] and references therein.

Starting from a $2n$-dimensional symplectic manifold (M, ω) with a Poisson structure $(C^\infty(M), \{\cdot,\cdot\}, \cdot)$, the first step is the introduction of a prequantization map $\mathcal{Q}$ of a Lie subalgebra of observables $\mathcal{O} \subset C^\infty(M)$, which is essentially a Lie representation of $\mathcal{O}$ by symmetric operators in some separable Hilbert space $\mathcal{H}$.

More specifically [174],

Definition B.0.3. Let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$ containing the constant function 1. A prequantization of $\mathcal{O}$ is a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space $\mathrm{Op}(D)$ of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space $\mathcal{H}$, such that for all $f, g \in \mathcal{O}$,

(Q1) $\mathcal{Q}(\{f, g\}) = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$,

(Q2) $\mathcal{Q}(1) = I$, and

(Q3) if the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D.

Prequantizations are usually easy to construct, cf. [137, 175, 176], however, prequantization representations tend to be flawed as the prequantization Hilbert spaces are usually "too big", in the sense that they cannot represent the phase spaces of a physically reasonable quantum systems [177]. There are different ways to modify the notion of prequantization, thus obtaining a genuine quantization.

Some versions [174] define it as a prequantization, not necessarily defined on the whole of $C^\infty(M)$, which is irreducible on a "basic set" $\mathfrak{b} \subset C^\infty(M)$ [178] (for example, in the group theoretical approach $\mathfrak{b}$ is realized as the Lie algebra of a symmetry group). Actually, proper quantization should yield an irreducible representation of this algebra $\mathfrak{b}$.

Particularly, we define [174]:

Definition B.0.4. A basic algebra of observables $\mathfrak{b}$ is a Lie subalgebra of $C^\infty(M)$ such that:

(B1) $\mathfrak{b}$ is finitely generated,

(B2) the Hamiltonian vector fields X_f, $f \in \mathfrak{b}$, are complete,

(B3) $\mathfrak{b}$ is transitive and separating, and

(B4) $\mathfrak{b}$ is a minimal Lie algebra satisfying these requirements.

A different approach to quantization is to require a prequantization $\mathcal{Q}$ to satisfy some "Von Neumann rule," the simplest being of the form

$$\mathcal{Q}(\phi \circ f) = \phi\big(\mathcal{Q}(f)\big) \tag{B.1}$$

for some distinguished observables $f \in C^\infty(M)$, and certain smooth functions $\phi \in C^\infty(\mathbb{R})$.

An other way to reduce it is by putting an additional geometric structure on the classical phase space, called polarization.
Indeed, after prequantization, the prequantum Hilbert space involves all the $2n$ coordinates of the symplectic manifold (M, ω) on which they are defined.
The idea is that one should select a Poisson-commuting set of n variables on the $2n$-dimensional phase space and consider functions (or, more properly, sections) that depend only on these n variables.
The n variables can be either real-valued, resulting in a position-style Hilbert space, or complex-valued, producing something like the Segal-Bargmann space [177]. A polarization is just a coordinate-independent description of such a choice of n Poisson-commuting functions.

Thus, informally, a "quantization" could be defined as a prequantization which incorporates one (or more) of the three additional requirements above (or possibly even others). For example, let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$, and suppose that $\mathfrak{b} \subset \mathcal{O}$ is a basic algebra of observables, then we can define [174]:

Definition B.0.5. A quantization of the pair $(\mathcal{O}, \mathfrak{b})$ is a prequantization $\mathcal{Q}$ of $\mathcal{O}$ on $\mathrm{Op}(D)$ satisfying

(Q4) $\mathcal{Q} \restriction \mathfrak{b}$ is irreducible,

(Q5) D contains a dense set of separately analytic vectors for $\mathcal{Q}(\mathfrak{b})$, and

(Q6) $\mathcal{Q} \restriction \mathfrak{b}$ is faithful.

B.1 Algebraic Canonical Quantization

In this section, following [89], we give a more technical definition of (Canonical) Quantization, based on the general concept of abstract *-algebras.
Particularly, in the first part of the section we, shortly, summarize the algorithm for Canonical Quantization of a Poisson algebra on a symplectic manifold.
In spirit of completeness, in the second part we point out the mathematical structures mentioned in the algorithm. However, our description is partial and the interested reader may refer to literature for more details, particularly [89] and references therein.

B.1.1 Canonical Quantization algorithm

Now, we briefly present the Canonical Quantization algorithm:

1. Classical Poisson *-subalgebra $\mathfrak{P}$:
   Choose a Poisson*-subalgebra of the full Poisson algebra of classical observables such that the classical system is still completely characterized by it.

2. Quantum *-algebra $\mathfrak{A}$:
   Following Dirac's correspondence principle, to each member of the the chosen set of classical observables, associate an operator on an abstract *-algebra $\mathfrak{A}$ such that Poisson brackets of classical observables are mapped to $i\hbar$ commutators of the corresponding operators.

3. Representations of $\mathfrak{A}$:
   Determine an explicit *-representation of the *-algebra $\mathfrak{A}$ in terms of linear operators on a Hilbert space.

B.1.2 Technical discussion

As we have seen, the Poisson algebra of all classical observables is too big an algebra to quantize (cf. Groenewold-van Hove's theorem, and other no-go theorems [179]) and is, therefore, important to select an appropriate Poisson subalgebra of $C^\infty(M)$. As we have seen the notion of complete set of observables together with other physical or mathematical considerations, for example related to the presence of natural symmetries (see for instance [89]) may guide us in selecting the Poisson subalgebra. In this section we denote by $\mathfrak{P}$ the appropriately chosen classical Poisson *-subalgebra of the quadruple $(C^\infty(M), \{\cdot,\cdot\}, \cdot, *)$.
Particularly, the Classical Poisson *-subalgebra $\mathfrak{P}$ is a Poisson *-subalgebra of the full Poisson algebra of classical observables such that the classical system is still completely characterized by it.

Definition B.1.1. We define the quantization map from the classical Poisson *-subalgebra $\mathfrak{P}$ to an abstract *-algebra of operators $\mathfrak{A}$ as

$$\mathcal{Q} : \mathfrak{P} \to \mathfrak{A}$$

such that

1. $\mathcal{Q}$ is a *-linear map, that is a linear map that preserves involution

   $$\mathcal{Q}(z_1 a + z_2 b) = z_1 \mathcal{Q}(a) + z_2 \mathcal{Q}(b)$$

   $$\mathcal{Q}(\bar{a}) = \mathcal{Q}(a)^*$$

   for all $z_1, z_2 \in \mathbb{C}$, $a, b \in \mathfrak{P}$, and where $\bar{\cdot}$ denotes complex conjugation (involution on $\mathfrak{P}$);

2. $\mathcal{Q}$ maps Poisson brackets to commutators according to the correspondence principle

$$[\mathcal{Q}(a), \mathcal{Q}(b)] = i\hbar \mathcal{Q}(\{a, b\}), \forall a, b \in \mathfrak{P} \tag{B.2}$$

where we have restored the reduced Planck's constant $\hbar$.

We observe that sometimes it may be necessary to relax condition (B.2), for example in deformed quantization where (B.2) is replaced by

$$[\mathcal{Q}(a), \mathcal{Q}(b)] = i\hbar \mathcal{Q}(\{a, b\}) + \mathcal{O}(\hbar^2) \quad \forall a, b \in \mathfrak{P} \tag{B.3}$$

We proceed our discussion by determining the *-algebra of operators $\mathfrak{A}$ from $\mathfrak{P}$.
The first step is to construct the algebraic tensor algebra $T(\mathfrak{P})$ over $\mathfrak{P}$:

$$T(\mathfrak{P}) := \mathbb{C} \oplus \mathfrak{P} \oplus \mathfrak{P} \oplus \cdots = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathfrak{P}^{\otimes n} \tag{B.4}$$

Then, one can show that the set:

$$\mathfrak{I} := \{a_1 \otimes b_1 - b_1 \otimes a_1 - i\{a_1, b_1\} : a_1, b_1 \in \mathfrak{P}\}, \tag{B.5}$$

is a two-sided ideal of $T(\mathfrak{P})$.

Lastly, we can define the abstract *-algebra of operators as:

Definition B.1.2. The quantum *-algebra $\mathfrak{A}$ is the quotient of the tensor algebra $T(\mathfrak{P})$ of the classical Poisson *-subalgebra by the two-sided ideal

$$\mathfrak{A} := T(\mathfrak{P})/\mathfrak{I}. \tag{B.6}$$

Finally, we can consider representations π of $\mathfrak{A}$ as a concrete algebra of (in general, unbounded) operators on some (dense subspace of a) Hilbert space $\mathcal{H}$. In particular, $\pi : \mathfrak{A} \to \mathrm{Aut}(\mathcal{H})$ is a linear *-homomorphism between $\mathfrak{A}$ and the automorphisms of $\mathcal{H}$, preserving commutators:

$$\begin{aligned}
\pi(\lambda A + \mu B) &= \lambda\pi(A) + \mu\pi(B)\,, \\
\pi(AB) &= \pi(A)\pi(B)\,, \\
\pi(A^*) &= \pi(A)^*\,, \\
\pi([A, B]) &= [\pi(A), \pi(B)]\,,
\end{aligned}$$

for all $A, B \in \mathfrak{A}$ and $\lambda, \mu \in \mathbb{R}$.

B.2 Geometric Quantization

In Section B we have seen that the complete set of observables is an irreducibility condition on the set of classical observables. Such irreducibility condition is fundamental for the quantization of symplectic manifold.

An other approach to derive the underlying quantum theory of a given classical system is geometric quantization, which we briefly review in this section, following [172]. The main difference with the Algebraic Quantization described above is that the irreducibility condition is implemented by the so-called polarization.

In order to geometric quantize a Poisson manifold we first need a Hilbert space (called prequantum Hilbert space) together with a quantization procedure for observables that exactly maps the classical Poisson brackets into commutators.

The prequantum Hilbert space can be introduced by the concept of line bundle on a real manifold:

Definition B.2.1. A line bundle X on a real manifold M is a vector bundle with $\mathbb{K} = \mathbb{C}$ and fibre dimension 1.

Let $\Gamma_X(M)$ be the set of smooth sections over a real manifold M of a line bundle X. Denote by $Vect(M, \mathbb{C})$ the space of smooth complex vector fields (so multiplication by complex numbers is included in the scalar multiplication) on M and by $C^\infty(M, \mathbb{C})$ the set of smooth complex-valued functions on M.

Definition B.2.2. A connection ∇ on the line bundle X is a map $\nabla : Vect(M, \mathbb{C}) \to End(\Gamma_X(M))$, that assigns to each $X \in Vect(M, \mathbb{C})$ an operator ∇_X on $\Gamma_X(M)$ such that

- $\nabla_{fX+Y} = f\nabla_X + \nabla_Y;\;\; f \in C^\infty(M, \mathbb{C}),\; X, Y \in Vect(M, \mathbb{C})$
- $\nabla_X(fs) = X(f)s + f\nabla_X s;\; X \in Vect(M, \mathbb{C}),\; s \in \Gamma_X(M)$

- $\nabla_X(s+s') = \nabla_X(s) + \nabla_X(s'); X \in Vect(M,\mathbb{C}), s, s' \in \Gamma_X(M)$

On a line bundle-with-connection we can define a Hermitian structure, i.e. an inner product such that

1. $\langle\cdot,\cdot\rangle : (\{m\}\times\mathbb{C})\times(\{m\}\times\mathbb{C})\times \simeq \mathbb{C}\times\mathbb{C}\to\mathbb{R}$
2. $s(m)\times s'(m)\mapsto\langle s,s'\rangle(m)\equiv\langle s(m),s'(m)\rangle$

A connection ∇ and a Hermitian structure $\langle\cdot,\cdot\rangle$ are called compatible if for every complex vector field X, $X \in Vec(M,\mathbb{C})$, and for every $s, s' \in \Gamma_X(M)$ we have

$$X\langle s,s'\rangle = \langle\nabla_{\bar{X}}s,s'\rangle + \langle s,\nabla_X s'\rangle$$

Here $\bar{X}$ denotes the formal complex conjugate of the complex vector field X. The formal complex conjugate of X satisfies by definition the following rules,

1. $\overline{X+Y} = \bar{X}+\bar{Y}$ with X,Y vector fields.
2. $\lambda\bar{X} = \overline{\bar{\lambda}X}$ with X a vector field, $\lambda\in\mathbb{C}$ and $\bar{\lambda}$ the complex conjugate of λ.

Furthermore, if we ask that the Hermitian structure is compatible with the connection, then we have a Hermitian line bundle-with-connection.

Moreover, given a Hermitian line bundle-with-connection, we can define the curvature of the connection as:

Definition B.2.3. The curvature 2-form of ∇ as the complex differential 2-form Ω on M determined by

$$\Omega(X,Y)s = \frac{i}{2}([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})s \quad X,Y\in Vect(M), s\in\Gamma_X(M).$$

Here $[\nabla_X,\nabla_Y] = \nabla_X\nabla_Y - \nabla_Y\nabla_X$ is a commutator bracket.

To make the prequantization construction rigorous one needs a Hermitian line bundle-with-connection over the classical phase space (the symplectic manifold M).
However, the existence of such a line bundle-with-connection is in general not guaranteed.
For such a line bundle-with-connection to exist a certain condition should be satisfied by the curvature 2-form Ω defined on it. This is the so-called integrability condition (IC), for details [172].
Suppose that the symplectic manifold (M,ω) is quantizable in the sense that ω satisfies the integrability condition (IC). Then it is possible to construct a Hermitian line bundle-with-connection X with curvature 2-form Ω, called the prequantum line bundle, from which we can define the prequantum Hilbert space as the space of all sections s of a prequantum line bundle X.

Definition B.2.4. We define the prequantization map $\mathcal{Q}$ as

$$\mathcal{Q} = -i\hbar\nabla_{X_f}s + fs \tag{B.7}$$

where f is a classical observable $f\in S\subset C^\infty(M,\mathbb{R})$, S is a subalgebra of $C^\infty(M,\mathbb{R})$ (that is, a subspace of $C^\infty(M,\mathbb{R})$ which is closed under pointwise multiplication of smooth real-valued functions on M), and X_f denotes the Hamiltonian vector field corresponding to the classical observable f.

In order to define a polarization, one needs the notion of distributions on a manifold M.

Definition B.2.5. For all $m \in M$ consider the subspaces $P_m \subset T_m M$. If m has an open neighbourhood $U \subset M$ such that for all $u \in U$ a set of r independent vector fields $\{X_j\}$ exists with $Span\{X_j\} = P_u$, then

$$P \equiv \coprod_{m \in M} P_m \subset TM$$

is called a distribution of dimension r on M.

Particularly, we are interested in the notion of completely integrable distribution P.

Definition B.2.6. If for every point on M, a coordinate chart U can be found such that a distribution P on M is spanned by just the derivatives with respect to the coordinates on U, then the distribution is called completely integrable.

With this information we can now define the polarization:

Definition B.2.7. Let (M, ω) be a symplectic manifold. A polarization P of (M, ω) is a maximally integrable distribution of $TM^{\mathbb{C}}$ such that P_m is a Lagrangian subspace of $T_m M^{\mathbb{C}}$ for all $m \in M$, where $TM^{\mathbb{C}} \equiv \amalg_{m \in M} T_m M^{\mathbb{C}} \equiv \amalg_{m \in M}(T_m M \otimes \mathbb{C})$ is the complexify of the tangent bundle TM of M

Finally,

Definition B.2.8. A polarized section of the prequantum line bundle X is a section s over M satisfying

$$\nabla_{\bar{X}} s = 0 \quad \forall X \in Vect(M, C; P)$$

We conclude this section describing a particular class of symplectic manifolds for which geometric quantization is fairly well understood, the Kähler manifolds. These manifolds have natural and well-behaved polarizations.

To understand what a Kähler manifold is we first need to define what an almost complex manifold and complex manifold are respectively. Particularly,

Definition B.2.9. An almost complex manifold (M, J) is a real manifold M that is equipped with a smooth real tensor field J such that at every point $m \in M$ the linear endomorphisms $J_m : T_m M \to T_m M$ satisfy $J_m \circ J_m = I_m$ with $I_m : T_m M \to T_m M$, mapping $v \mapsto v$ the identity operator on $T_m M$.

Moreover,

Definition B.2.10. Let (M, J) be an almost complex manifold. The smooth real tensor field J is called integrable if the Nijenhuis tensor $N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY][JX, JY]$ vanishes for all $X, Y \in Vect(M, \mathbb{R})$.

Subsequently,

Definition B.2.11. A complex manifold (M, J) is an almost complex manifold such that J is integrable. In this case, the smooth tensor field J is called the complex structure .

With these definitions we can give the following definition of a Kähler manifold:

Definition B.2.12. A Kähler manifold (M, J, ω) is a complex manifold (M, J) which at the same time is a symplectic manifold (M, ω) such that the symplectic form ω and complex structure J are compatible, that is

$$\omega(JX, JY) = \omega(X, Y) \;\; \forall X, Y \in Vect(M, \mathbb{R}).$$

Let us denote the space of polarized sections of the prequantum line bundle X over the Kähler manifold (M, J, ω) with respect to a Kähler polarization P by $\Gamma_X(M; P)$. As previously, denote the prequantum Hilbert space by H.

Definition B.2.13. The Hilbert space $\mathcal{H}_P$ is the space defined by $\mathcal{H}_P \equiv H \cap \Gamma_X(M; P)$. In other words, it consists of all square-integrable polarized sections of X.

Finally, one can introduce metaplectic correction to the operator corresponding to a polarization preserving observable (also known as the half-form correction), i.e. technical modifications of the above procedure that are necessary in the case of real polarizations and often convenient for complex polarizations, for details (see [136]).

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