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Combinatorial methods in Statistical Field Theory

Trees, loops, dimers and orientations vs. Potts and non-linear σ-models

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Introduction

In the novel by Danilo Kiš *Garden, Ashes*, the father of the protagonist spends all his life in the impossible goal of writing a complete timetable of the Jugoslavian train transport system. Although this paradigm is not making here its first appearence in literature (among the predecessors is of course Borges – is the recurrence of this theme partially autobiographic among world novelists?), and certainly not the most famous, it is emblematic in the self-evidence of its impossibility: train timetables change at a rate vanifying any systematisation effort.

The writing of this thesis has a similar hystory. The main topic is at the interface between critical phenomena in statistical mechanics, and tools in enumerative combinatorics, both aimed to be "beyond mean-field theory", whatever is the precise assertion of this expression. A paradigmatic system is the ground for original developments: Kirchhoff classical determinantal formula for counting the spanning trees of a given graph (i.e., from the point of view of statistical mechanics, a certain limit $q \to 0$ of the q-state Potts model).

For this system, a surprising number of connections with other fields of physics / mathematics / computer-science arises, in such a structural way that it would be hard omitting a discussion for each. Determinantal point processes, the Abelian Sandpile Model, the XXZ quantum spin chain at $\Delta = 0$, Logarithmic minimal models in CFT (this one being the first member of the family, $\mathcal{LM}(1,2)$), Temperley-Lieb and Hecke Algebras, Invariant theory of classical superalgebras, Loop Models, SLE (at $(\kappa, \tilde{\kappa}) = (8,2)$), Loop-erased Random Walks, algorithmic aspects of exact sampling (in particular Propp and Wilson algorithm), non-intersecting lattice paths, tiling problems and Gessel-Viennot formula (through Temperley bijection), complexity issues for the evaluation of Tutte polynomial (the Jaeger-Vertigan-Welsh plane), partitionability of Tutte polynomial, Schaeffer decomposition of planar maps into blossomed trees, and so on.

All of this, still, to be dealt with before going to the main research topic of this report, that is, which of these results, and how, can be extended to spanning forests, and understanding that this more general problem is related to a QFT being a OSP(1|2)

supersymmetric non-linear σ-model (spanning trees being the limit for the sphere radius going to infinity, where the non-linearity disappears), and understanding the asymptotic freedom of the model in two-dimensional regular lattices.

Going back to the paradigm of Danilo Kiš novel, the train lines changing all the time are not much here the scientific literature by itself, but my personal knowledge of all those fields that are connected to this core topic by such long shots. We did our best to elucidate all these intersting connections, and give original contributions, when possible, in our research of the last years. However, we are not able to include all of these different directions in this work, as each of them would deserve a separate report. So, after some motivational introduction and some "warm-up" example, we try to go straight to the point, and discuss the main axis Potts model – Uniform spanning forests – non-linear supersymmetric σ-model.

Hopefully, our world-comprehensive train timetable will see the light one of these days.

1

Graph theory

Graph Theory is the theory which encodes in a graphical way (say, in a pencil drawing) the basic concept of adjacency among elements of a set, without assuming any other mathematical structure.

A large number of real-life problems are more or less explicitly formulated over some graph. A conceptual step is to disentangle the purely geometrical relations among the elementary constituent of the problem, from the extra structure of the components, e.g. the group symmetry of the target space.

We will try to give a small intuition of the fact that, although in such an abstract setting, graph theory, starting from this elementary notion of adjacency, quickly cooks up other intimately related mathematical concepts (like linear independence and Matroid Theory), and gives rise to deep and elegant theories.

Again, our stress here is to structural characterizations. In this case, a special role is played by classes of graphs which are "minor-closed". Typical examples are forests, series-parallel graphs, and planar graphs. It often happens that a statistical-mechanics model is solvable on a class of graphs which is minor closed. This is not surprising if we interpret the edges as carrying a tunable "coupling" parameter, and describing a two-variable interaction. If, as happens in many cases (but not all, cfr. dimer coverings), the problem is stable under series-parallel reduction, then the stability of the class of solvable graphs follows easily.

Graph Minor Theory includes one of the most striking theorems of modern mathematics (the *Robertson-Seymour Theorem*, stating that every minor-closed class is characterizable through a finite list of forbidden minors [1]), yet another evidence of how far one can go in statistical field theory by reasonings involving only the geometrical structure of the problem.

In this chapter we partially follow the textbook by Diestel [2].

1.1 Basics in graph theory

A *graph* is a pair of sets, $G = (V, E)$ such that $V(G)$, the *vertex set*, is an (abstract) finite set of distinct elements v, and $E(G)$, the *edge set*, is a finite set, each element e being associated to an unordered pair of elements $(v, v') \in V$, the *endpoints* of the edge. As customary, cardinality of finite sets is denoted by $|\cdot|$; furthermore, explicitly for graphs, $|G|$ is a synonimous for $|V(G)|$, and $\|G\|$ for $|E(G)|$. If vertices v and v' are the endpoints of edge e, we also say that e joins v and v', and that the two vertices are *adjacent vertices*,

or *neighbours*. Furthermore, also an edge is said to be adjacent to its end-vertices. The *degree* of a vertex, $\deg(v)$, is defined as the number of edges adjacent to v.

In principle, multiple edges (i.e. distinct edges with the same end-pairs) and tadpoles (i.e. edges whose ends coincide)[†] may occur. Allowing these "pathological" situations is not relevant in many concrete applications, while cause some notational confusion (as, for example, we would not be allowed to use (ij) as a synonimous of an edge e with endpoints i and j). This leads us to the definition of *simple graphs*, as the graphs with neither tadpoles nor multiple edges.

However, a relevant case in which it is useful to deal with generic graphs, instead that simple ones, is when considering recurrences and reduction formulas, as in most of these circumstances the set of generic graphs happens to be stable, and the set of simple ones does not.

For $V' \subseteq V(G)$, the graph $G|_{V'}$ has V' as vertex-set, and $E' = \{e = (v, v') \,:\, v, v' \in V'\}$ as edge-set. The graph $G|_{V'}$ is also called the subgraph of G *induced* by V'.

A pair of vertices v and v' which are not neighbours, but have a common neighbour v'', are in a sense "second-nearest neighbours", and so on. Iterating the notion of adjacency, one is lead to define *paths*, and, as a special case, *cycles* (or loops, or circuits) on the graph, and a *distance* among vertices. Given $v, v' \in V(G)$, a path $\gamma_{v\to v'}$ on G, (the subscript reads for "from v to v'"), of length ℓ, is a sequence of ℓ edges $(e_1, \dots, e_\ell)$ and $\ell + 1$ vertices $(v \equiv v_0, v_1, \dots, v_{\ell-1}, v_\ell \equiv v')$ such that the endpoints of e_i are $\{v_{i-1}, v_i\}$. A *cycle* is a closed path, i.e. a path such that $v \equiv v'$.

Being connected is clearly an equivalence relation on $V(G)$. The classes determine induced subgraphs of G, called the *connected components*. Their number is denoted by the symbol $K(G)$. Thus, a graph is said to be *connected* if it has a single class, i.e. every pair of vertices is joined by at least one path on G.

A first simple statement is that, for a connected graph T, every pair of vertices is joined by *exactly* one path on T if and only if T does not contain any cycle. A graph with these properties is called a *tree*. A *forest* F is a collection of disjoint trees. If $K(F) = k$, we may write, with an allusive choice of letters, $F = \{T_1, \dots, T_k\}$, where the T_α's are the components of F.

The distance $d(v, w)$ among two vertices v, w in a connected graph is defined as the length of the shortest path which connect them. The maximum distance among vertex pairs in $V(G)$ is called the *diameter* of G, $\mathrm{diam}(G)$; similarly, the *radius* of a graph, $\mathrm{rad}(G)$, is given by an optimization over vertices:

$$\mathrm{diam}(G) = \max_{v,w\in V(G)^2} d(v, w)\,; \qquad \mathrm{rad}(G) = \min_{v\in V(G)} \max_{w\in V(G)} d(v, w)\,. \tag{1.1}$$

A vertex v realizing the minimum in this last expression is called a *central vertex* of the graph. Clearly, $\mathrm{rad}(G) \le \mathrm{diam}(G)$. Also, by triangular inequality, if v_1 and v_2 are such that $d(v_1, v_2) = \mathrm{diam}(G)$ and v_c is central, one easily have $2\,\mathrm{rad}(G) \ge d(v_1, v_c) + d(v_2, v_c) \ge d(v_1, v_2) = \mathrm{diam}(G)$, i.e., putting the two things together, $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$.

Both extremal cases have correspectives in "ordinary" geometries. For example $2\,\mathrm{rad} = \mathrm{diam}$ on a sphere or a hypercube (and a single central vertex exists), while $\mathrm{rad} = \mathrm{diam}$ on the d-dimensional torus (and all points are central).

A related concept is the *girth* of the graph, i.e. the minimum length of a cycle in G, composed all of distinct edges. It is easily proven that for every connected graph G which is not a tree, twice the diameter of the graph is strictly larger than the girth

[†]Edges with equal endpoints are commonly called *loops* in Graph Theory. However, at the light of the (bad) habit in physics of using "loop" as a synonimous for "cycle", we introduce a different name (mutuated from Feynman diagrams).

$$\mathrm{girth}(G) \leq 2\,\mathrm{diam}(G) + 1\,. \tag{1.2}$$

The proof is a simple absurd: take a cycle $C \subseteq G$ of length equal to the girth, then, if $\mathrm{girth}(G) \geq 2\,\mathrm{diam}(G)+2$, pairs of sufficiently-far vertices have distance equal to $\mathrm{diam}(G)+1$, if restricted to C, realized by a path P_c. But their distance on the whole graph is at most equal to the diameter, so there is a path P, not totally contained in P_c, realizing this distance. Then, the union of P and P_c contains a cycle of length smaller than the girth, which contraddicts the definition. Classifying the set of graphs realizing the inequality in tight way, known as *Moore Graphs*, is not at all an elementary task (cfr. [3]).

In principle, it is not required that all the vertices $\{v_0, v_1, \ldots, v_\ell\}$ of a path are distinct (or $\{v_0, v_1, \ldots, v_{\ell-1}\}$ for the case of a cycle). If this happens, we say that the path (resp. the cycle) is *self-avoiding*. For a collection of paths or cycles, when not conversely specified, self-avoidness is understood to hold on the union of all vertex sets (i.e., the paths also "avoid each other"). A path is said to be *non-backtraking* if $v_i \neq v_{i+2}$ for all $i = 0, \ldots, \ell - 2$ [‡]. Remark that this is a weaker local notion w.r.t. self-avoidness, but nonetheless it suffices for many purposes, for example, for a path on a tree, being non-backtracking is a sufficient condition for self-avoidness, or, for the definition of girth above, one can equivalently require minimality among non-backtracking cycles. Furthermore, in many theorems of existence, uniqueness, and so on, especially if constructive, typically self-avoidness is tacitally understood. On the contrary, for combinatorial or enumerative problems (and then also in statistical-mechanics applications), clearly it is crucial to distinguish among the two concepts.

The graphical representation of a graph G is typically done by representing vertices with dots or small bullets, and edges as curves which join the corresponding vertex-ends. Although the choice of where to locate dots and how to draw lines is irrelevant, in order to help visualization, one typically choose to draw "reasonably short" lines, and to avoid crossing of edges not in correspondence of a common adjacent vertex. This second task can be fulfilled if and only if the graph is *planar*, as the word suggests. The concept of planarity, its implications and peculiar features are introduced in a more formal way in Section 1.3.

A crucial point in Graph Theory is that, both in the naïve idea of "pencil drawing", and the formal definition through the sets of vertices and edges, a tedious arbitrariness in the representation appears. For example, one can naturally choose the abstract set V to be the set of integers, $V = \{1, 2, \ldots, |V|\}$, and this induces a labeling for the edges (say, with a superscript $(i,j)^1$, $(i,j)^2$, ... for non-simple graphs), but the sequence of integers for the vertex-labeling (among the $|V|!$ possible ones) is *not* given in a natural way. On the other side, all graph-theoretical definitions are carefully chosen in order to be "natural" and "instrinsic" to the geometrical structure purely induced by the local adjacency of the elementary components, i.e. invariant under the group $\mathcal{S}_{|V|}$ of label permutations, and thus, also the derived results are intrinsic in this sense.

At this aim, it is useful to allow for a vertex-labeling as above, and to define the proper quotient among equivalent structures, given by the concept of *graph isomorphism*. Two simple graphs G and G' are isomorphic if a bijection $\phi : V(G) \to V(G')$ exists such that $(v_1, v_2) \in E(G)$ if and only if $(\phi(v_1), \phi(v_2)) \in E(G')$ (for an example, cfr. figure 1.1)[††]. As always, this notion of isomorphism immediately induces a notion of *automorphism*, for $G \equiv G'$. The problem of recognizing whether two graphs are isomorphic, or of classifying

[‡]Here assume for notational clearness that $v \equiv v_0$ and $v' \equiv v_\ell$.

[††]The definition trivially extends to non-simple graphs: G and G' are isomorphic if a map $\phi : V(G) \to V(G')$ exists such that if k edges with ends (v_1, v_2) exist in $E(G)$, then also k edges with ends $(\phi(v_1), \phi(v_2))$ exist in $E(G')$.

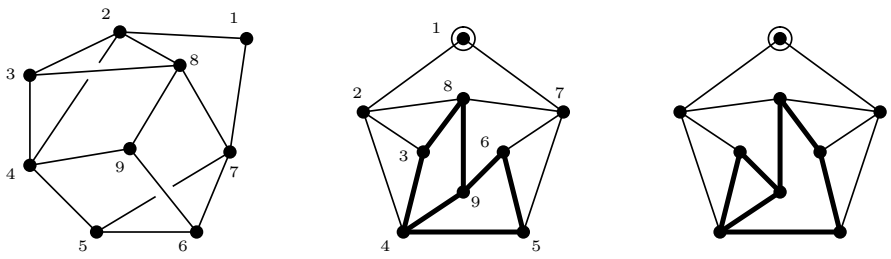


Fig. 1.1. On the left and center, two isomorphic graphs. A proof of the isomorphism is given by the labeling of the vertices: it suffices to check that the sets of induced edge labelings $\{(1,2),(1,7),(2,3),(2,4),(2,8),\dots,\}$ coincide. The representation on the center shows that the graph is planar, a thing that may have been not evident for the left-most representation. On the right, a graph which looks very similar, but is not isomorphic to the previous ones. A certificate of non-isomorphism is not obvious. A possible one could be: 1) notice that in both graphs there is a single vertex with degree 2 (the circled one): they must be one the image of the other under any isomorphism ϕ; 2) the sets of vertices at distance 2 or 3 from this special vertex contain six elements in both cases; 3) *but* the induced subgraphs (cfr. next section), denoted in bold lines, in the two cases are clearly non-isomorphic: for the graphs on the left-center, it consists of two squares sharing an edge, for the graph on the right, of a penthagon and a triangle sharing an edge.

the automorphism group of a given graph, although trivial in many cases (say, if $|G| \neq |G'|$, or $\|G\| \neq \|G'\|$, or the ordered sequence of $\{\deg(v)\}_{v\in V(G)}$ and $\{\deg(v')\}_{v'\in V(G')}$ do not coincide, and so on, we have a simple certificate of non-isomorphism), is a hard problem in general, and has lead to the emergence of interesting mathematics (cfr. [4]).

We will see in section 1.3 that a more restrictive notion of isomorphism can be naturally defined for a planar graph: in a sense, the isomorphism ϕ must preserve not only adjacency but also the planar embedding. More precisely, we deal there with isomorphism of *two-dimensional cell complexes*. This more restrictive notion, and the induced finite-dimensional feature on the search structure, make planar-isomorphism an "easy" problem, in the sense that a polynomial-time solution algorithm exists, and that the mathematics involved in its comprehension is not so severe. This feature, of showing a complexity gap between arbitrary graphs and arbitrary planar graphs, is common to other "generally hard" combinatorial problems on graph, and we will come back on the subject when dealing with Ising Model and Dimer Coverings, in Section 2.4.

Given a graph $G = (V, E)$, a *subgraph* $S \subseteq G$ is a pair (V', E') with $V' \subseteq V$ and $E' \subseteq E$, such that all the end-vertices in E' are in V'. This consistency check, although graphically clear, gives a complicated geometry to the space of subgraphs of a given graph (where the geometry is given by the inclusion order-relation). This is relaxed in two important special cases: a *(vertex-)induced subgraph*, already defined above, is determined by a vertex subset $V' \subseteq V$, with $E(S)$ being the set of edges $e \in E(G)$ such that both ends are in V' (in other words, it is the "maximal" subgraph with given V'); analogously, an *edge-induced subgraph* is defined only by an edge subset $E' \subseteq E$, with $V(S)$ being the union of the edges in E', seen as sets of cardinality two (in other words, it is the "minimal" subgraph of given E'); on the other side, a *spanning subgraph* is defined by an edge subset $E' \subseteq E$, with $V(S)$ being simply coincident with $V(G)$ (it is the "maximal" subgraph of given E'). In all these cases, S does not need to be connected. In particular, a spanning

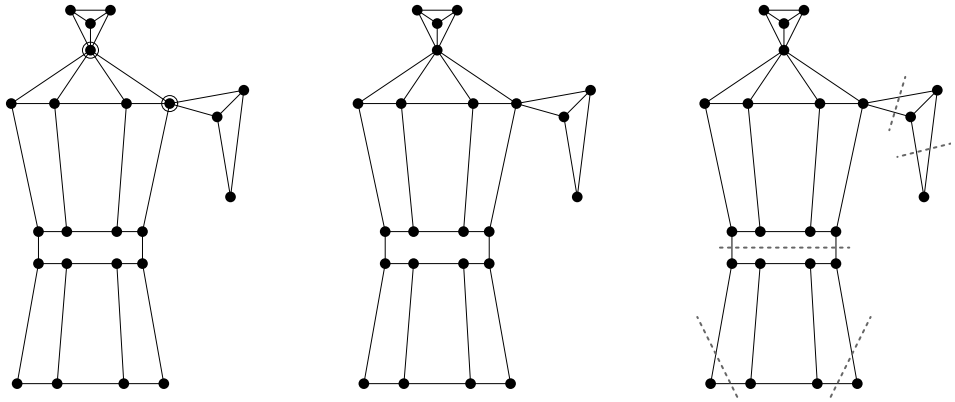


Fig. 1.2. In the middle, a graph, shaped like a moka espresso coffee maker, which well illustrates the notions of vertex- and edge-connectivity, and having $\kappa(G) = 1$ and $\lambda(G) = 2$. On the left, the two cutvertices are marked with circles. On the right the five edge-cutsets are denoted by dashed lines crossing the edges.

subgraph containing only a few edges constituting a single non-trivial component, formally contains also all the isolated vertices as single connected components.

A graph is said to be *bipartite* if a map $\sigma : V(G) \to \pm 1$ exists such that for each pair of adjacent vertices v, w we have $\sigma(v)\sigma(w) = -1$. If such a map exists, then their number is exactly $2^{K(G)}$. Otherwise, G must contain a cycle of odd length. Indeed, it is clear that the satisfaction of the constraint on a subgraph is a necessary condition of existence, and that and odd cycle is not bipartite; on the other side, if one starts choosing $\sigma(v)$ recursively on a whatever sequence $(v_1, \ldots, v_V)$ of vertices such that v_k is adjacent to some of the $\{v_1, \ldots, v_{k-1}\}$, and finds a contraddiction at step k, i.e. a pair of adjacent vertices v_k, v_h with the same sign, all the cycles on the induced subgraph with vertices $\{v_1, \ldots, v_k\}$ containing the edge (v_h, v_k) have odd degree.

The simple graph with n vertices $V = \{1, \ldots, n\}$ and all the possible $n(n-1)/2$ edges $E = \{(1,2), \ldots, (1,n), (2,3), \ldots, (n-1,n)\}$ is called the *complete graph* at n vertices, $\mathcal{K}_n$. The simple graph with $m + n$ vertices $V = \{1, \ldots, m + n\}$ and all the possible mn edges connecting one of the first m vertices to one of the last n is called the *complete bipartite graph* at m and n vertices of the two species, $\mathcal{K}_{m,n}$.

Given a connected graph $G = (V, E)$, a subset $X \subseteq V \cup E$ of its vertices and edges is said to be a *separating set* for G if a pair of vertices v, w not in X exists such that there is no path $\gamma_{v \to w}$ whose intersection with X is empty. The special cases of X composed of a single vertex or of a single edge are called respectively *cutvertex* and *bridge*. The minimal size of X among the separating sets of G composed of only vertices or of only edges are called respectively *vertex-* and *edge-connectivity*, and denoted as $\kappa(G)$ and $\lambda(G)$. Although these notions are somewhat more advanced w.r.t. the other ones introduced so far, they should not sound new to the physicist reader. Indeed, when exact resummations are considered in a Feynman diagrammatic expansion one often reduce the diagrams of interest to the family of *one-particle irreducible* ones, corresponding here to graphs G with $\lambda(G) > 1$.

The sets of induced subgraphs or of spanning subgraphs of a given graph are natural concepts for a statistical-mechanics study of a problem in which the graph plays the role of the physical lattice. This happens in two ways. First, there are many models in

which the subgraph itself is the interesting object (models for polymers, growth processes, percolation, aggregation). Furthermore, there are statistical-mechanics model, or even field theories, for which the degrees of freedom in the system, for example spin variables, are located on the elementary constituents of the graph (typically, on the vertices), the geometry of the interaction pattern is encoded in the graph-adjacency, and algebraic manipulations in the partition function lead to a sum over some family of subgraphs. In particular, the *a priori* measure of a statistical mechanics model is typically factorized on the single constituents, and both the ensembles corresponding to induced subgraphs and spanning subgraphs have a product form (indeed, they are naturally isomorphic to $\mathbb{Z}_2^{|G|}$ and $\mathbb{Z}_2^{\|G\|}$ respectively). For example, the partition function of a gas of particles (*lattice gas*, LG), with nearest-neighbour interaction potential u and chemical potential μ, is the generating function of induced subgraphs:

$$Z_{\mathrm{LG}} = \sum_{n\in\{0,1\}^V} e^{-\mu\sum n_i + u\sum_{\langle ij\rangle} n_i n_j} = \sum_{\substack{S\subseteq G\\ \text{vertex-induced}}} \exp\left(-\mu|S| + u\|S\|\right) ; \tag{1.3}$$

while the partition function of the Potts Model, introduced in Chapter 4, is shown to have a purely combinatorial reformulation in terms of spanning subgraphs

$$Z_{\mathrm{Potts}} = \sum_{\sigma\in\{1,\dots,q\}^V} \prod_{\langle ij\rangle} \left(1 + e^{\beta}\delta(\sigma_i,\sigma_j)\right) = \sum_{\substack{S\subseteq G\\ \text{spanning}}} q^{K(S)} e^{-\beta(\|G\|-\|S\|)} ; \tag{1.4}$$

and in particular, as $K(S) = 1$ only if $E(G \smallsetminus S)$ is not a separating set, the edge-connectivity of a graph is related to a zero-temperature limit of the free energy of the associated Potts model

$$\lambda(G) = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln \lim_{q\to 0} \frac{\partial}{\partial q} \frac{Z_{\mathrm{Potts}}(q,\beta;G)}{q} = \lim_{\substack{q\to 0\\ \beta\to\infty}} \frac{\partial}{\partial q} \frac{F_{\mathrm{Potts}}(q,\beta;G)}{q} . \tag{1.5}$$

We conclude with some dictionary correspondence with physics gergon, and especially statistical mechanics. It is frequent that coordinate-letters, such as x, y, ..., or latin letters, such as i, j, ..., are use to label vertices of a graph, and the notation $\langle ij\rangle$ is used to indicate that i and j are neighbours, i.e. that $(ij) \in E(G)$. Furthermore, "point" or "site" are often used as synonimous of vertex, and "bond", or "link", as synonimous of edge. The term "loop" is used as a synonimous of (self-avoiding) cycle, and this makes confusion with the mathematics use of loop for an edge with coinciding endpoints, which for this reason, we call here "tadpole".

1.2 Cyclomatic space and Euler formulas

We start now recognizing the natural emergence of some more mathematical structure on the graphs. Given the set of subgraphs (the reasoning extends also to the restriction to induced or spanning subgraphs), inclusion induces a (mathematical) lattice structure in this set, the meet and join operations being given by intersection and union operations among sets.

For the set $\mathcal{S}$ of spanning subgraphs, the symmetric difference of finite sets, $E_1 \vartriangle E_2 := (E_1 \cup E_2) \smallsetminus (E_1 \cap E_2)$, applied to edge-sets of the subgraphs, also induces a vector-space structure on $\mathbb{Z}_2$. Define $\deg_S(v)$ as the degree of vertex v in the subgraph S: a

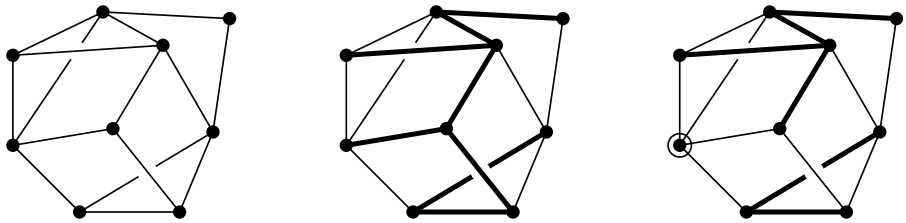


Fig. 1.3. A graph, a spanning tree and a spanning forest (an isolated vertex is highlighted for clarity).

vector subspace of $\mathcal{S}$ closed under the sum defined above is the space $\mathcal{L}$ of subgraphs L such that $\deg_L(v)$ is even for each vertex v. These are called *Eulerian subgraphs* This space has dimension $L(G)$ (in $\mathbb{Z}_2$), which is the number of *independent loops* in G, or *cyclomatic number*. Eulerian subgraphs consisting of a single connected component, and for which all the coordinations are either 0 or 2, are self-avoiding cycles. We already said that a definition of a tree is as a graph which contains no cycles; now we can restate this property saying that a tree is a connected graph T whose vector space $\mathcal{L}(T)$ has dimension zero.

A basis for $\mathcal{L}$ can be found, such that all the basis elements are cycles. This can be shown by the following construction: given the graph G, first consider the trivial graph G_0 such that $V(G_0) = V(G)$ and $E(G_0) = \varnothing$ (thus also $K(G_0) = |G|$ and $L(G_0) = 0$); then add the edges of G one by one, in a whatever sequence. At each step, add edge e_k to G_{k-1}, to obtain G_k. Two things can happen: if the new edge e_k connects two distinct components, $\Delta K = -1$, and the edge can not be contained in any loop subgraph (the intersection of the subgraph with each of the two previous connected components would be a subgraph with only one vertex of odd degree, and clearly in any graph the number of odd-degree vertices must be even), thus $\Delta L = 0$. If the new edge has both ends in the same components, $\Delta K = 0$, and a path γ exists on the previous component, joining the two ends. The union of this path and the new edge is a cycle, and a new independent loop, and can be chosen as a basis element. On the other side, the other loops containing e_k are such that their symmetric difference with the new loop is totally contained in G_{k-1}, and this proves that $\Delta L = 1$ (and not larger). Remark the corollary result, holding for each graph with V vertices and E edges: as at each step of our construction, either K decreases by one, or L increases by one, and for the graph $G_0 = (V, \varnothing)$, we have $K = V$ and $E = L = 0$, we conclude that

$$V + L = E + K\,. \tag{1.6}$$

This formula is known as *Euler formula* for generic graphs. In particular, for a forest, the condition $L(F) = 0$ gives the specialization $V = E + K$, and for a tree, the further condition $K(T) = 1$ gives $V = E + 1$. Examples are shown in figure 1.3.

The definition of the space $\mathcal{L}(G)$ of Eulerian subgraphs of G is one of a much broader family. Consider any abelian group H, with an additive notation. Take an arbitrary orientation of the edges (that is, for any edge $e = (ij)$ choose once and forever i to be the "tail" and j to be the "tip", or vice versa. In order to recall that E has such a structure, we use the notation $\mathbf{E}$ for this set. Consider now functions $f : \mathbf{E} \to H^E$, with the synonima $f(ji) = -f(ij)$. Such a function is called a *H-flow* on G if, for any cycle of the graph $\gamma = (i_1, i_2, \ldots, i_\ell, i_{\ell+1} \equiv i_1)$, the combination $f(i_1\, i_2) + f(i_2\, i_3) + \cdots + f(i_\ell\, i_1)$ is the

identity element of the group. Note that taking the cycle in reverse order does not change the definition. If the group has characteristic 2 (that is, it is $(\mathbb{Z}_2)^k$), then $g = -g$ and choosing an orientation is not necessary. We can thus define the space $\mathcal{L}(G; H)$ of H-flows over G, which is now a vector space $H^{L(G)}$, and the space $\mathcal{L}(G)$ above corresponds to $\mathcal{L}(G; \mathbb{Z}_2)$. Remark that the integer $L(G)$, intrinsic to the graph, is *the same* for all abelian groups.

Generating functions of flows, starting from Eulerian subgraphs, are also natural objects in Statistical Mechanics. They indeed abstract (to graphs, from regular lattices) the notion of irrotational abelian vector field.

A further extension can be done for *trace formulas* in non-abelian groups, instead that only using abelian groups. So say now that H is a (multiplicative) group, $f(ji) = f(ij)^{-1}$, and we have a map $\mathrm{tr}_\rho(\cdot)$ from H to a field with a notion of conjugation $(\cdot)^*$. This map is required to be cyclic invariant, i.e. $\mathrm{tr}_\rho(g_1 g_2 \cdots g_k) = \mathrm{tr}_\rho(g_2 \cdots g_k g_1)$, and unitary, $\mathrm{tr}_\rho(g) = \mathrm{tr}_\rho(g^{-1})^*$. called the *character of the representation* ρ. Then, flows are functions f such that $\mathrm{tr}_\rho\left(f(i_1\, i_2) f(i_2\, i_3) \cdots f(i_\ell\, i_1)\right)$ is the identity element of the field. Again, the properties we required ensure that the definition does not depend on the starting point of the cycle, and on the orientation.

This generalization is again natural in Statistical Mechanics. These flows abstract (to graphs, from regular lattices) the notion of irrotational non-abelian vector field. Typically, in physics the group is directly taken as a group of matrices, instead of abstracting to an arbitrary group, and then considering its characters and representations.

1.3 Planar graphs

A planar graph is a graph with the special property that an embedding on a genus-0 surface (a sphere) can be found such that there are no edge crossings.

Chosen such a representation, a new structure is identified: we have a set F of *faces*, as the set of elementary cycles We now explain this concept. A simple path on a surface of genus 0 identifies two disconnected regions. This is valid of course also for our discrete setting. A simple cycle is elementary if one of the two regions has no simple open paths with both endpoints on the cycle. If, conversely, it has paths with this property, these paths are said to be *chords* of the cycle. So, faces are cycles which, on one of the two sides, have no chords. The notion of adjacency can be given also for faces, stating that a face is adjacent to an edge, or a vertex, if the edge, or the vertex, enters the cycle corresponding to the face. Remark that each edge of G is adjacent to exactly two faces. Two faces are adjacent if they share one (or more) edges.

With this extra structure, the graph is raised to a two-dimensional cell complex. For these graphs, a notion of duality can be introduced. This is indeed an aspect, restricted to the special two-dimensional case, of *Poincaré duality* for cell complexes, and of *Hodge duality*, but here we will simply call it *planar duality*. It is also a special case of *Matroid duality* for graphical matroids, where one can show that, for planar graphs and only for this class, also the dual matroid is graphical.

Besides these scaring connections, a handwaving definition of the duality is very simple. We use *direct-* and *dual-* prefix for elements in the original and in the dual graph. The dual graph G^* of a graph G has vertices corresponding to the faces of G, (say, draw each dual-vertex as a point inside the corresponding direct-face), and, for every edge e of G, there is an edge e^* of G^* joining the two faces of G that contains e. The number of sides of a given face in the original graph G is equal to the degree of the corresponding vertex on the dual graph.

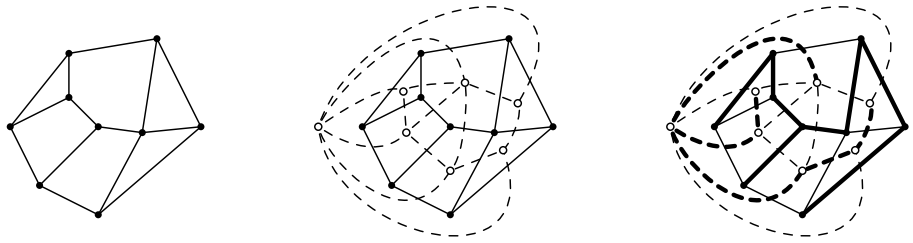


Fig. 1.4. A planar graph, its planar dual, and a pair of dual spanning subtrees.

This notion of duality has a natural extension to the set of spanning subgraphs, that we call *planar duality for subgraphs*. Given a spanning subgraph $S = (V, E')$ on G, its dual subgraph $S^* = (V^*, E'^*)$ is such that, for each pair of edge and dual-edge (e, e^*), either $e \in E'$ and $e^* \notin E'^*$, or $e \notin E'$ and $e^* \in E'^*$. In formulas

$$E(S^*) = E(G^*) \smallsetminus \big(E(S)\big)^* . \tag{1.7}$$

With this definition, it is clear that a cycle on the original subgraph corresponds to a connected component on the dual graph, and vice-versa, thus we have the following correspondences among subsets of spanning subgraphs, under the application of duality (D) on a graph:

$$\text{connected subgraphs} \quad \underset{D}{\longrightarrow} \quad \text{subgraphs without loops} \tag{1.8}$$

$$\text{subgraphs without loops} \quad \underset{D}{\longrightarrow} \quad \text{connected subgraphs} \tag{1.9}$$

Thus the set of spanning trees of a given planar graph has the special property that its defining characteristic is self-dual:

$$\{\text{spanning trees}\} = \left\{ \begin{array}{l} \bullet \text{ connected} \\ \bullet \text{ no loops} \end{array} \right\} \underset{D}{\longrightarrow} \left\{ \begin{array}{l} \bullet \text{ no loops} \\ \bullet \text{ connected} \end{array} \right\} = \{\text{spanning trees}\} . \tag{1.10}$$

No telling, the property of being spanning is preserved going from $S \subseteq G$ to its dual $S^* \subseteq G^*$. It results the the number of spanning trees is the same for a graph and its dual graph, and each spanning tree is in a canonical one-to-one correspondence with a spanning tree on dual. Remark that the obvious alternate bijection w.r.t. the one described by equation (1.7), namely

$$E(S^*) = \big(E(S)\big)^* , \tag{1.11}$$

would have had no special good property w.r.t. the "topological" quantities $K(S)$ and $L(S)$, as well as none of the two definitions, which is defined with no need of introducing the faces, has special good property with this respect if the underlying graph is not planar.

1.4 Graphs vs. hypergraphs

Hypergraphs are the generalization of graphs in which edges are allowed to contain more than two vertices. A *hypergraph* is a pair $G = (V, E)$, where V is a finite set and E is a collection (possibly empty) of subsets of V, each of cardinality ≥ 2. The elements of V are the *vertices* of the hypergraph G, and the elements of E are the *hyperedges* (the

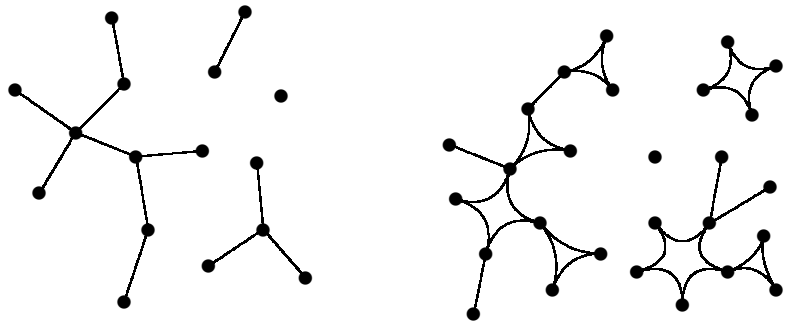


Fig. 1.5. A forest (left) and a hyperforest (right), each with four components. Hyperedges with more than two vertices are represented pictorially as star-like polygons.

prefix "hyper" can be omitted for brevity). Note that we forbid hyperedges of 0 or 1 vertices (some other authors allow these).[♯] We shall say that $A \in E$ is a *k-hyperedge* if A is a k-element subset of V. A hypergraph is called *k-uniform* if all its hyperedges are k-hyperedges. Thus, a graph is nothing other than a 2-uniform hypergraph.

The definitions of subgraphs, walks, cycles, connected components, trees, forests and unicyclics given in the previous sections for graphs generalize to hypergraphs. In certain cases, there are various possibilities, so that, in order to avoid confusions, we state here again these properties in the generalized framework.

A *walk* (of length $k \geq 0$) connecting v_0 with v_k in G is a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ such that all $v_i \in V$, all $e_i \in E$, and $v_{i-1}, v_i \in e_i$ for $1 \leq i \leq k$. Note that in general $\{v_{i-1}, v_i\} \subseteq e_i$, while for graphs we had equality.

A *path* in G is a walk in which $v_0, \ldots, v_k$ are distinct vertices of G and $e_1, \ldots, e_k$ are distinct edges of G. A *cycle* in G is a walk in which

(a) $v_0, \ldots, v_{k-1}$ are distinct vertices of G, and $v_k = v_0$
(b) $e_1, \ldots, e_k$ are distinct edges of G; and
(c) $k \geq 2$.

Again, the graph G is said to be *connected* if every pair of vertices in G can be connected by a walk. The *connected components* of G are the maximal connected subgraphs of G.

A *hyperforest* is a hypergraph that contains no cycles, while *hypertree* is a connected hyperforest. See Figure 1.5 for examples of a forest and a hyperforest.

The analogue of the Euler relation for graphs is the following (comparatively weaker) statement

Proposition 1.1. *Let $G = (V, E)$ be a hypergraph. Then*

$$\sum_{A \in E} (|A| - 1) \; - \; |V| \; + \; k(G) \; \geq \; 0 \,, \tag{1.12}$$

with equality if and only if G is a hyperforest.

[♯] Our definition of hypergraph is the same as that of McCammond and Meier [6]. It is also the same as that of Grimmett [7] and Gessel and Kalikow [8], except that these authors allow multiple edges and we do not: for them, E is a *multiset* of subsets of V (allowing repetitions), while for us E is a *set* of subsets of V (forbidding repetitions).

Proofs can be found, for instance, in [5, p. 392, Proposition 4] or [8, pp. 278–279, Lemma]. They are however elementary, and generalize the inductive argument used above for Euler relation.

Please note one important difference between graphs and hypergraphs: every connected graph has a spanning tree, but not every connected hypergraph has a spanning hypertree. Indeed, it follows from Proposition 1.1 that if G is a k-uniform connected hypergraph with n vertices, then G can have a spanning hypertree only if $k-1$ divides $n-1$. Of course, this is merely a necessary condition, not a sufficient one! In fact, the problem of determining whether there exists a spanning hypertree in a given connected hypergraph is NP-complete (hence computationally difficult), even when restricted to the following two classes of hypergraphs:

(a) hypergraphs that are linear (each pair of edges intersect in at most one vertex) and regular of degree 3 (each vertex belongs to exactly three hyperedges); or
(b) 4-uniform hypergraphs containing a vertex which belongs to all hyperedges, and in which all other vertices have degree at most 3 (i.e., belong to at most three hyperedges)

(see [138, Theorems 3 and 4]). The problem is polynomial for 3-uniform hypergraphs, but through a very sophisticated algorithm, in the wider context of 2-polymatroids, due to Lovász [9, 10].

Finally, let us discuss how a connected hypergraph can be built up one edge at a time. Observe first that if $G = (V, E)$ is a hypergraph without isolated vertices, then every vertex belongs to at least one edge (that is what "without isolated vertices" means!), so that $V = \bigcup_{A \in E} A$. In particular this holds if G is a connected hypergraph with at least two vertices. So let $G = (V, E)$ be a connected hypergraph with $|V| \geq 2$; let us then say that an ordering $(A_1, \ldots, A_m)$ of the hyperedge set E is a *construction sequence* in case all of the hypergraphs $G_\ell = \big(\bigcup_{i=1}^{\ell} A_i, \{A_1, \ldots, A_\ell\} \big)$ are connected ($1 \leq \ell \leq m$). An equivalent condition is that $\Big(\bigcup_{i=1}^{\ell-1} A_i \Big) \cap A_\ell \neq \varnothing$ for $2 \leq \ell \leq m$. We then have the following easy result:

Proposition 1.2. *Let $G = (V, E)$ be a connected hypergraph with at least two vertices. Then:*

(a) There exists at least one construction sequence.

(b) If G is a hypertree, then for any construction sequence $(A_1, \ldots, A_m)$ we have $\Big| \Big(\bigcup_{i=1}^{\ell-1} A_i \Big) \cap A_\ell \Big| = 1$ for all ℓ ($2 \leq \ell \leq m$).

(c) If G is not a hypertree, then for any construction sequence $(A_1, \ldots, A_m)$ we have $\Big| \Big(\bigcup_{i=1}^{\ell-1} A_i \Big) \cap A_\ell \Big| \geq 2$ for at least one ℓ.

Proof. (a) The "greedy algorithm" works: Let A_1 be *any* hyperedge; and at each stage $\ell \geq 2$, let A_ℓ be *any* hyperedge satisfying $\Big(\bigcup_{i=1}^{\ell-1} A_i \Big) \cap A_\ell \neq \varnothing$ (such a hyperedge has to exist, or else G fails to be connected).

(b) and (c) are then easy consequences of Proposition 1.1. □

1.5 Algebraic graph theory and the Laplacian of a graph

Algebraic Graph Theory [3] is a branch of Graph Theory in which algebraic methods are applied to problems about graphs. This is then adding new ingredients to the purely combinatorial approach of the previous sections. There are three main branches of algebraic graph theory, involving the use of linear algebra, the use of group theory, and the study of graph invariants. Our stress here is on those aspect, of all of these facets, which involve Statistical Mechanics, i.e. dealing algebraically with generating functions, and similar approaches.

Indeed, the whole story, central in these notes, of Tutte polynomial, started with the goal of finding a powerful algebraic invariant for graphs, that could characterize in a fine way families of isomorphic graphs. We recall that any graph-invariant, i.e. any function of the adjacency structure of the graph which is invariant w.r.t. the vertex labeling, divides the adjacency-matrix realizations of graphs into families which never split realizations of isomorphic graphs. So, the finer is the invariant, the stronger is the induced isomorphism test. This is very similar in spirit to the probably more known example of *knots invariants,* as described for example in [11].

A very basic isomorphism test for weighted graphs (with weights w_e – turning into ordinary isomorphism test if all w_e are equal), and also very "physical", is the following: put some unit height $h_i = 1$ on each vertex i of your graph, and let the system evolve "diffusively", that is,

$$\frac{\partial}{\partial t} h_i(t) = \sum_{j\sim i} w_{ij} \big(h_j(t) - h_i(t)\big)\,. \tag{1.13}$$

Remark that $\sum_i h_i(t)$ is constant in t. This is, of course, the space-discretized version of the *heat equation*, the partial differential equation which describes the distribution of heat (or variation in temperature) in a given isolated region over time. If the system is not isolated, and any infinitesimal element of height has the same probability of "evaporating" from the system, we would have

$$\frac{\partial}{\partial t} h_i(t) = -m h_i(t) + \sum_{j\sim i} w_{ij} \big(h_j(t) - h_i(t)\big)\,. \tag{1.14}$$

This kind of evolution is at sight independent from the labeling on the graph, so two graphs G_1 and G_2 are isomorphic only if the corresponding lists of heights $\{h_i^{(1)}(t)\}$ and $\{h_i^{(2)}(t)\}$ coincide at all times, up to reorderings (furthermore, the reordering may fix part of the isomorphism).

As is well known, the heat equation describes *Laplacian evolution*, that is, the equation above can be written in matrix form

$$\frac{\partial}{\partial t} h_i(t) = -\sum_j L_{ij} h_j(t)\,, \tag{1.15}$$

where L_{ij} is the discretized version of the Laplacian operator, that is the *Laplacian matrix* of the graph G, whose expression is easily deduced from comparison of (1.13) and (1.15)

$$L_{ij} = \begin{cases} -w_{ij} & \text{for } i \neq j, \\ \sum_{k\neq i} w_{ik} & \text{for } i = j. \end{cases} \tag{1.16}$$

This is a symmetric matrix with all row and column sums equal to zero. Since L annihilates the vector with all entries 1 (that is, the total heigh remains constant in the heat equation),

the determinant of this matrix is zero. If all weights are strictly positive, the remaining part of the spectrum is strictly positive [3, pag. 280].

Coming back to the isomorphism test, coincidence of the heights at all times is equivalent to the condition that the spectra of the Laplacian matrices associated to the two graphs are identical.

Actually, a stronger condition is necessary for isomorphism: that the Laplacian matrix of graph G_1 can be transformed into the Laplacian matrix of graph G_2 by conjugation with a permutation matrix (this is the very definition of isomorphism, after realizing that the Laplacian matrix contains all the information of the adjacency matrix), but such a property is not testable in an efficient way, while the spectrum is calculated in polynomial time, by algebraic techniques. So, Algebraic Graph Theory provides already a strong isomorphism test through methods that do not have a simple combinatorial restatement.

Of course, also statistical mechanics provides isomorphism tests: if the partition functions of a problem, when specialized to systems defined on graph G_1 and G_2, coincide, then G_1 and G_2 may be isomorphic. In particular, subgraph enumeration problems have this characteristic at sight. However, statistical mechanics models for which the partition function is hard to evaluate are not very effective test, and models (such as percolation) for which the partition function is blind to a large parte of the structure of the graph are not strong test. Surprisingly enough, the Laplacian test we sketched above coincides with a test in the form we have just described, at the light of Kirchhoff Matrix-Tree theorem: as the number of trees on a graph is related to the spectrum of the matrix, and the generating function of rooted spanning forests is the characteristic polynomial of the Laplacian matrix, the isomorphism test arising from the generating function of rooted spanning forests coincides with the one described by heat diffusion.

The problem of heat diffusion on a graph has also a combinatorial restatement in stochastc geometry in terms of random walks (Brownian motion when on the continuum), and has a long story of analytic approaches. In particular, an important role is played by the *heat kernel*, and the *Green function*, that is, in this context, the matrix

$$G(m) = (mI + L)^{-1} \,. \tag{1.17}$$

This matrix is the formal solution of the dissipative heat diffusion problem, starting from a condition of height concentrated on a single site. Then, by linearity of the equation, the set of solutions for height concentrated on a single site i, running over all sites, describes the evolution of an arbitrary initial condition. It is the Green function $G_{ij}(m)$ that is more strictly related to the formulation in terms of random walks, as it corresponds to the generating function for the random walks, of arbitrary length, starting at i and arriving at j, where the length is counted by the inverse powers of the mass. This statement is easily proven by the remark that the combinatorial local consistency required to the set of generating functions $\{G_{ij}(m)\}_{j\in V(G)}$ corresponds exactly to the equation $\sum_k G_{ik}(m)(m\delta_{kj} + L_{kj}) = \delta_{ij}$.

1.6 Subgraph enumeration: an overview of relationships

In the remainig part of this chapter, we recall, or give, the definion of a number of counting problems which are formulated in terms of subgraph enumerations over a given (weighted) graph, and can be related to statistical mechanics problems and, possibly, statistical field theories with 'ordinary' (local bosonic and/or fermionic) degrees of freedom. We

emphasize that fact that, essentially in all cases, a general formulation exists, in terms of multivariate (mostly multiaffine) generating functions over arbitrary weighted graphs (cfr. [12] for a discussion on the "multivariate philosophy").

As discussed in the introduction, the core subject of this report is the model of unrooted spanning forests. This model has the form described above, and is, of course, discussed here. Other systems in the list are models of interacting polymers, dimer models, and loop models.

There are natural reasons for the list of prototypal systems discussed here. They are, in a sense, a 'minimal' core set of different models, w.r.t. a surprisingly large number of interplays, bijections, specializations, similarity at the level of the solution tools, which emerge starting from the main topic of spanning forests.

As we said in the introduction, spanning forests are both a limit of Potts model, and of $O(n)$ non-linear σ-model. And, of course, spanning trees are a limit of spanning forests.

For what concerns the description of polymer models, we remark that rooted spanning trees are a limit of "dense phase" of branched polymers, and indeed, models describing spanning trees are sometimes named as "branched polymers" in the literature (as, for example, in [13] or in [14]). Differently from what happens for the forests, spanning trees and rooted spanning trees are ensembles with identical normalized measures, at least on graphs with undirected weights.

Furthermore, the BKAR (Brydges-Kennedy–Abdesselam-Rivasseau) forest-root formula [15, 16], and the Brydges-Imbrie dimensional reduction formula [17], which are naturally formulated in the setting of general branched polymer models, as a graphical expansion for Mayer-like cluster expansions, are somehow related to the forest enumerations, and the dimensional reduction mechanisms of Parisi-Sourlas kind, which are the main topic here.

For what concerns dimer models and Eulerian subgraphs, the motivations are again coming from different directions. Of course, a motivation comes from the Ising model, which is a special case ($q = 2$) of Potts model, and corresponds to a combinatorics of Eulerian subgraphs via high-temperature expansion. When on a cubic graph, through symmetric difference w.r.t. a given configuration, this expansion can be related to a formulation in terms of a dimer model.

On planar graphs, this system can be solved through the celebrated *Kasteleyn method* (in dimer formulation) [18, 19, 20], or the equivalent *Kac and Ward method* (in the Eulerian subgraph formulation) [21]. These solution methods are paradigmatic of non-trivial partition functions for which a pfaffian formula exists.

Pfaffians are easily related to determinants, so dimers in two dimensions are probably the most famous case of a "determinantal formula" in the statistical-mechanics community, while the Kirchhoff theorem for spanning trees is comparatively, (unfortunately, from a pedagogical point of view), a less widespread result. Furthermore, although the natural dimer occupation variables are not a determinantal process in general, a subset of the observables (including, among others, occupations on the same boundary face) give rise to a determinantal-process subsector of the full algebra of observables. So, dimers in two dimensions are also the most famous example in statistical mechanics of a determinantal process.

So, these properties relate dimer models to the model of spanning trees, for which the generating function is determinantal, for any graph, through Kirchhoff theorem, and the whole algebra of occupation numbers is a determinantal process.

In the dimer model, the role of planarity, and the emergence of simplicity for observables on the boundary, are also related to a different (and more powerful) determinantal

technique for the evaluation of the generating function, going under the name of *Gessel-Viennot* method. Not only this case makes more evident the determinantality of the process (reduced to the boundary), but also the fact that Gessel-Viennot paths follow a "fermionic statistic". We could say that, while Kirchhoff theorem allows to describe spanning trees as a theory of free fermions in $d = 2$, the Gessel-Viennot method allows to describe non-intersecting directed paths, i.e. certain dimer models, as a theory of free fermions in $1 + 1$-dimensions, thus making even more clear the interplay between the two subjects.

As a further motivation, we recall the Temperley bijection between ensembles of spanning trees and of dimer coverings, over certain pairs of related planar graphs, which can be extended to a full multivariate version on the "tree" side (giving only a subset of the set of natural weights, on the "dimer" side) [22, 23].

Last but not least, the relation between Ising and Eulerian subgraphs through high-temperature expansion is so strong that it generalizes to arbitrary $\mathbb{Z}_2$ spin systems (even with k-body interaction), in relation with Eulerian sub-hypergraphs of an appropriate hypergraph, thus giving a precise restriction, on the set of all sub-hypergraphs induced by the underlying $\mathbb{Z}_2$ symmetry of the original statistical-mechanic system. This mechanism is mimicked by the relation between OSP(1|2) non-linear σ-models with arbitrary interaction pattern and spanning hyperforests, providing a second example of non-trivial restriction of the ensemble of sub-hypergraphs, induced by the underlying, OSP(1|2) in this case, symmetry.

For what concerns loop models, besides the fact that Eulerian subgraphs coincide with a $n = 1$ loop model when on a cubic graph, the main motivation comes of course from the loop version of the $O(n)$ model, due to Nienhuis [24]. This combinatorial model raised a great interest in the literature, mainly because of the number of exact results, in regular two-dimensional lattices [25] and random planar graphs [26], and because of the emerging "universal" properties at the aim of describing universality classes and CFT's in two dimensions.

Although in many cases loop models arise quite naturally "as they are", we feel to propagandate the general idea that, at the aim of precise combinatorial bijections, models considering both loops and dimers arise in a wider set of circumstances. Although typically this does not change the universality class of the underlying system, specially fine-tuned cases exists for which new features emerge, in particular in the non-probabilistic regime in which either the topological weight n or the weights for dimer and loop occupations are not real positive. Examples of these features emerge, for example, in [26] and in [27].

Graph- and subgraph-orientations are certainly a more exotic example. However, we have a strong motivation in this direction, coming from a very general issue on Potts model. Subgraph-orientations arise in a certain sector of Potts model, which is not manifestly probabilistic in the random-cluster formulation (and, of course, is away from the original q-state description). A general issue for Potts model is which sectors of the phase diagram allow for any probabilistic formulation in terms of local variables. Forests arise as two special limits in the plane of parameters, one of which is at the edge of the FK probabilistic sector. Remarkably, the other limit is at the edge of another sector, written in terms of a new probabilistic formulation using subgraph-orientations. A more detailed description of these aspects is postponed to Chapter 5.

2

Statistical Mechanics and Critical Phenomena

Our aim here is to give a perspective on Statistical Mechanics and Critical Phenomena which fits in the framework depicted in the previous chapters, of problems defined on arbitrary, possibly weighted, graphs.

If the concepts of partition sum and generating function are clear synonimous, and some Statistical Mechanics jergon may be accepted *bona fide* by a mathematical reader, the concept of criticality is the one raising the strongest problems. Indeed, strictly speaking criticality may occur only in the limit of infinite degrees of freedom, so it may only emerge in *sequences* of larger and larger finite graphs, in which some signatures of the genuine infinite-volume criticality become more and more evident.

Furthermore, concepts which are central to the definition of criticality and symmetry breaking, such as the divergence of the correlation length and the Cluster Property, rely on a natural notion of translation invariance in the system, which needs to be reconsidered in the framework of arbitrary graphical structure (although easily extended to graphs which are "transitive" in any sense, i.e. have large automorphism groups). Even the mere choice of 'natural' sequences of graphs, allowing for a thermodynamic limit, may seem way too arbitrary in a context of generic graphical structures.

Here we pose a number of questions and give a number of tentative extended definitions, then we discuss these choices in some concrete examples.

2.1 Partition sums and equilibrium measures

Equilibrium Statistical Mechanics deals with thermodynamical systems with a large number of degrees of freedom. As a theory for homogeneous systems in an Euclidean D-dimensional space, it is well established (see for example [29]).

The thermodynamics is encoded in a 'small' set of global physical parameters (such as temperature, pressure, free energy density, ...), arising from average over the macroscopic number of variables (positions, momenta, spins, ...) which describe the microscopic components. A mathematical definition of a system at equilibrium is the datum of this set of external parameters a finite-dimensional space of configurations $X = X_0^N$, equipped with a reference measure $\mathrm{d}\mu_0(x) = \prod_i \mathrm{d}\mu_0'(x_i)$, and a Hamiltonian $\mathcal{H}(x) : X \to \mathbb{R}$. For each temperature $T = 1/\beta$, the Gibbs probability measure

$$\mathrm{d}\mu(x) = \frac{1}{Z_N} e^{-\beta \mathcal{H}_N(x)} \mathrm{d}\mu_0(x) \tag{2.1}$$

is stable under any local dynamics preserving detailed balance at given β, which is a standard request for a first modeling of the microscopic dynamics for systems at equilibrium.[†] The normalization factor

$$Z_N = \int \mathrm{d}\mu_0(x)\, e^{-\beta \mathcal{H}_N(x)} \tag{2.2}$$

is named *partition function*, and its logarithm is extensive in system size. This leads to the definition of the free energy F_N and the free energy density $f = F_N/N$

$$Z_N = \int_X \mathrm{d}x\, e^{-\beta \mathcal{H}_N(x)}\,; \qquad\qquad F_N = -\frac{1}{\beta} \ln Z_N\,. \tag{2.3}$$

A *physical state* for a given system corresponds to a linear functional over the space of observables $\{A\}$. The linearity property is implicit in functionals Ω of the form

$$\Omega(A) = \langle A \rangle_\Omega = \frac{\int_X \mathrm{d}\mu_\Omega(x)\, A(x)}{\int_X \mathrm{d}\mu_\Omega(x)}\,. \tag{2.4}$$

Note that, if ergodicity fails, the Gibbs measure is not guaranteed to be the only equilibrium measure on the system. Indeed, the infinite-volume limit (or *thermodynamic limit*) can lead to new phenomena forbidden in finite systems, namely the ergodicity under local dynamics can be broken, and the free-energy density of the system can be a non-analytic function of the physical parameters. The points in the space of parameters in which f is singular are called *critical points*. At these points, a *phase transition* can occur in the system.

If the system has a symmetry group G, that is, an action over configurations preserving both the measure and the Hamiltonian, i.e. for $g \in G$ and $x \in X$, $\mu_0(gx) = \mu_0(x)$ and $\mathcal{H}(gx) = \mathcal{H}(x)$, when the ergodicity is broken, in general a subgroup $H \subseteq G$ is preserved as a symmetry within each ergodicity bassin, and the quotient G/H acts as a permutation group among the bassins. If H is a proper subgroup, then we say that the system undergoes *Spontaneous Symmetry Breaking*.

The study of Critical Phenomena receives a boost from the *Universality Hypothesis*. This can be put as an hypothesis on physical grounds, which nowadays takes a stronger justification within the context of *Renormalization Group* [30]. The hypothesis states that the kind of singularity at the critical points is determined only by general properties of the configuration space and of the Hamiltonian (dimensionality of the underlying space, range of interaction, symmetry properties of the variables involved, i.e. both G and $H \subsetneq G$ groups above in the spontaneous symmetry breaking, ...).

This hypothesis justifies an abstract mathematical approach to critical phenomena: the study of idealized models reveals the critical properties also of the potentially complicated concrete physical systems which share the same universality characteristics of the model, i.e. which belong to the same *universality class*.

2.2 Spontaneous symmetry breaking in the Ising model

A prototype phase transition, the spontaneous magnetization of ferromagnetic materials below the Curie temperature, has provided a general lexicon in the field, for models describing the most diverse phenomenologies. The *Ising Model*, introduced to describe

[†] In formulas, the transition rates $W_{x\to x'}$ of the dynamics should satisfy $W_{x\to x'} \to 0$ for $|x - x'| \to \infty$ (locality) and $W_{x\to x'}/W_{x'\to x} = \exp[-\beta(\mathcal{H}(x') - \mathcal{H}(x))]$ (detailed balance).

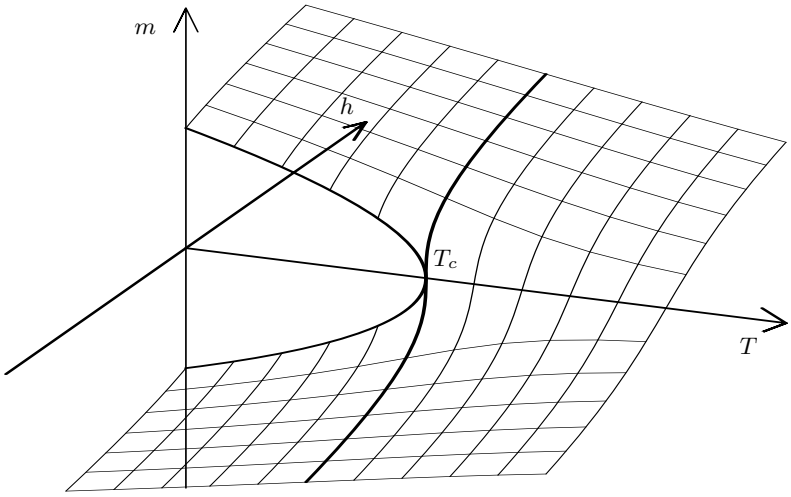


Fig. 2.1. Qualitative shape of the average magnetization $m(T, h)$ for the Ising Model in $d > 1$.

this kind of transition, has been a milestone in the theory of Critical Phenomena, since the acclaimed solution of the two-dimensional version of the problem by Onsager in 1944 [31].

We consider a regular lattice in d dimensions (to fix the ideas, think of a hypercubic lattice of side L), with magnetic elements (e.g. ions) on the $N = L^d$ vertices. Since we are interested in the magnetic behaviour, we only consider the degrees of freedom corresponding to the local magnetic induction neglecting the mechanical degrees of freedom of the crystal (e.g. vibrational modes around the equilibrium positions).

Universality suggests us to deal with quantized magnetic inductions, called *spins*,[‡] $\sigma_i \in \{\pm 1\}$. The configuration space is thus $\{\pm 1\}^N$, where N is the number of sites in the lattice. The Hamiltonian is

$$\mathcal{H}(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \,. \tag{2.5}$$

where $\langle i, j \rangle$ denotes pairs of first-neighbouring sites, $J > 0$ is the parameter defining the intensity of the interaction, and h is an external magnetic field. Consider the magnetization of the system as a function of temperature and magnetic field

$$m(T, h) = \langle m(\sigma) \rangle \,, \qquad\qquad m_N(\sigma) = \frac{1}{N} \sum_i \sigma_i \,. \tag{2.6}$$

In the thermodynamic limit, for $d > 1$, a critical point exists in the plane (T, h), for $h = 0$ and $T = T_c > 0$. The function $m(T, h)$ is analytic on the whole half-plane $T > 0$, with the exclusion of the line with zero external field and $T \in [0, T_c]$, and the original symmetry of the problem reflects into $m(T, h) = -m(T, -h)$. On the critical line there is a cut discontinuity: the limits $m_+(T) = \lim_{h \to 0^+} m(T, h)$ and $m_-(T) = \lim_{h \to 0^-} m(T, h)$

[‡]In analogy with the spin of a particle, which is quantized for a different and more fundamental reason.

do not coincide. This fact has strong implications on the equilibrium measures of the system. Consider the two limit measures

$$\mu_+^{(T)}(\sigma) = \lim_{h\to 0^+} \mu_{\text{Gibbs}}^{(T,h)}(\sigma)\,; \qquad \mu_-^{(T)}(\sigma) = \lim_{h\to 0^-} \mu_{\text{Gibbs}}^{(T,h)}(\sigma)\,. \tag{2.7}$$

A finite difference on the expectation value of an intensive quantity (as the average magnetization m) implies that the measures $\mu_+^{(T)}$ and $\mu_-^{(T)}$ do not coincide. This phenomenon is called *Spontaneous Symmetry Breaking*, in this case for the $\mathbb{Z}_2$ symmetry $\sigma \to -\sigma$ of the Hamiltonian (2.5) at $h = 0$.

Indeed, on the critical line, both measures are equilibrium measures for the system: two *pure phases* coexist. The set of equilibrium measures is a convex: any equilibrium measure can be written in one and only one way as a convex combination of extremal measures. The extremal measures on this convex correspond to pure phases. In this case, both $\mu_+^{(T)}$ and $\mu_-^{(T)}$ correspond to restrictions of the Gibbs measure to a region of the phase space (space of configurations).

In our case we can state

$$\mu_+^{(T)}(\sigma) \propto e^{-\frac{1}{T}\mathcal{H}_{h=0}(\sigma)}\theta(m(\sigma))\,; \qquad \mu_-^{(T)}(\sigma) \propto e^{-\frac{1}{T}\mathcal{H}_{h=0}(\sigma)}\theta(-m(\sigma))\,. \tag{2.8}$$

Note that this definition makes sense also at finite size, although stability under time evolution is obtained only in the thermodynamic limit, and derives from two facts

- local stability far from the θ discontinuity is a consequence of proportionality with Gibbs measure, and locality of the dynamics;
- the lack of stability near the edge of the θ is thermodynamically irrelevant, as the measure of the region of phase space with $m = \mathcal{O}(1/N)$ goes to zero exponentially with the size: indeed, far from the critical point, thermal fluctuations of the magnetization are of order $\mathcal{O}(N^{-1/2})$, and $|m(T)|$ is of order 1.

We could say that the finite-size measures (2.8) correspond to *"quasi"-pure phases*: pictorially speaking, they correspond to "valleys" of the phase space, whose border has a measure which goes to zero exponentially with the size, and thus become a "separated world" in the thermodynamic limit.

2.3 Characterization of pure phases and Cluster Property

The definition of pure phases as the ergodicity bassins of any local equilibrium dynamics for the system has various drawbacks. First, it makes reference to some dynamics, while being a concept that should be related to the structural properties of the Gibbs measure and the distance function on the configuration space, which are purely static concepts. Furthermore, the claimed invariance of the resulting bassins w.r.t. the choice of different local dynamics is not *a priori* clear. Finally, on a practical side, the characterization of the ergodicity bassins of a Markov chain on an exponentially-large space of configurations is in general hard to achieve.

A different approach to the definition of pure phases in a thermodynamical system is thus desiderable. Such an approach [32] is briefly discussed in this section.

One makes the physical assumption that the ground state of a system is perturbed only locally under local perturbations (say, measurements of a physical quantity), so measurements far away in space do not allow for non-trivial correlations. This property must be true only for a given ground state: when the system allows for different ground states

(and thus several "valleys" in a perturbation-theory approach), if we perform averages with a measure which is not extremal, correlations can originate from different expectation values of physical observables in different pure phases. Indeed, if we have two distinct pure phases, at least one physical observable must exist, such that its expectation value in the two phases is different. We already see at this level the emergence of a 'static' and '(multi-)local' characterization of a pure phase.

A formalization of the physical picture above (which goes under the name of *Cluster Property*), under general physical hypothesis, can be related to a definition of pure phase equivalent to the one in terms of ergodicity breaking we sketched above, and without the drawbacks we discussed.

Theorem 2.1 ((naïve) Cluster Property) *A given equilibrium measure μ on a physical system describes a pure phase if and only if, for each pair of local physical observables $\big(A(x), B(x)\big)$, the connected correlation function vanishes in the large distance limit:*

$$\forall \quad A, B \qquad \lim_{|x-y|\to\infty} \Big(\langle A(x)B(y)\rangle - \langle A(x)\rangle \langle B(y)\rangle \Big) = 0\,. \tag{2.9}$$

In the previous section, we defined some quantities using large finite systems, in order to guarantee a more precise mathematical control. In particular, we gave a hint on the concept of finite-system "quasi"-pure phases (in the Ising Model at $h = 0$ and $T < T_c$, the two measures (2.8), when considered at finite size): although only in the thermodynamic limit they have all the properties of pure phases (for instance, stability under a local dynamic), quantitative corrections to these properties are typically bounded by functions of the size (for instance, metastability lifetimes are expected to scale as $\tau \sim \exp(aN)$).

In particular, we expect a version of the Cluster Property to hold in finite-size:

Theorem 2.2 ((naïve) finite-size Cluster Property) *Consider a family of equilibrium measures μ_N on finite-size physical systems, which, w.r.t. expectation of a set of local observables in a given subvolume, converge to a measure μ in the large N limit. The limit measure μ corresponds to a pure phase if and only if for each pair of local physical observables $\big(A(x), B(x)\big)$, the connected correlation function, calculated w.r.t. the finite-size measures, vanishes in the large size, large distance limit, provided that some typical diameter $d(N)$ of the system is kept large w.r.t. the distance $|x-y|$ between the supports of the operators, and the operators are inside the subvolume*

$$\forall \quad A, B \qquad \lim_{\substack{|x-y|\to\infty \\ N\to\infty \\ |x-y|\ll d(N)}} \Big(\langle A(x)B(y)\rangle_N - \langle A(x)\rangle_N \langle B(y)\rangle_N \Big) = 0\,. \tag{2.10}$$

An intuition on the validity of Cluster Property is given by the example of the Ising Model. Consider the model at $h = 0$ and $T < T_c$, on a finite periodic lattice of size $N = L^D$. In this case $d(N) \sim L/2$. As all sites are equivalent under translations, we have

$$\frac{1}{N}\sum_x \langle \sigma_x \rangle_N \equiv \overline{m}_N = \langle \sigma_{x_0} \rangle_N \qquad \forall \quad x_0 \in \{1, \dots N\}\,. \tag{2.11}$$

From the discussion presented in the previous section, it is clear that any stable measure can be reconducted to convex combinations of the two (2.8), in the termodinamic limit.[††] Thus, we have a one-parameter family of measures

[††] At finite size, "quasi"-stable measures can be reconducted to convex combinations of the two (2.8), up to discrepancies in a part of the phase space, whose measure in any equilibrium distribution vanishes in the thermodynamic limit.

$$\mu_t(\sigma) = t\mu_-(\sigma) + (1-t)\mu_+(\sigma)\,, \qquad t \in [0,1]\,; \tag{2.12}$$

and, if $m = m_+(T)$ is the average magnetization in the phase μ_+, the average magnetization in the phase μ_t is given by $m(1-2t)$.

What happens is that, when $T < T_c$, the correlation length of the system is finite, thus long-range thermal fluctuations are suppressed. In the correlation function $\langle \sigma_{x_0}\sigma_x \rangle$, at large distance, either both spins are on average m, with almost decorrelated fluctuations (this happens with probability $(1-t)$), or both spins are on average $-m$, with almost decorrelated fluctuations (this happens with probability t). So the expectation value of the correlation function in this measure is

$$\lim_{|x-x_0|\to\infty} \langle \sigma_{x_0}\sigma_x \rangle_t = m^2\,, \tag{2.13}$$

for all choices of t. On the other side, both the one-point function $\langle \sigma_{x_0} \rangle$ and $\langle \sigma_x \rangle$ are on average $m(1-2t)$, so we have

$$\lim_{|x-x_0|\to\infty} \Big(\langle \sigma_{x_0} \rangle_t \langle \sigma_x \rangle_t \Big) = m^2(1-2t)^2 = m^2(1-4t(1-t))\,, \tag{2.14}$$

which coincides with equation (2.13) only in the two pure phase measures $t = \{0,1\}$.

To summarize, this section presents some simple non-rigorous hint on a mathematically delicate task, that is, characterizing pure phases of a thermodynamical system via the cluster property of correlation functions. The proof of the equivalence of this characterization with the more intuitive "valley" picture, and with stability under local dynamic, is out of our purposes, and is done in literature only for traditional ordered systems, with particular emphasis over S-matrix theory in Quantum Field Theory [32, 33].

2.4 Ising Model in two dimensions: Kramers-Wannier duality

When studying thermodynamics, phase transitions are often quickly addressed, in the form of Van Der Waals equation for the theory of liquid-vapour transition, and the related Maxwell "area" construction.

Within a statistical-mechanics, or a field-theoretical, perspective, we now know that unfortunately Van Der Waals theory is not a fully pedagogical example: it tacitly makes use of a mean-field approximation, and, although providing a reasonable picture of the transition, in the dimensionalities $d = 3$ or $d = 2$ of most of our experimental settings, it predicts a wrong set of critical exponents.

It tooks some while to hystorically understand this point. The first explicit proof that mean-field predictions were wrong at least in some cases, was through the analysis of an *exact* solution, namely, the celebrated Onsager solution of the 2D Ising Model. (The Gaussian Model discussed in Section 2.6, at least in its bosonic formulation and on simple lattices, was well-known, but inconclusive on this point as already knew to show the pathologies that we discuss in the pertinent section). The original proof is slightly convoluted, but a number of simplified versions appeared later on in the literature, and simple proofs exists nowadays also in basic textbooks (among which the fifth volume of Landau physics course, [35], and Feynman lectures on Statistical Mechanics, [36]).

Here we give a solution through Grassmann techniques, which is essentially equivalent to the combinatorial ones above, and introduces in a simple way a number of tools that will be used all along this work. We will also try to make a review of the connections

between this solution, and the various other ones, due to Kasteleyn [18], M. Kac and Ward [21], and Itzykson and Samuel [37, 38].

Consider a portion of a regular lattice in D dimensions (that is, a lattice having an invariance under a group of D-dimensional translations), for example the hypercubic lattice Λ in D dimension, with sides $L_1, \ldots, L_D$ and periodic boundary conditions. Sites are labeled with letters x, y, ..., and $x \in \{0, \ldots, L_1 - 1\} \times \ldots \times \{0, \ldots, L_D - 1\}$ is equivalent to $x + (n_1 L_1, \ldots, n_D L_D)$, with $n_i \in \mathbb{Z}$. Call $\hat{\mu}_i$ the versors of the lattice, then the $2D$ neighbours of site x are the sites in $\{x \pm \hat{\mu}_i\}_{i=1,\ldots,D}$. On each site x, we have a spin variable $\sigma_x \in \{\pm 1\}$. The Ising Hamiltonian and partition functions are given respectively by

$$\mathcal{H}(\sigma) = \sum_{x \in \Lambda} \sum_{i=1}^{D} -J \sigma_x \sigma_{x+\hat{\mu}_i} \; ; \qquad Z(\beta) = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)} \, . \tag{2.15}$$

The expression of the partition function is a sum over 2^V terms, with $V = L_1 \cdots L_D$ the volume of the lattice. Although, up to a trivial rescaling, it is a polynomial in the variable $e^{-2\beta J}$, its degree is huge (DV), and its basic analytic properties do not suffice to extract directly a scaling limit for large V

$$Z_V(\beta) = \exp\left(-\beta V f(\beta) + o(V)\right) \; ; \tag{2.16}$$

with $f(\beta)$ the free energy per site. On the contrary, we know that the non-trivial procedure of extracting this limit can spoil the analyticity of $Z(\beta)$ (and this happens indeed for one finite value β^* of the parameter β, if $D > 1$), leading to the interesting feature of critical phenomenon.

The source of the "leading" contribution to the partition sum, in some regimes, could however be of a simple nature, and could lead to the formulation of formal expansions. These expansions, in some lucky extraordinary cases like the one at hand, may allow for a complete exact resummation (this is the content of the Kac and Ward solution: the exact sum could be interpreted as a naïve expansion, with a sequence of correction terms that happen to have a closed expression, and allow for a resummation [39]).

In this section we show how to derive two of these expansions for the general Ising Model, for which the parameter is small respectively in a neighbourhood of zero and infinite temperature. It will turn out that these expansions are formally coincident in the case of the two-dimensional square lattice. This remark, first due to Kramers and Wannier [40], jontly with some assumptions checked *a posteriori* after the work of Onsager, provides a duality transformation which exchanges the parameter regions accessible through low-temperature and high-temperature expansions, and gives a first exact determination of the critical temperature, as the boundary among the two regions and fixed point of the duality involution.

We start from the expansion occurring at low temperatures. This is the most intuitive in two dimensions, but will happen to be structurally more complicated in arbitrary dimension, as it makes extensive use of the Euclidean structure of the underlying lattice. This expansion is the natural one for the polynomial of the partition function, written in terms of the parameter $t = e^{-2\beta J}$, the multiplier for the number of unsatisfied bonds.

The space of configurations $\sigma \in \{\pm 1\}^V$ has a trivial symmetry (that we could call a "global $\mathbb{Z}_2$ gauge"), the operation $I : \{\sigma_i\} \to \{-\sigma_i\}$. It is a symmetry because all the summands in the Hamiltonian are monomials in spin variables of even degree, so $\mathcal{H}(\sigma) = \mathcal{H}(I(\sigma))$. In particular, for the ferromagnetic system, the two ground states

are interchanged by this symmetry. The quotient of the full space of configuration by this symmetry is identified with the valid configurations of terms in the Hamiltonian which are unsatisfied (the pair of ferromagnetic ground states corresponding to the empty configurations).

The constraint defining this set of allowed configurations is in general complicated, and there are no simpler ways of expressing it, w.r.t. the original formulation with spin variables. An exception is when the graph describing the interaction pattern has a topological structure allowing for an extension to a D-dimensional cell complex, with D a "small" integer (*dimensionality* of the problem), even in the thermodynamic limit for the number of sites [41, 42, 43]. This is the case for planar graphs, where $D = 2$ and indeed the graph, when embedded on the plane, has also a natural notion of "faces", that is, the two-dimensional cells, besides vertices and edges, being respectively the zero- and one-dimensional cells. The planar duality of planar graphs is then recognized in this case as the two-dimensional specialization of Hodge duality for cell complexes. This is also the case for three-dimensional systems, although in this case we are required to introduce both *plaquettes* (the two-dimensional cells) and *elementary volumes* (the three-dimensional cells).

The cells in a cell complex may be arbitrary convex polytopes. However, in order to fix the ideas, it is always possible to think of hyper-tetrahedra only. Indeed, in the corresponding Ising problem, breaking a polytope into tetrahedra corresponds to add further couplings in the Hamiltonian up to make it "maximally D-dimensional", and then let these auxiliary couling go to zero. For example, for Ising model on the square lattice, we could add all the diagonals in one of the two directions, and produce a "triangulation" of the original lattice (triangles are of course the two-dimensional hyper-tetrahedra). Then, we can set to zero the couplings along these diagonals. A nice simplification coming from the exclusive use of tetrahedra is the fact that a D-dimensional hyper-tetrahedric cell in a D-dimensional cell complex has always exactly $D+1$ neighbouring $(D-1)$-dimensional cells.

In the language of cell complexes, it is easy to describe the constraint corresponding to valid configurations of unsatisfied bonds. Build up the cell complex corresponding to the interaction graph. Then, take its Hodge dual: now the spins are located in the D-dimensional cells, and the edge of interactions are $(D-1)$-dimensional cells. Subsets of the whole set of $(D-1)$-dimensional cells are a product space $\mathbb{Z}_2^{|E(G)|}$. This space has a canonical subspace of those elements having no boundary. These are exactly the configurations we are aiming to. This space has even a linear structure, as a vector space over $\mathbb{Z}_2$, where the sum corresponds to the symmetric difference of subsets in a given set.

For two-dimensional (dual) cell complexes in which all vertices have degree 3 (the dual of the "triangulations" discussed above) the allowed configurations are the ones corresponding to the "contours" of the spin domain, which are closed non-intersecting polygons on the lattice. An example is shown in Figure 2.2. These contours are called *Peierls contours*, as Sir Rudolf Peierls built an argument proving the existence of a phase transition in the two-dimensional Ising Model, based on the entropic analysis of this set of configurations [44]. For a generic two-dimensional cell complex for which one did not follow the prescription of extending it to a triangulation, we have vertices of arbitrary degree. Now the contours are not non-crossing in general. However, each vertex should be adjacent to an even number of edges in a Peierls contour, as these edges correspond to cyclic swaps of the spin sign. Beside this, we have no further constraint, so we see that vertices of odd degree are forbidden, and vertices of any even degree are all allowed with the same "topological" weight. So, if we call $\hat{G}$ the dual cell complex associated to G (that

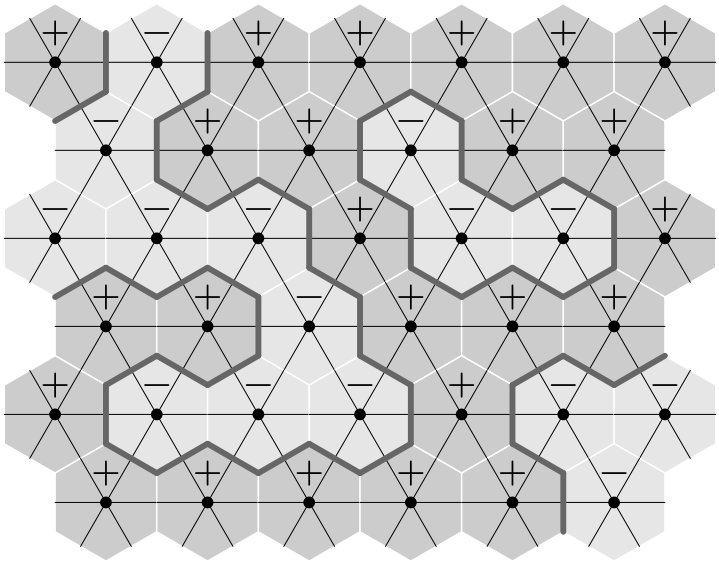


Fig. 2.2. A portion of a spin configuration on the triangular lattice, and the corresponding domain boundaries on the hexagonal lattice.

is, its planar dual in graph-theoretical sense), we have for the partition function

$$Z = 2 \sum_{\substack{\hat{H} \subseteq \hat{G} \\ \text{Eulerian}}} \prod_{\hat{e} \in E(\hat{H})} e^{-2\beta J_e} \,. \tag{2.17}$$

For three-dimensional (dual) cell complexes, in which all vertices have degree 4 (the dual of "tetrangulations"), the allowed configurations are the ones corresponding to non-crossing "bubble-shaped" surfaces, that is, surfaces componentwise homotopic to spheres or tori, without boundary. An example is shown in Figure 2.3. We could call these configurations *Peierls surfaces*. For regular graphs, the entropic analysis of Sir Rudolf Peierls can be extended to this set of configurations, as done for example in [45]. Calling S a generic valid configurations of Peierls surfaces, and $p(e)$, $e(p)$ the functions relating edges $e \in G$ with plaquettes $p \in \hat{G}$ (the three-dimensional dual cell complex), we could write

$$Z = 2 \sum_{S \text{ Peierls}} \prod_{p \in S} e^{-2\beta J_{e(p)}} \,. \tag{2.18}$$

Now we go back to the expression for the partition function (2.15), and perform a different expansion in which the perturative parameter is small for high temperatures. This expansion will work in the same way for arbitrary graphs, and will not make use of any geometrical "cell-complex" structure.

The key observation is that, as a monomial in spin variables can only take values in $\{-1, +1\}$, each function in such a variable can be stated as a binomial. That is, for every set I and every function $f(x)$, two values a and b exist such that

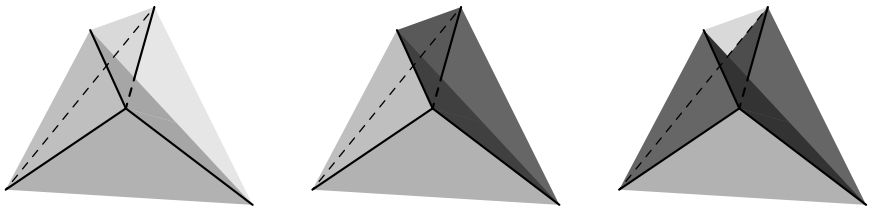


Fig. 2.3. The two possible non-trivial local configurations (up to permutation symmetry) of a 2-dimensional Peierls surfaces, in a neghbourhood of the point of incidence of 4 cells. Spins are located inside the cells. Left: the cell-complex before choosing a configuration. Center: the darker regions correspond to plaquettes in the Peierls surface, here in the case in which the top-right spin has a different sign w.r.t. the other three ones. Right: the case in which the top-right and top-left spins are equal among themselves, and different from the other two.

$$f\Big(\prod_{i\in I}\sigma_i\Big) = a\Big(1 + b\prod_{i\in I}\sigma_i\Big)\,. \tag{2.19}$$

$$a = \frac{f(1)+f(-1)}{2}\,; \qquad b = \frac{f(1)-f(-1)}{f(1)+f(-1)}\,; \tag{2.20}$$

with the exception of the degenerate case $f(1) = f(-1)$, for which we have $f\Big(\prod_{i\in I}\sigma_i\Big) = f(1)\prod_{i\in I}\sigma_i$. In particular, for each edge (ij) of the lattice,

$$\exp(-\beta J_{ij}\sigma_i\sigma_j) = (\cosh\beta J_{ij})(1 + \sigma_i\sigma_j\tanh\beta J_{ij})\,, \tag{2.21}$$

thus for a generic Hamiltonian with pairwise interactions, described by a graph $G(V,E)$ with weights J_{ij} on the edges (ij),

$$\mathcal{H}(\sigma;J) = \sum_{(ij)\in E(G)} -J_{ij}\sigma_i\sigma_j\,, \tag{2.22}$$

we can apply the technique above to the expression for the partition function, and obtain

$$\begin{aligned} Z(\beta J) = \sum_{\sigma} e^{-\beta\mathcal{H}(\sigma;J)} &= \prod_{(ij)\in E(G)}\cosh\beta J_{ij}\sum_{\sigma}\prod_{(ij)\in E(G)}(1+\sigma_i\sigma_j\tanh\beta J_{ij}) \\ &= \prod_{(ij)\in E(G)}\cosh\beta J_{ij}\sum_{E'\subseteq E(G)}\sum_{\sigma}\prod_{(ij)\in E'}(\sigma_i\sigma_j\tanh\beta J_{ij}) \\ &= 2^{|V(G)|}\prod_{(ij)\in E(G)}\cosh\beta J_{ij}\sum_{\substack{E'\subseteq E(G)\\ \text{Eulerian}}}\prod_{(ij)\in E'}\tanh\beta J_{ij}\,. \end{aligned} \tag{2.23}$$

The crucial step, in the last passage, was the simplification in the sum over all possible spin configurations, where we used a simple lemma: if $\{d_i\}$ is a set of integers, then

$$\sum_{\sigma}\prod_{i}\sigma_i^{d_i} = \begin{cases} 2^V & \text{all } d_i \text{ are even} \\ 0 & \text{some } d_i \text{ is odd} \end{cases} \tag{2.24}$$

The proof is quite simple: if d_i is odd, perform first the sum over variable σ_i, and recognize that it makes zero; otherwise if all d_i's are even, recognize that $\prod_i \sigma_i^{d_i} = 1$ identically.

As a result, up to a prefactor, we are left with a sum over Eulerian subsets of edges (or, equivalently, Eulerian spanning subgraphs), i.e. sets E' such that the degree of each vertex is even. We thus recognize the multivariate generating function of Eulerian subgraphs, defined in Section 1.2, with the identification $w_{ij} = \tanh \beta J_{ij}$

$$Z_{\text{Euler. subgr.}}(\{w_{ij}\}, G) = \sum_{\substack{E' \subseteq E(G) \\ \text{Eulerian}}} \prod_{(ij) \in E'} w_{ij} \,. \tag{2.25}$$

If the graph is the hypercubic lattice Λ with V vertices, with equal couplings, we have

$$Z(J) = (2 \cosh \beta J)^{DV} \sum_{\substack{E' \subseteq E(G) \\ \text{Eulerian}}} (\tanh \beta J)^{|E'|} \,. \tag{2.26}$$

If the graph G has only vertices of degree at most 3, the set of Eulerian subgraphs coincides with the set of self-avoiding loop configurations, and the dense limit is obtained for $\tanh J \to \infty$, i.e. $J \to i\pi/2$. In this case, if the graph admits a dimer covering, taking the complement of the Eulerian subgraphs we obtain a bijection with the perfect matchings of the graph.

If the graph G is a portion of the square lattice, for example in a thoroidal geometry, then G and $\hat{G}$ are the same graph, and we recognize that the low- and high-temperature expansion involve exactly the same set of combinatorial objects, with the same class of weights (cfr. equation (2.17) above) This gives an involutive identity for the partition function, and thus for the intensive free energy $f = F/|E(G)|$:

$$Z_{\text{sq. latt.}}(\beta, J) = 2 Z_{\text{Eul. subgr.}}(e^{-2\beta J}) = (\cosh \beta J)^{2V} Z_{\text{Eul. subgr.}}(\tanh \beta J) \,; \tag{2.27}$$

$$f(e^{-2\beta J}) = \ln(\cosh \beta J) + f(\tanh \beta J) \,; \tag{2.28}$$

so, the set of points where f is non-analytic must be a fixed point of the involution

$$\beta J = -\frac{1}{2} \ln(\tanh \beta J) \,. \tag{2.29}$$

This function has only one "physical" fixed point (i.e. for $\beta J \in \mathbb{R}$), that is

$$\beta J = \frac{1}{2} \ln(1 + \sqrt{2}) \tag{2.30}$$

and the Kramers-Wannier argument [40] reads that, if there is a single point of non-analyticity in the system, where the phase transtition occurs, the value of this point must be the fixed-point above.

2.5 Ising Model in two dimensions: solution with Grassmann variables

Here we give the exact solution of the Ising Model in two dimensions, for the triangular lattice, by using techniques of Grassmann variables. This section has been written in a very detailed and totally self-consistent way, references to other parts of the work are only included as side remarks and can be skipped in a first reading.

Consider the Ising Model on a portion of the triangular lattice. Denote with the label x a vertex on the lattice, and with $\{\hat{\mu}\}_{\mu=1,2,3}$ the three unit versors $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$,

such that the six neighbours of vertex x are at the six points $\{x \pm \hat{\mu}\}_{\mu=1,2,3}$. The couplings will be taken to be translationally invariant, and to depend only on the orientation μ of the edge. The Hamiltonian and partition function will be defined respectively as

$$\mathcal{H}(\sigma) = -\sum_{x,\mu} J_\mu(\sigma_x \sigma_{x+\hat{\mu}} - 1)\,; \qquad\qquad Z(\beta) = \sum_\sigma e^{-\beta\mathcal{H}(\sigma)}\,; \tag{2.31}$$

where the Hamiltonian has been shifted in order to have $\mathcal{H}(\sigma_\pm^{(\mathrm{g.s.})}) = 0$ for the two ground states $(\sigma_\pm^{(\mathrm{g.s.})})_x = \pm 1$ for all x, and positive otherwise. Some choice of boundary conditions will play a role only at a later stage of the analysis, and does not need to be done now.

As described in the previous section, up to fix the "global gauge", accounting for a factor 2 overall in Z, we can identify the valid configurations with the Peierls contours on the hexagonal lattice, figure 2.2, that is, self-avoiding loop configurations L. Calling ℓ_μ the number of edges in L intersecting a link with coupling J_μ, we have

$$Z(\beta) = 2\sum_L e^{-2\beta(J_1\ell_1 + J_2\ell_2 + J_3\ell_3)}\,. \tag{2.32}$$

The hexagonal unit-cell of the lattice can be transformed into a rhomboidal cell by use of the translational symmetry, and, as a rhomboidal lattice is isomorphic to a square lattice up to a linear transformation, this provides us with a simple "cartesian" reference for the elements of the lattice, as described in figure 2.4. Each cell contains two vertices and three edges, one per orientation. Label with x the points in the portion of $\mathbb{Z}^2$ containing our finite lattice, and with $\mathbf{e}_{1,2}$ the two versors, generators of translations on the rhomboidal lattice. The three edges of "cell x" are identified by pairs (x,μ), and the pairs of adjacent edges $e \sim e'$, ordered in such a way that the relative counter-clockwise angle is $\frac{2\pi}{3}$, are

$$A: \qquad (x,1) \sim (x,3) \qquad (x,3) \sim (x+\mathbf{e}_2, 2) \qquad (x+\mathbf{e}_2, 2) \sim (x,1) \tag{2.33a}$$

$$B: \qquad (x,3) \sim (x,2) \qquad (x,2) \sim (x+\mathbf{e}_1, 1) \qquad (x+\mathbf{e}_1, 1) \sim (x,3) \tag{2.33b}$$

The first and second line correspond to pairs of edges adjacent to the first and second vertex of the unit cell, named A and B in figure 2.4.

Now introduce a pair of Grassmann variables per edge, $\bar{\psi}_{x,\mu}$ and $\psi_{x,\mu}$. We claim that $\frac{1}{2}Z(\beta)$ is given by a Gaussian Grassmann integral,

$$\frac{1}{2}Z(\beta) = \int \mathcal{D}(\psi,\bar{\psi})\, e^{\mathcal{S}(\bar{\psi},\psi)}\,, \tag{2.34}$$

with the quadratic form $\mathcal{S}(\bar{\psi},\psi)$ given by

$$\mathcal{S}(\bar{\psi},\psi) = \mathcal{S}_{\mathrm{m}}(\bar{\psi},\psi) + \mathcal{S}_\nabla(\bar{\psi},\psi) + \mathcal{S}_\triangle(\bar{\psi},\psi)\,, \tag{2.35}$$

$$\mathcal{S}_{\mathrm{m}}(\bar{\psi},\psi) = \sum_{x,\mu} \bar{\psi}_{x,\mu}\psi_{x,\mu}\,; \tag{2.36}$$

$$\mathcal{S}_\nabla(\bar{\psi},\psi) = \sum_x \left(\bar{\psi}_{x,1}\bar{\psi}_{x,3} + \bar{\psi}_{x,3}\bar{\psi}_{x+\mathbf{e}_2,2} + \bar{\psi}_{x+\mathbf{e}_2,2}\bar{\psi}_{x,1}\right) \tag{2.37}$$

$$\mathcal{S}_\triangle(\bar{\psi},\psi) = \sum_x \left(t_3 t_2\, \psi_{x,3}\psi_{x,2} + t_2 t_1\, \psi_{x,2}\psi_{x+\mathbf{e}_1,1} + t_1 t_3\, \psi_{x+\mathbf{e}_1,1}\psi_{x,3}\right); \tag{2.38}$$

We used a shortcut for the coefficients $t_\mu = \exp(-2\beta J_\mu)$. Remark how we have a freedom on the choice of how to distribute these factors, due to the invariance $(\bar{\psi}_i, \psi_i) \to$

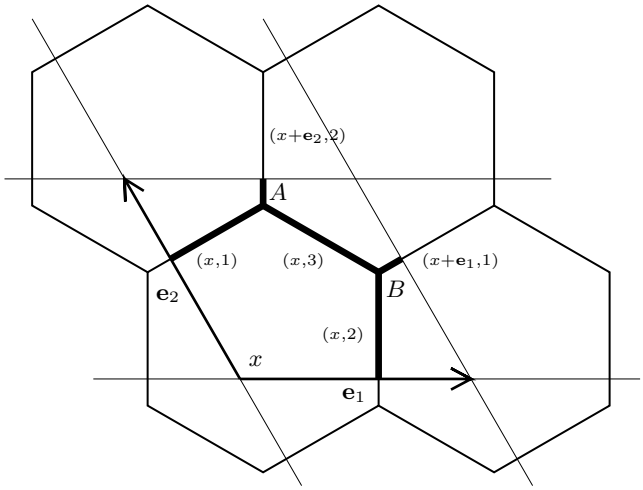


Fig. 2.4. A portion of the hexagonal lattice embedded into a rhomboidal lattice. Each cell contains two vertices and three edges. Here the two lattices have been slightly splitted for clarity of the drawing: it is intended that an edge is labeled after the cell in which is "almost totally" contained.

$(a\bar{\psi}_i, a^{-1}\psi_i)$ for $a \neq 0$. We adopted a choice which is asymmetric, but involves only integer powers of the t_μ's.

In the action above we recognize a diagonal "mass" term, $\mathcal{S}_{\mathrm{m}}$, and two off-diagonal parts, $\mathcal{S}_\triangledown$ and $\mathcal{S}_\triangle$, which are not charge-neutral, and are associated respectively to the triangles in the lattice pointing downwards (A-vertices) and upwards (B-vertices). These terms of the action contain one summand per pair of adjacent edges, and can be compared with the list (2.33).

If we expand this part of the action, we obtain configurations of markings for the half-edges of the hexagonal lattice: in each vertex, we can choose either to mark nothing (weight 1), or to mark two of the three adjacent edge terminations (weight 1 for A-vertices and $t_\mu t_{\mu'}$ for B-vertices in which edge-terminations μ and μ' are marked). A graphical representation of this combinatorics is shown in figure 2.5.

The integral of such a polynomial with the diagonal part, as explained in the Appendix A, restricts to those monomials which are "locally neutral", i.e. for each edge (x, μ) either both $\bar{\psi}_{x,\mu}$ and $\psi_{x,\mu}$ or none of them are present. Graphically, this means that, for each edge, either both or none of the two edge-terminations are marked, thus it is the integration which selects only "genuine" edge-marking configurations.

Because of the local rule discussed above, for the expansion of the off-diagonal part of the action, the valid configurations are collections of self-avoiding loops (corresponding to the marked edges), thus the surviving monomials are in bijection with the desired "loop configurations" L. Also, the factors t_μ, which are local on the edges, are easily seen to match with the ones required in the combinatorial sum over L. The only subtlety to check is the overall sign coming from the reordering of Grassmann variables. So, the proof of our claim is reduced to prove that this reordering never produces a minus sign.

As the variables on each vertex occur in pairs, these pairs can be freely commuted all along the monomial. In particular, we can factor the Grassmann monomials corre-

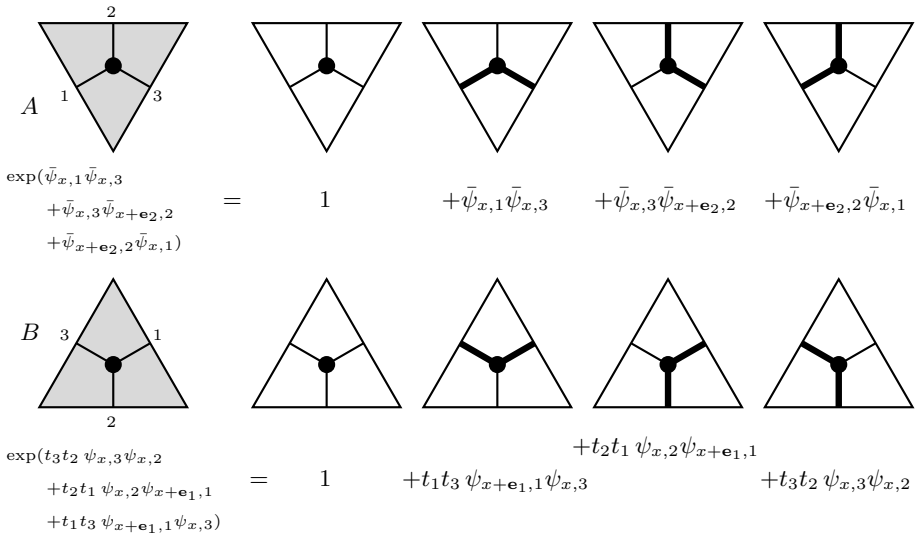


Fig. 2.5. Expansion of the off-diagonal part of the action: the resulting terms have a combinatorial interpretation as local marking of pairs of edges incident on a given vertex, the consistence of marking on the two extrema of each edge being provided by integration with the diagonal part of the action.

sponding to distinct loops, and concentrate on the sign of a single loop. Furthermore, we can reorder the pairs of variables coming from the vertices of the loop, for example, in counter-clockwise order starting from an arbitrary site, and relabel them into $1, \ldots, \ell$ for simplicity. Remark that the length ℓ of a loop is always even on this graph, which is bipartite. Similarly, relabel Grassmann variables on edge (i, j) as $\bar\psi_{i,j}$ and $\psi_{i,j}$. So, we have to reorder the sequence

$$(\pm\bar\psi_{\ell,1}\bar\psi_{1,2})\,(\pm\psi_{1,2}\psi_{2,3})\,(\pm\bar\psi_{2,3}\bar\psi_{3,4})\,(\pm\psi_{3,4}\psi_{4,5})\cdots(\pm\psi_{\ell-1,\ell}\psi_{\ell,1})\,; \tag{2.39}$$

where the sign in a $(\pm\psi_{i,j}\psi_{j,k})$ term depend from the way in which the pair $\psi_{i,j}\psi_{j,k}$ enters in the action, and it is $+$ if the edge (ij) precedes (jk) of a counter-clockwise rotation around j by an angle $\frac{2\pi}{3}$.

The loop identifies an interior and an exterior part, all the angles w.r.t. the interior part being either $\frac{2\pi}{3}$ (small angles) or $\frac{4\pi}{3}$ (large angles). As an example, in the easiest case, a loop surrounding a single hexagon, we have six small angles. We have a minus sign in (2.39) in correspondence of the large angles, and a plus for small angles, so that (2.39) becomes

$$(-1)^{\#\{\text{large angles}\}}\bar\psi_{\ell,1}\bar\psi_{1,2}\,\psi_{1,2}\psi_{2,3}\,\bar\psi_{2,3}\bar\psi_{3,4}\,\psi_{3,4}\psi_{4,5}\cdots\psi_{\ell-1,\ell}\psi_{\ell,1}\,; \tag{2.40}$$

Furthermore, the reordering of the 2ℓ fields in (2.40) produces a factor $(-1)^{\frac{\ell}{2}-1}$, as shown below

$$\overbrace{\bar\psi_{\ell,1}\,\underbrace{\bar\psi_{1,2}\psi_{1,2}}_{+}\,\underbrace{\psi_{2,3}\bar\psi_{2,3}}_{-}\,\underbrace{\bar\psi_{3,4}\psi_{3,4}}_{+}\,\underbrace{\psi_{4,5}\cdots}_{-}\quad\underbrace{\cdots\psi_{\ell-1,\ell}}_{+}\,\psi_{\ell,1}}^{+}\,; \tag{2.41}$$

(this overall -1 shift per loop in the exponent is a signature of the "fermionic" nature of these loops). So we get for (2.40)

$$(-1)^{\frac{\ell}{2}-1+\#\{\text{large angles}\}}\ \bar\psi_{1,2}\psi_{1,2}\,\bar\psi_{2,3}\psi_{2,3}\,\bar\psi_{3,4}\psi_{3,4}\cdots\bar\psi_{\ell,1}\psi_{\ell,1}\,. \tag{2.42}$$

Thus our claim is verified if (and only if) for each self-avoiding loop the combination

$$\ell/2-1+\#\{\text{large angles}\}$$

is even. Actually, it will turn out that the following stronger statement holds

Proposition 2.1 *In a self-avoiding loop on the hexagonal lattice,*

$$\ell/2-\#\{\textit{large angles}\}=3\,. \tag{2.43}$$

Indeed, we have

$$\ell=\#\{\text{edges}\}=\#\{\text{vertices}\}=\#\{\text{small angles}\}+\#\{\text{large angles}\}\,, \tag{2.44}$$

but, as overall we do exactly one counter-clockwise turn (because the loop does not cross itself, and the graph is planar), the number of left-turns must be larger than the number of right-turns by exactly 6, thus

$$\ell=6+\#\{\text{large angles}\}+\#\{\text{large angles}\}=6+2\#\{\text{large angles}\}\,, \tag{2.45}$$

which proves our proposition, and, in turns, also the original claim of equation (2.34).

Now that we have an expression for our partition function as a Gaussian Grassmann integral, the problem is essentially solved. We can go back to the expression for the action, and do some simplifications. We can easily perform the integration on edge-variables in a given direction (say $\mu=3$), that we highlight

$$\int\mathcal{D}(\psi,\bar\psi)\,\exp\Big(\mathcal{S}_{1,2}(\bar\psi_{x,1},\bar\psi_{x,2},\psi_{x,1},\psi_{x,2})+\sum_x\bar\psi_{x,3}\psi_{x,3} \\ +\sum_x\Big(\bar\psi_{x,1}\bar\psi_{x,3}+\bar\psi_{x,3}\bar\psi_{x+\mathbf{e}_2,2}+t_3t_2\,\psi_{x,3}\psi_{x,2}+t_1t_3\,\psi_{x+\mathbf{e}_1,1}\psi_{x,3}\Big)\Big)\,. \tag{2.46}$$

Integration is performed either by the general formulas of Appendix A, or completing the square and translating, i.e., for any Grassmann-odd combinations $\bar\eta$ and η,

$$\int\mathrm{d}\psi\mathrm{d}\bar\psi\,e^{\bar\psi\psi+\bar\eta\psi+\bar\psi\eta}=\int\mathrm{d}\psi\mathrm{d}\bar\psi\,e^{(\bar\psi+\bar\eta)(\psi+\eta)-\bar\eta\eta}=\int\mathrm{d}\psi\mathrm{d}\bar\psi\,e^{\bar\psi\psi-\bar\eta\eta}=e^{-\bar\eta\eta}\,. \tag{2.47}$$

In our specific case

$$\bar\psi_{x,3}\psi_{x,3}+\bar\psi_{x,1}\bar\psi_{x,3}+\bar\psi_{x,3}\bar\psi_{x+\mathbf{e}_2,2}+t_3t_2\,\psi_{x,3}\psi_{x,2}+t_1t_3\,\psi_{x+\mathbf{e}_1,1}\psi_{x,3} \\ =\Big(\bar\psi_{x,3}+t_3(t_1\,\psi_{x+\mathbf{e}_1,1}-t_2\,\psi_{x,2})\Big)\Big(\psi_{x,3}-\bar\psi_{x,1}+\bar\psi_{x+\mathbf{e}_2,2}\Big) \\ -t_3(\bar\psi_{x,1}-\bar\psi_{x+\mathbf{e}_2,2})(t_1\,\psi_{x+\mathbf{e}_1,1}-t_2\,\psi_{x,2})\,, \tag{2.48}$$

and thus the expression for the action, depending only on 4 fields per unit cell, is

$$\mathcal{S}(\bar\psi_{x,1},\bar\psi_{x,2},\psi_{x,1},\psi_{x,2})=\sum_x\Big(\bar\psi_{x,1}\psi_{x,1}+\bar\psi_{x,2}\psi_{x,2}+\bar\psi_{x+\mathbf{e}_2,2}\bar\psi_{x,1} \\ +t_2t_1\,\psi_{x,2}\psi_{x+\mathbf{e}_1,1}-t_3(\bar\psi_{x,1}-\bar\psi_{x+\mathbf{e}_2,2})(t_1\,\psi_{x+\mathbf{e}_1,1}-t_2\,\psi_{x,2})\Big)\,. \tag{2.49}$$

The Ising model on the square lattice corresponds to the case $J_3 = 0$ (i.e. $t_3 = 1$). In this case, the expression above for the action is slightly more symmetric after the rescaling

$$\psi_{x,1} \to \frac{1}{\sqrt{t_1}}\psi_{x,1}\,; \qquad \bar\psi_{x,1} \to \sqrt{t_1}\;\bar\psi_{x,1}\,; \tag{2.50}$$

$$\psi_{x,2} \to \frac{1}{\sqrt{t_2}}\psi_{x,2}\,; \qquad \bar\psi_{x,2} \to \sqrt{t_2}\;\bar\psi_{x,2}\,; \tag{2.51}$$

which gives

$$\mathcal{S}(\bar\psi,\psi) = \sum_x \Big(\bar\psi_{x,1}\psi_{x,1} + \bar\psi_{x,2}\psi_{x,2} - t_1\bar\psi_{x,1}\psi_{x+\mathbf{e}_1,1} - t_2\,\bar\psi_{x+\mathbf{e}_2,2}\psi_{x,2} \\ + \sqrt{t_1t_2}(\bar\psi_{x+\mathbf{e}_2,2}\bar\psi_{x,1} + \psi_{x,2}\psi_{x+\mathbf{e}_1,1} + \bar\psi_{x+\mathbf{e}_2,2}\psi_{x+\mathbf{e}_1,1} + \bar\psi_{x,1}\psi_{x,2})\Big)\,. \tag{2.52}$$

This expression is easily seen to be equivalent to equation (69) in [46] (where they have $t_1 = t_2$). Reverting to the full case of equation (2.49), we see that the action is trivially diagonalized in Fourier basis.[♯♯] The appropriate change of variables is thus

$$\bar\psi_{p,\mu} = \frac{1}{\sqrt{V}}e^{i\langle p,x\rangle}\bar\psi_{x,\mu}\,; \qquad \psi_{p,\mu} = \frac{1}{\sqrt{V}}e^{i(p,x)}\psi_{x,\mu}\,; \tag{2.53}$$

which is unitary and carries no Jacobian. The symbol $\langle p,x\rangle$ stands for the two-dimensional scalar product, and, for a parallelogram region of sides L_1 and L_2, the $p = (p_1,p_2)$'s are two-component vectors in the set $\left(\frac{2\pi}{L_1}\{0,1,\dots,L_1-1\}\right) \times \left(\frac{2\pi}{L_2}\{0,1,\dots,L_2-1\}\right)$ (with the *caveat* of the preceeding footnote). Use the shortcut $\xi = \exp(ip_1)$ and $\eta = \exp(ip_2)$, and recognize that the antisymmetric quadratic form in the action is block-diagonalized into blocks 4×4, labeled by the momenta p, each block being

$$\mathcal{S}(\bar\psi,\psi) = \frac{1}{2}\sum_p \begin{pmatrix}\bar\psi_{p,1} & \psi_{p,1} & \bar\psi_{p,2} & \psi_{p,2}\end{pmatrix} \begin{pmatrix} 0 & 1-t_1t_3\xi & -\eta & t_2t_3 \\ & 0 & -t_1,t_3\xi\eta & -t_1t_2\xi \\ & & 0 & 1-t_2t_3\eta \\ & \text{⅄} & & 0 \end{pmatrix} \begin{pmatrix}\bar\psi_{p,1}\\ \psi_{p,1}\\ \bar\psi_{p,2}\\ \psi_{p,2}\end{pmatrix} \tag{2.54}$$

where "⅄" stands for antisymmetric completion of the matrix. The expansion is trivial (even if not knowing the Pfaffian formula for real Grassmann Gaussian integral in Appendix A), and gives, through a series of simplifications that restore the symmetry in the three direction labels,

$$\exp(\mathcal{S}(\bar\psi,\psi)) = \prod_p \big(\bar\psi_{p,1}\psi_{p,1}\bar\psi_{p,2}\psi_{p,2}(1 - t_1t_3\xi - t_2t_3\eta - t_1t_2\xi\eta) + \text{lower orders}\big) \tag{2.55}$$

that is, after integration,

[♯♯] At this point, we would need to specify the boundary conditions: if we had periodic boundary condition on the torus, we would need to prove that loops with non-trivial winding have the appropriate sign, which is indeed false, and we would need to consider a modified set of weights, and take a certain combination of four of these modified expressions; if instead we work on a finite parallelogram, with open boundary conditions, then wuold have a slightly modified set of eigenfunctions. This would not however change anything in the leading part of the free energy, but only in the subleading corrections scaling only with the perimeter, so we neglect these subtleties here.

$$Z \sim \prod_p (1 - t_1 t_3 \xi - t_2 t_3 \eta - t_1 t_2 \xi \eta) \,. \tag{2.56}$$

The $\sim$ symbol is due, besides te factor 2 overall that we neglected, to our sloppiness in considering the boundary conditions. In the thermodynamic limit, at the leading order for the intensive free energy, we have

$$f = \int_{-\pi}^{\pi} \frac{\mathrm{d}p_1 \mathrm{d}p_2}{(2\pi)^2} \ln(1 - t_1 t_3 \xi - t_2 t_3 \eta - t_1 t_2 \xi \eta) \,. \tag{2.57}$$

We could use the discrete symmetries of the Brillouin zone on which we perfor our integrals, in order to show that the integral is indeed real, and get

$$f = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}p_1 \mathrm{d}p_2}{(2\pi)^2} \ln \left(1 - t_1 t_3 \cos p_1 - t_2 t_3 \cos p_2 - t_1 t_2 \cos(-p_1 - p_2)\right) , \tag{2.58}$$

or, in a more symmetric way,

$$f = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3}{(2\pi)^2} \delta(p_1 + p_2 + p_3) \ln \left(1 - (\cos p_1) t_2 t_3 - t_1 (\cos p_2) t_3 - t_1 t_2 (\cos p_3)\right) . \tag{2.59}$$

The expression (2.58) further simplifies in the case of a square lattice, that is $t_3 = 1$, giving

$$f = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}p_1 \mathrm{d}p_2}{(2\pi)^2} \ln \left(2 - (1 + t_1 \cos p_1)(1 + t_2 \cos p_2) + t_1 t_2 \sin p_1 \sin p_2\right) , \tag{2.60}$$

which is exactly the expression derived by Onsager. Coming from a regime of high temperature (small values of t_1 and t_2), the singularity arises at $p = 0$, in agreement with the fact that in a ferromagnetic system the susceptibility first diverges at zero momentum. So the critical value is found by the condition that the argument of the logarithm vanishes at $p = 0$,

$$2 = (1 + t_1)(1 + t_2) \,, \tag{2.61}$$

which, for $t_1 = t_2$, can further be recognized to be in agreement with the prediction of Kramers and Wannier.

2.5.1 Annotated bibliography

The first Onsager solution [31], obtained through a painful analysis of the transfer matrix, dates back to 1944. A remarkable simplification of the original Onsager method has been introduced by Kaufman in 1949 [34].

As remarked above, (2.52) appears already in Itzykson and Drouffe [46], as a Grassmann integral where it is referred to a paper by S. Samuel [38], and appears also in a paper by Itzykson [37]. From there on, Fourier diagonalization is nothing original, and is furthermore common to (almost) all flavours of the solution. However, in the case of Samuel-Itzykson solution, the expression is derived with methods of *Clifford Algebra*, and transfer matrix techniques.

Our derivation instead follows more closely in the spirit what is done for the related problem of Lozenge Tilings with the Pfaffian method of Kasteleyn, which is explained in detail e.g. in Kenyon [19]. The understanding of Kasteleyn solution at genus 1 is sketched already in the original paper by Kasteleyn. Also a hint on how to generalize the result to

higher genus is presented, but in a somewhat cryptic way. However, this must have been understood by the aware readers, as appears from the Regge Rasetti Zecchina approach [47, 48, 49], where they perform the even more ambitious task of exploiting further the mathematical structure in group lattices in order to reduce the number 4^g of distinct Pfaffians required for the evaluation of the partition function. The topic of describing the general genus is discussed also in Galluccio and Loebl [50], and the connection with spin structures and Dirac operators is discussed by Reshetikhin and Cimasoni [51, 52, 53].

The idea of considering a gas of self-avoiding loops appears in the Kac and Ward solution [21], which is among the most ancient combinatorial ones (it dates back to 1952). Their proof seems different from our derivation, but indeed almost every passage may be translated from one language to the other. In particular, in both cases a preminent role in the cancellation of the signs is played by the fact that cycles in a flat Euclidean space have a "winding" (that is, integration of the turning angles followed along the curve) which is an *odd* multiple of 2π if the loop is simple. A detailed derivation of the Kac and Ward method appears in Feynman lectures [36] and Landau 5-th volume of Theoretical Physics [35].

For a wider source of informations on the model, the reader is referred to Mc Coy and Wu book [54], while the italian-speaker readers are also suggested the recent Mussardo textbook [55].

2.6 The OSP($2n|2m$) model in the Gaussian case

Here we give an exact solution, on an arbitrary graph, of a Statistical Mechanics model in which the degrees of freedom have a highly non-trivial underlying symmetry group (rotations in a superspace of dimensions $2n|2m$).

This system is a special realization, within a wider class of systems with the symmetry above. It is such that the action is purely quadratic. So, the evaluation of the partition sum and of a number of observables boils down to Gaussian (and Grassmann-Gaussian) multidimensional integrals, and the relation between Laplacian and Green functions on a weighted graph (cfr. Section 1.5).

We will also specialize the discussion to particular families of graphs, more near to physically interpretable settings.

The reader has here the first approach in this work with "fermionic variables", i.e. Grassmann Algebra and Grassmann-Berezin integration, which will be a main topic in the following, starting from Chapter 7 on, and whose main properties are collected in Appendix A. However, at this introductory point, the reader not feeling comfortable with "superspaces" and anticommuting variables, can just set $m = 0$ along the whole section, the corresponding model having the symmetry of ordinary $O(2n)$ rotations in a $2n$-dimensional real Euclidean space, to which a n-dimensional complex structure has been given.

The triviality of the "Gaussian" solution method has a price: the nature of the emerging thermodynamical singularities is different from the one of the "generic" model with OSP($2n|2m$) symmetry (in the sense of universality described in Section 2.1), as the genuine criticality due to cooperative effects of fluctuations is here mixed with the possibility that the variables take unbounded values, which has to be rejected under physical grounds.

If, for example, a local variable is describing something like the local magnetization in the unit volume of the material at some coordinates, because of universality, this value may be modeled as a single ± 1 spin value, or a real value with some *a priori* measure,

e.g. symmetric under reversal if the system has this symmetry. Then, the interaction terms will be similarly modelized. However, the true system has a saturation limit: the magnetization in a given small value may not exceed a given threshold.

As a result, the solution of the given modelization may reasonably describe a genuine criticality, coming from cooperative effects in the thermodynamic limit, only if, in a whole neighbourhood of the pertinent critical values of the parameters, it predicts finite values for local magnetizations. Otherwise, the absence of a modelization of a threshold has to be interpreted as a lack in the description of the key "geometric" features of the model, w.r.t. universality.

In our model, we have fields $\boldsymbol{\sigma} = (\bar\phi^1, \phi^1, \cdots, \bar\phi^n, \phi^n; \bar\psi^1, \psi^1, \cdots, \bar\psi^m, \psi^m)$, and an orthosymplectic quadratic form g_{ab}, such that

$$|\boldsymbol{\sigma}|^2 = \boldsymbol{\sigma}^a g_{ab} \boldsymbol{\sigma}^b = 2\left(\bar\phi^1\phi^1 + \cdots + \bar\phi^n\phi^n\right) - 2\left(\bar\psi^1\psi^1 + \cdots + \bar\psi^m\psi^m\right) . \tag{2.62}$$

(The relative sign is choosen for convenience. It could be reabsorbed by a change of variables $\psi_i \to -\psi_i$ with Jacobian $(-1)^m$ per vector.) We consider a weighted graph G, with V vertices, and introduce a variable $\boldsymbol{\sigma}_i$ per vertex $i \in V(G)$, and study the properties of the integral

$$Z_{(2n|2m)}(G, w) := \int \prod_{i\in V} \mathrm{d}\mu(\boldsymbol{\sigma}_i)\, \exp\Big(-\frac{1}{2} \sum_{(ij)\in E} w_{ij} |\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j|^2 \Big) . \tag{2.63}$$

For real positive w's, and the purely bosonic case $m = 0$, this function describes the partition function of a system in which nearby degrees of freedom, valued in the target space $\mathbb{C}^n$, have a tendency of align towards the same value, thus, in physics jergon, a *ferromagnetic system*. The system has the anticipated underlying OSP$(2n|2m)$ symmetry if the measure $\mu(\boldsymbol{\sigma})$ does, that is, if $\mu(\boldsymbol{\sigma})$ only depends on $|\boldsymbol{\sigma}|$. We will assume that this is the case.

We need to modelize the threshold behaviour of the field, as described above. In order to be protected at all values of the w's, we should take a measure which is dumped more than as $\exp(-\text{const.}|\boldsymbol{\sigma}|^2)$ for large values of $|\boldsymbol{\sigma}|$, as we can get factors of the form $\exp(+\text{const.}|\boldsymbol{\sigma}|^2)$ from the interaction part of the Gibbs measure. A measure of the form

$$\mu(\boldsymbol{\sigma}) = \frac{1}{\pi^n} \exp\Big(-\frac{\mu}{2}|\boldsymbol{\sigma}|^2 - \frac{\lambda}{4}|\boldsymbol{\sigma}|^4 \Big) , \tag{2.64}$$

with $\lambda > 0$, would be just perfect. This model is indeed called "$\lambda\phi^4$", and is the prototype system in this class.

However this measure leads to non-trivial integrals, at difference with our goal of dealing only with Gaussian integrals. So we will just set $\lambda = 0$, i.e.

$$\mu(\boldsymbol{\sigma}) = \frac{1}{\pi^n} \exp\Big(-\frac{\mu}{2}|\boldsymbol{\sigma}|^2 \Big) , \tag{2.65}$$

and see the emerging features of this simplified case.

First of all, as a result of dropping the $|\boldsymbol{\sigma}|^4$ term, all the components of the vector decouple exactly (this has to be considered one of the pathologies of the model, as it trivializes the role of the undelying symmetry, in disagreement with the principles of universality). Indeed we get for the partition function

$$Z_{(2n|2m)}(G,w,\mu) = Z^n_{\rm bos}(G,w,\mu) Z^m_{\rm fer}(G,w,\mu)\,; \tag{2.66}$$

$$Z_{\rm bos}(G,w,\mu) = \int \prod_{i\in V} \frac{{\rm d}^2\phi_i}{\pi} \exp\big(-(\mu\bar\phi_i\phi_i + \bar\phi_i L_{ij}\phi_j)\big)\,; \tag{2.67}$$

$$Z_{\rm fer}(G,w,\mu) = \int \prod_{i\in V} {\rm d}\psi_i {\rm d}\bar\psi_i \, \exp\big(\mu\bar\psi_i\psi_i + \bar\psi_i L_{ij}\psi_j\big)\,. \tag{2.68}$$

Here we used a notation of summation over repeated indices, and recognized the appearence of the Laplacian matrix L associated to the graph G with weights w.

Then, (provided that $\mu I + L$ has all eigenvalues with positive real part, or that $n = 0$), a simple Gaussian integral gives

$$Z_{(2n|2m)}(G,w) = \det(\mu I + L)^{m-n}\,. \tag{2.69}$$

We know that, for any graph and real positive weights, L has definite spectral properties: it has all non-negative eigenvalues, and at least a vanishing one, corresponding to a null vector with constant entries. These facts suggest that the "critical" value for the mass is $\mu = 0$, reached from above. Call $\{p_i^2\}_{i=0}^{V-1}$ the eigenvalues of L, in increasing order, so that $p_0 = 0$ (this labeling is done by analogy with Fourier notation in regular geometries). The free energy thus reads

$$F = \ln Z = (m-n) \sum_{i=0}^{V-1} \log(\mu + p_i^2) \tag{2.70}$$

and, for a finite connected graph, it has a singularity of the form $(m-n)\log\mu$ for $\mu \ll p_1^2$. If instead we interchange the thermodynamic limit with the critical limit $\mu \searrow 0$, we have to study the limit $\mu \to 0$ of an integral of the form

$$F = (m-n) \int_0^{p_{\max}} {\rm d}\rho(p)\, \log(\mu + p^2)\,. \tag{2.71}$$

The singular behaviour is dominated by the integral near zero. If we assume an algebraic behaviour, and introduce some arbitrary cutoff Λ, we have

$$F_{\rm sing}(\mu) \propto (m-n) \int_0^{\Lambda} {\rm d}(p^2)\, (p^2)^{c-1} \log(\mu + p^2)\,, \tag{2.72}$$

which is convergent for $c > 0$. This is the form that we have, for example, in regular d-dimensional lattices, where there are $\sim p^{d-1}$ momenta of modulus $|p|$, so that ${\rm d}p\, p^{d-1} \sim {\rm d}(p^2)\,(p^2)^{\frac{d}{2}-1}$. The integral

$$\mathcal{I}_c(\mu) \int_0^1 {\rm d}x\, x^{c-1} \log(\mu + x) = -\frac{1}{c(c+1)\mu} {}_2F_1\big(1, 1+c; 2+c; -\tfrac{1}{\mu}\big) + \frac{\ln(1+\mu)}{c} \tag{2.73}$$

has a clear singular part for generic values of c:

$$\mathcal{I}_c(\mu) = \frac{\pi}{c\sin(\pi c)} \mu^c + (\text{Taylor series in } \mu) \tag{2.74}$$

while for integer values of c, where we have a divergence of the singular term in (2.74), we have to revert to (2.73) in order to recognize that there is a logarithmic correction in the singularity:

$$\mathcal{I}_c(\mu) = \frac{(-1)^c}{c}\mu^c \ln(\mu) + (\text{Taylor series in } \mu)\,. \tag{2.75}$$

We can now study a set of observables, as monomials in the field components. The expectation value of the most general monomial has the form

$$\begin{aligned}&\big\langle \bar\phi^1_{x(1,1)}\cdots\bar\phi^1_{x(1,p_1)}\phi^1_{y(1,1)}\cdots\phi^1_{y(1,q_1)}\ \cdots\ \bar\phi^n_{x(n,1)}\cdots\bar\phi^n_{x(n,p_n)}\phi^n_{y(n,1)}\cdots\phi^n_{y(1,q_n)}\\ &\quad\bar\psi^1_{z(1,1)}\cdots\bar\psi^1_{z(1,r_1)}\psi^1_{u(1,1)}\cdots\psi^1_{u(1,s_1)}\ \cdots\ \bar\psi^m_{z(n,1)}\cdots\bar\psi^m_{z(n,r_m)}\psi^m_{u(n,1)}\cdots\psi^m_{u(1,s_m)}\big\rangle\end{aligned} \tag{2.76}$$

where x, y,... with various indices are vertex labels, with possibly repeated entries, and p_α's, q_α's, r_α's and s_α's are non-negative integers telling the overall degree of fields with a given 'colour' index (and any 'coordinate' label) in the monomial.

First of all, remark that, even for the general invariant model (not necessarely Gaussian), in order to have a non-zero expectation it is required that $p_\alpha = q_\alpha$ for all $\alpha = 1,\dots,n$ and $r_\alpha = s_\alpha$ for all $\alpha = 1,\dots,m$. Then, specifically in the Gaussian model, because of the factorization of the Gibbs weight on the different components, we have

$$\begin{aligned}&\big\langle \bar\phi^1_{x(1,1)}\cdots\bar\phi^1_{x(1,p_1)}\phi^1_{y(1,1)}\cdots\phi^1_{y(1,p_1)}\ \cdots\ \bar\phi^n_{x(n,1)}\cdots\bar\phi^n_{x(n,p_n)}\phi^n_{y(n,1)}\cdots\phi^n_{y(1,p_n)}\\ &\quad\bar\psi^1_{z(1,1)}\cdots\bar\psi^1_{z(1,r_1)}\psi^1_{u(1,1)}\cdots\psi^1_{u(1,r_1)}\ \cdots\ \bar\psi^m_{z(n,1)}\cdots\bar\psi^m_{z(n,r_m)}\psi^m_{u(n,1)}\cdots\psi^m_{u(1,r_m)}\big\rangle\\ &= \big\langle \bar\phi^1_{x(1,1)}\cdots\bar\phi^1_{x(1,p_1)}\phi^1_{y(1,1)}\cdots\phi^1_{y(1,p_1)}\big\rangle\cdots\big\langle \bar\phi^n_{x(n,1)}\cdots\bar\phi^n_{x(n,p_n)}\phi^n_{y(n,1)}\cdots\phi^n_{y(1,p_n)}\big\rangle\\ &\quad\big\langle \bar\psi^1_{z(1,1)}\cdots\bar\psi^1_{z(1,r_1)}\psi^1_{u(1,1)}\cdots\psi^1_{u(1,r_1)}\big\rangle\cdots\big\langle \bar\psi^m_{z(n,1)}\cdots\bar\psi^m_{z(n,r_m)}\psi^m_{u(n,1)}\cdots\psi^m_{u(1,r_m)}\big\rangle\end{aligned} \tag{2.77}$$

where, furthermore, each average may be performed more simply in the corresponding 1-boson or 1-fermion reduced theory. Then we have, as a result of Wick theorem for bosons and fermions,

$$\big\langle \bar\phi_{x_1}\phi_{y_1}\cdots\bar\phi_{x_k}\phi_{y_k}\big\rangle = \operatorname{per}\big[\,\big\langle\bar\phi_{x_i}\phi_{y_j}\big\rangle\,\big]_{i,j=1,\dots,k}\,; \tag{2.78}$$

$$\big\langle \bar\psi_{x_1}\psi_{y_1}\cdots\bar\psi_{x_k}\psi_{y_k}\big\rangle = \det\big[\,\big\langle\bar\psi_{x_i}\psi_{y_j}\big\rangle\,\big]_{i,j=1,\dots,k}\,. \tag{2.79}$$

Here we use a notation such that $\big[\mathcal{A}(i,j)\big]_{i,j=1,\dots,k}$ is the $k\times k$ matrix A such that $A_{ij} = \mathcal{A}(i,j)$.

Furthermore, in particular the bosonic and fermionic two-point functions are equal (as required by the symmetry), and are given by

$$\big\langle \bar\phi_x\phi_y\big\rangle = \big\langle \bar\psi_x\psi_y\big\rangle = (\mu I + L)^{-1}_{xy} = G_{xy}(\mu)\,, \tag{2.80}$$

where $G_{xy}(\mu)$ is the coordinate-space free propagator of the graph at square-mass μ (cfr. Section 1.5).

The expressions for the expectation values of these operators are so easy that we can work out explicitly the effect of boundary conditions, selecting a "pure phase" with spontaneous magnetization in a component of the field, following Ruelle strategy, which we go to specialize to our system.

Consider a 'large' graph G, with a 'boundary' $\partial G \subset V(G)$, and some 'interior region' $X \in V(G)$. We will, as always, consider families of graphs realizing a thermodynamic limit, and we will require, in order for the pictorial meaning of the terms above to be reasonable, that the graph distance between X and ∂G grows in the sequence, with some power law in graph size, and that the size of ∂G grows, again with a power law in graph size with exponent strictly in the interval $(0,1)$. We will then concentrate on expectation

values of (microscopic) operators $O(\boldsymbol{\sigma})$ with support on X, and define averages "in the bosonic phase a" by introducing a (macroscopic) operator, with support on ∂G, which breaks the OSP$(2n|2m)$ symmetry by favouring, or discouraging, the expectation value of field components in the a direction.

The possibility that we will consider is the one of introducing parameters μ', μ'', with $\mu'' > 0$ and $\mu' < \mu''$ and, for sites $i \in \partial G$, deform the measure (2.65) into

$$\mu_a(\boldsymbol{\sigma}) = \frac{1}{\pi^n} \exp\Big(-(\mu'' - \mu')\bar{\phi}_a\phi_a - \mu'' \sum_{b\neq a} \bar{\phi}_b\phi_b + \mu'' \sum_{\alpha} \bar{\psi}_\alpha\psi_\alpha \Big) . \tag{2.81}$$

This corresponds to the product of the two macroscopic operators $B_a(\mu', \partial G)$ and $B_{\mathrm{sym}}(\mu'', \partial G)$ (we drop arguments when clear in the following)

$$B_a(\mu', \partial G) = \exp\Big(\mu' \sum_{i\in\partial G} \bar{\phi}_i^a\phi_i^a\Big) , \tag{2.82}$$

$$B_{\mathrm{sym}}(\mu'', \partial G) = \exp\Big((\mu'' - \mu) \sum_{i\in\partial G} |\boldsymbol{\sigma}_i|^2\Big) , \tag{2.83}$$

Clearly, $\mu' \in (0, \mu'')$ do favour the expectation of $|\phi^a|$ w.r.t. the other components, and acts ferromagnetically, while $\mu' \in (-\infty, 0)$ discourages this expectation, leaving in general a large residual symmetry and being the less interesting case. Also, we expect that the role of the symmetric operator B_{sym} is marginal in the symmetry-breaking mechanisms, and this operator has only been introduced in order to allow values $\mu' > \mu$ by taing $\mu'' \neq \mu$. We will understand this fact as included once and forever in the symmetric measure (and μ'' be a further thermodynamical parameter), and we will omit to write B_{sym} in expectation values (i.e. $\langle O\rangle = \int \mathrm{d}[\boldsymbol{\sigma}]\, O[\boldsymbol{\sigma}] B_{\mathrm{sym}}[\boldsymbol{\sigma}] \exp(\mathcal{S}[\boldsymbol{\sigma}])$).

So, given a microscopic observable, we define its expectation "in the phase a" in terms of these symmetric expectations

$$\langle O(\boldsymbol{\sigma})\rangle_a := \frac{\langle O(\boldsymbol{\sigma}) B_a\rangle}{\langle B_a\rangle} \tag{2.84}$$

This name does *not* imply that the "phase a" exists as a thermodynamic phase different from the symmetric phase, and symmetry breaking occurs, our aim being exactly to determine under which conditions over the graph and the thermodynamical parameters this occurs.

In particular, we want to evaluate the magnetization in a certain direction, in some point $x \in X$, as this is a viable order parameter for a ferromagnetic order/disorder transition. So we may consider observables as $\big\langle \bar{\phi}_x^b\phi_x^b \big\rangle_a$ and $\big\langle \bar{\psi}_x^b\psi_x^b \big\rangle_a$. A straightforward application of the statement (2.77) above on factorization of expectation values on different components shows that all these two-point functions coincide, as soon as the index in the trial operator is different from the index a of the tentative phase-selecting operator. So, a single independent interesting combination exists, namely, for any $b \neq a$ (assuming that $n \geq 2$), $\big\langle \bar{\phi}_x^a\phi_x^a - \bar{\phi}_x^b\phi_x^b \big\rangle_a$. Minor modifications are involved in the case $n = 1$ and $m > 0$. A reader used to field-theoretical notations may recognize the combination

$$\big\langle \bar{\phi}_x^a\phi_x^a - \bar{\phi}_x^b\phi_x^b \big\rangle_a = \big\langle (\bar{\phi}_x^a\phi_x^a - \bar{\phi}_x^b\phi_x^b) B_a \big\rangle = \big\langle : \bar{\phi}_x^a\phi_x^a : \, B_a \big\rangle , \tag{2.85}$$

that is, in the Wick expansion of the last operator, we forbid contraction of the first two fields.

So, in our Gaussian model, where expectation values are given by the Wick rule of pair contractions, we can easily understand the mechanism by which a spontaneous magnetization may arise. Indeed, the subtraction between a and b components produces a restriction of the sum to Wick contractions in which the two x-labeled fields are contracted with fields on the boundary. This forces the presence of two factors $\sim G_{xi}$, algebraically-decaying in the distance $d(x,i)$ (or exponentially, if $\mu \ll 1/d(x,i)^2$). However, the mere cardinality of the boundary (for the two choices of pairings) to boundary sites) may produce a "surface size" compensation, algebraic in system size, which furthermore is possibly enhanced significatively if cooperative effects for the operators on the boundary do arise (in a range between $|\partial G|$, for no cooperative effect, and $|\partial G|^2$, for perfect cooperative effects, but in a non-pathological system in which the order parameter is bounded). We will see now this mechanism at work through an analytic treatment. We will omit component indices as, besides the simple subtraction mechanism, all fields involved have the same index a.

First of all, perform a full Taylor expansion of the boundary operator

$$B_a = \sum_{\mathbf{k}\in\mathbb{N}^{\partial G}} \prod_{i\in\partial G} \frac{(\mu' \bar\phi_i \phi_i)^{k(i)}}{k(i)!} \tag{2.86}$$

Then, for every vector $\mathbf{k}$, consider the set of all valid Wick contractions (for complex bosons) of the resulting $2 + 2\sum_i k(i)$ fields (of which a complex pair located in x), with the only constraint that the two fields in x are not contracted together. Arrange these pairings into classes, labeled by ordered ℓ-uples of indices in ∂G, with $1 \le \ell \le \sum_i k(i)$, where the ℓ-uple $(i_1, \dots, i_\ell)$ corresponds to the pairings with the following structure:

- $\bar\phi_x$ is paired to the ϕ_{i_1} of one of the $k(i_1)$ monomials $(\bar\phi_{i_1}\phi_{i_1})$. We have a $k(i_1)$ combinatorial factor for the possible choices.
- the $\bar\phi_{i_1}$ field remained unpaired is paired to the ϕ_{i_2} of one of the $k(i_2) - \delta_{i_1,i_2}$ monomials $(\bar\phi_{i_2}\phi_{i_2})$. Remark how, by virtue of the factorials, in both cases of repeated or unrepeated indices we deal with the same cancellation of factors, and factors μ' appear overall and are easily handled. These are the key facts allowing for a recursion.
- finally, the $\bar\phi_{i_\ell}$ field remained unpaired is paired to the ϕ_x.

So we have that

$$\begin{aligned} \left\langle : \bar\phi_x \phi_x : \, B_a \right\rangle &= \sum_{\ell\ge 1} (\mu')^\ell \sum_{(i_1,\dots,i_\ell)\in\partial G^\ell} \left\langle \overbrace{\bar\phi_x \phi_{i_1}} \overbrace{\bar\phi_{i_1} \phi_{i_2}} \cdots \overbrace{\bar\phi_{i_\ell} \phi_x} B_a \right\rangle \\ &= \sum_{\ell\ge 1} (\mu')^\ell \sum_{(i_1,\dots,i_\ell)\in\partial G^\ell} G_{x i_1} G_{i_1 i_2} \cdots G_{i_\ell x} \left\langle B_a \right\rangle \, . \end{aligned} \tag{2.87}$$

Define the matrix $\hat G$ as the restriction of G_{ij} to $i,j \in \partial G$. Then we can perform the sum formally, and obtain the compact expression

$$\left\langle \bar\phi^a_x \phi^a_x - \bar\phi^b_x \phi^b_x \right\rangle_a = \mu' G_{xi} (I - \mu' \hat G)^{-1}_{ij} G_{jx} \, . \tag{2.88}$$

A case which is simpler to treat is the following: introduce the "boundary" portion of the lattice, ∂G, as above, but instead of modifying the measure on this set of vertices, introduce a new vertex 0, and connect vertices $i \in \partial G$ through some couplings w_{0i}. Then, modify the measure of site 0 only. The preference of 0 to align along a will influence the sites in ∂G in a first place, causing an effect similar to the one described above, but easier to treat. Indeed, we have (calling G the propagators in the new graph)

$$G_{x0} = \sum_{i \in \partial G} G_{xi} w_{i0} \,; \tag{2.89}$$

$$G_{00} = \sum_{i,j \in \partial G} w_{i0} G_{ij} w_{j0} \,; \tag{2.90}$$

and thus the simple expression

$$\left\langle \bar{\phi}^a_x \phi^a_x - \bar{\phi}^b_x \phi^b_x \right\rangle_a = \frac{\mu' G^2_{x0}}{1 - \mu' G_{00}} \,. \tag{2.91}$$

Now, for "regular" geometries, where typical distances among x and sites in ∂G, and among pairs of sites in ∂G, is $\sim r$, and assuming that the Green function decreases algebraically as $G_{xy} \sim |x-y|^{-\gamma}$, we have that (call w the typical value of w_{i0})

$$\mu' G^2_{x0} \sim \mu' w^2 |\partial G|^2 r^{-2\gamma} \,; \tag{2.92}$$

$$\mu' G_{00} \sim \mu' w^2 |\partial G|^2 r^{-\gamma} \,; \tag{2.93}$$

so, by tuning μ' and w arbitrarily, we can change only an overall adimensional parameter A, and are left with a definite algebraic dependence from r (in units of graph-distance)

$$\left\langle \bar{\phi}^a_x \phi^a_x - \bar{\phi}^b_x \phi^b_x \right\rangle_a \sim \frac{A r^{-\gamma}}{1 - A} \,. \tag{2.94}$$

From this we see that, even at zero mass μ, either the order parameter is algebraically small, with the same exponent γ of the Green function (for $A < 1$), or it diverges, becoming unbounded, because of a strong cooperative effect of the boundary with itself (for $A \nearrow 1$).

If the mass μ is finite, and much larger than $1/r^2$, then we have

$$\mu' G^2_{x0} \sim \mu' w^2 |\partial G|^2 \exp(-2r/\sqrt{\mu}) \,; \tag{2.95}$$

$$\mu' G_{00} \sim \mu' w^2 (|\partial G|^2 \exp(-r/\sqrt{\mu}) + \mathcal{O}(|\partial G|)) \,; \tag{2.96}$$

so that we have an exponentially small order parameter (or its artificial explosion for $\mu'' - \mu' \searrow 0$)

$$\left\langle \bar{\phi}^a_x \phi^a_x - \bar{\phi}^b_x \phi^b_x \right\rangle_a \sim \frac{A e^{-\frac{r}{\sqrt{\mu}}}}{1 - A} \,. \tag{2.97}$$

3

Graph-enumeration problems in $O(n)$ models

We have seen in Section 2.4 how both the low- and high-temperature expansions of Ising Model may produce a model of "loops". For the low-temperature case, we needed the underlying graph to be a planar triangulation, for the Peierls contours to be non-intersecting. For the high-temperature case we needed the graph to be of coordination three (although possibly non-planar) for Euclidean subgraphs coincide with loop configurations. Here we relate in a more systematic way certain loop models to statistical-mechanics models showing a $O(n)$ invariance, Ising corresponding to the $O(1)$ case under certain respects.

3.1 The $O(n)$ non-linear σ-model

For systems showing a critical behaviour, the universality class is mainly determined by the symmetry and dimensionality of the underlying geometry (the *lattice*), and the symmetry of the microscopical degrees of freedom (the *spins*).

One main chapter in this work is the *q-colour Potts Model* a generalization of Ising model to a system in which the underlying symmetry of the spin variables is the discrete permutation group $\mathcal{S}_q$, the spin-reversal symmetry of Ising model $\mathcal{S}_2 \cong \mathbb{Z}_2$ corresponding to $q = 2$.

Other generalizations, which explore different symmetry patterns, are possible. In particular, in connection with quantum field theory and methods of Renormalization Group, models with a set of *continuum* degrees of freedom are more tractable. Conversely, at the aim of giving a combinatorial interpretation of statistical mechanics models, discussed here, these models seem more far from this paradigm. The $O(n)$ non-linear σ-model, that we study here, is one of these other generalizations, which allows for a combinatorial interpretation in various regimes, and has interesting properties also from the QFT point of view.

Define S_n, the sphere having locally dimension $n-1$, as the set of points $x \in \mathbb{R}^n$ such that $|x| = 1$. The group of rotations, $O(n)$, leaves stable this set, and also preserves the measure locally, so it is a candidate symmetry group for fields "living" on the sphere, and having an uniform measure. Clearly, $O(1)$ is isomorphic to $\mathbb{Z}_2$, so we understand that a model with $O(n)$ symmetry, such that n could be considered as a parameter, would have the Ising model as the special case $n = 1$.

In this section we will consider such a model, and discuss possible approaches to the problem of finding an analytic continuation in n, and a relation to purely combinatorial counting problems. The main achievement on this subject is the work of Nienhuis on a very

peculiar realization of a $O(n)$-symmetric model, which presents an analytic continuation in n, has a combinatorial interpretation, and turns out to have a rich mathematical structure, especially in two dimensions. However, the definition has a certain number of drawbacks which make questionable the validity of the results for the expected universality class of $O(n)$-invariant models. In Chapter 8 we will show how, through a different approach, a combinatorial insight, in terms of spanning forests, for the model with OSP(1|2)-symmetry can be achieved. The super-group OSP(1|2) is in a sense isomorphic to $O(-1)$ (i.e. through the fermionic-bosonic correspondence, conjectured by Parisi and Sourlas [56]), thus such a study, beyond its importance *per se*, will provide us a tool for understanding the reliability of Nienhuis approach.

So, coming back to our point, the $O(n)$ model is the generalization of Ising model to systems in which the spin variables take value on a sphere in n dimensions,

$$\sigma_i \in \{\pm 1\} \qquad \longrightarrow \qquad \boldsymbol{\sigma}_i \in \{x \in \mathbb{R}^n : |x|^2 = 1\}\,, \tag{3.1}$$

and we already said that the Ising model corresponds to $n = 1$. The case $n = 2$ corresponds to angular variables, and deserves the special name of *XY-Model* [57]. It has an intuitive physical application to the systems with strong quantistic behaviour, in which the relevant degrees of freedom are the phases of localized wave-functions, and, in the simplest case of nearest-neighbour interaction on a regular two-dimensional lattice, shows an interesting duality which relates it to a model of charged particles with long-range interactions, called *Coulomb Gas* (see e.g.[25], or [28], chapt. 9), The case $n = 3$ is also called *classical Heisenberg model*, and could describe the interaction of classical three-dimensional dipoles, which may rotate, and are located at fixed positions in some lattice. So, among other physical circumstances, this could describe magnetic domains in an isotropic medium, this being a reason for the "ferromagnetism jaergon" for various statistical-mechanics quantities.

In order to understand the features of $O(n)$ models in an unified perspective, an analytic continuation to generic values of n would be useful. We have in mind the master example of Fortuin-Kasteleyn expansion for Potts Model, where the interaction is a linear combination of delta functions, which allows to restate the partition function as an expansion over subgraphs, where the parameter q counting the number of colours enters in an analytic way (cfr. Section 4.1). Thus it is auspicable to search, for a reasonably wide class of Hamiltonians, a combinatorial approach such that the parameter n of the dimension of the order-parameter space enters in an analytic way. Unfortunately, this program is fulfilled only for a very narrow class of models: the models with logarithmic action on a graph of degree 3.

Consider a graph $G = (V, E)$, and a set of V variables $\boldsymbol{\sigma}_i$ as in (3.1). The *a priori* measure over the variables is the uniform normalized Lesbesgue measure over the sphere. For any integrand function $\Phi(\boldsymbol{\sigma})$, one can integrate out the constraint by choosing a direction:

$$\frac{2\delta(\boldsymbol{\sigma}^2 - 1)}{|S_{n-1}|}\mathrm{d}^n\boldsymbol{\sigma}\,\Phi(\boldsymbol{\sigma}) = \frac{\mathrm{d}^{n-1}\boldsymbol{\pi}}{|S_{n-1}|}\theta(1-\boldsymbol{\pi}^2)\sum_{\epsilon=\pm 1}\Phi\big((\epsilon\sqrt{1-\boldsymbol{\pi}^2}, \epsilon\boldsymbol{\pi})\big)\,, \tag{3.2}$$

The measure is invariant under $O(n)$ transformations over the field. If Φ depends only on scalar products among k vectors, the product measure over these k vectors is invariant under *global* $O(n)$ transformations, that is, the same transformation is applied to all the vectors simultaneously.

This is clear at sight for the LHS of (3.2), but on the RHS this is somewat more implicit, and also is realized "non-linearly", as one of the (linear) elementary fields is now

replaced by the Taylor series of $\sqrt{1-\boldsymbol{\pi}^2}$. Models with this spherical measure are called *non-linear σ-models*.

Variables on a sphere are *not* the only choice of a measure with $O(n)$ symmetry. We could have had some non-trivial radial measure, that is a normalized measure over all $\mathbb{R}^n$ of the form $\mathrm{d}\mu(\boldsymbol{\sigma}) = \mathrm{d}\mu(|\boldsymbol{\sigma}|)$, The reason why variables on a sphere are specially studied is both that they locally (i.e. at small distances w.r.t. the curvature of the sphere) resemble free fields, and that, at fixed n, the possibility of locally solve the constraint could lead to special simplifications, as is the case, for example, of our analysis of Chapter 8.

As we discuss in Section 2.6, replacing the sphercal measure with a Gaussian measure would make the system much easier, but unfortunately, for a quadratic interaction measure, the system would also be pathological, as in some region of temperatures (including the critical one) the full measure would become unnormalizable, this fact mixing with the cooperative long-range behaviour of the order / disorder transition and spoiling universality.

It is a theorem of Invariant Theory that a generic function (thus, also the action) having global $O(n)$ invariance is a function involving only scalar products among the vectors, plus possibly a few "exceptional" invariants involving the whole set of vectors. So, with a small loss of generality, and appealing to locality properties of physical theories, we will concentrate on actions depending on scalar products $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ only, that, after solving the constraint, read

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \epsilon_i \epsilon_j \left(\sqrt{(1-\boldsymbol{\pi}_i^2)(1-\boldsymbol{\pi}_j^2)} + \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \right) . \tag{3.3}$$

A simple case is the one of a pairwise interaction, say on the first neighbours of a lattice, i.e. an action of the form

$$\mathcal{H}(\boldsymbol{\sigma}) = \sum_{\langle ij \rangle} W_{ij}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) , \tag{3.4}$$

where W_{ij}'s are arbitrary functions. Up to a shift overall of the action, we can set $W_{ij}(1) = 0$. This overall shift is performed in order to have $\mathcal{H} = 0$ on the fully aligned configuration, a property which makes sense because we have $|\boldsymbol{\sigma}| = 1$.

A further simplification comes from choosing these functions W_{ij} to be linear in the scalar products (i.e. quadratic in the fields), that is

$$\mathcal{H}(\boldsymbol{\sigma}) = \sum_{\langle ij \rangle} J_{ij}(1 - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) . \tag{3.5}$$

Other interesting possibilities include, for example, an action which is even w.r.t. each scalar product, such as

$$\mathcal{H}(\boldsymbol{\sigma}) = \frac{1}{2} \sum_{\langle ij \rangle} J_{ij}(1 - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)^2) . \tag{3.6}$$

This action would have also a local discrete $\mathbb{Z}_2$ gauge invariance $\mathcal{H}(\{\boldsymbol{\sigma}_i\}) = \mathcal{H}(\{\epsilon_i \boldsymbol{\sigma}_i\})$.

We could think to study the universality of actions in the family (3.4), or even more general which may involve k-point interactions. However, we should exclude the choices of actions that risk to be pathological for what concerns the Renormalization Group, such as actions which have an *infinite series* of operators of increasing order, if the coefficients are only "slowly" decreasing[†], as we know that, under a Renormalization-Group analysis,

[†]A potentially dangerous case is when the associated generating function has zero or finite radius of convergence.

an infinite series of operators of this kind could produce collective features that would not appear manifestly from the analysis of relevance of the single monomials in the action.

Nonetheless, because of its combinatorial implications, one is led to consider a model with these pathological features, the *Nienhuis Loop-Gas model*, with action

$$\mathcal{H}(\boldsymbol{\sigma}) = -\sum_{\langle ij\rangle} \ln(1 - J_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)\,. \tag{3.7}$$

Remark that the coupling J enters inside the logarithm, but the factor in front of the logarithm is *not* tunable, and instead fixed to -1. The reason for such a choice is that the resulting Gibbs weight $\exp(-\mathcal{H}(\boldsymbol{\sigma}))$ is a polynomial of bounded degree w.r.t. each variable, and the partition function only involves a finite number of moments of the *a priori* measure.

$$Z^{(n)}_{\text{Nienh.}}(\{J_{ij}\}, G) = \int \prod_{i\in V(G)} \mathrm{d}\mu(\boldsymbol{\sigma}_i) \prod_{(ij)\in E(G)} (1 - J_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \tag{3.8}$$

In particular, if the degree of the underlying lattice is three, and at the light of the parity of the measure, also for generic $O(n)$-invariant normalized measures on $\mathbb{R}_n$ (instead that confined on a sphere), only the second moment plays a role, and concentrating over site i, say with neighbours j_1, j_2 and j_3, (indices a_α represent components of the fields), we have

$$\int \mathrm{d}\mu(\boldsymbol{\sigma}_i)\Big(1 - J_{ij_1}\sum_{a_1=1}^{n} \sigma_i^{a_1}\sigma_{j_1}^{a_1}\Big)\cdots\Big(1 - J_{ij_3}\sum_{a_3=1}^{n} \sigma_i^{a_3}\sigma_{j_3}^{a_3}\Big) \tag{3.9}$$

Defining

$$\langle f(\boldsymbol{\sigma}_i)\rangle_0 = \int \mathrm{d}\mu(\boldsymbol{\sigma}_i) f(\boldsymbol{\sigma}_i)\,, \tag{3.10}$$

we simply get

$$\langle 1\rangle_0 = 1\,, \qquad \langle \sigma^a\rangle_0 = \langle \sigma^a\sigma^b\sigma^c\rangle_0 = 0\,, \qquad \langle \sigma^a\sigma^b\rangle_0 \propto \delta_{a,b}\,, \tag{3.11}$$

(in particular $\langle \sigma^a\sigma^b\rangle_0 = \frac{1}{n}\delta_{a,b}$ for the spherical measure (3.2)). We thus see that in the expansion of polynomials $(1 - J_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$ all the monomials that survive are those such that, for each vertex i of the lattice, either we took the factor 1 on all the three adjacent edges, or we took the factor 1 on one edge, and the factor $J\sigma_i^a\sigma_j^a$ for the other two, with *the same* component a. Globally, this forces *coloured loop configuations*, that is, subgraphs in which every component is a non-intersecting cycle, and has a "colour" $a \in \{1, \dots, n\}$ associated. Summing over possible colourings leaves with uncoloured loop configurations, each carrying a combinatorial "topological" factor n. This makes the partition function polynomial in n at sight, and leads to trhe definition of the loop-gas partition function

$$Z^{(n)}_{\text{Nienh.}}(\{J_{ij}\}, G) = \sum_{\substack{L\subseteq G\\ \text{loop config.}}} n^{k(L)} \prod_{(ij)\in E(G)} \frac{J_{ij}}{n}\,. \tag{3.12}$$

3.2 The limit $n \longrightarrow -1$ for the general case

The $O(n)$ non-linear σ-model described in the previous section, in the analytic continuation to $n = -1$, leads to the supersphere $RP(1|2)$ integration, i.e. the vector fields are now parametrized as

$$\boldsymbol{\sigma}_i = (\epsilon_i \sigma_i, \epsilon_i \bar{\psi}_i, \epsilon_i \psi_i) \tag{3.13}$$

and the expression for the measure is

$$\frac{2}{\mathcal{S}_n} \int_{\mathbb{R}^{-1}} \mathrm{d}\boldsymbol{\sigma}_i \, \delta(\boldsymbol{\sigma}_i^2 - 1) \quad \longrightarrow \quad \sum_{\epsilon_i = \pm 1} \int \frac{\mathrm{d}\psi_i \mathrm{d}\bar{\psi}_i}{2\sqrt{1 - 2\bar{\psi}_i \psi_i}} = \frac{1}{2} \sum_{\epsilon_i = \pm 1} \int \mathrm{d}\psi_i \mathrm{d}\bar{\psi}_i \, e^{\bar{\psi}_i \psi_i} \tag{3.14}$$

while the scalar product becomes

$$\begin{aligned} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j &= \epsilon_i \epsilon_j \left(\sigma_i \sigma_j + \bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i \right) \\ &= \epsilon_i \epsilon_j \left((1 - \bar{\psi}_i \psi_i)(1 - \bar{\psi}_j \psi_j) + (\bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i) \right) \end{aligned} \tag{3.15}$$

It follows that

$$(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)^k = (\epsilon_i \epsilon_j)^k \left[1 - k \left(\bar{\psi}_i \psi_i + \bar{\psi}_j \psi_j - \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j - \bar{\psi}_i \psi_j - \bar{\psi}_j \psi_i \right) \right] \tag{3.16}$$

Indeed, calling

$$f_{ij} = \bar{\psi}_i \psi_i + \bar{\psi}_j \psi_j - \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j - \bar{\psi}_i \psi_j - \bar{\psi}_j \psi_i \tag{3.17}$$

we have

$$(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)^k = (\epsilon_i \epsilon_j)^k \left[1 - k f_{ij} + \frac{k(k-1)}{2!} f_{ij}^2 - \frac{k(k-1)(k-2)}{3!} f_{ij}^3 + \cdots \right] \tag{3.18}$$

but

$$f_{ij}^2 = 2 \, \bar{\psi}_i \psi_i \, \bar{\psi}_j \psi_j + 2 \, \bar{\psi}_i \psi_j \, \bar{\psi}_j \psi_i = 0 \tag{3.19}$$

As a consequence we have that, for the general action (3.4), if we parametrize $W_{ij}(x)$ as

$$W_{ij}(x) = \sum_{k=1}^{\infty} \frac{\beta_k^{(ij)}}{k} (x^k - 1) \tag{3.20}$$

in this case $n = -1$ we can write (drop indices (ij) on W and β for simplicity)

$$W(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) = \sum_{k=1}^{\infty} \frac{\beta_k}{k} \left((\epsilon_i \epsilon_j)^k (1 - k f_{ij}) - 1 \right) \tag{3.21}$$

and, calling

$$\beta_{\text{even}} = \sum_{k=1}^{\infty} \beta_{2k} \qquad \beta_{\text{odd}} = \sum_{k=1}^{\infty} \beta_{2k-1} \qquad \gamma_{\text{odd}} = \sum_{k=1}^{\infty} \frac{\beta_{2k-1}}{2k-1} \tag{3.22}$$

we note that the whole function $W(x)$ is actually fully described by only these three parameters:

$$W(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) = -\left(\beta_{\text{even}} + \beta_{\text{odd}} \, \epsilon_i \epsilon_j \right) f_{ij} - \gamma_{\text{odd}} (1 - \epsilon_i \epsilon_j) \tag{3.23}$$

The expression for the partition function reads

$$Z = 2^{-V} \sum_{\epsilon} \int \mathcal{D}(\bar{\psi}, \psi) \, \exp \left[\sum_i \bar{\psi}_i \psi_i + \sum_{i<j} \left((\beta_{\text{even}}^{(ij)} + \beta_{\text{odd}}^{(ij)} \, \epsilon_i \epsilon_j) f_{ij} + \gamma_{\text{odd}}^{(ij)} (1 - \epsilon_i \epsilon_j) \right) \right] \tag{3.24}$$

From the combinatorics which relates the fermionic action partition function to the spanning-forest generating function (cfr. Chapter 8), we know that

$$\int \mathcal{D}(\bar{\psi},\psi)\,\exp\Big[\sum_i \bar{\psi}_i\psi_i + \sum_{i<j}\beta_{ij}f_{ij}\Big] = \sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} \tag{3.25}$$

and thus we can write

$$Z = 2^{-V}\sum_{\epsilon}\prod_{(ij)}\exp\Big(\gamma^{(ij)}_{\rm odd}(1-\epsilon_i\epsilon_j)\Big)\sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\Big(\beta^{(ij)}_{\rm even}+\beta^{(ij)}_{\rm odd}\,\epsilon_i\epsilon_j\Big) \tag{3.26}$$

Remark in particular that, if $W_{ij}(x)$ is an even function, then $\beta^{(ij)}_{\rm odd} = \gamma^{(ij)}_{\rm odd} = 0$ automatically, and we are left only with the fermionic action, the sum over spin variable being trivial.

$$\beta^{(ij)}_{\rm odd} = \gamma^{(ij)}_{\rm odd} = 0\,;\quad \beta^{(ij)}_{\rm even} = \beta_{ij}\ :$$
$$Z = 2^{-V}\sum_{\epsilon}\sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} = \sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} \tag{3.27}$$

This is a case on which we will concentrate later, but remark that we obtain this same value of the partition function in other two cases. The first is

$$\gamma^{(ij)}_{\rm odd} = 0\,;\quad \beta^{(ij)}_{\rm even} = \beta^{(ij)}_{\rm odd} = \beta_{ij}\ :$$
$$Z = 2^{-V}\sum_{\epsilon}\sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}2\beta_{ij}\delta(\epsilon_i,\epsilon_j) = \sum_{F\in\mathcal{F}}2^{k(F)}\prod_{(ij)\in E(F)}2\beta_{ij} = \sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} \tag{3.28}$$

i.e., all the spin variables in a given tree must have the same value, which is unconstrained (this accounts for the factor $2^{k(F)}$, where we denoted with $k(F)$ the number of connected components in $\mathcal{F}$), and we have the Euler relation $V = |E(F)| + k(F)$ as a consequence of the absence of loops in the forest. The second case is

$$\gamma^{(ij)}_{\rm odd} = 0\,;\quad \beta^{(ij)}_{\rm even} = -\beta^{(ij)}_{\rm odd} = \beta_{ij}\ :$$
$$Z = 2^{-V}\sum_{\epsilon}\sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}2\beta_{ij}\delta(\epsilon_i,-\epsilon_j) = \sum_{F\in\mathcal{F}}2^{k(F)}\prod_{(ij)\in E(F)}2\beta_{ij} = \sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} \tag{3.29}$$

where now we also use the fact that, as in a forest there are no loops, there are exactly two antiferromagnetic ground states in each connected component.

In the general case, it is still true that the partition function will be invariant, but in general it will not be anymore a polinomial in the β's.

Note that, in the case in which only $\beta_{\rm odd} = 0$, but $\gamma_{\rm odd} \neq 0$, again the spin and the fermionic degrees of freedom decouples, but the spin part now is a non-trivial Ising Model.

$$\beta^{(ij)}_{\rm odd} = 0\,;\quad \beta^{(ij)}_{\rm even} = \beta_{ij}\,;\quad \gamma^{(ij)}_{\rm odd} = \gamma_{ij}\ :$$

$$Z = Z_{\rm Ising}\cdot Z_{\rm Forests} \tag{3.30}$$

$$Z_{\rm Ising} = 2^{-V}\sum_{\epsilon}\exp\Big[\sum_{(ij)}\gamma_{ij}(1-\epsilon_i\epsilon_j)\Big] \tag{3.31}$$

$$Z_{\rm Forests} = \sum_{F\in\mathcal{F}}\prod_{(ij)\in E(F)}\beta_{ij} \tag{3.32}$$

The critical properties of the model are then exactly factorized, although, of course, sets of parameters $(\beta_{ij}, \gamma_{ij})$ exist such that the two partition functions are critical simultaneously.

A last case is the case $\gamma_{\text{odd}} = \beta_{\text{even}} = 0$, which surprisingly is totally trivial: if at least one connected component contains at least one site with odd coordination (for example, a leaf in a tree), the sum over the spin variable associated to that site produces a zero factor, thus only forests composed with trees with no leaves are admitted. Only the forest configuration with all isolated vertices survives, and the corresponding partition function is

$$Z = 2^{-V} \sum_{\epsilon} \sum_{F \in \mathcal{F}} \prod_{(ij) \in E(F)} \beta_{ij} \epsilon_i \epsilon_j = 1 \tag{3.33}$$

Remark that this result is not so intuitive as one could argue. For example, an original $O(n)$ action which gives this result in the $n = -1$ limit is the following

$$Z = \prod_i \frac{2}{\mathcal{S}_n} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\sigma}_i \, \delta(\boldsymbol{\sigma}_i^2 - 1) \, \exp\Big[- \sum_{(ij)} \frac{\beta_{ij}}{2} \Big(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)^3 \Big) \Big] \tag{3.34}$$

which is anti-ferromagnetic for β_{ij} positive. But, more interestingly, the action vanishes for $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \pm 1$, that is where the measure concentrates at $n = -1$, while for any value of n different from $1, -1, -3, \ldots$ the argument $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ takes support on the whole interval $[-1, 1]$ and the action is non-trivial.

3.3 Recovering the Nienhuis Model at $n = -1$

Nienhuis [24] adopts a precise choice of $O(n)$ action, in order to obtain a combinatorial connection between $O(n)$ model and a gas of self-avoiding loops, when the lattice is at most trivalent. This seems surprising, at a first look, as in our procedure the traditional $O(n)$ action is put in relation with a different combinatorial object: spanning forests, in case coupled with Ising variables. Furthermore, the value of the central charge of the critical theory in a two-dimensional regular lattice is different for self-avoiding loops and spanning forests.

Here we solve this apparent obstruction, obtaining the same result as a specialization of our general spanning-forests + Ising formulation of the previous section to this very special choice of lattice and action.

As we have seen, Nienhuis Model leads, after a few manipulations, to the study of configurations of "coloured" self-avoiding closed paths (loops) on the lattice, where the colours corresponds to the labels $1, \ldots, n$, or equivalently of (uncolored) loop configurations L, with a topological weight n per component. That is, the partition function is of the form

$$Z^{(n)}_{\text{Nienh.}} = \sum_L n^{k(L)} \prod_{(ij) \in E(L)} \frac{\beta_{ij}}{n} \tag{3.35}$$

Of particular interest for our purposes is the specialization to $n = -1$, to be intended as an "analytic continuation" w.r.t. the original formulation in terms of unitary vectors in n dimensions. We reverse the sign of the couplings for notational convenience:

$$Z^{(-1)}_{\text{Nienh.}} = \lim_{n \to -1} \int \prod_{i=1}^{V} \mathrm{d}^n \boldsymbol{\sigma}_i \frac{2\delta(\boldsymbol{\sigma}_i^2 - 1)}{\mathcal{S}_n} \prod_{(ij)} (1 - \beta_{ij} \, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) = \sum_L (-1)^{k(L)} \prod_{(ij) \in E(L)} \beta_{ij} \tag{3.36}$$

Now we want to obtain the same result as a special case of the general formula which expresses an $O(n)$ model with $n = -1$ with a whatever action as a model of Ising variables coupled to spanning forests on the graph, on the lines of equation (3.26). The partition function gives

$$Z^{(-1)}_{\text{Nienh.}} = \lim_{n\to -1} \int \prod_{i=1}^{V} \mathrm{d}^n \boldsymbol{\sigma}_i \frac{2\delta(\boldsymbol{\sigma}_i^2 - 1)}{\mathcal{S}_n} \prod_{(ij)} (1 - \beta_{ij}\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \tag{3.37}$$

$$= 2^{-V} \sum_{\epsilon} \int \mathcal{D}(\bar\psi, \psi)\, e^{\sum_i \bar\psi_i \psi_i} \prod_{(ij)} (1 - \beta_{ij}\epsilon_i\epsilon_j(1 - f_{ij})) \tag{3.38}$$

$$= 2^{-V} \sum_{\epsilon} \prod_{(ij)} (1 - \beta_{ij}\epsilon_i\epsilon_j) \int \mathcal{D}(\bar\psi, \psi)\, e^{\sum_i \bar\psi_i \psi_i + \sum_{(ij)} \frac{\beta_{ij}\epsilon_i\epsilon_j}{1-\beta_{ij}\epsilon_i\epsilon_j} f_{ij}} \tag{3.39}$$

(here we used $f_{ij}^2 = 0$)

$$= 2^{-V} \sum_{\epsilon} \prod_{(ij)} (1 - \beta_{ij}\epsilon_i\epsilon_j) \sum_{F\in\mathcal{F}} \prod_{(ij)\in E(F)} \frac{\beta_{ij}\epsilon_i\epsilon_j}{1 - \beta_{ij}\epsilon_i\epsilon_j} \tag{3.40}$$

$$= 2^{-V} \sum_{\epsilon} \sum_{F\in\mathcal{F}} \prod_{(ij)\in E(F)} \beta_{ij}\epsilon_i\epsilon_j \prod_{(ij)\notin E(F)} (1 - \beta_{ij}\epsilon_i\epsilon_j) \tag{3.41}$$

Now consider the expansion of the binomials $(1 - \beta_{ij}\epsilon_i\epsilon_j)$. Say that an edge is "black" if it is an edge of the forest, "red" if it comes from a term $-\beta_{ij}\epsilon_i\epsilon_j$ of the binomial, and "white" if it comes from a term 1 of the binomial. We use the fact that

$$\left\langle \epsilon_i^k \right\rangle = 1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \tag{3.42}$$

in order to sum over spin variables. This gives, instead that a complicated "Ising-like partition function", as it does in general, just an Eulerianity constraint: the number of black plus red edges adjacent to a given site must be even. So, the union of black and red edges form an Eulerian subgraph configuration, while the black edges alone must be a forest, that is they form no cycles.

When dealing with a lattice with all sites of coordination at most 3, the Eulerian subgraphs are just loop configurations, and forests contained within a loop configuration are constituted only of open paths.

We are left with

$$Z^{(-1)}_{\text{Nienh.}} = \sum_{\substack{L \text{ loop conf.} \\ F \text{ forest} \\ F\subseteq L}} \prod_{(ij)\in E(F)} \beta_{ij} \prod_{(ij)\in E(L\smallsetminus F)} (-\beta_{ij}) \tag{3.43}$$

Note that, given a loop of length l, the sum over admissible colourings in red and black edges produces a weight $\prod(\beta_{ij} - \beta_{ij}) - \prod \beta_{ij} = -\prod \beta_{ij}$, coming from a counting over the set $\{\text{red}, \text{black}\}^\ell \smallsetminus \{\text{black}\}^\ell$.

This is a special "fine-tuning" of a general structure: if black and red edges had weights respectively a_e and b_e, we would have had $\prod_{e\in\gamma}(a_e + b_e) - \prod_{e\in\gamma} b_e$, that is, for example for all a's equal and all b's equal, as a function of the length ℓ, we would have had the difference of two exponentials. With the fine-tuning $a = -b$ as in our case, we get a single exponential, with a "topological" minus sign overall per loop.

So we have

$$Z^{(-1)}_{\text{Nienh.}} = \sum_L (-1)^{k(L)} \prod_{(ij)\in E(L)} \beta_{ij} \tag{3.44}$$

in agreement with equation (3.36).

3.4 The Hintermann-Merlini–Baxter-Wu Model

We describe here a family of models of Ising type with three-spin interactions, defined on a class of planar graphs, which have a remarkable combinatorial structure. These models are related to $q = 4$ Potts and $n = 2$ Nienhuis loop-gas models for what concerns universality of the critical properties, while in the preliminary combinatorial approach we perform here, a curious mix of different techniques appropriate to Ising systems are employed, so that many methods discussed in Section 2.4 find here another application.

Consider a planar Euclidean triangulation G, i.e. a 2-dimensional cell complex, where all faces are triangles, and all vertices have even degree. Call V, E and F respectively the set of vertices, edges and faces. As a consequence of eulerianity, the graph is naturally and univocally tripartite, and we state $V = V_1 \cup V_2 \cup V_3$, $E = E_{12} \cup E_{23} \cup E_{31}$, and each $(ij) \in E_{ab}$ is such that $i \in V_a$ and $j \in V_b$. As a consequence, each $(ijk) \in F$ is such that $i \in V_a$, $j \in V_b$ and $k \in V_c$, while, for neighbouring faces, these vertex classes come alternatively in clockwise and counterclockwise order. We define $F = F_+ \cup F_-$ for these two classes.

Call $G_{ab} = G(V_a \cup V_b, E_{ab})$, and $G_c = (V_c, E_c)$ as the planar dual of G_{ab} (for $\{a, b, c\} = \{1, 2, 3\}$). This defines the set E_c, as the set of dual edges of E_{ab}. As a consequence of eulerianity of G, G_{ab} is (univocally) bipartite. Cfr. figure 3.1 for an example.

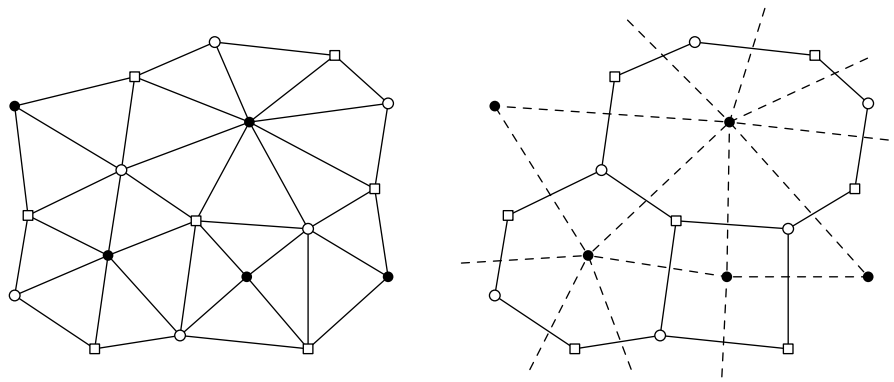


Fig. 3.1. A portion of a planar tripartite triangulation G (left), and the two associated lattices $G_{\square\circ}$ and $G_\bullet$ (right, respectively in solid and dashed lines).

Now we define on graph G a statistical mechanics model of Ising type. Let a set of coupling parameters J_{ijk} associated to faces. Introduce a set of spin variables on the vertices, $\{\sigma_i\}_{i\in V}$, $\sigma_i = \pm 1$, and define the Hamiltonian and partition function

$$\mathcal{H}_{G,J}(\sigma) = -\sum_{(ijk)\in F} J_{ijk}\, \sigma_i \sigma_j \sigma_k\,; \qquad Z_G(J) = \sum_\sigma \exp\big(-\mathcal{H}_{G,J}(\sigma)\big)\,. \tag{3.45}$$

The Hamiltonian in (3.45) has symmetry features which make this class of models appealing. Consider the involutions

$$R_{ab}: \quad \sigma_i \to \begin{cases} \sigma_i & i \notin V_a \cup V_b\,; \\ -\sigma_i & i \in V_a \cup V_b\,. \end{cases} \tag{3.46}$$

These transformations generate a simple abelian group (namely, the *Klein four-group*, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$), preserve the distance function $d(\sigma,\sigma') := \sum_{i\in V}(1-\delta_{\sigma_i,\sigma'_i})$, and also the Hamiltonian, that is

$$d(\sigma,\sigma') = d(R_{ab}\sigma, R_{ab}\sigma')\,; \qquad\qquad \mathcal{H}_{G,J}(\sigma) = \mathcal{H}_{G,J}(R_{ab}\sigma)\,, \tag{3.47}$$

while connecting configurations "far apart",

$$d(\sigma, R_{ab}\sigma) = |V_a| + |V_b|\,. \tag{3.48}$$

This suggests that, if an order-disorder phase transition occurs in the system, it may involve the spontaneous symmetry breaking of the Klein four-group, and thus be in the universality class of $q = 4$ ferromagnetic Potts model. At this aim it is necessary that the group orbits along the pure phases of the system, this being achieved, for example, in a limit of all strongly ferromagnetic couplings, for a large periodic lattice, and also, in a limit of all strongly anti-ferromagnetic couplings, as it suffices to consider the transformation $\sigma \to -\sigma$ in order to effectively change the sign of all the couplings.

By high-temperature expansion, the partition function in (3.45) is trivially related to the generating function of Eulerian sub-hypergraphs of G, seen as a 3-uniform hypergraph with hyper-edge set F:

$$\begin{aligned} Z_G(J) &= \Big(\prod_{(ijk)\in F} \cosh J_{ijk}\Big) \sum_{\sigma} \prod_{(ijk)\in F} (1 + \sigma_i\sigma_j\sigma_k \tanh J_{ijk}) \\ &= 2^{|V|}\Big(\prod_{(ijk)\in F} \cosh J_{ijk}\Big) \sum_{\substack{F'\subseteq F \\ \text{eulerian}}} \prod_{(ijk)\in F'} \tanh J_{ijk}\,; \end{aligned} \tag{3.49}$$

however this approach would not lead us very far, and we will instead use some specialties of planarity and eulerianity of G. For the sake of simplicity, we will also assume that J_{ijk} take only two distinct values, $J_{ijk} = J$ if $(ijk) \in F_+$ and $J_{ijk} = J'$ if $(ijk) \in F_-$.

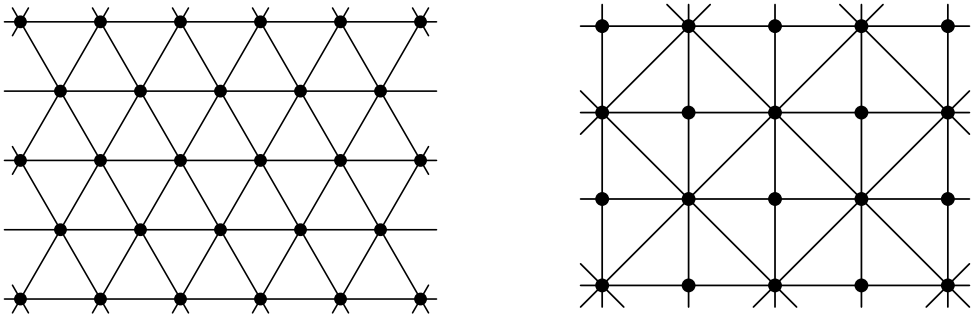
Two special choices of graph for the model above, studied in the literature, are the periodic triangular graph [58], and the "Union-Jack" graph, i.e. a square lattice with alternately-oriented diagonals [59] (cfr. figure 3.2).

For both these lattices, and overall for a certain characterized family of graphs, a different combinatorial expansion exists, in terms of pairs of interacting Eulerian subgraphs of (ordinary) edges on the graphs G_{ab}. If $J = J'$, the subgraphs interaction boils down to require edge-disjointness. In the Baxter-Wu case, also a $n = 2$ anisotropic Nienhuis loop model formulation is available. A duality, which interchanges the parameters counting the sizes of the two subsets, works in a way not dissimilar from the traditional Kramers-Wannier one, although in our final remarks we will stress certain conceptual differences.

For each edge $(ij) \in E_{ab}$, there is an associated pair of vertices $k, \ell \in V_c$ $(c \neq a, b)$ such that $(ijk) \in F_+$ and $(ij\ell) \in F_-$.

Now, choose a sublattice index (say, 3), and perform a high-temperature expansion in which triangles sharing a common edge $e \in E_{12}$ are paired in a single contribution. More

Fig. 3.2. A portion of the triangular lattice (left) and of the Union-Jack lattice (right).



precisely, for $(ij) \in E_{12}$, say that we have a quadrupole $(ij;k\ell)$ if (ijk) and $(ij\ell)$ are as above. Call $\sigma^{(a)}$ the set of spin variables with indices in V_a. Then we can write

$$\begin{aligned} Z &= \sum_{\sigma} \exp\Big(\sum_{(ij;k\ell)} (J\sigma_k + J'\sigma_\ell)\sigma_i\sigma_j \Big) \\ &= \sum_{\sigma^{(1)}} \prod_{(k\ell)\in E_3} \cosh(J\sigma_k + J'\sigma_\ell) \sum_{\sigma^{(1)},\sigma^{(2)}} \prod_{(ij;k\ell)} \Big(1 + \sigma_i\sigma_j \tanh(J\sigma_k + J'\sigma_\ell)\Big) \\ &= 2^{|V_1|+|V_2|} \sum_{\sigma^{(3)}} \prod_{(k\ell)\in E_3} \cosh(J\sigma_k + J'\sigma_\ell) \sum_{\substack{E'\subseteq E_{12} \\ \text{Eulerian}}} \prod_{\substack{(ij;k\ell) \\ (ij)\in E'}} \sigma_k \tanh(J + J'\sigma_k\sigma_\ell)\,; \end{aligned} \tag{3.50}$$

where we performed the traditional high-temperature expansion for spin systems, in terms of Eulerian subsets of edges, for spins on sublattice G_{12}, leaving spins on V_1 as parameters. Remark that the resulting factors lead to an effective pairwise Ising-like interacion among spin variables, for pairs corresponding to the edge-set E_3.

Remark that, for each of the factors in the rightmost product, we have

$$\sigma_k \tanh(J + J'\sigma_k\sigma_\ell) = \sigma_\ell \tanh(J' + J\sigma_k\sigma_\ell)\,, \tag{3.51}$$

so that we have an arbitrariness in the choice of the exposed spin factors.

We want now to perform a low-temperature expansion on these variables, calling E'' the set of edges corresponding to the Peierls contours of the spin configuration, i.e. the edges in the dual of G_3 (so, again on G_{12}) such that the facing spins have opposite signs. As true in general for Peierls contours on 2-body Ising systems defined on planar lattices, E'' must be an Eulerian subset of E_{12}.

However, we face at this point a difficulty, caused by these exposed spin factors, namely the product of σ_k for all quadruplets $(ij;k\ell)$ such that $(ij) \in E'$. Call k_0 a vertex at choice in V_3. The product of all these factors is of course in $\{\pm 1\}$, and is determined by the parity of the number of -1 factors. It is thus some function $\epsilon(E', E'', \sigma_{k_0})$. Remark that, as G_{12} is bipartite, each Eulerian set E' is adjacent to an even number of faces in F_+, so that, for each configuration of $\sigma^{(3)}$, the number of $\sigma_k = -1$–factor quadruplets, plus the one of $\sigma_k = +1$–factor quadruplets, is even, and this allows to conclude that

$$\epsilon(E', E'', +1) = \epsilon(E', E'', -1) =: \epsilon(E', E'')\,. \tag{3.52}$$

Remark however that $\epsilon(E', E'')$ is not in general a simple function, and is not either symmetric under exchange of E' and E''. So we can write, for our generic case

$$Z = 2^{|V_1|+|V_2|+1} \sum_{\substack{E', E'' \subseteq E_{12} \\ \text{Eulerian}}} \epsilon(E', E'') \, (\cosh(J + J'))^{|E_{12} \setminus (E' \cup E'')|} \\ (\cosh(J - J'))^{|E'' \setminus E'|} (\sinh(J + J'))^{|E' \setminus E''|} (\sinh(J - J'))^{|E' \cap E''|} \, . \tag{3.53}$$

Because of ϵ signs, even for $J \geq J'$ reals (so that $\sinh(J - J')$ has positive sign), the expression above is not probabilistic w.r.t. the configurations (E', E'').

It is now natural to ask if there are circumstances under which we can get rid of this undesired extra sign. In what follows we will prove two sufficient conditions:

Strong property: at least one set among V_1 and V_2 has all vertices of degree at most 4;

Weak property: at least one set among V_1 and V_2 has all vertices of degree at most 6, and $J = J'$.

Remark that, as in a large planar triangulation one has that the average vertex degree is $6 + \mathcal{O}(|V|^{-1})$, these conditions are not very restrictive on simple periodic lattices.

Remark that the first condition is satisfied in the Hintermann-Merlini case, while only the second one is satisfied in the Baxter-Wu case.

We start by proving the easier second condition. Say that vertices in V_2 have all degree at most 6. Indeed, if $J = J'$, we must have $E' \cap E'' = \varnothing$. Consider an arbitrary decomposition of E' and E'' into cycles (this always exists for Eulerian graphs). For each cycle of E', the Peierls contours E'' can cross the cycle only in correspondence of vertices of V_1 (because vertices in V_2 have degree at most 3 in G_{12}, and $E' \cap E'' = \varnothing$). This determines a decomposition of the cycle into an even number of open paths. But any open path in G_{12} connecting two vertices of V_1 must have even length. So an even number of factors σ_k are involved in each path, and they take all the same value (by construction, as no Peierls contour divides them), and the resulting sign is certainly $+1$. So $\epsilon(E', E'') = +1$ for all E', E'' such that $E' \cap E'' = \varnothing$, and we get in this case

$$Z = 2^{|V_1|+|V_2|+1} (\cosh 2J)^{|E_{12}|} \sum_{\substack{E', E'' \subseteq E_{12} \\ \text{disjoint, Eulerian}}} (\tanh 2J)^{|E'|} \left(\frac{1}{\cosh 2J} \right)^{|E''|} , \tag{3.54}$$

that is, calling for short

$$w_1(J) := \tanh 2J \, ; \qquad\qquad w_2(J) := \frac{1}{\cosh 2J} \, ; \tag{3.55}$$

and neglecting the analytic overall factor, we have a generating function

$$Z \sim \sum_{\substack{E', E'' \subseteq E_{12} \\ \text{disjoint, Eulerian}}} w_1^{|E'|} w_2^{|E''|} \, . \tag{3.56}$$

Remark that $w_1^2 + w_2^2 = 1$, thus for J positive real we could parametrize w_1 and w_2 as the cosine and sine of an angle in the interval $[0, \pi/2]$. Exchanging w_1 with w_2 is a symmetry of the generating function, which exchanges high and low values of J, with an unique fixed point J_* for the value $J_* = \frac{1}{2} \operatorname{arcsh} 1$. This allows us to state a Kramers-Wannier–type argument: for any family of periodic lattices allowing for a thermodynamic limit,

such that in the limit our spin model is critical, with an unique critical temperature, this should coincide with the fixed-point value J_*.

In the case of the triangular lattice, for any choice of the sublattice on which we perform the initial high-temperature expansion (they are all equivalent), we obtain an hexagonal lattice, and the condition is satisfied for both V_1 and V_2. As a result, E' and E'' are not only edge-disjoint, but also vertex-disjoint, and are both composed of disjoint cycles. If we consider $E' \cup E''$ as a whole, as the unique dynamical variable, we have a factor 2 per cycle, due to the associated double countings, and we recognize a realization of Nienhuis Loop Model on the hexagonal lattice, for $n = 2$.

The model is exactly solved, of course, throught Nienhuis solution for generic n [24], but also directly, through a Bethe-Ansatz on the transfer-matrix formulation of the partition function, and indeed, in both cases, the critical temperature found in the solution agrees with the value predicted by the duality. Numerical simulations have been performed, for generalizations analogous to the ones presented here [60].

Now we discuss the first condition, i.e. we will assume that all vertices of V_1 have degree at most 4, but we will consider generic J and J'. Now, vertices of V_1 inside G_{12} have exactly degree 2 (or, if smaller, the analysis trivializes), so that quadrangles $(ij;k\ell)$ come in pairs. Say that $i \in V_1$, then we have two quadrangles $(ij;k\ell)$ and $(j'i;k\ell)$. Edges (ij) and (ij') are either both in E', or none is, and similarly for E'', Consider a cycle decomposition of E', and say that (ij) and (ij') are in E'. Then, they are in the same cycle in any decomposition. If k is internal to the cycle in one quadrangle, then it is external in the other, so for each pair of quadrangles we get exactly a factor $\sigma_k\sigma_\ell \tanh(J + \sigma_k\sigma_\ell J')$, and, in the two possible cases according to E'' occupancies,

$$\big(\sigma_k\sigma_\ell \tanh(J + \sigma_k\sigma_\ell J')\big)^2 = \begin{cases} \tanh^2(J + J') & (ij), (ij') \notin E'' \\ -\tanh^2(J - J') & (ij), (ij') \in E'' \end{cases} \tag{3.57}$$

and the signs have been dealt locally. Also in the case $(ij), (ij') \notin E'$ we can combine the weigths for the two quadrangles, and get

$$\big(\cosh(J + \sigma_k\sigma_\ell J')\big)^2 = \begin{cases} \cosh^2(J + J') & (ij), (ij') \notin E'' \\ \cosh^2(J - J') & (ij), (ij') \in E'' \end{cases} \tag{3.58}$$

Collecting the factors in pairs corresponds to reduce in series the edges (ij) and (ij'), that is considering Eulerian subgraphs of the graph $\hat{G}_2$ such that $(jj') \in E(\hat{G}_2)$ iff (ij) and (ij') are as above in G_{12}. We get in this formulation

$$\begin{aligned} Z = 2^{|V_1|+|V_2|+1}(\cosh(J + J'))^{2|E(\hat{G}_2)|} \sum_{\substack{E', E'' \subseteq E(\hat{G}_2) \\ \text{disjoint, Eulerian}}} (\tanh^2(J + J'))^{|E' \smallsetminus E''|} \\ \left(\frac{\cosh^2(J - J')}{\cosh^2(J + J')}\right)^{|E'' \smallsetminus E'|} \left(-\frac{\sinh^2(J - J')}{\cosh^2(J + J')}\right)^{|E' \cap E''|}, \end{aligned} \tag{3.59}$$

that is, calling for short

$$w_1(J, J') := \tanh^2(J + J'); \tag{3.60}$$

$$w_2(J, J') := \frac{\cosh^2(J - J')}{\cosh^2(J + J')}; \tag{3.61}$$

$$w_3(J, J') := -\frac{\sinh^2(J - J')}{\cosh^2(J + J')}; \tag{3.62}$$

and neglecting the analytic overall factor, we have a generating function

$$Z \sim \sum_{\substack{E',E''\subseteq E(\hat{G}_2)\\ \text{Eulerian}}} w_1^{|E'\smallsetminus E''|}\, w_2^{|E''\smallsetminus E'|}\, w_3^{|E'\cap E''|}\,. \tag{3.63}$$

This time $w_1+w_2+w_3=1$, however for J and J' positive real and distinct, the parameter w_3 is negative, and the sum over configurations has no probabilistic interpretation. The parameters w_1, w_2, w_3 are all real positive if instead J is complex and J' is its complex conjugate. Indeed, for $J=\frac{r+i\theta}{2}$ we have

$$w_1=\tanh^2 r\,; \qquad w_2=\frac{\cos^2\theta}{\cosh^2 r}\,; \qquad w_3=\frac{\sin^2\theta}{\cosh^2 r}\,. \tag{3.64}$$

We can apply again the Kramers-Wannier procedure, and determine for which values of J and J' in $\mathbb{C}$ we have the symmetric values $w_1=w_2$, besides the already known values $J=J'=\pm J^*$. Indeed, now the equality condition reads $\pm(e^{2(J+J')}-1)=e^{2J}+e^{2J'}$, whose solution is

$$2J'=\begin{cases}\ln(e^{2J}+1)-\ln(e^{2J}-1) & J>0\\ \ln(1-e^{2J})-\ln(e^{2J}+1) & J<0\end{cases} \tag{3.65}$$

A case in which a further symmetry occurs is the one in which all vertices in V_1 have degree 4, and all vertices in $V_1\cup V_2$ have degree which is multiple of 4. This condition is satisfied, among others, by the Hintermann-Merlini Union Jack lattice. In this case, the graph $\hat{G}_2$ is itself Eulerian. If $J=J'$, and thus $E'\cap E''=\varnothing$, we have a partition function of the form

$$Z \sim \sum_{\substack{E',E''\subseteq E(\hat{G}_2)\\ \text{disjoint, Eulerian}}} w_1^{|E'|}\, w_2^{|E''|}\,. \tag{3.66}$$

Indeed, now also $E(\hat{G}_2)\smallsetminus(E'\cup E'')=:E'''$ is Eulerian, so the valid configurations are *partitions* of $E(\hat{G}_2)$ into triplets of edge-disjoint Eulerian subgraphs (E',E'',E'''), and we have not only a symmetry under $(w_1,w_2)\to(w_2,w_1)$, but also a symmetry under $(w_1,w_2)\to(w_1/w_2,1/w_2)$ and $(w_1,w_2)\to(1/w_1,w_2/w_1)$, up to trivial overall analytic factors. However, our "physical" weights, in which $w_{1,2}$ are parametrized in terms of J by hyperbolic trigonometric functions in the range $[0,1]$, explore only one sector of this new symmetry, unless again continuation to complex values of J is considered.

A remarkable property of the procedure is that, given the conditions for getting rid of the alternating signs, in principle three different reductions can be performed on the model, depending on the choice of sublattice G_{ab} we want to preserve in the reduction. Besides the triangular-lattice case, these choices lead in general to an expansion in terms of pairs of Eulerian subgraphs on different graphs, and it is interesting how the common 3-body Ising description relates partition functions otherwise apparently unrelated. We present some examples of this feature in figures 3.3 and 3.4.

We add a final consideration concerning duality. The Kramers-Wannier–type argument we used here involves only an "algebraic" manipulation, on the value of the coupling, and the underlying lattice does not change. This is a substantial difference w.r.t. the original Kramers-Wannier duality for Ising-like systems with pairwise interaction, where both a "reconstruction" of a 0-form from a closed 1-form is required (this procedure being delicate when the boundary conditions are periodic), and the planar duality (i.e. the geometrical duality of cell-complexes is required), such that the low-temperature and

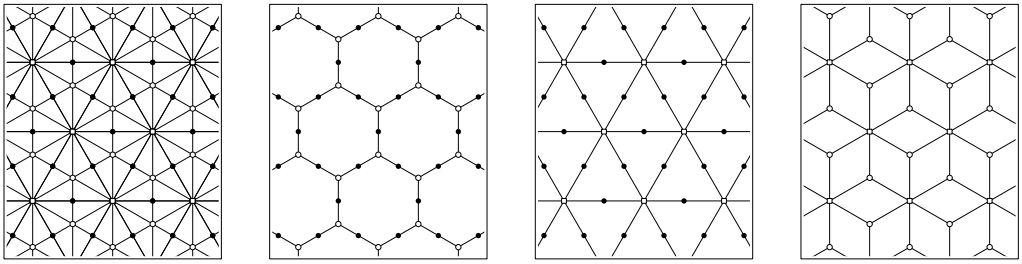


Fig. 3.3. An example of the "geometric" duality for the generating function of pairs of Euclidean subgraph, for a periodic lattice with degree homogeneous within vertex classes, given by $\{\deg_i\} = \{4, 6, 12\}$. On the left, we have a portion of G, then we have G_{12} and G_{13} which satisfy the strong property, and G_{23}, which satisfies the weak property.

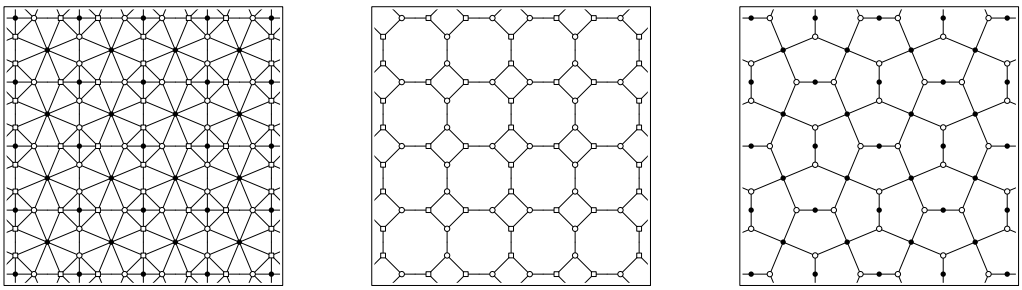


Fig. 3.4. Another example of the "geometric" duality for the generating function of pairs of Euclidean subgraph, for a periodic lattice with degree 6 within vertex classes 1 and 2, and degree 4 and 8 within class 3. On the left, we have a portion of G, then we have $G_{13} \simeq G_{23}$, and G_{12}. They both satisfy only the weak property.

high-temperature expansions relate in general the combinatorial expansion over Eulerian subgraphs on two *different* lattices, planar dual one of the other.

Surprisingly enough, it looks like we have also a kind of purely "geometrical" duality, independent from the algebraic one above, and due to the three choices for sublattices G_{ab}. In particular, if V_a satisfies either the weak or the strong condition for the cancellation of the alternating signs, the *both* G_{ab} and G_{ac} are valid choices for the expansion, and these two graphs are in general different, and the two generating functions (3.54) for the two lattices coincide, for the *same* value of the couplings J and J' in the two graphs.

Disentangling the geometric and the algebraic ingredients in the construction of dualities is a goal discussed in [43].

4

Graph-enumeration problems in the ferromagnetic Potts Model

4.1 Potts model and Fortuin-Kasteleyn expansion

Given the weighted graph $G = (V, E)$, with weight function $v : E(G) \to \mathbb{R}$ (or more generically in a commutative ring $\mathcal{R}$), and an integer $q \in \mathbb{N}$, the Potts Model partition function is defined as the sum over possible colourings of the vertices, $s : V(G) \to [q] = \{1, 2, \ldots, q\}$, with a "prize" v_{ij} if the endpoints of edge (ij) take the same colour. (Here $\delta(s, s')$ is Kronecher delta)

$$Z(G; v; q) = \sum_{s:V\to[q]} \prod_{(ij)\in E(G)} (1 + v_{ij}\delta(s_i, s_j)) \,. \tag{4.1}$$

A maybe more familiar formulation, and nearer to the traditional form of Ising partition function (the special case $q = 2$ of Potts) is

$$Z(G; v; q) = \sum_{s:V\to[q]} \exp\Big(\sum_{(ij)\in E(G)} J_{ij}\delta(s_i, s_j)\Big) \tag{4.2}$$

where $v_{ij} = e^{J_{ij}} - 1$. Indeed, as also exploited (in a slightly different way) in the high-temperature expansion for Ising, in Section 2.4, $e^{Jn} \equiv 1 + (e^J - 1)n$ if we restrict to $n \in \{0, 1\}$.

For real v's, we say that an edge (ij) is "ferromagnetic" if $v_{ij} > 0$, "antiferromagnetic" if $v_{ij} \in [-1, 0]$, and "non-physical" if $v_{ij} < -1$. This changes the ratio of weights for taking the same colour or not on neighbouring sites: in the first case, taking the same colour gives a factor larger than 1 (i.e. it is favoured), in the second case the factor is smaller than 1 up to the extreme case $v = -1$, where it is forbidden to have the same colour. In this case we reduce to the "colouring problem", a classical topic in Complexity Theory, and a first clear appearence of a "graphical combinatorial model" within the phase diagram of Potts model on arbitrary graphs.

The *colouring problem* is defined as follows: given a graph G, and an integer q (the number of colours), can we assign colours to the vertices in such a way that no neighbouring vertices take the same colour? This problem is trivial for $q = 1$ and 2, and known to be NP-complete for all integers $q \geq 3$. For its pictorial description, in particular 3-colouring has soon become a paradigm for NP-completeness within popular science, much more than the SAT problem, which, on its side, is more near to the original axioms of Turing machines and the concept of complexity completeness described in the seminal 1971 work by Cook [61]. No telling, it is much more widespread than the definition of the full Potts model / Tutte polynomial.

In the case of $v < -1$, the ratio between the two possibilities (equal or different colours at the two endpoints of an edge) is negative, so that the total weights associated to different colourings s do not take in general a definite sign, and the summation over s cannot be interpreted probabilistically as the normalization of a measure $\mu(s)$ over possible colourings s. This does not however exclude *a priori* that, in another set of local variables, a probabilistic interpretation of the summands can be given, this point being the main goal of Chapter 5.

More generally, we are interested to the classification of probabilistic combinatorial formulations of "as wider as possible" regions of the Potts phase diagram (in q and v, e.g. in large regular graphs with all equal weights $v_{ij} = v$), in terms of variables sharing interactions "as local as possible". This will lead us shortly to the definition of the (Fortuin-Kasteleyn) Random Cluster Model, and then to a discussion of the Antiferromagnetic Fortuin-Kasteleyn expansion, to a description in terms of acyclic orientations (based on results due to Greene and Zaslavsky, and to Stanley), and to the original formulation by Tutte in terms of spanning trees weighted by internal and external activities.

Then, complexity issues for numerical simulations, e.g. for Monte Carlo techniques, could be analysed as a side result of this study, while one expects that simulating non-probabilistic degrees of freedom would cause sign issues, producing an exponential decrease of the relevant statistics.

In the context of the present chapter, we say that a system (i.e. a pair (G, v)) is *ferromagnetic* if all v's are ferromagnetic, *antiferromagnetic* if all v's are antiferromagnetic, and *mixed* if some v's are ferromagnetic, some antiferromagnetic, and possibly some unphysical.

For the aim of probabilistic interpretations, the original q-colour formulation has a probabilistic interpretation for any system without unphysical weights, for any *integer* $q \geq 2$, and is trivial for $q = 1$. However restriction to integers is somewhat unsatisfactory, as we expect many features to be shared by different values of q, and an analytic extension could allow for an unified analysis.

In the case of our partition function (4.1) such a goal is feasible. Indeed, (4.1) is a polynomial in q. This fact is in a sense exceptional, and due to the special pattern of interaction through Kronecher delta's only. Other q-variable models in which a stronger $\mathbb{Z}_q$ mathematical structure is exploited (named *Potts gauge models* or *clock models*) can be proven not to have this property (cfr. for example [62]).

The possibility for analytic continuation for the ordinary Potts model is easily deduced through the so-called *Fortuin-Kasteleyn expansion* [63, 64], that we go to describe (see e.g. [12, Section 2.2]). For each binomial $1 + v_{ij}\delta(s_i, s_j)$ in the product in (4.1), corresponding to an edge $e \in E(G)$, say that the edge is *occupied* if the non-trivial monomial $v_{ij}\delta(s_i, s_j)$ is taken, and *non-occupied* if the trivial factor 1 is taken. So the expansion of the product induces a sum over all $2^{|E(G)|}$ possible subsets $E' \subseteq E(G)$ of occupied edges, which are naturally identified with the spanning subgraphs of G, i.e. $H = (V(G), E') \subseteq G$. One gets

$$Z(G; v; q) = \sum_{H \subseteq G} \sum_{s:V \to [q]} \prod_{(ij) \in E(H)} v_{ij} \prod_{(ij) \in E(H)} \delta(s_i, s_j) \tag{4.3}$$

The first product is just the natural multivariate "thermodynamic factor" associated with edge occupation, and is local, while the second product, which encodes all the dependence from q, is going to determine an extra factor in the weight of H, through summation over consistent s configurations, which will be non-local in edge occupations.

This case is particularly simple, as the delta factors imply that all the vertices in the same connected component have the same colour, while summation over sets of s_i's for

i's in disjoint components are exactly factorized. So summation over s produces a factor $q^{k(H)}$, where $k(H)$ is the number of components in H:

$$Z(G;v;q) = \sum_{H\subseteq G} q^{k(H)} \prod_{(ij)\in E(H)} v_{ij} \tag{4.4}$$

If the system is ferromagnetic and $q > 0$, also the variables describing H (i.e. occupation numbers on the edges) are a probabilistic set, as all the weights $W(H)$ described above, corresponding to the summands at a given H, are positive.

We see here a first non-trivial extension of the probabilistic region. Thus the Potts model has a probabilistic formulation:

- in colour variables s_i, if $q \in \mathbb{N}$ and $v_{ij} \geq -1$ for all edges (ij);
- in edge-occupation variables $n_e \in \{0,1\}$, if $q \in \mathbb{R}^+$ and $v_{ij} \geq 0$ for all edges (ij).

We will collect more positive results in the following sections.

Because of Euler relation

$$|V(G)| - k(G) = |E(G)| - L(G)\,, \tag{4.5}$$

where $L(G)$ is the cyclomatic number of G, one gets that for any factor ρ

$$Z(G;v;q) = \rho^{|V(G)|} \sum_{H\subseteq G} (q/\rho)^{k(H)}\, \rho^{L(H)} \prod_{(ij)\in E(H)} (v_{ij}/\rho) \tag{4.6}$$

Defining $\lambda = q/\rho$ and $w_{ij} = v_{ij}/\rho$ one has equivalently, up to an uninteresting prefactor $\lambda^{k(G)}\rho^{|V(G)|}$, the *Random Cluster* partition function

$$Z(G;w;\lambda,\rho) = \sum_{H\subseteq G} \lambda^{k(H)-k(G)}\, \rho^{L(H)} \prod_{(ij)\in E(H)} w_{ij} \tag{4.7}$$

This transformation is tautological, and of course preserves the probabilistic interpretation of the weights. It does that "at sight" if ρ is real positive. The limits $\lambda \to 0$ and $\rho \to 0$ select respectively *maximally-connected subgraphs* and *forests*, and the double limit $\lambda, \rho \to 0$ selects the intersection of these two sets, which are just spanning trees on G.

4.2 Potts model on a hypergraph

The procedure above is easily generalized to the case in which the underlying pattern of interaction, instead of being encoded by a graph, is described by a hypergraph $G = (V, E)$. We will typically denote by the letter $A \subseteq V$ a hyperedge $A \in E$, and, for $A = \{i_1, \ldots, i_k\}$ any set, we define the Kronecker delta δ_A as

$$\delta_A(s) = \begin{cases} 1 & \text{if } s_{i_1} = \cdots = s_{i_k} \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Then, for q a positive integer, the *q-state Potts model* on the hypergraph $G = (V, E)$ is defined as follows [7]: variables $s_i \in [q]$ are associated to the vertices $i \in V$ and the partition function generalizing (4.1) is

$$Z(G;v;q) = \sum_{s:V\to[q]} \prod_{A=\{i_1,\ldots,i_k\}\in E(G)} (1 + v_A \delta_A(s_{i_1}, \ldots, s_{i_k})) \tag{4.9}$$

where $\{v_A\}_{A\in E}$ are a set of couplings associated to the hyperedges of G.

We can then prove the Fortuin-Kasteleyn (FK) representation [63, 64] for the hypergraph Potts model [7], by following exactly the same method as is used for graphs in the previous section.

We start by writing

$$Z(G;v;q) = \sum_{s\colon V\to S} \prod_{A\in E} \left[1 + v_A \delta_A(s)\right] . \tag{4.10}$$

Now expand out the product over $A \in E$, and let $E' \subseteq E$ be the set of hyperedges for which the term $v_A\delta_A(s)$ is taken. Then perform the sum over configurations $\{s_i\}_{i\in V}$: in each connected component of the spanning subhypergraph (V, E') the color s_i must be constant, and there are no other constraints. Therefore,

$$Z(G;v;q) = \sum_{H\subseteq G} q^{k(H)} \prod_{A\in E(H)} v_A \,, \tag{4.11}$$

as was claimed.

Again the right-hand side of (4.11) is a polynomial in q; in particular, we can take it as the *definition* of the Potts-model partition function $Z_G(q, \mathbf{v})$ for noninteger q.

Now we want to perform the equivalent set of passages from (4.4) to (4.7), for the hypergraph case. Here we have to face the problem that "hyper-loops" are not univocally defined, as equivalent definitions for graphs become inequivalent for hypergraphs. In this context, it follows that the useful definition is as follows: for G a hypergraph, build an ordinary graph G' having the same vertex set, and each hyperedge A is replaced by an arbitrary spanning tree over A (multiple edges may possibly arise from this construction). Then $L(G')$ is independent from the chosen G', and can be taken as a definition of $L(G)$. Given this definition, we have a random-cluster formulation totally analogous to (4.4), namely, defining the rescaled variables $\lambda = q/\rho$ and $w_A = v_A/\rho^{|A|-1}$ one has, up to an uninteresting prefactor $\lambda^{k(G)}\rho^{|V(G)|}$, the *Random Cluster* partition function

$$Z(G;w;\lambda,\rho) = \sum_{H\subseteq G} \lambda^{k(H)-k(G)}\, \rho^{L(H)} \prod_{A\in E(H)} w_A \tag{4.12}$$

This transformation again preserves the probabilistic interpretation of the weights, and does that "at sight" if ρ is real positive. The limits $\lambda \to 0$ and $\rho \to 0$ select respectively *maximally-connected sub-hypergraphs* and *hyperforests*, and the double limit $\lambda, \rho \to 0$ selects the intersection of these two sets, which are just spanning hypertrees on G.

There is, however, a first important difference between the graph case and the hypergraph case: as discussed in [65], every connected graph has a spanning tree, but not every connected hypergraph has a spanning hypertree. Conversely, the set of hyperforests is always non-empty, as it contains at least the completely atomic forest, and the set of maximally connected sub-hypergraphs is non-empty, as it contains at least the full hypergraph.

Furthermore, complexity issues may be totally different for the broader hypergraph case. For example, as we will see, while counting the number of spanning trees on a graph is a polynomial probel, as a result of Kirchhoff Theorem, counting the spanning hypertrees of a hypergraph is a #P-complete problem at least for hypergraphs having hyperedges of degree 4 or higher.

4.3 The Random Cluster Model on planar graphs

In this section we want to state some special properties of the Random Cluster model when the undelying graph Λ is planar.

Given a connected planar graph $\Lambda = (V, E)$, a set of edge weights $\boldsymbol{w} = \{w_e\}_{e\in E(\Lambda)}$ and two parameters λ, ρ, we recall the *random cluster* partition function as

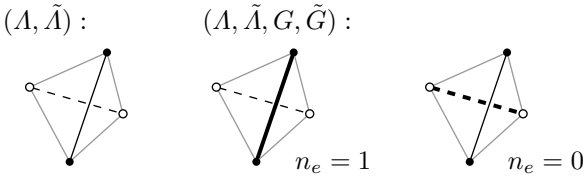
$$Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = \sum_{G\subseteq\Lambda} \lambda^{K(G)-1}\rho^{L(G)} \prod_{e\in E(G)} w_e \,, \tag{4.13}$$

where $K(G)$ and $L(G)$ are respectively the number of components and the cyclomatic number[†] of the spanning subgraph G.

It is well known that something special happens if Λ is a planar graph. For example, the value $\lambda\rho = 2$ corresponds to Ising Model, for which the partition function is known to be an algebraic expression (a Pfaffian, i.e. the square root of a determinant) for generic planar graph Λ and couplings $\{w_e\}$ (see for example Kac and Ward [21] or Kasteleyn [18]), while the approach extended to non-planar lattices produces a combination of terms growing exponentially with genus ($\sim 4^g$). We want thus to comment a few elementary distinctive properties of our system in the planar case, in order to have a flavour of what could get different.

For λ and ρ considered as "true" (e.g. complex) parameters (instead than formal commuting indeterminates, appearing in polynomial expressions), and given the invariance implied by Euler relation, discussed in equation (4.5), for arbitrary weights w_{ij} we could restrict to the three cases $\lambda = \rho = 0$, $\lambda = \rho \neq 0$ and $\lambda = 1$, $\rho = 0$ (or vice-versa). However, we prefer in this context to keep the invariance of equation (4.13), i.e. we use generic λ and ρ, in order not to suffer from the discontinuities of these fixing prescriptions.

As Λ is a planar connected graph, consider a fixed planar embedding. This defines the planar dual graph $\tilde\Lambda$, and a natural bijection between spanning subgraphs $G \subseteq \Lambda$ and $\tilde G \subseteq \tilde\Lambda$: $e \in E(G) \Leftrightarrow \tilde e \notin E(\tilde G)$ and vice-versa. Under this duality, we have $K(\tilde G) - 1 = L(G)$ and $L(\tilde G) = K(G) - 1$. In this case a graphical representation which makes the duality more clear involves simultaneously the quadruple $(\Lambda, \tilde\Lambda, G, \tilde G)$. We represent direct and dual vertices respectively by black and white bullets. Edges and dual-edges (resp. drawn in solid and dashed) come in pairs $(e, \tilde e)$, and identify tetragonal regions (whose perimeter will be drawn in gray). A given subgraph G will be represented by a thickening of the edges e which are occupied (i.e. with $n_e = 1$), and of the the dual-edges $\tilde e$ which are dual to non-occupied edges:



The planar embedding also describes a natural basis for the independent cycles, given by those cycles of G that on the planar representation have no chords. Symmetrically, it describes a natural concept of "exterior perimeter" for a connected component.[‡] This

[†]The cyclomatic number of a graph G is the dimension of the vector space of antisymmetric functions on $E(G)$ having zero divergence on all vertices (called *flows*). It is in a sense the number of "linearly independent" cycles on G.

[‡]These are two facts that need refinements for embeddings on surfaces of non-zero genus.

is a crucial point: on a non-planar graph, such natural prescriptions do not exist, and determining $L(G)$ and $K(G)$ as a result of a local algebra of loops, as we want to do in the next section, is a hard task, except for the trivial percolation case $\lambda\rho = 1$.[††]

(A FIGURE HERE, of what goes wrong for non-planar graphs)

Besides percolation, also the case $\rho = 0$ is an exception, and is accessible on non-planar graphs as well: indeed, in this case λ can be totally reabsorbed into the w's, and taken for example equal to 1, while any method for detecting a whatever cycle by means of local variables allows to restrict to configurations with $L(G) = 0$, even in absence of a natural prescription for counting the independent cycles. Not surprisingly, for the case $\lambda = \rho = 0$ (corresponding to spanning trees), even for a non-planar graph we have an algebraic expression for the partition function, in terms of a determinant (Kirchhoff Theorem). As another aspect of this fact, we will see in Chapter 10 that the pertinent Temperley-Lieb–like algebra restricted to $\rho = 0$ shows a new algebraic relation, which allows to describe the action of non-planar operators in terms of planar ones.

4.4 Leaf removal, series-parallel reduction and Y–Δ relation

In this section we describe some simple manipulations on the partition function of the Random Cluster Model, valid for general graphs but specially important for planar graphs, as they ultimately lead to the so-called *Yang-Baxter equation*, relevant to integrability for certain planar realizations of the model. Although our algebraic expressions can be made valid in the general framework, we thus choose to also exploit the planarity structure, and planar duality, in order both to give a deeper insight in the mechanisms mentioned above, and use the duality symmetry for reducing the case study.

Given a pair $(\Lambda, \tilde{\Lambda})$, at the aim of evaluating the partition function (4.13), vertices or dual-vertices of degree up to 2 can be easily decimated. Indeed, for a vertex v connected to Λ through a single edge e of weight w_e, we get

$$Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = (\lambda + w_e) Z_{\Lambda \setminus e}(\boldsymbol{w}; \lambda, \rho) \tag{4.14}$$

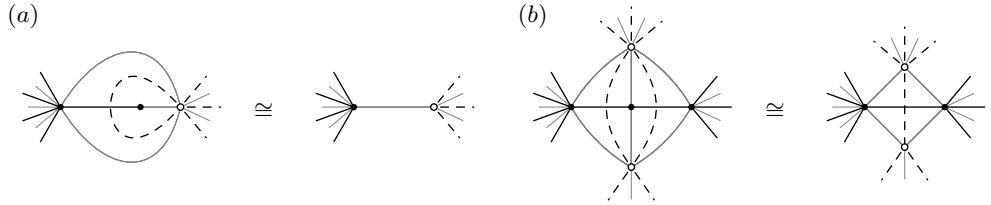
and similarly (better, dually), for an edge e having coinciding endpoints (a "loop", in some graph-theory terminology), we have

$$Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = (1 + \rho w_e) Z_{\Lambda \setminus e}(\boldsymbol{w}; \lambda, \rho) \tag{4.15}$$

The reduction of vertices or dual-vertices of degree 2 is similarly trivial in formulae, although it has some deeper reason, in connection to (generalized) series-parallel reduction

[††]For $K(G)$, if λ is integer, one could introduce some "colour" variables on the direct sites, constrained to be equal on sites connected on G, and totally unconstrained for disconnected clusters: the counting of allowed colouring configurations at fixed subgraph G produces thus a factor $\lambda^{K(G)}$. This is exactly the procedure which leads, from Potts model at integer q, to the Fortuin-Kasteleyn representation, corresponding, in the Potts parameters, to $\lambda = q$, $\rho = 1$ and $w_e = \exp(\beta J_e) - 1$. Similarly, if a finite field K of cardinality ρ exists (i.e. if ρ is a positive integer), one could consider the vector space of flows on K, i.e. of the antisymmetric functions from the oriented edges to this field, such that the divergence at each vertex vanishes: counting the allowed configurations of these "flow" variables one could produce a factor $\rho^{L(G)}$. Still, these methods give access only to positive integer values of ρ and λ (in particular $\rho = 1$ for the case of no auxiliary flux variables at all, or $\lambda = 1$ for the case of no auxiliary colour variables at all), so that only some special values of the invariant parameter $\lambda\rho$ are available. Furthermore, we expect that, beyond a threshold value for $\lambda\rho$, being 4 for 2-dimensional homogeneous systems and 2 in the mean-field limit $d = \infty$, there occurs only a phase transition of first kind.

Fig. 4.1. (a) the removal of an edge connected to a vertex of degree 1; (b) the reduction of two edges connected to a vertex of degree 2 into a single edge. Also dual elements are drawn, so that the pictures for the dual processes are immediately deduced.



and splitting formulas for Tutte polynomial [66] If a vertex v is connect to the rest of the graph Λ through the two edges e and e' of weights w_e and $w_{e'}$, with endpoints respectively u and u' (edges e and e' are said to be *in series* in such a case), in the Random Cluster Model we have four possibilities, for the four choices of $(n_e, n_{e'}) \in \{0,1\}^2$. In the three cases in which the occupancy number of the edges are not both 1 (with complexive weight $w_e + w_{e'} + \lambda$), the vertices u and u' are not connected through the path (u, e, v, e', u'), while in the fourth case, with weight $w_e w_{e'}$, they are connected through this path. The two variables n_e and $n_{e'}$ do not affect the patterns of loops and connection in G in any other way. So, at the aim of calculating Z, we can replace Λ by the simpler graph $\Lambda(\underline{e}\,.\,\underline{e'})$ where v has been removed, and e and e' are replaced by a single edge e'' connecting u to u'. The new weight $w_{e''}$ must be taken equal to the ratio $w_e w_{e'}/(w_e + w_{e'} + \lambda)$, while we get an overall factor $w_e + w_{e'} + \lambda$

$$Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = (\lambda + w_e + w'_e) Z_{\Lambda(\underline{e}\,.\,\underline{e'})}(\boldsymbol{w}'; \lambda, \rho) \tag{4.16}$$

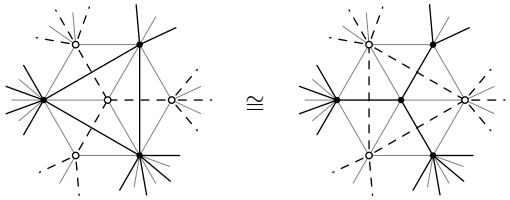
An analogous reasoning holds if two edges e and e' are *in parallel*, i.e. if they are double edges of the graph. Say that u and v are their common endpoints. Again, the four possible combinations of edge occupations split into 1-vs-3 cases for what concerns connectivity. If n_e and $n_{e'}$ are not both zero, (with complexive weight $w_e + w_{e'} + \rho w_e w_{e'}$), the vertices u and u' are connected through at least one among e and e', and we do not have to care about L, as the extra factor ρ in one of the three cases has already been taken into account in the combination above. In the fourth case, with weight 1, they are not connected through any of e and e'. The two variables n_e and $n_{e'}$ do not affect the patterns of loops and connection in G in any other way. So, at the aim of calculating Z, we can replace Λ by the simpler graph $\Lambda(e \| e')$ where e and e' are replaced by a single edge e''. The new weight $w_{e''}$ must be taken equal to the combination $w_e + w_{e'} + \rho w_e w_{e'}$, and no overall factor is needed in this case

$$Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = Z_{\Lambda(e \| e')}(\boldsymbol{w}'; \lambda, \rho) \tag{4.17}$$

These formulas are called of *generalized series-parallel* reductions, because in the case $\lambda = \rho = 0$ of spanning trees they reduce to the well-known formulas for combining resistences connected in series and parallel, with the identification $w_e = 1/R_e$ for resistors on the edges of the graph. Not accidentally, as Kirchhoff theorem originated from the study of the theory of electrical networks!

A drawing of the removal corresponding to equations (4.14) and (4.16) is shown in figure 4.1, while (4.15) and (4.17) come from the previous ones by interchanging solid and dashed lines.

Fig. 4.2. The process involved in the Y–Δ relation.



These procedures are algebraically elementary, and do not come as a big surprise. So we understand that, as we can decimate recursively the "leaves" and the "loops", and the series and parallel edges, in the pair $(\Lambda, \tilde{\Lambda})$, we can think that vertices and dual-vertices all have degree at least 3. One would guess that it is too much to require that further decimations can be performed exactly. What instead turns out is that we can push this sort of reasonings still a bit further: if certain triplets of edge weights, incident on the same vertex (*star*) or arranged on a triangle, satisfy a suitable relation, involving also λ and ρ, we can transform the "triangle" configuration into the "star" one, or vice-versa. Such a relation is called indeed "star-triangle", or Y–Δ (see [67], ch. 6). In the special cases $\lambda = \rho = 0$ and $\lambda\rho = 2$ (resp. spanning trees and Ising), the Y–Δ relation holds in a stronger form, for every value of the triple of parameters on the edges, so that we can think, for example, that Λ has no vertices of degree 3, or that it has no triangles.

Even in its "weak" form, in which the three coupling parameters have to be tuned on a certain two-dimensional manifold, the Y–Δ relation implies the possibility of certain manipulations that lead to prove the commutativity of a family of transfer matrices, i.e. matrices describing the construction of the partition function for a system defined on a strip. This will be the starting point for deducing properties of a system defined through a non-trivial transfer matrix. For example, at least conceptually, one can first diagonalise the easiest of the commuting partners in the family, and then study the original matrix in the thus discovered diagonalising basis. When the Y–Δ relation is used in this context, it is better known as *Yang-Baxter relation*.

Here we explore the Y–Δ relation in its easiest form, for a lattice system. We do that both in the Random Cluster formulation, and, in the special case of $\lambda = 1$ and $\rho = q$ integer, in the original Potts formulation, in order to highlight in the two cases the simplifications at the spanning-tree and Ising points.

In Potts representation, our conjectured equivalent subsystems are a "triangle" over variables s_1, s_2 and s_3, and a "star" which connect these variables to a central one s_0. The parameters are $w_{1,2,3}$ and $\tilde{w}_{1,2,3}$ in the two cases (remember $w = e^J - 1$ in Fortuin-Kasteleyn expansion). We have to match five equations, for the possibilities $s_1 = s_2 = s_3$, $s_1 = s_2 \neq s_3$ (plus cyclic symmetries) and all s_i's distinct, up to an overall normalization factor z. We thus get the equations

$$\begin{cases} A: \prod_\ell (1+w_\ell) = z\big(\prod_\ell (1+\tilde{w}_\ell) + (q-1)\big)\,; \\ B_i: \quad (1+w_i) = z\big((1+\tilde{w}_j)(1+\tilde{w}_k) + (1+\tilde{w}_i) + (q-2)\big)\,; \\ C: \quad\quad 1 = z\big(\sum_\ell (1+\tilde{w}_\ell) + (q-3)\big)\,; \end{cases} \tag{4.18}$$

that is, after expansion,

$$\begin{cases} A: \prod_\ell (1+w_\ell) = z\big(q + \sum_\ell \tilde{w}_\ell + \sum_{\ell\neq h} \tilde{w}_\ell \tilde{w}_h + \tilde{w}_i\tilde{w}_j\tilde{w}_k\big)\,; \\ B_i: \quad (1+w_i) = z\big(q + \sum_\ell \tilde{w}_\ell + \tilde{w}_j\tilde{w}_k\big)\,; \\ C: \quad 1 = z\big(q + \sum_\ell \tilde{w}_\ell\big)\,; \end{cases} \tag{4.19}$$

Ratios of eqs. B_i to C give that the combinations $w_i^{-1}\tilde{w}_j\tilde{w}_k$ are a symmetric function of indices (i,j,k) (namely, $q + \sum_\ell \tilde{w}_\ell = 1/z$), so that we can put $\tilde{w}_i = a/w_i$. Similarly, the combination of equations $A - \sum_i B_i + 2C$ gives the symmetric combination

$$\sum_{\substack{\ell,h \\ \ell\neq h}} w_\ell w_h + w_i w_j w_k = \frac{\tilde{w}_i\tilde{w}_j\tilde{w}_k}{q + \sum_\ell \tilde{w}_\ell} \tag{4.20}$$

So, we changed from the set (A, B_i, C) to the set $(A - \sum_i B_i + 2C, (B_i - C)/C, C)$, then dropped C by solving w.r.t. z, and used the definition of a in order to make the three eqs. in B_i equivalent, which gives

$$\begin{cases} a\big(q + \sum_\ell \tilde{w}_\ell\big) = \prod_\ell \tilde{w}_\ell \\ a^2\big(a + \sum_\ell \tilde{w}_\ell\big)\big(q + \sum_\ell \tilde{w}_\ell\big) = \big(\prod_\ell \tilde{w}_\ell\big)^2 \end{cases} \tag{4.21}$$

This forces $a = q$, and leaves with a single equation

$$q\Big(q + \sum_\ell \tilde{w}_\ell\Big) = \prod_\ell \tilde{w}_\ell \tag{4.22}$$

which, at fixed q, leads to a relation among the three parameters, e.g., in scaled variables $\tilde{w} = \sqrt{q}\tilde{w}'$,

$$\tilde{w}'_k = \frac{\sqrt{q} + \tilde{w}'_i + \tilde{w}'_j}{\tilde{w}'_i\tilde{w}'_j - 1}\,. \tag{4.23}$$

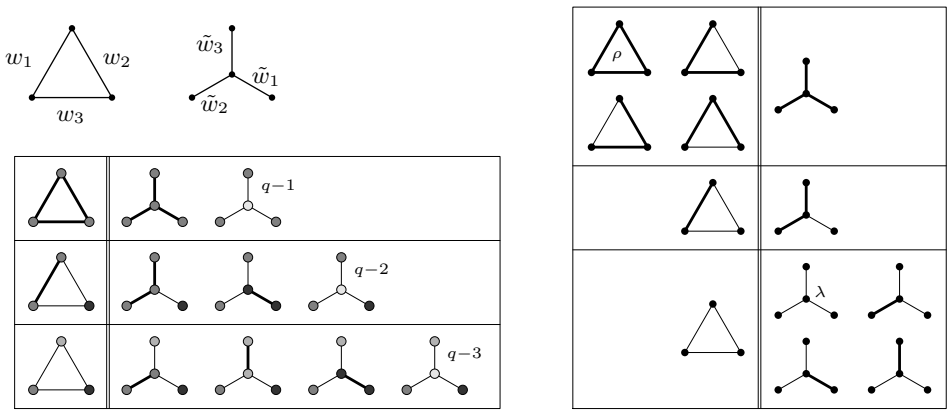
The case of Ising is special: we do not have to satisfy the equation C for the case of s_i all distinct, just because they are never all distinct! We get the smaller system (with a redefining of $z' = z/\prod_\ell(1+w_\ell)$)

$$\begin{cases} A: \quad 1/z' = \big(q + \sum_\ell \tilde{w}_\ell + \sum_{\ell\neq h} \tilde{w}_\ell\tilde{w}_h + \tilde{w}_i\tilde{w}_j\tilde{w}_k\big)\,; \\ B_iB_jB_k^{-1}: (1+w_k)^{-2} = z' \dfrac{\big(q+\sum_\ell \tilde{w}_\ell + \tilde{w}_i\tilde{w}_j\big)}{\big(q+\sum_\ell \tilde{w}_\ell + \tilde{w}_i\tilde{w}_k\big)\big(q+\sum_\ell \tilde{w}_\ell + \tilde{w}_j\tilde{w}_k\big)}\,; \end{cases} \tag{4.24}$$

so that z', and then the w_i's, are determined explicitly for any value of the $\tilde{w}_i$'s. A similar procedure exists for determining z' and $\tilde{w}_i$ from the w_i.

The "combinatorial" argument which excluded equation C can not be extended in analytical continuation (say, guessing that it is true for all $q \leq 2$ or other simple-minded ideas of this sort). A way of seeing this is through the counting of the "multiplicity" with whom all equations appear. The all-equal case (eq. A) has q occurrences for the q possible colours, the two-equal cases (eqs. B_i) have $q(q-1)$ occurrences, while the all-different case (eq. C) takes a factor $q(q-1)(q-2)$ – indeed they sum up to q^3, as they should. These factors are all polynomial and can be analytically extended, but, for example, the last one vanishes at $q = 2$ (so that 0 constraints are always satisfied for all choices of parameters), while for $q < 2$ just a non-zero overall factor changes sign. Something extra happens at $q = 1$, of course, where 4 of the 5 equations vanish, but we already know that, at the aim of evaluating the partition function, the percolation case $q = 1$ is trivial. A last "root" of the polynomial degeneracies in q is the case $q \to 0$. Here all the equations become

Table 4.1: Diagrams corresponding to the 5 equations in the Y–Δ relation, in Potts (colour) formulation, on the left, and in Random Cluster formulation, on the right.



singular, and one should find a proper regularization in order to perform the counting of the parameters.

A more transparent way of doing this is working in the Random Cluster formulation, where analytic continuation in q is well controlled. Now the variables are the occupations on the edges, and the five equations are w.r.t. the five possible connectivity patterns for sites $1, 2, 3$ in the 3-edge triangle and star subgraphs. The choice of (ρ, λ) coming out of Fortuin-Kasteleyn without manipulations with Euler Formula is $\rho = 1$ and $\lambda = q$, which would give directly our "good" combinations of before (not accidentally, as they were given by an inclusion-exclusion reasoning, as well as the core of the FK procedure). While, preserving the parameter ρ one would get

$$\begin{cases} A - \sum_i B_i + 2C : (w_i w_j + w_i w_k + w_j w_k) + \rho w_i w_j w_k = z \prod_\ell \tilde{w}_\ell \,; \\ B_i - C : \qquad\qquad w_i = z\, \tilde{w}_j \tilde{w}_k \,; \\ C : \qquad\qquad 1 = z\big(q/\rho + \sum_\ell \tilde{w}_\ell\big) \,. \end{cases} \tag{4.25}$$

Solving these equations at $\rho = 1$ would just produce again (4.23), while implementing a $q \to 0$ limit in such a way that both ρ and $\lambda = q/\rho$ vanish (thus producing the parameters for spanning trees) would give the simplified system

$$\begin{cases} A' : w_i w_j + w_i w_k + w_j w_k = z \prod_\ell \tilde{w}_\ell \,; \\ B'_i : \qquad\qquad w_i = z\, \tilde{w}_j \tilde{w}_k \,; \\ C' : \qquad\qquad 1 = z\big(\sum_\ell \tilde{w}_\ell\big) \,; \end{cases} \tag{4.26}$$

in which one equation is redundant, as can be checked from

$$\big(A'C' - (B'_1 B'_2 + B'_1 B'_3 + B'_2 B'_3)\big) : \qquad 0 = 0 \,. \tag{4.27}$$

Similarly to what is done above for the case of Ising, we can drop an equation (say, A'), and solve directly, first z through C', then the w_i's through the B'_i, for given arbitrary values of the $\tilde{w}_i$'s. Similarly it can be seen that none of the two limits $q \to 0$ with λ or ρ finite (resp. spanning forests and maximally connected subgraphs) can lead to a non-trivial relation, just because the combination (4.27) above would lead, in the two cases,

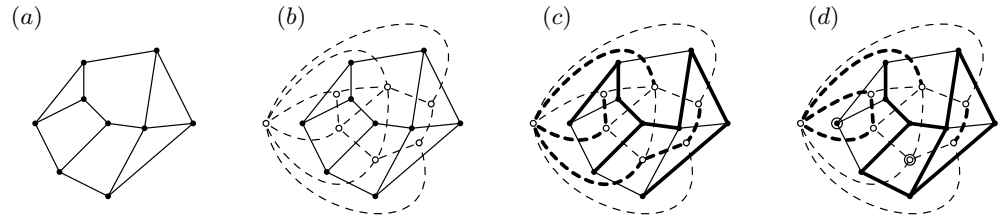


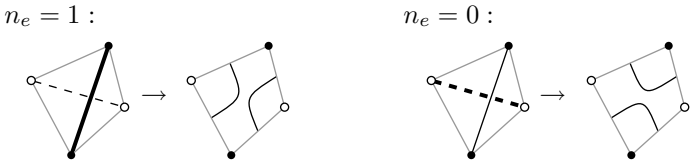
Fig. 4.3. (*a*) a planar graph Λ; (*b*) represented together with its dual $\tilde{\Lambda}$, the edges of the latter being dashed; (*c*) a quadruple $(\Lambda, \tilde{\Lambda}, G, \tilde{G})$ in which both G and $\tilde{G}$ are spanning trees on the respective graphs ($K(G) = K(\tilde{G}) = 1$, $L(G) = L(\tilde{G}) = 0$); (*d*) a quadruple $(\Lambda, \tilde{\Lambda}, G, \tilde{G})$ in which $K(G) = 2$ and $K(\tilde{G}) = 3$ (and then, $L(G) = 3 - 1 = 2$ and $L(\tilde{G}) = 2 - 1 = 1$).

to $\prod_\ell w_\ell = 0$ and $\prod_\ell \tilde{w}_\ell = 0$ respectively (i.e. the obvious limit solutions in which one side of the triangle disappears, and one leg of the star becomes infinitely strong, or in which two sides of the triangle and one leg of the star disappear, where the star and the triangle are indeed the same subgraph).

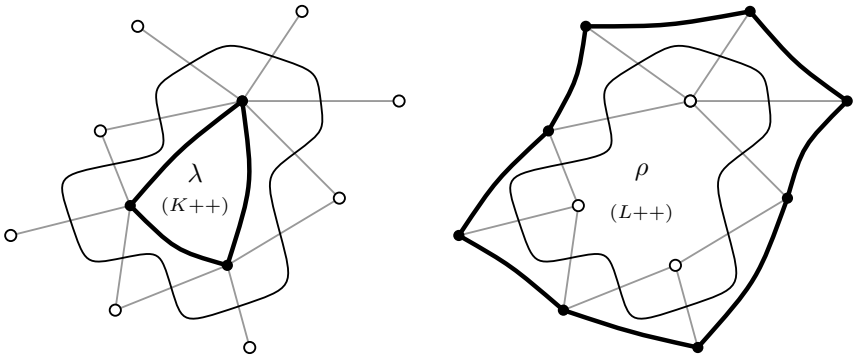
We will experience again this feature when solving the Y–Δ equation in Temperley-Lieb framework, in the next section.

4.5 Temperley-Lieb Algebra

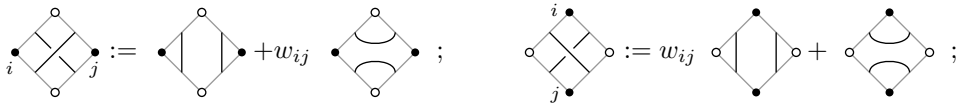
We have already seen how in the planar case one can adopt a graphical representation involving the quadruple $(\Lambda, \tilde{\Lambda}, G, \tilde{G})$, with pairs of dual edges $(e, \tilde{e})$ identifying tetragonal regions. At fixed Λ, a graph G is also identified by the following graphicl construction. In each tetragon, draw two arcs joining the midpoints of the four sides, such that they do not to cross the edge e, if e is occupied in G, or do not to cross $\tilde{e}$, if it is occupied in $\tilde{G}$.



These arc configurations identify globally some closed loops. If we consider them as oriented counter-clockwise, then such a loop either has all black left-neighbours and white right-neighbours or vice-versa. In the first case it surrounds in the tightest way a component of G, while in the second case it follows, in the tightest way from the inside, one of the "canonical" cyclomatic generators described above. So, in the two cases it contributes respectively to increase, by one unit, the number K of components or L of independent loops. There is no telling, the duality exchanges the two roles.



Following Temperley and Lieb [68], we want to describe in an algebraic way the fact that the deformation of loops which do not change the "topology" are not relevant to the weight $\lambda^K \rho^L$. Then, we will deform the lattice in such a way that a special direction is given, a (finite) set of parallel lines is introduced, and loops are constituted of segments on these lines, connected by "U-turns" among neighbouring lines (facing in pairs). Furthermore, all of this could be better discretized on a rectangular portion of the lattice $\mathbb{Z}^2$. One can always achieve such a scope for any finite planar lattice (in case by identifying single vertices with subtrees on $\mathbb{Z}^2$, i.e. forcing some occupation number of edges to be 1 or 0). For example, given the shortcuts[♯]



(4.28)

for the graph in figure 4.3 one could get what is depicted in figure 4.4.

We call *width* and *height* of the $\mathbb{Z}^2$ embedding of Λ the number of vertices in the horizontal and vertical directions. An interesting question is, for a given planar Λ, which is the value of the minimal width among possible embeddings, and if a constructive algorithm exists for building an embedding whose width is reasonably bounded by the parameters $|V(\Lambda)|$ and $|E(\Lambda)|$. We will not address this question here. We just state that, for "reasonable" graphs, it can be expected that the width is around $|V(\Lambda)|^{1/2}$, and that this is the case, by trivial constructions, for all planar regular lattices (e.g. finite or cylindrical portions of the square, triangular, hexagonal lattices and so on).

We will be now more clear on how the graphical construction above should be interpreted. Call n the width of the embedding. Then, the loop configurations onto the square lattice will "live" on $2n$ lines, that we label as $1, \ldots, 2n$ from left to right (dotted in the figure). Remark that direct (black) vertices are between lines $2i-1$ and $2i$, while dual (white) vertices are between lines $2i$ and $2i+1$.

We introduce a vector space of size C_n (the n-th Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$). Two equivalent representations of the basis vectors are through the set $NC(n)$ of *non-crossing partitions* of n ordered objects, and $LP(n)$ of *Link Patterns* of n non-crossing arcs

[♯]Remark that we have chosen, in a tetragon with direct sites in "North" and "South", to draw the SE-NW line above the SW-NE one. At this point this is just a graphical convention, and the reader should not need an explanation. Actually, a reason for this choice exists: the resulting diagram, interpreted as a knot, has a Kauffmann bracket which reproduces the Random Cluster partition function [11].

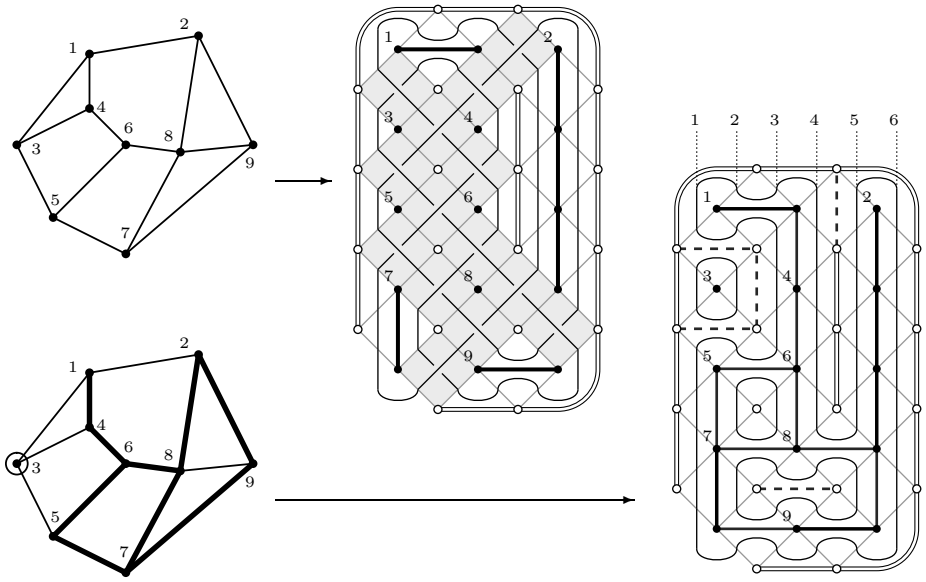
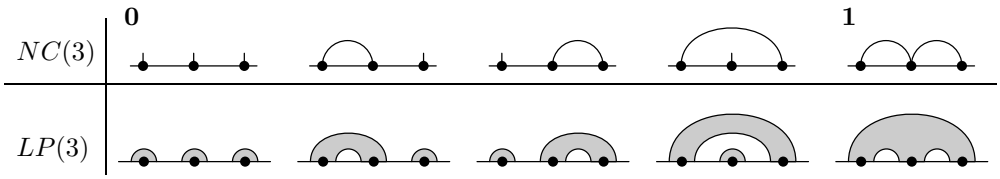


Fig. 4.4. Example of embedding in $\mathbb{Z}^2$ for the graph depicted in figure 4.3.

among $2n$ terminations ordered on a line. A precise definition, the bijection among the two classes, and other interesting properties (among which a graded lattice structure, and a relevant "duality") are discussed in Section 10.5. We do not discuss here the important representation as the global spin-0 eigenfunctions of n spin-1/2 particles.

At this point, as an example we describe this basis in the case $n = 3$ (allowing us to play with the example in figure 4.4). The **0** and **1** of the lattice are respectively the partitions with all atomic sets (i.e. the two link patterns with no nesting of arcs), and the one with a single large set. Even from the $n = 3$ drawing below, the most relevant facts should be clear from the drawings, however we will be more precise in the following.

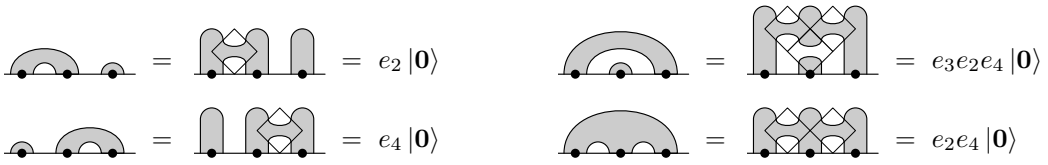


The pairs of facing U-turns can be seen as graphical operators acting on this basis. Their actions should be "bracketed" by the partition corresponding to all atomic sets, i.e. $\langle \mathbf{0}|\cdot|\mathbf{0}\rangle$, where $\langle \mathbf{A}|\mathbf{B}\rangle$ is normalized to $\lambda^K \rho^L$, K and L being the number of components and independent loops in the diagram on the plane composed by **B** and the image of **A** reflected by the horizontal line (in terms of non-crossing partitions, or the number of loops encircling a gray or a white region, in terms of link patterns shaded as in the drawing). Actually, all that we will need is that $\langle \mathbf{0}|\mathbf{B}\rangle$ is normalized to λ^K, with K being the number of components in **B**, i.e. n minus the height of **B** in the lattice.

As long as we draw vertical lines, no loops can be closed, so we can safely identify bundles of parallel vertical lines with an "identity" operator. Facing U-turns can act on $2n-1$ possible pairs of consecutive vertical lines. We call these operators $e_1, \ldots, e_{2n-1}$, where e_i acts on lines i and $i+1$. For clearness, the two facing U-turns of an operator

will often be drawn surrounded by a rhombus so that the endpoints of the arcs are on the midpoints of the sides (they are the "straightening" of the tetragons in the $(\Lambda, \tilde{\Lambda})$ construction). Clearly, operators e_i do not all commute, and the sequence is important. It is determined by the diagram, with, in our convention, operators on top acting on the right bracket first. The graphical choice of rhombi is such that operators that can be drawn at the same height with no obstruction do commute, so that there is no ambiguity in this construction.

All the elements of the basis can be represented as monomials in TL algebra, acting on $|\mathbf{0}\rangle$, first with some even element:



We can describe the action of the e's on the basis in a more complete way, more clearly if we represent a basis vector with the corresponding non-crossing partition $|\mathcal{C}\rangle$, with $\mathcal{C} = (C_1, \ldots, C_k) \in NC(n)$. Say that, if i is contained in the non-atomic component C_α of $\mathcal{C}$, then $\mathcal{C}_i = (C_1, \ldots, C_\alpha \smallsetminus i, \{i\}, \ldots, C_k)$, while if i and j are contained in distinct components C_α and C_β of $\mathcal{C}$, then $\mathcal{C}_{i,j} = (C_1, \ldots, C_\alpha \cup C_\beta, \ldots, C_k)$. Then we have

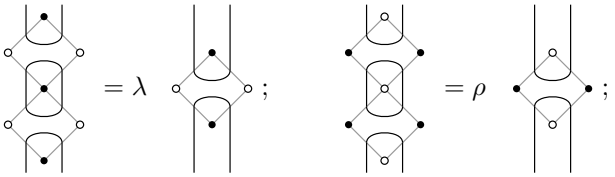
$$e_{2i}|\mathcal{C}\rangle = \begin{cases} \lambda\,|\mathcal{C}\rangle & \text{if } i \text{ is an atomic component} \\ |\mathcal{C}_i\rangle & \text{if } i \text{ is in a non-atomic component} \end{cases} \tag{4.29a}$$

$$e_{2i+1}|\mathcal{C}\rangle = \begin{cases} \rho\,|\mathcal{C}\rangle & \text{if } i \text{ and } i+1 \text{ are in the same component} \\ |\mathcal{C}_{i,i+1}\rangle & \text{if } i \text{ and } i+1 \text{ are in distinct components} \end{cases} \tag{4.29b}$$

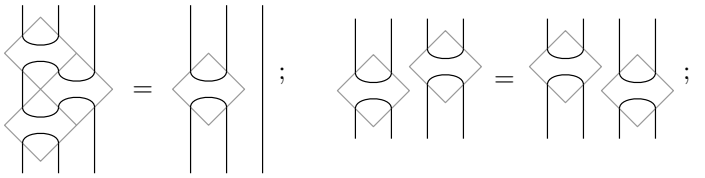
as is clear from a graphical representation in the fashion of the ones above. As a result, the action of any finite string of operators e, bracketed between any pair of basis elements, gives a quantity equal to $\lambda^K \rho^L$ for the corresponding diagram, through iterate application of the rules above. This proves that these rules alone, together with the representation of a graph as in figure 4.4, and with the definition (4.29), suffices to restate the partition function of the random cluster model on the given graph Λ. For example, for the graph in figure 4.4 we would have

$$\begin{aligned} Z_\Lambda(\boldsymbol{w}; \lambda, \rho) = \langle \mathbf{0}| &(w_{79} + e_2)e_4 \cdot (1 + w_{89}e_3)(1 + w_{29}e_5) \cdot (w_{78} + e_2)(w_{28} + e_4) \\ &\cdot (1 + w_{57}e_1)(1 + w_{68}e_3) \cdot (w_{56} + e_2) \cdot (1 + w_{35}e_1)(1 + w_{46}e_3) \\ &\cdot (w_{34} + e_2) \cdot (1 + w_{13}e_1)(1 + w_{14}e_3) \cdot e_2(w_{12} + e_4)|\mathbf{0}\rangle\,. \end{aligned} \tag{4.30}$$

What is more relevant is the fact that a few relations hold for the e's algebraically, in any position along the string. They describe the fact that continuous deformations of the loops do not change the weight, and isolated loops of one of the two kinds give a factor λ or ρ regardless from their position and shape on the lattice:



$$e_{2i}^2 = \lambda\, e_{2i}\,; \qquad e_{2i+1}^2 = \rho\, e_{2i+1}\,; \tag{4.31a}$$



$$e_i e_{i\pm1} e_i = e_i; \qquad [e_i, e_j] = 0 \text{ for } |i-j| \geq 2. \tag{4.31b}$$

Remarkably, these relations alone fully encode this concept of weight invariance, in the sense that they suffice to disentangle any finite string of operators e, arbitrarily long and even not bracketed between elements of the basis, into a factor $\lambda^K \rho^L$, times a monomial in the e's in a finite set of "irreducible" monomials. Pictorially, this is due to the fact that a whatever complicated diagram on a strip with n points, at the aims of our weight, can be summarized into the numbers K and L for loops not attached to the top- and bottom-borders, and the induced non-crossing partition of the $2n$ sites on the union of the two borders, where the latter can be reproduced by a combination of e's of length at most $\mathcal{O}(n^2)$.

The relations (4.31) above, in the case $\lambda = \rho$, define a *Temperley-Lieb Algebra* TL$(2n-1)$, i.e. with $2n-1$ generators on an "open chain". Similarly, one can think to a periodic TL$(2n)$ algebra, i.e. one in which we also have the generator e_{2n}, and notations $i \pm 1$ and $|i-j| = 1$ in the definition of TL must be intended modulo $2n$. Just, the generator e_{2n} does not actually appear explicitly in the operator corresponding to the partition function of the graph. The genuine planarity (instead of the common case of the cylindric geometry) avoids to consider the annoying extra relations causing also loops with non-trivial winding (like the one produced by $e_2 e_4 \cdots e_{2n} e_1 e_3 \cdots e_{2n-1}$) to get the proper weight.

However, the traditional (open or periodic) TL(n) algebra does not include the "even/odd" case in which $\rho = 0$ but $\lambda \neq 0$, corresponding, in the Random Cluster polynomial, to the counting of spanning forests. Also the corresponding degenerate "even/odd" Hecke algebra seems to be essentially uninvestigated.

We remark that, if we want to "raise" every relation with λ in the algebra to contain an independent parameter, i.e. some $e_i^2 = \lambda_i e_i$, we would face the consistency requirement

$$\begin{aligned} e_{i-1}\, e_{i+1}\, e_i\, e_{i-1}\, e_{i+1} &= e_{i+1}\, e_{i-1}\, e_i\, e_{i-1}\, e_{i+1} = e_{i+1}\, e_{i-1}\, e_{i+1} = \lambda_{i+1}\, e_{i-1}\, e_{i+1} \\ &= e_{i-1}\, e_{i+1}\, e_i\, e_{i+1}\, e_{i-1} = e_{i-1}\, e_{i+1}\, e_{i-1} = \lambda_{i-1}\, e_{i-1}\, e_{i+1} \end{aligned} \tag{4.32}$$

which forces $\lambda_{i+1} = \lambda_{i-1}$ for each i. Then, from the invariance discussed in the introduction, we realize that $\rho = 0$ and $\lambda \neq 0$ of the Random Cluster model, together with its dual, are also the *only* cases left aside by the traditional algebra.

4.6 Observables in colour and random-cluster representations

The physical interest on the Potts and Random Cluster models is not restricted to the evaluation of the partition function. It is well known that, for a sufficiently generalized model (with "current" terms), the partition function alone encodes a whole spectrum of "observables" (i.e. of evaluation of statistical averages e.g. of combinatorial quantities in the original system), by the application of some appropriate operator. For example, not only the free energy corresponds to the logarithm of the partition function (up to prefactors), but also a two-point correlation function will correspond to a derivative of the free energy in a generalized model where the Hamiltonian has two source terms, evaluated at zero sources. In the case of a field ϕ evalued on some (mathematical) field $\mathbb{K}$, with Hamiltonian $\mathcal{H}(\phi)$, the free energy is

$$F = \ln \sum_{\phi} \exp\left(-\mathcal{H}(\phi)\right) \tag{4.33}$$

and the two-point function $\langle \phi_x \phi_y \rangle$, defined as

$$\langle \phi_x \phi_y \rangle = \frac{\sum_\phi \phi_x \phi_y \exp\left(-\mathcal{H}(\phi)\right)}{\sum_\phi \exp\left(-\mathcal{H}(\phi)\right)} \tag{4.34}$$

is obtained through the generalized Hamiltonian $\mathcal{H}'(\phi, J) = \mathcal{H}(\phi) - \sum_x \phi_x J_x$, by realizing that

$$\frac{\sum_\phi \phi_x \phi_y \exp\left(-\mathcal{H}(\phi)\right)}{\sum_\phi \exp\left(-\mathcal{H}(\phi)\right)} = \left. \frac{\frac{\partial^2}{\partial J_x \partial J_y} \sum_\phi \exp\left(-\mathcal{H}(\phi) + \sum_z \phi_z J_z\right)}{\sum_\phi \exp\left(-\mathcal{H}(\phi) + \sum_z \phi_z J_z\right)} \right|_{J=0}, \tag{4.35}$$

that is

$$\langle \phi_x \phi_y \rangle = \left. \frac{\partial^2}{\partial J_x \partial J_y} F(J) \right|_{J=0}, \tag{4.36}$$

and similarly for higher-order correlation functions. Then, if certain inequalities on the growth of one-site momenta are verified, one is guaranteed on the uniqueness of the reconstruction of the (Gibbs symmetric) measure from these expectations alone, which thus constitute a basis for all the possible marginals in the system.

However, this "conceptual" encoding of the observables in the free energy of a "sufficiently generalized" model is not a practical tool, even on the most abstract side. Indeed, we could have positive results on the polynomial algorithmic complexity for evaluating F in a given model, which do not extend to the evaluation of the generalized $F(J)$, just because the latter is a too complicated object, but these positive results could extend to the evaluation of some "simple" observables, such as the two-point function (this is the case, for example, of some correlations for dimer models on planar graphs, or of boundary two-point functions in Ising Model on planar graphs).

So, a direct description of the generating function corresponding to unnormalized k-point functions, such as $Z \cdot \langle \phi_x \phi_y \rangle$, is of interest.

Furthermore, in order to understand at sight the features of criticality of the system, when defined on large homogeneous graphs, one would like to highlight the large-distance behaviour of k-point functions by mean of subtractions of the appropriate set of h-point functions (with $h \leq k$), called *connected* k-point function.

Given a measure μ_α, a property that characterizes it to be a pure phase is that, for any set of local observables $\phi^{(\alpha)}(x)$, and any set of $k + h$ points $x_1, \ldots, x_k, y_1, \ldots, y_h$ (partitioned in the two subsets of cardinalities h and k, both positive), we have the so-called *Cluster Property* for the averages performed with this measure (cfr. Section 2.3):

$$\lim_{\substack{r=\min(d(x_i,y_j)) \\ \to\infty}} \left\langle \phi^{(1)}_{x_1} \cdots \phi^{(k)}_{x_k} \hat{\phi}^{(1)}_{y_1} \cdots \hat{\phi}^{(h)}_{y_h} \right\rangle_\alpha - \left\langle \phi^{(1)}_{x_1} \cdots \phi^{(k)}_{x_k} \right\rangle_\alpha \left\langle \hat{\phi}^{(1)}_{y_1} \cdots \hat{\phi}^{(h)}_{y_h} \right\rangle_\alpha = 0\,. \tag{4.37}$$

Products of expectation values are however "new" objects in the algebra of observables: we have a "linear" space of expectation values of polynomials in the elementary fields, that is, operators "within a single bracket", and a "polynomial" algebra, where not only each bracket contains a polynomial in the algebra of observables, but also products of brackets are considered. As evident from (4.37), the Cluster Property lives on this larger space.

It is interesting to study which observables satisfy the Cluster Property without the non-linear term, in the high-temperature paramagnetic phase. That is, for which operators

the subtraction terms are zero, possibly as a consequence of the symmetry property of the order parameter in this phase. We can call *linearized cluster property* operators (LCP operators) such a subspace of operators in the linear space described above.

For example, in an homogeneous (e.g. vertex-transitive) system, for the 2-point function on any pair of local operators $\phi^{(1)}(x)$, $\phi^{(2)}(x)$, the obvious subtraction leading to a LCP operator is

$$\Big\langle \phi^{(1)}(x_1)\phi^{(2)}(x_2)\Big\rangle^{(\mathrm{conn.})} = \Big\langle \big(\phi^{(1)}(x_1) - \langle\phi^{(1)}\rangle_{\mathrm{symm}}\big)\big(\phi^{(2)}(x_2) - \langle\phi^{(2)}\rangle_{\mathrm{symm}}\big)\Big\rangle \,, \quad (4.38)$$

which satisfies manifestly the Cluster Property in its "linear form" (of course, in this case the use of $\langle\phi^{(1)}\rangle$ as a parameter, with average in the symmetric measure, is tautological w.r.t. our definition – we will encounter less trivial examples later on).

Besides subtractions, a normalization overall can be chosen (this is not compulsory, but generally adviced). If the system is such that the ground state can be easily identified as a fully ordered state (ferromagnetic ordering), at least in some regime of couplings, and the k-point function takes a definite non-zero value independently of positions in this limit, setting this limit value to 1 could be a reasonable normalization.

In our case of Potts Model, a set of observables which properly encode the global permutation symmetry is the one generated by the operators $\delta(s_x, s_y)$ [‡], and $\delta(s_x, a)$ for some colour $a \in [q]$, but the latters only in symmetric combinations, if we are interested in observables preserving the original symmetry (in a sense, the algebra "orthogonal" to the order parameters of Potts ferromagnetic transition).

So a first natural set of observables would be $\langle 1\rangle$, $\langle\delta(s_x, s_y)\rangle$, $\langle\delta(s_x, s_y, s_z)\rangle$, $\langle\delta(s_x, s_y, s_z, s_w)\rangle$, $\langle\delta(s_x, s_y)\delta(s_z, s_w)\rangle$, and so on.

Now we have to perform the appropriate subtractions and normalizations. The finite-temperature long-range extreme limit of a Kronecker delta in a q-state Potts model will always be

$$\lim_{r=\min d(x_i,x_j)\to\infty} \langle\delta(s_{x_1}, \cdots, s_{x_k})\rangle = \frac{1}{q^{k-1}} \,, \quad (4.39)$$

(as, for any of the q possible values for the first variable, one has exactly 1 out of q choices for any of the other ones), while in the $T = 0$ ferromagnetic limit, which is the limit we choose for normalization purposes, all expectations of delta's just give 1.

So the appropriate subtractions and rescalings are (just use $\delta_{xy\ldots}$ as a shortcut for $\delta(s_x, s_y, \ldots)$, and ellipses stand for terms deducible by symmetry)

$$(\delta_{xy})^* := \frac{q\delta_{xy} - 1}{q - 1} \,; \quad (4.40)$$

$$(\delta_{xyz})^* := \frac{q^2\delta_{xyz} - q\big(\delta_{xy} + \delta_{xz} + \delta_{yz}\big) + 2}{(q-1)(q-2)} \,; \quad (4.41)$$

$$(\delta_{xy}\delta_{zw})^* := \frac{q^2\delta_{xy}\delta_{zw} - q\big(\delta_{xy} + \delta_{zw}\big) + 1}{(q-1)^2} \,; \quad (4.42)$$

$$(\delta_{xyzw})^* := \frac{q^3\delta_{xyzw} - q^2\big(\delta_{xy}\delta_{zw} + \cdots\big) - q^2\big(\delta_{xyz} + \cdots\big) + 2q\big(\delta_{xy} + \cdots\big) - 6}{(q-1)(q^2 - 6q + 6)} \,; \quad (4.43)$$

and so on for higher-order correlation functions.

[‡] We generalize Kronecker delta to multi-variable case, such as $\delta(s_{x(1)}, \ldots, s_{x(k)}) := \delta(s_{x(1)}, s_{x(2)})\delta(s_{x(2)}, s_{x(3)}) \cdots \delta(s_{x(k-1)}, s_{x(k)})$.

Actually not all of these combinations are LCP operators, but only the ones consisting of a single k-point delta. For the others, the linearized cluster property (w.r.t. sets $(x_1,\dots,x_k)$ and $(y_1,\dots,y_h)$ as in (4.37)) holds only if a pair of separated variables (x_i, y_j) exists, with x_i and y_j in the same delta function in the observable.

We should explain more in detail how one fixes the coefficients in a $(\delta_{x_1\cdots x_k})^*$. Given S a subset of vertices of cardinality n, and π a partition of S with blocks π_μ and a number of blocks $|\pi|$, define

$$[\delta]_{S,\pi} = \prod_{\substack{\mu \\ \pi_\mu=\{x_1,\dots,x_k\}}} \delta_{x_1\cdots x_k}\,. \tag{4.44}$$

Say that the *degree* of $[\delta]_{S,\pi}$ is given by $|S|-|\pi|$ (it corresponds to the number of 2-point delta functions required to express the constraint).

Then, given the operator $[\delta]_{S,(S)}$ (i.e. with a single block), one expects *a priori* a general formula like

$$(\delta_{x_1\cdots x_n})^* := \frac{1}{P_n(q)}\Big(q^{n-1}\delta_{x_1\cdots x_n} - \sum_{\substack{\pi\in\Pi(n)\\ \pi\neq[n]}} (-q)^{n-|\pi|} a_n(\pi)\,[\delta]_{\{x_1,\dots,x_n\},\pi}\Big)\,, \tag{4.45}$$

where a single-point δ_x is intended as a synonimous of 1.

One has to choose coefficients $a_n(\pi)$ of subtraction by matching to zero the contribution of each partition of the vertices which does not consist of a single block, and then a polynomial at the denominator is fixing to 1 the contribution of the single-block partition. Alternate signs and powers of q are deduced by simple reasonings, but the exact expression for $a_n(\pi)$ requires a more subtle analysis.

For example, for the last equation (4.43) corresponding to the 4-point correlation function, one has to expect *a priori* a general formula like

$$(\delta_{xyzw})^* := \frac{1}{P(q)}\Big(q^3\delta_{xyzw} - aq^2\big(\delta_{xyz}+\cdots\big) - bq^2\big(\delta_{xy}\delta_{zw}+\cdots\big) + cq\big(\delta_{xy}+\cdots\big) - d\Big)\,. \tag{4.46}$$

We have to check all symmetry classes of 4-point partitions, which are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. We get for each of them:

$$\begin{aligned}
4: &\quad q^3 - 4aq^2 - 3bq^2 + 6cq - d\,;\\
3+1: &\quad q^2 - a(q^2+3q) - 3bq + c(3q+3) - d\,;\\
2+2: &\quad q^2 - 4aq - b(q^2+2) + c(2q+4) - d\,;\\
2+1+1: &\quad q - a(2q+2) - b(q+2) + c(q+5) - d\,;\\
1+1+1+1: &\quad 1 - 4a - 3b + 6c - d\,.
\end{aligned}$$

All the rows except the first one must be matched to 0 as polynomials, so we get a linear system, which has a solution given by $a=b=1$, $c=2$ and $d=6$. Then the polynomial $P(q)$ just corresponds to the first row, with the solution for the coefficients plugged in, so in this case $P(q) = q^3 - 7q^2 + 12q - 6$.

An analysis of the first cases seems to suggest that $a_n(\pi) = (|\pi|-1)!$, which is indeed true although not so trivial to prove. As a consequence, the coefficients of the polynomial $P(q)$ are the ones in Sloane's classification `A028246` (with alternate signs), or, more easily, the coefficients of $P(q)/(q-1)$ ($q=1$ is always a root of $P(q)$) are the Stirling numbers of second kind multiplied by our appropriate factorial $(|\pi|-1)!$, (still with alternated signs), classified in Sloane's `A019538`, that is, the number of partitions with a given number of blocks. So, the general formula for the subtracted operators is

$$(\delta_{x_1\cdots x_n})^* = \frac{1}{P_n(q)} \sum_{\pi\in\Pi(n)} (-1)^{|\pi|-1}(|\pi|-1)! q^{n-|\pi|}\, [\delta]_{\{x_i\},\pi}\,; \tag{4.47}$$

$$P_n(q) = q^{n-1} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{(-q)^{-(k-1)}}{(k-1)!}\,; \tag{4.48}$$

($[\delta]_{\{x_i\},\pi}$ is defined as in equation (4.44)). On the other side, a natural set of observables in the (ferromagnetic) Fortuin-Kasteleyn expansion are $\Gamma_{x_1\cdots x_k}$, which are the indicator function of the event that the k points are in the same connected component of the subgraph $H \subseteq G$. In this case, the term "connected" in connected components and in connected correlation functions does not constitute a verbal collision: all the single-monomial $\Gamma_{x_1\cdots x_k}$ are connected operators, in the sense that they are LCP operators, while all products of two or more delta factors, corresponding to the possibility of h-uples being in different components, are not LCP (i.e. not connected) also as operators.

Remarkably enough, the operators $\Gamma_{x_1\cdots x_k}$ in F-K formalism *do coincide* with the operators $(\delta_{x_1\cdots x_k})^*$ in the Potts colouring formalism. In a whatever other expansion for the partition function different from F-K or the original formulation, it will be our goal to devise also an expansion for these operators, as they are the quantities which encode the long-range behaviour of homogeneous systems at least in the ferromagnetic regime.

Another fact is that, for any $k \geq 2$, the combination

$$\mathcal{Q}_k(x_1,\dots,x_k) = \sum_{\alpha\in[q]} \prod_{i=1}^{k} \big(q\delta(s_{x_i},\alpha) - 1\big) \tag{4.49}$$

leads to a linear combination of subtracted k-point functions as above, through the general formula

$$\begin{aligned}\mathcal{Q}_k(x_1,\dots,x_k) = q^{k-1}\delta_{x_1\cdots x_k} - q^{k-2}\big(\delta_{x_1\cdots x_{k-1}} + \cdots\big) + q^{k-3}\big(\delta_{x_1\cdots x_{k-2}} + \cdots\big) \\ \cdots + (-1)^k q\big(\delta_{x_1x_2} + \cdots\big) - (-1)^k(k-1)\,.\end{aligned} \tag{4.50}$$

Indeed, this operator is just proportional to the natural connected one for $k = 2$ and 3, while for $k = 4$ it consists of a combination of $(\delta_{xyzw})^*$ and the three of $(\delta_{xy}\delta_{zw})^*$ (for $(xy|zw)$, $(xz|yw)$ and $(xw|yz)$). Similar combinations appear for larger values of k.

It should however be noted that these operators are not LCP for $k \geq 4$, for example the 4-point expectation value is subtracted with the symmetric combination $\langle\mathcal{Q}_2(x_1,x_2)\rangle\,\langle\mathcal{Q}_2(x_3,x_4)\rangle + \cdots$.

4.7 Positive and negative associativity

Consider the (ferromagnetic) F-K expansion as in (4.4), or better the Random Cluster expansion of (4.7). While in the Potts colouring language a natural set of observables which capture the long-range behaviour is the one generated by the $\delta(s_x, s_y)$ (better, their subtracted combinations as discussed in the previous section), in the F-K cluster language the natural observable is generated by two-point cluster connectivity, $\Gamma_{x,y}$. This statement has a counterpart also in the natural basis of local observables: the natural "local energy" observable in Potts language is $v_{xy}\delta(s_x, s_y)$ for $(xy) \in E(G)$, which, summed over all the edges, measures the internal energy of the system w.r.t. an arbitrary zero (here at an hypothetic antiferro ground state), the natural analogue of local-energy observable in cluster language is $v_e n_e := \chi(e \in E(H)) \in \{0, 1\}$.

A small possible confusion should be avoided here: although $\langle(\delta_{x,y,\ldots})^*\rangle \equiv \langle \Gamma_{x,y,\ldots}\rangle$, in statistical averages of the system in the two languages, it is *not* true that $\langle\delta(s_x, s_y)\rangle$ and $\langle n_{xy}\rangle$ are simply related, because x and y, even if first neighbours on G, can be connected in H through a long path that does not use (xy).

A surprising and quite relevant fact is that the algebra generated by the $\{n_e\}_{e\in E(G)}$ operators has a property called *positive associativity*, in a ferromagnetic system and in the regime $q \geq 1$, as proven first by Fortuin, Kasteleyn and Ginibrie [69]. It is indeed this deep original observation which motivates and gives a better insight on the otherwise not memorable easy algebraic manipulations involved in the derivation of the F-K expansion.

An event $\mathcal{E}$ is said *increasing* if

$$\chi(\mathcal{E} \text{ happens on } H) \geq \chi(\mathcal{E} \text{ happens on } H')$$

if $H \supseteq H'$ as subgraphs of G. The positive association property states that, for any pair of increasing events $\mathcal{E}$ and $\mathcal{E}'$,

$$\langle\chi(\mathcal{E} \wedge \mathcal{E}')\rangle \geq \langle\chi(\mathcal{E})\rangle\,\langle\chi(\mathcal{E}')\rangle\ . \tag{4.51}$$

Then, edge-occupations n_e provide a natural basis for increasing events (actually also the $\Gamma_{x,y,\ldots}$ are increasing events). So the simplest of the FKG relations states that, for any pair of edges e and f,

$$\langle n_e n_f\rangle \geq \langle n_e\rangle\,\langle n_f\rangle \tag{4.52}$$

for any ferromagnetic set of couplings and any $q \geq 1$. Clearly the inequality becomes an equality at $q = 1$, where all local degrees of freedom decouple. So it is natural to ask whether the property changes into some "negative association" in the remaining part of the probabilistic sector of F-K, i.e. v ferro and $q \in [0, 1)$. This issue is addressed for example in [70]. There it is shown the elementary fact that (4.51) can not be valid *tout-court* with inverted inequality at $q < 1$, just because any increasing event is positively associated to itself, and it is conjectured that a form of $q < 1$ (4.51) could hold if the two events are functions of two *disjoint* subsets of edge-occupation variables. In particular, a viable candidate benchmark is the claim that, in a ferromagnetic system at $q \leq 1$, for any pair of distinct edges e and f,

$$\langle n_e n_f\rangle \leq \langle n_e\rangle\,\langle n_f\rangle\ . \tag{4.53}$$

This fact is indeed known to hold for spanning trees, as a corollary of a broader result on matroid basis due to Feder and Mihail [71]. We will not show here this proof. The same fact is conjectured to hold for the whole region of ferromagnetic weights and $0 \leq q \leq 1$.

The present understanding of results on negative associativity for the random cluster model at $0 \leq q \leq 1$ is related to a property of polynomials called *strong Rayleigh property*, as elucidated in [72, 73, 74, 75].

Here we overtake a much easier task, to prove that, for any fixed weighted graph G, there exists a value $\epsilon > 0$ such that the "negative associativity" relation (4.53) holds for all values $1 - \epsilon \leq q < 1$. This is done by elementary means.

We just fix the notations by introducing the unnormalized random-cluster measure

$$\mu_q(H) = q^{k(H)} \prod_{(ij)\in E(H)} v_{ij} \tag{4.54}$$

and the partition function

$$Z_q(G) = \sum_{H\subseteq G} \mu_q(H)\,. \tag{4.55}$$

Given e and f two distinct edges of G, consider the "Rayleigh" expression

$$R_q(G;e,f) = Z_q(G)Z_q(G\smallsetminus\{e,f\}) - Z_q(G\smallsetminus\{e\})Z_q(G\smallsetminus\{f\}) \tag{4.56}$$

where it is understood that the weights $\{v_e\}_{e\in E(G)}$ are the same in the four graphs, except that for the dropped items.

As we said, as a special case of Fortuin, Kasteleyn and Ginibrie (FKG) inequalities, we have that for any G, any $e\neq f\in E(G)$, and any set of real non-negative weights, $R_q(G;e,f)\geq 0$ if $q\geq 1$, i.e. occupancies of edges e and f are positively (better, non-negatively) correlated. In particular, for the percolation case $q=1$ one trivially gets $R_q(G;e,f)=0$.

Conversely, it is not hard to see that, if the weights are strictly positive, e and f are in the same 2-connected component and $q>1$, then $R_q(G;e,f)>0$. Here we prove the following

Theorem 4.1. *Given a weighted graph G, with strictly positive weights, and two edges e and f in the same 2-connected component, there exists a value $\epsilon>0$ such that $R_q(G;e,f)<0$ for $1-\epsilon\leq q<1$.*

First of all, notice that, as Z is a polynomial in q, also R is. Then, as it vanishes at $q=1$, it must have at least one such root as a polynomial, so that also $R_q(G;e,f)/(q-1)$ is a polynomial. This allows to put the positive FKG result on positive associativity at $q>1$ and the conjecture on negative associativity at $0\leq q<1$ under the same roof:

Conjecture 4.2. Given a weighted graph G, with strictly positive weights, and two edges e and f in the same 2-connected component, then

$$\frac{R_q(G;e,f)}{q-1}>0 \tag{4.57}$$

for any $q\geq 0$.

This conjecture also contains a non-trivial statement on the de l'Hôpital limit for $q\to 1$, and what we will actually prove is the conjecture above, but for $q\geq 1-\epsilon$ for some positive value ϵ. We will consider a "perturbation" in $q-1$, calling $q=1+\epsilon$. We can always state

$$Z_q(G) = Z_{q'}(G)\left\langle (q/q')^{k(H)}\right\rangle_{q'} \tag{4.58}$$

that is, we evaluate $Z_q(G)$ as the unnormalized expectation value of $(q/q')^{k(H)}$, in the unnormalized measure $\mu_{q'}(H)$. This allows us to rewrite

$$\begin{aligned}\frac{R_q(G;e,f)}{q-1} &= \frac{R_{1+\epsilon}(G;e,f)}{\epsilon}\\ &= \frac{1}{\epsilon}\Big(Z_1(G)Z_1(G\smallsetminus\{e,f\})\left\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\right\rangle_{1;G}\left\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\right\rangle_{1;G\smallsetminus\{e,f\}}\\ &- Z_1(G\smallsetminus\{e\})Z_1(G\smallsetminus\{f\})\left\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\right\rangle_{1;G\smallsetminus\{e\}}\left\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\right\rangle_{1;G\smallsetminus\{f\}}\Big)\end{aligned} \tag{4.59}$$

Remark however that

$$Z_1(G)Z_1(G\smallsetminus\{e,f\}) = Z_1(G\smallsetminus\{e\})Z_1(G\smallsetminus\{f\}) = \frac{\prod_{g\in E(G)}(1+v_g)^2}{(1+v_e)(1+v_f)} \tag{4.60}$$

so that, up to a positive factor overall,

$$\begin{aligned}\frac{R_{1+\epsilon}(G;e,f)}{\epsilon} &\sim \frac{1}{\epsilon}\Big(\big\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\big\rangle_{1;G} \big\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\big\rangle_{1;G\smallsetminus\{e,f\}} \\ &\quad - \big\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\big\rangle_{1;G\smallsetminus\{e\}} \big\langle 1+\epsilon k(H)+\mathcal{O}(\epsilon^2)\big\rangle_{1;G\smallsetminus\{f\}} \Big) \\ &= \langle k(H)\rangle_{1;G} + \langle k(H)\rangle_{1;G\smallsetminus\{e,f\}} - \langle k(H)\rangle_{1;G\smallsetminus\{e\}} - \langle k(H)\rangle_{1;G\smallsetminus\{f\}} + \mathcal{O}(\epsilon)\end{aligned} \tag{4.61}$$

Call

$$R^{(0)}(G;e,f) = \langle k(H)\rangle_{1;G} + \langle k(H)\rangle_{1;G\smallsetminus\{e,f\}} - \langle k(H)\rangle_{1;G\smallsetminus\{e\}} - \langle k(H)\rangle_{1;G\smallsetminus\{f\}} \tag{4.62}$$

As a consequence of FKG at $\epsilon = 0^+$, we have $R^{(0)}(G;e,f) \geq 0$. In order to conclude our statement, we need to prove that, if e and f are 2-connected, then $R^{(0)}(G;e,f) > 0$.

Remark that, calling s and t the extrema of e,

$$\langle k(H)\rangle_{1;G} = \langle k(H)\rangle_{1;G\smallsetminus\{e\}} - \frac{v_e}{1+v_e}\langle s \nsim t\rangle_{1;G\smallsetminus\{e\}} \tag{4.63}$$

Then, calling s' and t' the extrema of f,

$$\begin{aligned}\langle s \nsim t\rangle_{1;G\smallsetminus\{e\}} &= \langle s \nsim t\rangle_{1;G\smallsetminus\{e,f\}} - \frac{v_f}{1+v_f} \\ &\quad \Big(\langle s\sim s', t\sim t', s\nsim t\rangle_{1;G\smallsetminus\{e,f\}} + \langle s\sim t', t\sim s', s\nsim t\rangle_{1;G\smallsetminus\{e,f\}}\Big)\end{aligned} \tag{4.64}$$

so that, in conclusion,

$$\begin{aligned}R^{(0)}(G;e,f) &= \frac{v_e}{1+v_e}\frac{v_f}{1+v_f} \\ &\quad \Big(\langle s\sim s', t\sim t', s\nsim t\rangle_{1;G\smallsetminus\{e,f\}} + \langle s\sim t', t\sim s', s\nsim t\rangle_{1;G\smallsetminus\{e,f\}}\Big)\,.\end{aligned} \tag{4.65}$$

As the probabilities all come with positive signs, we recover, as we should, that $R^{(0)}(G;e,f)$ is at sight non-negative. Then, we realize that 2-connectivity implies exactly that at least one of the two summands in parenthesis has at least one non-zero contribution, corresponding to the subgraph consisting of the two disjoint paths resulting from removing e and f from a cycle γ such that $e, f \in \gamma$. This completes the proof.

5

Graph-enumeration problems in the antiferromagnetic Potts Model

In this section we give a new representation of the Potts / Random-cluster / Tutte-polynomial generating function, which is probabilistic in a region of negative q and v's.

We start by reviewing a number of results concerning acyclic orientations of graphs, [76, 77, 78], in particular a theorem by Stanley [79] and two by Greene and Zaslavsky [80]. These results connect the problem of evaluating the (two-variate) Tutte polynomial at special integer points (or its derivative, when the value vanishes for any graph), to counting problems concerning acyclic orientations.

Then, we extend the method to the full multi-variate Tutte polynomial, and will in particular give a local-variable probabilistic description of the Potts Model in various sectors for which both the "colour" representation and the ordinary random cluster representation do not have this property.

5.1 Chromatic polynomial and acyclic orientations

The chromatic polynomial of a graph G, denoted by $P_G(q)$, is the restriction of the partition function $Z(G; v; q)$ defined in the previous sections to the case of all $v_e = -1$. For positive integer values of q, it counts the proper colourings of G, but, as a corollary of the fact that the whole Tutte polynomial is a polynomial, also $P_G(q)$ depends on q polynomially. In particular, as in the limit $q \gg |V(G)|$ almost all random colourings are good, this polynomial must be monic of degree $|V(G)|$. Other special values are that, because of the global symmetry under permutations on $[q]$, any graph G containing a k-clique has that $P_G(q)$ divides $q(q-1)\cdots(q-k+1)$. In particular, the chromatic polynomial of every graph has a root in 0, of every graph not consisting of a single vertex has a root in 1, and of every graph containing a triangle has a root in 2.

The chromatic polynomial has a deletion-contraction relation coming from the specialization of the one of Tutte polynomial, but in this specialization the relation has a simple combinatorial meaning. Consider a graph G and two vertices x and y such that $(xy) \notin E(G)$, and the polynomial at a finite integer q. The proper colouring functions ϕ on G are of two species: either $\phi(x) = \phi(y)$, or $\phi(x) \neq \phi(y)$. Restriction to the first case gives the counting of proper colourings in the graph in which vertices x and y are identified, while the second case gives the countings of the colourings in which also an edge constraint on x and y is added. Calling G' the graph $G \cup (xy)$, we have thus

$$P_G = P_{G' \setminus (xy)} = P_{G'} + P_{G' \bullet (xy)} \tag{5.1}$$

i.e. , solving w.r.t. G', and realizing that $G' \bullet (xy)$ has one vertex less, we get

$$(-1)^{|V(G)|} P_G = (-1)^{|V(G' \smallsetminus (xy))|} P_{G' \smallsetminus (xy)} + (-1)^{|V(G' \bullet (xy))|} P_{G' \bullet (xy)} \,. \tag{5.2}$$

Also the fact that the Tutte polynomial factorizes on 2-connected components has a natural restriction to P_G. Indeed, call G_1 and G_2 two graphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{i\}$. Then we have

$$P_G(q) = \frac{q-1}{q} P_{G_1}(q) P_{G_2}(q) \,. \tag{5.3}$$

Conversely, the problem of evaluating the chromatic polynomial on a 2-connected graph is in general hard, and worst-case NP-complete (actually even the evaluation at any integer $q \geq 3$ is worst-case NP-complete, corresponding to the celebrated *q-colouring problem*), and indeed this claim is a corollary of the full analysis of the complexity of Tutte polynomial on the Welsh plane). However the bad exponential upper bound on the complexity is *not* in the number of vertices $|V|$ or edges $|E|$ in the graph, but only on the cyclomatic number $L = E + K - V$. This happens because, as a corollary of the factorization on 2-components, the chromatic polynomial of a tree T with E edges is just $P_T(q) = q(q-1)^E$, and in L steps of deletion-contraction, always applied to edges which do not disconnect the graph (which are computationally fast to identify), one ends up with a linear combination of chromatic polynomials on trees.

The claims on complexity of the evaluation of the chromatic polynomial must be done more precise. Actually, evaluating $P_G(q = 2)$ is polynomial for trivial reasons (a graph is 2-colourable if and only if it is bipartite, a property which is trivially checked in linear time). Then, the problem of exactly evaluating the chromatic polynomial at any other q is #P-hard, but the weaker goal of finding a FPRAS (fully-polynomial approximation scheme) is maybe not so hard on a subset of the interval $[q_{\min}, \infty)$, where $q_{\min}$ is a non-integer value between 1 and 2 not interesting to our purposes Of course, at the values $q = 0$ and $q = 1$ one should understand the complexity of evaluating the first non-trivial term in the Taylor expansion near to the root. So the first integer points on the horizontal line going through 2-colouring in Welsh plane correspond to $q = 1 + \epsilon$, $q = \epsilon$, $q = -1$, $q = -2$, and so on. For the first two points of this list, a theorem by Greene and Zaslavsky states that the derivative of the chromatic polynomial in 1 and 0 is related to the counting of two classes of acyclic orientations on G. For q a non-zero negative integer, a theorem of Stanley relates the evaluation of the chromatic polynomial (which is always non-zero) to the counting of other families of acyclic orientations, and in particular at $q = -1$ we just have all the acyclic orientations with no other restrictions.

Given a graph G and an orientation $\phi : E(G) \to \{\pm 1\}$ (defined in terms of binary variables through some reference orientation of E), we call A_G the number of acyclic orientations of G, i.e. of orientations such that there is no oriented cycle. As customary, we define $\deg(v)$ as the degree of vertex v, and $\deg_{\text{in}}(v)$, $\deg_{\text{out}}(v)$ as the in- and out-degree, i.e. the number of adjacent edges which are oriented inbound and outbound w.r.t. v. For a pair (G, ϕ) and a vertex $v \in V(G)$, we say that v is a *source* if $\deg_{\text{in}}(v) = 0$, and a *sink* if $\deg_{\text{out}}(v) = 0$. We call $A_G^{x_1,\ldots,x_k|y_1,\ldots,y_h}$ the number of acyclic orientations which have exactly k sources in $x_1, \ldots, x_k$ and h sinks in $y_1, \ldots, y_h$, and with $A_G^{x_1,\ldots,x_k}$ the number of acyclic orientations which have exactly k sources in $x_1, \ldots, x_k$, which is of course the same number as the acyclic orientations with k sinks in $x_1, \ldots, x_k$.

The theorems we mostly need in the following, due to Greene-Zaslavsky [80] and Stanley [79], are three.

First we state a theorem for the limit $q \to 0$:

Theorem 5.1 (Greene-Zaslavsky $q \to 0$**)** *Given G connected with n vertices,*

$$P_G(q) = (-1)^{n-1} q\, A_G^x + \mathcal{O}(q^2) \tag{5.4}$$

for any vertex $x \in V(G)$.

or equivalently

Corollary 5.1 *Given G connected with n vertices,*

$$P_G(q) = \frac{(-1)^{n-1}}{n}\, q\, A'_G + \mathcal{O}(q^2) \tag{5.5}$$

where A'_G is the number of acyclic orientations of G with a single sink.

Then we state the theorem for the limit $q \to 1$:

Theorem 5.2 (Greene-Zaslavsky $q \to 1$**)** *Given G connected with n vertices,*

$$P_G(q) = (-1)^{n-1}(q-1)\, A_G^{x|y} + \mathcal{O}((q-1)^2) \tag{5.6}$$

for any $(xy) \in E(G)$.

or equivalently

Corollary 5.2 *Given G connected with n vertices and m edges,*

$$P_G(q) = \frac{(-1)^{n-1}}{m}\, (q-1)\, A''_G + \mathcal{O}((q-1)^2) \tag{5.7}$$

where A''_G is the number of acyclic orientations of G with a single source, a single sink, and source and sink are adjacent.

Finally we state the theorem for the value $q = -1$:

Theorem 5.3 (Stanley $q = -1$**)** *Given G connected with n vertices,*

$$P_G(-1) = (-1)^n A_G \tag{5.8}$$

where A_G is the number of acyclic orientations of G.

The theorem above is the first one of a family of more and more complicated statement, for all negative integer values of q, that we do not report here. So, overall we have a new family of isolated probabilistic points in the (q, v) plane, for $q = 1, 0, -1, -2, -3, \ldots$ and $v = -1$. We will extend this result further in the following sections.

The theorems above are now understood in a unified framework within the theory of Tutte-Gröthendieck invariants, as we also show in the following. First, we state a few preparatory lemmas, and other lemmas which are useful in the rest of the chapter.

For a digraph $\mathbf{G} = (G, \phi)$ and an edge $\mathbf{e} \in \mathbf{G}$, define the portion of $\mathbf{G}$ *downstream* w.r.t. $\mathbf{e}$ as the oriented subgraph induced by vertices and edges which can be reached from $\mathbf{e}$ through a directed path. The definition of upstream is analogous. The definition of downstream of a vertex v is just the union ov v, and the downsteam portions of all edges outgoing from v.

Remark that, if $\mathbf{G}$ is acyclic, the number of oriented paths starting from a given edge has always finite cardinality, as it is bounded by the number of self-avoiding paths on the unoriented G.

Lemma 5.1. *For any acyclic digraph (G, ϕ) and any oriented edge* **(ij)**, *the portion of G downstream of the edge is incident on a sink, and the portion upstream is incident to a source. In particular, any acyclic digraph has at least one sink and one source, and if it has a single sink i_0, this vertex is in the downstream of any edge.*

We do the proof by absurd. Consider the downstream component $D(e)$ out of an edge e. By construction, it certainly has a spanning oriented subtree T, with some number k of leaves. Then, in general, it will have some other edges in some set E', connecting vertices already in the tree. We want to prove that $D(e)$ is incident on some sink vertex, and we do that by absurd assuming that this is not the case. The only candidate sinks are the leaves of T, as all the other vertices have some outgoing edge by construction. Take a first leaf and label it ℓ_1, and call γ_1 the only oriented path on T from e to ℓ_1. As we want it not to be a sink, it must have at least one outgoing edge in E' within D. But, as we want the digraph to be acyclic, this outgoing edge can not point to a vertex in γ_1. As any vertex is in the oriented path from e to ℓ for at least one leaf ℓ, and as this vertex can not be in the path going to ℓ_1, it must be in the path going to some other leaf that we label ℓ_2. Similarly, we call γ_2 the path reaching ℓ_2 from e. As we want ℓ_2 not to be a sink, it must have at least one outgoing edge within D, reaching some vertex v. But v can not be neither on γ_1, nor on γ_2, otherwise it would produce a cycle. So it will be on the path oriented towards some other leaf ℓ_3 and so on. At the last leaf ℓ_k, we must have an outgoing edge within D, which however should point to a vertex v which is not in the union of $\gamma_1, \ldots, \gamma_k$. But this set is the whole tree T, so that we reach an absurd. □

Lemma 5.2. *If G does not consist of a single edge and is not 2-connected, and $(xy) \in E(G)$, then $A^{x|y}(G) = 0$.*

If the Greene-Zaslavsky theorem is given, this lemma is an elementary corollary: if the hypotheses above are not satisfied, then $(q-1)^2$ divides the chromatic polynomial, so that the derivative in $q = 1$ is zero, then also $A^{x|y}(G) = 0$.

However, this statement can be seen combinatorially, directly in the proper ensemble of acyclic orientations. Indeed, call G_1 and G_2 two graphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{i\}$. Both of G_1 and G_2 must be non-trivial for $A^{x|y}(G)$ being possibly non-zero, and also, they must be constituted of more than one edge. Indeed, any graph G with a vertex of degree 1 and different from the single-edge graph has $A^{x|y}(G) = 0$ through a simple reasoning: the leaf must be either a source or a sink, then its only neighbour must be either a sink or a source, and there are no other sources or sinks in the rest of the graph, but, if there are other edges, either their downstream or their upstream violates the statement of lemma 5.1.

Going back to our problem, as G_1 and G_2 share a single vertex, one among G_1 and G_2 does not contain either the source or the sink. Any acyclic orientation on G can be restricted to an orientation on G_1 and an orientation on G_2, both of which acyclic. The nature (sink / source / none) of the nodes in G, G_1 and G_2 does not change, except possibly for i. As, overall in G_1 and G_2, we need at least two sinks and two sources, but overall in G we have a single sink and a single source, the only possibility is that i is not a sink nor a source in G, but, when restricted to G_1 and G_2, it is respectively a sink and a source, and G_1 and G_2 did contain respectively the only source and sink of G (or vice versa). However, even in this case, the source and sink of G would not be neighbours. This completes the proof. □

Now we say something on the restriction to planar graphs. If G is planar, consider a planar embedding. This defines a set of *elementary cycles* (the ones with no chords w.r.t. the embedding).

We first state a trivial fact

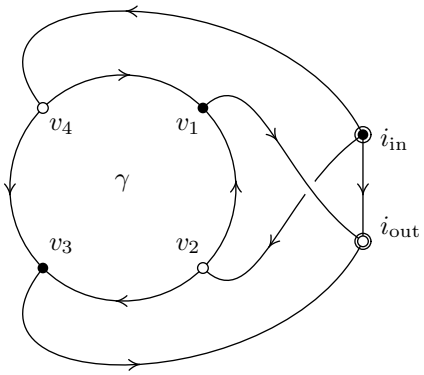
Lemma 5.3. *A planar 2-connected graphs is such that all elementary cycles have no other vertices nor edges inside.*

Indeed it can have no paths connecting two vertices on the cycle, otherwise it would not be elementary, but if it contains some component connected through a single vertex, then the graph would not be 2-connected.

Lemma 5.4. *Let a planar graph G, an orientation ϕ acyclic and single-source–single-sink, with adjacent source and sink (as in the ensemble for theorem 5.2), and an elementary cycle γ. Call V'_γ the set of vertices in γ being sources and sinks for the orientation restricted to γ. Then V'_γ has always cardinality 2.*

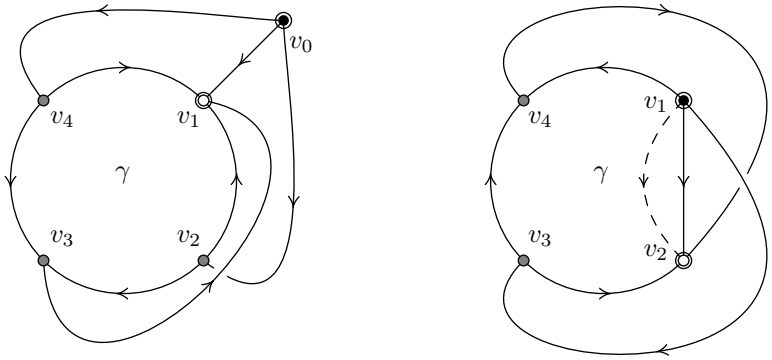
It is clear that, for any cycle even not elementary, and any orientation, V'_γ has always even cardinality, as sources and sinks alternate along the cycle. For an acyclic orientation, it cannot have cardinality 0, otherwise the cycle itself would be a circuit of the orientation. We then have to prove that it cannot have cardinality 4 or more in an elementary cycle, and for the class of orientations above. Assume by absurd that it does. We analise separately the three cases in which γ does not contain the source or the sink, it contains one of them, and it contains both.

In the case in which γ does not contain the source nor the sink, call $v_1, \ldots, v_4$ four vertices in V'_γ, alternatively source and sink, and consider the set of edges incident on the cycle. There must exist at least four of them, $e_1, \ldots, e_4$, alternatively oriented out- and in-bound, and incident on $v_1, \ldots, v_4$, because of the assumption that these vertices are not sources and sinks of the full orientation. This also tells us that we can take these four vertices with the appropriate orientations (outgoing, for the sinks of the cycle, and ingoing for the sources), and, as the cycle is elementary, we also know that these edges can be taken as going out of the cycle w.r.t. the planar embedding, and that each of them has in its downstream (resp. upstream) the single sink (resp. source) of the graph, through some path $\gamma_1, \ldots, \gamma_4$. But none of these paths γ_i can be incident on the cycle γ in any point between v_{i-1} and v_{i+1} (with labelings according to cyclic ordering, i.e. (mod4)), nor with γ_{i-1} or γ_{i+1} anywhere, otherwise it would make a circuit in the orientation. And finally, it is not even possible that all of the four paths reach their target in G, because the corresponding diagram of connectivity is a $K_{3,3}$ graph, and this would contraddict the hypothesis of planarity (cfr. drawing below).



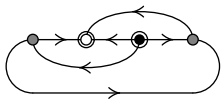
Now we go to the second case, in which one vertex (say v_1), is (say) the sink of the graph. Then, it must have a neighbour v_0 out of γ which is the only source in the graph. Then

v_2 and v_4 must have paths γ_2 and γ_4 connecting them to v_0, and v_3 must be connected to v_1 through a path γ_3 which does not enter the cycle, and does not intersect γ_2 and γ_4, and this is again in contraddiction with planarity (cfr. drawing below, left).



Finally, we go to the third case, of two vertices, say v_1 and v_2, being the source and the sink of the graph. In both cases in which the edge (v_1, v_2) is in the cycle, or it is outside, the proof is similar. We no know that there must be the two paths, γ_3 and γ_4, reaching respectively v_1 and v_2, which stay out of the cycle and do not cross each other, and this is again forbidden by planarity (cfr. drawing above, right, where the dashing denotes the two possibilities on $(v_1, v_2) \in \gamma$ or not). □

We used all along the proof the fact that the cycle is elementary w.r.t. the planar embedding. It is easy to see, through a counter-example, that this hypothesis cannot be relaxed, as the following picture shows.



5.1.1 Changing sink location in single-sink acyclic orientations

In the previous sections, for a graph G and a vertex i_0, we defined A_G^x to be the number of acyclic orientations in G with a single sink, located in x. Implicitly in the statement of Greene-Zaslavsky theorem is the fact that this number does not depend on the choice of x, although this fact is not obvious at sight. So we just denote by $A_G^\bullet$ this quantity.

It would be interesting, also for technical reasons discussed in Section 5.3, to understand this fact directly, instead that as a corollary of the theorem, and preferably through a bijection $\Phi_{G;i,i'}$ between configurations in the set corresponding to A_G^i and $A_G^{i'}$.

At the aim of finding a proper $\Phi_{G;i,i'}$, it is instructive to proceed by tries, starting from easy families of graphs.

If G is a tree, the choice of the sink corresponds to the choice of a root, and then taking the natural orientation. Of course, $A_G = 1$ in this case, and the map $\Phi_{G;i,i'}$ is the only possible one, which reverses the arrows along the path from i to i'. Analogous reasonings allow to concentrate only on the 2-connected components of G, so that the next relevant example is G being a cycle of length ℓ, for which $A_G = \ell - 1$. Indeed, we can and must have a single source on a vertex different fron the sink, and there are $\ell - 1$

choices for this. A choice for the map $\Phi_{G;i,i'}$ which works is as follows: if i' is not the source, then one adjacent edge is already inbound, and the other one is outbound and on a path which reaches i without passing through the source: reversing the arrows along this path provides the new configuration. If i' is instead the source, then both adjacent edges are outbound, and have disjoint paths which reaches i (they are just the two arcs of the cycle): reversing both paths provides the new configuration.

So, a tentative recipe seems to be inverting the inbound edges adjacent to i', along whole paths connecting them to i. Our lemma above states that at least one path with these properties exists for any edge. However, in less simple examples than tree and cycle graphs, two complicancies may occur: first, such a path is not unique in general, and it looks like we have to perform a non-canonical choice; second, if the path goes through some vertex with out-degree ≥ 2, it risks to make a cycle, and if i' itself has out-degree ≥ 2, we have two or more paths, which may have non-zero edge-intersection, and it is not clear what to do on the common parts. So our tentative recipe must be modified into some stronger theorem, hopefully with a statement which does not include non-canonical procedures. What happens is that all the complicancies above mix together into a coherent framework, such that the following surprisingly simple recipe holds

Theorem 5.4 *A valid map $\Phi_{G;i,i'}$ is the one which inverts the orientation of arrows on the subgraph $H \subseteq G$ downstream to i'.*

In order to prove the theorem we need to prove two things: first that all the appropriate constraints are satisfied in the new configuration, namely that: (1) i' is a sink; (2) i is not a sink; (3) any $i'' \notin \{i, i'\}$ is not a sink; (4) the new orientation is acyclic. Then, as a 5-th point, for proving that Φ is a bijection, it suffices to prove that $\Phi_{G;i,i'}\Phi_{G;i',i} = \mathrm{Id}$, i.e. that the subgraph H' constructed via $\Phi_{G;i',i}$ on the new configuration coincides with the first subgraph H.

The lemma 5.1 alone suffices to prove the points (1) and (2): as any edge has i in its downstream neighbourhood, this is true for any edge outgoing from i', so that all of them are flipped and i' becomes a sink. Furthermore, as i' is not a sink, it has at least one outgoing edge, which has at least one path reaching i, so at least one edge adjacent to i is flipped, so that i is not a sink anymore in the new configuration. Point (3) is also easy: every vertex different from i and i', with some adjacent edge which changes orientation, is an internal vertex for one or more paths from i' to i, so it is changing a non-zero in-degree with a non-zero out-degree, this implying that it cannot become a sink.

Point (4) is a bit more subtle, and is implied by the fact that we are taking the union of *all* the paths from i' to i. Indeed, assume by absurd that some cycle γ is generated by flipping the arrows in H. Then of course γ is neither fully contained in H, nor has empty intersection with it (otherwise it was already a cycle of G). The most general thing that can happen is that γ consists of $2k$ arcs, $k \geq 1$, alternately in H and not in H, and alternately oriented clockwise and counter-clockwise w.r.t. some planar representation of γ as a simple curve (so that swapping the orientation in H produces a cycle). Say that odd-labeled arcs 1, 3, ..., $2k-1$ are in H, and have inbound-outbound endpoints on γ respectively on vertices $(v_1 \to v_2)$, $(v_3 \to v_4)$, ..., $(v_{2k-1} \to v_{2k})$, while even-labeled arcs 2, 4, ..., $2k$ are not in H, and have inbound-outbound endpoints on γ on vertices $(v_3 \to v_2)$, $(v_5 \to v_4)$, ..., $(v_1 \to v_{2k})$. By the assumption that each odd-labeled arc is in H, there exists a path from i' to each odd vertex $\{v_{2j-1}\}_{j=1,\dots,k}$, and from each even vertex $\{v_{2j}\}_{j=1,\dots,k}$ to i. But this imples that also all of the even-labeled arcs should have been in H, as we have explicit paths from i' to i, by just connecting the path from i' to v_{2j}, the oriented arc on γ from v_{2j} to v_{2j+1} (w.r.t. cyclic ordering, i.e. $2k+1 \equiv 1$), and the path from v_{2k+1} to i, this causing an absurd.

Finally, we have to prove that map $\Phi_{G;i',i}$ is the inverse of $\Phi_{G;i,i'}$, by proving that the new subgraph H' coincides with the old one H. For sure, $H' \supseteq H$, as, if $H = \cup_\alpha \gamma_\alpha$ with γ_α a set of open oriented paths from i' to i, inverting H provides at sight that all paths γ_α are properly oriented from i to i'. So we have to check that H' has no new edges. Of course, it suffices to prove that it has no new edges among the ones adjacent to H (as induced subgraph). An edge not in H and adjacent to H must be adjacent through the "tip" of the arrow only, because, as a consequence of lemma 5.1, otherwise it would be in H.

After swapping H, the only edges adjacent to i and outbound are the ones which are in H. So H' coincides with their downstream component (because of the second sentence in the theorem), and it contains the whole H, as we already said. But, as we have now seen that all the edges adjacent to H and not in H are oriented inbound towards H, we also have that it can not be larger. This completes the proof. □

5.1.2 Changing edge location in single-source–single-sink acyclic orientations

A consequence of the second Greene-Zaslavsky theorem is that $A_G^{x|y}$ is independent on x and y as long as they are adjacent on G, and we call this quantity $A_G^{\cdot|\cdot}$. Again, we would like to understand this fact directly, and through a bijection $\Phi_{G;e,e'}$. It is not hard to do this, by using the result of the previous section: if e and e' are adjacent, say $e = (ij)$ and $e' = (jk)$, one can change the position of the only source (or sink) i into k through the map $\Phi_{G;i,k}$. We only need to check that in the resulting orientation j is still the only sink (or source). This is easily checked, as in $\Phi_{G;i,k}$ we are swapping the orientation on the subgraph $H \subset G$ corresponding to the downstream of k, but this graph does not contain j as it is a source, and a source is in the downstream only of itself. Furthermore, except that in k and i, no sources and sinks are created, as every other vertex in H has non-zero both in- and out-degree.

Then, for the generic case of e and e' not adjacent, as G is connected, we can always find a path connecting e to e'. If G is bipartite, so that all these paths have the "wrong" parity, we can reverse the whole orientation as a final step. This completes the proof of existence of a bijection.

5.2 The polynomials $P'_G(1)$, $P'_G(0)$ and $P_G(-1)$ under one roof

Now we prove the three theorems 5.1, 5.2 and 5.3, within the theory of Tutte-Gröthendieck invariants [81].

We have to check that, if $e \in E(G)$ is not a loop or an isthmus,

$$f(G) = f(G \smallsetminus e) + f(G \bullet e)\,, \tag{5.9}$$

while if e is a loop or an isthmus,

$$f(G) = f(e) f(G \smallsetminus e)\,. \tag{5.10}$$

Then the parameters $f(e)$ for the loop and the isthmus will fix the parameters x and y in Tutte formulation. Some extra care is due to the fact that, at $q = 0$ and $q = 1$, actually the polynomial vanishes so that we have to take the appropriate limit in the de l'Hôpital sense. At this aim, we introduce a formal variable ζ, and consider the generalizations

$$\tilde{P}_G(0) = \zeta^{k_2(G)} \prod_{G_\alpha} P'_{G_\alpha}(0)\,; \tag{5.11}$$

$$\tilde{P}_G(1) = \zeta^{k_2(G)+k(G)-1} \prod_{G_\alpha} P'_{G_\alpha}(1)\,; \tag{5.12}$$

where $k_2(G)$ is the number of 2-connected components in G, and G_α are the 2-connected components. The original definitions $P'_G(1)$ and $P'_G(0)$ are recovered from the term order ζ^1 in the polynomials above, which is also the "leading term" in a limit $\zeta \to 0$ (pictorially speaking, as ζ is just a formal indeterminate).

We start by analysing (5.10). If G has a loop, it makes a cycle regardless to the rest of the configuration, in all three cases of $q = 0, \pm 1$, so (5.10) is trivially satisfied with $f(\text{loop}) = y = 0$.

If G has an isthmus e, it cannot take part in a cycle for any orientation, so that the acyclic constraint is always satisfied. The three cases are different, For $q = -1$, there are no other constraints, so any pair of orientations on the two components incident on the two endpoints of e is valid. This leads to satisfy (5.10) with $f(\text{isthmus}) = x = 2$. For $q = 0$ and $q = 1$ we have $f(\text{isthmus}) = \zeta$, as only one orientation of the edge is legitimate, and G has an extra 2-connected component w.r.t. $G \smallsetminus e$.

Now we go to equation (5.9). If e is not a loop nor an isthmus, its endpoints x and y are distinct vertices on a connected graph $G' = G \smallsetminus e$.

For the case $q = -1$, consider $A(G')$. It takes three contributions: the first one, A_1, for orientations in which there exists a directed path from x to y, a second one, A_2, for orientations with a directed path from y to x, and a third one, A_3, for orientations in which there are no oriented paths between x and y. The fourth case, of both a directed path from x to y and one from y to x, is excluded because the orientation would have a cycle. So, $f(G \smallsetminus e) = A_1 + A_2 + A_3$. Now we can add the edge e with its orientation. In the sets A_1 and A_2, there is a single choice which does not make a cycle, while in the case A_3 both choices are valid, so that we get $f(G) = A_1 + A_2 + 2A_3$. Instead, if we contract edge (xy), we make a cycle both in set A_1 and A_2, while configurations in A_3 are valid, so that $f(G \bullet e) = A_3$, and relation (5.9) is satisfied.

For the case $q = 0$, we start by analysing the case in which the removal / contraction of e does not change the number of 2-connected components. We must have a single sink in the three relevant graphs G, $G \smallsetminus e$ and $G \bullet e$ (more precisely, in the restriction to the 2-connected component). Again we use $G' = G \smallsetminus e$ as a reference, and, thank to the invariance under sink relocation, we can assume that in all three graphs the sink is in x. As we are changing things only on one edge, the relevant quantities are $A^x_{G'}$ and $A^{xy}_{G'}$. Only $A^x_{G'}$ contributes to $f(G \smallsetminus e)$. As y has x in its downstream (because of Lemma 5.1), configurations in $A^x_{G'}$ make a cycle on $G \bullet e$, so they do not contribute on this graph. Conversely, by definition, configurations in $A^{xy}_{G'}$ all contribute. Finally, on G both $A^{xy}_{G'}$ and $A^x_{G'}$ contribute, and only with the edge directed towards x, so both with a coefficient 1, and relation (5.9) is satisfied at the leading order in ζ.

For the case $q = 1$, say that z is a neighbour of x (it can coincide with y or not, but we can assume that it does not coincide, because otherwise the graph G would be an isthmus, or have a double edge, and the counting of orientations is trivially reduced in this case, as multiple edges must be all oriented in the same way in order to not produce a cycle). We choose that in all three graphs x is the only sink and z is the only source. The relevant combinations on G' are the ones in which y can also be a sink or a source, that is $A^{x|z}_{G'}$, $A^{xy|z}_{G'}$ and $A^{x|yz}_{G'}$. Only the first one contributes to $G \smallsetminus e$, and only the second one contributes to $G \bullet e$. Both $A^{x|z}_{G'}$ and $A^{xy|z}_{G'}$ contribute to G, and only with the edge directed

towards x, while $A_{G'}^{x|yz}$ can not contribute even to G, because if we orient e towards x we have an extra source in y, and if we orient it towars y we lose the sink in x. Again relation (5.9) is satisfied at the leading order in ζ, and the theorems are proven, with $A(G) = (\partial/\partial\zeta)\, T_G(2+\zeta, 0)|_{\zeta=0}$. □

5.3 Antiferromagnetic Fortuin-Kasteleyn expansion

If one is interested in an expansion with a probabilistic interpretation of the new variables, in the antiferromagnetic case, and which achieves analytic continuation in q, one could be tempted to use the so-called *antiferromagnetic Fortuin-Kasteleyn expansion*, i.e. to write the partition function as

$$Z(G; v; q) = \sum_{s:V\to[q]} \prod_{(ij)\in E(G)} \big((1+v_{ij}) - v_{ij}(1-\delta(s_i, s_j))\big) \tag{5.13}$$

Now the two terms $1+v_{ij}$ and $-v_{ij}(1-\delta(s_i, s_j))$ are both positive in the antiferromagnetic regime, and one could hope that summation over s does not spoil this positivity. We will see how this happens for negative q. Indeed, we can define the parameters $x_{ij} = -v_{ij}/(1+v_{ij})$, which are real positive in an antiferro system, and get

$$Z(G; v; q) = \Big(\prod_{(ij)\in E(G)} (1+v_{ij})\Big) \sum_{s:V\to[q]} \prod_{(ij)\in E(G)} \big(1 + x_{ij}(1-\delta(s_i, s_j))\big) \tag{5.14}$$

So, we can drop the trivial prefactor for the moment, and expand the product of binomials in a fashion as above, getting

$$Z(G; v; q) \sim \sum_{H\subseteq G} \sum_{s:V\to[q]} \prod_{(ij)\in E(H)} x_{ij} \prod_{(ij)\in E(H)} (1-\delta(s_i, s_j)) \tag{5.15}$$

Again the first product is just the natural multivariate "thermodynamic factor" associated with edge occupation, while the second product encodes all the dependence from q, and is going to determine an extra factor in the weight of H, through summation over consistent s, and will be non-local in edge occupations, but factorized on the components $\{H_\mu\}$ of H. However now this factor is less trivial than the one for the ferromagnetic F-K. Actually, instead of being just q, it coincides with the well-known definition of the chromatic polynomial, where for any graph G the chromatic polynomial $P_G(q)$ counts the proper colouring of G with q colours

$$P_G(q) = \#\{s : V(G) \to [q] \;\big|\; s(i) \neq s(j) \quad \forall\ (ij) \in E(G)\} = Z(G; -1, q) \tag{5.16}$$

so that we can write

$$Z(G; v; q) = \Big(\prod_{(ij)\in E(G)} (1+v_{ij})\Big) \sum_{H\subseteq G} \prod_{(ij)\in E(H)} x_{ij} \prod_{\mu} P_{H_\mu}(q) \tag{5.17}$$

Unfortunately for the aims of building a probabilistic expansion, the chromatic polynomial of a connected graph does not have definite sign for all values of q. More precisely, it is monic and behaves as q^V for q large, it is free of roots for $q < 0$, has a single root at $q = 0$, and no other roots before $q = 1$. So $P_G(q)$ has a simple sign $(-1)^V$ only for $q < 0$. †

†Furthermore, the chromatic polynomial has a number of roots at $q = 1$ which is zero only if G has a single vertex, and otherwise is equal to the number of 2-connected componets in G, then it is free of zeroes up to 32/27 [82].

Even in the best-case range $q < 0$, the formula (5.17) is unsatisfactory, as except for integer negative values of q we do not have any efficient way either of evaluating $P_G(q)$, or even of describing it probabilistically by means of a further set of auxiliary variables (if q is a negative integer, the latter can be done by mean of a theorem by Stanley theorem, cfr. [78]).

Actually, if q is negative and the v's are negative and small enough, one could expect that, in a Monte Carlo Markov Chain on a regular euclidean lattice, a typical configuration has components which are both small and with a small number of loops, and, as the deletion-contraction method for evaluating the chromatic polynomial (by reduction to trees) has a complexity which is exponential in the number of loops in the largest 2-connected component, the heuristic complexity could be indeed polynomial in such a region.

However we expect that the nature of the transition, if any in this sector $q, v < 0$, is related to the emergence of a giant component, so this direct approach to the chromatic polynomial could suffer of an exponential slowing down when approaching criticality from the high-temperature region. So we want to do better than (5.17).

An exception to the conceptual hardness of the chromatic polynomial is the range of infinitesimal q, where the Greene-Zaslavsky theorem holds. Although A_G is still worst-case #P computable, at least, through the correspondence with a class of orientations, it can be represented with a set of local auxiliary variables, for any lattice and in any regime of parameters v, this being a practical tool in a Monte Carlo chain, especially at the light of positive results on the connectivity of the phase space [83]

So for q negative and infinitesimal and v finite we have for the partition function

$$Z(G; v; q) = (-1)^{|V(G)|}(-q) \prod_{(ij)} (1 + v_{ij}) \sum_{\substack{H \subseteq G \\ \text{connected}}} A^{\cdot}_{H} \prod_{(ij) \in E(H)} x_{ij} \; + \mathcal{O}(q^2) \tag{5.18}$$

while in a limit $q, v_{ij} \to 0$ with $v_{ij}/q = w_{ij}$ fixed, doing the appropriate limit by counting the powers of q in the factors v_{ij}, that is $x_{ij} = -qw_{ij} + \mathcal{O}(q^2)$ and $1 + v_{ij} = 1 + \mathcal{O}(q)$, we get

$$\begin{aligned} Z(G; v; q) &= (-1)^{|V(G)|} \sum_{H \subseteq G} \Big[(-q)^{k(H)} \prod_{(ij) \in E(H)} (-q w_{ij}) \prod_{\mu} A^{\cdot}_{H_\mu} \Big] (1 + \mathcal{O}(q)) \\ &= q^{|V(G)|} \sum_{H \subseteq G} \Big[(-q)^{L(H)} \prod_{(ij) \in E(H)} w_{ij} \prod_{\mu} A^{\cdot}_{H_\mu} \Big] (1 + \mathcal{O}(q)) \end{aligned} \tag{5.19}$$

where we used $|V(H)| - k(H) = |E(H)| - L(H)$. In the limit $q \to 0^-$, dropping the prefactor, we get

$$Z(G; v; q) \sim \sum_{\substack{H \subseteq G \\ \text{forests}}} \prod_{(ij) \in E(H)} w_{ij} \prod_{\mu} A^{\cdot}_{H_\mu} \tag{5.20}$$

So the surviving subgraphs H are forests, and each H_μ is a tree. But, in any tree T, $A^{\cdot}_{T}$ is trivially 1, so we end up with the partition function of unrooted spanning forests with weights w_{ij}.

So the chromatic polynomial has a simple sign $(-1)^{V-1}$ for $q \in [0, 1]$ (which in our expansion would require to count configurations with a factor $(-1)^{k(H)}$, spoiling the seeked probabilistic interpretation), then it requires the ability of counting 2-connected components in H in the interval $q \in [1, 32/27]$, and things only go worse for larger q, except for a value of q sufficiently large to determine that it is larger than the largest real zero of the polynomial.

This was indeed to be expected on a quite general ground: we know that in the limit $q, v_{ij} \to 0$ at $w_{ij} = v_{ij}/q$ fixed we have a symmetry between the set (q, v_{ij}) and $(-q, -v_{ij})$, as the partition function only depends on w's, and the system in the case of positive q and v's describes spanning forests, so the result above should be interpreted as a check of consistency of the procedure.

We would like to extend the treatment to finite negative q, exploiting further the theorem of Greene and Zaslavsky. At this aim we want to use a "Chayes-Machta" technique in order to split q into many infinitesimal summands, and use the theorem in each of these subclasses.

More precisely we write the partition function

$$Z(G; v; q) = \sum_{s:V\to[q]} \prod_{(ij)\in E(G)} \big(1 + v_{ij}\delta(s_i, s_j)\big) \tag{5.21}$$

performing some extra manipulations on each factor. For any set of conditions (event), say that χ(event) is 1 if the event is true and 0 otherwise. Consider a partition of $[q]$ into K subsets S_α, of cardinalities q_α. Clearly $\sum_{\alpha=1}^{K} q_\alpha = q$. Then $\sum_{s_i\in[q]}$ is a synonimous of the more explicit $\sum_{s_i} \chi(s_i \in [q])$, which we can now partitionate into $\sum_{s_i}\sum_\alpha \chi(s_i \in S_\alpha)$, i.e.

$$Z(G; v; q) = \sum_s \prod_i \Big(\sum_\alpha \chi(s_i \in S_\alpha)\Big) \prod_{(ij)\in E(G)} \big(1 + v_{ij}\delta(s_i, s_j)\big) \tag{5.22}$$

Then we want to manipulate each of the edge factors, writing

$$\begin{aligned}
1 + v\delta(s_i, s_j) &= 1 + \big(v - v/2\sum_\alpha \chi(s_i \in S_\alpha) - v/2\sum_\alpha \chi(s_j \in S_\alpha)\big) \\
&+ \big(v/2\sum_\alpha \chi(s_i, s_j \in S_\alpha) + v/2\sum_\alpha \chi(s_i, s_j \in S_\alpha) - v\sum_\alpha \chi(s_i, s_j \in S_\alpha)\big) \\
&+ \big(v\delta(s_i, s_j)\sum_\alpha \chi(s_i, s_j \in S_\alpha)\big) \\
&= (1 + v) + (-v/2)\sum_\alpha \chi(s_i \in S_\alpha, s_j \notin S_\alpha) \\
&+ (-v/2)\sum_\alpha \chi(s_j \in S_\alpha, s_i \notin S_\alpha) + (-v)\sum_\alpha \chi(s_i, s_j \in S_\alpha)(1 - \delta(s_i, s_j))
\end{aligned} \tag{5.23}$$

For v antiferromagnetic, remark that all the coefficients are positive.

The expansion of the factors $\sum_\alpha \chi(s_i \in S_\alpha)$ leads to the introduction of the "colouring" function $c : V \to [K]$, which says, for each $i \in V$, in which set S_α the variable s_i is contained. So we have

$$Z(G; v; q) = \sum_s \sum_{c:V\to[K]} \prod_i \chi(s_i \in S_{c(i)}) \prod_{(ij)\in E(G)} (\cdots) \tag{5.24}$$

where the dots stand for the long expression in (5.23) specialized to $v = v_{ij}$. However, the prefactors $\chi(s_i \in S_{c(i)})$ allow us to drop most of the terms from any of the edge-factors. For any edge (ij), if $c(i) = c(j) = \alpha$, the only terms which survive are the trivial one, and the one multiplying $\chi(s_i, s_j \in S_\alpha)$, so one gets $((1 + v_{ij}) - v_{ij}(1 - \delta(s_i, s_j)))$. If $c(i) = \alpha$ and $c(j) = \beta \neq \alpha$, the only terms which survive are the trivial one, the one with the factor $\chi(s_i \in S_\alpha, s_j \notin S_\alpha)$ and the one with the factor $\chi(s_j \in S_\beta, s_i \notin S_\beta)$, so one gets $((1 + v_{ij}) - 2v_{ij}/2) = 1$.

So, in the case $c(i) \neq c(j)$ we have a single monomial, and in the case $c(i) = c(j)$ we have a binomial. We can interpret the expansion of this expression in terms of a sum over subgraphs H which are "compatible with the colouring c", i.e. with edges $(ij) \in E(H)$ only if $c(i) = c(j)$.

For any edge (ij) such that $c(i) \neq c(j)$, we just have a factor 1. For any edge (ij) such that $c(i) = c(j)$, we have a factor $(1 + v_{ij})$ if $(ij) \notin E(H)$ and $-v_{ij}(1 - \delta(s_i, s_j))$ if $(ij) \in E(H)$.

In accord with the definition (5.16), at a given component H_μ with all vertices in the block $S_{c(\mu)}$, the sum over s of the product $\prod_{(ij)\in E(H_\mu)}(1-\delta(s_i, s_j))$ just gives $P_{H_\mu}(q_{c(\mu)})$. So we have

$$Z(G; v; q) = \sum_{c:V\to[K]} \sum_{\substack{H\subseteq G \\ \text{compatible} \\ \text{with } c}} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{\substack{(ij)\notin E(H) \\ c(i)=c(j)}} (1 + v_{ij}) \prod_{\mu} P_{H_\mu}(q_{c(\mu)}) \tag{5.25}$$

It should be noted that, up to now, we proven that this expression describes the q-colour Potts model for any integer $K \leq q$, and set of positive integers q_α summing up to q. However, as it is polynomial in all the q_α's, it should be interpreted as a viable analytic continuation in this set of parameters $\{q_\alpha\}_{\alpha=1,\dots,K}$, for any integer K. Furthermore, in any linear reparametrization of the q_α's which highlights the combination q, e.g. $\{q, q_1 - q_2, q_1 - q_3, \dots, q_1 - q_K\}$, we must have that the polynomial depends on q only. For example, taking all q_α to be the same, we must have that our expression above coincides with the (ferromagnetic) Fortuin-Kasteleyn expansion, for any integer K, replacing any $q_{c(\mu)}$ with q/K, and the result must be independent from the choice of the integer K, although this is not evident at sight.

If the system is antiferro, q is negative and all q_α's are negative, the expansion (5.25) is probabilistic (with an exposed factor $(-1)^V$), although it is at a first sight much more involuted than the one before, in (5.17).

Furthermore, it does not look much more convenient: again we are not able to determine with a fast procedure the values $P_{A_\mu}(q_{c(\mu)})$, and not even give a local-variable representation of them, and we have to deal with the complicated counting of edges not in H connecting sites in the same block.

In other words, commuting the sum over H and c, the compatibility constraint on c is just that $c(i) = c(j)$ for any i, j in the same component of H, and one would be tempted to think that, analogously to what happens in ordinary Fortuin-Kasteleyn, the summation is factorized on the possible colourings of each component, but the factors $(1 + v_{ij})$ spoil this fact, giving a different weight to neighbouring components when they get into the same block S_α.

Consider the case in which all q_α are equal. For a given subgraph H with $k(H)$ components, all factors $(-v)$ and some of the factors $(1 + v)$ are there for any consistent colouring, because the endpoints of the edge are in the same component of H (for $H \subseteq G$, we call $E_{\rm int}(H; G)$ the edges of G which are not edges of H, but such that both endpoints are in the same component of H – factors $(1 + v_e)$ for $e \in E_{\rm int}$ are the fixed ones).

Then, summation over the colourings c leads to a "coarse-grained" Potts model interaction, on an auxiliary graph $\tilde{G}$ where the vertices are labeled by the components, the number of colours is K, and the weight $\tilde{v}_{\mu\nu}$ is given by

$$\tilde{v}_{\mu\nu} = -1 + \prod_{\substack{(ij)\in E(G) \\ i\in V(H_\mu), j\in V(H_\nu)}} (1 + v_{ij}) \tag{5.26}$$

We thus have for the partition function

$$Z(G;v;q) = \sum_{H\subseteq G} Z(\tilde{G};\tilde{v};K) \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H;G)} (1+v_{ij}) \prod_{\mu} P_{H_\mu}(q/K) \tag{5.27}$$

However, what turns out is that in the limit $K\to\infty$ and all $q_\alpha \to 0$ (say, with $q_\alpha = q/K$ for all indices, but not necessarily), both the problems sketched above simplify, and the equivalent "reduced" Potts problem above simplifies into its trivial asymptotics of large number of colours.

Indeed, it is well known that for any finite-graph finite-weight Potts instance $(G;v)$ (say $|V(G)|=n$), the limit of large q is dominated by the configurations in which all the vertices have a different colour, and is a simple power at leading order (e.g. by inspection of the Fortuin-Kasteleyn expansion)

$$\frac{Z(G;v;q)}{q^n} = 1 + \mathcal{O}(v_{ij}/q) \tag{5.28}$$

and we want to apply this lemma to the factor $Z(\tilde{G};\tilde{v};K)$ in (5.27). So, in the large K limit above, the summation over c's just produces a factor $K^{k(H)}(1+\mathcal{O}(1/K))$, and we can avoid considering factors $(1+v)$ on edges which are not connecting vertices in the same component (the ones originating the $\tilde{v}$'s). We get

$$\begin{aligned} Z(G;v;q) = (-1)^{|V(G)|} \lim_{K\to\infty} \sum_{H\subseteq G} K^{k(H)} \big(1+\mathcal{O}\big(\tfrac{1}{K}\big)\big) \prod_{(ij)\in E(H)} (-v_{ij}) \\ \times \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij}) \prod_{\mu} \Big(-\frac{q}{K} A^{\bullet}_{H_\mu} + \mathcal{O}\big(\big(\tfrac{q}{K}\big)^2\big)\Big) \end{aligned} \tag{5.29}$$

Taking the limit is easy, and just gives the final formula

$$Z(G;v;q) = (-1)^{|V(G)|} \sum_{H\subseteq G} (-q)^{k(H)} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij}) \prod_{\mu} A^{\bullet}_{H_\mu} \tag{5.30}$$

Again this expression is probabilistic if the system is antiferromagnetic and q is negative. We understand it is a summation over subgraphs $H\subseteq G$, with some Gibbs weight which also includes a purely combinatorial factor related to the number of single-sink acyclic orientations in each component, which unfortunately is in general hard to compute:

$$Z(G;v;q) = (-1)^{|V(G)|} \sum_{H\subseteq G} W(H) \tag{5.31}$$

$$W(H) = (-q)^{k(H)} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij}) \prod_{\mu} A^{\bullet}_{H_\mu}\,. \tag{5.32}$$

Using auxiliary "orientation" variables, to be thermalized simultaneously to the edge-occupation variables, could be a viable tool for effectively sampling configurations through a Monte Carlo Markov Chain, also in the regions where we expect that the direct evaluation of $A^{\bullet}_{H}$ is computationally hard. So, if we denote by $\mathbf{H}$ a subgraph of G with an orientation on its edges, and by $\chi(\cdots)$ an indicator function over an event, we would have, for some prescription on how to take a "sink location" on a vertex set V', $\hat{i}(V')$

$$Z(G;v;q) = (-1)^{|V(G)|} \sum_{\mathbf{H}\subseteq G} W(\mathbf{H}) \tag{5.33}$$

$$W(\mathbf{H}) = (-q)^{k(H)} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij})\ \chi\left(\begin{array}{c}\mathbf{H}\text{ is acyclic;}\\ \text{sinks are in } \{\hat{i}(V_\mu)\}\end{array}\right). \tag{5.34}$$

Alternatively, we can average over all possible locations of the sink, and take the weights

$$W(\mathbf{H}) = (-q)^{k(H)} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij}) \prod_{\mu} \frac{1}{|V_\mu|} \, \chi\left(\begin{array}{c}\mathbf{H} \text{ is acyclic;}\\ \text{one sink per } H_\mu\end{array}\right) . \quad (5.35)$$

Now we pause a moment for analysing the computational effectiveness of a formula like (5.30). The factors $A^{\bullet}_G$ for the various components of our cluster expansions are crucial in the description of the rates of a Monte Carlo algorithm. One could make an arbitrary choice for fixing the location of i_0 for each component (e.g. the first vertex in some lexicographic order), as done in (5.34), but it is unlikely that such a rigid prescription would give a fast mixing of the chain, if not spoiling even the mere ergodicity (as it becomes hard to join together almost all pairs of components).

Alternatively, one can think of using the ensemble A'_G, of all acyclic single-sink orientations, which of course has a cardinality multiplied by the number of vertices in G, so that one has the reweightening of Gibbs factors as in (5.35). Then, even a single-edge update move would lead to a fast mixing within each component (as implied by [83]), and in particular to a fast diffusion of the sink on the component. Now, given two components H_μ and H_ν with an edge $e \in G$ connecting them, it does not happen anymore that the acceptance ratio for adding e to H is always zero because none of the two endpoints of e is the sink of its component.

However this acceptance could be unsatisfactory in a phase (or near to) in which we have giant components: say that the number of vertices in H_μ and H_ν are V_μ and V_ν respectively, then having a source at one of the endpoints is dumped by a factor approximatively of $1/V_\mu + 1/V_\nu$, plus a factor $(-v)$, and a number (possibly zero) of factors $(1+v)$ due to the difference $E_{\rm int}(H) \setminus (E_{\rm int}(H_\mu) \cup E_{\rm int}(H_\nu))$ But then, as we are working in (5.35) formulation, we should add a factor $V_\mu V_\nu/(V_\mu + V_\nu)$ for the relative weights $1/\prod_\mu V_\mu$ of the two configurations, which for large clusters can be very large. At equilibrium, the rare event of having a sink at one endpoint (dumped by $\sim 1/V_{\rm typ.}$), has a compensation from the last factor (going like $\sim V_{\rm typ.}$), but, in both directions, the acceptance rates for changing the connectivity support of pairs of large clusters are small of order $1/V_{\rm typ.}$.

Assume instead that we can boost the thermalization of the location of the sink, by defining some map $\Phi_{G;i,i'}$ being a bijection between the orientations with sink in i and the ones with sink in i'. In this case, instead of having, on one side, a $(1/V_{\rm typ.})$-rare event, and on the other side a $(1/V_{\rm typ.})$-dumped inverse rate, we would have rates in the two directions of order 1.

More precisely, within this assumption we can always think of thermalyzing the position of the sink in one step, in both components H_μ and H_ν. Of course, we do not really do this unless it is necessary, but this works at a conceptual level, as we now explain. Consider the resampling of edge $e = (ij)$, which is currently empty, and $i \in H_\mu$ and $j \in H_\nu$. We should calculate the acceptance rate for occupying e, oriented towards i or towards j. Of course our reasonings are symmetric, and we concentrate on i. Call H_ρ the potential new component $H_\mu \cup H_\nu \cup \{\mathbf{(ji)}\}$, and say that i_0 is the position of the sink in H_μ before the move.

With the naïve technique, the acceptance rate has a factor 1 or 0 with probabilities exactly $1/V_\mu$ and $1 - 1/V_\mu$, depending if $i = i_0(H_\mu)$. Now, we can replace these factors with their average, $1/V_\mu$, as we can think of having resampled the position of the sink. Then, *only if the move is accepted*, we apply $\Phi_{H_\mu;i_0,i}$, add the edge **(ji)**, then choose randomly a site $i' \in V(H_\rho)$, and finally apply $\Phi_{H_\rho;i,i'}$. If the move is not accepted, and we keep a dynamic list which associates to each vertex the size of its component in H,

we did a computational effort of order 1. Still, updating dynamically this list leads to a slowing down (cfr. however [84] for a speeding up of this procedure), but these updates are performed *only when the move is accepted*, and this is in general a milder source of computational slowing down.

This Monte Carlo technicality motivates further the study of the map $\Phi_{G;i,i'}$ (if the combinatorial interest of this bijection *per se* is not already evident), which has been performed in Section 5.1.1, with a simple result and a recipe which is realized through a computationally fast "greedy" exploration algorithm. It would be interesting to study the fractal dimension of the "downstream" subgraphs used in the bijection $\Phi_{G;i,i'}$, at criticality, similarly to what is done in [84], keeping in mind that, in the limit of spanning trees (which is reached also as a special limit of the present antiferromagnetic expansion), this gives the exponent for the chemical distance on uniform spanning trees, 5/4.

5.4 Antiferro F-K and acyclic orientations: a multilinearity proof

Here we give a different, and much simpler, proof of (5.30), due to Alan Sokal (on an e-mail dated Nov. 4th 2007).

We begin by classifying terms in the ordinary F-K expansion by the partitions corresponding to the supports of the different clusters. Call $\Pi(V)$ the set of partitions of a set V, and $|\pi|$ the number of blocks in the partition, so that $\pi = \{V_1, \dots, V_{|\pi|}\}$ is a typical element in this set. We have

$$Z(G; v; q) = \sum_{\pi \in \Pi(V)} q^{|\pi|} \prod_{\alpha=1}^{|\pi|} C(G|_{V_\alpha}, v|_{V_\alpha}) \,, \tag{5.36}$$

where $C(G; v)$ is the generating function for connected spanning subgraphs of G, i.e.

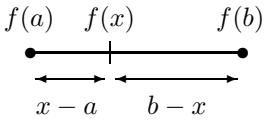
$$C(G; v) = \lim_{q \to 0} \frac{Z(G; v; q)}{q} = \frac{\partial}{\partial q} Z(G; v; q) \Big|_{q=0} \,. \tag{5.37}$$

Then we will use a lemma which exploits the multilinearity of the F-K partition function in all v_e's, i.e. the fact that, for any edge e, $Z(G; v; q) = Z_0 + v_e Z_1$ as a polynomial in v_e, for suitable coefficients Z_0 and Z_1 which polynomially depend on q and the other $v_{e'}$'s only.

Clearly, if $f(x)$ is a linear function on $\mathbb{R}$ and we know its value on two distinct points a and b, we simply have

$$f(x) = \frac{(x-a)f(b) + (b-x)f(a)}{b-a} = \frac{(x-a)f(b)}{b-a} + \frac{(x-b)f(a)}{a-b} \tag{5.38}$$

(where the second expression is less compact but symmetric under $a \leftrightarrow b$), i.e. the diagram

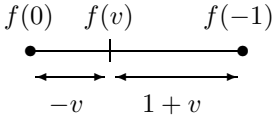


Applying this simple fact to all the parameters v_e of the Potts partition function gives

Lemma 5.5 (multilinearity of multivariate Tutte Polynomial). *For any set of vectors $\{v'_e\}$, $\{v''_e\}$, with $v'_e \neq v''_e$ for any edge e, consider the set of $2^{|E|}$ vectors $n = \{n_e\}$, $n_e \in \{v'_e, v''_e\}$, and take $\bar{n}$ as a synonimous for $v' + v'' - n$ (i.e. the vector of opposite choices w.r.t. n). Then we have*

$$Z(G; v; q) = \sum_n \left(\prod_e \frac{v_e - \bar{n}_e}{n_e - \bar{n}_e} \right) Z(G; n; q)\,. \tag{5.39}$$

If we specialize to $v'_e = 0$ and $v''_e = -1$, the barycentric combinations are encoded in the scheme



so that we get

$$Z(G; v; q) = \sum_{n \in \{0,-1\}^E} \prod_{e: n_e = 0} (1 + v_e) \prod_{e: n_e = -1} (-v_e)\; Z(G; n; q)\,. \tag{5.40}$$

We want to apply this lemma, in the form (5.40) above, to each of the factors $C(G|_{V_\alpha}, v|_{V_\alpha})$ in (5.36), written in the limit form of (5.37), namely

$$\lim_{q\to 0} \frac{Z(G; v; q)}{q} = \sum_{n \in \{0,-1\}^E} \prod_{e: n_e = 0} (1 + v_e) \prod_{e: n_e = -1} (-v_e)\; \lim_{q\to 0} \frac{Z(G; n; q)}{q}\,. \tag{5.41}$$

But now, as all n's are either 0 or -1, we just get for each term a chromatic polynomial on the spanning subgraph $H(n)$ whose edge-set is the subset of $E(G)$ with $n_e = -1$, i.e.

$$Z(G; n; q) = P_{H(n)}(q)\,; \tag{5.42}$$

and the limit expression $C(G; v)$ just gives, through Greene-Zaslavsky theorem

$$\lim_{q\to 0} \frac{Z(G; n; q)}{q} = P'_{H(n)}(0) = \begin{cases} (-1)^{|V(G)|-1} A_H & H \text{ is connected} \\ 0 & \text{o.w.} \end{cases} \tag{5.43}$$

Then we plug this statement on each of the factors in (5.36), and reabsorb the prefactors $(-1)^{|V(G|_{V_\alpha})|-1}$ into a $(-1)^{|V(G)|-|\pi|}$. Finally we realize that summation over π, times summation over subgraphs $\{H_\alpha\}$ such that $V(H_\alpha) = V_\alpha$, just corresponds to summation over spanning subgraphs $H \subseteq G$, with $|\pi|$ replaced by $k(H)$. This just gives again equation (5.30).

5.5 F-K for a system with antiferro and unphysical couplings

We know that by ordinary ferromagnetic F-K we have a probabilistic expansion of Potts model in analytic continuation in q, for $q \geq 0$ and all ferromagnetic couplings. Then the result of equation (5.30) implies a probabilistic expansion also in the range $q < 0$ and all antiferromagnetic couplings. Two important physical sectors are missing, $q > 0$ and v antiferro, and $q < 0$ and v ferro, which are interesting, especially in two dimensions, because the whole antiferro transition line is contained in these sectors. Further generalizing the results of the previous section to these sectors is expected to be a hard task, for the reasons we elucidated in the previous paragraphs.

But also, we do not have, up to now, a treatment for the "mixed" case, when there is a compresence of ferro, antiferro and unphysical couplings. Up to a certain point, this seems just to be a "technical" problem, of dealing with different kind of decompositions of $(1 + v\delta)$ according to the different situations, and we can probably reach some result with a small extra effort.

In particular, we want to prove here that a probabilistic expansion can be provided in the sector of $q < 0$, in a mixed system with antiferro and unphysical couplings, if the unphysical edges make a cycle-free subgraph of the underlying graph. This could seem quite a restrictive situation, but it has an easy and potentially interesting representative, in the Potts Model on the hypercubic lattice in D dimensions with free boundary conditions, with couplings v_{ij} depending on the edge direction, which are antiferro in all directions but one, and unphysical in the last one (for $D = 2$, e.g., $v_{\text{hor.}}$ antiferro and $v_{\text{vert.}}$ unphysical).

This is intersting at the aim of simulating Potts in three dimensions, where we know that there is a percolative phase transition in the spanning-forest model at a *finite* positive value of the coupling parameter $t = v/q$, that should correspond to a positive tilting w.r.t. the infinite slope at the origin in two-dimensional systems, e.g. of Baxter parabola for the square lattice. If we assume that this tilting is the only quantitative modification in an otherwise topologically analogous structure of "parabolas", then we expect a critical line in the $(q < 0, v < 0)$ quadrant, starting from the origin at a finite slope, then bending and crossing the vertical axis at some negative value $v_\star$, for a lattice with all equal weights v. However, if it happened that $v_\star < -1$, our antiferro expansion would not be sufficient to follow this full line. Nonetheless, when the v's, instead than equal, are dependent from the direction within a regular lattice, one has a behaviour similar to the presence of a single v, somewhat averaging among the various directions (this happens, for example, in the explicit anisotropic solution for the two-dimensional Ising model on the square lattice), so that the expansion we consider in this section could be a viable candidate for analysing this kind of situation.

Beside the "ferromagnetic" and the "antiferromagnetic" way of splitting the thermodynamic weight $\exp(J\delta(s, s'))$, that we followed in the previous sections, there is a third combination in breaking $1 + v\delta$ into two combinatorially-clear summands, namely $1 + v\delta = (1 + v)\delta + (1 - \delta)$. Then, factors $1 - \delta$ make components on which summation over s produces chromatic polynomials, while factors δ force the endpoints to have the same value, so that, if some "δ-edge" is present into a component H of "$(1 - \delta)$-edges", it acts as a contraction operation on the endpoints.

In order to be more precise, we need to introduce some notation. For a graph G and a set $E' \subseteq E(G)$, define $G \setminus E'$ the graph $(V(G), E(G) \setminus E')$ (i.e. the result of *deletion* of edges in E'). Then, define $G \bullet E'$ the graph obtained by shrinking iteratively the endpoints of edges in E' into a single vertex, unless the edge is a loop, in this case the edge is just dropped from E' (i.e. $G \setminus E'$ is the result of the *contraction* of edges in E'). Say that V' is the induced vertex-set of E'. It is evident that if (V', E') has some cycles, and (V', E'') is a spanning forest on (V', E') with the same number of components, $G \bullet E' \equiv G \bullet E''$. To see this, you can just first shrink all the edges in E'', and then realize that only loops have been left to contract.

So, for a Potts model F-K expansion, we can imagine to decompose the factors on a set $E_1 \subseteq E(G)$ as $(1+v)-v(1-\delta)$, and the factors in the complementary set $E_2 = E(G) \setminus E_1$ as $(1 + v)\delta + (1 - \delta)$. Then we have an expansion over sets $E' \subseteq E_1$ and $E'' \subseteq E_2$, such that

$$Z(G;v;q) = \sum_s \sum_{\substack{E'\subseteq E_1\\ E''\subseteq E_2}} \prod_{(ij)\in E'} (1+v_{ij}) \prod_{(ij)\in E_1\smallsetminus E'} (-v_{ij}(1-\delta(s_i,s_j))) \prod_{(ij)\in E''} (1+v_{ij})\delta(s_i,s_j) \prod_{(ij)\in E_2\smallsetminus E''} (1-\delta(s_i,s_j)) \quad (5.44)$$

For each pair (E',E''), summation over s produces the chromatic polynomial on the graph $G(E',E'') := (G\bullet E'')\smallsetminus E'$. Remark that the number of vertices in $G(E',E'')$ is easily deduced to be $|V(G)|-|E''|+L(E'') = |V(G)| - \mathrm{rank}(E'')$ (where rank is in matroidal sense). So we have

$$Z(G;v;q) \;=\; \sum_{\substack{E'\subseteq E_1\\ E''\subseteq E_2}} \prod_{(ij)\in E'} (1+v_{ij}) \prod_{(ij)\in E_1\smallsetminus E'} (-v_{ij}) \prod_{(ij)\in E''} (1+v_{ij})\; P_{G(E',E'')}(q) \quad (5.45)$$

So, in the regime $q<0$ where the sign of the chromatic polynomial only depends on the parity of the number of vertices in the graph, the sign of each summand (E',E'') is given by the parity of

- the number of ferro couplings in $E_1\smallsetminus E'$,
- plus the number of unphysical couplings in E',
- plus the number of unphysical couplings in E'',
- plus the number of vertices in the graph,
- minus the cardinality of E'',
- plus the cyclomatic number of E''.

If all the couplings in E_1 are antiferro, all the couplings in E_2 are unphysical, and E_2 has zero cyclomatic number (this being *a fortiori* true on each subset E''), the sign is the same for any summand. As anticipated, this provides a probabilistic expansion on a graph with a mixture of antiferro and unphysical weights, such that the unphysical edges make no cycles, and at $q<0$.

It is now easy to reproduce the reasonings of Section 5.3, and express the partition function in terms of orientations, instead of evaluations of the chromatic polynomial.

Indeed, combining equations (5.17) and (5.30) we have that, for any pair (G,v) and analytically in q,

$$\begin{aligned} Z(G;v;q) &= \sum_{H\subseteq G} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E(G)\smallsetminus E(H)} (1+v_{ij}) \prod_\mu P_{H_\mu}(q) \\ &= (-1)^{|V(G)|} \sum_{H\subseteq G} (-q)^{k(H)} \prod_{(ij)\in E(H)} (-v_{ij}) \prod_{(ij)\in E_{\mathrm{int}}(H)} (1+v_{ij}) \prod_\mu A^{\circ}_{H_\mu} \end{aligned} \quad (5.46)$$

On the other side, if in (5.45) one performs first summation over E'', and interpreting edges with a factor $(1-\delta)$ as edges of a subgraph H, one gets

$$\begin{aligned} Z(G;v;q) = &\sum_{E''\subseteq E_2} \prod_{(ij)\in E''} (1+v_{ij}) \\ &\times \sum_{\substack{H\subseteq G\bullet E''\\ E(H)\supseteq E_2\smallsetminus E''}} \prod_{(ij)\in E(G\bullet E'')\smallsetminus E(H)} (1+v_{ij}) \prod_{(ij)\in E(H)\smallsetminus(E_2\smallsetminus E'')} (-v_{ij}) \prod_\mu P_{H_\mu}(q) \end{aligned} \quad (5.47)$$

The factor on the second line can be interpreted as a realization of equation (5.46) in the form on the first line, for the graph $G\bullet E''$ and with a set of weights $\hat v$ defined as $\hat v_{ij} = v_{ij}$

if $(ij) \in E_1$, and $\hat{v}_{ij} = -1$ if $(ij) \in E_2 \setminus E''$. So we can use the lemma, and recognize that for E_2 cycle-free $|V(G \bullet E'')| = |V(G)| - |E''|$, to get

$$Z(G;v;q) = (-1)^{|V(G)|} \sum_{E'' \subseteq E_2} \prod_{(ij)\in E''} (-1 - v_{ij}) \times \sum_{H \subseteq G} (-q)^{k(H)} \prod_{(ij)\in E(H)} (-\hat{v}_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+\hat{v}_{ij}) \prod_{\mu} A^{\bullet}_{H_\mu} \quad (5.48)$$

Note in particular that if an edge of $E_2 \setminus E''$ is in $E_{\rm int}(H)$, we get a factor $(1+\hat{v}) = 0$, while in the other two cases, of being in $E(H)$ or of having endpoints on different components of H, it just gives a factor 1 (because $-\hat{v} = 1$). So we can move this constraint on the definition of the appropriate ensemble of H's.

Furthermore, instead of working on $G \bullet E''$, we can work in G but understand the components of E'' as "super-vertices" at the aims of $A^{\bullet}_H$ (i.e. in checking that the orientation has a single source per component, in evaluating the size of a component, if we prefer A'_H/n to $A^{\bullet}_H$, and in determining that there are no oriented cycles).

The concept of super-vertices for all purposes should be evident to the reader: it just acts analogously to a single vertex after the contraction of edges in E''. We denote by $A^*_{G,E}$ the set of acyclic orientations on G where components of E are treated as super-vertices. So we can write

$$Z(G;v;q) = (-1)^{|V(G)|} \sum_{E'' \subseteq E_2} \prod_{(ij)\in E''} (-1 - v_{ij}) \times \sum_{\substack{H \subseteq G \\ E(H) \supseteq E'' \\ E_{\rm int}(H) \cap E_2 = \varnothing}} (-q)^{k(H)} \prod_{(ij)\in E(H)\cap E_1} (-v_{ij}) \prod_{(ij)\in E_{\rm int}(H)} (1+v_{ij}) \prod_{\mu} A^*_{H_\mu;E''} \quad (5.49)$$

Also after these manipulations, if edges in E_2 are unphysical, edges in E_1 are antiferro, and $q < 0$, we get at sight a definite sign for each configuration of local variables (E'', H). The advantage of this restatement is that, beyond providing a probabilistic expansion in the regime discussed above, also provides a set of local variables with computable Gibbs factors, which is thus viable for a Monte Carlo Markov Chain.

5.6 Observables in the antiferro F-K expansion

We want to understand how to relate the traditional cluster observables of the Random Cluster Model to the different family of clusters we have in the expression (5.30). We start with the first relevant observable,

$$\begin{aligned} \langle \gamma_{xy} \rangle &= \text{prob}(x \text{ and } y \text{ are in the same component in ferro F-K}) \\ &= \left\langle \frac{q\delta(s_x,s_y) - 1}{q-1} \right\rangle = \left\langle 1 - \frac{q}{q-1}(1 - \delta(s_x,s_y)) \right\rangle , \end{aligned} \quad (5.50)$$

where the first line is done in F-K formulation, and the second line in the original Potts "colouring" formulation. Through the Potts formulation, it is clear that the operator $\delta(s_x, s_y)$ can be seen as replacing the original graph G with the one obtained identifying x and y:

$$Z(G;v;q)\,\langle \gamma_{xy} \rangle = \frac{1}{q-1}\Big(qZ(G \bullet (xy);v;q) - Z(G;v;q) \Big) . \quad (5.51)$$

Actually, the whole operator γ_{xy} is related to adding to the graph an extra edge (xy) with an effective parameter $-v_{\rm eff}/(1+v_{\rm eff}) = -q/(q-1)$, i.e. $v_{\rm eff} = -q$. However the form of equation (5.51) is convenient at least in order to understand that $\langle\gamma_{xy}\rangle$ goes to zero at large distances if the clusters of H are small.

Indeed, consider the expansion as in (5.34), with a choice of $\hat{i}(V')$ such that $\hat{i}(V') = x$ if $x \in V'$ and $\hat{i}(V') = y$ if $y \in V'$ and $x \notin V'$, with a whatever prescription in the other case. Then, in the expansion for graph G, we have three subensembles for subgraphs H:

$\mathcal{H}_{(xy)}$: vertices x and y are in the same component H_0 of H; x is the sink;
$\mathcal{H}_{x|y}$: vertices x and y are in different components H_x and H_y of H, and these components are adjacent on G through the edge-set E' (i.e. edges in E' have one endpoint in H_x and one in H_y); x and y are the sink the resp. components;
$\mathcal{H}_{x||y}$: vertices x and y are in different components H_x and X_y of H, and these components are not adjacent on G; x and y are the sink the resp. components.

Summation over H in $Z(G)$ is then splitted into the three restricted sums. For each sum, as much as possible, we want to understand bijections among the configurations appearing in $Z(G)$ and the ones appearing in the $Z(G \bullet (xy))$, and hopefully write the correlation function γ_{xy} in terms of three observables:

$$\begin{aligned} Z(G;v;q)\,\langle\gamma_{xy}\rangle = &\sum_{\mathbf{H}\in\mathcal{H}_{(xy)}} W(\mathbf{H})\mathcal{O}_{(xy)}(\mathbf{H}) + \sum_{\mathbf{H}\in\mathcal{H}_{x|y}} W(\mathbf{H})\mathcal{O}_{x|y}(\mathbf{H}) \\ &+ \sum_{\mathbf{H}\in\mathcal{H}_{x||y}} W(\mathbf{H})\mathcal{O}_{x||y}(\mathbf{H})\,. \end{aligned} \tag{5.52}$$

A natural bijection among configurations is to consider $\mathbf{H}$ and $\mathbf{H} \bullet (xy)$, which indeed correspond to the same set of edge occupations/orientations, just with the difference that connectivity, the constraint of a single source per component and acyclicity must be understood in the adjacency pattern induced by G and by $G \bullet (xy)$.

We have to ask ourselves two questions: are there valid configurations $\mathbf{H}$ which are not valid as $\mathbf{H} \bullet (xy)$? are there valid configurations $\mathbf{H} \bullet (xy)$ which are not valid as $\mathbf{H}$?

Our choice of $\hat{i}(V')$ has been done in order to make these questions as easy as possible. First we work out the contribution of all valid $\mathbf{H}$, and of all the $\mathbf{H} \bullet (xy)$ induced by the contraction. At the end, we consider possible extra contributions coming from valid $\mathbf{H} \bullet (xy)$ which are not induced by the contraction.

For $\mathbf{H}$ in ensembles $\mathcal{H}_{x|y}$ and $\mathcal{H}_{x||y}$, both x and y are the sink of their component. These configurations are in natural bijection with $\mathbf{H} \bullet (xy)$, as in this case $x \equiv y$ is the only source of the component $H_x \cup H_y$, and there are no issues of acyclicity because no cycle can go through a sink. The number of components in the configuration $H \bullet (xy)$ is smaller by one than on H, so we have a relative factor $1/(-q)$. Then, the number of vertices in $G \bullet (xy)$ is also smaller by one than on G, this giving a factor (-1). Finally we have a relative factor q in the two generating functions appearing in combination (5.51).

The factors of $(-v)$'s are the same in $H \bullet (xy)$ and H. Furthermore, in the ensemble $\mathcal{H}_{x||y}$ also the factors of $(1+v)$'s are the same in $H \bullet (xy)$ and H, this producing an exact cancellation (or, in other words, $\mathcal{O}_{x||y}(\mathbf{H}) = 0$).

In the ensemble $\mathcal{H}_{x|y}$ instead we have a relative factor $\prod_{e\in E'}(1+v_e)$, which, up to some factors q, is the only thing than can not be reabsorbed into the Gibbs weight, so that we get

$$\mathcal{O}_{x|y}(\mathbf{H}) = \frac{1 - \prod_{e\in E'}(1+v_e)}{1-q} \tag{5.53}$$

which is positive in the antiferro region of v and $q < 0$.

In the ensemble $\mathcal{H}_{(xy)}$ we have a realization of the first of our questions: $H \bullet (xy)$ is never a valid configuration, because y is not a sink, so it has at least one outgoing edge, and because of lemma 5.1 this must cause a cycle after identifying x and y. So only H contributes, and we just get $\mathcal{O}_{(xy)}(\mathbf{H}) = 1/(1-q)$.

Now consider instead configurations $\mathbf{H}'$ valid on $G \bullet (xy)$. Again the component H_0 containing $x \equiv y$ has its only sink on x. What does it happen when we split x from y and go back to G? If x is a cutset of H_0, both x and y will be the single sink of their component. Otherwise the image of H_0 is still connected, but contains two sinks, so that the image of $\mathbf{H}'$ is not valid on G. This answers the second of our questions, and adds a further contribution to the expectation value.

Call $A_G^{x,y}$ the number of acyclic orientations of G with exactly two sinks, one located in x and one in y. Remark that, even if G is connected, this number can be zero, while this was not true for $A_G^{\cdot}$. Because of what we have seen, we have an extra summand in $\mathcal{O}_{(xy)}(\mathbf{H})$

$$\mathcal{O}_{(xy)}(\mathbf{H}) = \frac{1}{1-q} - \frac{q}{1-q}\frac{A_{H_0}^{x,y}}{A_{H_0}} \tag{5.54}$$

where the prefactor is due to the $-q/(q-1)$ in (5.51), and to the fact that $G \bullet (xy)$ has one less vertex, which changes the overall sign in the expansion.

So, complexively we can write

$$Z(G;v;q)\,\langle \gamma_{xy} \rangle = \frac{1}{1-q}\Big(\sum_{H \in \mathcal{H}_{(xy)}} W(H)\Big(1 - q\frac{A_{H_0}^{x,y}}{A_{H_0}}\Big) + \sum_{H \in \mathcal{H}_{x|y}} W(H)\Big(1 - \prod_{e \in E'}(1+v_e)\Big)\Big)\,. \tag{5.55}$$

The observable $A_{H_0}^{x,y}/A_{H_0}$ looks hard to simulate, and will probably require a Worm technique. Nonetheless, we should stress that $A_G^{x,y}$ has a simple "deletion-contraction" formula, namely

$$A_G^{x,y} = A_{G \cup (xy)} - A_G\,, \tag{5.56}$$

as can be seen by a simple argument: any acyclic orientation on $G \cup (xy)$, say with single sink in x, is still acyclic if the arrow **(yx)** is removed, but vertex y could have become a sink or not, so this suggests the combination $A_{G\cup(xy)} = A_G + A_G^{x,y}$. In order to see that it is true, we have to check that, given any acyclic orientation in the ensembles corresponding respectively to A_G and $A_G^{x,y}$, adding the edge **(yx)** gives a valid orientation in $G \cup (xy)$. This is true because the only thing which could lead to reject the configuration is the production of a cycle, but as we are adding an arrow pointing towards the sink, no cycles can use this edge.

Two remarks are in order. First realize that for v antiferro and q negative all the contributions have positive sign, so there are no subtle cancellations among the terms of different nature, at most some of them are subleading. Furthermore, we want to stress that, although the clusters in this expansion has a different nature w.r.t. the ones in ordinary F-K, there is some residual of the interpretation of $\langle \gamma_{xy} \rangle$ in terms of cluster connectivities, in particular from the fact that $\mathcal{O}_{x||y} = 0$.

Of course, in the forest limit $q, v \to 0$ at v/q fixed, also the case in which H_x and H_y are adjacent does not contribute, because the extra factors $(1+v)$ converge to 1. Then, the $1/(1-q)$ in the prefactor goes to 1, and $q/(1-q)$ goes to zero, so we recover the fact that $\langle \gamma_{xy} \rangle$ coincides with cluster connectivity, as expected by the symmetry $(q, v) \to (-q, -v)$ of this point of the phase diagram.

Remark that, although we have proven (5.55) in the formulation of (5.34) and with a specific prescription on $\hat{i}(V')$, the result is independent from this, so it must be valid for any formulation, e.g. for (5.35) and for (5.32).

The only serious complicancy in the expression for $\langle \gamma_{xy} \rangle$ seems to be the appearence of the new counting object $A_G^{x,y}$, for two-sink acyclic orientations. It is a natural question of what will happen in more complicated expectation values, such as $\langle \gamma_{x_1 x_2 \ldots x_k} \rangle$. It is not hard to guess that countings of up to k-sink acyclic orientations, $A_G^{x_1,\ldots,x_k}$, will be required. The true question is: is that all, or we will need some extra, more complicated, enumeration? The answer is, unfortunately, negative, as one could show by analysing the three-point function. However, we do not perform this analysis here.

6

The phase diagram of Potts Model

In this chapter we briefly discuss some aspects of the phase diagram of the Potts model, in regular D-dimensional geometries, with special emphasis to the neighbourhood of the spanning-forest regime, for which we make a conjecture on the general-D behaviour of the RG flow.

There exist two convenient parametrization for the description of the phase diagram. The one most used in the physics community uses parameters q and $v = e^J - 1$. Its main advantages are the fact that the RG flow follows simple vertical lines, and that the ferromagnetic, antiferromagnetic and unphysical regions are simple horizontal strips. Its drawbacks are the fact that, in two dimensions, planar duality is not a simple symmetry of the diagram, and that the double limit $q \to 0$ and $v \to 0$, which presents different interesting regimes, collapses here to a single point, and thus requires the introduction of a "zoomed inset" for illustrating the situation in this case.

Another parametrization, mostly used in the mathematical literature on the Tutte polynomial, makes use of q/v and v. The x and y variables in the Tutte polynomial, and in the weights for loop and isthmus graphs which are central to the Tutte-Grothendrieck invariant description of the polynomial, correspond to this parametrization up to an unitary shift. Advantages and drawbacks are essentially exchanged w.r.t. the previous parametrization. The main advantages are the simple description, in two dimensions, of planar duality, as a symmetry of reflection along the diagonal; then, the $q \to 0$ limit is fully described by the union of the two axes: spanning forests and spanning connected subgraphs are described by the horizontal and vertical axis respectively, while spanning trees are at the intersection, i.e. at the origin. Its drawbacks are the fact that the RG flow, and, for a given dimensionality D, the limit value of q for having a second-order transition, are now hyperbolas. Still, the ferromagnetic, antiferromagnetic and unphysical regions are simple horizontal strips.

6.1 Potts phase diagram in $D = 2$

The phase diagram of the model in two dimensions is known exactly from the work of Baxter, though methods of Integrable Systems [67] (see also [85]), in the case of the square lattice. The picture is expected to be universal at least in the probabilistic ferromagnetic region, of both q and v positive. The antiferromagnetic region is expected to have less universal properties. Its characteristics are however further elucidated, as for example in [86, 87, 88].

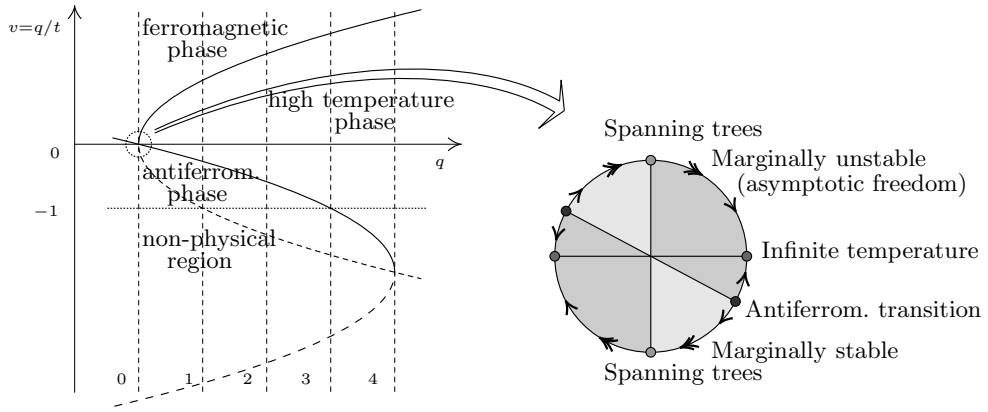


Fig. 6.1. Phase diagram of the Potts Model in the (q, v) plane (where $v = e^J - 1$), for the infinite square lattice, as deduced from Baxter solution. The curves are exact parabolas, with symmetry axis parallel to the x-axis, vertices in $(0, 0)$ and $(4, 2)$ respectively, and each parabola crossing the vertex of the other one. At any fixed $0 < q < 4$, there are six critical points: two trivial ones, at $v = 0$ and $v = \infty$, correspond to infinite and zero temperature respectively, are attractive for the RG flow; for $0 < q \leq 3$, two 'physical' transitions, for $v > 0$ finite and $-1 < v < 0$, corresponding respectively to the ferromagnetic and anti-ferromagnetic order/disorder phase transition (if $3 < q < 4$, the antiferro transition point falls into the unphysical region). These two transitions are repulsive for the RG flow, as always happens for order/disorder transitions w.r.t. the thermal parameter. The other two branches of the parabolas correspond to unphysical trasitions, in a region where we do not have a local probabilistic interpretation (except for the special point $(q, v) = (1, -1)$, cfr. Theorem 5.2): among these, the unphysical branch of the ferromagnetic parabola is attractive for the RG flow.
The picture in the $q = 0$ case becomes singular, and has to be analyzed with more care, as there are different limits. This is done, for the $q \to 0$ and $v \to 0$ limit, in the inset on the right, describing thus the restriction of the phase diagram to the case of spanning forests. A full parametrization of this case is covered by any half-circle, as the circle has an invariance under inversion. We have three fixed points, of which the one corresponding to spanning trees is a "double root" of the beta function of the model, and presents a marginal flow, as discussed all along the body of the text.

The robust (universal) characteristics of the diagram, valid for any regular two-dimensional graph, should be the fact that the "ferromagnetic parabola" turns into a curve, which is convex as a function $q(v)$ and has a minimum in the point $(q, v) = (0, 0)$, and the "antiferromagnetic parabola" is a curve crossing this point with negative slope, and having a maximum as a function $q(v)$ for $q = 4$. The slope of the antiferromagnetic parabola in the triangular lattice has been approached in [85, 89] (respectively, with a numerical approach of transfer matrix on a strip, and an analytic approach based on RG). Further informations on the phase diagram can be extracted from the exact solution on random planar graphs [90, 91] and KPZ correspondence. Similarly, we can extract informations on the spanning-forest portion of the diagram through the relation with the $O(n)$ model at $n = -1$, and the exact solution of $O(n)$ on random planar graphs [26, 92, 93, 94, 27], and again KPZ correspondence. However we do not discuss here the solution of statistical mechanics models on Random Planar Graphs.

We also omit here a discussion on the special values of q, named *Behara numbers*, for which the field content of the model has peculiar characteristics, as this is not specially concerned with our main aim of discussing the neighbourhood of the spanning-forest limit

in the phase diagram. Further details on the phase diagram are given in the caption of figure 6.1.

We also give a picture of the phase diagram in the Tutte parametrization, with arrows describing the RG flow, for comparison with the results at other dimensionalities (cfr. figure 6.2).

6.2 Potts phase diagram in $D = 1$

The Potts Model partition function, in F-K form, is trivially summed over on a 1-dimensional closed chain, and we do the exercise here only in order to make a comparison of the resulting phase diagram with the informations we have at other dimensionalities.

Remark that all the configurations except $H = G$ have $L = 0$, so that we can treat separately this contribution, and for the rest use the factorized form for Potts on a tree

$$Z_\ell(q, v) = (v + q)^\ell + (q - 1)v^\ell \tag{6.1}$$

(the last term is the correction). In the limit $q \to 0$, the choice $\rho = 0$ gives

$$Z_\ell(\lambda, \rho = 0, w) = \frac{(w + \lambda)^\ell - w^\ell}{\lambda} \tag{6.2}$$

while $\lambda = 0$ gives

$$Z_\ell(\lambda = 0, \rho, w) = \ell w^{\ell-1} + \rho w^\ell \tag{6.3}$$

and the double limit gives

$$Z_\ell(\lambda = \rho = 0, w) = \ell w^{\ell-1} . \tag{6.4}$$

Not only the partition function is summed exactly (this property being shared, for example, also by a strip of finite width, through transfer matrix technique), but more relevant to our purposes, the RG flow is exact on this subsystem. This means that the length-ℓ chain with parameters v and q has a partition function which *coincides* with the one of a chain of a shorter length ℓ', up to a free-energy shift, and a modification of the parameter v into some v'. This is just the goal of Renormalization Group, which on harder systems can be performed only in an approximated way, this making things complicated. As we will see, the study of the full RG flow boils down to the analysis of a simple iterated map.

Say that $\ell/\ell' = 1 + \epsilon$, with ϵ positive. Matching the two partition functions up to a free-energy shift gives

$$\frac{(v' + q)^{\ell'}}{(v + q)^\ell} = \frac{(v')^{\ell'}}{v^\ell} , \tag{6.5}$$

that is

$$v' = \frac{q v^{1+\epsilon}}{(q + v)^{1+\epsilon} - v^{1+\epsilon}} . \tag{6.6}$$

Three fixed points exist for each value of q and ϵ, namely $v = 0$, $v = -q/2$ and $v = -q$. Of course, $v \to \infty$ must be checked separately, and it is easily verified that this limit is repulsive, i.e. we recover the well-known fact that there is no other fixed point besides the infinite-temperature one ($v = 0$) in the ferromagnetic regime.

Again we prefer to work in coordinates v and q/v. In the variable q/v the fixed points are 0, -1, -2 and $(\pm)\infty$, of which -1 and ∞ are attractive, 0 and -2 repulsive, and the flow in the full plane is the only possible one compatible with the statements above, and the fact that it is constrained on the lines of constant q.

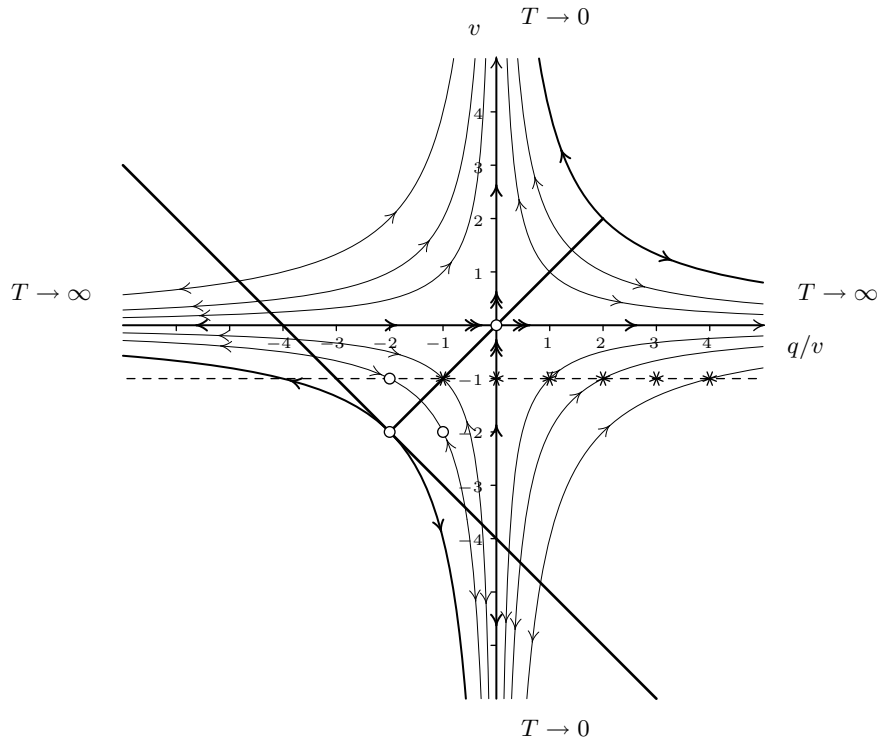


Fig. 6.2. Phase diagram for Potts model on the 2-dimensional square lattice, after Baxter solution, in the plane $(q/v, v)$. Duality corresponds to reflection w.r.t. the main diagonal. The predictions are restricted to the range $q \leq 4$, where the criticalities are all of second order. The bold straight lines correspond to the reparametrization of Baxter parabolas (in (q, v) plane, cfr. figure 6.1), where the model is exactly solvable by integrability. The hyperbola $q = 1$, corresponding to percolation, and the points $(0, 0)$ (corresponding to spanning trees), $(-2, -1)$ (2-colouring), $(-2, -2)$ and $(-1, -2)$ (two other special points with a convoluted combinatorial description) are exactly solvable on any graph (these points are denoted with bullets). The hyperbola $q = 2$, corresponding to Ising, is exactly solvable on any planar graph. All the other points are worst-case #P-complete, as understood by Jaeger, Vertigan and Welsh [81]. The positive horizontal and vertical axes are respectively spanning forests and connected subgraphs.
The dashed line is the boundary of the negative-q antiferro regime (towards unphysical), where the probabilistic Antiferro F-K is available. Asterisks corresponds to enumerations of acyclic orientations on the whole graph (no sum over subgraphs), the first two via Greene-Zaslavsy, the others via Stanley. Integer-valued points on the left of $(-2, -1)$ correspond instead to "traditional" colouring problems, for $q \geq 3$ integer. Single-source–single-sink cyclic orientations ($q = 1$) are expected to show critical behaviour, as they are on the pseudo-physical branch of Baxter parabolas, while 3-colouring is expected to be critical, because on the antiferro branch of Baxter parabolas. Thin lines highlight the RG flow of the system, which is constrained on curves of constant q (we draw $q = (\pm)1, 2, 4$). Remark the marginality of the flow near to $(0, 0)$.

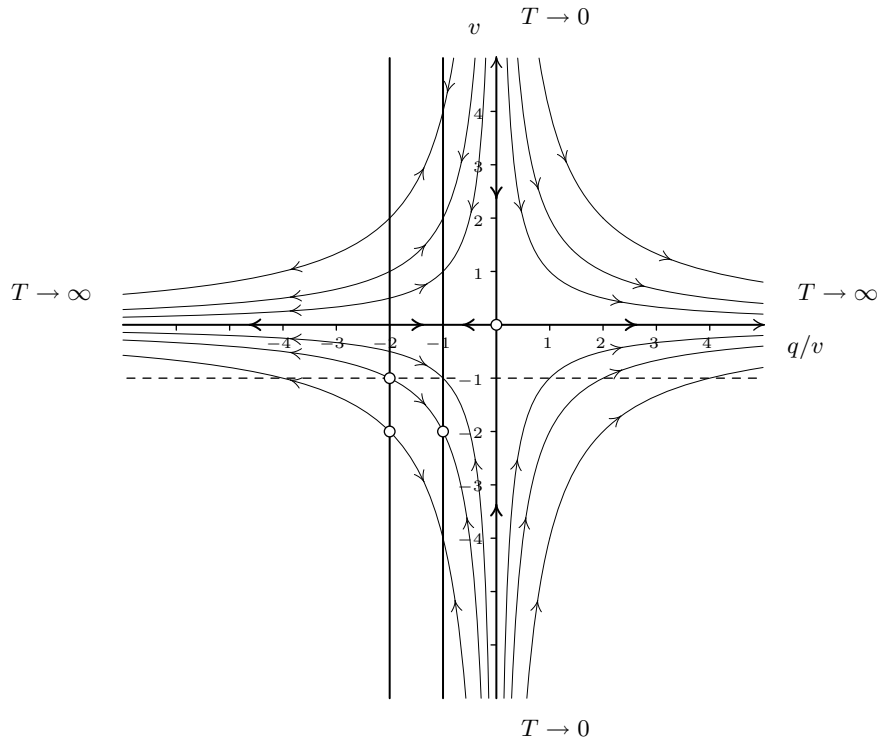


Fig. 6.3. Phase diagram for the Potts model on the 1-dimensional chain. The point $(0, 0)$ corresponds to spanning trees, and the positive horizontal and vertical axes are respectively spanning forests and connected subgraphs. The dashed line is the boundary of the antiferro region.
Thin lines highlight the RG flow of the system, which is constrained on curves of constant q (we draw $q = (\pm)1, 2, 4$).

Some further analysis is required for the possible $q \to 0$ limits. For $\rho = 0$, it suffices to remark that the simultaneous scaling of w and λ leads to (the $q \to 0$ limit of) the general case. For $\lambda = 0$ we get $w' = w/(1 + \epsilon)$, which flows from $w = \pm\infty$ towards $w = 0$, always relevantly. For the trees, the absence of more than a single monomial leaves any scale unfixed, and we can understand this limit as a further fixed point. Summing up, we get the phase diagram in figure 6.3.

6.3 Potts phase diagram in infinite dimension

Here we give the finite-degree mean-field solution of Potts Model, in Fortuin-Kasteleyn representation (for which the mean-field approximation is well-controlled at least in the whole probabilistic sector, at difference with the colour-variable formulation). According to the general theory of Critical Phenomena, the universal set of the results should be interpreted as describing the model in Euclidean D-dimensional lattices, for D sufficiently large (namely, above the so-called *upper critical dimension*). With abuse of language, this case is often called as the description of a model at infinite dimension.

We use the Bethe-Peierls mean-field expansion, a refined version of bare mean-field which accounts for Onsager "cavity corrections", and is exact on tree graphs [95]. We

take the assumption that criticality is of percolative nature, for the Random Cluster description, and that, in the percolative phase, there is a single gigantic cluster. Such a picture is quite general and well understood for problems of this kind (the mean-field fixed-degree assumption is optimally described by graphs of Erdös-Renyi type, for which probabilistic analysis are available, cfr. for example [96]).

So we consider a graph G, and a Random Cluster model on it, with subgraphs $S \subseteq G$. In agreement with the cavity idea, we define the probability p for the following event: that chosen at random an edge (ij) of the graph, and one of its two terminations (say, i), the vertex i is connected to the gigantic component of the subgraph S (if any) in the restriction to $G \setminus (ij)$.

If G has fixed degree $k+1$, with $k \geq 2$, and can be considered approximatively (oriented-)edge-transitive, then this probability satisfies a simple approximated self-consistent relation (with corrections, due to the average effect of the loops in the graph, algebraically depressed with a negative power of the size N, except that at criticality). We choose a Random Cluster parametrization with $\lambda = 1$ and $\rho = q$, and weight w on the edges. W.r.t. the case $\lambda = q$ and $\rho = 1$, with edge-weight v, we have $v = w\rho$.

Consider the k vertices ℓ_α adjacent to i in $G \setminus (ij)$. If any two of them are not in the gigantic component, then they are almost surely in distinct components (again up corrections algebraically depressed with N), so, in the 2^k choices for the vertices being in the gigantic component or not, in $G \setminus \{(ij), (i\ell_1), \dots, (i\ell_k)\}$ (determined still by p up to size corrections), and the 2^k choices for the edges $(i\ell_\alpha)$ being in S or not, we can easily handle the only non-locality in the weight, being the number of factors ρ: we produce cycles only through multiple occupancies of edges incident on vertices in the gigantic component. We thus get

$$\begin{aligned} \frac{p}{1-p} &= \frac{\sum_{h=0}^{k} \binom{k}{h} p^h (1-p)^{k-h} \frac{1}{\rho}\big((1+\rho w)^h - 1\big)(1+w)^{k-h}}{\sum_{h=0}^{k} \binom{k}{h} p^h (1-p)^{k-h} (1+w)^{k-h}} \\ &= \frac{\big(p(1+\rho w) + (1-p)(1+w)\big)^k - \big(p + (1-p)(1+w)\big)^k}{\rho \left(p + (1-p)(1+w)\right)^k} \\ &= \frac{1}{\rho}\left[\left(1 + \rho \frac{pw}{1+(1-p)w}\right)^k - 1\right]. \end{aligned} \tag{6.7}$$

The minimal non-trivial case $k = 2$ leads to the simplest equations. We get as possible solutions

$$p = 0\,; \qquad\qquad p = \frac{2 - \rho - \sqrt{(1-\rho)4/w^2 + \rho^2}}{2 - 2\rho}\,; \tag{6.8}$$

(we dropped another solution, with opposite sign in the square root, because it leads at sight to the unfeasible fact that $p > 1$, when real). In the plane q/v vs. v, as chosen in Section 6.2 (i.e. $1/w$ vs. $w\rho$, in these notations), the resulting diagram is symmetric under rotations of π (because the solution depends on w only through w^2, and the rotation corresponds to change sign in w but not in ρ). We identify three qualitative regions: one in which the discriminant Δ is negative, and $p = 0$ is the only real solution; one in which a real solution for $p \neq 0$ exists, but gives $p < 0$, and is thus unacceptable, and one in which $p \in [0, 1]$, and a gigantic component exists (this can be seen either by stability arguments, or by an analysis of the resulting free-energy in the two corresponding trial points in the Landau-Ginzburg functional).

The percolation transition is of second order, if going from $p < 0$ to $p \in [0, 1]$, or of first order, if going from $\Delta < 0$ to $p \in [0, 1]$. We have first order if and only if $q > 2$ (compare with the fact in dimension 2, for which we have first order if $q > 4$). The transition lines are $q/v = \pm 1$, with $v < 2$ and $v > -2$ in the two cases, for the second-order line, and $q/v = \frac{4+v^2}{4v}$, for $v > 2$ and $v < -2$, for the first-order line. Remark that, of the two second-order endpoints $(1, 2)$ and $(-1, -2)$, the negative one is a special point in the Jaeger-Vertigan-Welsh classification.

The RG flow is uniquely determined from this set of transition lines, in combination with the general fact that the flow runs on curves of constant q (hyperbolas, in our parametrization), and that zero and infinite temperatures, corresponding to the four directions along the axes, are attractive fixed points. The resulting flow is described in figure 6.4. In particular, the origin, corresponding to spanning trees, is attractive along the horizontal direction (spanning forests), and repulsive along the vertical direction (maximally-connected spanning subgraphs), contrarily to what happens in the analysis for $D = 1$.

6.4 A conjecture on the flow near the Spanning Tree point

Ou analysis of these three exactly solved cases, together with the numerical analysis of [97], suggests a definite picture concerning the flow of the Renormalization Group near to the origin in $(q/v, v)$ plane, i.e. near to the fixed point of Spanning Trees.

We conjecture the following:

- For $D < 2$, the flow runs along hyperbolas in the four quadrants, the axes being the separatrix of the flows. It is *repulsive* along forests, and *attractive* along connected subgraphs, so it is attractive along the q/v axis;
- For $D > 2$, the flow runs along hyperbolas in the four quadrants, the axes being the separatrix of the flows. It is *attractive* along forests, and *repulsive* along connected subgraphs, so it is attractive along the v axis;
- For $D = 2$, a critical line crosses the Spanning Tree point, staying in the quadrants with positive q. This line is *repulsive* for $v > 0$ and *attractive* for $v < 0$. In these quadrants, the flow is on hyperbolas, and is attracted to or repelled by the asymptotes in both directions, according to the behaviour described above. In the other two quadrants, the flow is on hyperbolas, and is attracted towards "north-east". The axes are still separatrices of the flow, with the Spanning Tree point being marginal, and marginally repulsive and attractive respectively for positive and negative semi-axes.

This picture is illustrated in Figure 6.5.

There is another suggestion, of theoretical nature, towards our conjecture. This comes from the Propp-Wilson algorithm for generating spanning trees uniformly [98], and already at the level of Pemantle theorem [99]. These results connect the ensemble of spanning trees with loop-erased random walks in the graph. In particular, given two points x and y on a vertex-transitive weighted graph, the probability that on a random spanning tree from the ensemble the path connecting x to y is γ coincides with its measure in the ensemble of loop-erased random walks, started in x and stopped the first time hitting y. A similar theorem exists for non–vertex-transitive graphs, having a non-uniform sum of incident weights, but a certain asymmetric reweightening of the transition rates in the diffusion kernel has to be considered.

It is well known that random walks on regular Euclidean lattices change structurally between $D \leq 2$ and $D > 2$ (see for example [46, chapt. 1]). Indeed, for $D \leq 2$, the random walk is *recurrent*, that is, in the infinite space $\mathbb{Z}^D$, with probability 1 it comes back to

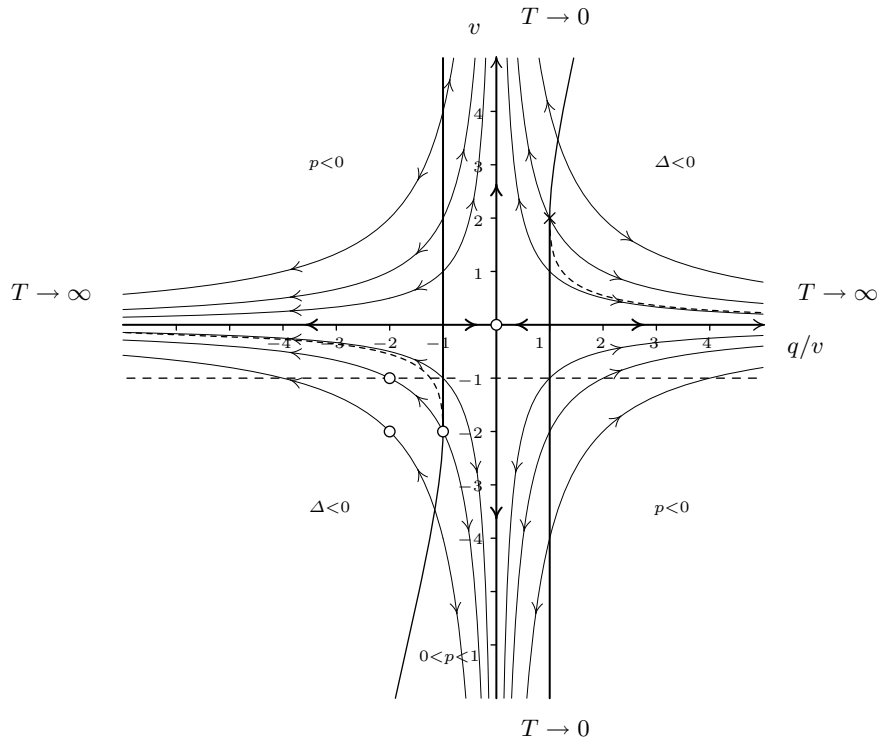


Fig. 6.4. Phase diagram for the Potts model on degree-3 random graphs, which should correspond to the limit of infinite dimensionality of Euclidean models. The point $(0, 0)$ corresponds to spanning trees, and the positive horizontal and vertical axes are respectively spanning forests and connected subgraphs. The dashed line is the boundary of the antiferro region. Thin lines highlight the RG flow of the system, which is constrained on curves of constant q (we draw $q = (\pm)1, 2, 4$).
Up to rotation symmetry, there are three regions, in which respectively there exists a percolative phase with a gigantic component ($0 < p < 1$), there is no valid solution for p, because the self-consistent equation gives $p < 0$, and there is no valid solution for p, because the self-consistent equation has a negative discriminant $\Delta = (1 - \rho)4/w^2 + \rho^2$. Beyond $q = 2$ the phase transition becomes of first order (as, on the transition, $\mathfrak{Im}(p) \to 0$ with $\mathfrak{Re}(p) > 0$), while below $q = 2$ it is of second order (as, on the transition, $\mathfrak{Re}(p) \to 0$).

the starting point infinitely many times, while for $D > 2$ it is *transient*, that is, with probability 1 it comes back to the starting point only a finite number of times, before "escaping to infinity". As a corollary, for $D \leq 2$, a random walk starting from any point $r \neq 0$ will certainly hit 0 at some time, while for $D > 2$, a random walk starting from any point $r \neq 0$ will have a non-zero probability of "escaping to infinity" without having ever hitten 0 (this handwaving picture has a well-understood rigorous formulation).

As always happens in statistical mechanics, an uniform sampler for a model at some value t^* of some thermodynamic parameter t provides a sampler for the same model at $t = t^* + \delta t$, which is inefficient as $\sim \exp(-\text{const.} N \delta t)$ w.r.t. the original sampler (here N is the size of the system). So, for a whole region of the phase diagram of Potts model infinitely near to the point of spanning trees, the statistical properties of the Propp-Wilson algorithm encode the typical behaviour of the configurations.

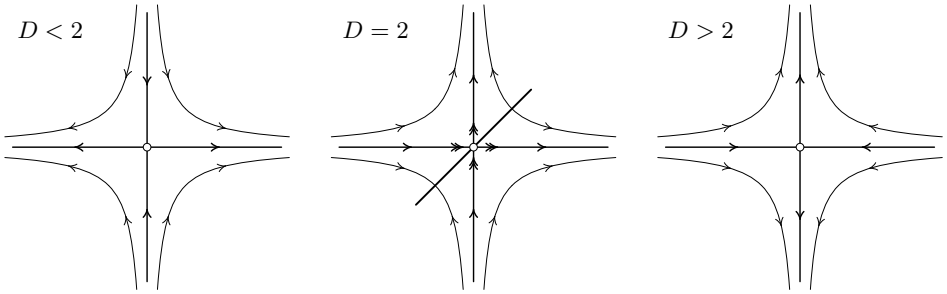


Fig. 6.5. Description of the conjectural RG flow for Potts Model near the point of Spanning Trees, at any dimension. Left, middle and right describe respectively $0 < D < 2$, $D = 2$ and $D > 2$.

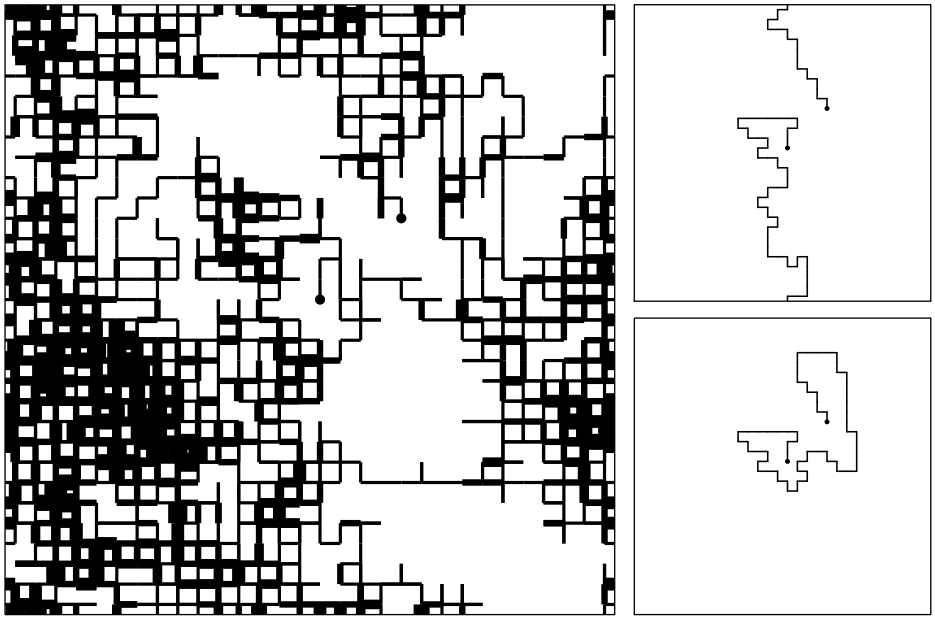


Fig. 6.6. On the left, a random walk on a square lattice of side 30 and periodic boundary conditions, with endpoints at relative distance $(4, 4)$. On the right, on top the associated loop-erased random walk, on bottom, the loop-erased random walk obtained by removing last the cycles with non-trivial winding. In the random walk, edges traversed more than once are drawn with proportionally enhanced thickness.

Our aim is to motivate, through this sort of reasonings, that a typical forest with a number of components $o(N)$ has almost surely a gigantic component for $D > 2$, and does not for $D < 2$. Such a claim would combine with the idea that, if there exists a transition for spanning forests at a finite real positive value of the parameter, this is of percolative nature (that is, going towards the $t = \infty$ point corresponding to the atomic forest, at

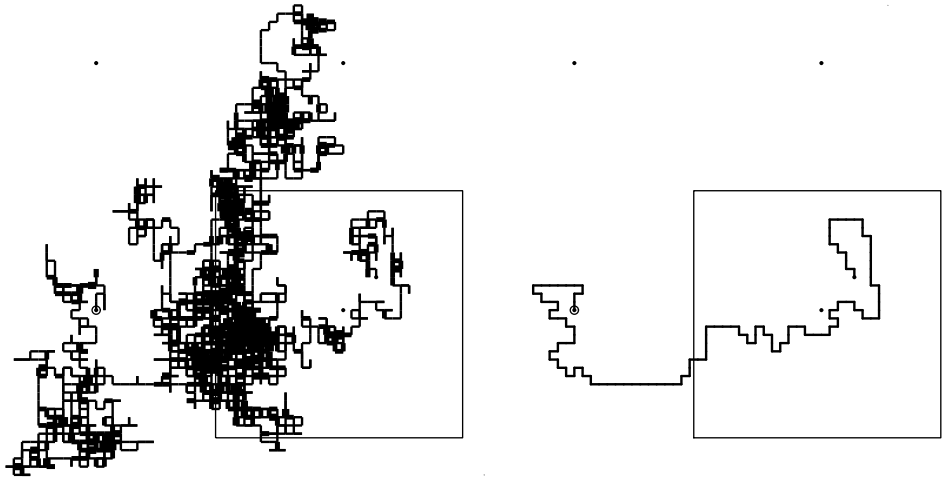


Fig. 6.7. On the left, the same random walk as in figure 6.6, but drawn on the whole $\mathbb{Z}^2$. Multiple images of the arrival point are shown, and the one which is hit by the path is marked with a circle. On the right, the resulting loop-erasure on $\mathbb{Z}^2$. In the random walk, edges traversed more than once are drawn with proportionally enhanced thickness.

some critical value t^* the gigantic component disappears). The two claims together would strongly suggest the picture we have given above, on the RG flow of the model.

Consider working on a finite large lattice, with all sides of length L (say, the origin is in the middle) and periodic boundary conditions, and consider a random walk starting at some point r, and arrested the first time it hits the origin. Assume $|r| \ll L$. Then, if and only if the random walk is transient, also the loop-erased random walk (arrested at the origin) is transient, as the position of the endpoint is independent from the loop-erasure procedure. A recurrent random walk would *a fortiori* imply a recurrent LERW, and thus r is almost surely connected to the origin through a path having zero winding number and a length of order $|r|^\gamma$ (γ being some fractal dimension of the LERW in dimension D, being $\frac{5}{4}$ for $D = 2$), so that the correction to the configuration required in order to generate a forest "infinitesimally near to a tree" would break this path with a probability related to this length, with some proportionality, and all the components will have similar sizes.

A transient random walk would instead imply a non-zero probability that the LERW connecting r to the origin has non-zero winding number and a length of order L^γ. This class of sites has a totally different probability of being disconnected from the origin (much higher, as it is somewhat proportional to L). As a result, comparatively, given the goal number of connected components in the algorithm producing an uniform spanning forest from an uniform spanning tree, there will be many small components (many nearby points are easily disconnected) and a single gigantic one (the remaining of the original tree, after the other small components have been cut off).

All of this argument is handwaving. For example, it is not even clear why, if a random walk has finite probability of having a certain winding number, also its loop erasure should have a non-zero probability. It is however possible that an expert probabilist could cook up a proof out of a similar collection of arguments, and part of the answer may be already contained in [100, 101, 102].

We just conclude this discussion by illustrating further our ideas, with an example of random walk in the plane, starting at r and absorbed at all points in $L\mathbb{Z} \times L\mathbb{Z}$ (figure 6.6), together with its loop-erasure, in the two cases in which the walk is seen as in the whole plane, or on the torus, though a quotient (figure 6.7).

7

Uniform spanning trees

We review here the basic facts about Kirchhoff Theorem, and deduce a first set of non-trivial informations on the related model of statistical mechanics, corresponding to spanning trees on a weighted graph (or, more simply for an unweighted graph, uniform spanning trees).

In particular, we study the distribution of the coordination number in the ensemble of spanning trees of a given graph. In the case of hypercubic lattices with periodic boundary conditions, we provide the explicit solution both for the probability distribution on a single site, and for generic connected n-point correlation functions.

The leading behaviour of such quantities is already well known in the literature, and obtained, surprisingly, from a detour on the Abelian Sandpile Model, by Dhar and collaborators. Anyhow, the formulas at the basis of these quantities are implicitly determined by a distinct well-known fact, that edge occupations in uniform spanning trees make a discrete determinantal process (cfr. for example [108]). Such a property will implicitly appear in our treatment.

For what concerns the probability distribution, in the infinite volume limit, but also in the finite volume in dimensions less than 3, the solution is expressed in terms of only one lattice integral. Interestingly enough, in two dimensions this integral allows an expansion in terms of modular forms. A class of modular properties of Eisenstein Series is reduced to symmetry properties of the problem. To our knowledge, the study of this peculiar property is new. Our aim is also to relate this subtle aspect to the results of Section 7.5, concerning forests of unicyclics in toroidal geometries, where similar features emerge, but this time at the leading order.

For what concerns connected correlators, we obtain a formula in terms of the free propagators, connecting the points in a single cycle, summed over possible cycles. No subtraction from unconnected correlators is necessary. The asymptotic behaviour for large distances is obtained, and recognized to have the properties of conformal correlators, both in dimension two, and in generic dimension for 2-point and 3-point functions.

7.1 Statistical averages in the ensemble of Uniform Spanning Trees

Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E. Call $\mathcal{S}$ the set of spanning subgraphs of G, i.e. the set of subgraphs $S = (V, E')$ with $E' \subseteq E$. Call $\mathcal{T} \subseteq \mathcal{S}$ the subset of spanning subgraphs which are connected and contain no cycles. Remark that $\mathcal{T} \neq \varnothing$ iff G is connected. Since here on we will restrict to this case.

Associate to each edge e a weight w_e. For $i \neq j$, let $w_{ij} = w_{ji}$ be the sum of w_e over all edges e that connect i to j. The (weighted) Laplacian matrix L for the graph G is then defined by $L_{ij} = -w_{ij}$ for $i \neq j$, and $L_{ii} = \sum_{k \neq i} w_{ik}$. This is a symmetric matrix with all row and column sums equal to zero. Since L annihilates the vector with all entries 1, its determinant is zero. If all weights are strictly positive, also the remaining part of the spectrum is strictly positive [3, pag. 280].

The Matrix-tree Theorem states that the determinant of L restricted to the space orthogonal to the zero mode coincides with the generating polynomial $Z(G)$ of spanning trees in G, up to a trivial factor:

Theorem 7.1 (Krichhoff Matrix-Tree Theorem). *Let G be a connected weighted graph with V vertices, and w_e the indeterminate (weight) associated to edge e. Then*

$$Z_{\mathrm{Tree}}(G; w) := \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} w_e = \frac{1}{|V|} \det{}' L \,, \tag{7.1}$$

where the sum runs over all spanning trees T in G and $\det'$ denotes the determinant in the space orthogonal to the constant function, i.e. the product of the non-zero eigenvalues of L.

We do not prove this theorem, as various proof of much more general statements appear in Chapter 8. In the following we denote simply by $Z(G)$ the generating function $Z_{\mathrm{Tree}}(G; w)$.

The Matrix-tree theorem, combined to standard generating-function techniques, or lemmas in linear algebra concerning Schur complements, allows to determine a large number of statistical properties for spanning trees, where average is intended with the measure

$$\mu(T) = \frac{1}{Z(G)} \prod_{e \in T} w_e \,. \tag{7.2}$$

We will illustrate this in a number of cases for "local" observables. At this aim, we introduce an algebra of local operators, inspired by combinatorial quantities: for each spanning tree $T \in \mathcal{T}$ and each edge $e \in E(G)$, define $\theta_e(T)$ as the quantity valued 1 if $e \in E(T)$ and 0 otherwise, and the complementary $\overline{\theta}_e(T) = 1 - \theta_e(T)$; analogously, for each vertex $i \in V(G)$ define $c_i(T)$ the coordination of i over T, i.e. $c_i(T) = c$ if c edges of T is adjacent to vertex i. Clearly, calling $E(i) \subset E(G)$ the set of edges in G adjacent to i, this new quantity is a simple combination of the previous ones,

$$c_i(T) = \sum_{e \in E(i)} \theta_e(T) \,. \tag{7.3}$$

Then, we introduce a name for the statistical averages of generic monomials in the algebra of θ_e's,

$$q_{\{e_1,\dots,e_n\}} = \langle \theta_{e_1} \cdots \theta_{e_n} \rangle \;; \tag{7.4}$$

$$q^{\mathrm{conn}}_{\{e_1,\dots,e_n\}} = \langle \theta_{e_1} \cdots \theta_{e_n} \rangle^{\mathrm{conn}} \;; \tag{7.5}$$

where superscript "conn" stands for *connected*, i.e. with all subtractions understood, e.g. for the two-point and three-point functions

$$q^{\mathrm{conn}}_{\{e_1,e_2\}} = q_{\{e_1,e_2\}} - q_{\{e_1\}} q_{\{e_2\}} \;; \tag{7.6}$$

$$\begin{aligned} q^{\mathrm{conn}}_{\{e_1,e_2,e_3\}} = {} & q_{\{e_1,e_2,e_3\}} - q_{\{e_1\}} q_{\{e_2,e_3\}} - q_{\{e_2\}} q_{\{e_1,e_3\}} - q_{\{e_3\}} q_{\{e_1,e_2\}} \\ & + 2\, q_{\{e_1\}} q_{\{e_2\}} q_{\{e_3\}} \;; \end{aligned} \tag{7.7}$$

and the general formula is expressed in terms of a sum over set-partitions.† Clearly, as $|E(T)| = |V(T)| - 1 = |V(G)| - 1$ for all spanning trees, in the unweighted case, if G is edge-transitive (i.e. all edges are equivalent) the one-point functions are trivial

$$q_{\{e\}} = \frac{|V(G)| - 1}{|E(G)|} \qquad \forall\ e \in E\,. \tag{7.9}$$

We don't need to introduce also names for the statistical averages of generic monomials in the algebra of c_i's, as they are trivially related to the q_S's above

$$\langle c_1 \cdots c_n \rangle = \sum_{\substack{e_1,\ldots,e_n \\ e_a \in E(i_a)}} \langle \theta_{e_1} \cdots \theta_{e_n} \rangle\ ; \tag{7.10a}$$

$$\langle c_1 \cdots c_n \rangle^{\mathrm{conn}} = \sum_{\substack{e_1,\ldots,e_n \\ e_a \in E(i_a)}} \langle \theta_{e_1} \cdots \theta_{e_n} \rangle^{\mathrm{conn}}\ . \tag{7.10b}$$

On the other side, the joint probability distribution for the *exact values* of the coordinations are not trivially reduced to the algebra of θ_e, especially if we have arbitrarily large degrees in the original graph G‡. So, it is thus worth defining the quantities

$$p_i(c) = \langle \delta_{c_i,c} \rangle = \mathrm{prob}(c_i = c)\,; \tag{7.12}$$

$$p_{i,j}(c,c') = \left\langle \delta_{c_i,c}\delta_{c_j,c'} \right\rangle = \mathrm{prob}(c_i = c,\ c_j = c')\,; \tag{7.13}$$

$$p^{\mathrm{conn}}_{i,j}(c,c') = \left\langle \delta_{c_i,c}\delta_{c_j,c'} \right\rangle^{\mathrm{conn}} = p_{i,j}(c,c') - p_i(c)p_j(c')\,; \tag{7.14}$$

and so on for joint probability distributions of coordinations of a generic number of sites. Again, the coordination correlation functions can be stated as combinations of these quantities

$$\langle c_1 \cdots c_n \rangle^{\mathrm{conn}} = \sum_{k_1,\ldots,k_n} k_1 \cdots k_n\ p^{\mathrm{conn}}_{\{i_1,\ldots,i_n\}}(k_1,\ldots,k_n)\,. \tag{7.15}$$

Our aim is to evaluate these statistical averages. The key observation to the formal solution of this problem is that, even if we are interested only in the uniform case, using the stronger multivariate formulation of the Matrix-Tree Theorem allows us to introduce all the required multipliers for having the generating function for a set of expectation values. And also that, if the obervables are located at k points, with k of order 1, even for arbitrary lattice size N, the perturbation to the Laplacian matrix due to the introduction of the multipliers is of low rank (of order k, independent from N).

†Call $P \in \Pi(S)$ a partition of set S into sets $S_1, \ldots, S_{|P|}$, the general formula is

$$q_S^{\mathrm{conn}} = \sum_{P \in \Pi(S)} (-1)^{|P|-1}(|P|-1)! \prod_{i=1}^{|P|} q_{S_i}\,. \tag{7.8}$$

‡Of course, we have for a vertex i of degree d in G

$$\langle \delta_{c_i,c} \rangle = \sum_{\substack{A \subseteq E(i) \\ |A| = c}} \left\langle \prod_{e \in A} \theta_e \prod_{e \in E(i) \smallsetminus A} (1 - \theta_e) \right\rangle = \sum_{A \subseteq E(i)} \binom{|A|}{c} (-1)^{|A|-c} \left\langle \prod_{e \in A} \theta_e \right\rangle\,, \tag{7.11}$$

but, contrarily to equations 7.10, this is not restating a k-point function as a linear combination of k-point functions, but as a combination of up to (dk)-point functions.

More explicitly, if we perturb the Laplacian matrix modifying the weights w_e into $w_e(1+u_e)$ with $u_e \neq 0$ on a certain subset E' of the edges, the measure (7.2) is modified with an extra factor per occupied edge

$$\mu'(T) \propto \prod_{e \in T} w_e(1+u_e)\,. \tag{7.16}$$

Calling $L + \delta L^{(\mathrm{edge})}$ the new Laplacian matrix, we have

$$\frac{\det'(L+\delta L^{(\mathrm{edge})})}{\det' L} = \Big\langle \prod_{e \in E'} (1+\theta_e u_e) \Big\rangle = \sum_{E'' \subseteq E'} \Big(\prod_{e \in E''} u_e \Big) q_{E''}\,. \tag{7.17}$$

In a similar way we can obtain the joint probability distributions for site coordinations. If one perturbes the Laplacian matrix modifying the weights w_{ij} into $(1+t_i)w_{ij}(1+t_j)$, with $t_i \neq 0$ on a certain subset I of the vertices, trees with coordination c_i on i are weighted with an extra factor $(1+t_i)^{c_i}$, and the measure (7.2) is modified into

$$\mu'(T) \propto \Big(\prod_{e \in T} w_e \Big) \Big(\prod_i (1+t_i)^{c_i} \Big)\,. \tag{7.18}$$

Calling $L + \delta L^{(\mathrm{site})}$ the new Laplacian matrix, we have

$$\frac{\det'(L+\delta L^{(\mathrm{site})})}{\det' L} = \Big\langle \prod_{i \in I} (1+t_i)^{c_i} \Big\rangle = \sum_{\{c_i\}_{i \in I}} \mathrm{prob}(\{c_i\}_{i \in I}) \prod_{i \in I} (1+t_i)^{c_i}\,. \tag{7.19}$$

In particular, when the parameter t_i is non-null for a single i, we have

$$\frac{\det'(L+\delta L^{(\mathrm{site})})}{\det' L} = \sum_c p_i(c)(1+t_i)^c\,, \tag{7.20}$$

while for t_i and t_j non-zero we have

$$\frac{\det'(L+\delta L^{(\mathrm{site})})}{\det' L} = \sum_{c,c'} p_{i,j}(c,c')(1+t_i)^c (1+t_j)^{c'}\,. \tag{7.21}$$

In a similar fashion, we can obtain expressions for a generic joint probability of site coordinations.

In all these cases, the matrices $L + \delta L$ and L are Laplacian, with a zero-mode in correspondence with the constant vector. Call L_0 and δL_0 the restrictions to the orthogonal space: then L_0 is invertible, and we have

$$\frac{\det'(L+\delta L)}{\det' L} = \frac{\det(L_0 + \delta L_0)}{\det L_0} = \det(I + {L_0}^{-1} \delta L_0) = \exp \mathrm{tr} \ln(I + {L_0}^{-1} \delta L_0)\,. \tag{7.22}$$

Say $N = |V|$ is the number of vertices and U is an unitary matrix which diagonalizes L, such that the first new base-vector is the zero-mode, i.e.

$$U L U^{-1} = \mathrm{diag}(0, \lambda_1, \dots, \lambda_{N-1})\,; \qquad \lambda_p > 0 \quad \text{for } p > 0; \tag{7.23}$$

$$\delta L_{p,q} := U(\delta L) U^{-1}\,, \qquad \delta L_{p,q} = 0 \quad \text{if } p = 0 \text{ or } q = 0. \tag{7.24}$$

In these notations we can write

$$\exp \operatorname{tr} \ln(I + {L_0}^{-1}\delta L_0) = \exp \sum_{\ell=1}^{\infty} \frac{(-1)^{1+\ell}}{\ell} Q_\ell \,, \tag{7.25}$$

with

$$Q_\ell = \operatorname{tr}({L_0}^{-1}\delta L_0)^\ell = \sum_{\substack{(p_1,\ldots,p_\ell) \\ \in\{1,\ldots,N-1\}^\ell}} \prod_i \frac{\delta L_{p_i,p_{i+1}}}{\lambda_{p_i}} \,. \tag{7.26}$$

Remark the omission of $p_i = 0$ in the summation.

Besides the generality of this mathematical tool, the statistical mechanics interest is on *fixed* observables, for a family of lattices of size N, in the thermodynamic limit $N \to \infty$. Suppose that we recognize that, for a given observable, the matrix $\delta L_{p,q}$ has a rank r independently from N. Then we could restate $\delta L_{p,q}$ as a sum of factorized contributions, i.e.

$$\delta L_{p,q} = \sum_{k=1}^{r} f_k(p) g_k(q) \,. \tag{7.27}$$

Then, we can interchange summations in the expansion of Q_ℓ

$$Q_\ell = \sum_{(p_1,\ldots,p_\ell)} \sum_{(k_1,\ldots,k_\ell)} \prod_i \frac{f_{k_i}(p_i) g_{k_i}(p_{i+1})}{\lambda_{p_i}} = \sum_{(k_1,\ldots,k_\ell)} \prod_i \sum_{p\in\{1,\ldots,N-1\}} \frac{g_{k_{i-1}}(p) f_{k_i}(p)}{\lambda_p} \,, \tag{7.28}$$

and defining the quantities

$$M_{hk} = \sum_{p\in\{1,\ldots,N-1\}} \frac{f_h(p) g_k(p)}{\lambda_p} \tag{7.29}$$

we have $Q_\ell = \operatorname{tr}(M^T)^\ell = \operatorname{tr} M^\ell$, as traces of matrices of dimension r, at all lattice sizes, each entry of the matrix being a specific lattice sum (involving $\mathcal{O}(N)$ summands), and then

$$\frac{\det'(L + \delta L)}{\det' L} = \exp \sum_{\ell=1}^{\infty} \frac{(-1)^{1+\ell}}{\ell} Q_\ell = \det(I + M) \,. \tag{7.30}$$

Expressions like (7.29) are in general "good candidates" in order to perform a thermodynamic limit, as the sums converge to integrals, over the spectrum of eigenvalues of L. Furthermore, this sort of integrals often leads to power series in the inverse-size parameter $1/N$, which even allow to systematically control finite-size corrections. when this procedure is understood, expression (7.30) is also a power series in $1/N$, the leading term being the infinite-size limit of the quantity of interest.

This program can be fulfilled in particular in the case of periodic hypercubic lattices. For a d-dimensional hypercubic lattice of size $N = L_1 L_2 \cdots L_d$ with periodic boundary conditions, the unitary matrix $U_{p,x}$ is the Fourier kernel

$$U_{p,x} = \prod_{k=1}^{d} L_k^{-\frac{1}{2}} e^{i p_k x_k} = N^{-\frac{1}{2}} e^{i p\cdot x} \,. \tag{7.31}$$

Variables $p_k = \frac{2\pi n_k}{L_k}$ with $n_k \in \{0, 1, \cdots, L_k - 1\}$ are the momenta on the lattice, $p \cdot x$ is the scalar product, and the zero-mode corresponds to $p = 0$. In the case of $w_{ij} = 1$ for i, j first-neighbours, and 0 otherwise, the eigenvalues of L are

$$\lambda_p = \sum_{i=1}^{d}(2-2\cos p_i) = \sum_{i=1}^{d} 4\sin^2\left(\frac{p_i}{2}\right) = \widehat{p}^{\,2} \tag{7.32}$$

where we introduced the standard notation $\widehat{p}_i = 2\sin\left(\frac{p_i}{2}\right)$ and $\widehat{p}^{\,2} = \sum_i \widehat{p}_i^{\,2}$. Summations over $p \in \{1,\dots,N-1\}$, appearing in the previous section, will be simply denoted as $\int_p$ since here on, and will be understood that at finite volume

$$\int_p = \frac{1}{N}\sum_{n_1=0}^{L_1-1}\cdots\sum_{n_d=0}^{L_d-1} \qquad (n_1,\cdots,n_d) \neq (0,\cdots,0) \tag{7.33}$$

while at infinite volume

$$\int_p = \int_{-\pi}^{\pi}\frac{\mathrm{d}p_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{\mathrm{d}p_d}{2\pi} = \int\frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\,. \tag{7.34}$$

For what concerns the calculation of expectation values of edge-occupation operators, when non-zero perturbation parameters $u_1,\dots,u_n$ are introduced in correspondence of edges $e_1,\dots,e_n$, the perturbation of the Laplacian Matrix takes the form

$$\delta L_{x,y} = \sum_{a=1}^{n} u_a\,\delta L_{x,y}(e_a)\,. \tag{7.35}$$

If the two endpoints of edge e are $\mathbf{r}$ and $\mathbf{r}+\widehat{\eta}_\alpha$, adopt the more versatile notation

$$e \equiv (\mathbf{r},\alpha,+1) \equiv (\mathbf{r}+\widehat{\eta}_\alpha,\alpha,-1)\,. \tag{7.36}$$

The function $\delta L_{x,y}(e) = \delta L_{x,y}(\mathbf{r},\alpha,\tau)$, for $1\le\alpha\le D$ and $\tau=\pm1$, is given by

$$\delta L_{x,y}(\mathbf{r},\alpha,\tau) = \begin{cases} 1 & x=y\,;\ \ x,y\in\{\mathbf{r},\mathbf{r}+\tau\widehat{\eta}_\alpha\}\,; \\ -1 & x\neq y\,;\ \ x,y\in\{\mathbf{r},\mathbf{r}+\tau\widehat{\eta}_\alpha\}\,; \\ 0 & \text{otherwise.}\end{cases} \tag{7.37}$$

It is easily found for $\delta L_{p,q}$

$$\delta L_{p,q}(\mathbf{r},\alpha,\tau) = \frac{1}{N}\left(1-e^{i\tau p_\alpha}\right)\left(1-e^{-i\tau q_\alpha}\right)e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}}\,; \tag{7.38}$$

$$\delta L_{p,q} = \sum_{a=1}^{n} u_a \delta L_{p,q}(\mathbf{r}_a,\alpha_a,\tau_a)\,. \tag{7.39}$$

We have that $\delta L_{p,q}$ is in the general form (7.27), with n summands. According to the general result of equations (7.17) and (7.30), the correlation functions of edge-occupation are restated as the determinant of a matrix $I+uM$ of dimension n:

$$\left\langle\prod_e(1+\theta_e u_e)\right\rangle = \det(I+uM)\,; \tag{7.40}$$

$$u_{ab} = u_a\delta_{ab}\,, \qquad M_{ab} = \int_p\left(1-e^{i\tau_a p_{\alpha_a}}\right)\left(1-e^{-i\tau_b p_{\alpha_b}}\right)\frac{e^{i\mathbf{p}\cdot(\mathbf{r}_a-\mathbf{r}_b)}}{\widehat{p}^{\,2}}\,. \tag{7.41}$$

As a specialization of equation (7.17), $\langle\overline{\theta}_1\cdots\overline{\theta}_n\rangle$ is given by the determinant of $I+uM$ with all $u_a=-1$, and $\langle\theta_1\cdots\theta_n\rangle$ is given by the coefficient of $\det(I+uM)$ linear in all u_a:

$$\left\langle \overline{\theta}_1 \cdots \overline{\theta}_n \right\rangle = \det(I+uM)|_{u_a=-1} = \det(I-M)\,; \tag{7.42}$$

$$\langle \theta_1 \cdots \theta_n \rangle = \frac{\partial^n}{\partial u_1 \cdots \partial u_n} \det(I+uM)\Big|_{u_a=0} = \det M\,; \tag{7.43}$$

and, more generally,

$$\left\langle \overline{\theta}_1 \cdots \overline{\theta}_m \theta_{m+1} \cdots \theta_n \right\rangle = (-1)^m \det M'\,; \tag{7.44}$$

$$M'_{ab} = \begin{cases} M_{ab} - 1 & a = b \le m\,; \\ M_{ab} & \text{otherwise.} \end{cases} \tag{7.45}$$

For what concerns the calculation of site-coordination joint probability distributions, when non-zero perturbation parameters $t_1, \ldots, t_n$ are introduced in correspondence of vertices $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$, the perturbation of the Laplacian Matrix takes the form

$$\delta L_{x,y} = \sum_{v=1}^{n} t_v\, \delta L_{x,y}(\mathbf{r}_v) \tag{7.46}$$

with the function $\delta L_{x,y}(\mathbf{r}_v)$ given by

$$\delta L_{x,y}(\mathbf{r}) = \sum_{\substack{\alpha=1,\ldots,d \\ \tau=\pm 1}} \delta L_{x,y}(\mathbf{r},\alpha,\tau) = \begin{cases} 2d & x = y = \mathbf{r}\,; \\ 1 & x - \mathbf{r} = \pm\widehat{\eta}_\alpha\,;\ \ x = y\,; \\ -1 & x - y = \pm\widehat{\eta}_\alpha\,;\ \ x = \mathbf{r} \text{ or } y = \mathbf{r}\,; \\ 0 & \text{otherwise.} \end{cases} \tag{7.47}$$

This proves immediately that $\delta L_{p,q}$ is in the general form (7.27), with $2dn$ summands, although a small algebraic manipulation shows out to be fruitful. Introducing the definitions

$$f_+(x) = \cos\left(\tfrac{x}{2}\right)\,; \qquad\qquad f_-(x) = \sin\left(\tfrac{x}{2}\right)\,; \tag{7.48}$$

one has

$$\sum_{\tau=\pm 1} (1 - e^{i\tau p_\alpha})(1 - e^{-i\tau q_\alpha}) = \frac{1}{2}\widehat{p}_\alpha \widehat{q}_\alpha \cos(p_\alpha - q_\alpha) = \frac{1}{2} \sum_{\sigma=\pm 1} \widehat{p}_\alpha \widehat{q}_\alpha f_\sigma(p_\alpha) f_\sigma(q_\alpha)\,. \tag{7.49}$$

Within these two parametrizations, the full expression for the perturbed Laplacian reads

$$\delta L_{p,q} = \frac{1}{N} \sum_{v=1}^{n} \sum_{\substack{\alpha=1,\ldots,d \\ \tau=\pm 1}} t_v \left((1 - e^{i\tau p_\alpha}) e^{i\mathbf{p}\cdot\mathbf{r}_v}\right) \left((1 - e^{-i\tau q_\alpha}) e^{-i\mathbf{q}\cdot\mathbf{r}_v}\right) \tag{7.50a}$$

$$= \frac{1}{N} \sum_{v=1}^{n} \sum_{\substack{\alpha=1,\ldots,d \\ \sigma=\pm 1}} 2t_v \left(\widehat{p}_\alpha f_\sigma(p_\alpha) e^{i\mathbf{p}\cdot\mathbf{r}_v}\right) \left(\widehat{q}_\alpha f_\sigma(q_\alpha) e^{-i\mathbf{q}\cdot\mathbf{r}_v}\right)\,. \tag{7.50b}$$

According to the general result of equations (7.19) and (7.30), the correlation functions of site-coordinations are restated as the determinant of a matrix $I + tM$ of dimension $2dn$

$$\left\langle \prod_v (1+t_v)^{c_v} \right\rangle = \det(I + tM)\,; \tag{7.51}$$

where again the matrix t is diagonal, and valued t_v in correspondence of an entry $2d(v-1) < h \leq 2dv$. The elements of M in the two parametrizations (7.50) are defined as

$$M_{kh} =: M_{\alpha,\tau,v;\alpha',\tau',v'} = \int_p \left(1 - e^{i\tau_a p_{\alpha_a}}\right)\left(1 - e^{-i\tau_b p_{\alpha_b}}\right) \frac{e^{i\mathbf{p}\cdot(\mathbf{r}_a - \mathbf{r}_b)}}{\widehat{p}^2}\,; \tag{7.52a}$$

$$M_{kh} =: M_{\alpha,\sigma,v;\alpha',\sigma',v'} = 2\int_p \frac{1}{\widehat{p}^2}\left(\widehat{p}_\alpha f_\sigma\left(p_\alpha\right) e^{i\mathbf{p}\cdot\mathbf{r}}\right)\left(\widehat{p}_{\alpha'} f_{\sigma'}\left(p_{\alpha'}\right) e^{-i\mathbf{p}\cdot\mathbf{r}'}\right)\,. \tag{7.52b}$$

7.2 Coordination distribution for finite-dimensional lattices

Here we analyse the "1-point" expectations $p_i(c)$, in various physical circumstances (mainly in hypercubic lattices in arbitrary dimension D). Although we motivated the fact that the observables θ_e are somewhat more fundamental than the $\delta_{c_i,c}$, we start with the latter because, as we already said, in most concrete applications the 1-point function associated to θ_e is trivial in the ensemble of spanning trees. An isolated exception, of an interesting physical system with a non-trivial 1-point function $\langle\theta_e\rangle$, is the case of a hypercubic lattice in dimension D, an anisotropic weights along the D directions. We will also analyse this problem in the following.

So, we start by concentrating on a hypercubic lattice in dimension D, with uniform weights, and try to evaluate the probability distribution for the coordination of the site at the origin. For simplicity, we shall denote by t the only non-vanishing perturbation parameter $t_{\mathbf{0}}$. The perturbation of the Laplacian reads

$$\delta L_{x,y} = t\,\delta L_{x,y}(\mathbf{0}) = \begin{cases} 2dt & x = y = 0\,; \\ t & x = y\,; \quad |x| = 1\,; \\ -t & |x-y| = 1\,; \quad x = 0 \text{ or } y = 0\,; \\ 0 & \text{otherwise.} \end{cases} \tag{7.53}$$

We find for $\delta L_{p,q}$

$$\delta L_{p,q} = \frac{t}{N}\left(\widehat{p}^2 + \widehat{q}^2 - \widehat{p-q}^{\,2}\right) = \frac{2t}{N}\sum_{\alpha=1}^{d}\sum_{\sigma=\pm1} \widehat{p}_\alpha f_\sigma(p_\alpha)\,\widehat{q}_\alpha f_\sigma(q_\alpha) \tag{7.54}$$

According to the general result of equations (7.30), and using the definition (7.52b), the generating function for vertex coordination is restated as the determinant of a matrix $I + M$ of dimension $2d$, a generic element of M being

$$M_{kh} =: M_{\alpha,\sigma;\alpha',\sigma'} = 2t\int_p \frac{1}{\widehat{p}^2}\widehat{p}_\alpha\widehat{p}_{\alpha'} f_\sigma\left(p_\alpha\right) f_{\sigma'}\left(p_{\alpha'}\right)\,. \tag{7.55}$$

As elements of $I + M$ are of the form $\delta_{hk} + t\,m_{hk}$, the generating function for coordination is a polynomial of degree $2d$, which is indeed the maximum allowed coordination.

In particular we have for $\alpha = \alpha'$

$$\frac{1}{2t}M_{\alpha,+;\alpha,+} = \int_p \frac{4\sin^2\left(\frac{p_\alpha}{2}\right)\cos^2\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} = \int_p \frac{\widehat{p}_\alpha^2}{\widehat{p}^2}\left(1 - \frac{\widehat{p}_\alpha^2}{4}\right) =: B_d(\alpha) \tag{7.56}$$

$$\frac{1}{2t}M_{\alpha,+;\alpha,-} = \int_p \frac{4\sin^3\left(\frac{p_\alpha}{2}\right)\cos\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} = 0 \tag{7.57}$$

$$\frac{1}{2t}M_{\alpha,-;\alpha,-} = \int_p \frac{4\sin^4\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} = \frac{1}{4}\int_p \frac{\widehat{p}_\alpha^4}{\widehat{p}^2} =: A_d(\alpha) \tag{7.58}$$

while for $\alpha \neq \alpha'$

$$\frac{1}{2t} M_{\alpha,+;\alpha',+} = \int_p \frac{4 \sin\left(\frac{p_\alpha}{2}\right) \cos\left(\frac{p_\alpha}{2}\right) \sin\left(\frac{p_{\alpha'}}{2}\right) \cos\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} = 0 \tag{7.59}$$

$$\frac{1}{2t} M_{\alpha,+;\alpha',-} = \int_p \frac{4 \sin\left(\frac{p_\alpha}{2}\right) \cos\left(\frac{p_\alpha}{2}\right) \sin^2\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} = 0 \tag{7.60}$$

$$\frac{1}{2t} M_{\alpha,-;\alpha',-} = \int_p \frac{4 \sin^2\left(\frac{p_\alpha}{2}\right) \sin^2\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} = \frac{1}{4} \int_p \frac{\widehat{p}_\alpha^2 \widehat{p}_{\alpha'}^2}{\widehat{p}^2} =: C_d(\alpha, \alpha') \tag{7.61}$$

Special combinations of these quantities, via a cancellation of the $\widehat{p}^2$ denominator, lead to relations to which we will give a combinatorial interpretation:

$$1 - \frac{1}{N} = \int_p 1 = \int_p \frac{\sum_\alpha \widehat{p}_\alpha^2}{\widehat{p}^2} = \sum_{\alpha=1}^{d} \left(A_d(\alpha) + B_d(\alpha) \right) ; \tag{7.62a}$$

$$\frac{1}{2} = \frac{1}{4} \int_p \widehat{p}_\alpha^2 = \frac{1}{4} \int_p \frac{\widehat{p}_\alpha^2 \left(\sum_{\alpha'} \widehat{p}_{\alpha'}^2 \right)^2}{\widehat{p}^2} = A_d(\alpha) + \sum_{\alpha' \neq \alpha} C_d(\alpha, \alpha') \, . \tag{7.62b}$$

The relations above have a certain number of implications, and show simplifications in specific cases. First remark that, as a consequence of (7.62b), the vector $\mathbf{w}_{\alpha,\sigma} = \delta_{\sigma,-1}$ is an eigenvector of M with eigenvalue $1/2$. This fact implies that $I + 2tM$ has an eigenvalue $1 + t$, and this, at the level of the generating function, implies that $p(0) = 0$, as was to be argued for spanning trees on any non-trivial lattice.

Furthermore, we know that for a tree of N vertices $\overline{c} = 2E(T)/V(T) = 2 - 2/N$ (where $\bar{\cdot}$ stands here for an average over the sites), and, as each spanning tree has the same number of vertices of the whole lattice, $\overline{c} = 2 - 2/N$ identically for each tree in the ensemble, and then also on the statistical average. This fact extends immediately also for the coordination at the origin, for our graphs, that are vertex-transitive, and in this form it emerges also from our formulas: indeed, remark that

$$\begin{aligned} \langle c \rangle &= \sum_c c \, p(c) = \frac{\partial}{\partial t} \sum_c (1+t)^c p(c) \Big|_{t=0} = \frac{\partial}{\partial t} \det(I + 2tM) \Big|_{t=0} \\ &= \frac{\partial}{\partial t} \exp \operatorname{tr} \ln(I + 2tM) \Big|_{t=0} = \operatorname{tr} \frac{2M}{I + 2tM} \exp \operatorname{tr} \ln(I + 2tM) \Big|_{t=0} \\ &= 2 \operatorname{tr} M = 2 \sum_\alpha \Big(A_d(\alpha) + B_d(\alpha) \Big) = 2 - \frac{2}{N} \, , \end{aligned} \tag{7.63}$$

where last equation is a consequence of equation (7.62a). Remark how we recovered the relation not only at the leading order, but at all orders in $1/N$.

A further simplification occurs when all L_i are equal (or, possibly, all of them are sent to infinity, in this case read the results as $\frac{1}{N} \equiv 0$). Then, the quantities A_d, B_d and C_d become independent from their arguments $1 \leq \alpha \leq D$, and equations (7.62) reduce to

$$1 - \frac{1}{N} = d(A_d + B_d) \, ; \tag{7.64a}$$

$$\frac{1}{2} = A_d + (d-1) C_d \, . \tag{7.64b}$$

For the remaining part of this paragraph, we shall restrict to this last case. So the matrix M has the form

$$M = 2t \left(\begin{array}{ccc|ccc} B & 0 & 0 \cdots & & & \\ 0 & B & 0 & & & \\ 0 & 0 & B & & 0 & \\ \vdots & & \ddots & & & \\ \hline & & & A & C & C \cdots \\ & & & C & A & C \\ & 0 & & C & C & A \\ & & & \vdots & & \ddots \end{array} \right) \tag{7.65}$$

and we are interested in $\det(I+M)$. The spectum of M has d eigenvalues $2tB_d$, $d-1$ eigenvalues $2t(A_d - C_d)$ and one eigenvalue $2t(A_d + (d-1)C_d) = t$, so we have

$$\frac{\det'(L+\delta L)}{\det' L} = (1+t)(1+2tB_d)^d (1+2t(A_d-C_d))^{d-1}$$

which, compared with the initial expression (7.20) gives

$$\sum_c t^{c-1} p(c) = (1+2(t-1)B_d)^d (1+2(t-1)(A_d-C_d))^{d-1} \,. \tag{7.66}$$

Curiously, at the light of (7.64), only one independent parameter, stated as a lattice integral, determines the d values for the probabilities $p(c)$, and thus its determination is equivalent to the determination of any non-trivial expectation of coordinate probability. For example, for the infinite d-dimensional system we have

$$4d(d-1)\left\langle 5 - c^2 \right\rangle = \frac{1}{d}(dA_d - 1)^2 + \frac{1}{d-1}(dA_d - \tfrac{1}{2})^2 \,, \tag{7.67}$$

as can be seen through the determination of the only relevant average, $\langle c(c-1) \rangle$, by differentiating the generating function twice, and setting $t \to 1$. Remark that the RHS is manifestly positive, so that this equation proves that $\left\langle c^2 \right\rangle < 5$ for the infinite hypercubic lattice in any physical dimension.

This result should be seen as yet another manifestation of the fact that results for large dimensionality converge to the results for the "mean field" situation of the complete graph. Indeed, for $\mathcal{K}_n$ it is easy to show, by direct diagonalization of the Laplacian matrix, even if perturbed, that

$$\sum_c p(c) t^c = t \left(\frac{n-1+t}{n} \right)^{n-2} \xrightarrow[n\to\infty]{} t e^{t-1} \tag{7.68}$$

from which it is easy to derive $\left\langle c^2 \right\rangle = 5$,[††] and even at finite size

$$\left\langle c^2 \right\rangle = 5 \left(1 - \frac{6}{5n} \right) \left(1 - \frac{1}{n} \right) \tag{7.69}$$

exactly.

It is a natural conjecture to state that 5 is an upper limit for $\left\langle c^2 \right\rangle$ in any vertex-transitive connected graph, even weighted (but with weights preserving the vertex-transitiveness), the optimal at size n being realized by the complete graph. This would be a very tight bound, as the minimum value of $\left\langle c^2 \right\rangle$ for graphs of size n is $4 - \mathcal{O}(1/n)$, and is attained, for example, among vertex-transitive graphs, by large cycles, a cycle of length n giving $\left\langle c^2 \right\rangle = 4 - \frac{6}{n}$.

[††] More generally, it is easy to see that, for our Poissonian limit distribution, and any $s \geq 1$, $\langle (c-1)(c-2)\cdots(c-s) \rangle = 1$, which easily gives any fixed momentum.

7.2.1 The case of anisotropic weights

The whole technique illustrated above can be easily extended to the case of anisotropic weights, i.e. to the case in which $w_{xy} = v_\alpha$ if x and y are neighbouring sites along direction α, and 0 otherwise. The eigenvalues of the unperturbed Laplacian change into

$$\lambda_p = \sum_{i=1}^{d} v_i \widehat{p}_i^2 \,, \tag{7.70}$$

and we will denote $\lambda_p = \widehat{p}^2$ also in this generalized case, with abuse of notation, in order to keep a stronger analogy between the formulas in the isotropic and anisotropic cases. The perturbation of the Laplacian for the coordination at point r modifies into

$$\delta L_{p,q}(\mathbf{r}) = \frac{2}{N} \sum_{\alpha=1}^{d} 4 v_\alpha \sin\left(\frac{p_\alpha}{2}\right) \sin\left(\frac{q_\alpha}{2}\right) \cos\left(\frac{p_\alpha - q_\alpha}{2}\right) e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}} \,, \tag{7.71}$$

while the expressions for the matrix elements $M_{\alpha,\sigma;\alpha',\sigma'}$ change by a factor $\sqrt{v_\alpha v_{\alpha'}}$. Of relations (7.62), only relation (7.62b) is modified

$$\frac{1}{2} = A_d(\alpha) + \sum_{\alpha' \neq \alpha} \sqrt{\frac{v_{\alpha'}}{v_\alpha}} C_d(\alpha, \alpha') \,, \tag{7.72}$$

and this leads to a different eigenvector $\mathbf{w}_{\alpha,\sigma} = \sqrt{v_\alpha}\delta_{\sigma,-1}$, but always with eigenvalue $1/2$ as required by consistency with the combinatorial fact $p(0) = 0$.

In this case, the expectation $\langle \theta_{\mathbf{r},\alpha,+} \rangle$ depends in a sensible way from α. We get

$$\langle \theta_{\mathbf{r},\alpha,+} \rangle = \int_p \frac{v_\alpha \widehat{p}_\alpha^2}{\sum_\beta v_\beta \widehat{p}_\beta^2} \,. \tag{7.73}$$

The result is particularly simple in two dimensions, where we get

$$\langle \theta_{\mathbf{r},x,+} \rangle = \frac{2}{\pi} \arctan \frac{v_x}{v_y} \,. \tag{7.74}$$

7.2.2 The case $d = 1$

As a first application let us consider the trivial case $d = 1$. Here for every volume

$$A_1 = \frac{1}{2} \,; \qquad B_1 = \frac{1}{2}\left(1 - \frac{2}{N}\right) \,; \qquad C_1 = 0 \,; \tag{7.75}$$

therefore

$$\sum_c t^{c-1} p(c) = \frac{2}{N} + t\left(1 - \frac{2}{N}\right) \tag{7.76}$$

so that

$$p(1) = \frac{2}{N} \,; \qquad p(2) = 1 - \frac{2}{N} \,. \tag{7.77}$$

Indeed, each spanning tree is the whole lattice but an edge. Thus two vertices have coordination 1 and all the other $N - 2$ coordination 2.

More convincing is the case of a $2 \times L$ strip, extended in the x direction. Consider the anisotropic problem, with weight $v_x = 1$ and $v_y = v$. The two vertical bonds of the lattice at a given value of x can never be occupied simultaneously, and this also implies that $p(c) \neq 0$ for $c = 1, 2, 3$ only. We have only two momenta in direction y, with $\widehat{p}_y = 0, 2$ respectively for $n_y = 0, 1$. This implies that $B(y)$ vanishes, as $\widehat{p}_y^2 - \widehat{p}_y^4/4 = 0$ for both values, and that $A(y)$ is a one-dimensional lattice sum

$$A(y) = \frac{1}{2L}\frac{1}{4}{\sum_{n_x,n_y}}' \frac{v\widehat{p}_y^4}{v\widehat{p}_y^2 + \widehat{p}_x^2} = \frac{1}{2L}\frac{1}{4}\sum_{n_x=0}^{L-1} \frac{4v}{4v + \widehat{p}_x^2} = \frac{1}{2}\sqrt{\frac{v}{v+1}}\coth(L\,\mathrm{arcsh}\,\sqrt{v})\,. \quad (7.78)$$

Using relations (7.62a) and (7.72), and introducing the recurrent combination

$$g(L) = -v + 2(v+1)A(y) \quad (7.79)$$

we find

$$\sum_c t^{c-1}p(c) = (1 + (t-1)g(L))\left(1 + (t-1)\left(1 - g(L) - \tfrac{1}{L}\right)\right)\,. \quad (7.80)$$

For an infinite strip $g = -v + \sqrt{v(v+1)}$, and

$$p(1) = p(3) = g(1-g)\,; \qquad p(2) = g^2 + (1-g)^2\,. \quad (7.81)$$

The probabilities $p(1)$ and $p(3)$ are equal, as a consequence of $\langle c\rangle = 2$, and of the fact that only coordinations 1, 2 and 3 are allowed. By simple algebra, we find the only parameter left

$$\frac{p(2)}{p(1)} = \frac{g}{1-g} + \frac{1-g}{g} = \frac{1+2v}{\sqrt{v+v^2}}\,. \quad (7.82)$$

This result is in agreement with what can be easily found by a transfer-matrix technique.

7.2.3 The case $d = 2$

The case $d = 2$ is special under many aspects. First, remark that in the isotropic 2-dimensional case, for a generic rectangular geometry, equation (7.62b) implies that

$$A_2(x) = A_2(y)\,, \quad (7.83)$$

At infinite volume

$$A_2 = \frac{1}{\pi} \quad (7.84)$$

and

$$\sum_c t^{c-1}p(c) = [2A_2 + t\,(1 - 2A_2)]^2\,[2 - 4A_2 + t(4A_2 - 1)] \quad (7.85)$$

so that

$$\sum_c t^{c-1}p(c) = \left[\frac{2}{\pi} + t\left(1 - \frac{2}{\pi}\right)\right]^2\left[2 - \frac{4}{\pi} + t\left(\frac{4}{\pi} - 1\right)\right] \quad (7.86)$$

and the probabilities for the coordinations are

$p(1)$	$\frac{8\,(\pi-2)}{\pi^3}$	0.294545
$p(2)$	$\frac{4(12-9\pi+2\pi^2)}{\pi^3}$	0.446990
$p(3)$	$\frac{2\,(\pi-2)\left(12-6\pi+\pi^2\right)}{\pi^3}$	0.222385
$p(4)$	$\frac{(4-\pi)\,(\pi-2)^2}{\pi^3}$	0.036080

(7.87)

At finite volume interesting features emerge. We must have the same result in the rectangular geometries (L_1, L_2) and (L_2, L_1). Let $\rho = L_2/L_1$. We have computed (cfr. Section 7.4) the first correction to $A_2(1)$ and $B_2(1)$ in inverse volume N by keeping fixed the aspect-ratio ρ. For A_2 and the combination $A_2 + B_2$ the result is

$$A_2(1) = \frac{1}{\pi} + \frac{(2\pi)^3}{480}\frac{1}{N^2}\,\rho^2 E_4(i\rho) + O(N^{-3})\,; \tag{7.88}$$

$$A_2(1) + B_2(1) = \frac{1}{2} - \frac{2\pi}{12}\frac{1}{N}\,\rho E_2(i\rho) + +O(N^{-2})\,; \tag{7.89}$$

where $E_2(z)$ and $E_4(z)$ are *normalized Eisenstein series* [103], defined for k positive even integer as

$$G_k(z) := {\sum_{m,n}}' \frac{1}{(mz+n)^k}\,; \qquad E_k(z) := \lim_{w\to i\infty}\frac{G_k(z)}{G_k(w)} = \frac{1}{2\zeta(k)}G_k(z)\,; \tag{7.90}$$

where z is a complex number with positive imaginary part, and summation is over $\mathbb{Z}^2$ but not $(0,0)$. For $k \geq 4$ these functions have the remarkable property that

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k E_k(z) \tag{7.91}$$

if $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ belongs to $SL_2(\mathbb{Z})$, the group of 2×2 integral unimodular matrices acting on the upper complex half-plane, so that

$$E_k\left(-\frac{1}{z}\right) = z^k E_k(z) \tag{7.92}$$

that is E_k is a *modular form of weight* k. As a consequence

$$\rho^2 E_4\left(i\rho\right) = \frac{1}{\rho^2}E_4\left(i\frac{1}{\rho}\right)\,. \tag{7.93}$$

For $k = 2$, the series (7.90) is not absolutely convergent, and one should give a prescription on the order of summation:

$$E_2(z) = \frac{1}{2\zeta(2)}\sum_m {\sum_n}' \frac{1}{(mz+n)^2}\,, \tag{7.94}$$

where each summation is over $\mathbb{Z}$, and the prime stands for "$n \neq 0$ if $m = 0$". This prescription, asymmetric under $m \leftrightarrow n$, causes an anomalous modular relation

$$\rho E_2(i\rho) + \frac{1}{\rho}E_2\left(\frac{i}{\rho}\right) = \frac{12}{2\pi}\,. \tag{7.95}$$

The relations (7.83) and (7.62a) have consequences on the possible expressions for lattice integrals, order by order in $1/N$ expansion. Define

$$A_2(1) = a_0 + \frac{1}{N}a_1(\rho) + \frac{1}{N^2}a_2(\rho) + \dots \tag{7.96}$$

$$B_2(1) = b_0 + \frac{1}{N}b_1(\rho) + \frac{1}{N^2}b_2(\rho) + \dots \tag{7.97}$$

then for each $k \geq 1$,

$$a_k(\rho) = a_k(1/\rho)\,; \tag{7.98}$$

$$2a_k(\rho) + b_k(\rho) + b_k(1/\rho) = -\delta_{k,1}\,. \tag{7.99}$$

Relation (7.98) for $k = 2$ is an alternative proof of (7.93), while relation (7.99) for $k = 1$ is an alternative proof of (7.95). Similar but more involved modular properties arise at higher orders of both relations.

This sort of *quasi–modular-invariance* emerges also in other lattice problems for which the continuum limit depends on the relative orientation of lattice axes w.r.t. finite geometry sides. This was noted, for example [104, 105], for Cardy percolation formulas [106]. For the aware reader, we recall the trivial geometric conditions on percolation crossing probabilities on the triangular lattice (also here $\rho = L_2/L_1$)

$$\Pi_h(\rho) + \Pi_h(1/\rho) = 1\,; \qquad\qquad \Pi_{h,v}(\rho) = \Pi_{h,v}(1/\rho)\,; \tag{7.100}$$

which resemble our (7.99) and (7.98), that is

$$B_2(x) + B_2(y) = 2C_2(x,y) - \frac{1}{N}\,; \qquad\qquad A_2(x) = A_2(y)\,; \tag{7.101}$$

Also in the case of Cardy formula, although only a $\rho \leftrightarrow 1/\rho$ symmetry on a rectangular geometry is expected, modular forms emerge in a way which suggests a hidden larger symmetry in the problem.

From equations (7.88) and (7.89) it is easily derived that first- and second-order corrections to the probability distribution have the peculiar form

$$\begin{aligned} p^{(N)}(c) &= p_0(c) - \frac{1}{N} p_1(c) \frac{2\pi}{12} \left(\rho E_2(i\rho) + \frac{1}{\rho} E_2\Big(i\frac{1}{\rho}\Big) \right) \\ &\quad + \frac{1}{N^2} \left(p_{2,1}(c)\, E_2(i\rho) E_2\Big(i\frac{1}{\rho}\Big) + p_{2,2}(c) \rho^2 E_4(i\rho) \right) + O(N^{-3}) \\ &= p_0(c) - \frac{1}{N} p_1(c) + \frac{1}{N^2} \left(p_{2,1}(c) f_{2,1}(\rho) + p_{2,2}(c) f_{2,2}(\rho) \right) + O(N^{-3})\,. \end{aligned} \tag{7.102}$$

In particular, we get the striking result that, at the first perturbative order, the correction coefficients are independent from the aspect ratio ρ, while, at the second order, depend on ρ only through two functions, $f_{2,1}(\rho)$ and $f_{2,2}(\rho)$, expressed in terms of modular forms and separately invariant under $\rho \leftrightarrow 1/\rho$. Remarkably, we have under $\rho \to \rho + i$:

$$\rho^2 E_4(i\rho) \quad \longrightarrow \quad \left(\frac{\rho + i}{\rho} \right)^2 \rho^2 E_4(i\rho)\,; \tag{7.103}$$

$$E_2(i\rho) E_2\Big(i\frac{1}{\rho}\Big) \quad \longrightarrow \quad \left(\frac{\rho + i}{\rho} \right)^2 E_2(i\rho) E_2\Big(i\frac{1}{\rho}\Big) + i \left(\frac{\rho + i}{\rho} \right) \frac{12}{2\pi} E_2(i\rho)\,. \tag{7.104}$$

The precise perturbation coefficients $p_1(c)$ are given in table 7.2.3, while $p_{2,1}(c)$ and $p_{2,2}(c)$ are given in table 7.2.3. Note in particular that, as expected from the fact that at all orders $\langle c \rangle = 2 - 2/N$,

$$\sum_c c\, p_1(c) = -2\,; \qquad\qquad \sum_c c\, p_{2,1}(c) = \sum_c c\, p_{2,2}(c) = 0\,. \tag{7.105}$$

$p_1(1)$	$\frac{8\,(\pi-2)}{\pi^2}$	0.92534
$p_1(2)$	$-\frac{4(4-\pi)(\pi-3)}{\pi^2}$	− 0.04926
$p_1(3)$	$-\frac{2\,(24-16\pi+3\pi^2)}{\pi^2}$	− 0.67750
$p_1(4)$	$-\frac{2\,(4-\pi)\,(\pi-2)}{\pi^2}$	− 0.19858

Table 7.1: Finite-size first-order pertubation coefficients for coordination probabilities of spanning trees on the square lattice.

$p_{2,1}(1)$	$\frac{2\pi(\pi-2)}{9}$	0.796982	$p_{2,2}(1)$	$\frac{4\pi(\pi-3)}{15}$	0.118620
$p_{2,1}(2)$	$-\frac{\pi\,(5\pi-12)}{9}$	− 1.294323	$p_{2,2}(2)$	$-\frac{2\pi(\pi-3)(6-\pi)}{15}$	− 0.169533
$p_{2,1}(3)$	$\frac{4\pi(\pi-3)}{9}$	0.197701	$p_{2,2}(3)$	$-\frac{4\pi(\pi-3)^2}{15}$	− 0.0167958
$p_{2,1}(4)$	$\frac{\pi(4-\pi)}{9}$	0.299641	$p_{2,2}(4)$	$\frac{2\pi(\pi-3)(\pi-2)}{15}$	0.0677081

Table 7.2: Finite-size second-order pertubation coefficients for coordination probabilities of spanning trees on the square lattice.

7.2.4 The case $d = 4$

In the special case of $d = 4$, interest on the high-precision numerical value of the integral C_4 already arose in the context of lattice Feynman integrals (where it has been denoted by Z_1). Let us use what has been reported in [107]

$$Z_1 \sim 0.10778131353987400134339155 0 \tag{7.106}$$

We get

$$A_4 = \frac{1}{2} - 3Z_1 \tag{7.107}$$

$$B_4 = -\frac{1}{4} + 3Z_1 \tag{7.108}$$

$$C_4 = Z_1 \tag{7.109}$$

We recover for the probabilities of the coordination numbers

$$\begin{aligned} p \sim \{&0.3398850257, 0.3966063441, 0.1982970907, 0.0550688383, \\ &0.0091738514, 0.0009167550, 0.0000508846, 0.0000001210\}\,. \end{aligned} \tag{7.110}$$

7.3 Correlation functions for edge-occupations and coordinations

We start with a general fact concerning connected and unsubtracted k-point functions. Suppose to have a theory with an algebra of local observables $\phi_i(x)$, defined on an infinite

graph, or a family of graphs of increasing size, with a group of translation invariance, so that taking a limit of $|x_i - x_j| \to \infty$ makes sense. Suppose to be in a pure phase, and thus that Cluster Property holds. Also suppose to know that a class of k-point functions have a (quasi-)determinantal expression, that is, calling $\ell(\sigma)$ the number of cycles in a permutation σ, an expression of the form

$$\langle \phi_{i_1}(x_1)\phi_{i_2}(x_2)\cdots\phi_{i_k}(x_k)\rangle = K \sum_{\sigma\in\mathcal{S}_k} t^{\ell(\sigma)} \prod_{i=1}^{k} \mathcal{K}^{(i,\sigma(i))}(x_i - x_{\sigma(i)}) \tag{7.111}$$

for a set of functions $\mathcal{K}^{(i,j)}(x)$, at any k-uple of points $\{x_i\}$ (K is some constant). If $t = -1$ we deal with a *determinantal process*, while if $t = +1$ we have a *permanental process* [108].

The connected k-point function associated to the average of a product of k fields is the full average, subtracted by a combination, over the partitions of the factors in the product, of the averaging performed separately on products within the blocks of the partition. Each non-trivial partition P participates the subtraction mechanism with some coefficient (that has a definite expression: it is just $(-1)^{|P|-1}(|P|-1)!$).

When on a system with translation invariance, this combination has also an alternate definition, which will prove useful in the following. It is the only combination of the form above such that, for every set $A \subseteq [k]$ with both A and $A^c = [k] \setminus A$ non-empty,

$$\lim_{|y|\to\infty} \Big\langle \Big(\prod_{a\in A} \phi_{i_a}(x_a)\Big)\Big(\prod_{b\in A^c} \phi_{i_b}(x_b - y)\Big)\Big\rangle^{\text{conn.}} = 0\,. \tag{7.112}$$

This property leads to the following proposition:

Proposition 7.2. *Under the conditions above,*

$$\langle \phi_{i_1}(x_1)\phi_{i_2}(x_2)\cdots\phi_{i_k}(x_k)\rangle^{\text{conn.}} = Kt \sum_{\substack{\sigma\in\mathcal{S}_k \\ \ell(\sigma)=1}} \prod_{i=1}^{k} \mathcal{K}^{(i,\sigma(i))}(x_i - x_{\sigma(i)})\,. \tag{7.113}$$

The proof is just by recognizing that any permutation consisting of more cycles fails the condition (7.112) for A being a block of the partition induced by the cycles of the permutation, and is minimal with this property w.r.t. the partial ordering of partitions.

It would be easy to extend the reasoning to k-point functions being λ-determinants (defined as in Robbins and Rumsey [109, 110]), where connected k-point functions would correspond to a sum restricted to connected $k \times k$ Alternating Sign Matrices, that is matrices in this ensemble which are not block-decomposable by a permutation of rows and columns.

Remark that equations (7.40) and (7.41) give an expression exactly of the form just described, in the simplest case of a determinantal process. In particular, the expression for M_{ab} is a function depending on lattice positions through $\mathbf{r}_a - \mathbf{r}_b$ only.

So, applying the general proposition above, in this case the corresponding *connected* correlation functions can be formulated directly as a restricted sum, in which only permutations constituted of a single cycle are considered.

$$\langle \theta_1 \cdots \theta_n \rangle^{\text{conn}} = (-1)^{n+1} \sum_{\substack{\pi\in\mathcal{S}_n \\ \text{one-cycle}}} \prod_{a=1}^{n} M_{a\pi(a)}\,. \tag{7.114}$$

Now we go back to the explicit expression (7.41). Recall the definition of the lattice propagator [111]

$$G(\mathbf{r}) := \int_p \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\widehat{p}^2}\,, \tag{7.115}$$

and define as usual the right- and left-derivatives on the lattice as

$$\nabla^+_\alpha f(\mathbf{r}) = f(\mathbf{r}+\widehat{\eta}_\alpha) - f(\mathbf{r})\,; \qquad \nabla^-_\alpha f(\mathbf{r}) = f(\mathbf{r}) - f(\mathbf{r}-\widehat{\eta}_\alpha)\,. \tag{7.116}$$

When the function f depends on more than one variable (say, on a set of $\{r_v\}$), a further label $\nabla_{\alpha,v}$ will specify the variable of derivation. Within these notations, a matrix element M_{ab} (7.41) is given by

$$M_{ab} = \tau_a\tau_b \nabla^{\tau_a}_{\alpha_a,a}\nabla^{\tau_b}_{\alpha_b,b} G(\mathbf{r}_a-\mathbf{r}_b) = -\tau_a\tau_b\nabla^{\tau_a}_{\alpha_a}\nabla^{\tau_b}_{\alpha_b} G(\mathbf{r}_a-\mathbf{r}_b)\,, \tag{7.117}$$

where in the last passage derivatives are intended w.r.t. $\mathbf{r}=\mathbf{r}_a-\mathbf{r}_b$; formula (7.40) (and thus (7.10b)) become, in the more interesting case of connected expectations,

$$\langle \theta_{e_1}\cdots\theta_{e_n}\rangle^{\text{conn}} = -\sum_{\substack{\pi\in\mathcal{S}_n\\ \text{one cycle}}}\prod_v \nabla^{\tau_v}_{\alpha_v}\nabla^{\tau_{\pi(v)}}_{\alpha_{\pi(v)}} G(\mathbf{r}_v-\mathbf{r}_{\pi(v)})\,; \tag{7.118a}$$

$$\langle c_{\mathbf{r}_1}\cdots c_{\mathbf{r}_n}\rangle^{\text{conn.}} = -\sum_{\substack{\pi\in\mathcal{S}_n\\ \text{one cycle}}}\sum_{\alpha_v,\tau_v}\prod_v \nabla^{\tau_v}_{\alpha_v}\nabla^{\tau_{\pi(v)}}_{\alpha_{\pi(v)}} G(\mathbf{r}_v-\mathbf{r}_{\pi(v)})\,. \tag{7.118b}$$

For the case of expectation in the algebra of c_i's, we still have the properties that

$$p_{i_1,\dots,i_k}(c_1,\dots,c_k) = 0 \qquad \text{if any } c_i = 0 \tag{7.119a}$$

$$\sum_{c_k} p_{i_1,\dots,i_k}(c_1,\dots,c_k) = \left(2-\tfrac{2}{N}\right) p_{i_1,\dots,i_{k-1}}(c_1,\dots,c_{k-1})\,. \tag{7.119b}$$

We sketch briefly how this happens. We have for $p_{i_1,\dots,i_k}(c_1,\dots,c_k)$ a determinantal expression involving the matrix elements, for $\alpha=\alpha'$

$$M_{\alpha,+;\alpha,+}(\mathbf{r}) = \int_p \frac{4\sin^2\left(\frac{p_\alpha}{2}\right)\cos^2\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: B_d(\alpha,\mathbf{r}) \tag{7.120a}$$

$$M_{\alpha,+;\alpha,-}(\mathbf{r}) = \int_p \frac{4\sin^3\left(\frac{p_\alpha}{2}\right)\cos\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: E_d(\alpha,\mathbf{r}) \tag{7.120b}$$

$$M_{\alpha,-;\alpha,-}(\mathbf{r}) = \int_p \frac{4\sin^4\left(\frac{p_\alpha}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: A_d(\alpha,\mathbf{r}) \tag{7.120c}$$

and for $\alpha\neq\alpha'$

$$M_{\alpha,+;\alpha',+}(\mathbf{r}) = \int_p \frac{4\sin\left(\frac{p_\alpha}{2}\right)\cos\left(\frac{p_\alpha}{2}\right)\sin\left(\frac{p_{\alpha'}}{2}\right)\cos\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: D_d(\alpha,\alpha',\mathbf{r}) \tag{7.120d}$$

$$M_{\alpha,+;\alpha',-}(\mathbf{r}) = \int_p \frac{4\sin\left(\frac{p_\alpha}{2}\right)\cos\left(\frac{p_\alpha}{2}\right)\sin^2\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: F_d(\alpha,\alpha',\mathbf{r}) \tag{7.120e}$$

$$M_{\alpha,-;\alpha',-}(\mathbf{r}) = \int_p \frac{4\sin^2\left(\frac{p_\alpha}{2}\right)\sin^2\left(\frac{p_{\alpha'}}{2}\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} =: C_d(\alpha,\alpha',\mathbf{r}) \tag{7.120f}$$

where A_d, B_d and C_d generalize the single-site expressions by taking as argument the distance vector among the two sites under consideration, and D_d, E_d and F_d are similarly introduced.

Special combinations of quantities defined in (7.120), via a cancellation of the $\widehat{p}^2$ denominator, lead to the relations that are responsible for the properties (7.119). In particular, for the functions

$$R_1(\mathbf{r}) = \delta_{\mathbf{r},\mathbf{0}} - \frac{1}{N} \tag{7.121}$$

$$R_2^{(\alpha)}(\mathbf{r}) = \frac{1}{2}\,\delta_{\mathbf{r},\mathbf{0}} - \frac{1}{4}\,(\delta_{\mathbf{r},+\widehat{\eta}_\alpha} + \delta_{\mathbf{r},-\widehat{\eta}_\alpha}) \tag{7.122}$$

$$R_3^{(\alpha)}(\mathbf{r}) = \frac{i}{4}\,(\delta_{\mathbf{r},+\widehat{\eta}_\alpha} - \delta_{\mathbf{r},-\widehat{\eta}_\alpha}) \tag{7.123}$$

we have

$$R_1(\mathbf{r}) = \int_p \frac{\sum_\alpha \widehat{p}_\alpha^2}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} = \sum_{\alpha=1}^{d} (A_d(\alpha,\mathbf{r}) + B_d(\alpha,\mathbf{r})) \tag{7.124}$$

$$R_2^{(\alpha)}(\mathbf{r}) = \frac{1}{4}\int_p \widehat{p}_\alpha^2 e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{4}\int_p \frac{\widehat{p}_\alpha^2\left(\sum_{\alpha'}\widehat{p}_{\alpha'}^2\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} = A_d(\alpha,\mathbf{r}) + \sum_{\alpha'\neq\alpha} C_d(\alpha,\alpha',\mathbf{r}) \tag{7.125}$$

$$\begin{aligned} R_3^{(\alpha)}(\mathbf{r}) &= \frac{1}{2}\int_p \sin p_\alpha e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{2}\int_p \frac{\sin p_\alpha\left(\sum_{\alpha'}\widehat{p}_{\alpha'}^2\right)}{\widehat{p}^2} e^{i\mathbf{p}\cdot\mathbf{r}} \\ &= E_d(\alpha,\mathbf{r}) + \sum_{\alpha'\neq\alpha} F_d(\alpha,\alpha',\mathbf{r}) \end{aligned} \tag{7.126}$$

One of the consequences of the statements above is that, at least if all $|\mathbf{r}_v - \mathbf{r}_{v'}| > 1$, the vectors $\mathbf{W}^{(v)}$ such that $(\mathbf{W}^{(v)})_j = 1$ if $(2v-1)d < j \leq 2vd$ and 0 otherwise, are eigenvectors with eigenvalues $1+t_v$ (this proves that no c_v can be zero). We give no proof of the other relation, but this can be done on similar grounds to what has already been shown.

7.3.1 Continuum limit

We come back to the analysis of equations (7.118), and consider the continuum limit. At leading order for large $r_{vv'} = |\mathbf{r}_v - \mathbf{r}_{v'}|$, we can replace the expression for the lattice Green function with its limit, solving the Laplacian equation in continuum Euclidean space with a Dirac delta. Call $\mathbf{n} = \mathbf{r}/r$, we have

$$\partial_\alpha \partial_{\alpha'} G(\mathbf{r}) = C_d \frac{n_\alpha n_{\alpha'}}{r^d}(1 + \mathcal{O}(r^{-2})) \tag{7.127}$$

with C_d an universal constant depending only on the dimension of the lattice, given by

$$C_d = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}}\,. \tag{7.128}$$

In this framework, equations (7.118) become

$$\langle \theta_{e_1} \cdots \theta_{e_n} \rangle^{\text{conn}} = -{C_d}^n \left(\prod_v \frac{(\mathbf{n}_{v,v+1})_{\alpha_v} (\mathbf{n}_{v,v+1})_{\alpha_{v+1}}}{(r_{v,v+1})^d} + \underset{\text{permut.}}{\text{other}} \right); \tag{7.129a}$$

$$\langle c_{\mathbf{r}_1} \cdots c_{\mathbf{r}_n} \rangle^{\text{conn.}} = -(2C_d)^n \left(\prod_v \frac{(\mathbf{n}_{v-1,v} \cdot \mathbf{n}_{v,v+1})}{(r_{v,v+1})^d} + \underset{\text{permut.}}{\text{other}} \right); \tag{7.129b}$$

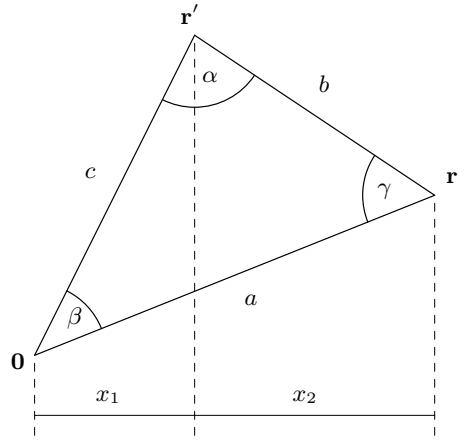


Fig. 7.1. Definition of the geometrical quantities involved in equations (7.130–7.133).

Some concrete examples for $k = 2$ and $k = 3$ are

$$\langle \theta_{\mathbf{0},\alpha} \theta_{\mathbf{r},\alpha'} \rangle^{\text{conn.}} \simeq -C_d^2 \frac{n_\alpha^2 n_{\alpha'}^2}{r^{2d}} \tag{7.130}$$

$$\langle c_{\mathbf{0}} c_{\mathbf{r}} \rangle^{\text{conn.}} \simeq -\frac{4C_d^2}{r^{2d}} \tag{7.131}$$

$$\langle c_{\mathbf{0}} c_{\mathbf{r}} c_{\mathbf{r}'} \rangle^{\text{conn.}} \simeq 16 C_d^3 \frac{\cos\alpha \, \cos\beta \, \cos\gamma}{(a\,b\,c)^d} \tag{7.132}$$

$$\left\langle \theta_{\mathbf{0},\hat{1},+} \theta_{\mathbf{r},\hat{1},+} \theta_{\mathbf{r}',\hat{1},+} \right\rangle^{\text{conn.}} \simeq -2 C_d^3 \frac{(x_1 x_2 (x_1 + x_2))^2}{(a\,b\,c)^{d+2}} \tag{7.133}$$

where α, β, γ and a, b, c are respectively the angles and the side lengths of the triangle corresponding to the three points, and x_1, x_2 and $x_1 + x_2$ are the (unsigned) projections of the three distance vectors along direction $\hat{1}$ (cfr. figure 7.1).

In particular, in dimension 2, we can exploit the fact that

$$\mathbf{r}_{ij}{}^2 = \overline{z}_{ij} z_{ij} \, ; \qquad\qquad 2\, \mathbf{r}_{ij} \cdot \mathbf{r}_{jk} = \overline{z}_{ij} z_{jk} + z_{ij} \overline{z}_{jk} \, ; \tag{7.134}$$

in order to write

$$\langle c_{\mathbf{r}_1} \cdots c_{\mathbf{r}_n} \rangle^{\text{conn.}} = -{C_d}^n \left(\prod_v \frac{(\overline{z}_{v-1,v} z_{v,v+1} + \text{c.c.})}{(\overline{z}_{v,v+1} z_{v,v+1})^{3/2}} + \begin{matrix} \text{other} \\ \text{permut.} \end{matrix} \right) ; \tag{7.135}$$

7.4 Integration of 2-dimensional lattice integrals at finite volume

For the calculation of lattice sums of the form

$$\frac{1}{L_1 L_2} \sum_{n_1=0}^{L_1-1} \sum_{n_2=0}^{L_2-1} \frac{f(p)}{\widehat{p}^2} \tag{7.136}$$

we shall follow the method outlined in the Appendix of [111]. If the numerator is a function of the first moment only, the sum over the second direction can be performed exactly as

$$\frac{1}{L_2}\sum_{n_2=0}^{L_2-1}\frac{f(p_1)}{\widehat{p}_1^2+\widehat{p}_2^2}=\frac{f(p_1)}{\widehat{p}_1\sqrt{4+\widehat{p}_1^2}}\coth\left[L_2\,\mathrm{arcsh}(\widehat{p}_1/2)\right]$$
$$=\frac{f(p_1)}{\widehat{p}_1\sqrt{4+\widehat{p}_1^2}}\left(1+\frac{2}{e^{2L_2\,\mathrm{arcsh}(\widehat{p}_1/2)}-1}\right). \tag{7.137}$$

The lattice integrals of our interest, $A_2(1)$ and $B_2(1)$ are of the form above, with the choice

$$A_2(1)+B_2(1)\ :\quad f(p_1)=\widehat{p}_1^2\,;\qquad\qquad A_2(1)\ :\quad f(p_1)=\widehat{p}_1^4\,; \tag{7.138}$$

so we will consider the general expression

$$\mathcal{I}_k=\frac{1}{L_1L_2}\sum_{n_1=0}^{L_1-1}\sum_{n_2=0}^{L_2-1}\frac{\widehat{p}_1^{2k}}{\widehat{p}^2}=\frac{1}{L_1}\sum_{n_1=0}^{L_1-1}\frac{\widehat{p}_1^{2k-1}}{\sqrt{4+\widehat{p}_1^2}}\left(1+\frac{2}{e^{2L_2\,\mathrm{arcsh}(\widehat{p}_1/2)}-1}\right) \tag{7.139}$$

for $k\geq 1$, being understood that $A_2(1)=\mathcal{I}_2$ and $B_2(1)=\mathcal{I}_1-\mathcal{I}_2$. The sum is more conveniently computed separately in the two contributions

$$\mathcal{I}_k^{(1)}=\frac{1}{L_1}\sum_{n_1=0}^{L_1-1}\frac{\widehat{p}_1^{2k-1}}{\sqrt{4+\widehat{p}_1^2}}\,; \tag{7.140}$$

$$\mathcal{I}_k^{(2)}=\frac{1}{L_1}\sum_{n_1=0}^{L_1-1}\frac{2\widehat{p}_1^{2k-1}}{\sqrt{4+\widehat{p}_1^2}}\frac{1}{e^{2L_2\,\mathrm{arcsh}(\widehat{p}_1/2)}-1}\,. \tag{7.141}$$

On the first term, by using the Euler-Mac Laurin expansion at order n,

$$\mathcal{I}_k^{(1)}=\mathcal{I}_k^{\infty}-\frac{1}{2\pi}\sum_{h=1}^{n-1}\frac{B_{2h}}{(2h)!}\left(\frac{2\pi}{L_1}\right)^{2h}\frac{\mathrm{d}^{2h-1}}{\mathrm{d}p^{2h-1}}\frac{\widehat{p}^{2k-1}}{\sqrt{4+\widehat{p}^2}}\bigg|_{p=0}+O(L_1^{-2n})\,, \tag{7.142}$$

where we called $\mathcal{I}_k^{\infty}$ the continuum limit of the expression,

$$\mathcal{I}_k^{\infty}:=\int_0^{2\pi}\frac{dp}{2\pi}\frac{\widehat{p}^{2k-1}}{\sqrt{4+\widehat{p}^2}}\,. \tag{7.143}$$

The coefficients B_{2h} are the Bernoulli numbers, defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=1-\frac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\cdots\,, \tag{7.144}$$

the explicit expression for $2n\geq 2$ being

$$B_{2n}=\frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\zeta(2n)\,. \tag{7.145}$$

Introducing the Taylor coefficients

$$\frac{\widehat{p}^{2k-1}}{\sqrt{4+\widehat{p}^2}}=\sum_h a_{k,h}p^{2h-1}\,, \tag{7.146}$$

we have

$$\mathcal{I}_k^{(1)} = \mathcal{I}_k^{\infty} - \frac{1}{2\pi}\sum_{h=1}^{n-1}\frac{B_{2h}a_{k,h}}{2h}\left(\frac{2\pi}{L_1}\right)^{2h} + O(L_1^{-2n})\,. \tag{7.147}$$

This series turns out to be asymptotic, and thus must be intended at fixed n, for large L.[‡‡]

For the second term, $\mathcal{I}_k^{(2)}$, remark that $\widehat{p} > 0$ in the whole interval $p \in [0,\pi]$, but the two endpoints. Alternatively, if one translates the interval of integration to $p \in [-\pi/2,\pi/2]$, i.e. sums over $n_1 = -L_1/2,\ldots,L_1/2-1$, when perturbing for small values of $\widehat{p}$, one can restrict the attention to expansions around $p=0$. In particular, for the argument of the exponential in (7.141), consider the Taylor expansion

$$2L_2\,\mathrm{arcsh}\left(\sin\frac{p}{2}\right) = L_2 p\sum_{j=0}^{\infty} b_j p^{2j} = 2\pi\rho n\left(1+\sum_{j=1}^{\infty} b_j p^{2j}\right), \tag{7.148}$$

and thus we can write

$$\frac{1}{e^{2L_2\,\mathrm{arcsh}(\widehat{p}_1/2)}-1} = \frac{1}{e^{2\pi\rho'(n)n}-1}\,, \qquad \rho'(n) = \rho\left(1+\sum_{j=1}^{\infty} b_j p^{2j}\right). \tag{7.149}$$

We expand the n-dependence of $\rho'(n)$ via a further Taylor expansion around $\rho'(n)=\rho$. With the new set of coefficients

$$\sum_{j=\ell}^{\infty} b_{\ell,j}\,p^{2j} = \frac{1}{\ell!}\left(\sum_{i=1}^{\infty} b_i\,p^{2i}\right)^{\ell}, \tag{7.150}$$

we have

$$\frac{1}{e^{2L_2\,\mathrm{arcsh}(\widehat{p}_1/2)}-1} = \sum_{\ell=0}^{\infty}\sum_{j=\ell}^{\infty} b_{\ell,j}\,p^{2j}\rho^{\ell}\frac{\partial^{\ell}}{\partial\rho^{\ell}}\frac{1}{e^{2\pi\rho n}-1} \tag{7.151}$$

At finite ρ, the series coefficients of the sum over n show an exponential dumping, thus, if we neglect the truncation of the sum to $n=\pm L_1/2$, the error is exponentially small in L_1, and then negligible at all orders in our polynomial expansion in inverse system-size. Using also the definition (7.146), and introducing the new coefficients

$$a_{k,h,\ell} = \sum_{j=\ell}^{h-k} a_{k,h-j}b_{\ell,j}\,, \qquad \sum_h a_{k,h,\ell}p^{2h-1} = \frac{\widehat{p}^{\,2k-1}}{\sqrt{4+\widehat{p}^{\,2}}}\cdot\frac{1}{\ell!}\left(\frac{2}{p}\,\mathrm{arcsh}\,\frac{\widehat{p}}{2}-1\right)^{\ell} \tag{7.152}$$

we have

$$\mathcal{I}_k^{(2)} = \frac{1}{\pi}\sum_{h=k}^{\infty}\sum_{\ell=0}^{h-k}\left(\frac{2\pi}{L_1}\right)^{2h} a_{k,h,\ell}\,\rho^{\ell}\frac{\partial^{\ell}}{\partial\rho^{\ell}}\sum_{n_1=1}^{\infty}\frac{n^{2h-1}}{e^{2\pi\rho n}-1} \tag{7.153}$$

where we recognize a definition of the Eisenstein series $G_{2h}(\tau)$, or, alternatively, of the normalized Eisenstein series $E_{2h}(\tau)$

$$\sum_{n=1}^{\infty}\frac{n^{2h-1}}{e^{2\pi\rho n}-1} = \frac{B_{2h}}{2h}\left(1-\frac{G_{2h}(i\rho)}{2\zeta(2h)}\right) = \frac{B_{2h}}{2h}(1-E_{2h}(i\rho)) \tag{7.154}$$

[‡‡]Up to polynomial prefactors, the asymptotics of Bernoulli numbers is given by $|B_{2n}| \sim (2n)!(2\pi)^{-2n}$, while the one of $a_{k,n}$ for fixed k and $n\to\infty$, deducible at the complex-plane singularity nearest to the origin, is given by $|a_{k,n}| \sim (\mathrm{arcsh}\,1)^{-2n}$, so, up to polynomial prefactors, the n-th term of the Euler-McLaurin series scales as $(2n)!(L_1\,\mathrm{arcsh}\,1)^{-2n}$.

$k=1:$

h \ ℓ	0	1	2
1	1		
2	$-\frac{2}{12}$	$-\frac{1}{12}$	
3	$\frac{19}{480}$	$\frac{7}{288}$	$\frac{1}{288}$

$k=2:$

h \ ℓ	0	1	2
2	1		
3	$-\frac{3}{12}$	$-\frac{1}{12}$	
4	$\frac{27}{480}$	$\frac{9}{288}$	$\frac{1}{288}$

Table 7.3: Coefficients $a_{k,h,\ell}$ for the first few values of the indices.

The constant coefficient in (7.154) does not play a role for $\ell \geq 1$. On the other side, for $\ell = 0$, cancels out exactly with the contribution in the expression for $\mathcal{I}_k^{(1)}$, at equation (7.147), when one remarks that $a_{k,h,0} = a_{k,h}$. With the more symmetric notation $L_1^2 = V/\rho$, we have

$$\mathcal{I}_k - \mathcal{I}_k^\infty = -\frac{1}{\pi} \sum_{h=k}^{n-1} \frac{B_{2h}}{2h} \frac{(2\pi)^{2h}}{V^h} \sum_{\ell=0}^{h-k} a_{k,h,\ell}\, \rho^{\ell+h} \frac{\partial^\ell}{\partial \rho^\ell} E_{2h}(i\rho) + O(L_1^{-2n})\,. \tag{7.155}$$

With the explicit expression (7.145) for Bernoulli numbers, and calling $\alpha_{k,h,\ell} := (-1)^{h-1}(2h-1)!\, a_{k,h,\ell}$, we can alternatively write

$$\mathcal{I}_k - \mathcal{I}_k^\infty = -\frac{1}{\pi} \sum_{h=k}^{n-1} \sum_{\ell=0}^{h-k} \alpha_{k,h,\ell}\, \frac{\rho^{h+\ell}}{V^h} \frac{\partial^\ell}{\partial \rho^\ell} G_{2h}(i\rho) + O(L_1^{-2n})\,. \tag{7.156}$$

Using the definition (7.152), the full generating function for coefficients $a_{k,h,\ell}$ is

$$\frac{\omega \widehat{p} \cdot \exp\left[\lambda\left(\frac{2}{p}\operatorname{arcsh}\frac{\widehat{p}}{2} - 1\right)\right]}{(1-\omega\widehat{p}^{\,2})\sqrt{4+\widehat{p}^{\,2}}} = \sum_{k=1}^{\infty} \sum_{h=k}^{\infty} \sum_{\ell=0}^{h-k} a_{k,h,\ell}\, \omega^k p^{2h-1} \lambda^\ell\,. \tag{7.157}$$

At this point, we can write explicitly the first terms of the expansion for the integrals of interest, $\mathcal{I}_1$ and $\mathcal{I}_2$

$$\begin{aligned} -\pi(\mathcal{I}_1 - \mathcal{I}_1^\infty) = {} & \frac{\rho}{V} G_2(i\rho) + \frac{\rho^2}{V^2}\left(1 + \frac{1}{2}\rho\frac{\partial}{\partial\rho}\right) G_4(i\rho) \\ & + \frac{\rho^3}{V^3}\left(\frac{19}{4} + \frac{35}{12}\rho\frac{\partial}{\partial\rho} + \frac{5}{12}\rho^2\frac{\partial^2}{\partial\rho^2}\right) G_6(i\rho) + \cdots \end{aligned} \tag{7.158}$$

$$\begin{aligned} -\pi(\mathcal{I}_2 - \mathcal{I}_2^\infty) = {} & -6\frac{\rho^2}{V^2} G_4(i\rho) - \frac{\rho^3}{V^3}\left(30 + 10\rho\frac{\partial}{\partial\rho}\right) G_6(i\rho) \\ & - \frac{\rho^4}{V^4}\left(\frac{567}{2} + \frac{315}{2}\rho\frac{\partial}{\partial\rho} + \frac{35}{12}\rho^2\frac{\partial^2}{\partial\rho^2}\right) G_8(i\rho) + \cdots \end{aligned} \tag{7.159}$$

Relations (7.99) and (7.98) imply that, for all values of V, i.e. order by order in the perturbative expansion, the following equalities must hold

$$\mathcal{I}_1(\rho) + \mathcal{I}_1\left(\frac{1}{\rho}\right) - 2\mathcal{I}_1^\infty = \frac{1}{V} \tag{7.160}$$

$$\mathcal{I}_2(\rho) = \mathcal{I}_2\left(\frac{1}{\rho}\right) \tag{7.161}$$

7.5 Forests of unicyclic subgraphs and the coupling with a gauge field

The Matrix-Tree theorem, if the determinant of the Laplacian matrix is expressed as a Gaussian Grassmann integral, gives a field-theoretical description of the combinatorial model of (rooted) spanning trees in terms of a free (massless) scalar complex fermion. The introduction of a mass term in the action, which would leave the theory quadratic, corresponds combinatorially to a theory of rooted spanning forests, i.e. forests counted with the product of the sizes of the components. This is a trivial theory from the field-theoretical perspective, and not a specially interesting combinatorial model, and furthermore can be reproduced steadly in the framework of spanning trees, by introducing a single extra vertex, so we should not consider further this direction. There is however another natural direction for "perturbing" the free-fermion theory by leaving it free, which correspond to promote lattice derivatives to covariant derivatives, introducing an antisymmetric vector field on the edges of the graph. For each configuration of the vector field, the combinatorial expansion now produces a sum over spanning subgraphs, in which each component is unicyclic (that is, it has a single cycle). The weight of the component depends locally on edge weights, times a single "topological" factor, related to the circuitation of the vector field along the cycle. As only circuitations contribute to the determination of the Gibbs measure, we have a large "gauge invariance" in the definition of this field (in the lattice formulation of Helmholtz decomposition for a vector field into irrotational and divergence-free parts, only the divergence-free degrees of freedom play a role, so the gauge group of invariance is the sum of an arbitrary irrotational field). If the vector field is "dynamical", that is, it has some measure on which we should integrate, we have a non-trivial interacting theory, as we have vertices involving the vector field, and two fermions, simultaneously. If conversely the vector field is fixed once and forever ("statical"), the theory is free and the partition function is a determinant. We will be interested here in this case.

So we now give a more precise definition of the model we will study. Consider a graph Λ with V vertices. Define $\mathcal{T}$ as the set of spanning trees on Λ, i.e. the subset of spanning subgraphs T of Λ which are both connected and without loops. Say that a graph U is *unicyclic* if it is connected and has a single cycle, denoted as $\gamma(U)$ (note that an unicyclic graph has as many vertices as edges), and a *forest of unicyclics* if each of its components is an unicyclic (also a forest of unicyclics has as many vertices as edges). Define $\mathcal{U}$ as the set of spanning forests of unicyclic on Λ.

For many lattices, a natural notion of *winding number* $\mathbf{n}(\gamma)$ of a cycle $\gamma \subseteq \Lambda$ is defined. This happens when one has a natural determination of an embedding on a d-dimensional manifold with a non-trivial first homotopy group. In particular, if the manifold is a d-dimensional torus, the winding number $\mathbf{n} \in \mathbb{Z}^d$ is "abelian", and just counts the windings around the periodicities of the torus (regardless from the cyclic ordering in which they are performed). Say that for our lattice the function $\mathbf{n}(\gamma)$ is defined, and, for U an unicyclic, the notation $\mathbf{n}(U) \equiv \mathbf{n}(\gamma(U))$ is adopted. Introduce a set of weights $c(\mathbf{n})$. We can consider the partition functions

$$Z_{\mathcal{T}} = \sum_{T \in \mathcal{T}} \prod_{e \in E(T)} w_e \,; \tag{7.162}$$

$$Z_{\mathcal{U}}(c(\mathbf{n})) = \sum_{\substack{U \in \mathcal{U} \\ U=(U_1,\dots,U_k)}} \prod_{e \in E(U)} w_e \prod_{i=1}^{k} c(\mathbf{n}(U_i)) \,. \tag{7.163}$$

Of particular interest is the case $c(\mathbf{0}) = 0$, for which the only unicyclic configurations which survive are the ones in which each connected component has a cycle with non-trivial homotopy. Thus, in a limit of large size of the lattice, with the topology of the underlying manifold kept fixed, we may expect that the partition function $Z_{\mathcal{U}}$ is a "boundary deformation" of the pure spanning-tree partition function $Z_{\mathcal{T}}$, i.e. that $Z_{\mathcal{U}}/Z_{\mathcal{T}}$ scales subexponentially with lattice volume. In order to keep notations simple, in the following we will concentrate on the uniform case $w_e = 1$ for any $e \in E(\Lambda)$, but extension to the more general case would be easy.

7.5.1 Extension of the Matrix-Tree theorem to forests of unicyclics

For Λ embedded in a d-dimensional torus[‡‡], an extension of Kirchhoof Matrix-Tree theorem allows to calculate the quantity (7.163) as a simple determinant in the specific case of weights

$$c_{\boldsymbol{\epsilon}}(\mathbf{n}) = 2\Big(1 - \cos\sum_{i=1}^{d} \epsilon_i n_i\Big) \tag{7.164}$$

or, defining $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{d} v_i w_i$,

$$c_{\boldsymbol{\epsilon}}(\mathbf{n}) = 2\left(1 - \cos(\boldsymbol{\epsilon} \cdot \mathbf{n})\right) \tag{7.165}$$

A valid choice for a modified Laplacian matrix $L(\boldsymbol{\epsilon})$ such that $Z_{\mathcal{U}}(\boldsymbol{\epsilon}) = \det L(\boldsymbol{\epsilon})$, is the matrix such that $L_{x,x} = 2d$, $L_{x,x\pm\hat{\eta}_j} = -\exp(\pm i\epsilon_j/N_j)$ and $L_{x,y} = 0$ otherwise. So we have a constant linear vector field, realized on the lattice as the projection along the embedding of the edge of a fixed vector ϵ, this being locally both irrotational and divergence-free, but having a non-trivial circuitation along the fundamental cycles of the torus, given exactly by the parameters ϵ_i (this is why we divided by N_i, in the definition of L_{xy}).

So we claim that

Proposition 7.3.

$$Z_{\mathcal{U}}(c_{\boldsymbol{\epsilon}}(\mathbf{n})) = \det L(\boldsymbol{\epsilon})\,. \tag{7.166}$$

A complete proof of a wider generalization of this statement is postponed to Section 8.2. However, we can sketch here a specialization of that proof. First, consider a restatement of $\det L(\boldsymbol{\epsilon})$ as a Grassmann-Berezin integral

$$\det L(\boldsymbol{\epsilon}) = \int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} L \psi} \tag{7.167}$$

The expansion of the exponential produces a diagrammatic of terms $(\bar{\psi}\psi)_x 1_{x\pm\hat{\eta}_j}$ (type-1) and $-e^{\pm i\epsilon_j/N_j}\bar{\psi}_x\psi_{x\pm\hat{\eta}_j}$ (type-2), which put a "marked arrow" on the edges $(x, x \pm \hat{\eta}_j)$ of the lattice. Say that arrows on the marked edges are oriented with the tail on the site having the field $\bar{\psi}$.

Each resulting monomial in Grassmann fields, in order to have non-null integration, must be originated combinatorially from a configuration of arrows satisfying a number of local constraints. We must have at each site, beyond an arbitrary number of unmarked and in-coming type-1 edges (giving Grassmann factors 1), either one out-coming type-1

[‡‡] Not necessarily as a cell complex: e.g. for the two-dimensional case we could have Λ a portion of the square lattice with *both* diagonals.

edge (contributing a whole factor $\bar{\psi}\psi$), or two type-2 edges, of which one in-coming and the other out-coming (contributing respectively the factors ψ and $\bar{\psi}$).

Because of this diagrammatics, type-2 edges can only form closed self-avoiding cycles, while type-1 edges can take part of self-avoiding cycles, or oriented arborescences in-coming towards a type-1 or type-2 cycle. So, we can have only configurations of forests of unicyclic subgraphs, spanning the lattice.

Summing over each cycle type and orientation, we will have four contributions per component. Type-1 cycles give a contribution of 1 per orientation, while type-2 cycles give a minus sign because of the fermionic nature of the loop (coming from commutation of the fields), and a phase $e^{\pm i\boldsymbol{\epsilon}\cdot\mathbf{n}}$, with $\mathbf{n}$ the winding number of the cycle, and the two signs corresponding to the two possible orientations of the cycle. This gives the weight in equation (7.165).

The bosonic-fermionic loop cancellation, which plays a crucial role in the ordinary Matrix-Tree theorem, is preserved here exactly in simple cycles, with $\mathbf{n} = \mathbf{0}$, this being a property of coupling to vector fields which are locally irrotational.

7.5.2 Calculation of the partition function on the torus

Obviously, the matrix L can be diagonalized through Fourier Transform, using discrete momenta $p_i = 2\pi\ell_i/N_i$ as the new degrees of freedom, with, for example, $\ell_i \in \{-N_i/2, \dots, N_i/2-1\}$ (if all lattice sizes are even). Adopt the definitions $\widehat{v}_i = 2\sin(v_i/2)$, $\widehat{v}^2 = \sum_i \widehat{v}_i^2$ and $\boldsymbol{\epsilon}_N = (\epsilon_1/N_1, \dots, \epsilon_d/N_d)$. We have

$$\begin{aligned} L_{p,q}(\boldsymbol{\epsilon}) &= V\delta_{p,q} 2\Big(d - \sum_i \cos\Big(p_i + \frac{\epsilon_i}{N_i}\Big)\Big) \\ &= V\delta_{p,q}\Big[\widehat{p}^2 + 2\sum_i \left(\cos p_i \left(1 - \cos\frac{\epsilon_i}{N_i}\right) + \sin p_i \sin\frac{\epsilon_i}{N_i}\right)\Big] \\ &= V\delta_{p,q}\Big(\widehat{p}^2 + \widehat{\epsilon}_N \cdot \widehat{2p + \epsilon_N}\Big) \end{aligned} \tag{7.168}$$

then the partition function is

$$Z_{\mathcal{U}}(\boldsymbol{\epsilon}) = \det L(\boldsymbol{\epsilon}) = V^V \prod_p \Big(\widehat{p}^2 + \widehat{\epsilon}_N \cdot \widehat{2p + \epsilon_N}\Big) \tag{7.169}$$

and, removing the bulk contribution,

$$\frac{Z_{\mathcal{U}}(\boldsymbol{\epsilon})}{Z_{\mathcal{T}}} = V\widehat{\epsilon}_N^2 \prod_{p\neq 0} \left(1 + \frac{\widehat{\epsilon}_N \cdot \widehat{2p + \epsilon_N}}{\widehat{p}^2}\right) \tag{7.170}$$

we are left with an expression for the shift in free energy between the ensemble of spanning trees, and the one of forests of unicyclics with parameter $\boldsymbol{\epsilon}$

$$\delta F(\boldsymbol{\epsilon}) := \ln Z_{\mathcal{U}}(\boldsymbol{\epsilon}) - \ln Z_{\mathcal{T}} = \ln(V\widehat{\epsilon}_N^2) + \sum_{p\neq 0} \ln\left(1 + \frac{\widehat{\epsilon}_N \cdot \widehat{2p + \epsilon_N}}{\widehat{p}^2}\right). \tag{7.171}$$

Note that the weights $c_{\boldsymbol{\epsilon}}(\mathbf{n})$ are invariant under $\boldsymbol{\epsilon} \to \boldsymbol{\epsilon} + 2\pi\mathbf{m}$, so we can restrict our attention to the domain $\boldsymbol{\epsilon} \in [-\pi, \pi]^d$, and assume that, in the large volume limit, $(\epsilon_N)_i$ scales as N_i^{-1}.

In the large volume limit, the prefactor $V\widehat{\epsilon}_N^2$ scales as

$$V\widehat{\epsilon}_N^2 = \Big(\prod_j N_j\Big)\Big(\sum_i \frac{\epsilon_i^2}{N_i^2}\Big)\,\big(1+\mathcal{O}(N^{-2})\big) \tag{7.172}$$

and in particular, in the case $d=2$, with $\rho = N_y/N_x$ kept fixed in the limit,

$$V\widehat{\epsilon}_N^2 = \rho\epsilon_x^2 + \frac{1}{\rho}\epsilon_y^2 + \mathcal{O}(N^{-2})\,. \tag{7.173}$$

Then, we have to deal with the other summands in (7.171). This is a definite sum for any finite torus of sides $\{N_1, N_2, \dots, N_d\}$, however it is not clear *a priori* how to deal with a scaling allowing to reach the continuum limit. The intuition is that the parameters encoded in $\widehat{\epsilon}_N$ carry factors $1/N_i$, so that, for most summands, we should just analyse the expansion of the logarithm in powers of ϵ. However, the leading term in the expansion is odd in the momentum variables p_i, so that it will eventually vanish in the sum. In order to highlight this feature and single out the truly dominant term in the expansion, we symmetrize the sum as

$$\begin{aligned}\sum_{p\neq 0} \ln\left(1+\frac{\widehat{\epsilon}_N\cdot\widehat{2p+\epsilon_N}}{\widehat{p}^2}\right) &= \frac{1}{2}\sum_{p\neq 0}\ln\left(1+\frac{\widehat{\epsilon}_N\cdot\widehat{2p+\epsilon_N}}{\widehat{p}^2}\right)\left(1-\frac{\widehat{\epsilon}_N\cdot\widehat{2p-\epsilon_N}}{\widehat{p}^2}\right)\\ &= \frac{1}{2}\sum_{p\neq 0}\ln\left(1+2\frac{\widehat{\epsilon}_N^2}{\widehat{p}^2}-4\frac{\sum_i(\widehat{\epsilon}_N)_i^2\widehat{p}_i^2}{\widehat{p}^2}-\frac{\sum_{i,j}(\widehat{\epsilon}_N)_i(\widehat{\epsilon}_N)_j(\widehat{2p+\epsilon_N})_i(\widehat{2p+\epsilon_N})_j}{\widehat{p}^4}\right).\end{aligned} \tag{7.174}$$

There are two regimes of summands. If $|p| \gg 1/N$, then we can simplify our rational expressions, which are homogeneous of degree zero in variables ϵ_i, p_i altogether (that is, their are ratios of homogeneous polynomials with the same degree). In each rational function, the terms ϵ_i/N_i are systematically smaller than the p_i's, and, at the leading order in $1/N^2$, it suffices to keep the smallest available degree in ϵ_i's. This gives

$$\begin{aligned}&\frac{1}{2}\ln\left(1+2\frac{\widehat{\epsilon}_N^2}{\widehat{p}^2}-4\frac{\sum_i(\widehat{\epsilon}_N)_i^2\widehat{p}_i^2}{\widehat{p}^2}-\frac{\sum_{i,j}(\widehat{\epsilon}_N)_i(\widehat{\epsilon}_N)_j(\widehat{2p+\epsilon_N})_i(\widehat{2p+\epsilon_N})_j}{\widehat{p}^4}\right)\\ &= \left(\frac{\widehat{\epsilon}_N^2}{\widehat{p}^2}-2\frac{\sum_i(\widehat{\epsilon}_N)_i^2\widehat{p}_i^2}{\widehat{p}^2}-\frac{1}{2}\frac{\sum_i(\widehat{\epsilon}_N)_i^2(\widehat{2p})_i^2}{\widehat{p}^4}\right)\big(1+\mathcal{O}\big((|p|N_i)^{-2}\big)\big)+\text{terms odd in } p_i\text{'s}\,.\end{aligned} \tag{7.175}$$

Instead, the terms having all the p_i's of order $1/N_i$ lead to rational functions with numerators and denominators of order 1, and have to be dealt completely. However, another form of simplication can be done, as at the leading order in $1/N^2$ we can omit all $\widehat{\cdot}$'s, and deal with a sum of rational polynomial functions of lattice integers. Call $N = (\prod_j N_j)^{1/d}$ the geometric average of the lengths of the torus, and $\rho_i = N_i/N$ the set of aspect ratios, in a redundant parametrization which does not break the d-dimensional permutation symmetry ($\prod_i \rho_i = 1$). We still preserve the quick notation $\epsilon^2 = \sum_i(\rho_i\epsilon_i)^2$ and $p^2 = \sum_i(\rho_i p_i)^2$ (and, still, $p^{2k} \equiv (p^2)^k$). Then, the leading part of the summand reads

$$\begin{aligned}&\frac{1}{2}\ln\left(1+2\frac{\widehat{\epsilon}_N^2}{\widehat{p}^2}-4\frac{\sum_i(\widehat{\epsilon}_N)_i^2\widehat{p}_i^2}{\widehat{p}^2}-\frac{\sum_{i,j}(\widehat{\epsilon}_N)_i(\widehat{\epsilon}_N)_j(\widehat{2p+\epsilon_N})_i(\widehat{2p+\epsilon_N})_j}{\widehat{p}^4}\right)\\ &= \frac{1}{2}\ln\left(1+2\frac{\epsilon^2}{p^2}-\sum_{i,j}\frac{\epsilon_i\epsilon_j(4p_ip_j-\epsilon_i\epsilon_j)}{(\rho_i\rho_j)^2\,p^4}+\mathcal{O}\big(N_i^{-2}\big)\right).\end{aligned} \tag{7.176}$$

So, for example, we could combine the two regimes, and split the sum onto the set of momenta with all $p_i < \alpha\sqrt{N}$ (*small momenta*), and the set with some $p_i > \alpha\sqrt{N}$ (*large momenta*), each summand, in the appropriate expansion, being described up to order $1/N$.

In dimension $d > 2$, the sum for large momenta is dominant w.r.t. the one for small momenta, and, accordingly, the resulting lattice integrals are non-singular. Indeed we have for (7.175), up to an overall factor $\prod_i N_i$ connecting lattice sums to continuous integrals over angular variables,

$$\widehat{\epsilon}_N^2 \int_p \frac{1}{\widehat{p}^2} - 2\sum_i (\widehat{\epsilon}_N)_i^2 \int_p \frac{\widehat{p}_i^2}{\widehat{p}^2} - \frac{1}{2}\sum_i (\widehat{\epsilon}_N)_i^2 \frac{4\widehat{p}_i^2 - \widehat{p}_i^4}{\widehat{p}^4} \,. \tag{7.177}$$

Then, we recognize various standard lattice integrals, which allows us to symmetrize the expressions above into

$$\widehat{\epsilon}_N^2 \left(-\frac{2}{d} + \left(1 - \frac{2}{d}\right) \int_p \frac{1}{\widehat{p}^2} + \frac{1}{2} \int_p \frac{\widehat{p}_1^4}{\widehat{p}^4} \right) . \tag{7.178}$$

That is, similarly to what happens in (7.172), reintroducing the overall volume factors,

$$\Big(\prod_j N_j\Big)\Big(\sum_i \frac{\epsilon_i^2}{N_i^2}\Big) \left(-\frac{2}{d} + \left(1 - \frac{2}{d}\right) \int_p \frac{1}{\widehat{p}^2} + \frac{1}{2} \int_p \frac{\widehat{p}_1^4}{\widehat{p}^4} \right) . \tag{7.179}$$

This expression indeed combines exactly with the single summand in (7.172) that we treated separately, giving finally for the shift of the intensive free energy

$$\frac{\delta F(\boldsymbol{\epsilon})}{\prod_j N_j} = \Big(\sum_i \frac{\epsilon_i^2}{N_i^2}\Big) \left(\frac{d-2}{d} \left(1 + \int_p \frac{1}{\widehat{p}^2}\right) + \frac{1}{2} \int_p \frac{\widehat{p}_1^4}{\widehat{p}^4} \right) \big(1 + \mathcal{O}(N_i^{-2})\big) \,. \tag{7.180}$$

We recognize, as claimed heuristically at the beginning of the section, that this free-energy shift, being due to a boundary effect, scales subextensively. Then, we also see a precise behaviour: the dependence from ϵ is "spherical" at the leading order, and is given by the factor $\sum_i \epsilon_i^2/N_i^2 = N^{-2}\sum_i (\epsilon_i/\rho_i)^2$.

We recall again that the result above is derived by neglecting the contributions of "small momenta", which is legitimate only for $d > 2$. We see a signature of this fact from the resulting formula, where we have a "$0 \cdot \infty$" singularity to deal with, coming from $(d-2)\int_p \frac{1}{\widehat{p}^2}$. We recall the reason why the integral $\int_p \frac{1}{\widehat{p}^2}$ is divergent in two dimensions. The dependence from the dimension is easily made analytic through the representation

$$\frac{1}{\widehat{p}^2} = \int_0^\infty e^{-t\widehat{p}^2} \tag{7.181}$$

which gives an expression in terms of Bessel functions

$$\int_p \frac{1}{\widehat{p}^2} = \int_0^\infty \mathrm{d}t \, \Big(e^{-\frac{t}{2}} I_0(\tfrac{t}{2})\Big)^d \tag{7.182}$$

and the combination $e^{-\frac{t}{2}} I_0(\frac{t}{2})$ is regular near $t = 0$, but decreases only as $1/\sqrt{\pi t}$ for large t, this making the integral divergent for $d \leq 2$. It is easy to isolate the most singular contribution for $d = 2 + \delta$, and $\delta \searrow 0$, and realize that

$$\lim_{\delta \to 0^+} \frac{\delta}{2+\delta} \int_0^\infty \mathrm{d}t \, \Big(e^{-\frac{t}{2}} I_0(\tfrac{t}{2})\Big)^{2+\delta} = \lim_{\delta \to 0^+} \frac{\delta}{2+\delta} \int_0^\infty \mathrm{d}t \, (\pi t)^{-1-\frac{\delta}{2}} = \frac{1}{\pi} \,. \tag{7.183}$$

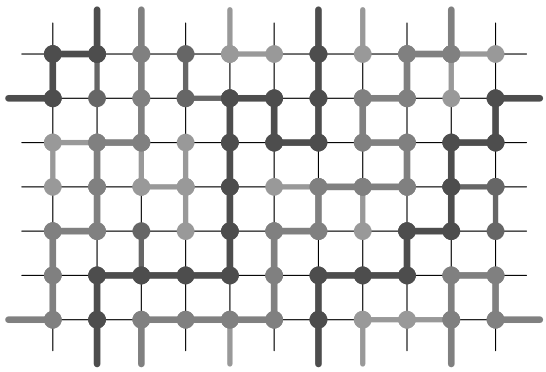


Fig. 7.2. A forest of unicyclics on a portion of the square lattice (11×7) with periodic boundary conditions. Here we have two components, both with winding $\mathbf{n} = (1, 2)$.

This limit combine with the expression for the other lattice integral in the expression (7.180),

$$\int_p \frac{\widehat{p}_1^4}{\widehat{p}^4} = \frac{1}{\pi}\,, \tag{7.184}$$

so that the quantity (7.180) calculated in the limit $d \to 2$ in "dimensional reduction", gives

$$\frac{\delta F(\boldsymbol{\epsilon})}{N_x N_y} = \frac{3}{2\pi}\Big(\frac{\epsilon_x^2}{N_x^2} + \frac{\epsilon_y^2}{N_y^2}\Big)\,\big(1 + \mathcal{O}(N_i^{-2})\big) + \text{small momenta contrib.} \tag{7.185}$$

We will discuss more extensively the two-dimensional case in the next section, where we also illustrate some other facts that are special to lattices planarly embedded on the torus.

7.5.3 Specific topological properties of dimension 2

In the case of dimension 2, and lattices which can be embedded on a genus-1 surface without edge-crossings, a certain number of new features arise in the problem. A typical configuration in such a context is shown in figure 7.2.

First, remark that topologically non-trivial self-avoiding cycles γ can only take "prime" winding numbers $\mathbf{n}(\gamma)$, i.e. winding numbers $\mathbf{n} = (n_x, n_y)$ such that $\gcd(n_x, n_y) = 1$.[♯] Call $\mathbb{Z}_2^*$ the subset of $\mathbb{Z}^2$ containing the pairs (n_x, n_y) of relatively prime integers.

Then, remark that, if a forest of unicyclics contains more than one connected component, all with non-trivial topology, then the associated winding numbers are all equal (otherwise the corresponding cycles would cross, which contraddicts the assumption that the lattice is planarly embedded on the torus).

This allows us to state that

$$Z_{\mathcal{U}}(\boldsymbol{\epsilon}) = \sum_{k=1}^{\infty} \sum_{\mathbf{n}\in\mathbb{Z}_2^*} g_{k,\mathbf{n}} \left(c_{\boldsymbol{\epsilon}}(\mathbf{n})\right)^k ; \tag{7.186}$$

$$g_{k,\mathbf{n}} = \#\{U \in \mathcal{U} \,:\, U \text{ has } k \text{ components } U_i, \text{ all with } \mathbf{n}(U_i) = \mathbf{n}\}\,. \tag{7.187}$$

[♯]The function gcd, defined on $\mathbb{N}^2$ in the obvious way, and on $\mathbb{N}^k$ recursively, has a natural extension to $\mathbb{Z}^2$ (and thus, recursively, to $\mathbb{Z}^k$). First, state that $\gcd(n_1, n_2) = \gcd(|n_1|, |n_2|)$, and then, for $n_1 \geq 0$, $\gcd(n_1, 0) = n_1$.

Table 7.4. The number of unicyclic forests on the 3×3 portion of the square lattice compactified on the torus, in each family $\{\mathbf{n}, k\}$, denoted as $\mathbf{n}^k$ (the last row is the number of spanning trees, given for comparison). Only one representative per symmetry class is given, and entries not listed have no configurations.

$(n_x, n_y)^k$	# configs.
$(1,0)^1$	9090
$(1,0)^2$	222
$(1,0)^3$	1
$(1,1)^1$	1272
$(1,2)^1$	3
trees	11664

Furthermore, this sum can be restricted to a finite number of summands on a torus having edge-cuts of size N_x and N_y along the elementary cycles in the x and y directions. Namely, we must have $|kn_x| \leq N_x$ and $|kn_y| \leq N_y$. As there is a parity invariance in the definition of the ensemble for $g_{k,\mathbf{n}}$, we could consider a restriction to the pairs (n_x, n_y) with $n_x > 0$, or $n_x = 0$ and $n_y > 0$. Say that this convention is understood in the following sum, in the function $\theta_+((n_x, n_y))$

$$Z_{\mathcal{U}}(\boldsymbol{\epsilon}) = \sum_{k=1}^{\infty} \sum_{|n_x| \leq \lfloor N_x/k \rfloor} \sum_{|n_y| \leq \lfloor N_y/k \rfloor} \delta(\gcd(n_x, n_y) = 1)\, \theta_+((n_x, n_y))\, g_{k,\mathbf{n}} \left(c_{\boldsymbol{\epsilon}}(\mathbf{n})\right)^k ; \tag{7.188}$$

where we denote $\delta(a = b)$ instead of the Kronecker delta $\delta_{a,b}$ in order to avoid multiple subscripts. This triple sum, with the strong primality constraint, is more easily encoded into a double sum, if we understand the synonima, for $(m_x, m_y) \in \mathbb{Z}^2 \setminus (0,0)$ (with the parity redundance discussed above) of $k = \gcd(m_x, m_y)$ and $(n_x, n_y) = (m_x, m_y)/k$:

$$Z_{\mathcal{U}}(\boldsymbol{\epsilon}) = \sum_{\substack{(m_x,m_y) \in \{0,1,\ldots,N_x\} \\ \times \{0,1,\ldots,N_y\} \setminus (0,0)}} \theta_+((m_x, m_y))\, g_{k,\mathbf{n}} \left(c_{\boldsymbol{\epsilon}}(\mathbf{n})\right)^k . \tag{7.189}$$

If one could solve this equation extracting all the non-vanishing expressions $g_{k,\mathbf{n}}$, one could then make the statistical sum (7.163) for an arbitrary set of weights $c(\mathbf{n})$ (still restricted to the most interesting case of $c(\mathbf{0}) = 0$), or even a more refined counting with a dependence also on the number of connected components of U.

The formula above, together with the explicit expression (7.171), allows to describe a very precise polynomial numerical recipe for this problem. We can evaluate $Z_{\mathcal{U}}(\boldsymbol{\epsilon})$ for a set of order V of different vectors $\boldsymbol{\epsilon}$, taking a time $\mathcal{O}(V^3)$ per point, and then solve the linear system corresponding to (7.189), for the indeterminates $g_{k,\mathbf{n}}$. This system has size V, so its solution takes a time of order V^3. As a result, the overall complexity of the procedure is of order V^4.

Application to a small example (the square lattice of size 3×3) gives the results in Table 7.4.

8

OSP(1|2) non-linear σ-model and spanning hyperforests

In this chapter we consider a Grassmann algebra over the vertex set of a hypergraph G. Our aim is to show that a class of Grassmann integrals permits an expansion in terms of spanning hyperforests, possibly interacting under various patterns.

A special case provides the generating functions for (unrooted) spanning (hyper)forests. Variations of the method easily provide analogues for rooted hyperforests, and for spanning hypertrees, among other things.

All these results are generalizations of Kirchhoff's matrix-tree theorem. Most importantly, we show that the class of integrals describing unrooted spanning (hyper)forests is induced by a theory with an underlying OSP(1|2) supersymmetry, and in particular that *the most general* OSP(1|2)-invariant non-linear σ-model leads to a combinatorial expansion in terms of hyperforests only, in a way similar to how the most general ($\mathbb{Z}_2$-invariant) Ising Model leads to a combinatorial expansion in terms of Eulerian subgraphs only.

The role of a certain subalgebra of the Grassmann Algebra is elucidated, and a set of useful properties are proven and exploited. The key fact in the existence of these striking properties is that this subalgebra corresponds to the algebra of OSP(1|2)-invariant functions, where the OSP(1|2) supersymmetry is realized in a non-linear way. These facts will be discussed in greater detail in the forthcoming chapters.

8.1 Introduction

Kirchhoff's matrix-tree theorem [112, 113, 114] and its generalizations [115, 116, 117], which express the generating polynomials of spanning trees and rooted spanning forests in a graph as determinants associated to the graph's Laplacian matrix, play a central role in electrical circuit theory [118, 119] and in certain exactly-soluble models in statistical mechanics [120].

Like all determinants, those arising in Kirchhoff's theorem can be rewritten as Gaussian integrals over fermionic (Grassmann) variables. Indeed, the use of Grassmann–Berezin calculus [129] has provided an interesting short-cut toward the classical matrix-tree result as well as generalizations thereof [117, 130]. For instance, Abdesselam [117] has obtained in a simple way the recent pfaffian-tree theorem [131, 132, 133] and has generalized it to a hyperpfaffian-cactus theorem.

In [130] we proved a generalization of Kirchhoff's theorem, in which a large class of combinatorial objects are represented by suitable *non-Gaussian* Grassmann integrals. In particular, we showed how the generating function of spanning forests in a graph, which

arises as the $q \to 0$ limit of the partition function of the q-state Potts model [134, 135, 85, 12], can be represented as a Grassmann integral involving a quadratic (Gaussian) term together with a special nearest-neighbor four-fermion interaction. Furthermore, this fermionic model possesses an OSP(1|2) supersymmetry.

This fermionic formulation is also well-suited to the use of standard field-theoretic machinery. For example, in [130] we obtained the renormalization-group flow near the spanning-tree (free-field) fixed point for the spanning-forest model on the square lattice, and in [89] this was extended to the triangular lattice.

In the present chapter we will review these results, but in a wider context, on the lines of [136]. We indeed extend the fermionic representation of spanning forests from graphs to hypergraphs (hypergraphs are described briefly in section 1.4). We shall show here how the generating function of spanning hyperforests in a hypergraph, which arises as the $q \to 0$ limit of the partition function of the q-state Potts model on the hypergraph [7], can be represented as a Grassmann integral involving a quadratic term together with special multi-fermion interactions associated to the hyperedges. Once again, this fermionic model possesses an OSP(1|2) supersymmetry. This extension from graphs to hypergraphs is thus not only natural, but actually sheds light on the underlying supersymmetry.

Let us begin by recalling briefly the combinatorial identities proven in [130], which come in several levels of generality. Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E. To each edge e we associate a weight w_e, which can be a real or complex number or, more generally, a formal algebraic variable; we then define the *(weighted) Laplacian matrix* $L = (L_{ij})_{i,j \in V}$ for the graph G by

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum\limits_{k \neq i} w_{ik} & \text{if } i = j \end{cases} \tag{8.1}$$

We introduce, at each vertex $i \in V$, a pair of Grassmann variables ψ_i, $\bar{\psi}_i$, which obey the usual rules for Grassmann integration ([129, 137], see also Appendix A). Our identities show that certain Grassmann integrals over ψ and $\bar{\psi}$ can be interpreted as generating functions for certain classes of combinatorial objects on G.

Our most general identity concerns the operators Q_Γ associated to arbitrary connected subgraphs $\Gamma = (V_\Gamma, E_\Gamma)$ of G via the formula

$$Q_\Gamma = \left(\prod_{e \in E_\Gamma} w_e \right) \left(\prod_{i \in V_\Gamma} \bar{\psi}_i \psi_i \right) \ . \tag{8.2}$$

(Note that each Q_Γ is even and hence commutes with the entire Grassmann algebra.) We prove the very general identity

$$\int \mathcal{D}(\psi, \bar{\psi}) \ \exp\left[\bar{\psi} L \psi + \sum_\Gamma t_\Gamma Q_\Gamma \right] \ = \sum_{\substack{H \text{ spanning} \subseteq G \\ H = (H_1, \ldots, H_\ell)}} \left(\prod_{e \in H} w_e \right) \prod_{\alpha=1}^{\ell} W(H_\alpha) \ , \tag{8.3}$$

where the sum runs over spanning subgraphs $H \subseteq G$ consisting of connected components $(H_1, \ldots, H_\ell)$, and the weights $W(H_\alpha)$ are defined by

$$W(H_\alpha) = \sum_{\Gamma \prec H_\alpha} t_\Gamma \ , \tag{8.4}$$

where $\Gamma \prec H_\alpha$ means that H_α contains Γ and contains no cycles other than those lying entirely within Γ.

Let us now specialize (8.3) to the case in which $t_\Gamma = t_i$ when Γ consists of a single vertex i with no edges, $t_\Gamma = u_e$ when Γ consists of a pair of vertices i, j linked by an edge e, and $t_\Gamma = 0$ otherwise. We then have

$$\begin{aligned}\int \mathcal{D}(\psi,\bar\psi)\, \exp\Big[\bar\psi L\psi \,+\, \sum_i t_i \bar\psi_i\psi_i \,+\, \sum_{\langle ij\rangle} u_{ij} w_{ij} \bar\psi_i\psi_i\bar\psi_j\psi_j\Big] \\ = \sum_{\substack{F\,\in\,\mathcal{F}(G)\\ F\,=\,(F_1,\ldots,F_\ell)}} \Big(\prod_{e\in F} w_e\Big) \prod_{\alpha=1}^{\ell} \Big(\sum_{i\in V(F_\alpha)} t_i + \sum_{e\in E(F_\alpha)} u_e\Big)\,,\end{aligned} \tag{8.5}$$

where the sum runs over spanning forests F in G with components $F_1, \ldots, F_\ell$; here $V(F_\alpha)$ and $E(F_\alpha)$ are, respectively, the vertex and edge sets of the tree F_α.

If we further specialize (8.5) to $u_e = -\lambda$ for all edges $e$and $t_i = \lambda$ for all vertices i, we obtain

$$\begin{aligned}\int \mathcal{D}(\psi,\bar\psi)\, \exp\Big[\bar\psi L\psi \,+\, \lambda\sum_i \bar\psi_i\psi_i \,-\, \lambda\sum_{\langle ij\rangle} w_{ij} \bar\psi_i\psi_i\bar\psi_j\psi_j\Big] \\ = \sum_{F\in\mathcal{F}(G)} \Big(\prod_{e\in F} w_e\Big)\, \lambda^{k(F)} = \lambda^{|V|} \sum_{F\in\mathcal{F}(G)} \Big(\prod_{e\in F} \frac{w_e}{\lambda}\Big)\end{aligned} \tag{8.6}$$

where $k(F)$ is the number of component trees in the forest F; this is the generating function of (unrooted) spanning forests of G. Remark how one crucially exploits the fact that, for any tree T, $|V(T)| - |E(T)| = 1$. Furthermore, as discussed in [130] and in more detail in Section 8.6 below, the model (8.6) possesses an OSP(1|2) invariance.

If, by contrast, in (8.5) we take $u_{ij} = 0$ but $\{t_i\}$ general, we obtain

$$\int \mathcal{D}(\psi,\bar\psi)\, \exp\Big[\bar\psi L\psi \,+\, \sum_i t_i \bar\psi_i\psi_i\Big] \;=\; \sum_{\substack{F\,\in\,\mathcal{F}(G)\\ F\,=\,(F_1,\ldots,F_\ell)}} \Big(\prod_{e\in F} w_e\Big) \prod_{\alpha=1}^{\ell} \Big(\sum_{i\in V(F_\alpha)} t_i\Big)\,, \tag{8.7}$$

which is the formula representing *rooted* spanning forests (with a weight t_i for each root i) as a fermionic Gaussian integral (i.e., a determinant) involving the Laplacian matrix (this formula is a variant of the so-called "principal-minors matrix-tree theorem").

Remark however how this formula is comparatively much less interesting than (8.6), at least from the point of view of statistical field theory, as it corresponds in this perspective, to the obvious massive perturbation of the "free (massless) complex scalar fermion" theory, implicit in the Grassmann formulation of Kirchhoff Theorem, while (8.6) shows a specially tuned non-trivial interaction, corresponding to "switching" a non-zero curvature in an underlying non-linear σ-model.

Here we shall not attempt to find the hypergraph analogue of the general formula (8.3), but shall limit ourselves to finding analogues of (8.5)–(8.7). The formulae to be presented here thus express the generating functions of unrooted or rooted spanning hyperforests in a hypergraph in terms of Grassmann integrals. In particular, the hypergraph generalization of (8.6) possesses the same OSP(1|2) supersymmetry that (8.6) does (see Section 8.6). Reasonings *a posteriori* on the symmetry property of the resulting expression will suggest indeed that an extension of the general formula (8.3) is not even desiderable, as it would hardly be a 'natural' object.

The proof given here of all these identities is purely algebraic (and astonishingly simple); the crucial ingredient is to recognize the role and the rules of a certain Grassmann

subalgebra (see Section 8.3). It turns out (Section 8.6) that this subalgebra is nothing other than the algebra of OSP(1|2)-invariant functions, though this is far from obvious at first sight. The unusual properties of this subalgebra (see Lemma 8.1) thus provide a deeper insight into the identities derived in [130] as well as their generalizations to hypergraphs, and indeed provide an alternate proof of (8.6) and (8.7). Pictorially, we can say that it is the underlying supersymmetry that is responsible for the cancellation of the cycles in the generating function, leaving only those spanning (hyper)graphs that have no cycles, namely, the (hyper)forests.

In particular, the limit of spanning hyper*trees*, which is easily extracted from the general expression for (rooted or unrooted) hyperforests, corresponds in the OSP(1|2)-invariant σ-model to the limit in which the radius of the supersphere tends to infinity, so that the nonlinearity due to the curvature of the supersphere disappears. However, the action is in general still non-quadratic, so that the model is not exactly soluble. (This is no accident: even the problem of determining whether there *exists* a spanning hypertree in a given hypergraph is NP-complete [138]).

Only in the special case of ordinary graphs is the action purely quadratic, so that the partition function is given by a determinant, corresponding to the statement of Kirchhoff's matrix-tree theorem.

The OSP(1|2)-invariant fermionic models discussed in [130] and the present analysis can be performed in three equivalent ways:

- As purely fermionic models, in which the supersymmetry is somewhat hidden.
- As σ-models with spins taking values in the unit supersphere in $\mathbb{R}^{1|2}$, in which the supersymmetry is manifest.
- As $O(n)$ non-linear σ-models, as defined in Chapter 3. The spins taking values in the unit sphere of $\mathbb{R}^n$, we should understand the model as analytically continued to $n = -1$.

The first two formulations (and their equivalence) are discussed in Section 8.6. Further aspects of this equivalence – notably, the role played by the Ising variables arising in (8.95) and neglected here – will be discussed in more detail elsewhere [139], and are partially presented in Section 3.2.

In Chapter 9 we will discuss the Ward identities associated to the OSP(1|2) supersymmetry, and their relation to the combinatorial identities describing the possible connection patterns among the (hyper)trees of a (hyper)forest.

The method proposed in the present paper has additional applications not considered here. With a small further effort, a class of Grassmann integrals wider than (8.60)/(8.74) – allowing products $\prod_\alpha f^{(\lambda)}_{C_\alpha}$ in the action in place of the single operators $f^{(\lambda)}_A$ – can be handled. Once again one obtains a graphical expansion in terms of spanning hyperforests, where now the weights have a more complicated dependence on the set of hyperedges, thus permitting a description of certain natural interaction patterns among the hyperedges of a hyperforest (see the concluding remarks at the end of Section 8.4). This extended model is, in fact, the *most general* Hamiltonian that is invariant under the OSP(1|2) supersymmetry.

8.2 Graphical approach to generalized matrix-tree theorems

In this section we shall give a "graphical" proof of the classical matrix-tree theorem as well as a number of extensions thereof, by interpreting in a graphical way the terms of a

formal Taylor expansion of an action belonging to the even subalgebra of a Grassmann algebra. (We require the action to belong to the even subalgebra in order to avoid ordering ambiguities when exponentiating a sum of terms.)

This is the widest generalization of the original approach in [130], which makes appeal to visualization and combinatorics for facts that are now better understood on purely algebraic grounds. It could be however of some utility, by providing a different perspective, especially in the extensions where the underlying symmetry is broken and the algebraic techniques become comparatively weaker.

So, some of these extensions of the matrix-tree theorem will be discussed in the forthcoming sections where they are proven by an "algebraic" method, based on Lemma 8.1 and its corollaries. Other more exotic extensions, described here with an eye to future work, could also be proven by simple suitable variants of the algebraic technique, that we leave to the reader.

Curiously enough, it turns out that the more general is the fact we want to prove, the *easier* is the proof; indeed, the most general facts ultimately become almost tautologies on the rules of Grassmann algebra and integration. The only extra feature of the most general facts is that the "zoo" of graphical combinatorial objects has to become wider (and wilder).

So, in this exposition we shall start by describing the most general situation, and then show how, when special cases are chosen for the parameters in the action, a corresponding simplification occurs also in the combinatorial interpretation.

Consider a hypergraph $G = (V, E)$ as defined in Section 1.4, i.e. V is a finite set and E is a set of subsets of V, each of cardinality at least 2, called *hyperedges*. As usual we introduce a pair $\psi_i, \bar\psi_i$ of Grassmann generators for each $i \in V$. We shall consider actions of the form

$$\mathcal{S}(\psi, \bar\psi) = \sum_{A \in E} \mathcal{S}_A(\psi, \bar\psi) \;, \tag{8.8}$$

where

$$\mathcal{S}_A(\psi, \bar\psi) = w^*_A \tau_A + \sum_{i \in A} w_{A;i}\, \tau_{A \setminus i} + \sum_{\substack{i,\, j \,\in\, A \\ i \,\neq\, j}} w_{A;ij}\, \psi_i \bar\psi_j \tau_{A \setminus \{i,j\}} \tag{8.9}$$

and $\tau_A = \prod_{i \in A} \bar\psi_i \psi_i$. Please note that the form (8.9) resembles the definition (8.25) of $f_A^{(\lambda)}$ the same monomials appear, but now each one is multiplied by an independent indeterminate. Thus, for each hyperedge A of cardinality k we have $k^2 + 1$ parameters: w^*_A, $\{w_{A;i}\}_{i \in A}$ and $\{w_{A;i,j}\}_{(i \neq j) \in A}$. [We have chosen, for future convenience, to write the last term in (8.25) as $+\psi_i \bar\psi_j$ rather than $-\bar\psi_i \psi_j$.]

Please note that, for $|A| > 2$, all pairs of terms in $\mathcal{S}_A(\psi, \bar\psi)$ have a vanishing product, because they contain at least $2(2|A| - 2) = 4|A| - 4$ fermions in a subalgebra (over A) that has only $2|A|$ distinct fermions. As a consequence, we have in this case

$$\exp[\mathcal{S}_A(\psi, \bar\psi)] = 1 + \mathcal{S}_A(\psi, \bar\psi) \,. \tag{8.10}$$

On the other hand, if $|A| = 2$ (say, $A = \{i, j\}$), we have two nonvanishing cross-terms:

$$(w_{A;i}\, \bar\psi_j \psi_j)\, (w_{A;j}\, \bar\psi_i \psi_i) = w_{A;i} w_{A;j}\, \bar\psi_i \psi_i \bar\psi_j \psi_j \tag{8.11a}$$

$$(w_{A;ij}\, \psi_i \bar\psi_j)\, (w_{A;ji}\, \psi_j \bar\psi_i) = -w_{A;ij} w_{A;ji}\, \bar\psi_i \psi_i \bar\psi_j \psi_j \tag{8.11b}$$

where the minus sign comes from commutation of fermionic fields. So we can write in the general case

$$\exp[\mathcal{S}_A(\psi,\bar\psi)] = 1 + \widehat{\mathcal{S}}_A(\psi,\bar\psi) \;, \tag{8.12}$$

where $\widehat{\mathcal{S}}_A(\psi,\bar\psi)$ is defined like $\mathcal{S}_A(\psi,\bar\psi)$ but with the parameter w_A^* replaced by

$$\widehat{w}_A^* = \begin{cases} w_A^* + w_{A;i} w_{A;j} - w_{A;ij} w_{A;ji} & \text{if } A = \{i,j\} \\ w_A^* & \text{if } |A| \ge 3 \end{cases} \tag{8.13}$$

Consider now a Grassmann integral of the form

$$\int \mathcal{D}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \exp\left[\sum_i t_i \bar\psi_i \psi_i + \sum_{A \in E} \mathcal{S}_A(\psi,\bar\psi) \right] , \tag{8.14}$$

where $\mathbf{t} = (t_i)_{i \in V}$ are parameters, $I = (i_1, i_2, \ldots, i_k) \in V^k$ and $J = (j_1, j_2, \ldots, j_k) \in V^k$ are ordered k-tuples of vertices, and

$$\mathcal{O}_{I,J} \;:=\; \bar\psi_{i_1} \psi_{j_1} \cdots \bar\psi_{i_k} \psi_{j_k} \tag{8.15}$$

[cf. (8.72)]. Here the $i_1, \ldots, i_k$ must be all distinct, as must the $j_1, \ldots, j_k$, but there can be overlaps between the sets $\mathsf{I} = \{i_1, i_2, \ldots, i_k\}$ and $\mathsf{J} = \{j_1, j_2, \ldots, j_k\}$.[†] We intend to show that (8.14) can be interpreted combinatorially as a generating function for rooted oriented[‡] spanning sub(hyper)graphs of G, in which each connected component is either a (hyper-)tree or a (hyper-)unicyclic. In the case of a unicyclic component, the rest of the component is oriented towards the cycle, and no vertex from $\mathsf{I} \cup \mathsf{J}$ lies in the component. In the case of a tree component, either (a) no vertex from $\mathsf{I} \cup \mathsf{J}$ is in the component, and then there is either a special "root" vertex or a "root" hyperedge, all the rest of the tree being oriented towards it, or (b) the component contains a single vertex from $\mathsf{I} \cap \mathsf{J}$, which is the root vertex, and the tree is again oriented towards it, or (c) the component contains exactly one vertex from I and one from J, a special oriented path connecting them, and all the rest is oriented towards the path. The weight of each configuration is essentially the product of t_i for each root $i \notin \mathsf{I} \cup \mathsf{J}$ and an appropriate weight ($\widehat{w}_A^*$, $w_{A;i}$ or $w_{A;ij}$) for each occupied hyperedge, along with a $-$ sign for each unicyclic using the $w_{A;ij}$'s and a single extra $\pm$ sign corresponding to the pairing of vertices of I to vertices of J induced by being in the same component. (This same sign appeared already in Section 8.5.)

Kirchhoff's matrix-tree theorem arises when all the hyperedges A have cardinality 2 (i.e. G is an ordinary graph), $\mathsf{I} = \mathsf{J} = \{i_0\}$ for some vertex i_0, all $t_i = 0$, all $w_A^* = 0$, and $w_{A;i} = w_{A;ij} = w_A$. The principal-minors matrix-tree theorem is obtained by allowing $\mathsf{I} = \mathsf{J}$ of arbitrary cardinality k, while the all-minors matrix-tree theorem is obtained by allowing also $\mathsf{I} \neq \mathsf{J}$. Rooted forests with root weights t_i can be obtained by allowing $t_i \neq 0$. On the other hand, unrooted forests are obtained by taking all $t_i = \lambda$, $\mathsf{I} = \mathsf{J} = \varnothing$, $w_A^* = -\lambda w_A$ and the rest as above. [More generally, unrooted hyperforests are obtained by taking all $t_i = \lambda$, $\mathsf{I} = \mathsf{J} = \varnothing$, $w_A^* = -\lambda(|A|-1) w_A$ and the rest as above.] The sequences I and J are used mainly in order to obtain expectation values of certain connectivity patterns in the relevant ensemble of spanning subgraphs.

Let us now prove all these statements, and give precise expressions for the weights of the configurations, which until now have been left deliberately vague in order not to overwhelm the reader.

[†]Please note the distinction between the *ordered* k-tuple $I = (i_1, i_2, \ldots, i_k)$, here written in italic font, and the *unordered* set $\mathsf{I} = \{i_1, i_2, \ldots, i_k\}$, here written in sans-serif font.

[‡]We shall define later what we mean by "orienting" a hyperedge A: it will correspond to selecting a single vertex $i \in A$ as the "outgoing" vertex.

Factors coming from $\mathcal{O}_{I\cup V',J\cup V'}$		
$\bar\psi_i\psi_i$	i ◉	root vertex
$\bar\psi_i$	i ○	sink vertex
ψ_i	i ×	source vertex

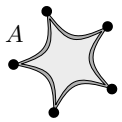
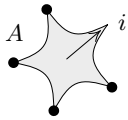
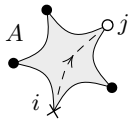
Factors coming from $\prod \widehat{\mathcal{S}}_A$		
τ_A		root hyperedge
$\tau_{A\setminus i}$		pointing hyperedge
$\psi_i\bar\psi_j\tau_{A\setminus\{i,j\}}$		dashed hyperedge

Table 8.1. Graphical representation of the various factors in the expansion (8.17).

We start by manipulating (8.14), exponentiating the action to obtain

$$\int \mathcal{D}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \left(\prod_{i\in V}(1+t_i\bar\psi_i\psi_i)\right)\left(\prod_{A\in E}(1+\widehat{\mathcal{S}}_A)\right) \tag{8.16}$$

or, expanding the last products,

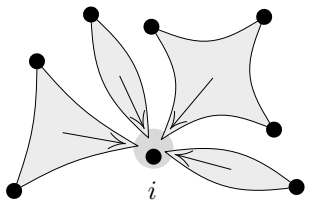
$$\sum_{\substack{V'\subseteq V\setminus(I\cup J)\\ E'\subseteq E}} \left(\prod_{i\in V'} t_i\right) \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{O}_{I\cup V',J\cup V'} \left(\prod_{A\in E'} \widehat{\mathcal{S}}_A\right), \tag{8.17}$$

where $I\cup V'$ consists of the sequence I followed by the list of elements of V' in any chosen order, and $J\cup V'$ consists of the sequence J followed by the list of elements of V' in the *same* order.

We now give a graphical representation and a fancy name to each kind of monomial in the expansion (8.17), as shown in Table 8.1.

Please note that in this graphical representation a solid circle • corresponds to a factor $\bar\psi_i\psi_i$, an open circle ○ corresponds to a factor $\bar\psi_i$, and a cross × corresponds to a factor ψ_i.

According to the rules of Grassmann algebra and Grassmann–Berezin integration, we must have in total exactly one factor $\bar\psi_i$ and one factor ψ_i for each vertex i. Graphically this means that at each vertex we must have either a single • or else the superposed pair ⊗ (please note that in many drawings we actually draw the ○ and × slightly split, in order to highlight which variable comes from which factor). At each vertex i we can have an arbitrary number of "pointing hyperedges" pointing towards i, as they do not carry any fermionic field:

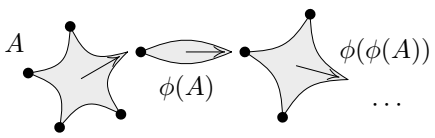


Aside from pointing hyperedges, we must be, at each vertex i, in one of the following situations (Figure 8.1):

(a) If $i \in V'$ or $i \in \mathsf{I} \cap \mathsf{J}$ [resp. cases (a) and (b) in the figure], the quantity $\mathcal{O}_{I\cup V',J\cup V'}$ provides already a factor $\bar{\psi}_i\psi_i$; therefore, no other factors of $\bar{\psi}_i$ or ψ_i should come from the expansion of $\prod \widehat{\mathcal{S}}_A$.
(b) If $i \in \mathsf{I} \smallsetminus \mathsf{J}$, the quantity $\mathcal{O}_{I\cup V',J\cup V'}$ provides already a factor $\bar{\psi}_i$; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide ψ_i, i.e. we must have one dashed hyperedge pointing from i.
(c) If $i \in \mathsf{J} \smallsetminus \mathsf{I}$, the quantity $\mathcal{O}_{I\cup V',J\cup V'}$ provides already a factor ψ_i; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide $\bar{\psi}_i$, i.e. we must have one dashed hyperedge pointing towards i.
(d) If $i \notin \mathsf{I} \cup \mathsf{J} \cup V'$, then the quantity $\mathcal{O}_{I\cup V',J\cup V'}$ provides neither $\bar{\psi}_i$ nor ψ_i; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide both $\bar{\psi}_i$ and ψ_i, so that at i we must have one of the following configurations:
   a) a non-pointed vertex of a pointing hyperedge;
   b) a vertex of a dashed hyperedge that is neither of the two endpoints of the dashed arrow;
   c) a vertex of a root hyperedge;
   d) two dashed hyperedges, one with the arrow incoming, one outgoing.

Having given the local description of the possible configurations at each vertex i, let us now describe the possible global configurations. Note first that at each vertex we can have at most two incident dashed arrows, and if there are two such arrows then they must have opposite orientations. As a consequence, we see that dashed arrows must either form cycles, or else form open paths connecting a source vertex of $\mathsf{I} \smallsetminus \mathsf{J}$ to a sink vertex of $\mathsf{J} \smallsetminus \mathsf{I}$. Let us use the term *root structures* to denote root vertices, root hyperedges, cycles of dashed hyperedges, and open paths of dashed hyperedges.

As for the solid arrows in the pointing hyperedges, the reasoning is as follows: If a pointing hyperedge A points towards i, then either i is part of a root structure as described above, or else it is a non-pointed vertex of another pointing hyperedge $\varphi(A)$. We can follow this map iteratively, i.e. go to $\varphi(\varphi(A))$, and so on:



Because of the finiteness of the graph, either we ultimately reach a root structure, or we enter a cycle. Cycles of the "dynamics" induced by φ correspond to cycles of the pointing hyperedges. We now also include such cycles of pointing hyperedges as a fifth type of root structure (see Figure 8.2 for the complete list of root structures).

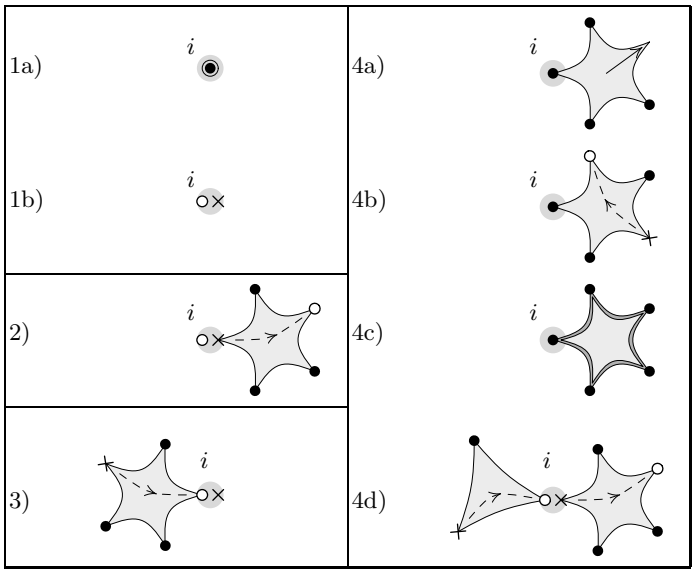


Fig. 8.1. Possible ways of saturating the Grassmann fields on vertex i (indicated by the gray disk).

All the rest is composed of pointing hyperedges, which form directed arborescences, rooted on the vertices of the root structures. In conclusion, therefore, the most general configuration consists of a bunch of disjoint root structures, and a set of directed arborescences (possibly reduced to a single vertex) rooted at its vertices, such that the whole is a spanning subhypergraph H of G.

As each root structure is either a single vertex, a single hyperedge, a (hyper-)path or a (hyper-)cycle, we see that each connected component of H is either a (hyper-)tree or a (hyper-)unicyclic. Furthermore, all vertices in $\mathsf{I} \cup \mathsf{J}$ are in the tree components, and each tree contains either one vertex from I and one from J (possibly coincident) or else no vertices at all from $\mathsf{I} \cup \mathsf{J}$.

We still need to understand the weights associated to the allowed configurations. Clearly, we have a factor $w_{A;i}$ per pointing hyperedge in the arborescence. Root vertices coming from V' have factors t_i, and root hyperedges have factors $\widehat{w}^*_A$. Cycles $\gamma = (i_0, A_1, i_1, A_2, \ldots, i_\ell = i_0)$ of the dynamics of φ (*bosonic cycles*) have a weight $w_{A_1;i_1} \cdots w_{A_\ell;i_\ell}$. All the foregoing objects contain Grassmann variables only in the combination $\bar\psi_i \psi_i$, and hence are commutative. Finally, we must consider the dashed hyperedges, which contain "unpaired fermions" ψ_i and $\bar\psi_j$, and hence will give rise to signs coming from anticommutativity. Let us first consider the dashed cycles $\gamma = (i_0, A_1, i_1, A_2, \ldots, i_\ell = i_0)$, and note what happens when reordering the fermionic fields:

$$\begin{aligned} &(w_{A_1;i_\ell i_1} \psi_{i_\ell} \bar\psi_{i_1})(w_{A_2;i_1 i_2} \psi_{i_1} \bar\psi_{i_2}) \cdots (w_{A_\ell;i_{\ell-1} i_\ell} \psi_{i_{\ell-1}} \bar\psi_{i_\ell}) \\ &\quad = -w_{A_1;i_\ell i_1} w_{A_2;i_1 i_2} \cdots w_{A_\ell;i_{\ell-1} i_\ell}\, \bar\psi_{i_1} \psi_{i_1} \cdots \bar\psi_{i_\ell} \psi_{i_\ell} \end{aligned} \tag{8.18}$$

because ψ_{i_ℓ} had to pass through $2\ell - 1$ fermionic fields to reach its final location. This is pretty much the result one would have expected, *but* we have an overall minus sign, irrespective of the length of the cycle (or its parity), which is in a sense "non-local", due to the fermionic nature of the fields ψ and $\bar\psi$. For this reason we call a dashed cycle a *fermionic cycle*.

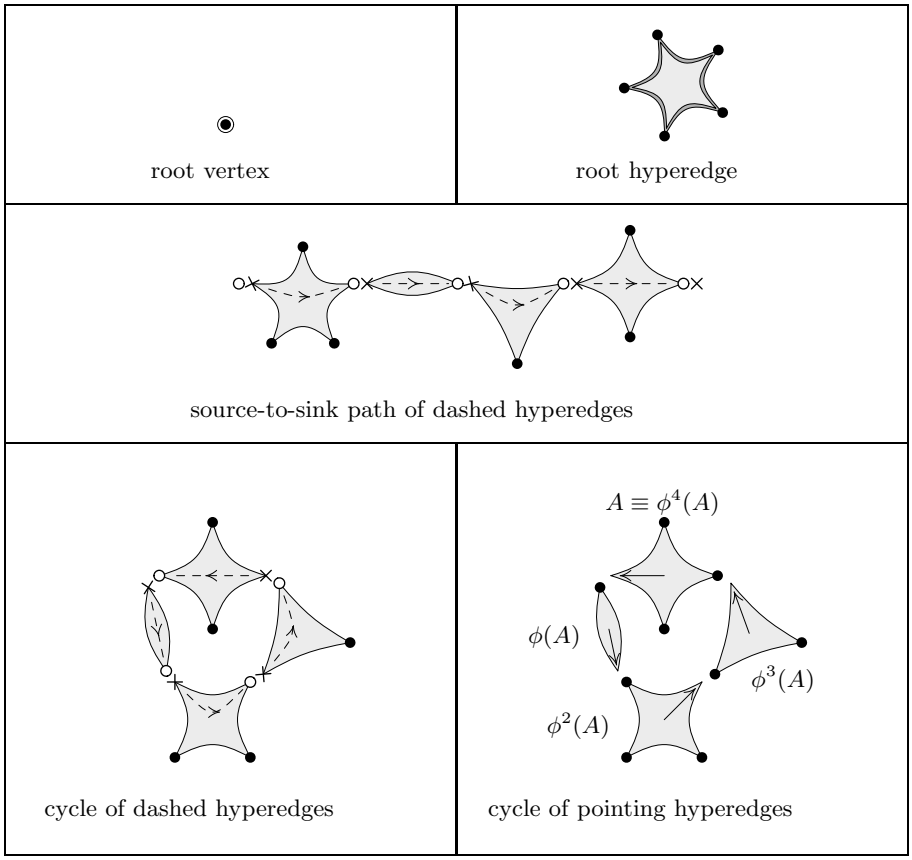


Fig. 8.2. The five kinds of root structures.

A similar mechanism arises for the open paths of dashed hyperedges $\gamma = (i_0, A_1, i_1, A_2, \dots, i_\ell)$, where i_0 is the source vertex and i_ℓ is the sink vertex. Here the weight $w_{A_1;i_0 i_1} w_{A_2;i_1 i_2} \cdots w_{A_\ell;i_{\ell-1} i_\ell}$ multiplies the monomial $\psi_{i_0} \bar\psi_{i_1} \psi_{i_1} \bar\psi_{i_2} \psi_{i_2} \cdots \bar\psi_{i_{\ell-1}} \psi_{i_{\ell-1}} \bar\psi_{i_\ell}$, in which the only unpaired fermions are ψ_{i_0} and $\bar\psi_{i_\ell}$. in this order. Now the monomials for the open paths must be multiplied by $\mathcal{O}_{I,J}$, and each source (resp. sink) vertex from an open path must correspond to a vertex of I (resp. J). This pairing thus induces a permutation of $\{1, \dots, k\}$, where $k = |\mathsf{I}| = |\mathsf{J}|$: namely, i_r is connected by an open path to $j_{\pi(r)}$. We then have

$$\left(\prod_{r=1}^{k} \bar\psi_{i_r} \psi_{j_r} \right) \left(\prod_{r=1}^{k} \psi_{i_r} \bar\psi_{j_{\pi(r)}} \right) , \tag{8.19}$$

where the first product is $\mathcal{O}_{I,J}$ and the second product comes from the open paths. This can easily be rewritten as

$$\prod_{r=1}^{k} \bar{\psi}_{i_r} \psi_{j_r} \psi_{i_r} \bar{\psi}_{j_{\pi(r)}} = \prod_{r=1}^{k} \bar{\psi}_{i_r} \psi_{i_r} \bar{\psi}_{j_{\pi(r)}} \psi_{j_r} \tag{8.20a}$$

$$= \left(\prod_{r=1}^{k} \bar{\psi}_{i_r} \psi_{i_r} \right) \left(\prod_{r=1}^{k} \bar{\psi}_{j_{\pi(r)}} \psi_{j_r} \right) \tag{8.20b}$$

$$= \operatorname{sgn}(\pi) \left(\prod_{r=1}^{k} \bar{\psi}_{i_r} \psi_{i_r} \right) \left(\prod_{r=1}^{k} \bar{\psi}_{j_r} \psi_{j_r} \right) . \tag{8.20c}$$

Putting everything together, we see that the Grassmann integral (8.14) can be represented as a sum over rooted oriented spanning subhypergraphs $\mathbf{H}$ of G, as follows:

- Each connected component of H (the unoriented subhypergraph corresponding to $\mathbf{H}$) is either a (hyper-)tree or a (hyper-)unicyclic.
- Each (hyper-)tree component contains either one vertex from I (the *source vertex*) and one from J (the *sink vertex*, which is allowed to coincide with the source vertex), or else no vertex from $\mathsf{I} \cup \mathsf{J}$. In the latter case, we choose either one vertex of the component to be the *root vertex*, or else one hyperedge of the component to be the *root hyperedge*.
- Each unicyclic component contains no vertex from $\mathsf{I} \cup \mathsf{J}$. As a unicyclic, it necessarily has the form of a single (hyper-)cycle together with (hyper-)trees (possibly reduced to a single vertex) rooted at the vertices of the (hyper-)cycle.
- Each hyperedge other than a root hyperedge is *oriented* by designating a vertex $i(A) \in A$ as the *outgoing vertex*. These orientations must satisfy following rules:
  (i) each (hyper-)tree component is directed towards the sink vertex, root vertex or root hyperedge,
  (ii) each (hyper-)tree belonging to a unicyclic component is oriented towards the cycle, and
  (iii) the (hyper-)cycle of each unicyclic component is oriented consistently.
  Thus, in each (hyper-)tree component the orientations are fixed uniquely, while in each unicyclic component we sum over the two consistent orientations of the cycle.

The *weight* of a configuration $\mathbf{H}$ is the product of the weights of its connected components, which are in turn defined as the product of the following factors:

- Each root vertex i gets a factor t_i.
- Each root hyperedge A gets a factor $\widehat{w}^*_A$.
- Each hyperedge A belonging to the (unique) path from a source vertex to a sink vertex gets a factor $w_{A;ij}$, where j is the outgoing vertex of A and i is the outgoing vertex of the preceding hyperedge along the path (or the source vertex if A is the first hyperedge of the path).
- Each hyperedge A that does not belong to a source-sink path or to a cycle gets a factor $w_{A;i(A)}$ [recall that $i(A)$ is the outgoing vertex of A].
- Each oriented cycle $(i_0, A_1, i_1, A_2, \ldots, i_\ell = i_0)$ gets a weight

$$\prod_{\alpha=1}^{\ell} w_{A_\alpha;i_\alpha} - \prod_{\alpha=1}^{\ell} w_{A_\alpha;i_{\alpha-1}i_\alpha} . \tag{8.21}$$

- There is an overall factor $\operatorname{sgn}(\pi)$.

Finally, we consider some special cases of the formulas above. The contribution from unicyclic components cancels out whenever $\prod_{\alpha=1}^{\ell} w_{A_\alpha;i_\alpha} = \prod_{\alpha=1}^{\ell} w_{A_\alpha;i_{\alpha-1}i_\alpha}$ for every oriented cycle $(i_0, A_1, i_1, A_2, \ldots, i_\ell = i_0)$. In particular, this happens if $w_{A;ij} = w_{A;j}$ for all A and all $i, j \in A$. More generally, it happens if $w_{A;ij} = w_{A;j} \exp(\phi_{A;ij})$ where ϕ has "zero circulation" in the sense that $\sum_{\alpha=1}^{\ell} \phi_{A_\alpha;i_{\alpha-1}i_\alpha} = 0$ for every oriented cycle $(i_0, A_1, i_1, A_2, \ldots, i_\ell = i_0)$. Physically, ϕ can be thought of as a kind of "gauge field' to which the fermions $\psi, \bar\psi$ are coupled; the zero-circulation condition means that ϕ is gauge-equivalent to zero. Note, finally, that if $w_{A;i} w_{A;j} = w_{A;ij} w_{A;ji}$ for all $i, j \in A$, then $\widehat{w}^*_A = w^*_A$.

At the other extreme, if we take all $t_i = 0$, all $\widehat{w}^*_A = 0$ and $I = J = \varnothing$, then all tree components disappear, and we are left with only unicyclics.

In certain "symmetric" circumstances, we can combine the contributions from tree components having the same set of (unoriented) hyperedges but different roots, and obtain reasonably simple expressions. In particular, suppose that the weights $w_{A;i}$ are independent of i (let us call them simply w_A), and consider a tree component T that does not contain any vertices of $\mathsf{I} \cup \mathsf{J}$. Then we can sum over all choices of root vertex or root hyperedge, and obtain the weight

$$\left(\prod_{A \in E(T)} w_A \right) \left(\sum_{i \in V(T)} t_i + \sum_{A \in E(T)} \frac{\widehat{w}^*_A}{w_A} \right) . \tag{8.22}$$

A further simplification occurs in two cases:

- If all $t_i = t$ and all $\widehat{w}^*_A = 0$, then the second factor in (8.22) becomes simply $t|V(T)|$: we obtain forests of *vertex-weighted trees*.
- If all $t_i = t$ and $\widehat{w}^*_A = t(1 - |A|)w_A$ for all A, then the second factor in (8.22) becomes simply t (by virtue of Proposition 1.1) and we obtain *unrooted* forests.

Recall, finally, that if we also take $w_{A;ij} = w_A$ for all A and all $i, j \in A$, then the unicyclic components cancel and $\widehat{w}^*_A = w^*_A$, so that (8.22) reduces to (8.67).

It is instructive to consider the special case in which G is an ordinary graph, i.e. each hyperedge $A \in E$ is of cardinality 2. If we further take all $w^*_A = 0$, then the quantity in the exponential of the functional integral (8.14) is a quadratic form $\mathcal{S}(\psi, \bar\psi) + \sum_{i \in V} t_i \bar\psi_i \psi_i = \bar\psi M \psi$, with matrix

$$M_{ij} = \begin{cases} t_i + \sum_{k \neq i} w_{\{i,k\};k} & \text{if } i = j \\ -w_{\{i,j\};ji} & \text{if } i \neq j \end{cases} \tag{8.23}$$

Our result for $I = J = \varnothing$ then corresponds to the "two-matrix matrix-tree theorem" of Moon [116, Theorem 2.1] with $r_{ik} = w_{\{i,k\};k}$ for $i \neq k$, $r_{ii} = 0$, $s_{ij} = w_{\{i,j\};ij}$ for $i \neq j$ and $s_{ii} = -t_i$.[††]

8.3 A Grassmann subalgebra with unusual properties

From this section on, we develop our algebraic techniques, in a way which is independent from the discussion of the previous section.

[††]There is a slight notational difference between us and Moon [116]: he has the bosonic and fermionic cycles going in the same direction, while we have them going in opposite directions. But this does not matter, because $\det(M) = \det(M^{\mathrm{T}})$. Our "transposed" notation was chosen in order to make more natural the definitions of correlation functions in Section 8.5.

Let V be a finite set of cardinality n. For each $i \in V$ we introduce a pair ψ_i, $\bar\psi_i$ of generators of a Grassmann algebra (with coefficients in $\mathbb{R}$ or $\mathbb{C}$). We therefore have $2n$ generators, and the Grassmann algebra (considered as a vector space over $\mathbb{R}$ or $\mathbb{C}$) is of dimension 2^{2n}.

For each subset $A \subseteq V$, we associate the monomial $\tau_A = \prod_{i\in A} \bar\psi_i\psi_i$, where $\tau_\varnothing = 1$. Please note that all these monomials are even elements of the Grassmann algebra; in particular, they commute with the whole Grassmann algebra. Clearly, the elements $\{\tau_A\}_{A\subseteq V}$ span a vector space of dimension 2^n. In fact, this vector space is a subalgebra, by virtue of the obvious relations

$$\tau_A\,\tau_B = \begin{cases} \tau_{A\cup B} & \text{if } A\cap B = \varnothing \\ 0 & \text{if } A\cap B \neq \varnothing \end{cases} \tag{8.24}$$

Let us now introduce another family of even elements of the Grassmann algebra, also indexed by subsets of V, which possesses very interesting and unusual properties. For each subset $A \subseteq V$ and each number λ (in $\mathbb{R}$ or $\mathbb{C}$), we define the Grassmann element

$$f_A^{(\lambda)} \;=\; \lambda(1-|A|)\tau_A + \sum_{i\in A} \tau_{A\smallsetminus i} \;-\; \sum_{\substack{i,\,j\,\in\,A \\ i\,\neq\,j}} \bar\psi_i\psi_j\tau_{A\smallsetminus\{i,j\}}\,. \tag{8.25}$$

(The motivation for this curious formula will be explained in Section 8.6.) For instance, we have

$$f_\varnothing^{(\lambda)} = \lambda \tag{8.26a}$$

$$f_{\{i\}}^{(\lambda)} = 1 \qquad \text{for all } i \tag{8.26b}$$

$$\begin{aligned} f_{\{i,j\}}^{(\lambda)} &= -\lambda\bar\psi_i\psi_i\bar\psi_j\psi_j + \bar\psi_i\psi_i + \bar\psi_j\psi_j - \bar\psi_i\psi_j - \bar\psi_j\psi_i \\ &= -\lambda\bar\psi_i\psi_i\bar\psi_j\psi_j + (\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j) \end{aligned} \tag{8.26c}$$

and in general

$$f_{\{i_1,\ldots,i_k\}}^{(\lambda)} = \lambda(1-k)\tau_{\{i_1,\ldots,i_k\}} + \sum_{\alpha=1}^{k} \tau_{\{i_1,\ldots,\not{i}_\alpha,\ldots,i_k\}} - \sum_{\substack{1\,\leq\,\alpha,\,\beta\,\leq\,k \\ \alpha\,\neq\,\beta}} \bar\psi_{i_\alpha}\psi_{i_\beta}\tau_{\{i_1,\ldots,\not{i}_\alpha,\ldots,\not{i}_\beta,\ldots,i_k\}}\,. \tag{8.27}$$

(Whenever we write a set $\{i_1,\ldots,i_k\}$, it is implicitly understood that the elements $i_1,\ldots,i_k$ are all distinct.) Clearly, each $f_A^{(\lambda)}$ is an even element in the Grassmann algebra, and in particular it commutes with all the other elements of the Grassmann algebra.

The definition (8.25) can also be rewritten as[♯♯]

$$f_A^{(\lambda)} = \Big(\lambda(1-|A|) + \sum_{i,j\in A} \partial_i\bar\partial_j\Big)\tau_A = \Big(\lambda(1-|A|) + \partial\bar\partial\Big)\tau_A \tag{8.28}$$

[♯♯] We "missed" this reformulation, which has been pointed out to us by an anonymous referee. More generally, for a long while we tried to avoid as much as possible the introduction of Clifford Algebra, which instead, at the light of the connection with Clifford representation of a degenerate Temperley-Lieb Algebra described in Section 10.2, shows up to be a natural and fruitful context. See also later for a proof of a "Grassmann" fact that stongly simplifies in the wider "Clifford" context.

where $\partial_i = \partial/\partial\psi_i$ and $\bar\partial_i = \partial/\partial\bar\psi_i$ are the traditional anticommuting differential operators satisfying $\partial_i\psi_j = \delta_{ij}$, $\partial_i\bar\psi_j = 0$, $\bar\partial_i\bar\psi_j = \delta_{ij}$, $\bar\partial_i\psi_j = 0$ and the (anti-)Leibniz rule, while $\partial = \sum_{i\in V}\partial_i$ and $\bar\partial = \sum_{i\in V}\bar\partial_i$.

Let us observe that

$$f_A^{(\lambda)}\,\tau_B = \begin{cases} \tau_{A\cup B} & \text{if } |A\cap B| = 1 \\ 0 & \text{if } |A\cap B| \geq 2 \end{cases} \tag{8.29}$$

as an immediate consequence of (8.24) [when $A\cap B = \{k\}$, only the second term in (8.25) with $i = k$ survives]. Note, finally, the obvious relations

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}f_A^{(\lambda)} = (1-|A|)\tau_A \tag{8.30}$$

and

$$f_A^{(\lambda)} - f_A^{(\lambda')} = (\lambda-\lambda')(1-|A|)\tau_A\ . \tag{8.31}$$

We are interested in the subalgebra of the Grassmann algebra that is generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V, for an arbitrary fixed value of λ.[‡] The key to understanding this subalgebra is the following amazing identity:

Lemma 8.1. *Let $A, B \subseteq V$ with $A\cap B \neq \varnothing$. Then*

$$f_A^{(\lambda)}\,f_B^{(\lambda)} = \begin{cases} f_{A\cup B}^{(\lambda)} & \textit{if } |A\cap B| = 1 \\ 0 & \textit{if } |A\cap B| \geq 2 \end{cases} \tag{8.32}$$

More generally,

$$f_A^{(\lambda)}\,f_B^{(\lambda')} = \begin{cases} f_{A\cup B}^{(\lambda'')} & \textit{if } |A\cap B| = 1 \\ 0 & \textit{if } |A\cap B| \geq 2 \end{cases} \tag{8.33}$$

where λ'' is the weighted average

$$\lambda'' = \frac{(|A|-1)\lambda + (|B|-1)\lambda'}{|A|+|B|-2} = \frac{(|A|-1)\lambda + (|B|-1)\lambda'}{|A\cup B|-1}\ . \tag{8.34}$$

Remarkably the three equations (8.24), (8.29) and (8.32) will fall under one roof, in the generalization we will use to describe a theory with OSP(1|2n), in Chapter 11.

First proof. Call $A\cup B = C$ and $A\cap B = D$. For the first part of the lemma, D has cardinality 1, say $D = \{i\}$. We then have

$$\begin{aligned} f_A^{(\lambda)} = \Big[\lambda(1-|A|)\tau_A + \sum_{j\in A\smallsetminus i}\tau_{A\smallsetminus j} - \sum_{\substack{j_1,\,j_2\,\in\, A\,\smallsetminus\, i \\ j_1\neq j_2}}\bar\psi_{j_1}\psi_{j_2}\tau_{A\smallsetminus\{j_1,j_2\}}\Big] \\ - \sum_{j\in A\smallsetminus i}\bar\psi_i\psi_j\tau_{A\smallsetminus\{i,j\}} - \sum_{j\in A\smallsetminus i}\bar\psi_j\psi_i\tau_{A\smallsetminus\{i,j\}} + \tau_{A\smallsetminus i} \end{aligned} \tag{8.35}$$

$$=: a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i + a_0\ , \tag{8.36}$$

[‡] One can also consider the smaller subalgebras generated by the elements $f_A^{(\lambda)}$ as A ranges over some collection $\mathcal{S}$ of subsets of V.

where the four terms correspond to the expansion with respect to the dependence on $\bar\psi_i$ and ψ_i. A similar expansion can be performed for $f_B^{(\lambda')}$. Then, in the product $f_A^{(\lambda)} f_B^{(\lambda')}$, some terms vanish because $\bar\psi_i^2 = \psi_i^2 = 0$, and we are left with

$$f_A^{(\lambda)} f_B^{(\lambda')} = (a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i + a_0)(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i + b_0) \tag{8.37}$$

$$\begin{aligned} &= a_0b_0 + a_0(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i) + b_0(a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i) \\ &\quad + a\bar b\bar\psi_i\psi_i + b\bar a\bar\psi_i\psi_i \ . \end{aligned} \tag{8.38}$$

Substituting the expressions for the various quantities a_1, a_0, a, $\bar a$ and b_1, b_0, b, $\bar b$, we find

$$a_0b_0 = \tau_{C\smallsetminus i} \tag{8.39}$$

$$a_0(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i) = \lambda'(1-|B|)\tau_C + \sum_{j\in B\smallsetminus i}\tau_{C\smallsetminus j} \tag{8.40}$$

$$- \sum_{\substack{j_1,j_2\in B\smallsetminus i\\ j_1\neq j_2}} \bar\psi_{j_1}\psi_{j_2}\tau_{C\smallsetminus\{j_1,j_2\}} - \sum_{j\in B\smallsetminus i}\bar\psi_i\psi_j\tau_{C\smallsetminus\{i,j\}} - \sum_{j\in B\smallsetminus i}\bar\psi_j\psi_i\tau_{C\smallsetminus\{i,j\}} \tag{8.41}$$

$$b_0(a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i) = \text{same as preceding, with } A\leftrightarrow B \text{ and } \lambda\leftrightarrow\lambda' \tag{8.42}$$

$$a\bar b\bar\psi_i\psi_i = -\sum_{j_1\in A\smallsetminus i}\sum_{j_2\in B\smallsetminus i}\bar\psi_{j_2}\psi_{j_1}\tau_{C\smallsetminus\{j_1,j_2\}} \tag{8.43}$$

$$b\bar a\bar\psi_i\psi_i = \text{same as preceding, with } A\leftrightarrow B \tag{8.44}$$

Reordering the summands, and observing that $(1-|A|)+(1-|B|) = 2-|A\cap B|-|A\cup B| = 1-|C|$, we have

$$\lambda'(1-|B|)\tau_C + \lambda(1-|A|)\tau_C = \lambda''(1-|C|)\tau_C \tag{8.45a}$$

$$\tau_{C\smallsetminus i} + \sum_{j\in A\smallsetminus i}\tau_{C\smallsetminus j} + \sum_{j\in B\smallsetminus i}\tau_{C\smallsetminus j} = \sum_{j\in C}\tau_{C\smallsetminus j} \tag{8.45b}$$

and

$$\begin{aligned} &\left[\sum_{j\in A\smallsetminus i}(\bar\psi_i\psi_j + \bar\psi_j\psi_i)\tau_{C\smallsetminus\{i,j\}} + \sum_{\substack{j_1,\,j_2\,\in\, A\,\smallsetminus\, i\\ j_1\,\neq\, j_2}}\bar\psi_{j_1}\psi_{j_2}\tau_{C\smallsetminus\{j_1,j_2\}}\right] - [A\leftrightarrow B] \\ &- \sum_{j_1\in B\smallsetminus i}\sum_{j_2\in A\smallsetminus i}(\bar\psi_{j_1}\psi_{j_2} + \bar\psi_{j_2}\psi_{j_1})\tau_{C\smallsetminus\{j_1,j_2\}} = -\sum_{\substack{j_1,\,j_2\,\in\, C\\ j_1\,\neq\, j_2}}\bar\psi_{j_1}\psi_{j_2}\tau_{C\smallsetminus\{j_1,j_2\}} \ . \end{aligned} \tag{8.46}$$

Collecting the right-hand sides, we obtain $f_C^{(\lambda'')}$, and thus prove the first part of the lemma.

For the second part of the lemma, we now have $|D|\geq 2$. Note first that, inside $f_A^{(\lambda)}$ and $f_B^{(\lambda')}$, the monomials of lowest degree with respect to the variables in D have degree $2|D|-2$, so the lowest possible degree in the product is $4|D|-4$. But the highest-degree monomial in the Grassmann algebra over D has degree $2|D|$; so if $|D|>2$, then each element in the expansion of the product must be zero, and we are done. In case $|D|=2$ (say, $D=\{i_1,i_2\}$), then in the expansion of the product $f_A^{(\lambda)}f_B^{(\lambda')}$, only pairs of terms in which both factors have degree 2 over D can contribute:

$$f_A^{(\lambda)}\, f_B^{(\lambda')} = [\tau_{A\setminus i_1} + \tau_{A\setminus i_2} - (\bar\psi_{i_1}\psi_{i_2} + \bar\psi_{i_2}\psi_{i_1})\tau_{A\setminus\{i_1,i_2\}} + \text{higher degree}] \times [\tau_{B\setminus i_1} + \tau_{B\setminus i_2} - (\bar\psi_{i_1}\psi_{i_2} + \bar\psi_{i_2}\psi_{i_1})\tau_{B\setminus\{i_1,i_2\}} + \text{higher degree}] \tag{8.47a}$$

$$= \tau_{A\setminus i_1}\tau_{B\setminus i_2} + \tau_{A\setminus i_2}\tau_{B\setminus i_1} + \bar\psi_{i_1}\psi_{i_2}\tau_{A\setminus\{i_1,i_2\}}\bar\psi_{i_2}\psi_{i_1}\tau_{B\setminus\{i_1,i_2\}} + \bar\psi_{i_2}\psi_{i_1}\tau_{A\setminus\{i_1,i_2\}}\bar\psi_{i_1}\psi_{i_2}\tau_{B\setminus\{i_1,i_2\}} \tag{8.47b}$$

$$= \tau_C\,(1 + 1 - 1 - 1) \tag{8.47c}$$

$$= 0\,. \tag{8.47d}$$

This completes the proof.[#] □

Second proof.

We are grateful to an anonymous referee for suggesting the following simple and elegant proof using the differential operators ∂ and $\bar\partial$:

Since $\partial^2 = \bar\partial^2 = 0$, we have

$$\big(\partial\bar\partial\tau_A\big)\,\big(\partial\bar\partial\tau_B\big) = \partial\bar\partial\,\big(\tau_A\partial\bar\partial\tau_B\big) = \partial\bar\partial\,\big(\tau_B\partial\bar\partial\tau_A\big)\ , \tag{8.48}$$

so that

$$f_A^{(\lambda)}\, f_B^{(\lambda')} = \lambda(1-|A|)\tau_A\partial\bar\partial\tau_B + \lambda'(1-|B|)\tau_B\partial\bar\partial\tau_A + \lambda\lambda'(1-|A|)(1-|B|)\tau_A\tau_B + \partial\bar\partial\,\big(\tau_A\partial\bar\partial\tau_B\big)\ . \tag{8.49}$$

If $|A\cap B| \geq 1$, then $\tau_A\tau_B = 0$ and

$$\tau_A\partial\bar\partial\tau_B = \tau_B\partial\bar\partial\tau_A = \begin{cases} \tau_{A\cup B} & \text{if } |A\cap B| = 1 \\ 0 & \text{if } |A\cap B| \geq 2 \end{cases} \tag{8.50}$$

This proves (8.33). □

As a first consequence of Lemma 8.1, we have:

Corollary 8.2. *Let $A \subseteq V$ with $|A| \geq 2$. Then the Grassmann element $f_A^{(\lambda)}$ is nilpotent of order 2, i.e.*

$$\big(f_A^{(\lambda)}\big)^2 = 0\,.$$

[#]The four terms of the expansion (8.47) derive from the fact that if $A\cap B = \{i_1, i_2\}$, then the hypergraph $(A\cup B, \{A, B\})$ contains one cycle (i_1, A, i_2, B, i_1) with two possible orientations, whose fermionic fields can be saturated either by paired factors (say, $\tau_{A\setminus i_1}\tau_{B\setminus i_2}$) or by unpaired factors (say, $\bar\psi_{i_1}\psi_{i_2}\tau_{A\setminus\{i_1,i_2\}}\bar\psi_{i_2}\psi_{i_1}\tau_{B\setminus\{i_1,i_2\}}$). In the former case no minus sign comes from the reordering, while in the latter case a minus sign comes from the reordering of the unpaired fermions, $\bar\psi_{i_1}\psi_{i_2}\bar\psi_{i_2}\psi_{i_1} = -\bar\psi_{i_1}\psi_{i_1}\bar\psi_{i_2}\psi_{i_2}$, which is typical for fermionic loops in Feynman graphs. So, in a sense, we have a cancellation among fermionic and bosonic loops, exactly as happens in the Feynman expansion of supersymmetric theories. See Section 8.6 below for a full explanation of the underlying supersymmetry.

In the graphical formalism of Section 8.2, this fermionic-bosonic cancellation is utilized explicitly for cycles of all lengths. In the present algebraic formalism, by contrast, this cancellation arises explicitly in (8.47) only for cycles of length 2; longer cycles are handled implicitly by repeated use of Lemma 8.1.

In particular, a product $\prod_{i=1}^m f_{A_i}^{(\lambda)}$ vanishes whenever there are any repetitions among the $A_1, \ldots, A_m$. So we can henceforth concern ourselves with the case in which there are no repetitions; then $E = \{A_1, \ldots, A_m\}$ is a *set* (as opposed to a *multiset*) and $G = (V, E)$ is a hypergraph.

By iterating Lemma 8.1 and using Proposition 1.2, we easily obtain:

Corollary 8.3. *Let $G = (V, E)$ be a* connected *hypergraph. Then*

$$\prod_{A \in E} f_A^{(\lambda)} = \begin{cases} f_V^{(\lambda)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases} \tag{8.51}$$

More generally,

$$\prod_{A \in E} f_A^{(\lambda_A)} = \begin{cases} f_V^{(\lambda_\star)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases} \tag{8.52}$$

where $\lambda_\star$ is the weighted average

$$\lambda_\star = \frac{\sum\limits_{A \in E} (|A| - 1)\lambda_A}{\sum\limits_{A \in E} (|A| - 1)} = \frac{\sum\limits_{A \in E} (|A| - 1)\lambda_A}{\Big| \bigcup\limits_{A \in E} A \Big| - 1} \ . \tag{8.53}$$

We are now ready to consider the subalgebra of the Grassmann algebra that is generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V. Recall first that a *partition* of V is a collection $\mathcal{C} = \{C_\gamma\}$ of disjoint nonempty subsets $C_\gamma \subseteq V$ that together cover V. We denote by $\Pi(V)$ the set of partitions of V. If V has cardinality n, then $\Pi(V)$ has cardinality $B(n)$, the n-th *Bell number* [121, pp. 33–34]. We remark that $B(n)$ grows asymptotically roughly like $n!$ [122, Sections 6.1–6.3].

The following corollary specifies the most general product of factors $f_A^{(\lambda)}$. Of course, there is no need to consider sets A of cardinality 1, since $f_{\{i\}}^{(\lambda)} = 1$.

Corollary 8.4. *Let E be a collection (possibly empty) of subsets of V, each of cardinality ≥ 2.*

(a) If the hypergraph $G = (V, E)$ is a hyperforest, and $\{C_\gamma\}$ is the partition of V induced by the decomposition of G into connected components, then $\prod\limits_{A \in E} f_A^{(\lambda)} = \prod\limits_\gamma f_{C_\gamma}^{(\lambda)}$. More generally, $\prod\limits_{A \in E} f_A^{(\lambda_A)} = \prod\limits_\gamma f_{C_\gamma}^{(\lambda_\gamma)}$, where λ_γ is the weighted average (8.53) taken over the hyperedges contained in C_γ.

(b) If the hypergraph $G = (V, E)$ is not a hyperforest, then $\prod\limits_{A \in E} f_A^{(\lambda)} = 0$, and more generally $\prod\limits_{A \in E} f_A^{(\lambda_A)} = 0$.

Proof. It suffices to apply Corollary 8.3 separately in each set C_γ, where $\{C_\gamma\}$ is the partition of V induced by the decomposition of G into connected components. □

It follows from Corollary 8.4 that any polynomial (or power series) in the $\{f_A^{(\lambda)}\}$ can be written as a linear combination of the quantities $f_{\mathcal{C}}^{(\lambda)} = \prod_\gamma f_{C_\gamma}^{(\lambda)}$ for partitions $\mathcal{C} = \{C_\gamma\} \in \Pi(V)$.

Using the foregoing results, we can simplify the Boltzmann weight associated to a Hamiltonian of the form

$$\mathcal{H} = -\sum_{A \in E} w_A f_A^{(\lambda)} \,. \tag{8.54}$$

Corollary 8.5. *Let $G = (V, E)$ be a hypergraph (that is, E is a collection of subsets of V, each of cardinality ≥ 2). Then*

$$\exp\left(\sum_{A \in E} w_A f_A^{(\lambda)}\right) = \sum_{\substack{F \in \mathcal{F}(G) \\ F = (F_1, \ldots, F_\ell)}} \left(\prod_{A \in F} w_A\right) \prod_{\alpha=1}^{\ell} f_{V(F_\alpha)}^{(\lambda)} \,, \tag{8.55}$$

where the sum runs over spanning hyperforests F in G with components $F_1, \ldots, F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α. More generally,

$$\exp\left(\sum_{A \in E} w_A f_A^{(\lambda_A)}\right) = \sum_{\substack{F \in \mathcal{F}(G) \\ F = (F_1, \ldots, F_\ell)}} \left(\prod_{A \in F} w_A\right) \prod_{\alpha=1}^{\ell} f_{V(F_\alpha)}^{(\lambda_\alpha)} \,, \tag{8.56}$$

where λ_α is the weighted average (8.53) taken over the hyperedges contained in the hypertree F_α.

Proof. Since the $f_A^{(\lambda_A)}$ are nilpotent of order 2 and commuting, we have

$$\exp\left(\sum_{A \in E} w_A f_A^{(\lambda_A)}\right) = \prod_{A \in E} \left(1 + w_A f_A^{(\lambda_A)}\right) \tag{8.57a}$$

$$= \sum_{E' \subseteq E} \left(\prod_{A \in E'} w_A\right) \left(\prod_{A \in E'} f_A^{(\lambda_A)}\right) . \tag{8.57b}$$

Using now Corollary 8.4, we see that the contribution is nonzero only when (V, E') is a hyperforest, and we obtain (8.55)/(8.56). □

In chapter 10 we shall study in more detail the Grassmann subalgebra that is generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V. In the present section we have seen that any element of this subalgebra can be written as a linear combination of the quantities $f_{\mathcal{C}}^{(\lambda)} = \prod_\gamma f_{C_\gamma}^{(\lambda)}$ for partitions $\mathcal{C} = \{C_\gamma\} \in \mathit{\Pi}(V)$. It turns out that the quantities $f_{\mathcal{C}}^{(\lambda)}$ are linearly *dependent* (i.e., an overcomplete set) as soon as $|V| \geq 4$. We shall show, in fact, that a vector-space basis for the subalgebra in question is given by the quantities $f_{\mathcal{C}}^{(\lambda)}$ as $\mathcal{C}$ ranges over all *non-crossing* partitions of V (relative to any fixed total ordering of V). It follows that the vector-space dimension of this subalgebra is given by the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$, where $n = |V|$. This is vastly smaller than the Bell number $B(n)$, which is the dimension that the subspace would have if the $f_{\mathcal{C}}^{(\lambda)}$ were linearly independent. [Indeed, one can see immediately that the $\{f_{\mathcal{C}}^{(\lambda)}\}$ must be linearly dependent for all sufficiently large n, simply because the entire Grassmann algebra has dimension only $4^n \ll B(n)$.] It also turns out that all the relations among the $\{f_{\mathcal{C}}^{(\lambda)}\}$ are generated (as an ideal) by the elementary relations $R_{abcd} = 0$, where

$$\begin{aligned} R_{abcd} = \lambda f_{\{a,b,c,d\}}^{(\lambda)} - f_{\{b,c,d\}}^{(\lambda)} - f_{\{a,c,d\}}^{(\lambda)} - f_{\{a,b,d\}}^{(\lambda)} - f_{\{a,b,c\}}^{(\lambda)} \\ + f_{\{a,b\}}^{(\lambda)} f_{\{c,d\}}^{(\lambda)} + f_{\{a,c\}}^{(\lambda)} f_{\{b,d\}}^{(\lambda)} + f_{\{a,d\}}^{(\lambda)} f_{\{b,c\}}^{(\lambda)} \end{aligned} \tag{8.58}$$

and a, b, c, d are distinct vertices. This will be discussed in greater detail in Chapter 10.

8.4 Grassmann integrals for counting spanning hyperforests

For any subset $A \subseteq V$ and any vector $\mathbf{t} = (t_i)_{i \in V}$ of vertex weights, let us define the integration measure

$$\mathcal{D}_{A,\mathbf{t}}(\psi, \bar{\psi}) \;:=\; \prod_{i \in A} \mathrm{d}\psi_i \, \mathrm{d}\bar{\psi}_i \; e^{t_i \bar{\psi}_i \psi_i} \,. \tag{8.59}$$

Our principal goal in this section is to provide a combinatorial interpretation, in terms of spanning hyperforests, for the general Grassmann integral ("partition function")

$$Z = \int \mathcal{D}(\psi, \bar{\psi}) \, \exp\Bigl[\sum_i t_i \bar{\psi}_i \psi_i + \sum_{A \in E} w_A f_A^{(\lambda)} \Bigr] \tag{8.60a}$$

$$= \int \mathcal{D}_{V,\mathbf{t}}(\psi, \bar{\psi}) \, \exp\Bigl[\sum_{A \in E} w_A f_A^{(\lambda)} \Bigr] \,, \tag{8.60b}$$

where $G = (V, E)$ is an arbitrary hypergraph (that is, E is an arbitrary collection of subsets of V, each of cardinality ≥ 2) and the $\{w_A\}_{A \in E}$ are arbitrary hyperedge weights. We also handle the slight generalization in which a separate parameter λ_A is used for each hyperedge A.

Our basic results are valid for an arbitrary vector $\mathbf{t} = (t_i)_{i \in V}$ of "mass terms". However, as we shall see, the formulae simplify notably if we specialize to the case in which $t_i = \lambda$ for all $i \in V$. This is not an accident, as it corresponds to the case in which the action is OSP(1|2)-invariant (see Section 8.6).

We begin with some formulae that allow us to integrate over the pairs of variables $\psi_i, \bar{\psi}_i$ one at a time:

Lemma 8.6. *Let $A \subseteq V$ and $i \in V$. Then:*

(a) $$\int \mathrm{d}\psi_i \, \mathrm{d}\bar{\psi}_i \; e^{t_i \bar{\psi}_i \psi_i} \, \tau_A = \begin{cases} \tau_{A \setminus i} & \text{if } i \in A \\ t_i \tau_A & \text{if } i \notin A \end{cases}$$

(b) $$\int \mathrm{d}\psi_i \, \mathrm{d}\bar{\psi}_i \; e^{t_i \bar{\psi}_i \psi_i} \, f_A^{(\lambda)} = \begin{cases} f_{A \setminus i}^{(\lambda)} + (t_i - \lambda) \tau_{A \setminus i} & \text{if } i \in A \\ t_i f_A^{(\lambda)} & \text{if } i \notin A \end{cases}$$

Proof. (a) is obvious, as is (b) when $i \notin A$. To prove (b) when $i \in A$, we write

$$f_A^{(\lambda)} \;=\; \lambda(1 - |A|)\tau_A + \sum_{j \in A} \tau_{A \setminus j} \;-\; \sum_{\substack{j,k \in A \\ j \neq k}} \bar{\psi}_j \psi_k \tau_{A \setminus \{j,k\}} \tag{8.61}$$

and integrate with respect to $\mathrm{d}\psi_i \, \mathrm{d}\bar{\psi}_i \; e^{t_i \bar{\psi}_i \psi_i}$. We obtain

$$\lambda(1 - |A|)\tau_{A \setminus i} + t_i \tau_{A \setminus i} + \sum_{j \in A \setminus i} \tau_{A \setminus \{i,j\}} \;-\; \sum_{\substack{j,k \in A \setminus i \\ j \neq k}} \bar{\psi}_j \psi_k \tau_{A \setminus \{i,j,k\}} \tag{8.62}$$

(in the last term we must have $j, k \neq i$ by parity), which equals $f_{A \setminus i}^{(\lambda)} + (t_i - \lambda)\tau_{A \setminus i}$ as claimed. □

Applying Lemma 8.6 repeatedly for i lying in an arbitrary set $B \subseteq V$, we obtain:

Corollary 8.7. *Let $A, B \subseteq V$. Then*

$$\int \mathcal{D}_{B,\mathbf{t}}(\psi,\bar\psi)\, f_A^{(\lambda)} = \left(\prod_{i\in B\smallsetminus A} t_i\right)\left[f^{(\lambda)}_{A\smallsetminus B} + \Big(\sum_{i\in B\cap A}(t_i-\lambda)\Big)\tau_{A\smallsetminus B}\right]. \tag{8.63}$$

In particular, for $B = A$ we have

$$\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)\, f_A^{(\lambda)} = \lambda + \sum_{i\in A}(t_i-\lambda)\,. \tag{8.64}$$

Proof. The factors t_i for $i \in B\smallsetminus A$ follow trivially from the second line of Lemma 8.6(b). For the rest, we proceed by induction on the cardinality of $B\cap A$. If $|B\cap A| = 0$, the result is trivial. So assume that the result holds for a given set B, and consider $B' = B\cup\{j\}$ with $j \in A\smallsetminus B$. Using Lemma 8.6(a,b) we have

$$\begin{aligned}
&\int \mathrm{d}\psi_j\,\mathrm{d}\bar\psi_j\; e^{t_j\bar\psi_j\psi_j}\left[f^{(\lambda)}_{A\smallsetminus B} + \Big(\sum_{i\in B\cap A}(t_i-\lambda)\Big)\tau_{A\smallsetminus B}\right] \\
&\quad= f^{(\lambda)}_{(A\smallsetminus B)\smallsetminus\{j\}} + (t_j-\lambda)\tau_{(A\smallsetminus B)\smallsetminus\{j\}} + \Big(\sum_{i\in B\cap A}(t_i-\lambda)\Big)\tau_{(A\smallsetminus B)\smallsetminus\{j\}} \qquad (8.65a)\\
&\quad= f^{(\lambda)}_{A\smallsetminus B'} + \Big(\sum_{i\in B'\cap A}(t_i-\lambda)\Big)\tau_{A\smallsetminus B'}\,, \qquad (8.65b)
\end{aligned}$$

as claimed. □

Applying (8.64) once for each factor C_α, we have:

Corollary 8.8. *Let $\{C_\alpha\}$ be a partition of V. Then*

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\prod_\alpha f^{(\lambda_\alpha)}_{C_\alpha} = \prod_\alpha\Big(\lambda_\alpha + \sum_{i\in C_\alpha}(t_i-\lambda_\alpha)\Big)\,. \tag{8.66}$$

The partition function (8.60) can now be computed immediately by combining Corollaries 8.5 and 8.8. We obtain the main result of this section:

Theorem 8.9. *Let $G = (V,E)$ be a hypergraph, and let $\{w_A\}_{A\in E}$ be hyperedge weights. Then*

$$\begin{aligned}
&\int \mathcal{D}(\psi,\bar\psi)\,\exp\left[\sum_i t_i\bar\psi_i\psi_i + \sum_{A\in E} w_A f_A^{(\lambda_A)}\right] \\
&\quad= \sum_{\substack{F\,\in\,\mathcal{F}(G)\\ F=(F_1,\dots,F_\ell)}}\left(\prod_{A\in F} w_A\right)\prod_{\alpha=1}^{\ell}\left(\sum_{i\in V(F_\alpha)} t_i - \sum_{A\in E(F_\alpha)}(|A|-1)\lambda_A\right), \qquad (8.67)
\end{aligned}$$

where the sum runs over spanning hyperforests F in G with components $F_1,\dots,F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α.

Proof. We apply (8.66), where (according to Corollary 8.5) λ_α is the weighted average (8.53) taken over the hyperedges contained in the hypertree F_α. Then

$$\lambda_\alpha + \sum_{i \in V(F_\alpha)} (t_i - \lambda_\alpha) = \sum_{i \in V(F_\alpha)} t_i - \lambda_\alpha(|V(F_\alpha)| - 1) \tag{8.68a}$$

$$= \sum_{i \in V(F_\alpha)} t_i - \sum_{A \in E(F_\alpha)} (|A| - 1)\lambda_A \; . \tag{8.68b}$$

□

If we specialize (8.67) to $t_i = \lambda$ for all vertices i, and $\lambda_A = \lambda$ for all hyperedges A, we obtain:

Corollary 8.10. *Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A \in E}$ be hyperedge weights. Then*

$$\int \mathcal{D}(\psi, \bar\psi) \exp\left[\lambda \sum_i \bar\psi_i \psi_i + \sum_{A \in E} w_A f_A^{(\lambda)}\right] = \sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} w_A\right) \lambda^{k(F)} \tag{8.69a}$$

$$= \lambda^{|V|} \sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} \frac{w_A}{\lambda^{|A|-1}}\right) , \tag{8.69b}$$

where the sum runs over spanning hyperforests F in G, and $k(F)$ is the number of connected components of F.

This is the generating function of *unrooted* spanning hyperforests, with a weight w_A for each hyperedge A and a weight λ for each connected component. Note that the second equality in (8.69) uses Proposition 1.1. In this formulation it is clear that, as a generating function in the variables w (i.e. for arbitrary weights), the parameter λ only accounts for a rescaling of the weights, so that it could be set to 1, except for the case of spanning trees, where a limit $\lambda \to 0$ at fixed w is required.

If we specialize (8.67) to $\lambda_A = 0$, we obtain an expression for rooted forests

Corollary 8.11. *Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A \in E}$ be hyperedge weights. Then*

$$\int \mathcal{D}(\psi, \bar\psi) \exp\left[\sum_i t_i \bar\psi_i \psi_i + \sum_{A \in E} w_A f_A^{(0)}\right] = \sum_{\substack{F \in \mathcal{F}(G) \\ F = (F_1, \dots, F_\ell)}} \left(\prod_{A \in F} w_A\right) \prod_{\alpha=1}^{\ell} \Big(\sum_{i \in V(F_\alpha)} t_i\Big) , \tag{8.70}$$

where the sum runs over spanning hyperforests F in G with components $F_1, \dots, F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α.

This is the generating function of *rooted* spanning hyperforests, with a weight w_A for each hyperedge A and a weight t_i for each root i.

Finally, returning to the case in which $t_i = \lambda$ for all i, we can obtain a formula more general than (8.69) in which the left-hand side contains an additional factor $f_{\mathcal{C}}^{(\lambda)} = \prod_\gamma f_{C_\gamma}^{(\lambda)}$, where $\mathcal{C} = \{C_\gamma\}$ is an arbitrary family of disjoint nonempty subsets of V. Indeed, it suffices to differentiate (8.69) with respect to all the weights w_{C_γ}.[†††] We obtain:

[†††] If the sets C_γ do not happen to belong to the hyperedge set E, it suffices to adjoin them to E and give them weight $w_{C_\gamma} = 0$. Indeed, there is no loss of generality in assuming that G is the complete hypergraph on the vertex set V, i.e. that *every* subset of V of cardinality ≥ 2 is a hyperedge.

Corollary 8.12. *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A \in E}$ be hyperedge weights, and let $\mathcal{C} = \{C_\gamma\}$ be a family of disjoint nonempty subsets of V. Then*

$$\int \mathcal{D}(\psi, \bar{\psi}) \left(\prod_\gamma f_{C_\gamma}^{(\lambda)} \right) \exp\left[\lambda \sum_i \bar{\psi}_i \psi_i + \sum_{A \in E} w_A f_A^{(\lambda)} \right]$$

$$= \sum_{F \in \mathcal{F}(G;\mathcal{C})} \left(\prod_{A \in F} w_A \right) \lambda^{k(F) - \sum_\gamma (|C_\gamma| - 1)} , \tag{8.71}$$

where $\mathcal{F}(G;\mathcal{C})$ denotes the set of spanning hyperforests in G that do not contain any of the $\{C_\gamma\}$ as hyperedges and that remain hyperforests (i.e., acyclic) when the hyperedges $\{C_\gamma\}$ are adjoined.

Indeed, to deduce Corollary 8.12 from Corollary 8.10 by differentiation, it suffices to observe that, by Proposition 1.1, the number of connected components in the hyperforest obtained from F by adjoining the hyperedges $\{C_\gamma\}$ is precisely $k(F) - \sum_\gamma (|C_\gamma| - 1)$.

For instance, if $\prod_\gamma f_{C_\gamma}^{(\lambda)}$ consists of a single factor f_C, then $\mathcal{F}(G; \{C\})$ consists of the spanning hyperforests in which all the vertices of the set C belong to different components. Similarly, if $\prod_\gamma f_{C_\gamma}^{(\lambda)}$ consists of two factors $f_{C_1} f_{C_2}$ with $C_1 \cap C_2 = \varnothing$, then $\mathcal{F}(G; \{C_1, C_2\})$ consists of the spanning hyperforests in which each component contains at most one vertex from C_1 and at most one vertex from C_2. The conditions get somewhat more complicated when there are three or more sets C_γ.

It is possible to obtain an analogous extension of Theorem 8.9 by the same method, but the weights get somewhat complicated, precisely because we lose the opportunity of using Proposition 1.1 in a simple way.

One can also generalize (8.60) to allow products $f_{\mathcal{C}}^{(\lambda)} = \prod_\alpha f_{C_\alpha}^{(\lambda)}$ in the exponential (i.e., in the *action*) in place of the single operators $f_A^{(\lambda)}$, with corresponding coefficients $w_{\mathcal{C}}$. These generalized integrals likewise lead to polynomials in the variables $\{w_{\mathcal{C}}\}$ such that the union of the families $\mathcal{C}_j$ arising in any given monomial is the set of hyperedges of a hyperforest. However, the simultaneous presence of certain sets of hyperedges in the hyperforest now gets extra weights. This generalized model has a few applications (for example, it can be used in order to study a *replicated* system, i.e. for evaluating statistical averages of *overlap* observables), but, more relevantly, when $t_i \lambda$ for all i it is conceptually important, as it corresponds to the *most general* OSP(1|2)-invariant action (see Section 8.6).

8.5 Extension to correlation functions

In the preceding section we saw how the partition function (8.60) of a particular class of fermionic theories can be given a combinatorial interpretation as an expansion over spanning hyperforests in a hypergraph. In this section we will extend this result to give a combinatorial interpretation for a class of Grassmann integrals that correspond to (unnormalized) correlation functions in this same fermionic theory; we will obtain a sum over partially rooted spanning hyperforests satisfying particular connection conditions.

Given ordered k-tuples of vertices $I = (i_1, i_2, \ldots, i_k) \in V^k$ and $J = (j_1, j_2, \ldots, j_k) \in V^k$, let us define the operator

$$\mathcal{O}_{I,J} \;:=\; \bar\psi_{i_1}\psi_{j_1}\cdots\bar\psi_{i_k}\psi_{j_k}\;, \tag{8.72}$$

which is an even element of the Grassmann algebra. Of course, the $i_1, i_2, \ldots, i_k$ must be all distinct, as must the $j_1, j_2, \ldots, j_k$, or else we will have $\mathcal{O}_{I,J} = 0$. We shall therefore assume henceforth that $I, J \in V^k_{\neq}$, where $V^k_{\neq}$ is the set of ordered k-tuples of *distinct* vertices in V. Note, however, that there can be overlaps between the sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$. Note finally that $\mathcal{O}_{I,J}$ is antisymmetric under permutations of the sequences I and J, in the sense that

$$\mathcal{O}_{I\circ\sigma, J\circ\tau} = \mathrm{sgn}(\sigma)\;\mathrm{sgn}(\tau)\,\mathcal{O}_{I,J} \tag{8.73}$$

for any permutations σ, τ of $\{1, \ldots, k\}$.

Our goal in this section is to provide a combinatorial interpretation, in terms of partially rooted spanning hyperforests satisfying suitable connection conditions, for the general Grassmann integral ("unnormalized correlation function")

$$[\mathcal{O}_{I,J}] = Z\langle\mathcal{O}_{I,J}\rangle = \int \mathcal{D}(\psi,\bar\psi)\,\mathcal{O}_{I,J}\,\exp\left[\sum_i t_i\bar\psi_i\psi_i + \sum_{A\in E} w_A f_A^{(\lambda)}\right] \tag{8.74a}$$

$$= \int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\,\mathcal{O}_{I,J}\,\exp\left[\sum_{A\in E} w_A f_A^{(\lambda)}\right]. \tag{8.74b}$$

The principal tool is the following generalization of (8.64):

Lemma 8.13. *Let $A \subseteq V$, and let $I = (i_1, i_2, \ldots, i_k) \in A^k_{\neq}$ and $J = (j_1, j_2, \ldots, j_k) \in A^k_{\neq}$. Then*

$$\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)\,\mathcal{O}_{I,J}\,f_A^{(\lambda)} = \begin{cases} \lambda + \sum_{i\in A}(t_i - \lambda) & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases} \tag{8.75}$$

Proof. The case $k = 0$ is just (8.64). To handle $k = 1$, recall that

$$f_A^{(\lambda)} \;=\; \lambda(1-|A|)\tau_A + \sum_{\ell\in A}\tau_{A\setminus l} \;-\; \sum_{\substack{\ell,m\in A\\ \ell\neq m}} \bar\psi_\ell\psi_m\tau_{A\setminus\{l,m\}}\,. \tag{8.76}$$

Now multiply $f_A^{(\lambda)}$ by $\bar\psi_i\psi_j$ with $i, j \in A$, and integrate with respect to $\mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)$. If $i = j$, then the only nonzero contribution comes from the term $\ell = i$ in the single sum, and $\bar\psi_i\psi_i\tau_{A\setminus i} = \tau_A$, so the integral is 1. If $i \neq j$, then the only nonzero contribution comes from the term $\ell = j$, $m = i$ in the double sum, and $(\bar\psi_i\psi_j)(-\bar\psi_j\psi_i)\tau_{A\setminus\{i,j\}} = \tau_A$, so the integral is again 1.

Finally, if $|I| = |J| = k \geq 2$, then every monomial in $\mathcal{O}_{I,J}f_A^{(\lambda)}$ has degree $\geq 2|A| - 2 + 2k > 2|A|$, so $\mathcal{O}_{I,J}f_A^{(\lambda)} = 0$. □

Of course, it goes without saying that if $m(\psi,\bar\psi)$ is a monomial of degree k in the variables ψ_i $(i \in A)$ and degree k' in the variables $\bar\psi_i$ $(i \in A)$, and k is *not* equal to k', then $\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)\,m(\psi,\bar\psi)\,f_A^{(\lambda)} = 0$.

Now go back to the general case $I = (i_1, i_2, \ldots, i_k) \in V^k_{\neq}$ and $J = (j_1, j_2, \ldots, j_k) \in V^k_{\neq}$, let $\mathcal{C} = \{C_\alpha\}_{\alpha=1}^m$ be a partition of V, and consider the integral

$$\mathcal{I}(I,J;\mathcal{C}) \;:=\; \int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \prod_{\alpha=1}^{m} f_{C_\alpha}^{(\lambda)} \,. \tag{8.77}$$

The integral factorizes over the sets C_α of the partition, and it vanishes unless $|I \cap C_\alpha| = |J \cap C_\alpha|$ for all α; here $I \cap C_\alpha$ denotes the subsequence of I consisting of those elements that lie in C_α, kept in their original order, and $|I \cap C_\alpha|$ denotes the length of that subsequence (and likewise for $J \cap C_\alpha$). So let us decompose the operator $\mathcal{O}_{I,J}$ as

$$\mathcal{O}_{I,J} = \sigma(I,J;\mathcal{C}) \prod_{\alpha=1}^{m} \mathcal{O}_{I\cap C_\alpha, J\cap C_\alpha} \,, \tag{8.78}$$

where $\sigma(I,J;\mathcal{C}) \in \{\pm 1\}$ is a sign coming from the reordering of the fields in the product. Applying Lemma 8.13 once for each factor C_α, we see that the integral (8.77) is nonvanishing only if $|I \cap C_\alpha| = |J \cap C_\alpha| \le 1$ for all α: that is, each set C_α must contain either one element from I and one element from J (possibly the same element) or else no element from I or J. Let us call the partition $\mathcal{C}$ *properly matched* for (I,J) when this is the case. (Note that this requires in particular that $m \ge k$.) Note also that for properly matched partitions $\mathcal{C}$ we can express the combinatorial sign $\sigma(I,J;\mathcal{C})$ in a simpler way: it is the sign of the unique permutation π of $\{1,\dots,k\}$ such that i_r and $j_{\pi(r)}$ lie in the same set C_α for each r ($1 \le r \le k$). (Note in particular that when $\{i_1,i_2,\dots,i_k\} \cap \{j_1,j_2,\dots,j_k\} \equiv S \neq \varnothing$, the pairing π has to match the repeated elements [i.e., $i_r = j_{\pi(r)}$ whenever $i_r \in S$], since a vertex cannot belong simultaneously to two distinct blocks C_α and C_β.) We then deduce immediately from Lemma 8.13 the following generalization of Corollary 8.8:

Corollary 8.14. *Let $I,J \in V_{\neq}^k$ and let $\mathcal{C} = \{C_\alpha\}$ be a partition of V. Then*

$$\begin{aligned} &\int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \prod_{\alpha} f_{C_\alpha}^{(\lambda)} \\ &\quad = \begin{cases} \operatorname{sgn}(\pi) \displaystyle\prod_{\alpha:\, |I\cap C_\alpha|=0} \Big(\lambda + \sum_{i\in C_\alpha} (t_i - \lambda)\Big) & \text{if } \mathcal{C} \text{ is properly matched for } (I,J) \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{8.79}$$

where π is the permutation of $\{1,\dots,k\}$ such that i_r and $j_{\pi(r)}$ lie in the same set C_α for each r.

We can now compute the integral (8.74) by combining Corollaries 8.5 and 8.14. If $G = (V,E)$ is a hypergraph and G' is a spanning subhypergraph of G, let us say that G' is *properly matched* for (I,J) [we denote this by $G' \sim (I,J)$] in case the partition of V induced by the decomposition of G' into connected components is properly matched for (I,J). We then obtain the main result of this section:

Theorem 8.15. *Let $G = (V,E)$ be a hypergraph, let $\{w_A\}_{A\in E}$ be hyperedge weights, and let $I,J \in V_{\neq}^k$. Then*

$$\begin{aligned} &\int \mathcal{D}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \exp\left[\sum_i t_i \bar\psi_i \psi_i + \sum_{A\in E} w_A f_A^{(\lambda)}\right] \\ &\quad = \sum_{\substack{F\in\mathcal{F}(G) \\ F\sim(I,J) \\ F=(F_1,\dots,F_\ell)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A\in F} w_A\right) \prod_{\alpha:\, |I\cap V(F_\alpha)|=0} \left(\lambda + \sum_{i\in V(F_\alpha)} (t_i - \lambda)\right), \end{aligned} \tag{8.80}$$

where the sum runs over spanning hyperforests F in G, with components $F_1, \ldots, F_\ell$, that are properly matched for (I, J), and $V(F_\alpha)$ is the vertex set of the hypertree F_α; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component F_α for each r.

If we specialize (8.80) to $t_i = \lambda$ for all vertices i, we obtain:

Corollary 8.16. *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A \in E}$ be hyperedge weights. and let $I, J \in V_{\neq}^k$. Then*

$$\begin{aligned} &\int \mathcal{D}(\psi, \bar\psi)\, \mathcal{O}_{I,J}\, \exp\left[\lambda \sum_i \bar\psi_i \psi_i + \sum_{A \in E} w_A f_A^{(\lambda)}\right] \\ &\qquad = \sum_{\substack{F \in \mathcal{F}(G) \\ F \sim (I,J)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A \in F} w_A\right) \lambda^{k(F)-k} \qquad (8.81a) \\ &\qquad = \lambda^{|V|-k} \sum_{\substack{F \in \mathcal{F}(G) \\ F \sim (I,J)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A \in F} \frac{w_A}{\lambda^{|A|-1}}\right), \qquad (8.81b) \end{aligned}$$

where the sum runs over spanning hyperforests F in G that are properly matched for (I, J), and $k(F)$ is the number of connected components of F; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component of F for each r.

This is the generating function of spanning hyperforests that are rooted at the vertices in I, J and are otherwise unrooted, with a weight w_A for each hyperedge A and a weight λ for each unrooted connected component.

If, on the other hand, we specialize (8.80) to $\lambda = 0$, we obtain:

Corollary 8.17. *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A \in E}$ be hyperedge weights, and let $I, J \in V_{\neq}^k$. Then*

$$\begin{aligned} &\int \mathcal{D}(\psi, \bar\psi)\, \mathcal{O}_{I,J}\, \exp\left[\sum_i t_i \bar\psi_i \psi_i + \sum_{A \in E} w_A f_A^{(0)}\right] \\ &\qquad = \sum_{\substack{F \in \mathcal{F}(G) \\ F \sim (I,J) \\ F = (F_1, \ldots, F_\ell)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A \in F} w_A\right) \prod_{\alpha\colon |I \cap V(F_\alpha)| = 0} \left(\sum_{i \in V(F_\alpha)} t_i\right), \qquad (8.82) \end{aligned}$$

where the sum runs over spanning hyperforests F in G, with components $F_1, \ldots, F_\ell$, that are properly matched for (I, J), and $V(F_\alpha)$ is the vertex set of the hypertree F_α; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component F_α for each r.

This is the generating function of rooted spanning hyperforests, with a weight w_A for each hyperedge A and a weight t_i for each root i *other than* those in the sets I, J.

Let us conclude by making some remarks about the *normalized* correlation function $\langle \mathcal{O}_{I,J} \rangle$ obtained by dividing (8.74) by (8.60). For simplicity, let us consider only the two-point function $\langle \bar\psi_i \psi_j \rangle$. We have

$$\langle \bar\psi_i \psi_j \rangle = \left\langle \gamma_{ij} \left(\lambda + \sum_{k \in \Gamma(i)} (t_k - \lambda) \right)^{-1} \right\rangle , \tag{8.83}$$

where the expectation value on the right-hand side is taken with respect to the "probability distribution"[†††] on spanning hyperforests of G in which the hyperforest $F = (F_1, \ldots, F_\ell)$ gets weight

$$Z^{-1} \left(\prod_{A \in F} w_A \right) \prod_{\alpha=1}^{\ell} \left(\lambda + \sum_{k \in V(F_\alpha)} (t_k - \lambda) \right) , \tag{8.84}$$

γ_{ij} denotes the indicator function

$$\gamma_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ belong to the same component of } F \\ 0 & \text{if not} \end{cases} \tag{8.85}$$

and $\Gamma(i)$ denotes the vertex set of the component of F containing i. The factor $\Big(\lambda + \sum_{k \in \Gamma(i)} (t_k - \lambda) \Big)^{-1}$ in (8.83) arises from the fact that in (8.67) each component gets a weight $\lambda + \sum_{k \in \Gamma(i)} (t_k - \lambda)$, while in (8.80) only those components *other than* the one containing i and j get such a weight. So in general the correlation function $\langle \bar\psi_i \psi_j \rangle$ is *not* simply equal to (or proportional to) the connection probability $\langle \gamma_{ij} \rangle$. However, in the special case of Corollaries 8.10 and 8.16 — namely, all $t_i = \lambda$, so that we get unrooted spanning hyperforests with a "flat" weight λ for each component — then we have the simple identity

$$\langle \bar\psi_i \psi_j \rangle = \lambda^{-1} \langle \gamma_{ij} \rangle . \tag{8.86}$$

Combinatorial identities generalizing (8.86), and their relation to the Ward identities arising from the OSP(1|2) supersymmetry, will be discussed elsewhere [140].

8.6 The role of OSP(1|2) symmetry

In [130] we have shown how the fermionic theory (8.6) emerges naturally from the expansion of a theory with bosons and fermions taking values in the unit supersphere in $\mathbb{R}^{1|2}$, when the action is quadratic and invariant under rotations in OSP(1|2). Here we would like to discuss this fact in greater detail, and extend it to the hypergraph fermionic model (8.69).

We begin by introducing, at each vertex $i \in V$, a superfield $\mathbf{n}_i := (\sigma_i, \psi_i, \bar\psi_i)$ consisting of a bosonic (i.e., real) variable σ_i and a pair of Grassmann variables $\psi_i, \bar\psi_i$. We equip the "superspace" $\mathbb{R}^{1|2}$ with the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j \;:=\; \sigma_i \sigma_j + \lambda(\bar\psi_i \psi_j - \psi_i \bar\psi_j) , \tag{8.87}$$

where $\lambda \neq 0$ is an arbitrary real parameter.

The infinitesimal rotations in $\mathbb{R}^{1|2}$ that leave invariant the scalar product (8.87) form the Lie superalgebra osp(1|2) [141, 142, 143]. This algebra is generated by two types of

[†††] We write "probability distribution" in quotation marks because the "probabilities" will in general be complex. They will be true probabilities (i.e., real numbers between 0 and 1) if the hyperedge weights w_A are nonnegative real numbers.

transformations: Firstly, we have the elements of the sp(2) subalgebra, which act on the field as $\mathbf{n}_i' = \mathbf{n}_i + \delta\mathbf{n}_i$ with

$$\delta\sigma_i = 0 \tag{8.88a}$$
$$\delta\psi_i = -\,\alpha\,\psi_i + \gamma\,\bar\psi_i \tag{8.88b}$$
$$\delta\bar\psi_i = +\,\alpha\,\bar\psi_i + \beta\,\psi_i \tag{8.88c}$$

where α, β, γ are bosonic (Grassmann-even) global parameters; it is easily checked that these transformations leave (8.87) invariant. Secondly, we have the transformations parametrized by fermionic (Grassmann-odd) global parameters $\epsilon, \bar\epsilon$:

$$\delta\sigma_i = -\lambda^{1/2}(\bar\epsilon\psi_i + \bar\psi_i\epsilon) \tag{8.89a}$$
$$\delta\psi_i = \lambda^{-1/2}\,\epsilon\,\sigma_i \tag{8.89b}$$
$$\delta\bar\psi_i = \lambda^{-1/2}\,\bar\epsilon\,\sigma_i \tag{8.89c}$$

(Here an overall factor $\lambda^{-1/2}$ has been extracted from the fermionic parameters for future convenience.) To check that these transformations leave (8.87) invariant, we compute

$$\delta(\mathbf{n}_i\cdot\mathbf{n}_j) = (\delta\sigma_i)\sigma_j + \sigma_i(\delta\sigma_j) + \lambda\big[(\delta\bar\psi_i)\psi_j + \bar\psi_i(\delta\psi_j) - (\delta\psi_i)\bar\psi_j - \psi_i(\delta\bar\psi_j)\big] \tag{8.90a}$$
$$= -\lambda^{1/2}(\bar\epsilon\psi_i + \bar\psi_i\epsilon)\sigma_j - \lambda^{1/2}(\bar\epsilon\psi_j + \bar\psi_j\epsilon)\sigma_i + \lambda^{1/2}\big[\bar\epsilon\psi_j\sigma_i + \bar\psi_i\epsilon\sigma_j - \epsilon\bar\psi_j\sigma_i - \psi_i\bar\epsilon\sigma_j\big] \tag{8.90b}$$
$$= 0\,. \tag{8.90c}$$

In terms of the differential operators $\partial_i = \partial/\partial\psi_i$ and $\bar\partial_i = \partial/\partial\bar\psi_i$, the transformations (8.88) can be represented by the generators

$$X_0 = \sum_{i\in V}(\bar\psi_i\bar\partial_i - \psi_i\partial_i) \tag{8.91a}$$
$$X_+ = \sum_{i\in V}\bar\psi_i\partial_i \tag{8.91b}$$
$$X_- = \sum_{i\in V}\psi_i\bar\partial_i \tag{8.91c}$$

corresponding to the parameters α, β, γ, respectively, while the transformations (8.89) can be represented by the generators

$$Q_+ = \lambda^{-1/2}\sum_{i\in V}\sigma_i\partial_i + \lambda^{1/2}\sum_{i\in V}\bar\psi_i\frac{\partial}{\partial\sigma_i} \tag{8.92a}$$
$$Q_- = \lambda^{-1/2}\sum_{i\in V}\sigma_i\bar\partial_i - \lambda^{1/2}\sum_{i\in V}\psi_i\frac{\partial}{\partial\sigma_i} \tag{8.92b}$$

corresponding to the parameters $\epsilon, \bar\epsilon$, respectively. (With respect to the notations of [143] we have $X_\pm = L_\mp$, $X_0 = -2L_0$ and $Q_\pm = \mp 2iR_\mp$.) These transformations satisfy the commutation/anticommutation relations

$$[X_0, X_\pm] = \pm 2X_\pm \qquad [X_+, X_-] = X_0 \tag{8.93a}$$
$$\{Q_\pm, Q_\pm\} = \pm 2X_\pm \qquad \{Q_+, Q_-\} = X_0 \tag{8.93b}$$
$$[X_0, Q_\pm] = \pm Q_\pm \qquad [X_\pm, Q_\pm] = 0 \qquad [X_\pm, Q_\mp] = -Q_\pm \tag{8.93c}$$

Note in particular that $X_\pm = Q_\pm^2$ and $X_0 = Q_+Q_- + Q_-Q_+$. It follows that any element of the Grassmann algebra that is annihilated by $Q_\pm$ is also annihilated by the entire osp(1|2) algebra.

Now let us consider a σ-model in which the superfields $\mathbf{n}_i$ are constrained to lie on the unit supersphere in $\mathbb{R}^{1|2}$, i.e. to satisfy the constraint

$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \sigma_i^2 + 2\lambda\bar\psi_i\psi_i = 1\,. \tag{8.94}$$

We can *solve* this constraint by writing

$$\sigma_i = \pm(1 - 2\lambda\bar\psi_i\psi_i)^{1/2} = \pm(1 - \lambda\bar\psi_i\psi_i)\,, \tag{8.95}$$

exploiting the fact that $\psi_i^2 = \bar\psi_i^2 = 0$. Let us henceforth take only the + sign in (8.95), neglecting the other solution (the role played by these neglected Ising variables will be considered in more detail elsewhere [139]), so that

$$\sigma_i = 1 - \lambda\bar\psi_i\psi_i\,. \tag{8.96}$$

We then have a purely fermionic model with variables $\psi, \bar\psi$ in which the sp(2) transformations continue to act as in (8.88) while the fermionic transformations act via the "hidden" supersymmetry

$$\delta\psi_i = \lambda^{-1/2}\,\epsilon\,(1 - \lambda\bar\psi_i\psi_i) \tag{8.97a}$$

$$\delta\bar\psi_i = \lambda^{-1/2}\,\bar\epsilon\,(1 - \lambda\bar\psi_i\psi_i) \tag{8.97b}$$

All of these transformations leave invariant the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; 1 - \lambda(\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j) + \lambda^2\bar\psi_i\psi_i\bar\psi_j\psi_j\,. \tag{8.98}$$

The generators $Q_\pm$ are now defined as

$$Q_+ = \lambda^{-1/2}\sum_{i\in V}(1 - \lambda\bar\psi_i\psi_i)\partial_i = \lambda^{-1/2}\partial - \lambda^{1/2}\sum_{i\in V}\bar\psi_i\psi_i\partial_i \tag{8.99a}$$

$$Q_- = \lambda^{-1/2}\sum_{i\in V}(1 - \lambda\bar\psi_i\psi_i)\bar\partial_i = \lambda^{-1/2}\bar\partial - \lambda^{1/2}\sum_{i\in V}\bar\psi_i\psi_i\bar\partial_i \tag{8.99b}$$

where we recall the notations $\partial = \sum_{i\in V}\partial_i$ and $\bar\partial = \sum_{i\in V}\bar\partial_i$.

Let us now show that the polynomials $f_A^{(\lambda)}$ defined as in (8.28) are OSP(1|2)-invariant, i.e. are annihilated by all elements of the osp(1|2) algebra. As noted previously, it suffices to show that the $f_A^{(\lambda)}$ are annihilated by $Q_\pm$. Applying the definitions (8.99), we have

$$Q_-\tau_A = \lambda^{-1/2}\bar\partial\tau_A \tag{8.100}$$

and hence

$$Q_+Q_-\tau_A = \lambda^{-1}\partial\bar\partial\tau_A - |A|\tau_A\,, \tag{8.101}$$

so that

$$f_A^{(\lambda)} = \lambda(1 + Q_+Q_-)\tau_A\,. \tag{8.102}$$

The next step is to compute $Q_+f_A^{(\lambda)}$: since

$$\begin{aligned} Q_+(1 + Q_+Q_-) &= Q_+ + Q_+^2Q_- = Q_+ + X_+Q_- \\ &= Q_+ + [X_+, Q_-] + Q_-X_+ = Q_+ - Q_+ + Q_-X_+ = Q_-X_+ \end{aligned} \tag{8.103}$$

by the relations (8.93b)/(8.93c), while it is obvious that $X_+\tau_A = 0$, we conclude that $Q_+ f_A^{(\lambda)} = 0$, i.e. $f_A^{(\lambda)}$ is invariant under the transformation Q_+. A similar calculation of course works for Q_-.[††‡]

In fact, the OSP(1|2)-invariance of $f_A^{(\lambda)}$ can be proven in a simpler way by writing $f_A^{(\lambda)}$ explicitly in terms of the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j \in A$. Note first that

$$f_{\{i,j\}}^{(\lambda)} = -\lambda \bar\psi_i \psi_i \bar\psi_j \psi_j + (\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j) \tag{8.104a}$$

$$= \frac{1}{\lambda}\,(1 - \mathbf{n}_i \cdot \mathbf{n}_j) \tag{8.104b}$$

$$= \frac{(\mathbf{n}_i - \mathbf{n}_j)^2}{2\lambda}\ . \tag{8.104c}$$

By Corollary 8.3, we obtain

$$f_{\{i_1,i_2,\dots,i_k\}}^{(\lambda)} = \frac{1}{\lambda^{k-1}}\,(1 - \mathbf{n}_{i_1} \cdot \mathbf{n}_{i_2})\,(1 - \mathbf{n}_{i_2} \cdot \mathbf{n}_{i_3}) \cdots (1 - \mathbf{n}_{i_{k-1}} \cdot \mathbf{n}_{i_k}) \tag{8.105a}$$

$$= \frac{1}{(2\lambda)^{k-1}}\,(\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2\,(\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \cdots (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2\ . \tag{8.105b}$$

Note the striking fact that the right-hand side of (8.105) is invariant under all permutations of $i_1, \dots, i_k$, though this fact is not obvious from the formulae given, and is indeed false for vectors in Euclidean space $\mathbb{R}^N$ with $N \neq -1$. Moreover, the path $i_1, \dots, i_k$ that is implicit in the right-hand side of (8.105) could be replaced by *any tree* on the vertex set $\{i_1, \dots, i_k\}$, and the result would again be the same (by Corollary 8.3).

It follows from (8.104)/(8.105) that the subalgebra generated by the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j \in V$ is identical with the subalgebra generated by the $f_A^{(\lambda)}$ for $A \subseteq V$, for any $\lambda \neq 0$. Therefore, the *most general* OSP(1|2)-symmetric Hamiltonian depending on the $\{\mathbf{n}_i\}_{i \in V}$ is precisely the one discussed in the Remark at the end of Section 8.4, namely in which the action contains all possible products $f_{\mathcal{C}}^{(\lambda)} = \prod_\alpha f_{C_\alpha}^{(\lambda)}$, where $\{C_\alpha\}$ is a partition of V.

Finally, we need to consider the behavior of the integration measure in (8.60), namely

$$\mathcal{D}_{V,\mathbf{t}}(\psi, \bar\psi) = \prod_{i \in V} \mathrm{d}\psi_i\, \mathrm{d}\bar\psi_i\ e^{t_i \bar\psi_i \psi_i}\ , \tag{8.106}$$

under the supersymmetry (8.97). In general this measure is *not* invariant under (8.97), but in the special case $t_i = \lambda$ for all i, it *is* invariant, in the sense that

$$\int \mathcal{D}_{V,\lambda}(\psi, \bar\psi)\ \delta F(\psi, \bar\psi) = 0 \tag{8.107}$$

for any function $F(\psi, \bar\psi)$. Indeed, $\mathcal{D}_{V,\lambda}(\psi, \bar\psi)$ is invariant more generally under *local* supersymmetry transformations in which separate generators $\epsilon_i, \bar\epsilon_i$ are used at each vertex i. To see this, let us focus on one site i and write $F(\psi, \bar\psi) = a + b\psi_i + c\bar\psi_i + d\bar\psi_i\psi_i$ where a, b, c, d are polynomials in the $\{\psi_j, \bar\psi_j\}_{j \neq i}$ (which may contain both Grassmann-even and Grassmann-odd terms). Then

[††‡] Again, we are grateful to an anonymous referee for suggesting this proof.

$$\delta F = \lambda^{1/2} \left[b\epsilon_i\sigma_i + c\bar{\epsilon}_i\sigma_i + d\left(\bar{\epsilon}_i\sigma_i\psi_i + \bar{\psi}_i\epsilon_i\sigma_i\right)\right] \tag{8.108a}$$

$$= \sigma_i\, \lambda^{1/2} \left[b\epsilon_i + c\bar{\epsilon}_i + d(\bar{\epsilon}_i\psi_i + \bar{\psi}_i\epsilon_i)\right] . \tag{8.108b}$$

Since $\sigma_i = e^{-\lambda\bar{\psi}_i\psi_i}$, this cancels the factor $e^{t_i\bar{\psi}_i\psi_i}$ from the measure (since $t_i = \lambda$) and the integral over $d\psi_i\, d\bar{\psi}_i$ is zero (because there are no $\bar{\psi}_i\psi_i$ monomials). Thus, the measure $\mathcal{D}_{V,\mathbf{t}}(\psi,\bar{\psi})$ is invariant under the local supersymmetry at site i whenever $t_i = \lambda$. If this occurs for all i, then the measure is invariant under the global supersymmetry (8.89).

The OSP(1|2)-invariance of $\mathcal{D}_{V,\lambda}(\psi,\bar{\psi})$ can be seen more easily by writing the manifestly invariant combination

$$\delta(\mathbf{n}_i^2 - 1)\, d\mathbf{n}_i = \delta(\sigma_i^2 + 2\lambda\bar{\psi}_i\psi_i - 1)\, d\sigma_i\, d\psi_i\, d\bar{\psi}_i \tag{8.109a}$$

$$= e^{\lambda\bar{\psi}_i\psi_i}\, \delta\big(\sigma_i - (1 - \lambda\bar{\psi}_i\psi_i)\big)\, d\sigma_i\, d\psi_i\, d\bar{\psi}_i\ , \tag{8.109b}$$

where the factor $e^{\lambda\bar{\psi}_i\psi_i}$ comes from the inverse Jacobian. Integrating out σ_i from (8.109b), we obtain $e^{\lambda\bar{\psi}_i\psi_i}\, d\psi_i\, d\bar{\psi}_i$.

8.7 A determinantal formula for f_A

The main purpose of this section is to prove the following determinantal formula for $f_A^{(\lambda)}$.

Theorem 8.18. *For $A = \{1,\dots,n\}$, and $\mathbf{n}_i = (1 - \lambda\bar{\psi}_i\psi_i, \bar{\psi}_i, \psi_i)$, we have*

$$\det\left(\mathbf{n}_i \cdot \mathbf{n}_j\right)_{i,j=1}^n = n!\, f_A^{(\lambda)} . \tag{8.110}$$

Along the way we will obtain a rather more general graphical representation of certain determinants.

Let $A = (a_{ij})_{i,j=1}^n$ be a matrix whose elements belong to a commutative ring R. The determinant is defined as usual by

$$\det A = \sum_{\pi\in\Pi_n} \mathrm{sgn}(\pi) \prod_{i=1}^n a_{i\pi(i)}\ , \tag{8.111}$$

where the sum runs over permutations π of $[n] := \{1,\dots,n\}$, and

$\mathrm{sgn}(\pi) = (-1)^{\#(\text{even cycles of }\pi)}$ is the sign of the permutation π.

We begin with a formula for the determinant of the sum of two matrices in terms of minors, which ought to be well known but apparently is not[‡††]:

Lemma 8.19. *Let A and B be $n \times n$ matrices whose elements belong to a commutative ring R. Then*[‡††]

[‡††]This formula can be found in [123, pp. 162–163, Exercise 6] and [124, pp. 221–223]. It can also be found — albeit in an ugly notation that obscures what is going on — in [123, pp. 145–146 and 163–164] [125, pp. 31–33] [126, pp. 281–282]; and in an even more obscure notation in [127, p. 102, item 5]. We remark that an analogous formula holds (with the same proof) in which all three occurrences of determinant are replaced by permanent and the factor $\epsilon(I,J)$ is omitted.

[‡††]The determinant of an empty matrix is of course defined to be 1. This makes sense in the present context even if the ring R lacks an identity element: the term $I = J = \varnothing$ contributes $\det B$ to the sum (8.112), while the term $I = J = [n]$ contributes $\det A$.

$$\det(A+B) = \sum_{\substack{I,\,J\,\subseteq\,[n]\\ |I|\,=\,|J|}} \epsilon(I,J)\,(\det A_{IJ})(\det B_{I^cJ^c})\;, \tag{8.112}$$

where $\epsilon(I,J) = (-1)^{\sum_{i\in I} i + \sum_{j\in J} j}$ is the sign of the permutation that takes II^c into JJ^c (where the sets I, I^c, J, J^c are all written in increasing order).

PROOF. Using the definition of determinant and expanding the products, we have

$$\det(A+B) = \sum_{\pi\in\Pi_n} \mathrm{sgn}(\pi) \sum_{I\subseteq[n]} \prod_{i\in I} a_{i\pi(i)} \prod_{\ell\in I^c} b_{\ell\pi(\ell)}\;. \tag{8.113}$$

Define now $J = \pi[I]$. Then we can interchange the order of summation:

$$\det(A+B) = \sum_{\substack{I,\,J\,\subseteq\,[n]\\ |I|\,=\,|J|}} \sum_{\substack{\pi\,\in\,\Pi_n\\ \pi[I]\,=\,J}} \mathrm{sgn}(\pi) \prod_{i\in I} a_{i\pi(i)} \prod_{\ell\in I^c} b_{\ell\pi(\ell)}\;. \tag{8.114}$$

Suppose now that $|I| = |J| = k$, and let us write $I = \{i_1,\ldots,i_k\}$ and $J = \{j_1,\ldots,j_k\}$ where the elements are written in increasing order, and likewise $I^c = \{\ell_1,\ldots,\ell_{n-k}\}$ and $J = \{m_1,\ldots,m_{n-k}\}$. Let $\pi'\in\Pi_k$ and $\pi''\in\Pi_{n-k}$ be the permutations defined so that

$$\pi(i_\alpha) = j_\beta \longleftrightarrow \pi'(\alpha) = \beta \tag{8.115a}$$

$$\pi(\ell_\alpha) = m_\beta \longleftrightarrow \pi''(\alpha) = \beta \tag{8.115b}$$

It is easy to see that $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi')\,\mathrm{sgn}(\pi'')\epsilon(I,J)$. The formula then follows by using twice again the definition of determinant. □

Corollary 8.20. *Let A and B be $n\times n$ matrices whose elements belong to a commutative ring R. Then $\det(A+\lambda B)$ is a polynomial in λ of degree at most $\mathrm{rank}(B)$, where "rank" here means determinantal rank (i.e. the order of the largest nonvanishing minor).*

PROOF. This is an immediate consequence of the formula (8.112), since all minors of B of size larger than its rank vanish by definition. □

Next recall the traditional graphical representation of the determinant:

Lemma 8.21. *Let $C = (c_{ij})_{i,j=1}^n$ be a matrix whose elements belong to a commutative ring R. Then*

$$\det(-C) = \sum_{\mathbf{G}} (-1)^{\#(\text{cycles of }\mathbf{G})} \prod_{ij\in E(\mathbf{G})} c_{ij}\;, \tag{8.116}$$

where the sum runs over all permutation digraphs $\mathbf{G}$ on the vertex set $\{1,2,\ldots,n\}$, i.e. all directed graphs in which each connected component is a directed cycle (possibly of length 1).

PROOF. This is an immediate consequence of (8.111) and the fact that $(-1)^{\#(\text{even cycles of }\pi)}$ $= (-1)^{\#(\text{cycles of }\pi)}(-1)^{\#(\text{odd cycles of }\pi)}$. □

Now let $\mathbf{a} = (a_i)_{i=1}^n$ and $\mathbf{b} = (b_i)_{i=1}^n$ be a pair of vectors with elements in the ring R. The main result of this appendix is the following generalization of Lemma 8.21:

Lemma 8.22. *Let $C = (c_{ij})_{i,j=1}^n$ be a matrix whose elements belong to a commutative ring R, and let $\mathbf{a} = (a_i)_{i=1}^n$ and $\mathbf{b} = (b_i)_{i=1}^n$ be vectors with elements in R. Then*

$$\det(\mathbf{a}\mathbf{b}^{\mathrm{T}} - C) = \det(-C) + \sum_{\mathbf{G}} (-1)^{\#(\text{cycles of } \mathbf{G})}\, b_{s(\mathbf{G})}\, a_{t(\mathbf{G})} \prod_{ij \in E(\mathbf{G})} c_{ij}\ , \tag{8.117}$$

where the sum runs over all directed graphs $\mathbf{G}$ on the vertex set $\{1, 2, \ldots, n\}$ in which one connected component is a directed path (possibly of length 0, i.e. an isolated vertex) from source $s(\mathbf{G})$ to sink $t(\mathbf{G})$ and all the other connected components are directed cycles (possibly of length 1).

PROOF. Introduce an indeterminate λ and let us compute $\det(\lambda \mathbf{a}\mathbf{b}^{\mathrm{T}} - C)$, working in the polynomial ring $R[\lambda]$, by substituting $c_{ij} - \lambda a_i b_j$ in place of c_{ij} in (8.116). The term of order λ^0 is $\det(-C)$, which is given by (8.116). In the term of order λ^1, one edge ij in $\mathbf{G}$ carries a factor $-a_i b_j$ and the rest carry matrix elements of C. Setting $\mathbf{G}' = \mathbf{G} \setminus ij$, we see that $\mathbf{G}'$ has one less cycle than $\mathbf{G}$ [thereby cancelling the minus sign] and has a path running from source $s(\mathbf{G}') = j$ to sink $t(\mathbf{G}') = i$. Dropping the prime gives (8.117). Terms of order λ^2 and higher vanish by Corollary 8.20 because $\mathbf{a}\mathbf{b}^{\mathrm{T}}$ has rank 1. □

Corollary 8.23. *Let $C = (c_{ij})_{i,j=1}^n$ be a matrix whose elements belong to a commutative ring-with-identity element R, and let E be the $n \times n$ matrix with all elements 1. Then*

$$\det(E - C) = \det(-C) + \sum_{\mathbf{G}} (-1)^{\#(\text{cycles of } \mathbf{G})} \prod_{ij \in E(\mathbf{G})} c_{ij}\ , \tag{8.118}$$

where the sum runs over all directed graphs $\mathbf{G}$ on the vertex set $\{1, 2, \ldots, n\}$ in which one connected component is a directed path (possibly of length 0, i.e. an isolated vertex) and all the other connected components are directed cycles (possibly of length 1).

The following result is an immediate consequence of Lemma 8.21 and Corollary 8.23:

Corollary 8.24. *Let $C = (c_{ij})_{i,j=1}^n$ be a matrix whose elements belong to a commutative ring-with-identity-element R and satisfy $c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{k-1} i_k} c_{i_k i_1} = 0$ for all $i_1, \ldots, i_k$ ($k \geq 1$), and let E be the $n \times n$ matrix with all elements 1. Then*

$$\det(E - C) = \sum_{\mathbf{P}} \prod_{ij \in E(\mathbf{P})} c_{ij}\ , \tag{8.119}$$

where the sum runs over all directed paths $\mathbf{P}$ on the vertex set $\{1, 2, \ldots, n\}$. (There are $n!$ such contributions.)

PROOF. The hypotheses on C lead to the vanishing of all terms containing at least one cycle (including cycles of length 1). Therefore, the only remaining possibility is a single directed path. □

Let us now specialize Corollary 8.24 to the case in which the commutative ring R is the even subalgebra of our Grassmann algebra, and the matrix C is given by

$$c_{ii} = 0 \tag{8.120}$$

$$c_{ij} = c_{ji} = \lambda f^{(\lambda)}_{\{i,j\}} \qquad \text{for } i \neq j \tag{8.121}$$

The hypothesis $c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{k-1} i_k} c_{i_k i_1} = 0$ is an immediate consequence of Corollary 8.3. Moreover, by equation (8.104b) we have $(E - C)_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$. In the expansion (8.119) we obtain $n!$ terms, each of which is of the form λ^{n-1} times $\prod_{ij \in E(\mathbf{P})} f^{(\lambda)}_{\{i,j\}}$ for some directed path $\mathbf{P}$ on the vertex set $\{1, 2, \ldots, n\}$. But by Corollary 8.3, each such product equals $f^{(\lambda)}_{\{1,\ldots,n\}}$, so this proves the determinantal formula in Theorem 8.18.

We remark that the determinant of a matrix of inner products is commonly called a *Gram determinant* [128, p. 110], and has a preminent role in the construction of the invariant theory for the corresponding set of vectors [144].

9

Ward identities for the $O(N)$ σ-model and connectivity probabilities

We review the consequences of global rotational invariance of models with $O(N)$ symmetry, namely Ward Identities, with special concern on the model in the limit $N \to -1$, i.e. with invariance under OSP(1|2). As we know from the previous chapter, this model is perturbatively equivalent to a generating function of spanning (hyper-)forests on a suitable (hyper-)graph. We show how all the Ward Identities have a precise interpretation as conservation of probability for connectivity events in the forest formulation.

9.1 Introduction

Universality in critical phenomena of Statistical Mechanics is ruled by symmetry principles. Besides some remarkable exceptions, symmetry alone is not sufficient to fully determine the properties of a given system, but nonetheless it typically gives access to a number of strong facts, often not obvious *a priori*, with relatively small effort and good elegance. Understanding what is implied uniquely by symmetry principles, and what conversely is specific of the given system, or of the given pattern of interaction, is a first key step in fully understanding the properties of a system.

This is in particular true for the system we are concentrating on in this work: the generating function of spanning (hyper-)forests of a given (hyper-)graph. The immediate combinatorial description of the model relates to a limit $q \to 0$ of Potts Model on a (hyper-)graph in Fortuin-Kasteleyn (random cluster) formulation. More surprisingly, a field-theoretical representation of the generating function also relates to a limit $N \to -1$ of the non-linear spherical $O(N)$ σ-model (or, more precisely, of a σ-model with variables valued on a projective "supersphere" $\mathrm{RP}^{1|2}$, with one bosonic and two fermionic components, equipped with the standard orthosymplectic metric).

We have discussed so far how the underlying OSP(1|2) invariance of the model in the latter formulation is responsible for the "fine-tuning" of two parameters in the Hamiltonian, which allows to count unrooted spanning forests (while the generalized model at a generic point is in the universality class of rooted spanning forests, which on a regular lattice is trivially a theory of a massive complex fermion).

We also discussed how the analytical continuation (in N) of the RG calculations for the $O(N)$ σ-model remarkably predicts in two dimensions the asymptotic freedom of the model in a neighbourhood of the fixed point of massless complex fermion, corresponding to (the fully integrable $c = -2$ CFT of) uniform spanning trees.

As the OSP(1|2) symmetry is an underlying principle of the generic problem, non only at arbitrary euclidean dimension but also on a generic (hyper-)graph, the features, like

asymptotic freedom, that are genuinely two-dimensional, cannot be ultimate consequence only of this. So, at the aim of understanding these mechanisms, it is of some preliminary interest to understand what can *not* be the only responsible, because it is just implied by symmetry.

The systematic study of the consequences of the symmetry on a statistical field theory goes under the name of *Ward Identity classification*. This is typically a very preliminary step in the study of a field theory, and tacitly assumed to be a standard part of the job in modern research. We stress in this chapter how, in our present context, beside the classification itself of the Ward Identities being indeed a simple exercise, understanding the consequences of these identities at the level of the combinatorial language of spanning forests gives a certain extra insight which was not evident. It is furthermore surprising that all the infinite family of Ward Identities can be classified in terms of an infinite family of identities for probabilities of events, corresponding to the normalization of the sum of probabilities of mutually excluding facts.

These relations are valid, as they should, for a generic measure having support on the ensemble of spanning (hyper-)forests on a (hyper-)graph. Moreover, as the description of the events crucially relies on the fact that, if two vertices are in the same component, there is a unique simple path connecting them, we can expect that these same relations do not hold for measures describing more generic classes of spanning subgraphs. This is in agreement with the purely algebraic fact, stressed in the previous chapter, that *the most general* (perturbative) theory with OSP(1|2) symmetry induces a partition function that can be naturally rewritten as an expansion over suitable weighted spanning hyperforests on some hypergraph.

9.2 Lattice notation

This work is mainly devoted to "combinatorial models" in Statistical Field Theory, and most of the systems that we analyse are described on some finite discrete graph (or hypergraph). Ward Identities are however a technology mostly developed in the context of Quantum Field Theory in continuum space. The easiest point of contact between the two subjects is, of course, the Euclidean lattice realization of the QFT under investigation, compactified on some finite thorus: while now we deal with a precise finite graph, if the lattice spacing is "small enough" and the radia of compactifications are "large enough", and in particular near to a critical temperature where the characteristic length ξ is diverging, the lattice features should only appear as benefits, of providing both infrared and ultraviolet regularization of possibly diverging quantities of the continuum.

On a given graph, we are used to sums or products, over vertices or edges (or faces, if the graph is planar). At the aim of making the notations more similar to the ones in the continuum, we will adopt here some allusive synonima. Say that the degrees of freedom of the system are associated to the vertices of the graph, and encoded by some field ϕ, and that the interaction pattern is described by the comparison of the values of the field on adjacent vertices. Then the "graph notation" partition function could be of the form

$$Z = \sum_{\phi \in X^{V(G)}} \prod_{(ij) \in E(G)} W_{ij}(\phi_i, \phi_j)\,, \tag{9.1}$$

and, if W_{ij} is non-zero everywhere, and depends on the fields through the "difference" (in some sense), we can also write

$$Z = \sum_{\phi \in X^{V(G)}} \exp\Big(\sum_{(ij)\in E(G)} S_{ij}(\phi_i - \phi_j)\Big) . \tag{9.2}$$

In "lattice notation" we will write

$$\int [d\phi] \quad \sim \quad \sum_{\phi \in X^{V(G)}} ; \tag{9.3}$$

$$\int d^d x \quad \sim \quad \sum_{i\in V} ; \tag{9.4}$$

$$\int d^d x F[\partial_\mu g(x)] \quad \sim \quad \sum_{i\in V} \sum_{j\sim i} F[g(j) - g(i)] \tag{9.5}$$

$$\sim \quad 2 \sum_{(ij)\in E} F[g(j) - g(i)] \qquad \text{if } F[g] = F[-g]. \tag{9.6}$$

In particular, the "Laplacian" operator L (or Δ) reads

$$\tfrac{1}{2} \int d^d x \; fLg \quad \sim \quad \sum_{(ij)\in E} \big(f(i) - f(j)\big)\big(g(i) - g(j)\big) . \tag{9.7}$$

9.3 Ward identities for the $O(N)$ σ-model

In this section we will derive the Ward identities due to $O(N)$-invariance in the non-linear σ-model, essentially following the derivation in the Zinn-Justin book [137]. And we shall put them in the form of identities among invariant correlation functions. The field $\phi(x)$ on each site x has N components, that we arrange in a component $\sigma(x)$ and a vector of $N-1$ components $\boldsymbol{\pi}(x)$. This choice is done to highlight the structure of the perturbative treatment, around a vacuum configuration $\sigma_*(x) = 1$, $\boldsymbol{\pi}_*(x) = \mathbf{0}$.

The partition function (in "lattice notation") is defined as

$$Z := \int [d\boldsymbol{\pi} d\sigma \delta(\boldsymbol{\pi}^2 + \sigma^2 - 1)] \exp\left[-\frac{1}{2g} \int d^d x \, [(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2] \right] \tag{9.8}$$

We generalize it to include a source term. Again we use a different name for the source H coupled to σ, and the source $\boldsymbol{J}$ coupled to $\boldsymbol{\pi}$, i.e.

$$Z[\boldsymbol{J}, H] := \int [d\boldsymbol{\pi} d\sigma \delta(\boldsymbol{\pi}^2 + \sigma^2 - 1)] \\ \exp\left[-\frac{1}{2g} \int d^d x \, [(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2] + \frac{1}{g} \int d^d x \, (\boldsymbol{J} \cdot \boldsymbol{\pi} + H \, \sigma) \right] . \tag{9.9}$$

As

$$d\sigma \delta(\boldsymbol{\pi}^2 + \sigma^2 - 1) = \frac{1}{2\sqrt{1 - \boldsymbol{\pi}^2}} \left[\delta\left(\sigma - \sqrt{1 - \boldsymbol{\pi}^2} \right) + \delta\left(\sigma + \sqrt{1 - \boldsymbol{\pi}^2} \right) \right] \tag{9.10}$$

we have that

$$(\partial_\mu \sigma)^2 = \left[\frac{\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi}}{\sqrt{1 - \boldsymbol{\pi}^2}} \right]^2 \tag{9.11}$$

This choice preserves at sight the residual $O(N-1)$ invariance of $\boldsymbol{\pi}$, while non-trivial informations can be deduced by the nonlinear transformation which preserves the action

$$\begin{cases} \delta\boldsymbol{\pi} = \mathbf{w}\sqrt{1-\boldsymbol{\pi}^2} \ = \mathbf{w}\,\sigma \\ \delta\sigma \ = -\,\mathbf{w}\cdot\boldsymbol{\pi} \end{cases} \tag{9.12}$$

which depends on a $(N-1)$-component vector $\mathbf{w}$, and mixes σ with $\boldsymbol{\pi}$ degrees of freedom.

We can thus define the action

$$S(\boldsymbol{\pi};\boldsymbol{J},H) := \frac{1}{2}\int d^dx\,[(\partial_\mu\boldsymbol{\pi})^2 + (\partial_\mu\sigma)^2] - \int d^dx\,(\boldsymbol{J}\cdot\boldsymbol{\pi} + H\,\sigma) \tag{9.13}$$

where σ is a function of $\boldsymbol{\pi}$ as described above, and as customary

$$Z[\boldsymbol{J},H] = \int \left[\frac{d\boldsymbol{\pi}}{\sqrt{1-\boldsymbol{\pi}^2}}\right] \exp\left\{-\frac{1}{g}S(\boldsymbol{\pi};\boldsymbol{J},H)\right\} \tag{9.14}$$

The transformation

$$\boldsymbol{\pi} = \boldsymbol{\pi}' + \delta\boldsymbol{\pi} = \boldsymbol{\pi}' + \frac{\delta S(\boldsymbol{\pi}';\boldsymbol{J},H)}{\delta H}\mathbf{w} = \boldsymbol{\pi}' - \sigma'\mathbf{w} \tag{9.15}$$

can be implemented in the functional integral. The change of variables has a Jacobian which is cancelled by the variation of the measure (as should because of the original symmetry of the model, before solving w.r.t. σ), the classical action is invariant, the only change is in the source terms. We get

$$0 = \int \left[\frac{d\boldsymbol{\pi}}{\sqrt{1-\boldsymbol{\pi}^2}}\right] \int d^dx\ [\sigma(x)\boldsymbol{J}(x) - H(x)\boldsymbol{\pi}(x)] \exp\left\{-\frac{1}{g}S(\boldsymbol{\pi};\boldsymbol{J},H)\right\} \tag{9.16}$$

that is

$$\int d^dx\ \left[\boldsymbol{J}(x)\frac{\delta}{\delta H(x)} - H(x)\frac{\delta}{\delta\boldsymbol{J}(x)}\right]\ Z\,[\boldsymbol{J},H] = 0\,. \tag{9.17}$$

This is the "generating function" of all Ward identities, where $\boldsymbol{J}$ and H are the corresponding multipliers, that is, the family of Ward identities corresponds to setting to zero the coefficient of any possible monomial in $\boldsymbol{J}$ and H inside (9.17), and assuming a Taylor expansion of $Z[\boldsymbol{J},H]$, into (unnormalized connected) k-point functions. In other words, Ward identities are extracted by taking a certain number of derivatives w.r.t. $\boldsymbol{J}$ and H, and then setting them to zero. The first few cases are now described explicitly.

Let us first take the i-component of this relation and a derivative with respect to $J^j(y)$

$$\int d^dx\ \left[\delta^{ij}\,\delta(x-y)\frac{\delta}{\delta H(x)} + J^i(y)\frac{\delta^2}{\delta J^j(y)\delta H(x)} - H(x)\frac{\delta^2}{\delta J^j(y)\delta J^i(x)}\right]\ Z\,[\boldsymbol{J},H] = 0 \tag{9.18}$$

(where δ^{ij} is a Kronecker delta on component indices, while $\delta(x-y)$ is a delta on the geometrical positions of the fields, Kronecker or Dirac respectively if we deal with the theory on a lattice or in the continuum). At $\boldsymbol{J} = H = 0$ and $i = j$ this tells us that

$$\langle\sigma(y)\rangle = 0 \tag{9.19}$$

Analogously, by taking a derivative of (9.17) with respect to $H(y)$ at vanishing sources, we easily get

$$\langle\boldsymbol{\pi}(y)\rangle = \mathbf{0} \tag{9.20}$$

By taking a derivative in (9.18) with respect to $H(z)$

$$0 = \int d^d x \left[\delta^{ij}\, \delta(x-y) \frac{\delta^2}{\delta H(x) \delta H(z)} + J^i(y) \frac{\delta^3}{\delta J^j(y) \delta H(x) \delta H(z)} \right. \\ \left. + H(x) \frac{\delta^3}{\delta J^j(y) \delta J^i(x) \delta H(z)} - \delta(x-z) \frac{\delta^2}{\delta J^j(y) \delta J^i(x)} \right] Z\left[\boldsymbol{J}, H\right] \tag{9.21}$$

at vanishing sources we arrive at

$$\delta^{ij} \left\langle \sigma(y)\sigma(z) \right\rangle = \left\langle \pi^i(y) \pi^j(z) \right\rangle \tag{9.22}$$

and by taking $i = j$ and summing on i

$$(N-1) \left\langle \sigma(y)\sigma(z) \right\rangle = \left\langle \boldsymbol{\pi}(y) \cdot \boldsymbol{\pi}(z) \right\rangle \tag{9.23}$$

By taking one more derivative in (9.18) with respect to $H(u)$ at vanishing sources we get

$$\delta^{ij} \left\langle \sigma(y)\sigma(z)\sigma(u) \right\rangle = \left\langle \pi^i(z) \pi^j(y) \sigma(u) \right\rangle + \left\langle \pi^i(u) \pi^j(y) \sigma(z) \right\rangle \tag{9.24}$$

which for $i = j$ and summing on i becomes

$$(N-1) \left\langle \sigma(y)\sigma(z)\sigma(u) \right\rangle = \left\langle \left(\boldsymbol{\pi}(z) \cdot \boldsymbol{\pi}(y)\right) \sigma(u) \right\rangle + \left\langle \left(\boldsymbol{\pi}(u) \cdot \boldsymbol{\pi}(y)\right) \sigma(z) \right\rangle \tag{9.25}$$

And by taking instead one more derivative with respect to $H(t)$ at vanishing sources we get

$$\delta^{ij} \left\langle \sigma(y)\sigma(z)\sigma(u)\sigma(t) \right\rangle = \left\langle \pi^i(z) \pi^j(y) \sigma(u) \sigma(t) \right\rangle \\ + \left\langle \pi^i(u) \pi^j(y) \sigma(z) \sigma(t) \right\rangle + \left\langle \pi^i(t) \pi^j(y) \sigma(z) \sigma(u) \right\rangle \tag{9.26}$$

which for $i = j$ and summing on i becomes

$$(N-1) \left\langle \sigma(y)\sigma(z)\sigma(u)\sigma(t) \right\rangle = \left\langle \left(\boldsymbol{\pi}(z) \cdot \boldsymbol{\pi}(y)\right) \sigma(u) \sigma(t) \right\rangle \\ + \left\langle \left(\boldsymbol{\pi}(u) \cdot \boldsymbol{\pi}(y)\right) \sigma(z) \sigma(t) \right\rangle + \left\langle \left(\boldsymbol{\pi}(t) \cdot \boldsymbol{\pi}(y)\right) \sigma(z) \sigma(u) \right\rangle \,. \tag{9.27}$$

As a last example, let us put $H = 0$ in (9.21) and take two more derivatives with respect to $J^r(u)$ and $J^s(t)$ to get at $\boldsymbol{J} = 0$

$$0 = - \left\langle \pi^i(z) \pi^j(y) \pi^r(u) \pi^s(t) \right\rangle + \delta^{ij} \left\langle \sigma(y)\sigma(z) \pi^r(u) \pi^s(t) \right\rangle \\ + \delta^{ir} \left\langle \pi^j(y) \pi^s(t) \sigma(u) \sigma(z) \right\rangle + \delta^{is} \left\langle \pi^j(y) \pi^r(u) \sigma(t) \sigma(z) \right\rangle \tag{9.28}$$

therefore, for example, by taking $i = j$ and $r = s$ and by summation on i and r

$$\left\langle \left(\boldsymbol{\pi}(y) \cdot \boldsymbol{\pi}(z)\right) \left(\boldsymbol{\pi}(u) \cdot \boldsymbol{\pi}(t)\right) \right\rangle = (N-1) \Big(\left\langle \sigma(y)\sigma(z) \left(\boldsymbol{\pi}(u) \cdot \boldsymbol{\pi}(t)\right) \right\rangle \\ + \left\langle \sigma(u)\sigma(z) \left(\boldsymbol{\pi}(y) \cdot \boldsymbol{\pi}(t)\right) \right\rangle + \left\langle \sigma(t)\sigma(z) \left(\boldsymbol{\pi}(y) \cdot \boldsymbol{\pi}(u)\right) \right\rangle \Big) \,. \tag{9.29}$$

9.4 Ward identities for the generating function of spanning forests

As we have derived in the previous chapter, the generating functional for the forests is

$$Z := \int [d\psi d\bar{\psi}] \exp\left[\bar{\psi} L \psi + t\bar{\psi}\psi + \frac{t}{2}(\bar{\psi}\psi) L (\bar{\psi}\psi)\right] \tag{9.30}$$

where L is the laplacian operator, and again "lattice notations" are used. We find convenient to make the rescaling

$$(\psi, \bar{\psi}) \to \frac{1}{\sqrt{t}}(\psi, \bar{\psi}) \tag{9.31}$$

so that

$$Z := t^V \int [d\psi d\bar{\psi}\, e^{\bar{\psi}\psi}] \exp\left\{\frac{1}{t}\left[\bar{\psi} L \psi + \frac{1}{2}(\bar{\psi}\psi) L (\bar{\psi}\psi)\right]\right\} \tag{9.32}$$

We can recognize that this is obtained from the generating functional of the $O(N)$ non-linear σ-model by the replacement

$$\boldsymbol{\pi}(x) \cdot \boldsymbol{\pi}(y) \to \bar{\psi}(x)\psi(y) - \psi(x)\bar{\psi}(y) \tag{9.33}$$

indeed

$$\frac{1}{2}\boldsymbol{\pi} L \boldsymbol{\pi} \quad \to \quad \bar{\psi} L \psi \tag{9.34}$$

$$\sigma \quad \to \quad 1 - \bar{\psi}\psi = e^{-\bar{\psi}\psi} \tag{9.35}$$

$$\frac{1}{2}\sigma L \sigma \quad \to \quad \frac{1}{2}(1 - \bar{\psi}\psi) L (1 - \bar{\psi}\psi) = \frac{1}{2}\bar{\psi}\psi L \bar{\psi}\psi \tag{9.36}$$

where the last replacement is justified by the fact that the vector $(1, 1, \ldots, 1)$ is a zero mode of L both on the left and on the right. Again here we neglect the role of "Ising spins" corresponding to the two possible determinations of the square root defining σ.

Introducing anti-commuting sources for the fermionic components, we have also

$$\boldsymbol{J} \cdot \boldsymbol{\pi} \to \bar{J}\psi + \bar{\psi} J$$

so that

$$Z[J, \bar{J}, H] = t^V \int [d\psi d\bar{\psi}\, e^{\bar{\psi}\psi}] \exp\left\{\frac{1}{t}\left[\bar{\psi} L \psi + \frac{1}{2}(\bar{\psi}\psi) L (\bar{\psi}\psi) + \bar{J}\psi + \bar{\psi} J + H(1 - \bar{\psi}\psi)\right]\right\} \tag{9.37}$$

which makes clear what is the part due to the integration measure, which is independent from t, and what is the *classical action*.

We are interested in the transformations

$$\delta\psi = \epsilon\, \sigma = \epsilon(1 - \bar{\psi}\psi) \tag{9.38}$$

$$\delta\bar{\psi} = \bar{\epsilon}\, \sigma = \bar{\epsilon}(1 - \bar{\psi}\psi) \tag{9.39}$$

where both ϵ and $\bar{\epsilon}$ are anticommuting variables, and replace the vector $\mathbf{w}$, and therefore

$$\delta\sigma = -\delta(\bar{\psi}\psi) = -\delta\bar{\psi}\, \psi + \bar{\psi}\, \delta\psi = -\bar{\epsilon}\, \psi + \bar{\psi}\, \epsilon\,. \tag{9.40}$$

The third generator of OSP(1|2), such that

$$\delta\psi = \lambda\bar{\psi}\,, \qquad \delta\bar{\psi} = -\lambda\psi\,, \qquad \delta\sigma = 0\,, \tag{9.41}$$

just induces charge symmetry, which only gives the trivial property

$$\left\langle \bar{\psi}_1 \cdots \bar{\psi}_a \psi_{a+1} \cdots \psi_{a+b} \right\rangle \propto \delta_{a,b}$$

(this is the analogue of the trivial residual $O(N-1)$ symmetry in the previous section).

Once more this leaves invariant the classical action and the change of the measure is exactly compensated by the Jacobian of the transformation and therefore in the functional integral ther is only a change due to the source terms as

$$\delta\left[\bar{J}\psi+\bar{\psi}J+H(1-\bar{\psi}\psi)\right]=(-\sigma\,\bar{J}+H\,\bar{\psi})\,\epsilon+\bar{\epsilon}\,(\sigma\,J+H\,\psi) \tag{9.42}$$

so that

$$0=\int d^dx\left[\bar{J}(x)\frac{\partial}{\partial H(x)}+H(x)\frac{\partial}{\partial J(x)}\right]\,Z[J,\bar{J},H] \tag{9.43}$$

$$0=\int d^dx\left[J(x)\frac{\partial}{\partial H(x)}-H(x)\frac{\partial}{\partial \bar{J}(x)}\right]\,Z[J,\bar{J},H] \tag{9.44}$$

These equations are equivalent under charge conjugation (with a minus sign coming from the fermionic nature of the J, $\bar{J}$, ψ, $\bar{\psi}$'s), so we can consider only the first one. Ward identities are obtained considering the action of a certain number of derivaties w.r.t. the source fields, taken at vanishing external sources. In order to have a non-vanishing contribution from the first summand, we must contract the monomial $\bar{J}(x)$ with a derivative $\partial/\partial\bar{J}(y)$, while on the second summand we need at least one derivative $\partial/\partial\bar{J}(y)$ in order to have charge neutrality. So, up to a relabeling of the sites, without loss of generality we can restrict to the derivative of (9.43) with respect to $\bar{J}(y)$:

$$0=\int d^dx\left[\delta(x-y)\frac{\partial}{\partial H(x)}-\bar{J}(x)\frac{\partial^2}{\partial\bar{J}(y)\partial H(x)}+H(x)\frac{\partial^2}{\partial\bar{J}(y)\partial J(x)}\right]\,Z[J,\bar{J},H]\,. \tag{9.45}$$

This equation, taken at zero sources, tells us that

$$\left\langle\bar{\psi}\psi(y)\right\rangle=1 \tag{9.46}$$

(where $\bar{\psi}\psi(y)$ is a shorthand for $\bar{\psi}(y)\psi(y)$) which is of course the identity corresponding with (9.19). Let's interpret combinatorially the expectation above, in the setting developed in Section 8.5. Given the 'rules' developed, this identity simply means that, as the forests are spanning, each point must belong to exactly one tree in every configuration.

Let us put $\bar{J}=J=0$ in (9.45)

$$0=\int d^dx\left[\delta(x-y)\frac{\partial}{\partial H(x)}+H(x)\frac{\partial^2}{\partial\bar{J}(y)\partial J(x)}\right]\,Z[J,\bar{J},H]\Big|_{\bar{J}=J=0} \tag{9.47}$$

If we take a derivative of (9.47) with respect to $H(z)$ at vanishing sources we get

$$0=\left[\frac{\partial^2}{\partial H(y)\partial H(z)}+\frac{\partial^2}{\partial\bar{J}(y)\partial J(z)}\right]\,Z[J,\bar{J},H]\Big|_{J=\bar{J}=H=0} \tag{9.48}$$

that is

$$-\left\langle\sigma(y)\sigma(z)\right\rangle=\left\langle\bar{\psi}(z)\psi(y)\right\rangle \tag{9.49}$$

But as $\sigma=1-\bar{\psi}\psi$ and at the light of (9.46) this becomes

$$1-\left\langle\bar{\psi}\psi(y)\bar{\psi}\psi(z)\right\rangle=\left\langle\bar{\psi}(z)\psi(y)\right\rangle \tag{9.50}$$

which has a very simple interpretation, as $\left\langle\bar{\psi}\psi(y)\bar{\psi}\psi(z)\right\rangle$ is the probability that the points y and z do not belong to the same tree, $\left\langle\bar{\psi}(z)\psi(y)\right\rangle$ is instead the probability that they belong to the same tree in the forest.

If instead of (9.43) we use (9.44), we would obtaine a similar relation with z and y interchanged, that is $\langle\bar\psi(y)\psi(z)\rangle$ in the right-hand side. The average of the two identities is exactly reproducing (9.23) when $N=-1$.

Analogously we can derive a 3-point Ward identity

$$-\langle\sigma(y)\sigma(z)\sigma(u)\rangle = \langle\bar\psi(z)\psi(y)\sigma(u)\rangle + \langle\bar\psi(u)\psi(y)\sigma(z)\rangle \tag{9.51}$$

Once more, substituting $\sigma = 1-\bar\psi\psi$ and using (9.46) and (9.49), on the left hand side we get

$$-\langle\sigma(y)\sigma(z)\sigma(u)\rangle = \langle\bar\psi\psi(y)\bar\psi\psi(z)\bar\psi\psi(u)\rangle - 1 + \langle\bar\psi(y)\psi(z)\rangle + \langle\bar\psi(u)\psi(y)\rangle + \langle\bar\psi(z)\psi(u)\rangle \tag{9.52}$$

while on the right

$$\begin{aligned}&\langle\bar\psi(z)\psi(y)\sigma(u)\rangle + \langle\bar\psi(u)\psi(y)\sigma(z)\rangle\\ &\quad= \langle\bar\psi(z)\psi(y)\rangle + \langle\bar\psi(u)\psi(y)\rangle - \langle\bar\psi(z)\psi(y)\bar\psi\psi(u)\rangle - \langle\bar\psi(u)\psi(y)\bar\psi\psi(z)\rangle\end{aligned} \tag{9.53}$$

so that

$$\begin{aligned}\langle\bar\psi\psi(y)\bar\psi\psi(z)\bar\psi\psi(u)\rangle &= 1 - \langle\bar\psi(z)\psi(u)\rangle - \langle\bar\psi(z)\psi(y)\bar\psi\psi(u)\rangle - \langle\bar\psi(u)\psi(y)\bar\psi\psi(z)\rangle\\ &= \langle\bar\psi\psi(z)\bar\psi\psi(u)\rangle - \langle\bar\psi(z)\psi(y)\bar\psi\psi(u)\rangle - \langle\bar\psi(u)\psi(y)\bar\psi\psi(z)\rangle\end{aligned} \tag{9.54}$$

which means that the three points y,z,u are in different trees if z and u are in different trees and y is neither in the same tree of z nor of u.

A different series of identities is derived by taking in (9.45) two more derivatives respect to $J(u)$ and $\bar J(t)$. At zero sources we get

$$\langle\bar\psi(y)\psi(u)\sigma(t)\rangle - \langle\bar\psi(t)\psi(u)\sigma(y)\rangle = 0 \tag{9.55}$$

which is restated as

$$\langle\bar\psi(y)\psi(u)\rangle + \langle\bar\psi(t)\psi(u)\bar\psi(y)\psi(y)\rangle = \langle\bar\psi(t)\psi(u)\rangle + \langle\bar\psi(y)\psi(u)\bar\psi(t)\psi(t)\rangle \tag{9.56}$$

that is, given site u, the probability that it is connect to at least one among y and t can be stated in two ways: either as the probability of being connected to y, regardless to t, plus the probability of being connected to t and not connected to y, or as the analogous statement with t and y interchanged.

9.5 All Ward identities

Here we find all possible non-trivial Ward identities, i.e., as we discussed in the previous section, the ones which come from equation (9.45), and are not implied just by charge conservation. We use the following short-hand notations

$$H_i = H(x_i)\,; \qquad \bar J_i = \bar J(x_i)\,; \qquad J_i = J(x_i)\,; \tag{9.57}$$

$$D_i = \frac{\partial}{\partial H(x_i)}\,; \qquad \overline{\partial}_i = \frac{\partial}{\partial \bar J(x_i)}\,; \qquad \partial_i = \frac{\partial}{\partial J(x_i)}\,. \tag{9.58}$$

and recall that H_i and D_i are commuting variables, while J_i, $\bar J_i$, ∂_i and $\overline{\partial}_i$ are anticommuting.

The set of all Ward identities can be classified by taking an arbitrary number of derivatives in (9.45), taken at zero external sources. As equation (9.45) is already charge-neutral, identities satisfying charge conservation must involve the same number of $\overline{\partial}$ and ∂, and thus are, for some k-uple of vertices $(i_1, i_2, \dots, i_k)$, possibly with repetitions, and an integer $h \leq k/2$,

$$0 = D_{i_k} \cdots D_{i_{2h}} \overline{\partial}_{i_{2h-1}} \partial_{i_{2h-2}} \cdots \overline{\partial}_{i_3} \partial_{i_2} \left[D_{i_1} + \sum_{i_0 \in V} \left(-\bar{J}_{i_0} \overline{\partial}_{i_1} D_{i_0} + H_{i_0} \overline{\partial}_{i_1} \partial_{i_0} \right) \right] Z[J, \bar{J}, H] \Big|_{J,\bar{J},H=0} . \tag{9.59}$$

In order to enlight the notation, we make a relevant abuse of language, by replacing operators X_{i_a} by X_a. Recall however that the string $(i_1, i_2, \dots, i_k)$ possibly allows repetitions, despite what is suggested by this condensed notation. So, for example, we will rewrite the above expression as

$$0 = D_k \cdots D_{2h} \overline{\partial}_{2h-1} \partial_{2h-2} \cdots \overline{\partial}_3 \partial_2 \left[D_1 + \sum_{i_0} \left(-\bar{J}_0 \overline{\partial}_1 D_0 + H_0 \overline{\partial}_1 \partial_0 \right) \right] Z[J, \bar{J}, H] \Big|_{J,\bar{J},H=0} . \tag{9.60}$$

An equivalent basis is the one in which we use the operators $\widetilde{D}_i := \alpha + \beta D_i$ instead of D_i, with $\beta \neq 0$ (it is equivalent at sight, as it corresponds to recombine linearly the equations through a triangular system). The case $\widetilde{D}_i = D_i$ is such that $\widetilde{D}_i Z = \sigma_i Z$, while the one $\widetilde{D}_i = 1 - D_i$ is such that $\widetilde{D}_i Z = \bar{\psi}_i \psi_i Z$ (after our rescaling of t). This is the only choice of α/β which produces the insertion of an operator of homogeneous degree, a feature that we will see being desiderable. Also, this choice forbids almost completely repetitions in the string $(i_1, i_2, \dots, i_k)$, thus making the condensed notation *a posteriori* more legitimate.

Reintroducing t one would have

$$(1 - t\, D_i) Z(t) = \bar{\psi}_i \psi_i Z(t) ; \qquad t\, \overline{\partial}_i Z(t) = \psi_i Z(t) ; \qquad t\, \partial_i Z(t) = -\bar{\psi}_i Z(t) . \tag{9.61}$$

So we consider the expression

$$0 = \widetilde{D}_k \cdots \widetilde{D}_{2h} \overline{\partial}_{2h-1} \partial_{2h-2} \cdots \overline{\partial}_3 \partial_2 \left[D_1 + \sum_{i_0} \left(-\bar{J}_0 \overline{\partial}_1 D_0 + H_0 \overline{\partial}_1 \partial_0 \right) \right] Z[J, \bar{J}, H] \Big|_{J,\bar{J},H=0} . \tag{9.62}$$

We want to commute all $\overline{\partial}$ and ∂'s through the factors $\bar{J}_0$ and H_0. A consequence of the (anti-)commutation relations of the elementary operators gives, for an arbitrary integrand $I(\bar{\psi}_0, \psi_0)$,

$$\sum_{i_0} \widetilde{D}_i H_0 I(\bar{\psi}_0, \psi_0) = \sum_{i_0} H_0 \widetilde{D}_i I(\bar{\psi}_0, \psi_0) + \beta I(\bar{\psi}_i, \psi_i) \tag{9.63}$$

$$\sum_{i_0} \overline{\partial}_i \partial_j \bar{J}_0 I(\bar{\psi}_0, \psi_0) = \sum_{i_0} \bar{J}_0 \overline{\partial}_i \partial_j I(\bar{\psi}_0, \psi_0) - \partial_j I(\bar{\psi}_i, \psi_i) \tag{9.64}$$

Given the notation

$$A_k A_{k-1} \cdots \underbrace{A_h} \cdots A_1 := A_k A_{k-1} \cdots A_{h+1} A_{h-1} \cdots A_1 \tag{9.65}$$

commuting all $\overline{\partial}$ and ∂'s using (9.64), and all $\widetilde{D}_i$'s using (9.63), gives

$$
\begin{aligned}
0 = \Bigg(& \sum_{a=0}^{h-1} \widetilde{D}_k \cdots \widetilde{D}_{2h} \overline{\partial}_{2h-1} \partial_{2h-2} \cdots \underbrace{\overline{\partial}_{2a+1}} \cdots \partial_2 \overline{\partial}_1 D_{2a+1}\, Z[J,\bar{J},H] \\
& + \beta \sum_{b=2h}^{k} \widetilde{D}_k \cdots \underbrace{\widetilde{D}_b} \cdots \widetilde{D}_{2h} \overline{\partial}_{2h-1} \partial_{2h-2} \cdots \overline{\partial}_1 \partial_b\, Z[J,\bar{J},H] \\
& - \sum_{i_0} (\bar{J}_0 D_0 + H_0 \partial_0) \widetilde{D}_k \cdots \widetilde{D}_{2h} \overline{\partial}_{2h-1} \partial_{2h-2} \cdots \overline{\partial}_1\, Z[J,\bar{J},H] \Bigg) \Bigg|_{J,\bar{J},H=0} .
\end{aligned}
\tag{9.66}
$$

When we take these equations at vanishing sources, the last term vanishes. Using the relations (9.61) (at $t = 1$ for simplicity, but generalization is immediate) we end up with identities involving the expectation values of fermionic operators, on the theory with no sources. For the case $\widetilde{D} = D$ we have

$$
\begin{aligned}
0 = & \sum_{a=0}^{h-1} \left\langle \sigma_k \cdots \sigma_{2h} \sigma_{2a+1}\, \bar{\psi}_{2h-1} \psi_{2h-2} \cdots \underbrace{\bar{\psi}_{2a+1}} \cdots \bar{\psi}_1 \right\rangle \\
& + \sum_{b=2h}^{k} \left\langle \sigma_k \cdots \underbrace{\sigma_b} \cdots \sigma_{2h}\, \bar{\psi}_{2h-1} \psi_{2h-2} \cdots \bar{\psi}_1\, \psi_b \right\rangle ,
\end{aligned}
\tag{9.67}
$$

where we recall that σ_i is here a shortcut for $1 - \bar{\psi}_i \psi_i$, while if $\widetilde{D} = 1 - D$ we have

$$
\begin{aligned}
0 = & \sum_{a=0}^{h-1} \left\langle \bar{\psi}_k \psi_k \cdots \bar{\psi}_{2h} \psi_{2h}\, (1 - \bar{\psi}_{2a+1} \psi_{2a+1})\, \bar{\psi}_{2h-1} \psi_{2h-2} \cdots \underbrace{\bar{\psi}_{2a+1}} \cdots \bar{\psi}_1 \right\rangle \\
& - \sum_{b=2h}^{k} \left\langle \bar{\psi}_k \psi_k \cdots \underbrace{\bar{\psi}_b \psi_b} \cdots \bar{\psi}_{2h} \psi_{2h}\, \bar{\psi}_{2h-1} \psi_{2h-2} \cdots \bar{\psi}_1\, \psi_b \right\rangle .
\end{aligned}
\tag{9.68}
$$

In this case we see the advantage of working with $\widetilde{D} = 1 - D$: all the terms have almost the same degree (up to a difference of at most 2, due to the operator D explicitly appearing in (9.45)), so that they can be interpreted as combinatorics of connectivities among (almost) the same set of points.

Recall that, in the string $(i_1, i_2, \ldots, i_k)$ corresponding to the operators descibed above, vertices i_a with a odd and smaller than $2h$ correspond to fields $\bar{\psi}_{i_a}$, vertices i_a with a even and smaller than $2h$ correspond to fields ψ_{i_a}, and vertices i_a with $a \geq 2h$ correspond to the composite fields $\bar{\psi}_{i_a} \psi_{i_a}$. So, as a special feature of this OSP(1|2) model (of being expanded in terms of nilpotent fields), and of the choice $\widetilde{D} = 1 - D$ (so that also the composite field is nilpotent), we see that the mere argument in the Ward identity is zero, thus trivializing the statement, if any two indices are equal, except possibly for pairs (i_a, i_b), with a odd and b even, and both smaller than $2h$. However, as the product of the two contributions is in this case equal to the composite field $\bar{\psi}_i \psi_i$, also this case can be dropped, as it gives no new identities.

We have two easy relations among the expectation values of fermionic operators. If $f(\bar{\psi}, \psi)$ is a function of $(\bar{\psi}_i \psi_i)$'s only, we have

$$
\begin{aligned}
& \left\langle \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_h} \psi_{j_h} f(\bar{\psi}, \psi) \right\rangle = \epsilon(\sigma\tau) \left\langle \bar{\psi}_{i_{\sigma(1)}} \psi_{j_{\tau(1)}} \cdots \bar{\psi}_{i_{\sigma(h)}} \psi_{j_{\tau(h)}} f(\bar{\psi}, \psi) \right\rangle \qquad \forall\ \sigma, \tau \in \mathcal{S}_h\,; \\
& \left\langle \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_h} \psi_{j_h} f(\bar{\psi}, \psi) \right\rangle = \left\langle \bar{\psi}_{j_1} \psi_{i_1} \cdots \bar{\psi}_{j_h} \psi_{i_h} f(\bar{\psi}, \psi) \right\rangle\,;
\end{aligned}
$$

In order to avoid redundancies, we exploit the first of these invariances and determine a canonical representative for these classes, the one such that $i_\alpha < i_{\alpha+1}$ and $j_\alpha < j_{\alpha+1}$ In terms of these representatives, we can write equations (9.68) as

$$\sum_{a=0}^{h-1}(-1)^a\Big\langle \bar\psi_1\psi_2\cdots\bar\psi_{2a-1}\psi_{2a}\,\bar\psi_{2a+3}\psi_{2a+2}\cdots\bar\psi_{2h-1}\psi_{2h-2}\,\bar\psi_{2h}\psi_{2h}\cdots\bar\psi_k\psi_k\Big\rangle$$
$$=\sum_{a=0}^{h-1}(-1)^a\Big\langle \bar\psi_1\psi_2\cdots\bar\psi_{2a-1}\psi_{2a}\,\bar\psi_{2a+3}\psi_{2a+2}\cdots\bar\psi_{2h-1}\psi_{2h-2}\,\bar\psi_{2a+1}\psi_{2a+1}\,\bar\psi_{2h}\psi_{2h}\cdots\bar\psi_k\psi_k\Big\rangle$$
$$-(-1)^h\sum_{b=2h}^{k}\Big\langle \bar\psi_1\psi_2\cdots\bar\psi_{2h-3}\psi_{2h-2}\bar\psi_{2h-1}\,\psi_b\,\bar\psi_{2h}\psi_{2h}\cdots\bar\psi_{b-1}\psi_{b-1}\bar\psi_{b+1}\psi_{b+1}\,\bar\psi_k\psi_k\Big\rangle\,. \tag{9.69}$$

9.6 Combinatorial meaning of Ward Identities

Consider the set $\Pi(S)$ of partitions of the set S. If the set has cardinality k, we may want to recall this by a subscript, and write $\Pi_k(S)$.

Call P a partition in this set (for example, a partition of $\{1,2,3,4,5,6\}$ could be $\{1,3,4\},\{2\},\{5,6\}$).

For every set of k points $x_1,\,\ldots,\,x_k$ in our graph, we can decompose the partition function of our system in the form

$$Z(t)=\sum_{P\in\Pi_k(\{x_1,\ldots,x_k\})} Z(t;P)\,, \tag{9.70}$$

where $Z(t)$ is the partition function corresponding to the sum over all spanning forests, and $Z(t;P)$ is the one in which the sum is restricted to forests such that the pattern of connection of our k points is given by partition P. We also define

$$\mathrm{prob}(P)=\frac{Z(t;P)}{Z(t)}\,. \tag{9.71}$$

According to the results of Section 8.5, we have that, for a generic theory involving scalar products of $RP^{1|2}$ supervectors, expectations of monomials in the fields in the points of set $S\subseteq V$ (or a subset) can be written as a linear combination of the partial partition function $Z(t;P)$ for $P\in\Pi(S)$. More precisely, if I, J and L are disjoint subsets of S, with $|I|=|J|$, and $I=\{i_1,\ldots,i_h\}$, $J=\{j_1,\ldots,j_h\}$ and $L=\{\ell_1,\ldots,\ell_{k'-2h}\}$, defining the observable $O(S;I,J,L)$

$$O(S;I,J,L)=\bar\psi(x_{i_1})\psi(x_{j_1})\cdots\bar\psi(x_{i_h})\psi(x_{j_h})\,\big(\bar\psi\psi\big)(x_{\ell_1})\cdots\big(\bar\psi\psi\big)(x_{\ell_{k'-2h}})\,, \tag{9.72}$$

one has

$$\langle O(S;I,J,L)\rangle=\sum_{P\in\Pi_k(S_k)} M^{(k)}_{OP}\;\mathrm{prob}(P) \tag{9.73}$$

$$M^{(k)}_{OP}=\sum_{\pi\in\mathcal{S}_h}\epsilon(\pi)\,\delta\Big(P|_{I\cup J\cup L};\;\{i_1,j_{\pi(1)}\},\cdots,\{i_h,j_{\pi(h)}\},\{\ell_{2h+1}\},\cdots,\{\ell_{k'}\}\Big)\,; \tag{9.74}$$

where $\mathcal{S}_h$ is the symmetric group over h objects, and $\delta(P;P')$ is a Kronecker delta, for the partitions P, P' in the same set $\Pi(I\cup J\cup L)$ to coincide.

In other words, an operator $\langle O\rangle$ takes contributions from connectivity patterns on S such that, restricted to points in $I\cup J\cup L$, have the following properties: the points

in L are all in distinct components; the points in I and J are collected pairwise, each component containing one site from I and one from J.

The sign due to commutation of fermionic fields is globally encoded by the signature of the pairing. Remark that the coefficients $M^{(k)}_{OP}$ for each pair (O,P) of operator and partition are either zero or ± 1.

The matrix $M^{(k)}_{OP}$ has a certain dimension, that we now calculate: as P runs over $\mathit{\Pi}(S)$, the number of columns is B_k, the k-th Bell number; similarly, O runs over triplets of disjoint subsets of S, in which $|I| = |J|$, so it is given by

$$\sum_{h,k'} \frac{k!}{(h!)^2(k'-2h)!(k-k')!} = \begin{cases} \binom{2k-1}{k-1} & k \text{ odd;} \\ \binom{2k-1}{k-1} + \binom{k-1}{k/2-1} & k \text{ even.} \end{cases} \tag{9.75}$$

so it roughly goes like 4^k (it would just be 4^k if it were not for the constraint $|I| = |J|$). So, in general $M^{(k)}_{OP}$ has a left- and a right-kernel, and for large k the right-kernel must have a large dimension, as $B_k \sim k! \gg 4^k$. However, we will prove in Chapter 10 first that this matrix has a rank which is bounded by $C_k = \frac{1}{k+1}\binom{2k}{k}$, the k-th Catalan number, and then, much more hardly, that is has exactly this rank. If we also count only once the pairs of distinct operators differing by a full charge-conjugation, we get half the expression in (9.75), plus 2^{k-2}. This does not change the fact that the number of rows in $M^{(k)}_{OP}$ scales approximatively like a central binomial, that is $\sim \text{const.} 4^k/\sqrt{k}$, while the Catalan numbers have an extra algebraic dumping, $\sim \text{const.} 4^k/(k\sqrt{k})$. So we deduce that also the left-kernel of $M^{(k)}_{OP}$ tends to grow for large k.

It is then clear that vectors $\mathbf{v} = \{v_O\}$ in the (left-)kernel of $M^{(k)}$ can be stated combinatorially as vanishing combinations of probabilities of events. However, they also correspond to linear relations among expectation values of fermionic operators, in the fashion of equations (9.69). What we want to prove now is that indeed *all* equations (9.69), involving k points, correspond to vectors in the left-kernel of $M^{(k)}$, and thus, if reformulated combinatorially in the language of Section 8.5, can be stated by probabilistic means only in the spanning-forest formulation.

In order to do this, we need some definitions. We essentially want to follow the structure above, of operators identified by a triplet (I,J,L) of subsets of S, but there is no loss of generality in assuming that these sets have some canonical labeling. So, given the integers k and h, with $h \leq (k+1)/2$, call I_0, J and B the sets

$$I_0 = \{1, 3, \ldots, 2h-1\}\,; \tag{9.76}$$

$$J = \{2, 4, \ldots, 2h-2\}\,; \tag{9.77}$$

$$B = \{2h, 2h+1, \ldots, k\}\,. \tag{9.78}$$

Remark that $|I_0| = h$ while $|J| = h-1$. This definition is such that in all the summands appearing in (9.69) the sets (I', J', L') identifying an operators "almost" coincide with I_0, J and B (that is, they differ by at most one element).

We want to show how these small modifications combine in order to state the LHS and RHS of (9.69) into two equivalent combinations of the same probabilistic event.

For (I', J', B') a partition of $\{1, \ldots, k'\}$ into three blocks, with $|I'| = |J'| = h'$, $I' = \{i_\alpha\}$, $J' = \{j_\alpha\}$ and $B' = \{\ell_\beta\}$ with the natural ordering of $\{1, \ldots, k'\}$, we define

$$W(I', J', B') := \sum_{\pi \in \mathcal{S}_{h'}} \epsilon(\pi)\, \text{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_{\alpha=1,\ldots,h'}\ ,\ \{\ell_\beta\}_{\beta=1,\ldots,k'-2h'}) \tag{9.79}$$

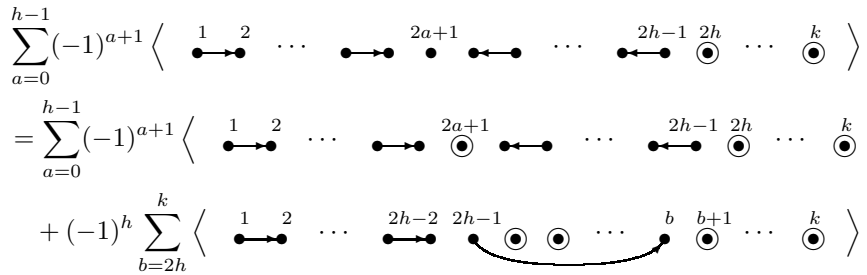
and, for S' a set disjoint from $I' \cup J' \cup B$,

$$W(I', J', B'; S') := \sum_{\pi \in \mathcal{S}_{h'}} \epsilon(\pi) \, \mathrm{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_{\alpha=1,\dots,h'} \, , \, \{\ell_\beta\}_{\beta=1,\dots,k'-2h'} \, , S') \tag{9.80}$$

Furthermore, for $x \in X \subseteq \{1, \dots, n\}$ for some n, define $\mathrm{pos}(x, X)$ the position of x in X w.r.t. the natural ordering. Our Ward Identities (9.69) read in these new notations as

$$\begin{aligned} &\sum_{i_0 \in I_0} (-1)^{\mathrm{pos}(i_0, I_0)} W(I_0 \smallsetminus i_0, J, B) \\ &\quad = \sum_{i_0 \in I_0} (-1)^{\mathrm{pos}(i_0, I_0)} W(I_0 \smallsetminus i_0, J, B \cup i_0) + (-1)^h \sum_{b \in B} W(I_0, J \cup b, B \smallsetminus b) \end{aligned} \tag{9.81}$$

(apparently, there is a minus sign of difference w.r.t. (9.69), which is due to the fact that $\mathrm{pos}(x, X)$ gives 1 on the first element, while it would have been "natural" to start counting from 0, as index a does in summations). The pictorial representation of (9.81) is



(9.82)

Here arrows denote the "canonical" matching of I' to J' (that is, the α-th element of I' is matched to the α-th of J'), taken as a reference for the signature $\epsilon(\pi)$. Circled dots denote singleton sets in the partition. On the LHS, the only uncircled dot means that the connectivity of that point is not specified in $W(I', J', B')$. Remark however that this drawing is done only in order to help the eye, and is not compulsory in our purely algebraic proof.

Recognize how on the LHS we have connectivity patterns over only $k-1$ of the k points of interest, i.e., in the notation above for observables $O(S; I, J, L)$, we have $(I \cup J \cup L) \subsetneq S_k$. Instead, on the RHS we have connectivity patterns over all the k points. So we want to express the $(k-1)$-point terms as combinations of k-point terms, accordingly to the possible connection of the last point.

Say we are in the condition of definition (9.79) (just dropping all primes for brevity), and that $a \notin I \cup J \cup B$. Then clearly we have

$$\begin{aligned} W(I, J, B) &= \sum_{\pi \in \mathcal{S}_h} \epsilon(\pi) \, \mathrm{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_\alpha \, , \, \{\ell_\beta\}_\beta \,) \\ &= \sum_{\pi \in \mathcal{S}_h} \epsilon(\pi) \, \mathrm{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_\alpha \, , \, \{\ell_\beta\}_\beta \, , \{a\}) && \text{①} \\ &\quad + \sum_{\pi \in \mathcal{S}_h} \epsilon(\pi) \sum_{\bar\beta=1}^{k-2h} \mathrm{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_\alpha \, , \, \{\ell_\beta\}_{\beta \neq \bar\beta} \, , \{a, \ell_{\bar\beta}\}) && \text{②} \\ &\quad + \sum_{\pi \in \mathcal{S}_h} \epsilon(\pi) \sum_{\bar\alpha=1}^{h} \mathrm{prob}(\{i_\alpha, j_{\pi(\alpha)}\}_{\alpha \neq \bar\alpha} \, , \, \{\ell_\beta\}_\beta \, , \{i_{\bar\alpha}, a, j_{\pi(\bar\alpha)}\}) && \text{③} \end{aligned} \tag{9.83}$$

We use "numbered bullets" ①, ② and ③ for making future reference to the various terms in this sum.

Indices α and β at pedices of sets has the obvious domains $1 \le \alpha \le h$, and $1 \le \beta \le k - 2h$, unless otherwise specified.

The term ① is of the form of equation (9.79), and ② and ③ of equation (9.80). However, while ① and ② are immediately recognized as of this form, and correspond respectively to

$$① : \quad W(I, J, B \cup a)\,; \qquad\qquad ② : \quad \sum_{b\in B} W(I, J, B \smallsetminus b; \{a, b\})\,; \tag{9.84}$$

in ③ the signature factors are messed up. In ③, call π' the restriction of π to the sets $I \smallsetminus i_{\bar\alpha}$ and $J \smallsetminus j_{\pi(\bar\alpha)}$. Then we have

$$\epsilon(\pi) = \epsilon(\pi')\,(-1)^{\mathrm{pos}(i_{\bar\alpha},I)+\mathrm{pos}(j_{\pi(\bar\alpha)},J)} \tag{9.85}$$

and the sum over $i_{\bar\alpha}$ and π is equivalent to a double sum, over i and j, times a sum over π' (which is on a smaller symmetric group than π), so that ③ is equivalent to

$$③ : \quad \sum_{i\in I}\sum_{j\in J}(-1)^{\mathrm{pos}(i,I)+\mathrm{pos}(j,J)}\; W(I \smallsetminus i, J \smallsetminus j, B; \{i, a, j\})\,.$$

Collecting all the three contributions we get

$$\begin{aligned} W(I, J, B) = {} & W(I, J, B \cup a) + \sum_{b\in B} W(I, J, B \smallsetminus b; \{a, b\}) \\ & + \sum_{i\in I}\sum_{j\in J}(-1)^{\mathrm{pos}(i,I)+\mathrm{pos}(j,J)}\; W(I \smallsetminus i, J \smallsetminus j, B; \{i, a, j\})\,. \end{aligned} \tag{9.86}$$

Now we can apply this formula to all the terms on the LHS of (9.81). Again, we will need to handle three terms

$$\begin{aligned} & \sum_{i_0\in I_0} (-1)^{\mathrm{pos}(i_0,I_0)} W(I_0 \smallsetminus i_0, J, B) \\ & = \sum_{i_0\in I_0} (-1)^{\mathrm{pos}(i_0,I_0)} W(I_0 \smallsetminus i_0, J, B \cup i_0) && ① \\ & \quad + \sum_{i_0\in I_0}\sum_{b\in B}(-1)^{\mathrm{pos}(i_0,I_0)}\; W(I_0 \smallsetminus i_0, J, B \smallsetminus b; \{i_0, b\}) && ② \\ & \quad + \sum_{i_0\in I_0}\sum_{i_1\in I_0\smallsetminus i_0}\sum_{j\in J}(-1)^{\mathrm{pos}(i_0,I_0)+\mathrm{pos}(i_1,I_0\smallsetminus i_0)+\mathrm{pos}(j,J)} \\ & \qquad W(I_0 \smallsetminus \{i_0, i_1\}, J \smallsetminus j, B; \{i_0, i_1, j\}) && ③ \end{aligned} \tag{9.87}$$

First realize how the terms in ③ simplify pairwise, because of antisymmetry in $i_0 \leftrightarrow i_1$ (indeed, if $i_0 > i_1$ then $\mathrm{pos}(i_1, I_0 \smallsetminus i_0) = \mathrm{pos}(i_1, I_0)$, otherwise $\mathrm{pos}(i_1, I_0 \smallsetminus i_0) = \mathrm{pos}(i_1, I_0) - 1$).

Then, looking now at the RHS of (9.81) we realize that ① simplifies with the first family of terms. So we are left with proving that the contribution ② equals the second family of terms of the RHS. This actually happens term by term in the two sums over b, i.e.

$$\sum_{i_0\in I_0} (-1)^{\mathrm{pos}(i_0,I_0)}\; W(I_0 \smallsetminus i_0, J, B \smallsetminus b; \{i_0, b\}) = (-1)^h W(I_0, J \cup b, B \smallsetminus b) \tag{9.88}$$

The reason for this is again due to a formula like (9.85), where now π is the permutation in $\mathcal{S}_h$ on the right, and π' is its restriction to $I_0 \setminus i_0$ and $J \equiv (J \cup b) \setminus b$ on the left. The (9.85) reads in the new labels

$$\epsilon(\pi) = \epsilon(\pi')\,(-1)^{\mathrm{pos}(i_0, I_0) + \mathrm{pos}(b, J \cup b)} \tag{9.89}$$

Factors $(-1)^{\mathrm{pos}(i_0, I_0)}$ cancel on the two sides, while one recognize that, within our labeling of I_0, J and B, $\mathrm{pos}(b \in J \cup b) = |J| + 1 = h$ always.

This completes the proof of all the Ward Identites (9.69), when the involved expectation values for the fields in the theory are restated in terms of probabilities of connectivity events in the spanning-forest formulation. □

10

The ideal of relations in Forest Algebra

We have seen in Chapter 8 how the set $\{f_{\mathcal{C}}\}_{\mathcal{C}\in\Pi(V)}$, i.e. for $\mathcal{C}$ partitions of the vertex set, is a basis for the linear space generated by the polynomial algebra of the f_{ij}'s ($= 1-\mathbf{n}_i\cdot\mathbf{n}_j$).

Here we show how linear reduction from $\{f_{\mathcal{C}}\}_{\mathcal{C}\in\Pi(n)}$ to $\{f_{\mathcal{C}}\}_{\mathcal{C}\in NC(n)}$, the *non-crossing partitions*, is achieved via a single 4-point relation $R_{abcd}=0$, which involves 8 summands (4 with positive sign, 4 negative), and 8 fermionic variables, $\{\bar\psi,\psi\}_{a,b,c,d}$. This pattern will be a *leitmotiv* in the following.

Consider also the operator in Clifford Algebra $p_a=\int \mathrm{d}\psi_a\,\mathrm{d}\bar\psi_a\,\exp(\lambda\bar\psi_a\psi_a)=\partial_a\overline{\partial}_a(1+\lambda\bar\psi_a\psi_a)$, corresponding to one-site integration with the appropriate OSP(1|2)-invariant measure. The sets $\{p_i, f_{i,i+1}\}$ form an (even/odd) Temperley-Lieb Algebra, with the identification $p_i\equiv e_{2i}$ and $f_{i,i+1}\equiv e_{2i+1}$. It is "even/odd" in the sense that $e_k^2=\lambda_{\mathrm{parity}(k)}e_k$, with $\lambda_{\mathrm{even}}=\lambda$ and $\lambda_{\mathrm{odd}}=0$. This algebra allows to write the Transfer Matrix for spanning forests on a planar lattice, as acting on a basis of planar connectivity patterns, naturally identified with Link Patterns. The relation $R_{abcd}=0$ is "equivalent" to a 3-point relation $R^b_{ac}=0$, where also p_b is involved. It still involves 8 summands (4 positive, 4 negative) and 8 fermions, $\{\bar\psi,\psi\}_{a,c}$ and $\{\bar\psi,\psi,\overline{\partial},\partial\}_b$. Here equivalence between two statements is intended, at a heuristic level, by the fact that reductions from one relation to the other are achieved by more elementary facts than the truthness of (any of) the two statements, and at a more abstract algebraic level.

An operator B_{ab} is further introduced. Its action is that of swapping a and b subscripts in the Clifford-algebra operators, namely $B_{ab}\psi_a=\psi_bB_{ab}$, $B_{ab}\partial_a=\partial_bB_{ab}$, and similarly with $a\leftrightarrow b$ and with charge-conjugate operators. It is similar but different from the traditional braid operator b_i of Temperley-Lieb Algebra. In a sense, it is a "thickened" braid operator (swapping two pairs of points, instead that two points, in Link Pattern representation). It is fermionically represented, e.g., as

$$B_{ab}=\exp\big(-i\tfrac{\pi}{2}[(\psi_a-\psi_b)(\partial_a-\partial_b)+\mathrm{c.c.}]\big)\,.$$

The 3-point relation $R^b_{ac}=0$ is equivalent to a 2-point relation $R^{ab}=0$, involving f_{ab}, p_a, p_b and B_{ab}. It still involves 8 summands (4 positive, 4 negative) and 8 fermions, $\{\bar\psi,\psi,\overline{\partial},\partial\}_{a,b}$.

Relations R^{ab}, R^b_{ac} or R_{abcd} are ultimately the algebraic tool which makes Temperley-Lieb Algebra (extended to include also f_{ab} and B_{ab} for non-consecutive indices) able to describe the transfer matrix for problems of spanning forests on non-planar lattices. Such an algebraic structure appears to be new, or at least it has never been investigated up to its full range of possibilities (neither this is done here), although it is partially related to the "Join-and-detach" algebra in Salas and Sokal [145].

We expect that this algebra may play a role in the description of the planar Potts model, in the $\rho = 0$ and $\lambda \neq 0$ regime of spanning forests (and its dual counterpart of connected spanning subgraphs), analogously to how Temperley-Lieb algebra is central to the understanding of critical planar Potts model, for $\lambda = \rho = \sqrt{q}$.

10.1 Scalar products on $RP^{1|2}$ and $R_{abcd} = 0$

We recall briefly how the algebra of f_A's arised from scalar products of OSP(1|2)-invariant unit vectors. Consider vectors on the $RP^{1|2}$ supersphere:

$$\mathbf{n} = (\sigma; \bar{\psi}, \psi)\,; \qquad \|\mathbf{n}\|^2 = 1\,; \qquad g_{\alpha\beta} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & \lambda \\ 0 & -\lambda & 0 \end{array}\right)\,. \tag{10.1}$$

Solving the bosonic component σ w.r.t. the fermionic ones gives $\sigma = \sqrt{1 - 2\lambda\bar{\psi}\psi} = 1 - \lambda\bar{\psi}\psi$ if we neglect the second determination of the square root (or if the action is even w.r.t. the fields in each site i). The measure over the sphere reduces to a measure over only fermionic components, with a Jacobian in the form of a mass term

$$\int_{RP^{1|2}} \mathrm{d}\mathbf{n} \qquad \longrightarrow \qquad \int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\; e^{\lambda\bar{\psi}\psi} \tag{10.2}$$

Then, the scalar product among two such vectors on the $RP^{1|2}$ supersphere, shifted by its component of zero degree in Grassmann Algebra, is given by

$$f_{ij}^{(\lambda)} = \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{\lambda} = (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) - \lambda\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\,, \tag{10.3}$$

an expression which is well defined also in the limit $\lambda \to 0$, where it becomes purely quadratic, and factorizes into a product of binomials.

Given n vectors $\{\mathbf{n}_i\}_{i\in V}$ on the $RP^{1|2}$ supersphere, with $|V| = n$, we are interested in the polynomial algebra generated by the set of possible scalar products, $\{\mathbf{n}_i \cdot \mathbf{n}_j\}_{i,j\in V}$, as it describes the set of all the functions which are invariant under global OSP(1|2) rotations.

This algebra is clearly equivalent to the one generated by the linear combinations $\{f_{ij}^{(\lambda)}\}_{i,j\in V}$, with $i \neq j$. Thanks to the algebraic Lemma 8.1, and the observation after Corollary 8.4, we know that this coincides with the linear span of the expressions $f_{\mathcal{C}}^{(\lambda)}$, with $\mathcal{C}$ a partition of $[n]$ into disjoint sets $A_1, \ldots, A_k$, called *blocks* (we call $\Pi(n)$ the set of partitions of n objects). These expressions were defined as

$$f_{\mathcal{C}}^{(\lambda)} = f_{A_1}^{(\lambda)} \cdots f_{A_k}^{(\lambda)}\,; \tag{10.4}$$

$$f_A^{(\lambda)} = \Big[\lambda(1 - |A|) + \Big(\sum_{v\in A} \partial_v\Big)\Big(\sum_{v\in A} \bar{\partial}_v\Big)\Big] \prod_{v\in A} \bar{\psi}_v\psi_v\,, \tag{10.5}$$

and it is understood that $f_{\{i,j\}}^{(\lambda)} \equiv f_{ij}^{(\lambda)}$.

The basis of the $\{f_{\mathcal{C}}^{(\lambda)}\}_{\mathcal{C}\in\Pi(n)}$ could be in principle redundant, i.e. there could be linear relations among them. This is indeed what happens for $n \geq 4$. Such a phenomenon is evident from a simple counting argument: the number of partitions is called *Bell number*,

B_n, and its leading order behaviour is $B_n \sim n!$ [†], while the charge-neutral subalgebra of the Grassmann algebra over n complex fields, i.e. the algebra where the only allowed monomials have as many $\bar\psi$'s as ψ's, has dimension $\binom{2n}{n}$, which has leading behaviour only $\sim 4^n$ (the whole Grassmann Algebra has dimension exactly 4^n).

Indeed, a single first linear relation appears at $n=4$, and, for $V=\{a,b,c,d\}$, reads

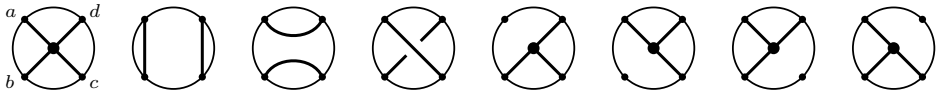
$$R_{abcd} := \lambda f_{abcd} + (f_{ab}f_{cd} + f_{ac}f_{bd} + f_{ad}f_{bc}) - (f_{abc} + f_{abd} + f_{acd} + f_{bcd}) = 0\,. \quad (10.6)$$

Here we used a compact notation, e.g. f_{abc} in place of $f^{(\lambda)}_{\{a,b,c\}}$, and dropped the λ superscript, while it is understood in the whole section that λ is considered as a generic commuting indeterminate[‡].

Recalling that $f_\varnothing = \lambda$ and $f_i = 1$ for all atomic sets i, the relation above can be read

$$R_{abcd} = f_\varnothing f_{abcd} + (f_{ab}f_{cd} + f_{ac}f_{bd} + f_{ad}f_{bc}) - (f_{abc}f_d + f_{abd}f_c + f_{acd}f_b + f_{bcd}f_a)\,; \quad (10.7)$$

where it now appears as a linear combinations of all possible partitions (even singular) of $\{a,b,c,d\}$ into two sets. This formulation is more suitable for extension to OSP$(1|2n)$ symmetry, cfr. Chapter 11. The relation can also be graphically represented by denoting each summand with the connection patterns it describes:



λf_{abcd} $\quad f_{ab}f_{cd}$ $\quad f_{ad}f_{bc}$ $\quad f_{ac}f_{bd}$ $\quad -f_{bcd}$ $\quad -f_{acd}$ $\quad -f_{abd}$ $\quad -f_{abc}$

The proof of this relation is achieved via explicit expansion of the various terms in Grassmann monomials, and shortened by a clever use of the evident permutation symmetry. It is postponed to section 10.4.1.

We will further prove that this relation is the only independent one, by proving the following striking statement: *for any ordering of the n points of V, a basis of linearly independent objects in $\{f^{(\lambda)}_{\mathcal{C}}\}_{\mathcal{C}\in\Pi(n)}$ is the set $\{f^{(\lambda)}_{\mathcal{C}}\}_{\mathcal{C}\in NC(n)}$, where $NC(n)$ is the set of non-crossing partitions. For any partition $\mathcal{C}\in\Pi(n)\setminus NC(n)$, one can reduce $f_{\mathcal{C}}$ to a linear combination in the basis by iterated application of $R_{abcd}=0$* (cfr. the statement of Theorem 10.3). This is done in Section 10.5.

10.2 Even/odd Temperley-Lieb Algebra and $R^b_{ac}=0$

We have seen in (10.6) a relation of the form $R_{abcd}=0$, "living" on a 4-point complex Grassmann algebra, but still we lack a deeper interpretation for it, besides the bare proof by algebraic check of the explicit fermionic expressions, of section 10.4.1.

Here we prove that this relation is equivalent to a "three-point" relation, involving also the derivative operators ∂_i and $\overline{\partial}_i$.

Consider the combination

$$p^{(\lambda)}_i := \partial_i\overline{\partial}_i(1+\lambda\bar\psi_i\psi_i) = \partial_i\overline{\partial}_i + \lambda(1-\psi_i\partial_i)(1-\bar\psi_i\overline{\partial}_i) = \int \mathrm{d}\psi_i \mathrm{d}\bar\psi_i\, e^{\lambda\bar\psi_i\psi_i} \quad (10.8)$$

[†] In the sense that $\ln B_n = n\ln n + o(n\ln n)$.

[‡] I.e. we always work with expressions which are polynomial both in Grassmann fields *and* in λ: we never "divide by λ" (unless in a polynomial having an overall factor λ), or "take $\sqrt{\lambda}$" or other things of this sort.

which, in the right-most formulation, is at sight OSP(1|2) symmetric, and satisfies the following algebraic relations (here we drop again $\cdot^{(\lambda)}$ superscripts)

$$p_i^2 = \lambda p_i\,; \qquad [p_i, p_j] = 0\,; \tag{10.9a}$$

$$[p_i, f_A] = 0 \qquad \text{if } i \notin A; \qquad \big(p_i f_A\big) = f_{A\smallsetminus i} \qquad \text{if } i \in A. \tag{10.9b}$$

The expression $\big(p_i f_A\big)$ with parenthesis stands for the fact that the derivatives in p_i acts only on the fermions explicitly appearing in the expression at its right, in the same sense as $\big(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\big) g(x)$ is different from $\frac{\mathrm{d}}{\mathrm{d}x} f(x) g(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x) g(x)\big)$. Note however that, as p_i is not linear in derivative operators (but quadratic), there is no easy Leibniz rule, i.e. $p_i f_A \Phi \neq \big(p_i f_A\big)\Phi + f_A p_i \Phi$. Combinations like $[p_i, f_A]$ for $i \in A$ are more complicated, and are briefly discussed in Section 10.2.1.

As a consequence of the relations above, and jointly with the relations involving only the f's

$$[f_A, f_B] = 0\,; \qquad f_A f_B = \begin{cases} f_{A\cup B} & \text{if } |A \cap B| = 1; \\ 0 & \text{if } |A \cap B| \geq 2; \end{cases} \tag{10.10}$$

(the leftmost one is obvious, the rightmost one is the content of Lemma 8.1), we get the following useful relations

$$p_i\, f_{i\cup A}\, p_i = f_A\, p_i\,; \tag{10.11}$$

$$f_{i\cup A}\, p_i\, f_{i\cup B} = \begin{cases} f_{i\cup A\cup B} & |A \cap B| = 1\,; \\ 0 & |A \cap B| \geq 2\,; \end{cases} \tag{10.12}$$

where it is understood that $i \notin A, B$.

At this point it is intersting to concentrate our attention first to a special subset of the relations stated above. Choose a (linear or cyclic) total ordering of the indices, and restrict the f_A's to the set of $\{f_{i\,i+1}\}$: the pertinent relations are

$$p_i^2 = \lambda p_i\,; \qquad f_{i\,i+1}^2 = 0\,; \tag{10.13a}$$

$$[p_i, p_j] = [f_{i\,i+1}, f_{j\,j+1}] = 0\,; \qquad [p_i, f_{j\,j+1}] = 0 \qquad \text{if } i \neq j, j+1; \tag{10.13b}$$

$$p_i\, f_{i\,i\pm1}\, p_i = p_i\,; \qquad f_{i\,i\pm1}\, p_i\, f_{i\,i\pm1} = f_{i\,i\pm1}\,. \tag{10.13c}$$

The expert reader will recognize a structure which closely resembles the one of a Temperley-Lieb Algebra. At the aim of make this similarity even clearer, we make the identification $p_i \to e_{2i}$ and $f_{i\,i+1} \to e_{2i+1}$ and restate the equations above as

$$e_i^2 = \begin{cases} \lambda e_i & i \text{ even}; \\ 0 & i \text{ odd}; \end{cases} \tag{10.14}$$

$$[e_i, e_j] = 0 \qquad \text{if } i \neq j-1, j+1; \tag{10.15}$$

$$e_i e_{i\pm1} e_i = e_i\,; \tag{10.16}$$

that is, exactly a Temperley-Lieb Algebra, except for the fact that it is a kind of "even-odd" generalization

$$e_i^2 = \lambda e_i \qquad \longrightarrow \qquad e_{2i}^2 = \lambda_{\text{even}} e_{2i}\,; \qquad e_{2i+1}^2 = \lambda_{\text{odd}} e_{2i+1}\,;$$

with $\lambda_{\text{even}} = \lambda$ and $\lambda_{\text{odd}} = 0$. It is easily seen that this is *the only* consistent generalization of the form $e_i^2 = \lambda_i e_i$. Indeed we have the consistency requirement

$$\begin{aligned} e_{i-1}\, e_{i+1}\, e_i\, e_{i-1}\, e_{i+1} = e_{i+1}\, e_{i-1}\, e_i\, e_{i-1}\, e_{i+1} = e_{i+1}\, e_{i-1}\, e_{i+1} = \lambda_{i+1}\, e_{i-1}\, e_{i+1} \\ = e_{i-1}\, e_{i+1}\, e_i\, e_{i+1}\, e_{i-1} = e_{i-1}\, e_{i+1}\, e_{i-1} = \lambda_{i-1}\, e_{i-1}\, e_{i+1} \end{aligned} \tag{10.17}$$

which forces $\lambda_{i+1} = \lambda_{i-1}$ for each i. Furthermore, as we have that equations $e_i e_{i\pm1} e_i = e_i$ are invariant under $e_{2i} \to a\, e_{2i}$ and $e_{2i+1} \to a^{-1}\, e_{2i+1}$, for a nonzero, while $\lambda_{\rm even} \to a\lambda_{\rm even}$ and $\lambda_{\rm odd} \to a^{-1}\lambda_{\rm odd}$ under this transformation, we see that our case of $\lambda_{\rm even} \neq 0$ and $\lambda_{\rm odd} = 0$ (or vice versa) is the only one not implicitly contained in the λ-homogeneous analysis.

This is not a surprise. The Temperley-Lieb algebra at $\lambda = \sqrt{q}$ describes the Transfer Matrix formalism of the (FK representation of the) Potts Model with q colours, on a strip of the square lattice. Euler formula is crucially used for $q \neq 0$, in order to write $q^{K(S)} \to \sqrt{q}^{K(S)+L(S)}$ up to a rescaling of the weights and a factor overall, and even/odd bubbles e^2_{2i}, e^2_{2i+1} correspond to an unit increase of K or of L, when the contours described by the e's are interpreted as surrounding the cluster components.

The special limit corresponding to spanning forests, instead, involves a precise $q \to 0$ prescription, such that loops are forbidden, $L(S) = 0$, while connected components $K(S)$ are counted with a finite factor λ.

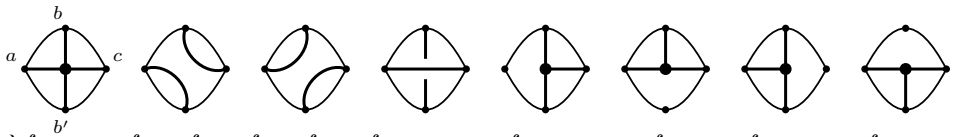
We recall that, beyond the general interest on the generating function for spanning forests, the case of the regular 2-dimensional system is of special importance, as it leads to an asymptotically-free renormalizable system of statistical mechanics.

At the aim of identifying this (generalized) Temperley-Lieb algebra structure, we only used the properties of Lemma 8.1 and the analogous ones for p_i. We never used the basic relation (10.6) above.

Here we state, and prove later on in this Chapter, the following relation:

$$R^b_{ac} = \lambda f_{abc} + f_{ac} + f_{bc} p_b f_{ab} + f_{ab} p_b f_{bc} - \{p_b, f_{abc}\} - f_{ab} - f_{bc} = 0\,, \tag{10.18}$$

whose single diagrams are graphically represented as follows (b and b' represent site label b respectively before and after the application of the "transfer matrix", in a formalism in which operators $f_{i\,i+1}$ and p_i, which do not commute, are applied in a precise sequence)



λf_{abc} $\quad f_{ab}p_b f_{bc}$ $\quad f_{bc}p_b f_{ab}$ $\quad f_{ac}$ $\quad -f_{bc}$ $\quad -p_b f_{abc}$ $\quad -f_{ab}$ $\quad -f_{abc}p_b$

We will also show that $R_{abcd} = 0$ and $R^b_{ac} = 0$ are "equivalent statements", in the sense described above.

The relation (10.18) is furthermore rephrased into the statement that there exists a (unique) way of expressing f_{ac} in terms of f_{ab}, f_{bc} and p_b only:

$$f_{ac} = -\lambda f_{ab} f_{bc} + p_b f_{ab} f_{bc} + f_{ab} f_{bc} p_b + f_{ab} + f_{bc} - f_{bc} p_b f_{ab} - f_{ab} p_b f_{bc}\,. \tag{10.19}$$

Remark that, if a, b and c are consecutive w.r.t. the ordering, the relation above allows to recursively write $f_{i\,i+k}$ for arbitrary $k \geq 2$ in terms of $\{p_j, f_{j\,j+1}\}$, i.e. in terms of operators already in T-L Algebra.

Proof of (10.18)→(10.6): Multiply the right side of equation (10.18) on the right by f_{bd}, and apply the properties (10.9) of p_b, in order to get the relation (10.6).

The direct proof of (10.18) is done in section 10.4.3.

10.2.1 Some other commutators

Here we give some other remarkably simple commutators for elements in the algebra generated by p_i's and f_{ij}'s, besides the ones already implied by relations (10.9).

Indeed, relation (10.19) can be rewritten as

$$[[p_b, f_{ab}], f_{bc}] = -f_{ab} - f_{bc} + f_{ac} + \lambda f_{abc} \tag{10.20}$$

while more directly, for $a = c$, only using T-L:

$$[[p_b, f_{ab}], f_{ab}] = p_b f_{ab}^2 - 2 f_{ab} p_b f_{ab} + f_{ab}^2 p_b = -2 f_{ab} \tag{10.21}$$

and these are the only nontrivial cases involving a single operator p. Furthermore we have

$$\begin{aligned} [[p_a, f_{ab}], [p_b, f_{ab}]] &= \big(p_a f_{ab} p_b f_{ab} - f_{ab} p_a p_b f_{ab} - p_a f_{ab}^2 p_b + f_{ab} p_a f_{ab} p_b\big) + \big(a \leftrightarrow c\big) \\ &= p_a f_{ab} p_b f_{ab} + f_{ab} p_a f_{ab} p_b - p_b f_{ab} p_a f_{ab} - f_{ab} p_b f_{ab} p_a \\ &= [p_a, f_{ab}] - [p_b, f_{ab}] \end{aligned} \tag{10.22}$$

again by mean only of T-L.

The expression for $[[p_a, f_{ab}], p_b]$, the only non-trivial one remaining which involves two p_i operators, is not specially simple. With the definition of B_{ab} of the following section, and a relation (10.54) that we will prove, we can write a somewhat more condensed expression

$$[[p_a, f_{ab}], p_b] = (p_a + p_b) B_{ab} - \{p_a p_b, f_{ab}\} \,. \tag{10.23}$$

10.3 Braid generators and $R^{ab} = 0$

In Section 10.1 we introduced the relation (10.6), $R_{abcd} = 0$ (without proof), with supports on 4 points, which involves 8 summands (4 with positive sign, 4 negative), and 8 fermionic variables, $\{\bar\psi, \psi\}_{a,b,c,d}$. The relation can be interpreted as disentangling crossing partitions into linear combinations of non-crossing terms.

In Section 10.2 we stated that this relation is equivalent to a 3-point relation $R^b_{ac} = 0$, where also derivatives on site b are involved. It still includes 8 summands (4 positive, 4 negative) and 8 fermions, $\{\bar\psi, \psi\}_{a,c}$ and $\{\bar\psi, \psi, \overline{\partial}, \partial\}_b$. The operator in Clifford Algebra p_b acting on site b has an interpretation as completing, together with terms $f_{i\,i+1}$, an (extension of) Temperley-Lieb Algebra, while the relation is interpreted as disentangling f_{ab} for non-consecutive (a, b) into a polynomial in p_i's and $f_{i\,i+1}$'s.

Given this structure, one is then induced to guess that both the 4-point relation $R_{abcd} = 0$ and the 3-point relation $R^b_{ac} = 0$ are equivalent to some 2-point relation $R^{ab} = 0$, still involving 8 summands (4 positive, 4 negative) and 8 fermions, $\{\bar\psi, \psi, \overline{\partial}, \partial\}_{a,b}$. It turns out that this is the case, but we need to define a further operator, with a simple operatorial behaviour both as acting on connectivity patterns, and within the Clifford Algebra.

The easiest way of describing this operator, that we call B_{ab}, is indeed through its action on polynomials in Clifford Algebra:

$$B_{ab} \Phi(\bar\psi_a, \psi_a, \bar\psi_b, \psi_b, \overline{\partial}_a, \partial_a, \overline{\partial}_b, \partial_b, \ldots) = \Phi(\bar\psi_b, \psi_b, \bar\psi_a, \psi_a, \overline{\partial}_b, \partial_b, \overline{\partial}_a, \partial_a, \ldots) B_{ab} \,, \tag{10.24}$$

i.e. it exchanges labels a and b in fermions, both "variables" and "derivatives". It is maybe surprising that it can be implemented *within* the Clifford Algebra, and in a simple way.

For example, we have, in "polynomial" and "exponential" form, (as always, c.c. stands for charge conjugation)

$$\begin{aligned} B_{ab} &= \big(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\big) \cdot \big(\text{c.c.}\big) \\ &= \exp\Big[- \frac{i\pi}{2}\Big((\psi_a - \psi_b)(\partial_a - \partial_b) + \text{c.c.}\Big)\Big] . \end{aligned} \tag{10.25}$$

The proof is by direct investigation of the action over a single fermion, and is simplified by the property of factorization of "holomorphic" and "anti-holomorphic" parts (the two factors are separately commuting), and the symmetry of these factors under $a \leftrightarrow b$:

$$\begin{aligned} &\big(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\big)\, \psi_a = \psi_a + (\psi_a - \psi_b)(\psi_a\partial_a - \psi_a\partial_b - 1) = \psi_b + \psi_a\psi_b(\partial_a - \partial_b), \\ &\psi_b \big(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\big) = \psi_b + \psi_a\psi_b(\partial_a - \partial_b)\,, \\ &\big(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\big)\, \partial_a = \partial_a + (\psi_a - \psi_b)\partial_b\partial_a\,, \\ &\partial_b \big(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\big) = \partial_b + (\psi_a\partial_b - \psi_b\partial_b + 1)(\partial_a - \partial_b) = \partial_a + (\psi_a - \psi_b)\partial_b\partial_a\,. \end{aligned}$$

Already at the abstract level of "swap" operator, for the operators B_{ab}, f_{ab} and p_a we must have

$$B_{ab}^2 = 1\,; \qquad B_{ab}B_{bc} = B_{ac}B_{ab} = B_{bc}B_{ac}\,; \tag{10.26}$$

$$B_{ab}p_a = p_bB_{ab} \qquad B_{ab}B_{bc}B_{ab} = B_{ac} = B_{bc}B_{ab}B_{bc}\,; \tag{10.27}$$

$$[B_{ab}, f_{ab}] = 0\,; \qquad B_{ab}f_{ac} = f_{bc}B_{ab}\,. \tag{10.28}$$

A further useful statement is

$$B_{ab}f_{ab}\Phi(\psi, \bar\psi) = f_{ab}\Phi(\psi, \bar\psi)\,, \tag{10.29}$$

which is valid for any Φ in Grassmann (but not in Clifford) Algebra, because f_{ab} symmetrizes the dependence on variables with index a and b when acting on an element in Grassmann algebra. This claim is checked easily on the few possible terms:

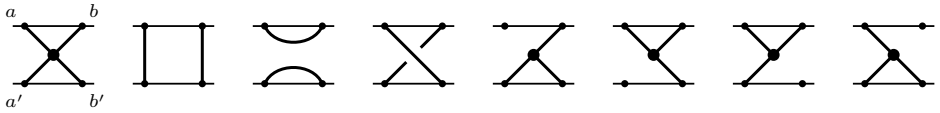
$$f_{ab}1 = f_{ab}\,; \qquad f_{ab}\psi_a = (\bar\psi_a - \bar\psi_b)\psi_a\psi_b\,; \tag{10.30}$$

$$f_{ab}\bar\psi_a\psi_a = -f_{ab}\bar\psi_a\psi_b = \tau_{ab}\,; \qquad f_{ab}\psi_a\psi_b = 0\,. \tag{10.31}$$

Given the definition of B_{ab}, we are now ready to state our aforementioned two-point relation:

$$R^{ab} = 1 + B_{ab} + \lambda f_{ab} + f_{ab}p_ap_bf_{ab} - \{p_a + p_b, f_{ab}\} = 0\,. \tag{10.32}$$

Again, the terms in the relation have a useful diagrammatic represention (indices without and with primes denote site labels before and after the action of the operators, e.g. in a "Transfer Matrix")



λf_{ab} $\quad 1 \quad$ $f_{ab}p_ap_bf_{ab}$ $\quad B_{ab} \quad$ $-f_{ab}p_a$ $\quad -p_af_{ab}$ $\quad -p_bf_{ab}$ $\quad -f_{ab}p_b$

The relation (10.32) above can be interpreted as expressing the swapping operator B_{ab} (in the unique possible way) as a polynomial in f_{ab}, p_a and p_b:

$$B_{ab} = \{p_a + p_b, f_{ab}\} - 1 - \lambda f_{ab} - f_{ab}p_ap_bf_{ab}\,. \tag{10.33}$$

This is stating in a different way a property already mentioned at the end of Section 10.2, concerning equation (10.19): we had there that the "planar" (even/odd) Temperley-Lieb Algebra was allowing to deal also with non-planar graphs, because we could restate links between non-adjacent sites though $f_{i\,i+k}$ for $k > 1$, and then use (10.19) iteratively for reducing to a polynomial in the "planar" subalgebra. Here, similarly, as we can represent $B_{i\,i+1}$ in the "planar" subalgebra, we could restate links between non-adjacent sites by swapping the site indices up to making them adjacent, apply $f_{i\,i+1}$, and then swapping them again in the original positions.

This is yet another indication that equations like (10.32) and (10.18) are "equivalent" in some deep sense.

10.4 Proof of the basic relations

We start by giving a very simple lemma, which is used many times in the following and is also of separate interest:

Lemma 10.1. *Let A a set of indices, i an index in A, and j an index not in A. Consider the Clifford Algebra over the set $A \cup \{j\}$. Let Φ_A be a polynomial in the Clifford Algebra over the set A. If*

$$\Phi_A f_{ij} = 0 \tag{10.34}$$

then also

$$\Phi_A = 0\,. \tag{10.35}$$

Proof: the polynomial $\Phi_A f_{ij}$ is Grassmann in variable j, so it can be expanded as $P_0 + \bar{\psi}_j P + \bar{P}\psi_j + P_1 \bar{\psi}_j \psi_j$. From the explicit definition of f_{ij}, we have, in particular, $P_0 = \Phi_A \bar{\psi}_i \psi_i$ and $P_1 = \Phi_A(1 - \lambda \bar{\psi}_i \psi_i)$. By hypothesis we have $P_0 = P_1 = 0$, so also $\Phi_A = P_1 + \lambda P_0 = 0$. □

10.4.1 Algebraic proof of $R_{abcd} = 0$

We prove here that $R_{abcd} = 0$ as a polynomial in Grassmann Algebra. This is a considerably easier task to perform, w.r.t. the other forms of the basic relation, $R^b_{ac} = 0$ and $R^{ab} = 0$, for which the task has to be performed in Clifford Algebra, an effort overwhelming the simplification coming from the fact that less points are involved.

As we anticipated, we can exploit the symmetry of the expression R_{abcd} under permutations of its four arguments, and avoid to build a full matrix of $M_{m,\mathcal{C}}$ (the coefficients by which the monomial m appears in $f_{\mathcal{C}}$) We restrict the set of possible monomials to one monomial per orbit under the permutations in $\mathcal{S}_4$ acting on $\{a, b, c, d\}$. The relation is proven if we check that the sum along each row is zero. The resulting table, showing this claimed property, is

	λf_{abcd}	$f_{ab}f_{cd}$	$f_{ac}f_{bd}$	$f_{ad}f_{bc}$	$-f_{abc}$	$-f_{abd}$	$-f_{acd}$	$-f_{bcd}$
$\lambda^2\tau_{abcd}$	-3	$+1$	$+1$	$+1$	0	0	0	0
$\lambda\tau_{abc}$	$+1$	-1	-1	-1	$+2$	0	0	0
$\lambda\bar{\psi}_a\psi_b\tau_{cd}$	-1	$+1$	0	0	0	0	0	0
τ_{ab}	0	0	$+1$	$+1$	-1	-1	0	0
$\bar{\psi}_a\psi_b\tau_c$	0	-1	0	0	$+1$	0	0	0
$\bar{\psi}_a\psi_b\bar{\psi}_c\psi_d$	0	$+1$	0	-1	0	0	0	0

A few remarks are in order. First, check that we enumerated all relevant monomials: they must be charge-neutral, and of degree between 4 and 8 (as a partition $\mathcal{C}$ with k sets has monomials of degree between $2n-2k$ and $2n$, and here all partitions involve up to 2 sets). Then, the homogeneity (in $\deg_\psi - \deg_\lambda$) fixes the dependence from λ.

A more relevant line is the last one: indeed, in all other rows the cancellations were occurring among terms of different partition classes, the coefficients inside each class being equal or analogous. Up to chosing the proper coefficients for the f's, such a cancellation was relatively easy to achieve (although, however, non-trivially realized, as we have far more symmetry classes of monomials than symmetry classes of partitions in the table). On the last line, instead, the cancellation is among two partitions on the same class, and relies on the first consequence of anticommutation when restricted to the charge-neutral subalgebra, namely that $\bar\psi_a\psi_b\bar\psi_c\psi_d = -\bar\psi_a\psi_d\bar\psi_c\psi_b$. There would have been no bosonic analogue of this.

10.4.2 Proof of $R^{ab} = 0$ acting on Grassmann Algebra

We can prove that relation (10.32) $R^{ab} = 0$ holds as an operator acting in Grassmann Algebra by testing its action on any monomial in ψ_a, $\bar\psi_a$, ψ_b, $\bar\psi_b$, and exploiting the symmetry of R^{ab} both under exchange of a with b and charge conjugation in order to reduce the number of checks. Remark that this is proving a weaker fact w.r.t. the true property, of $R^{ab} = 0$ being zero as a polynomial in Clifford Algebra. It is however useful in order to have an intuition on the "coicidence" it implies, by counting the symmetry classes of monomials w.r.t. the symmetry classes of operators. A full proof through explicit check of the action on possible Clifford monomials would be possible, but quite more painful. Comparatively, we found that arguments as in Lemma 10.1 are much more effective for proving statements in Clifford Algebra. In Section 10.4.4 we obtain an alternate proof, through equivalence with the other forms of the basic relation, $R_{abcd} = 0$ and $R^b_{ac} = 0$.

So, we get the following table, and the proof is achieved by checking that, on each column, the sum of entries in the first 4 rows equals the sum on the remaining 4 rows.

	1	ψ_a	$\bar\psi_a\psi_a$	$\psi_a\psi_b$	$\bar\psi_a\psi_a\psi_b$	τ_{ab}	$\bar\psi_a\psi_b$
1	1	ψ_a	$\bar\psi_a\psi_a$	$\psi_a\psi_b$	$\bar\psi_a\psi_a\psi_b$	τ_{ab}	$\bar\psi_a\psi_b$
B_{ab}	1	ψ_b	$\bar\psi_b\psi_b$	$-\psi_a\psi_b$	$-\bar\psi_b\psi_a\psi_b$	τ_{ab}	$\bar\psi_b\psi_a$
λf_{ab}	λf_{ab}	$\lambda(\bar\psi_a-\bar\psi_b)\psi_a\psi_b$	$\lambda\tau_{ab}$	0	0	0	$\lambda\tau_{ab}$
$f_{ab}p_ap_bf_{ab}$	λf_{ab}	0	f_{ab}	0	0	0	f_{ab}
p_af_{ab}	1	ψ_b	$\bar\psi_b\psi_b$	0	0	0	$\bar\psi_b\psi_b$
p_bf_{ab}	1	ψ_a	$\bar\psi_a\psi_a$	0	0	0	$\bar\psi_a\psi_a$
$f_{ab}p_a$	λf_{ab}	0	f_{ab}	0	$(\bar\psi_a-\bar\psi_b)\psi_a\psi_b$	τ_{ab}	0
$f_{ab}p_b$	λf_{ab}	$\lambda(\bar\psi_a-\bar\psi_b)\psi_a\psi_b$	$\lambda\tau_{ab}$	0	0	τ_{ab}	0

10.4.3 Algebraic proof of $R^b_{ac} = 0$

Here we give a direct algebraic proof of the relation (10.19)

Theorem 10.1 *Given three distinct points a, b, c, the*

$$f_{ac} = -\lambda f_{ab}f_{bc} + p_bf_{ab}f_{bc} + f_{ab}f_{bc}p_b + f_{ab} + f_{bc} - f_{bc}p_bf_{ab} - f_{ab}p_bf_{bc} \tag{10.36}$$

holds in Clifford Algebra.

Proof: First we reorder a bit the terms, writing

$$R^b_{ac} = (f_{bc} - f_{ac})(1 - p_b f_{ab}) + f_{ab}(1 - p_b f_{bc}) + f_{abc}(p_b - \lambda) = 0\,. \tag{10.37}$$

Then we remark that in Clifford Algebra the rightmost factors in the summands above have an expansion in which all terms have at least one derivative

$$p_b - \lambda = -\lambda(\psi_b\partial_b + \bar\psi_b\overline{\partial}_b) + (1 + \lambda\bar\psi_b\psi_b)\partial_b\overline{\partial}_b\,; \tag{10.38}$$

$$\begin{aligned} 1 - p_b f_{ab} &= \big(1 + (\bar\psi_b - \bar\psi_a)\overline{\partial}_b\big)\big(1 + (\psi_b - \psi_a)\partial_b\big) - 1 \\ &= (\psi_b - \psi_a)\partial_b + (\bar\psi_b - \bar\psi_a)\overline{\partial}_b - (f_{ab} + \lambda\tau_{ab})\partial_b\overline{\partial}_b\,. \end{aligned} \tag{10.39}$$

So we have for R^b_{ac} an expression of the form $P\partial_b + \bar P\overline{\partial}_b + P_1\partial_b\overline{\partial}_b$, where P and $\bar P$ are expressions easily related. We thus just need to prove that $P = P_1 = 0$.

The term P_1 is the simplest: we just get

$$P_1 = (f_{abc} + \lambda\tau_{abc})(1 - 1 - 1 + 1) = 0\,. \tag{10.40}$$

The term P will make us work more. We get

$$P = (f_{bc} - f_{ac})(\psi_b - \psi_a) + f_{ab}(\psi_b - \psi_c) - \lambda f_{abc}\psi_b\,. \tag{10.41}$$

This expression is simplified through the fact, valid for $i \in A$:

$$f_A\psi_i = \Big[\lambda(1 - |A|)\tau_A + \Big(\sum_{j\in A}\partial_j \sum_{k\in A}\overline{\partial}_k\tau_A\Big)\Big]\psi_i = \Big(\partial_i \sum_{k\in A}\overline{\partial}_k\tau_A\Big)\psi_i = \Big(\sum_{k\in A}\overline{\partial}_k\tau_A\Big)\,. \tag{10.42}$$

This allows to write

$$P = -\big(f_{ab}\psi_c + \text{cyclic}\big) + \big((\overline{\partial}_a + \overline{\partial}_b)\tau_{ab} + \text{cyclic}\big) - \lambda(\overline{\partial}_a + \overline{\partial}_b + \overline{\partial}_c)\tau_{abc} \tag{10.43}$$

which, remarkably, is manifestly symmetric in a, b, c (the built-in symmetry was only in a, c). The expansion of the first term gives

$$\begin{aligned} f_{ab}\psi_c &= -\lambda\tau_{ab}\psi_c + (\tau_a + \tau_b)\psi_c - \bar\psi_a\psi_b\psi_c - \bar\psi_b\psi_a\psi_c \\ &= -\lambda\overline{\partial}_c\tau_{abc} + \overline{\partial}_c(\tau_{ac} + \tau_{bc}) - \bar\psi_a\psi_b\psi_c - \bar\psi_b\psi_a\psi_c\,. \end{aligned} \tag{10.44}$$

The terms explicitly involving $\overline{\partial}_c$ simplify with their homologous in (10.43), while the remaining ones simplify in the cyclic sum, as can be arranged as three summands of the form $\bar\psi_a\psi_b\psi_c + \bar\psi_a\psi_c\psi_b$. This completes the proof.

As a final remark, note that equation (10.36) makes sense also if $a = c$, where it is understood that, if $f_{ij} = -\lambda\bar\psi_i\psi_i\bar\psi_j\psi_j + (\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j)$, then $f_{ii} = -\lambda(\bar\psi_i\psi_i)^2 + (\bar\psi_i - \bar\psi_i)(\psi_i - \psi_i) = 0$. Indeed we have

$$f_{aa} = 0 = -\lambda f_{ab}^2 + p_b f_{ab}^2 + f_{ab}^2 p_b + 2f_{ab} - 2f_{ab}p_b f_{ab} = 2(f_{ab} - f_{ab}p_b f_{ab}) \tag{10.45}$$

and the relation becomes a restatement of the "Temperley-Lieb" relation $f_{ab} = f_{ab}p_b f_{ab}$.

10.4.4 Algebraic proof of $R_{abcd} = 0 \;\Rightarrow\; R^b_{ac} = 0 \;\Rightarrow\; R^{ab} = 0$

We now prove that relations (10.18) $R^b_{ac} = 0$ and (10.32) $R^{ab} = 0$ hold algebraically in Clifford Algebra, by deducing them, in sequence, from $R_{abcd} = 0$, which has been proven in Section 10.4.1.

In particular we will show that

$$R_{abcd}p_b f_{bd} = R^b_{ac} f_{bd}\,; \tag{10.46}$$

$$R^b_{ac}p_a f_{ac} = R^{ab} f_{ac}\,; \tag{10.47}$$

these equations implying our statements above by virtue of Lemma 10.1.

Equation (10.46) does not need many comments, as the equivalence is proven by only mean of the basic relations described in (10.13). The explicit calculation is however instructive, as it shows how the various terms in the two equation are matched one with the other, and no linear combinations are involved (as suggested by the pictorial representations of the relations shown in the previous sections).

The evaluation of $R^b_{ac}p_a f_{ac} = 0$ has similar features, for the exception of a single term, that has to be matched, in R^{ab}, with the newly-defined operator B_{ab}, so that we now follow the calculation in detail. We have

$$\begin{aligned} R^b_{ac}p_a f_{ac} = {} & f_{ac}p_a f_{ac} + \lambda f_{ab}f_{ac}p_a f_{ac} + f_{bc}p_b f_{ab}p_a f_{ac} + f_{ab}p_b f_{bc}p_a f_{ac} \\ & - p_b f_{ab}f_{ac}p_a f_{ac} - f_{ab}f_{ac}p_b p_a f_{ac} - f_{ab}p_a f_{ac} - f_{bc}p_a f_{ac} \end{aligned} \tag{10.48}$$

All of these terms except for the third one, that we treat separately, are either of the form of a factor depending only on sites a and b, multiplied by f_{ac} on the right (terms 7 and 8), or are reduced to this form by mean of the basic relations described in (10.13) (terms 1, 2, 4, 5 and 6), and give respectively

$$f_{ac}p_a f_{ac} = f_{ac}\,; \tag{10.49}$$

$$\lambda f_{ab}f_{ac}p_a f_{ac} = \lambda f_{ab}f_{ac}\,; \tag{10.50}$$

$$f_{ab}p_b f_{bc}p_a f_{ac} = f_{ab}p_b p_a f_{bc}f_{ac} = f_{ab}p_b p_a f_{ab}f_{ac}\,; \tag{10.51}$$

$$p_b f_{ab}f_{ac}p_a f_{ac} = p_b f_{ab}f_{ac}\,; \tag{10.52}$$

$$f_{ab}f_{ac}p_b p_a f_{ac} = f_{ab}p_b f_{ac}p_a f_{ac} = f_{ab}p_b f_{ac}\,. \tag{10.53}$$

The third summand should be treated by the following lemma:

Lemma 10.2. *The relations*

$$B_{ab}p_a = p_b f_{ab}p_a\,; \tag{10.54}$$

$$B_{ab}f_{ac} = f_{bc}p_b f_{ab}p_a f_{ac}\,; \tag{10.55}$$

hold in Clifford Algebra.

Indeed, if this lemma is proven, from (10.55) we end up verifying equation (10.47), and thus our main claim.

So now we prove Lemma 10.2. First we show that (10.54) implies (10.55), through the simple manipulation:

$$f_{bc}p_b f_{ab}p_a f_{ac} - B_{ab}f_{ac} = f_{bc}p_b f_{ab}p_a f_{ac} - B_{ab}f_{ac}p_a f_{ac} = f_{bc}\big(p_b f_{ab} - B_{ab}\big)p_a f_{ac}\,. \tag{10.56}$$

Then, recalling that $p_a = \partial_a\overline{\partial}_a(1+\lambda\bar{\psi}_a\psi_a)$, we will prove (10.54) by proving the stronger statement

$$\big(p_b f_{ab} - B_{ab}\big)\partial_a\overline{\partial}_a = 0 \tag{10.57}$$

(as polynomials in Clifford Algebra). Rewrite this expression as

$$\big(p_b f_{ab} - B_{ab}\big)\partial_a\overline{\partial}_a = -\big((1 - p_b f_{ab}) - (1 - B_{ab})\big)\partial_a\overline{\partial}_a \tag{10.58}$$

In the right side we can use both the definition (10.25) and the combination (10.39)

$$(1-p_b f_{ab})-(1-B_{ab}) = (\psi_b-\psi_a)\partial_a + (\bar\psi_b-\bar\psi_a)\overline{\partial}_a \\ -(\bar\psi_b-\bar\psi_a)(\psi_b-\psi_a)\big(\partial_b\overline{\partial}_b-(\partial_b-\partial_a)(\overline{\partial}_b-\overline{\partial}_a)\big)\,, \quad (10.59)$$

but now at sight in each term above there is at least a factor ∂_a or a factor $\overline{\partial}_a$, multiplied on the right, so that, in the combination of (10.58), the product must vanish. □

10.4.5 Algebraic proof of $R^{ab}=0 \;\Rightarrow\; R_{abcd}=0$

Here we prove that $R_{abcd}=0$ assuming that $R^{ab}=0$, thus, combining with the results in Section 10.4.4, "closing the cycle" of direct implications among the various formulations of the basic relation. The idea is similar to the one used in Section 10.4.4, however now we do not need the Lemma 10.1. Indeed, we just prove that

$$R^{bc} f_{ab} f_{cd} = R_{abcd} \quad (10.60)$$

as polynomials in Clifford Algebra, by only mean of the basic relations described in (10.13).

In fact, R^{bc} reads

$$R^{bc} = 1 + B_{bc} + \lambda f_{bc} + f_{bc} p_b p_c f_{bc} - \{p_b + p_c, f_{bc}\} = 0 \quad (10.61)$$

Multiplying this expression on the right by $f_{ab}f_{cd}$, and performing some simplifications through (10.13), we get

$$B_{bc} f_{ab} f_{cd} = f_{ac} f_{bd}\,; \quad (10.62)$$
$$\lambda f_{bc} f_{ab} f_{cd} = \lambda f_{abcd}\,; \quad (10.63)$$
$$f_{bc} p_b p_c f_{bc} f_{ab} f_{cd} = f_{bc} p_b p_c f_{abcd} = f_{bc} f_{ad}\,; \quad (10.64)$$
$$p_b f_{bc} f_{ab} f_{cd} = p_b f_{abcd} = f_{acd}\,; \quad (10.65)$$
$$f_{bc} p_b f_{ab} f_{cd} = f_{bc} f_{cd} = f_{bcd}\,. \quad (10.66)$$

All other terms are either trivial or analogous. At the end, the eight summands are exactly the ones in R_{abcd}.

Clearly because of the permutation symmetry of R_{abcd}, we would have obtained the same result e.g. from $R^{ab} f_{ac} f_{bd}$. We performed the choice above in order to preserve the natural ordering of the points, in the correspondence among the eight terms in the two expressions R_{abcd} and R^{bc}, that again we want to stress. In particular, with the choice above, the "crossing" term B_{bc} is mapped onto the "crossing" term $f_{ac}f_{bd}$.

10.4.6 Yet another proof of $R_{abcd}=0$

Here we give another proof of the relation (10.6), $R_{abcd}=0$, which will be suitable for extension to the OSP(1|2n) case, in Chapter 11.

Indeed, define $R_{a|b_1 b_2 \cdots b_k|c}$ as

$$R_{a|b_1 b_2 \cdots b_k|c} = \prod_{i=1}^{k} (f_{ab_i} - f_{b_i c})\,. \quad (10.67)$$

Then we easily recognize that

$$R_{abcd} = \big(p_e R_{a|bcd|e}\big) \tag{10.68}$$

where on the RHS we intend that we apply the derivatives inside p_e to the purely-Grassmann polynomial $R_{a|bcd|e}$. Indeed, a stronger statement w.r.t. R_{abcd} holds, as also $R_{a|bcd|e}$ is vanishing.

This is not hard to recognize, by first inspecting the combination

$$f_{ab} - f_{bc} = (\bar\psi_a\psi_a - \bar\psi_c\psi_c)(1 - \lambda\bar\psi_b\psi_b) - (\bar\psi_a - \bar\psi_c)\psi_b - \bar\psi_b(\psi_a - \psi_c) \tag{10.69a}$$

$$= \frac{1 - \lambda\bar\psi_b\psi_b}{2}\big((\bar\psi_a - \bar\psi_c)(\psi_a + \psi_c - 2\psi_b) + (\bar\psi_a + \bar\psi_c - 2\bar\psi_b)(\psi_a - \psi_c)\big) \tag{10.69b}$$

$$\begin{aligned} &= (\bar\psi_a - \bar\psi_c)\big(- \psi_b + \tfrac{1}{2}(\psi_a + \psi_c)(1 - \lambda\bar\psi_b\psi_b)\big) \\ &\quad + \big(- \bar\psi_b + \tfrac{1}{2}(\bar\psi_a + \bar\psi_c)(1 - \lambda\bar\psi_b\psi_b)\big)(\psi_a - \psi_c) \end{aligned} \tag{10.69c}$$

this showing (more explicitly in the last formulation (10.69c)) that $f_{ab} - f_{bc}$ is linear in the two nilpotent variables $\bar\psi_a - \bar\psi_c$ and $\psi_a - \psi_c$. Then our claim follows, by realizing that we have three such factors in $R_{a|b_1b_2b_3|c}$.

10.5 The basis of non-crossing partitions

Now we want to prove a set of claims done in Section 10.1, on the fact that relation (10.6) $R_{abcd} = 0$ is the only independent one in the algebra of scalar products $\{f_{ij}\}$, besides the "forest" relation Lemma 8.1, and other things concerning the linear space of this algebra, collected in the following

Theorem 10.3. *For V a set of cardinality n, and $\mathbf{n}_i = (1 - \lambda\bar\psi_i\psi_i; \bar\psi_i, \psi_i)$, the algebra generated by $\{\mathbf{n}_i \cdot \mathbf{n}_j\}_{i,j\in V}$, i.e. the linear span of $\{f_{\mathcal{C}}\}_{\mathcal{C}\in\Pi(n)}$, has dimension given by the n-th Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$. Given an arbitrary total ordering of the n indices (i.e. identifying V with $[n]$), a non-redundant basis is given by the set $\{f_{\mathcal{C}}\}_{\mathcal{C}\in NC(n)}$, where $NC(n) \subseteq \Pi(n)$ denotes the non-crossing partitions of n totally-ordered objects. Each element $f_{\mathcal{C}}$ for $\mathcal{C} \in \Pi(n) \setminus NC(n)$ is linearly expanded in this basis by only mean of relation (10.6).*

Remark that C_n is smaller than $\binom{2n}{n}$, and coincident with B_n up to $n = 3$, as needed.

We denoted by $NC(n) \subseteq \Pi(n)$ the set of non-crossing partitions, i.e. partitions $\mathcal{C}$ such that there exists no pair of blocks $A, B \in \mathcal{C}$, and quadruplet of elements (a, b, a', b'), with $a < b < a' < b'$, $a, a' \in A$ and $b, b' \in B$.

The theorem is clearly composed of more parts. So, we give for granted the well-known fact that the cardinality of $NC(n)$ is C_n, and split the theorem into two lemmas. First, one in which we prove that the set of $\{f_{\mathcal{C}}\}_{\mathcal{C}\in NC(n)}$ spans the whole algebra.

Lemma 10.4. *Each element $f_{\mathcal{C}}$ for $\mathcal{C}$ crossing can be replaced by a linear combination of non-crossing contributions, by iterate application of relation $R_{abcd} = 0$.*

Then, one in which we prove that no further linear relations hold among the $f_{\mathcal{C}}$'s non-crossing

Lemma 10.5. *There are no linear relations among the $\{f_{\mathcal{C}}\}_{\mathcal{C}\in NC(n)}$.*

It appears evident that lemmas 10.4 and 10.5 together imply theorem 10.3.

The first lemma 10.4 will be proven through an "algorithmic" restatement, lemma 10.7, that we present later on, as it requires the introduction of some further notations.

The second lemma, 10.5, is a corollary of the more explicit

Lemma 10.6. *There exist a partial ordering $\prec$ over $NC(n)$, and a map $m(\mathcal{C})$ from the set of noncrossing partitions to charge-neutral Grassmann monomials, such that $m(\mathcal{C})$ appears with non-vanishing coefficient in $f_{\mathcal{C}}$, and with coefficient zero in all other $f_{\mathcal{C}'}$'s for $\mathcal{C}'$ non-crossing and $\mathcal{C}' \not\prec \mathcal{C}$.*

Indeed, we start considering the easier proof that Lemma 10.6 implies Lemma 10.5. Then we introduce Lemma 10.7, together all the required definitions, and we prove that it implies Lemma 10.4. Then we prove in sequence the two lemmas 10.7 and 10.6. The logic of lemma-splitting is depicted in the following scheme:

$$\left.\begin{array}{l}\text{Lem. } 10.7 \Rightarrow \text{Lem. } 10.4\\ \text{Lem. } 10.6 \Rightarrow \text{Lem. } 10.5\end{array}\right\} \text{Lem. } 10.4 + \text{Lem. } 10.5 \Rightarrow \text{Theor. } 10.3$$

Proof that Lemma 10.6 $\Rightarrow$ Lemma 10.5: Indeed, call $M_{m,\mathcal{C}}$ the coefficients with whom a monomial m appears in a basis element $f_{\mathcal{C}}$. They form a matrix of dimension $\binom{2n}{n} \times \frac{1}{n+1}\binom{2n}{n}$. There are no further linear relations if this matrix has maximal rank C_n. This statement is implied by the result of lemma 10.6 above. Consider the minor containing only monomials $m(\mathcal{C})$. As a byproduct of lemma 10.6, $m(\mathcal{C})$'s for different $\mathcal{C}$'s are distinct, and thus this minor is a square matrix of dimension C_n. Order the $\mathcal{C}$'s in some way respecting the partial ordering $\prec$, and put the set of $\{\mathcal{C}\}$ and of $\{m(\mathcal{C})\}$ in the same sequence. This minor will thus be lower triangular, with nonzero coefficients on the diagonal, and therefore of maximal rank. □

Consider a spanning forest F over $\mathcal{K}_n$, and "$<$" being used for the ordering of $[n] \equiv V(\mathcal{K}_n)$. The number of crossings $X(F)$ is defined as the number of quadruplets $a, b, c, d \in [n]$, ordered as $a < b < c < d$, such that (ac) and (bd) are edges of F.

A useful graphical representation is the following: put the n vertices equally spaced along the border of a disk, and represent each edge $(ij) \in E(F)$ as an arc of circle, internal to the disk and orthogonal to its border at both endpoints. This prescription is such that the "algebraic" definition of crossings coincides with the "visual" one on the diagram.

A partition $\mathcal{C}(F) \in \Pi(n)$ is clearly identified by the subsets of vertices in the same component, and it is trivially seen that a partition $\mathcal{C}$ is noncrossing if and only if there exists a forest F such that $\mathcal{C}(F) = \mathcal{C}$ and $X(F) = 0$. The "canonical" representation $F_{\text{can}}(\mathcal{C})$ is the one such that, if $A = (a_1, \ldots, a_k)$ is in $\mathcal{C}$, with the a_i's ordered, then the edges of F in the component A are $(a_1a_2), (a_2a_3), \ldots, (a_{k-1}a_k)$. It is such that $X(F_{\text{can}}(\mathcal{C})) = 0$ iff $\mathcal{C} \in NC(n)$ (cfr. figure 10.1 for an example). Remark that we deliberately choose not to define an analogue function $X(\mathcal{C})$ for the number of crossings in a partition. A definition consistent with the one of $X(F)$ is possible, but not used in the proofs.[††]

We will use $f(F)$ as a shortcut for $f_{\mathcal{C}(F)}$, and thus, by mean of the "forest" Lemma 8.1, $f(F) = \prod_{(ij)\in E(F)} f_{ij}$. The number $X(F)$ of crossings (a, b, c, d) in F takes two contributions, the ones for which (ac) and (bd) are in the same component of F (*self-crossings*), and the ones for which they are in disjoint components (*true crossings*). We

[††]The good one would be, for $\mathcal{C} = (A_1, \ldots, A_k)$ and $A_\alpha = (a_1^{(\alpha)}, \ldots, a_{\ell(\alpha)}^{(\alpha)})$,

$$X(\mathcal{C}) = \#\{(\alpha, \beta; i, j) : a_1^{(\alpha)} < a_j^{(\beta)} < a_{i+1}^{(\alpha)} < a_{j+1}^{(\beta)}\}\,.$$

With this definition we would have

$$X(\mathcal{C}) = \min_{F:\mathcal{C}(F)=\mathcal{C}} \big(X(F)\big) = X(F_{\text{can}}(\mathcal{C}))\,.$$

$$\begin{aligned}\mathcal{C} = \{&\{1,6,8,9,13\},\\ &\{2,4,5\},\{3\},\{7\},\\ &\{10\},\{11,12\},\{14\},\\ &\{15\},\{16,17\},\{18\},\\ &\{19,23,24\},\\ &\{20,21,22\}\}\end{aligned}$$

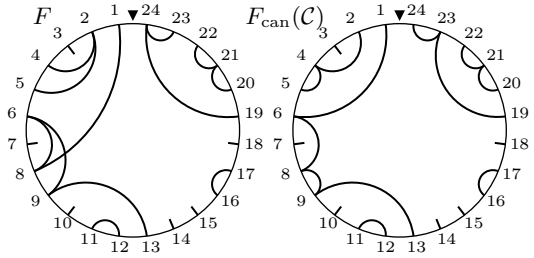


Fig. 10.1. A non-crossing partition $\mathcal{C}$, a forest F such that $\mathcal{C}(F) = \mathcal{C}$, and the canonical forest $F_{\rm can}(\mathcal{C})$. Remark that, while $F_{\rm can}$ has no crossings, the F in the example has a self-crossing.

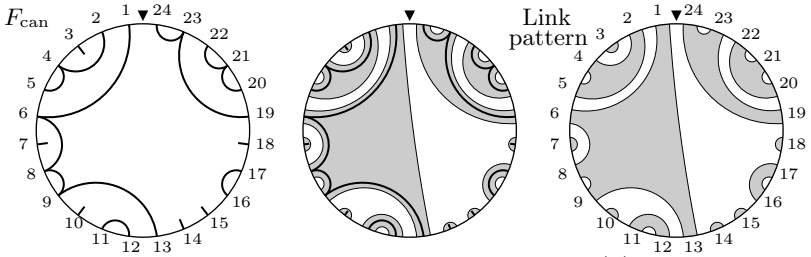


Fig. 10.2. Correspondence among non-crossing partitions $\mathcal{C} \in NC(n)$ and configurations of Link patterns among $2n$ points, through the representation of $\mathcal{C}$ with the canonical forest $F_{\rm can}$ (left). On the link pattern, the arcs identify regions that we colour alternately of white and gray in order to enhance visualization (right). The colours are chosen in such a way that the interval between $2n$ and 1 (the one pointed by the small arrow) is in white. The positions at which the gray regions touch the disk identify the components of $\mathcal{C}$.

define the *length* $\ell(a,b,c,d)$ of a crossing (a,b,c,d) as $\ell = d - a$. Both $X(F)$ and ℓ are positive integers, bounded by a function of n ($X(F) < n^2/2$, $\ell < n$). We will use in the following two consequences of these facts: that for a forest with $X > 0$ there must be at least one crossing with minimum length, and that, if we have an algorithm such that at each step, if $X > 0$, it decreases by a positive integer, this algorithm must halt in a finite number of steps with $X = 0$.

Consider a forest F with $X(F) > 0$. We want to manipulate the edges of the forest, finding a finite set of new forests $\{F_i\}$ and coefficients $\{z_i\}$ (polynomial in λ) with

$$f(F) = \sum_i z_i\, f(F_i)\,; \qquad\qquad \forall\, i \quad X(F_i) < X(F)\,. \tag{10.70}$$

Say the number of forests F_i is at most k (with k fixed), then, for any given forest, it would suffice to iterate the procedure above (for a number of steps bounded by $k^{X(F)}/(k-1)$, and remember that $X(F)$ is integer-valued and bounded by $n^2/2$) in order to algorithmically implement the claim of lemma 10.4.

Given a crossing (a,b,c,d) of F, call $F' = F \setminus \{(ac),(bd)\}$. Introduce the seven forests

$$F_{ab,cd} = F' \cup \{(ab),(cd)\} \qquad F_{ad,bc} = F' \cup \{(ad),(bc)\} \tag{10.71a}$$

$$F_{abc} = F' \cup \{(ab),(bc)\} \qquad F_{abd} = F' \cup \{(ab),(bd)\} \tag{10.71b}$$

$$F_{acd} = F' \cup \{(ac),(cd)\} \qquad F_{bcd} = F' \cup \{(bc),(cd)\} \tag{10.71c}$$

$$F_{abcd} = F' \cup \{(ab),(bc),(cd)\} \tag{10.71d}$$

We have that

Lemma 10.7. *With the definitions above,*

$$X(F_{ab,cd}),\ X(F_{ad,bc}) < X(F)\,. \tag{10.72}$$

Furthermore, if (a,b,c,d) is of minimum length among the crossings of F, also

$$X(F_{abcd}),\ X(F_{abc}),\ X(F_{abd}),\ X(F_{acd}),\ X(F_{bcd}) < X(F)\,. \tag{10.73}$$

Proof that Lemma 10.7 ⇒ Lemma 10.4: If (a,b,c,d) is a self-crossing, and the two edges (ac) and (bd) are connected through a path from c to d, or a path from a to b, the forest $F_{ad,bc} = F' \cup \{(ad),(bc)\}$ corresponds to the same partition, $\mathcal{C}(F) = \mathcal{C}(F_{ad,bc})$, while $X(F_{ad,bc}) < X(F)$ by lemma 10.7. Analogously, if the two edges are connected through a path from a to d, or from b to c, the forest $F_{ab,cd} = F' \cup \{(ab),(cd)\}$ corresponds to the same partition, $\mathcal{C}(F) = \mathcal{C}(F_{ab,cd})$, while $X(F_{ab,cd}) < X(F)$ by lemma 10.7. So we have an expression like in equation (10.70), with a single forest F_i.

The two statements are indeed equivalent, after realizing that in this part of the proof we still have a cyclic symmetry over $\mathbb{Z}_n$.

Instead, if (a,b,c,d) is a true crossing, the defined forests (10.71) correspond to the seven other partitions, besides $\mathcal{C}(F)$, involved by the relation

$$f(F')\,R_{abcd} = 0\,, \tag{10.74}$$

implied by our basic relation $R_{abcd} = 0$, so that we can state

$$f(F) = -\lambda f(F_{abcd}) - f(F_{ab,cd}) - f(F_{ad,bc}) + f(F_{abc}) + f(F_{abd}) + f(F_{acd}) + f(F_{bcd})\,. \tag{10.75}$$

So again we have an expression like in equation (10.70), now with seven forests F_i.

As in each forest F with $X(F) > 0$, we always have a self-crossing, or a true crossing of minimum length, thanks to the observation above we conclude that lemma 10.7 implies lemma 10.4, the induced algorithm makes use only of relation (10.6). □

Proof of Lemma 10.7: We define the four sets $S_1, \ldots, S_4$:

$$S_1 = \{i \in [n] : a < i < b\} \qquad S_2 = \{i \in [n] : b < i < c\} \tag{10.76a}$$

$$S_3 = \{i \in [n] : c < i < d\} \qquad S_4 = \{i \in [n] : i < a \vee i > d\} \tag{10.76b}$$

and the six non-negative integers $\{P_{x,y}(F)\}_{1 \le x < y \le 4}$, corresponding to the number of edges in F with endpoints one in S_x, one in S_y.

Equation (10.72) is implied by the two (cyclically-symmetric) observations that

$$X(F) - X(F_{ab,cd}) = 1 + 2P_{2,4} > 0 \tag{10.77a}$$

$$X(F) - X(F_{ad,bc}) = 1 + 2P_{1,3} > 0\,. \tag{10.77b}$$

Equation (10.73) is implied by two facts. The first one is that $P_{1,2} = P_{2,3} = 0$ if (a,b,c,d) is of minimum length. Indeed, otherwise, if it were $P_{1,2} > 0$, we would have an edge $(a'c')$ with $a' \in S_1$ and $c' \in S_2$, but then (a',b,c',d) would be a crossing of F, and $d - a' < d - a$. The reasoning is analogous (and reflection-symmetric) for $P_{2,3}$. The second fact consists in checking that, given $P_{1,2} = P_{2,3} = 0$, also the following expressions are strictly positive

$$X(F) - X(F_{abcd}) = 1 - P_{1,2} - P_{2,3} + P_{2,4} = 1 + P_{2,4} > 0 \tag{10.78a}$$

$$\begin{aligned} X(F) - X(F_{abc}) &= 1 - P_{1,2} + P_{1,3} + P_{2,4} + P_{3,4} \\ &= 1 + P_{1,3} + P_{2,4} + P_{3,4} > 0 \end{aligned} \tag{10.78b}$$

$$X(F) - X(F_{abd}) = 1 - P_{1,2} + P_{2,3} + P_{2,4} = 1 + P_{2,3} + P_{2,4} > 0 \tag{10.78c}$$

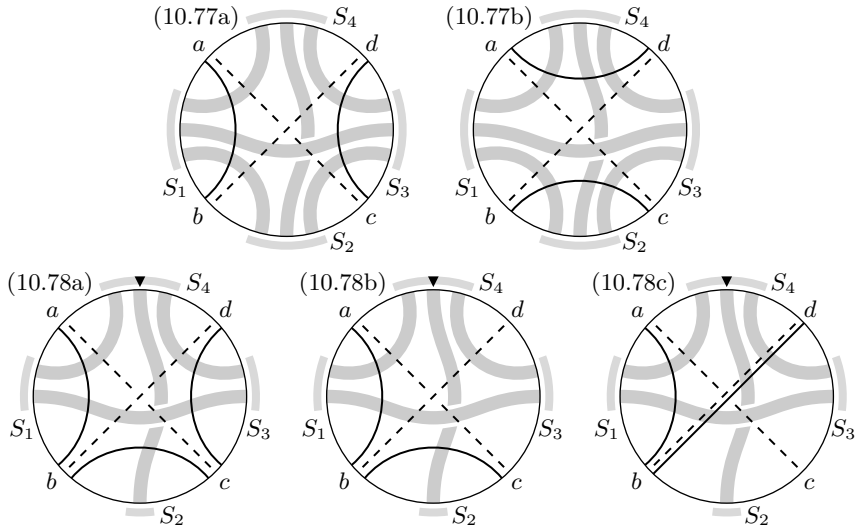


Fig. 10.3. Schemes for deducing the coefficients of the various $P_{x,y}$ in equations (10.77) and (10.78). Thick gray lines inside the disks stands for the set of arcs among regions S_x and S_y. The pairs $S_1 \leftrightarrow S_2$ and $S_2 \leftrightarrow S_3$ are omitted in equations (10.78.x) because we know that $P_{1,2} = P_{2,3} = 0$. Thin lines denote the edges between points a, b, c, d (dashed: before; solid: after). The contribution of $P_{x,y}$ in an equation is given by the number of crossings of the corresponding gray strip with dashed lines, minus crossings with solid lines.

(other two cases, F_{bcd} and F_{acd}, are implicit because of the symmetry under reflection). A graphical illustration of equations (10.77) and (10.78) is given in figure 10.3. □

Proof of Lemma 10.6. We want to define a certain partial ordering $\prec$ on the set of non-crossing partitions $\mathcal{C} \in NC(n)$, and a function $m(\mathcal{C})$ from $NC(n)$ to the set of charge-neutral monomials in the Grassmann Algebra over n indices, such that $M_{m,\mathcal{C}'} \neq 0$ only for $\mathcal{C}' \preceq \mathcal{C}$. In the case $\mathcal{C}' = \mathcal{C}$ we further have $M_{m,\mathcal{C}} = \pm 1$.

We could give straight our definition for $m(\mathcal{C})$, but instead we prefer to motivate it "pictorially", instead of giving it just out of the blue. Closely inspect the expression for $f_{\mathcal{C}}$, (10.4) and (10.5). Consider that, as we want to introduce some partial ordering $\prec$ on $NC(n)$, it is easy to guess that we will exploit at this purpose the natural total ordering on the n indices. Pairs of "unpaired fermions" $\bar{\psi}_i \psi_j$, $i \neq j$, in $f_{\mathcal{C}}$, may come at most one per block $A \in \mathcal{C}$. They somehow characterize the shape of the blocks, and in particular, the unpaired fermions $\bar{\psi}_{a_1} \psi_{a_k}$, coming from the action of $\partial_{a_1} \bar{\partial}_{a_k}$ in the expression (10.5), are "optimal" at the aim of determine the extension of block $A = (a_1, \ldots, a_k)$. So, it is reasonable to guess that these "extremal" terms are optimally capturing the structure of the partition, and we are thus led to tentatively define

$$m(\mathcal{C}) = \prod_{\substack{A \in \mathcal{C} \\ A=(a_1,\ldots,a_k)}} \left(\partial_{a_1} \bar{\partial}_{a_k} \tau_A\right) = \left(\prod_{\substack{A \in \mathcal{C} \\ A=(a_1,\ldots,a_k)}} \partial_{a_1} \bar{\partial}_{a_k} \right) \tau_{[n]} \,. \tag{10.79}$$

Remark that under this guess, as we do not make use of the terms $\lambda(1 - |A|)\tau_A$ in (10.5), and all the other coefficients in f_A are ± 1, the coefficients $M_{m(\mathcal{C}),\mathcal{C}}$ are ± 1 as claimed.

In principle, we do not need to prove separately that monomials $m(\mathcal{C})$ for different $\mathcal{C}$ are distinct, as it is implicit in proving the whole lemma. However it is instructive

to see how this comes out. Given a monomial m in the image of the map $m(\mathcal{C})$, the inverse map $\mathcal{C}(m)$ comes from a precise algorithmic procedure, and is thus unique. A good representation which allows to invert the map is achieved via the Dyck Path representation of the canonical diagram for $F_{\rm can}(\mathcal{C})$. A Dyck Path of length $2n$ is a height function $\{h(i)\}_{i=1,\ldots,2n+1}$, such that $h(1) = h(2n+1) = 0$, $h(i) \geq 0$ and $h(i+1) - h(i) = \pm 1$. This is a well-known representation for Link Pattern configurations of n arcs: $\Delta h = +1$ (resp. -1) in correspondence of a left (resp. right) termination of an arc. It is also well-known the direct bijection, also with no need of intermediary Dyck Path, among configurations of Link patterns and of non-crossing partitions. It is depicted in an example in Figure 10.2, and again, in linear representation, in Figure 10.4.

Similar classical identifications exists among a Dyck path and a non-crossing partitions, or a Dyck path and the canonical forest associated to the partition.

For these latter, given a Dyck Path representing $\mathcal{C}$, the identification rules are:

- An arc (ij) (with $i < j$) is in the forest $F_{\rm can}(\mathcal{C})$ iff $h(2i) = h(2j) = h_0$ and $h(s) > h_0$ for all $2i < s < 2j$.
- Two points i, j (with $i < j$) are in the same component of $\mathcal{C}$ iff $h(2i) = h(2j) = h_0$ and $h(s) \geq h_0$ for all $2i \leq s \leq 2j$.

Similarly easy, although specific to our problem and thus not in the literature, is the correspondence with the monomial $m(\mathcal{C})$:

- The field $\bar{\psi}_i$ is in m iff $h(2i+1) - h(2i) = +1$, while the field ψ_i is in m iff $h(2i) - h(2i-1) = -1$.

All these correspondences are depicted on an example in figure 10.4.

Now we have to prove the second statement, i.e. that $m(\mathcal{C})$ does not enter in the expansion of any other $\mathcal{C}'$, noncrossing and not preceding $\mathcal{C}$. It is now time to define our partial ordering $\prec$.

Given a monomial m, define its *dipole* $d(m)$ as

$$d(m) = \sum_{i:\psi_i \in m} i - \sum_{j:\bar{\psi}_j \in m} j\,, \tag{10.80}$$

which is exactly the dipole moment of the charge distribution along a bar of length n, if ψ_i puts a unit of positive charge at position i, and $\bar{\psi}_j$ puts a negative charge at j. Our pictorial argument motivating the choice for the function $m(\mathcal{C})$ can be rephrased now as the choice of the monomial in the expansion of $f_{\mathcal{C}}$ maximizing the dipole, as we will see in a while.

Similarly, given a noncrossing partition $\mathcal{C} = (A_1, \ldots, A_k)$, define its *polarizability* $p(\mathcal{C})$ as

$$p(\mathcal{C}) = \sum_{\alpha=1}^{k} \big(\max(A_\alpha) - \min(A_\alpha)\big) = \sum_{(ij)\in F_{\rm can}(\mathcal{C})} |i-j|\,. \tag{10.81}$$

We claim that, (as suggested by the names), for each pair $(m, \mathcal{C})$ such that $M_{m,\mathcal{C}} \neq 0$, $-p(\mathcal{C}) \leq d(m) \leq p(\mathcal{C})$, and that $d(m) = p(\mathcal{C})$ iff $m = m(\mathcal{C})$. Indeed, monomials m in the expansion of $f_{\mathcal{C}}$ are products of terms coming from the expansion of f_A's, for A separate blocks of $\mathcal{C}$. Analogously to $p(\mathcal{C})$, also $d(m)$ for m in the expansion of $f_{\mathcal{C}}$ is additive in the blocks, so that we can concentrate on a single block A. This is the crucial point, as the claim witrhin a single block is trivial: for $|A| \geq 2$, the contribution to the dipole of a term in the expansion of f_A is 0 if we take the term $\lambda(1-|A|)\tau_A$, and $j - i$ if we take the term

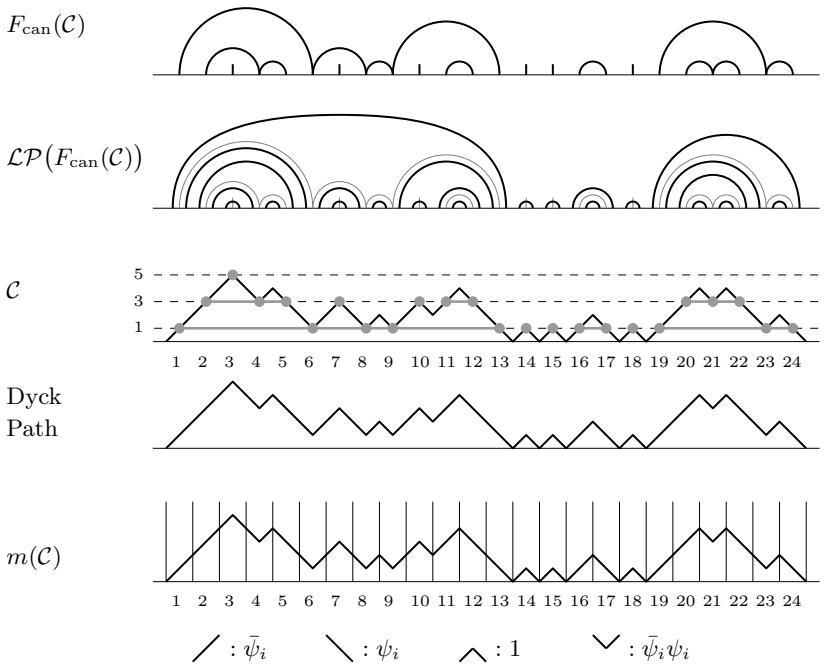


Fig. 10.4. The constructive method for going from $F_{\rm can}(\mathcal{C})$ to $m(\mathcal{C})$ and back, via the Dyck Path representation. The example of partition is the same as in figure 10.1. The forest is represented on a line, instead that on a disk, with obvious correspondence. On the drawing corresponding to $\mathcal{C}$, the sets are read from the positions of gray bullets connected by a gray segment.

$\partial_i\bar\partial_j\tau_A$; for A atomic, $A=\{i\}$, there is a single term $\partial_i\bar\partial_i\tau_i=1$. So, in both cases, the maximum is $\max(A)-\min(A)$, and is achieved only by the term that appears in $m(\mathcal{C})$.

This proves that $d(m(\mathcal{C}))=p(\mathcal{C})$, and otherwise m can appear in the expansion of a $\mathcal{C}'$ only if $d(m)<p(\mathcal{C}')$. So the partial ordering induced by the polarizability

$$\mathcal{C}\prec\mathcal{C}' \qquad\Longleftrightarrow\qquad p(\mathcal{C})<p(\mathcal{C}') \tag{10.82}$$

makes the game. This completes the proof of lemma 10.6, and thus of the whole theorem 10.3. □

10.6 Jordan-Wigner transformation

Everything we stated above in Clifford Algebra can be easily rephrased in terms of SU(2) spins by mean of the Jordan-Wigner transformation. A large number of references and historical comments on the Jordan-Wigner transformation, and especially its relation to Temperley-Lieb algebra, can be found in [146, 147].

Consider a set of indices $i=0,1,\dots,n-1$, and define the Pauli matrices

$$s^+=\begin{pmatrix}0&1\\0&0\end{pmatrix}\;;\qquad s^-=\begin{pmatrix}0&0\\1&0\end{pmatrix}\;;\qquad s^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\;; \tag{10.83}$$

satisfying the multiplication table

$$(s^{\pm})^2 = 0\,; \qquad (s^z)^2 = I\,; \qquad s^{\pm}s^{\mp} = \tfrac{1}{2}(I \pm s^z)\,; \qquad s^z s^{\pm} = -s^{\pm}s^z = \pm s^{\pm}\,. \tag{10.84}$$

Then consider the tensor product of n copies of $\mathbb{C}^2$, where subscript $i \in \{0, \ldots, n-1\}$ in Pauli matrices corresponds to the label of the $\mathbb{C}^2$ subspace. Define the shortcut

$$\pi_{i,j} = \prod_{\min(i,j)<k<\max(i,j)} s^z_k\,; \qquad\qquad \pi_i \equiv \pi_{-1,i}\,; \tag{10.85}$$

where it is understood that $\pi_{i,i\pm1} = 1$. It is easy to see that the following prescription reproduces the rules of Clifford Algebra:

$$\bar\psi_i = \pi_{2i}s^-_{2i}\,; \qquad \overline{\partial}_i = \pi_{2i}s^+_{2i}\,; \qquad \psi_i = \pi_{2i+1}s^+_{2i+1}\,; \qquad \partial_i = -\pi_{2i+1}s^-_{2i+1}\,; \tag{10.86}$$

and that the "charge" grading in Clifford Algebra corresponds to the total magnetization $S^z := \sum_k s^z_k$. In particular all charge-neutral bilinears (i.e. all quadratic monomials of the form $\bar\psi_i\psi_j$, $\partial_i\overline{\partial}_j$, $\bar\psi_i\overline{\partial}_j$ or $\psi_i\partial_j$) correspond to monomials in Pauli matrices that preserve the value of S^z. These bilinears all involve a pair of expressions π_i, so that the result has a small degree as a polynomial in $s_i^{z,+,-}$ if $|i-j|$ is small. More precisely

$$\bar\psi_i\psi_j = s^-_{2i}s^+_{2j+1}\pi_{2i,2j+1}\,; \qquad\qquad \partial_i\overline{\partial}_j = s^-_{2i+1}s^+_{2j}\pi_{2i+1,2j}\,; \tag{10.87}$$

$$\bar\psi_i\overline{\partial}_j = \begin{cases} \frac{1}{2}(I - s^z_{2i}) & i = j \\ s^-_{2i}s^+_{2j}\pi_{2i,2j} & i \neq j \end{cases} \qquad \psi_i\partial_j = \begin{cases} -\frac{1}{2}(I + s^z_{2i}) & i = j \\ s^-_{2i+1}s^+_{2j+1}\pi_{2i+1,2j+1} & i \neq j \end{cases} \tag{10.88}$$

From these "finite-order" building block is easy to derive Jordan-Wigner expressions for our Temperley-Lieb operators p_i, $f_{i,i+1}$ and $B_{i,i+1}$, and then, for example, the transfer matrix corresponding to a weighted regular periodic bidimensional lattice at choice, on a strip geometry.

10.7 Completeness of $\{\langle f_{\mathcal{C}}\rangle\}_{\mathcal{C}\in NC[n]}$

We have seen how any expression depending on f_{ij}'s only can be expanded as a linear combination of $f_{\mathcal{C}}$'s, by mean of Lemma 8.1. This holds, then, also for the exponential of our action, which thus has the form

$$\exp(S(\psi,\bar\psi)) =: Z(\psi,\bar\psi) =: \sum_{\mathcal{C}\in\Pi(n)} z(\mathcal{C}) f^{(\lambda)}_{\mathcal{C}}\,. \tag{10.89}$$

This defines, for a given weighted hypergraph, the expressions $z(\mathcal{C})$, which are 'numbers' w.r.t. Grassmann Algebra, and polynomials in the edge weights.

Furthermore, as a result of what is shown in Section 10.5, we can combine the coefficients into a complete and non-redundant basis, for $\mathcal{C}$ non-crossing partitions (w.r.t. some given fixed ordering) of the vertex set:

$$Z(\psi,\bar\psi) = \sum_{\mathcal{C}\in NC(n)} z'(\mathcal{C}) f^{(\lambda)}_{\mathcal{C}}\,, \tag{10.90}$$

e.g. by applying the algorithm implicit in Section 10.5 to the expression on the RHS of (10.89). As always, we can omit the (λ) superscript, and take $\lambda = 1$.[‡]

[‡]Recall that, as both the f_{ij}'s and the measure are homogeneous w.r.t. the combinations $\deg_\psi - \deg_{\bar\psi}$ and $\deg_\psi + \deg_{\bar\psi} - 2\deg_\lambda$, using this second relation it is easy to reintroduce λ's at the end, just by power-counting.

The partition function, and similarly the expectation value of a function $A(\psi,\bar\psi)$, are defined through integration on the $RP^{1|2}$ supersphere:

$$Z = \int \mathcal{D}(\psi,\bar\psi) e^{\sum_i \bar\psi_i\psi_i} \sum_{\mathcal{C}\in\Pi(n)} z(\mathcal{C}) f_{\mathcal{C}}\,; \tag{10.91}$$

$$Z\left\langle A(\psi,\bar\psi)\right\rangle = \int \mathcal{D}(\psi,\bar\psi) e^{\sum_i \bar\psi_i\psi_i} A(\psi,\bar\psi) \sum_{\mathcal{C}\in\Pi(n)} z(\mathcal{C}) f_{\mathcal{C}}\,. \tag{10.92}$$

Remark that, even in the case in which $Z=0$, in which $\left\langle A(\psi,\bar\psi)\right\rangle$ could be not defined, the bare integral $Z\left\langle A(\psi,\bar\psi)\right\rangle$ has a definite meaning.

From the homogeneity of $\deg_\psi - \deg_{\bar\psi}$, expectation values of monomials in Grassmann Algebra can be non-zero only if there are as many $\bar\psi$'s as ψ's (condition of *charge neutrality*), so we can restrict to linear combinations of charge-neutral monomials. A discussion of these expectation values is done in Section 8.5. Here we further restrict to functions $A(\psi,\bar\psi)$ being products of f's.

As we have B_n (Bell number n) different combinations $f_{\mathcal{C}}$, with $\mathcal{C}\in\Pi(n)$, we also have B_n possible observables of the form $Z\langle f_{\mathcal{C}}\rangle$. However, already at an algebraic level, before taking averages, only a set of C_n elements $f_{\mathcal{C}}$, with $\mathcal{C}\in NC(n)$ (w.r.t. some ordering of the points), are linearly independent. So it is in particular true that *at most* the C_n expressions $Z\langle f_{\mathcal{C}}\rangle$ for $\mathcal{C}\in NC(n)$ are independent.

These expectation values are then linear combinations of the coefficients $z(\mathcal{C})$ defined in (10.89), and can be probabilistically interpreted as linear combinations of probabilities for connectivity patterns among the n vertices, not only from the general discussion of Section 8.5, done for arbitrary charge-neutral fermionic observables, but at the light of a more powerful interpretation which is special to OSP(1|2)-invariant observables. Indeed we have

$$Z\langle f_{\mathcal{C}}\rangle = \sum_{\mathcal{C}'\in\Pi(n)} \langle\,\mathcal{C}\,|\,\mathcal{C}'\,\rangle\, z(\mathcal{C}')\,, \tag{10.93}$$

where

$$\langle\,\mathcal{C}\,|\,\mathcal{C}'\,\rangle := \int \mathcal{D}_n(\psi,\bar\psi) e^{\sum_{i=1}^n \bar\psi_i\psi_i} f_{\mathcal{C}} f_{\mathcal{C}'} = \begin{cases} 1 & \mathcal{C}\cup\mathcal{C}' \text{ is a hyperforest;} \\ 0 & \mathcal{C}\cup\mathcal{C}' \text{ contains cycles;} \end{cases} \tag{10.94}$$

These statements are a consequence, for example, of equation (8.64) in Corollary 8.7.

Thus, if we consider the redundant set of expectation values $Z\langle f_{\mathcal{C}}\rangle$ for $\mathcal{C}\in\Pi(n)$, the matrix $Q^{(0)}_{\mathcal{C},\mathcal{C}'} := \langle\,\mathcal{C}\,|\,\mathcal{C}'\,\rangle$ on $\Pi(n)\otimes\Pi(n)$ is a symmetric square matrix of dimension B_n, with all entries 0 and 1. And, as we said above, we know that this matrix has at most rank C_n. Let's call $Q_{\mathcal{C},\mathcal{C}'}$ the matrix $Q^{(0)}$ restricted to the space $NC(n)\otimes NC(n)$ (where we adopted the same total ordering in the two spaces).

Now, it is legimitate to ask ourselves two questions:

- has the matrix $Q^{(0)}_{\mathcal{C},\mathcal{C}'}$ *exactly* rank C_n?
- if so, is the minor $Q_{\mathcal{C},\mathcal{C}'}$ of rank C_n? In other words, are the set of $\{\langle f_{\mathcal{C}}\rangle\}_{\mathcal{C}\in NC(n)}$ a complete set of observables for the theory?

In case of positive answer, we would have a remarkable consequence: the complicated combinatorial facts in Section 8.5, for generic observables, in many concrete applications could be traded for the simplest description (10.94) of invariant observables. This is somewhat an intuitive fact, as in a theory having an invariance under the action of a group, expectation of a whatever observable coincides with the expectation of the same

observable averaged over the action of the group (with the invariant Haar measure), which is by construction an invariant expression. However at the moment we are not aware of any simple procedure for performing this averaging for an expression in Grassman Algebra (using of course the correspondence $(\sigma; \bar{\psi}, \psi) \equiv (1 - \lambda\bar{\psi}\psi; \bar{\psi}, \psi)$), so that we cannot show constructively the claim above, and have to revert to the question above concerning matrix $Q_{\mathcal{C},\mathcal{C}'}$.

Indeed, it happens that both of the questions itemized above are answered, in a positive way, by a theorem of Ko and Smolinsky [148], and later, constructively, by Di Francesco, Golinelli and Guittier [149] (cfr. in particular eq. (5.13) and following sentences) [♯]. In order to find a contact with this result, a few easy observations are required.

First define the quantity $\|\mathcal{C}\|$ (in the range $1, \ldots, n$) as the number of blocks in $\mathcal{C}$. Then, it is easily seen that $Q_{\mathcal{C},\mathcal{C}'} = 0$ if $\|\mathcal{C}\| + \|\mathcal{C}'\| > n + 1$. Furthermore, there are as many $\mathcal{C}$ with $\|\mathcal{C}\| = k$ than with $\|\mathcal{C}\| = (n + 1) - k$ (this is a consequence of the duality, cfr. for example Section 10.5, and in particular Figure 10.2), so that we can conclude that matrix Q is block-triangular, with block row- and column-indices labeled by $\|\mathcal{C}\|$ and $\|\mathcal{C}'\|$. This implies that its spectrum is fully determined through the diagonal square blocks. But, reintroducing λ's, we have that elements in a block are all either zero or equal to $\lambda^{n+2-\|\mathcal{C}\|-\|\mathcal{C}'\|}$, so that, for every value of λ, $1/\lambda$ times the restriction of $Q_{\mathcal{C},\mathcal{C}'}$ to the diagonal blocks coincides with $\lim_{q\to 0} \mathcal{G}_n(q) = \mathcal{G}'_n(0)$ in [149], so that their result, worked out for the ordinary "isotropic" T-L algebra, immediately extends to forests, that is our degenerate "even/odd" T-L algebra. This is indeed, remarkably, one of the few non-trivial facts that, in the even/odd T-L, can be inferred directly from the analogous result in the ordinary T-L algebra.

[♯] D.B. Wilson has brought this result to our attention, by explaining it at a seminar we were attending. I should say that, before being aware of this, I tried myself to prove the small corollary described below, without success.

11

OSP(1|2n) vector models and Spanning Forests

We show here how a family of OSP(1|2n)-invariant field theories on an arbitrary weighted graph G leads to the generating function of unrooted spanning forests on G, in a way similar to what happens for OSP(1|2) theories, described in Chapter 8. Parisi-Sourlas correspondence relates these models to the analytic continuation of $O(n)$ vector models to n odd negative integers, descendents of Ising by dimensional reduction in target space.

This continues our work on Grassmann representations of the abstract Forest Algebra, that has been started with the exploration of the OSP(1|2)-invariant theory. Although none of these representations are faithful, the quotients they induce are nested one into the other, and are more and more weak (we make these statements precise through a number of theorems).

11.1 The abstract Forest Algebra

Consider a set V of n points, and two commuting indeterminates λ and ρ. Consider the ring generated by the further 2^n indeterminates x_A, for $A \subseteq V$, and the relations:

$$[x_A, x_B] = 0 \tag{11.1}$$

$$x_A x_B = \rho^{|A\cap B|-1} x_{A\cup B} \qquad \text{if } A \cap B \neq \varnothing \tag{11.2}$$

This already implies that $x_A \rho^{-|A|+1}$ is idempotent for any $A \neq \varnothing$. Remark that $x_\varnothing$ is a central element.

The analysis of the relations above, in particular, shows that the polynomial ring generated by these indeterminates is a finite-dimensional linear space, of polynomials in the single central indeterminate $x_\varnothing$, that we take as a synonimous of λ. That is, if we call $\mathit{\Pi}(V)$ the set of partitions of a set V, the algebra above coincides with the linear span of the generators, labeled by a subset $V' \subseteq V$, a partition $P = (A_1, \dots, A_k) \in \mathit{\Pi}(V')$, and an integer s

$$X_{V',P,s} := x_\varnothing^s \prod_{A_i \in P} x_{A_i} \,. \tag{11.3}$$

This description allows us to partition the possible elements, for example, according to the two parameters V' and s, and a simple inspection shows that the product of an element with parameters (s_1, V_1') with an element with parameters (s_2, V_2') is within the class with parameters $(s_1 + s_2, V_1' \cup V_2')$. In particular, the class with $V' = V$ is an ideal. It is

a principal ideal, as it is generated by the single element $\prod_{i\in V} x_{\{i\}}$. We will take $x_{ijk\ldots}$ as shortcuts for $x_{\{i,j,k,\ldots\}}$ when clear.

Now consider also the n linear operators y_i, for $i \in V$, acting on the algebra above, satisfying the relations

$$[y_i, y_j] = 0 \tag{11.4}$$
$$[y_i, x_A] = 0 \qquad \text{if } i \notin A \tag{11.5}$$
$$y_i \circ x_A = x_i x_{A\smallsetminus i} \qquad \text{if } i \in A \tag{11.6}$$

In particular, applying (11.6) twice we get that the operator y_i^2 coincides with $x_\varnothing y_i \equiv \lambda y_i$ when acting on an element of the algebra as in (11.3) such that $i \in V'$.

This algebraic structure is underlying the Random Cluster Model described in Section 4.1, or better its broader variant on hypergraphs, of Section 4.2. For example, one easily sees that

$$\Big(\prod_{i\in V(G)} y_i\Big)\Big(\prod_{A\in E(G)} (1 + w_A x_A)\Big)\Big(\prod_{i\in V(G)} x_i\Big) = Z_{\mathrm{RC}}(G; w; \lambda, \rho)\Big(\prod_{i\in V(G)} x_i\Big) \tag{11.7}$$

and much better, by exploiting the "detach" action of operators y_i, one could express the same partition function acting through an algebra in a much smaller number of vertices, being the "band-width" of the graph, instead that the cardinality of $V(G)$ (cfr. Section 4.3 for an explanation of how this works on planar graphs).

Things get quite simplified, with a small loss in generality on the algebraic side, if we restrict our attention to the principal ideal described above, of elements containing the factor $\prod_{i\in V(G)} x_i$. In this case we can promote the y_i, from operators acting on the algebra of the x_A's, to generators of the algebra (as in this extension the algebra is no more commutative, we specify that we consider the *left* ideal, of elements having $\prod_{i\in V(G)} x_i$ multiplied *on the right*). We can naturally identify the resulting algebra with the one having the further relations

$$x_i = 1 \qquad \text{for all } i \in V; \tag{11.8}$$
$$y_i x_A y_i = y_i x_{A\smallsetminus i} \qquad \text{for } i \in A; \tag{11.9}$$

and in particular, if $A = \{i\}$,

$$y_i^2 = \lambda y_i\,. \tag{11.10}$$

In this algebra, the expression (11.7) above simplifies into

$$\Big(\prod_{i\in V(G)} y_i\Big)\Big(\prod_{A\in E(G)} (1 + w_A x_A)\Big) = Z_{\mathrm{RC}}(G; w; \lambda, \rho)\,. \tag{11.11}$$

It turns out that, by virtue of the redundance of the (λ, ρ) definition of the Random Cluster Model, that is, the implication of Euler formula, we could restrict our attention to only two cases: $\rho = 1$ and $\rho = 0$. The first one describes the largest part of the phenomenology of Potts Model (it suffices that $q \neq 0$), and is sometimes called *Join and Detach Algebra* (cfr. [145]). The second one is a specially simple case when also $\lambda = 0$, and corresponds to spanning trees, but it produces a peculiar "degenerate" algebraic structure otherwise, where it corresponds to spanning forests.

So, we define *Forest Algebra* the abstract algebra above, with the choices $\rho = 0$ and the identification $x_i = 1$ for all i. This algebra encodes completely the abstract property of "cycle-free connectivity" intrinsic to forests, and is suitable for the description

of general hyperforests models on hypergraphs. Our aim would be to study representations of this algebra, and through these deduce sensible results on the corresponding statistical-mechanics models.

At the light of this abstract definition, we now interpret Lemma 8.1 and equations (10.9) as proving that the set of $\{f_A\}$ and $\{p_i\}$ provides a representation of this algebra, as a subalgebra of the complex Clifford Algebra over the sites in V, containing only elements which are OSP(1|2) invariant (w.r.t. the non-linear realization of this symmetry, in Clifford Algebra).

We also understand the basic relations $R_{abcd} = R^b_{ac} = R^{ab} = 0$, discussed all along Chapter 10, as the illustration and certification that this concrete Clifford subalgebra is not the full Forest Algebra, but a quotient.

This is reminiscent of what happens for the integrable XXZ quantum spin chain. Certain polynomials in the spin-1/2 Pauli matrices on sites i and $i+1$, corresponding to the local summands in the Hamiltonian of the XXZ quantum spin chain, provide a representation of the elements e_i in a Hecke Algebra, which is however a quotient, as in particular we have the stronger T-L relation $e_i e_{i\pm1} e_i - e_i = 0$ instead of just the corresponding Hecke relation $e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}$.

However, we have other representations, even still in the spin-chain language, but with "higher spin", or higher-dimensional rotation groups (SU(N) instead of the SU(2) of physical spins), or better, in the appropriate language of Quantum Groups, to $U_q(su(n))$, as described in [150, 151, 152, 153] (also extension to $U_q(su(n|m))$ is possible, cfr. [154]).

For these representations, especially for the linear family $U_q(su(n))$, we learn an interesting lesson: for these groups, we have quotiens of Hecke which are "nested" with each other, that is, the ideal of relations that describes the quotient of the $U_q(su(n+1))$ representation of Hecke is a sub-ideal of the one of $U_q(su(n))$. In particular, if the length L of the spin chain is fixed, a value n exists such that the $U_q(su(n))$ representation of Hecke on $L-1$ generators is faithful (and thus also all the ones with $n' \geq n$ are).

This instructive analogy allows us to clearly motivate the work of this chapter, concerning OSP(1|2n) systems: not only they, interestingly, provide other Grassmann expressions for the generating function of spanning forests; not only, the larger n is, the larger is the set of probabilistic observables that can be addressed through fermionic operators in the theory; but also, as the algebraic ground of this last property, we are producing a family of "nested" quotiens of the abstract Forest Algebra defined above, with the property that, if n is large enough w.r.t. V, then the representation is faithful.

This fact has also some resemblance in a "threshold mechanism", among n and V, occurring for the Gram matrix of V unit vectors on the n-dimensional sphere (that is, having the ordinary "bosonic" $O(n)$ invariance). If n is sufficiently larger than V, the measure over (the off-diagonal elements of) the Gram matrix is a continuous Lesbesgue measure over $[-1,1]^{V(V-1)/2}$. If instead n is smaller than V (as it happens for "thermodynamical" systems, in which $V \to \infty$ at fixed n), the measure over the set of matrix elements is a distribution, and a number of delta function arise stating that the $\frac{V(V-1)}{2}$ upper-diagonal entries of the matrix are not independent.

11.2 A reminder on the OSP(1|2)–Spanning-Forest correspondence

Given a weighted graph $G = (V, E)$, with weights w_{ij} on edges $(ij) \in E$, we denote by $\mathcal{F}(G)$ the set of spanning forests on G, i.e. the subgraphs of G with no cycles. We show in

[130, 155] how a non-linear σ-model theory with OSP(1|2) symmetry, plus discrete spin-inversion, reproduces the (weighted) generating function for spanning forests, namely

$$\mathcal{N}^{-1}\int\Big(\prod_i \mathrm{d}_{(1|2)}\mathbf{n}_i\Big)e^{\frac{1}{2}\sum_{(ij)}w_{ij}(1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2)}=\sum_{F\in\mathcal{F}(G)}\prod_{(ij)\in E(F)}w_{ij}\,; \tag{11.12}$$

where notations are better explained in the following. Our main goal here is to show that the relation above is the first one of a family, running over theories with OSP(1|2n) symmetry, with n a positive integer, more specifically

$$\mathcal{N}^{-1}\int\Big(\prod_i \mathrm{d}_{(1|2n)}\mathbf{n}_i\Big)e^{\frac{(2n-1)!!}{(2n)!!}\sum_{(ij)}w_{ij}(1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2)^n}=\sum_{F\in\mathcal{F}(G)}\prod_{(ij)\in E(F)}w_{ij}\,. \tag{11.13}$$

In both cases the unpleasant "quartic" factor $(1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2)/2$ can be replaced just by the "quadratic" $1-\mathbf{n}_i\cdot\mathbf{n}_j$, at the price that the correspondence holds only at a perturbative level (however notice that this simplification is not of big advantage, as the non-linear constraint does anyhow make the theory non-free).

This OSP(1|2n) generalization is not immediate: many algebraic propositions which were checked just "by hands" in the OSP(1|2) case, now require some technique to be handled at generic values of n. We find that methods of exponential generating functions, always with the same basic ingredients, appear crossways along the various new proofs. What, in the OSP(1|2)–Spanning-Forest correspondence, could have seem an accidental coincidence of numerical values, helped by the strong truncations due to fermions' nilpotence, here is highlighted as an instrumental coincidence of the involved analytic functions, so suggesting that the correspondence is indeed an intimate mechanism.

All the proofs in this paper are purely combinatorial, simple, rigorous and start from first principles, so they should have more the status of a "mathematical proof", than of a "physical argument". Still, a motivation to this work relies on subtle and non-rigorous points, such as

- understand the analytic continuation of models with $O(n)$ symmetry (besides the special class of logarithmic Nienhuis models [24]);
- understand the validity, in $O(n)$ models, of the perturbative expansion around a nematic state;
- understand the correctness of the Parisi-Sourlas dimensional reduction procedure [56] in this case.

Within this framework of conjectural ideas, OSP(1|2n) models are the family of descendents, under dimensional reduction, of Ising Model (i.e. $O(1)$), and are the only ones such that the measure over angular degrees of freedom becomes singular, more precisely a distribution concentrated on the two "poles" $\cos\theta=\pm1$. So it is not totally surprising that these models, and most probably only these among the full $O(n)$ family, admits for an all-temperature purely combinatorial description of the partition function and of a large algebra of observables.

Physical observables are an interesting point. We show in Sections 8.5 and 9.6 how the k-point correlation functions in the fermionic theory have a probabilistic interpretation in the combinatorial model, as probabilities of connectivity patterns among the k sites of the lattice, and in Section 10.7 how this is simplified for invariant operators, but we also show (cfr. Section 10.1, equation (10.6) on the relation $R_{abcd}=0$) that not all the event probabilities are accessible through this correspondence. We now show how the same fact holds for these more general OSP(1|2n) model: actually, the larger n is, the larger is the

family of event probabilities reproduced by the correspondence, and for every event there exists a value of n such that the corresponding theory with this number of components (or larger) can give access to its probability.

In Chapter 9 we also discuss how, in the OSP(1|2) theory, the Ward Identities originating by the spherical symmetry are interpreted as probabilistic statements (i.e. different rephrasings of the same event, or sum of probabilities for a complete disjoint set of events equal 1), true already in the abstract Forest Algebra and not related to the quotient. Similar mechanisms arise here.

This chapter, besides the mathematical discussion of the previous section, is organized in a very pragmatical way, and gives in the following the Clifford Algebra tools required to understand the extension of all these statements to OSP($1|2n$) theories. In Section 11.3 we recall the vector notations for OSP($1|2n$)-invariant spherical theories, while in Section 11.4 we summarize the results of this chapter. Then, in Sections 11.5 and 11.6 we give the proofs respectively of the main theorem concerning the generating function, and of the necessary lemmas. Finally, in Section 11.7 we show how the procedure of section 11.5 generalizes to the probabilistic comprehension of correlation functions in the fermionic theory.

11.3 Unit vectors in $\mathbb{R}^{1|2n}$

On each site i of our lattice, we have a target space composed of a real variable ϕ_i and $2n$ Grassmann variables $\bar\psi_i^1, \psi_i^1, \dots, \bar\psi_i^n, \psi_i^n$. A vector $\mathbf{n}_i = \{n_i^a\}$ collects all the components, and the scalar product is given by the orthosymplectic form

$$\mathbf{n}_i \cdot \mathbf{n}_j := n_i^a g_{ab} n_j^b ; \qquad g_{ab} := \left(\begin{array}{c|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 1 & & & \\ 0 & -1 & 0 & & 0 & \\ \vdots & & & \ddots & & \\ 0 & & 0 & & 0 & 1 \\ 0 & & & & -1 & 0 \end{array}\right) ; \tag{11.14}$$

that is

$$\mathbf{n}_i = (\phi_i, \bar\psi_i^1, \psi_i^1, \dots, \bar\psi_i^n, \psi_i^n) ; \tag{11.15}$$

$$\mathbf{n}_i \cdot \mathbf{n}_j = \phi_i \phi_j + \sum_{a=1}^{n} (\bar\psi_i^a \psi_j^a + \bar\psi_j^a \psi_i^a) . \tag{11.16}$$

Superscripts $a \in [n]$ in fermionic fields should not be confused with exponents (which essentially do not appear all along the paper on fermionic fields, as, by nilpotence, $(\psi_i^a)^2 = 0$).

The $\mathbb{R}^{1|2n}$ supersphere (of radius 1) is defined by the constraint $|\mathbf{n}_i|^2 = 1$. Call $\bar{\boldsymbol\psi}_i$ and $\boldsymbol\psi_i$ the n-uples of fermionic components, and $\bar{\boldsymbol\psi}_i \boldsymbol\psi_i = \sum_a \bar\psi_i^a \psi_i^a$. Solving the spherical constraint w.r.t. the bosonic variable ϕ gives

$$\phi_i = \sqrt{1 - 2\bar{\boldsymbol\psi}_i \boldsymbol\psi_i} = - \sum_{S \subseteq [n]} (2|S| - 3)!! \prod_{a \in S} \bar\psi_i^a \psi_i^a . \tag{11.17}$$

In the Taylor expansion of the square root we used the well-known formula

$$\sqrt{1-2x} = \sum_{k\geq 0} -\frac{(2k-3)!!}{k!} x^k \,; \tag{11.18}$$

where, as appropriate, $1!! = (-1)!! = -(-3)!! = 1$. Define the formal integration over the supersphere of dimension $(1|2n)$

$$\mathrm{d}_{(1|2n)}\mathbf{n} := \prod_{a=1}^{n} \left(\mathrm{d}\psi^a \mathrm{d}\bar{\psi}^a\right) \mathrm{d}\phi \; 2\delta(\mathbf{n}^2 - 1) = \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}})\, \mathrm{d}\phi \, 2\delta(\mathbf{n}^2 - 1)\,, \tag{11.19}$$

where as customary integration in $\mathrm{d}\phi$ runs over $\mathbb{R}$, and $\mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}})$ stands for the canonical Grassmann integration $\mathrm{d}\psi^n \mathrm{d}\bar{\psi}^n \cdots \mathrm{d}\psi^1 \mathrm{d}\bar{\psi}^1$ with formal rule $\int \mathrm{d}\psi \;\; \psi = 1$ and $\int \mathrm{d}\psi \;\; = 0$. The Jacobian of the integration of the bosonic degree of freedom gives

$$\int \mathrm{d}\phi_i \, 2\delta(\mathbf{n}^2 - 1) = \frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i}} \tag{11.20}$$

and it is intended that ϕ_i in the integrand must be replaced by the expression $\phi(\bar{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i)$ in eq. (11.17). Exponentiating the Jacobian, and using the expansion

$$-\tfrac{1}{2}\ln(1-2x) = \sum_{k\geq 1} \frac{2^{k-1}}{k} x^k \,, \tag{11.21}$$

gives

$$\frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i}} = \exp\left[-\tfrac{1}{2}\ln(1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i)\right] = \exp\left[\sum_{\substack{S\subseteq[n] \\ S\neq\varnothing}} (2|S|-2)!! \prod_{a\in S} \bar{\psi}_i^a \psi_i^a\right]. \tag{11.22}$$

Alternatively, we can write

$$\frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i}} = \sum_{S\subseteq[n]} (2|S|-1)!! \prod_{a\in S} \bar{\psi}_i^a \psi_i^a \,; \tag{11.23}$$

where we used an analogue of (11.18)

$$\frac{1}{\sqrt{1-2x}} = \sum_{k\geq 0} \frac{(2k-1)!!}{k!} x^k \,. \tag{11.24}$$

In principle we have two determinations in the square root, so that the true integration over the full supersphere reads

$$\int \mathrm{d}_{(1|2n)}\mathbf{n}_i \; F(\mathbf{n}_i) := \sum_{\epsilon_i = \pm 1} \int \mathcal{D}(\boldsymbol{\psi}_i, \bar{\boldsymbol{\psi}}_i) \frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i}} \; F\big(\epsilon_i(\phi(\bar{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i), \bar{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i)\big)\,. \tag{11.25}$$

We can define a "perturbative" region of integration, analytic continuation of what, in finite integer dimension, is the half-sphere with $|\phi| > 0$, accessible by expansion around the polarized state $\mathbf{n} = (1, 0, 0, \ldots)$. This corresponds to neglect all the signs ϵ_i in the action, or, equivalently, consider only the perturbation around the nematic state $\epsilon_i = +1$ for all i.

$$\int^{(+)} \mathrm{d}_{(1|2n)}\mathbf{n}_i \, F(\mathbf{n}_i) := \int \mathcal{D}(\boldsymbol{\psi}_i, \bar{\boldsymbol{\psi}}_i) \frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i}} \, F\big((\phi(\bar{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i), \bar{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i)\big) \,. \tag{11.26}$$

For future convenience we define the "volume" of the supersphere $\mathbb{R}^{1|2n}$:

$$\begin{aligned} \Omega_{(1|2n)} &:= \int \big(\mathrm{d}_{(1|2n)}\mathbf{n}\big) \, 1 = \sum_{\epsilon=\pm 1} \int \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}}) \frac{1}{\sqrt{1 - 2\bar{\boldsymbol{\psi}}\boldsymbol{\psi}}} \\ &= 2 \int \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}}) \sum_{S \subseteq [n]} (2|S| - 1)!! \prod_{a \in S} \bar{\psi}^a \psi^a = 2(2n-1)!! \,, \end{aligned} \tag{11.27}$$

and similarly

$$\Omega^{(+)}_{(1|2n)} := \int^{(+)} \big(\mathrm{d}_{(1|2n)}\mathbf{n}\big) \, 1 = (2n-1)!! \,. \tag{11.28}$$

This result is in agreement with the analytic continuation of the traditional measure of the sphere, done by mean of Parisi-Sourlas correspondence [56]. Indeed the spherical integration of a unit vector in $\mathbb{R}^n$ has a normalization

$$\Omega_n = \frac{\int \mathrm{d}^n\mathbf{n} \, e^{-\frac{|\mathbf{n}^2|}{2}}}{\int_0^\infty \mathrm{d}r \, r^{n-1} e^{-\frac{r^2}{2}}} = \frac{(2\pi)^{\frac{n}{2}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \,, \tag{11.29}$$

while, using Parisi-Sourlas correspondence, so that the $2n$ fermionic degrees of freedom "count as" $-2n$ bosonic degrees of freedom, we have

$$\Omega_{(1|2n)} = \frac{\int \mathrm{d}\phi \, \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}}) \, e^{-\frac{\phi^2}{2} - \sum_a \bar{\psi}^a \psi^a}}{\int_0^\infty \mathrm{d}r \, r^{(1-2n)-1} e^{-\frac{r^2}{2}}} = \frac{(-1)^n \sqrt{2\pi}}{2^{-\left(\frac{1}{2}-n\right)-1}\Gamma\left(\frac{1}{2}-n\right)} = 2(2n-1)!! \,. \tag{11.30}$$

11.4 The results

In [155], where we study the case of OSP(1|2), for a set V we introduce the quantities

$$\tau_V := \prod_{i \in V} \bar{\psi}_i \psi_i \,; \tag{11.31}$$

$$f_V^{(\lambda=1)} := (1 - |V|)\tau_V + \sum_{i \in V} \tau_{V \smallsetminus i} - \sum_{(i \neq j) \in V} \bar{\psi}_i \psi_j \tau_{V \smallsetminus \{i,j\}} \,. \tag{11.32}$$

With abuse of notations, we are used to write f_i, f_{ij}, ... for $f_{\{i\}}$, $f_{\{i,j\}}$, ... when clear.

W.r.t. the notations in [155], in all this paper we specialize to the case $\lambda = 1$ in order to enlight the already involved notation. So we drop the superscript λ from $f_V^{(\lambda)}$. Instead, now that we have n complex fermions per site, we need an index a in order to specify the component, so that we have quantities τ_V^a and f_V^a, for the analogs of the quantities above in the variables $\{\bar{\psi}_i^a, \psi_i^a\}$. Remark in particular that $\tau_\varnothing^a = f_\varnothing^a = f_i^a = 1$, while all the other ones are nilpotent of degree 2 (trivially by power counting for all of them, except for f_{ij}'s, where a cancellation occurs).

Three lemmas are instrumental to our purposes. A first one is

Lemma 11.1. *The quantity* $(1 - \mathbf{n}_i \cdot \mathbf{n}_j)$*, where* $\mathbf{n}$*'s are vectors on the* $\mathbb{R}^{1|2n}$ *supersphere and* ϕ*'s are intended as in eq. (11.17), is nilpotent of order* $n+1$*, i.e.* $(1 - \mathbf{n}_i \cdot \mathbf{n}_j)^{n+1} = 0$*.*

Remark: $(1-\mathbf{n}_i\cdot\mathbf{n}_j)$ is a polynomial in a Grassmann Algebra of dimension $4n$, and every monomial has degree at least 2, so it was obvious by simple power counting that it is nilpotent with some order at most $2n+1$. The non-trivial fact is that some cancellations occur, such that it is nilpotent "only" of order $n+1$.

So, it can be interesting to look at the largest degree k such that $(1-\mathbf{n}_i\cdot\mathbf{n}_j)^k$ is non-vanishing. The next lemma states that this value is exactly n, and that what remains has a peculiar form.

Lemma 11.2.

$$\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n}{n!}=\sum_{S\subseteq[n]}c(|S|)\prod_{a\in S}\tau_{ij}^a\prod_{a\in[n]\smallsetminus S}f_{ij}^a\,; \tag{11.33}$$

where the coefficients $c(k)$ are given by the exponential generating function

$$\hat{c}(x):=\sum_k\frac{c(k)}{k!}x^k=e^x\sqrt{1-2x}\,. \tag{11.34}$$

The importance of this lemma relies on the algebraic properties of expressions as in the RHS of (11.33) described by the following Lemma 11.3. Before writing it down, we introduce some notation. For a set of coefficients $\{C(k)\}_{k\geq 0}$, with exponential generating function $g(x)$

$$g(x)=\sum_k\frac{x^k}{k!}C(k)\,, \tag{11.35}$$

and, for an arbitrary set V of vertices, we define the quantity

$$Q_V(g):=\sum_{S\subseteq[n]}C(|S|)\prod_{a\in S}\tau_V^a\prod_{a\in[n]\smallsetminus S}f_V^a\,. \tag{11.36}$$

The Q_V's at fixed V have a trivial additivity property,

$$Q_V(g_1)+Q_V(g_2)=Q_V(g_1+g_2)\,, \tag{11.37}$$

where it is understood that $(g_1+g_2)(x)=g_1(x)+g_2(x)$. But the crucial fact relies on the multiplicative property encoded in the lemma

Lemma 11.3. *For V_1 and V_2 not disjoint, and $Q_{V_1}(g_1)$, $Q_{V_2}(g_2)$ as described above, we have*

$$Q_{V_1}(g_1)\,Q_{V_2}(g_2)=\begin{cases}Q_{V_1\cup V_2}(g_1g_2) & |V_1\cap V_2|=1\\ 0 & |V_1\cap V_2|\geq 2\end{cases} \tag{11.38}$$

(now it is undersood that $(g_1g_2)(x)=g_1(x)g_2(x)$). See, for example, how the expression in Lemma 11.2 is $Q_{ij}(e^x\sqrt{1-2x})$, and, as within Lemma 11.3 one can write

$$Q_{ij}(e^x\sqrt{1-2x})=Q_{ij}(e^x)Q_i(\sqrt{1-2x})=Q_{ij}(1)Q_i(e^x)Q_i(\sqrt{1-2x}) \tag{11.39}$$

one easily get the rephrasings of Lemma 11.2

$$\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n}{n!}=\phi_i\sum_{S\subseteq[n]}\prod_{a\in S}\tau_{ij}^a\prod_{a\in[n]\smallsetminus S}f_{ij}^a \tag{11.40a}$$

$$=\phi_i e^{\bar{\boldsymbol{\psi}}_i\boldsymbol{\psi}_i}\prod_a f_{ij}^a\,. \tag{11.40b}$$

Lemma 11.3 puts under one roof, and greatly generalizes, the three equations (8.24, 8.29, 8.32), and represents in great generality the crucial abstract identity in Forest Algebra, (11.2) at $\rho = 0$ (we are however not aware of any natural generalization to arbitrary values of ρ).

The combination of (11.20), (11.22) and (11.33) implies the following purely fermionic formulation of the OSP(1|2n) theory, where we exponentiate all the factors, and then we can recognize the expression at the exponent as the fermionic action $\mathcal{S}(\bar\psi, \psi)$

$$\begin{aligned}\int^{(+)} \Big(\prod_i \mathrm{d}_{(1|2n)}\mathbf{n}_i\Big)\, \exp\Big[\frac{1}{n!}\sum_{(ij)} w_{ij}(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n\Big] &= \int \mathcal{D}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}})\exp\big[\mathcal{S}(\bar\psi,\psi)\big] \\ = \int \mathcal{D}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}})\exp\Big[\sum_i \sum_{\substack{S\subseteq[n]\\ S\neq\varnothing}} (2|S|-2)!!\prod_{a\in S}\tau_i^a + \sum_{(ij)} w_{ij}\sum_{S\subseteq[n]} c(|S|)\prod_{a\in S}\tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a\Big]&.\end{aligned} \tag{11.41}$$

The expression above is an intermediate step in order to prove the main theorem

Theorem 11.4.

$$\int^{(+)}\Big(\prod_i \mathrm{d}_{(1|2n)}\mathbf{n}_i\Big) e^{\frac{1}{n!}\sum_{(ij)} w_{ij}(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n} = \sum_{F\in\mathcal{F}(G)} \big((2n-1)!!\big)^{K(F)} \prod_{(ij)\in E(F)} w_{ij}\,; \tag{11.42}$$

where $K(F)$ is the number of components in F. It is a simple remark that, because of an Euler formula

$$K(F) + |E(F)| = |V(G)|\,, \tag{11.43}$$

the factor $\big((2n-1)!!\big)^{K(F)}$ in the generating function for spanning forests is easily reabsorbed into a redefinition of the w's, and a normalization overall

$$\int^{(+)} \Big(\prod_i \frac{\mathrm{d}_{(1|2n)}\mathbf{n}_i}{(2n-1)!!}\Big)\, e^{\frac{(2n-1)!!}{n!}\sum_{(ij)} w_{ij}(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n} = \sum_{F\in\mathcal{F}(G)}\prod_{(ij)\in E(F)} w_{ij}\,. \tag{11.44}$$

Also remark that a factor $1/(2n-1)!! = 1/\Omega^{(+)}_{(1|2n)}$, as a normalization per each perturbative integration over a $\mathbb{R}^{1|2n}$ unit vector, is quite natural.

We can also use lemma 11.1 in order to state

$$\begin{aligned}\big(1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2\big)^n &= (1-\mathbf{n}_i\cdot\mathbf{n}_j)^n(1+\mathbf{n}_i\cdot\mathbf{n}_j)^n \\ &= (1-\mathbf{n}_i\cdot\mathbf{n}_j)^n\big(2-(1-\mathbf{n}_i\cdot\mathbf{n}_j)\big)^n = 2^n(1-\mathbf{n}_i\cdot\mathbf{n}_j)^n\,.\end{aligned} \tag{11.45}$$

Analogously to what happens in the case of OSP(1|2), and as discussed in [155], the combination $1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2$ is convenient if we want to work in the non-perturbative framework, as the theory is then effectively defined by unit vectors on the projective supersphere, instead that the full supersphere. Indeed, reintroducing the nematic variabes $\{\epsilon_i\}$, we have that the combination $1-\mathbf{n}_i\cdot\mathbf{n}_j$ is not invariant, while the squared one is:

$$1-(\mathbf{n}_i\cdot\mathbf{n}_j) \quad\longrightarrow\quad 1-\epsilon_i\epsilon_j\,(\mathbf{n}_i\cdot\mathbf{n}_j)\,; \tag{11.46}$$

$$1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2 \quad\longrightarrow\quad 1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2\,. \tag{11.47}$$

This allows us to write a non-perturative version of the theorem

Corollary 11.5.

$$\int \Big(\prod_i \frac{\mathrm{d}_{(1|2n)}\mathbf{n}_i}{2(2n-1)!!} \Big) e^{\frac{(2n-1)!!}{(2n)!!}\sum_{(ij)} w_{ij}\left(1-(\mathbf{n}_i\cdot\mathbf{n}_j)^2\right)^n} = \sum_{F\in\mathcal{F}(G)} \prod_{(ij)\in E(F)} w_{ij}\,. \quad (11.48)$$

Again remark that the normalization factor $1/(2(2n-1)!!) = 1/\Omega_{(1|2n)}$ is quite natural.

From here on we will work in the perturbative framework, and drop the superscript $(+)$ from integrals where clear.

11.5 Proof of the theorem

Here we prove Theorem 11.4 assuming the Lemmas 11.2 and 11.3, which are proven, together with Lemma 11.1, in section 11.6.

As anticipated, we use the expression in equation (11.41), but we use the form (11.23) instead of (11.22) for the factors coming from the Jacobian.

So we consider the expansion of the action

$$\Big(\prod_i \Big(\sum_{S\subseteq[n]} (2|S|-1)!! \prod_{a\in S} \tau_i^a \Big) \Big) \exp\Big[\sum_{(ij)} w_{ij} \sum_{S\subseteq[n]} c(|S|) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a \Big]\,. \quad (11.49)$$

Remark that each summand in the exponential is nilpotent (of order 2), and that all the terms proportional to a given w_{ij} have vanishing cross-product, so that we can write

$$\prod_i \Big(\sum_{S\subseteq[n]} (2|S|-1)!! \prod_{a\in S} \tau_i^a \Big) \prod_{(ij)} \Big(1 + w_{ij} \sum_{S\subseteq[n]} c(|S|) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a \Big)\,. \quad (11.50)$$

Call $E'\subseteq E(G)$ the set of edges (i.e. pairs (ij)) in which the summand proportional to w_{ij} has been taken in the last product, that is expand the expression above on the right into

$$\prod_{(ij)} \Big(1 + w_{ij} \sum_{S\subseteq[n]} c(|S|) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a \Big) = \sum_{E'\subseteq E(G)} \Big(\prod_{(ij)\in E'} w_{ij} \Big) \prod_{(ij)\in E'} \Big(\sum_{S\subseteq[n]} c(|S|) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a \Big)\,. \quad (11.51)$$

Recalling that $f_i = 1$, and in the language of Lemma 11.3, we recognize that the whole action (11.50) (also included the factors from the Jacobian) has the form

$$\sum_{E'\subseteq E(G)} \prod_{(ij)\in E'} w_{ij} \prod_{i\in V(G)} Q_i\Big(\frac{1}{\sqrt{1-2x}} \Big) \prod_{(ij)\in E'} Q_{ij}\big(e^x \sqrt{1-2x} \big)\,, \quad (11.52)$$

where we used the expression for $\hat{c}(x)$, described in Lemma 11.2. Lemma 11.3 implies that, if E' contains any cycle, the product corresponding to this term vanishes in Grassmann Algebra (as it is seen e.g. by multiplying first the factors Q_{ij} for edges (ij) ordered along the cycle). So we can restrict the attention to sets E' identifying spanning forests $F = (V, E')$ on G.

For each forest F, we have a "thermodynamic factor" $\prod_{(ij)\in E(F)} w_{ij}$, and a factor involving fermions, which is factorized on the single components. So, if $\{V_\alpha\}$ is the partition

of $V(G)$ induced by the components of the forest $F = \{T_\alpha\}$, for each tree $T_\alpha = (V_\alpha, E_\alpha)$ we have a quantity

$$\prod_{i\in V_\alpha} Q_i\left(\frac{1}{\sqrt{1-2x}}\right) \prod_{(ij)\in E_\alpha} Q_{ij}\big(e^x\sqrt{1-2x}\big) = Q_{V_\alpha}\left(\frac{e^{x(|V_\alpha|-1)}}{\sqrt{1-2x}}\right), \tag{11.53}$$

as we can see by iterate application of Lemma 11.3, for example, by first multiplying all the factors $Q_{ij}(\cdot)$ on a sequence of edges corresponding to a leaf-construction of the tree, and finally multiplying all the factors $Q_i(\cdot)$ in whatever order, and, of course, recalling the Euler relation that, for a tree, $|V| = |E| + 1$.

We now see well how the statement of Theorem 11.4, involving the Grassmann integration of $\exp\big[\mathcal{S}(\bar\psi,\psi)\big]$ as defined in (11.41), has indeed a stronger formulation which holds for the integrand itself, even before performing the integral. For a forest F of the vertex-set V, define $P(F) \in \Pi(V)$ the partition corresponding to the vertex-sets of the connected components of F. We then have

$$\exp\big[\mathcal{S}(\bar\psi,\psi)\big] = \sum_{F\in\mathcal{F}(G)} \Big(\prod_{e\in E(F)} w_e\Big) \prod_{A\in P(F)} Q_A\left(\frac{e^{x(|A|-1)}}{\sqrt{1-2x}}\right). \tag{11.54}$$

This intermediate step will be useful when considering correlation functions, as in that case we have to multiply our operator *before* performing Grassmann integration, and a statement on the integral alone would have been insufficient at that purpose.

In order to prove the theorem, we have to prove that, given a set $V \equiv [k]$ of vertices of cardinality $k \geq 1$, the quantity

$$\mathcal{I}(n,k) := \int \mathcal{D}_{n\times[k]}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}}) Q_{[k]}\left(\frac{e^{x(k-1)}}{\sqrt{1-2x}}\right) \tag{11.55}$$

is independent from k and given by

$$\mathcal{I}(n,k) = (2n-1)!!\,. \tag{11.56}$$

We will equivalently prove, for every $k \geq 1$, a statement on the generating function:

$$\sum_n \frac{x^n}{n!}\mathcal{I}(n,k) = \sum_n \frac{x^n}{n!}(2n-1)!! = \frac{1}{\sqrt{1-2x}}\,. \tag{11.57}$$

Call $\tilde c(h,k)$ the coefficients of the generating function

$$\frac{e^{x(k-1)}}{\sqrt{1-2x}} = \sum_h \frac{x^h}{h!}\tilde c(h,k)\,, \tag{11.58}$$

and recall that

$$\int \mathcal{D}_{[k]}(\psi,\bar\psi)\tau_{[k]} = 1\,; \qquad\qquad \int \mathcal{D}_{[k]}(\psi,\bar\psi) f_{[k]} = 1-k\,. \tag{11.59}$$

This gives

$$\begin{aligned}\int \mathcal{D}_{n\times[k]}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}}) Q_{[k]}\left(\frac{e^{x(k-1)}}{\sqrt{1-2x}}\right) &= \sum_{S\subseteq[n]} \tilde c(|S|,k)\,(1-k)^{n-|S|} \\ &= \sum_{h=0}^{n}\binom{n}{h}\tilde c(h,k)\,(1-k)^{n-h}\,,\end{aligned} \tag{11.60}$$

so that, in generating function,

$$\sum_n \frac{x^n}{n!}\mathcal{I}(n,k) = \sum_n \frac{x^n}{n!}\sum_{h=0}^{n}\binom{n}{h}\tilde{c}(h,k)\,(1-k)^{n-h}$$
$$= \sum_h \frac{x^h}{h!}\tilde{c}(h,k)\;\sum_\ell \frac{x^\ell}{\ell!}\,(1-k)^\ell = \frac{e^{x(k-1)}}{\sqrt{1-2x}}\,e^{-x(k-1)} = \frac{1}{\sqrt{1-2x}}\,, \quad (11.61)$$

as was to be proven. □

11.6 Proof of the lemmas

We start by giving a proof of Lemma 11.3. We recall the rules of the algebra of f_A's and τ_A's, for generic sets A, which are discussed in Section 8.3. Indeed, as we said above, the lemma itself is the natural OSP(1|2n) generalization of the three equations (8.24, 8.29, 8.32), in the case $\lambda = 1$.

Componentwise, we get

$$|V_1 \cap V_2| = 1: \qquad f^a_{V_1} f^a_{V_2} = f^a_{V_1\cup V_2} \qquad f^a_{V_1}\tau^a_{V_2} = \tau^a_{V_1\cup V_2} \qquad \tau^a_{V_1}\tau^a_{V_2} = 0 \quad (11.62)$$
$$|V_1 \cap V_2| \geq 2: \qquad f^a_{V_1} f^a_{V_2} = 0 \qquad f^a_{V_1}\tau^a_{V_2} = 0 \qquad \tau^a_{V_1}\tau^a_{V_2} = 0 \quad (11.63)$$

So, the second part of the lemma (i.e. when $|V_1 \cap V_2| \geq 2$) is obvious, while for the first part we get

$$\Big(\sum_{S_1\subseteq[n]} c_1(|S_1|)\prod_{a\in S_1}\tau^a_{V_1}\prod_{a\in\overline{S_1}} f^a_{V_1}\Big)\Big(\sum_{S_2\subseteq[n]} c_2(|S_2|)\prod_{a\in S_2}\tau^a_{V_2}\prod_{a\in\overline{S_2}} f^a_{V_2}\Big)$$
$$= \sum_{\substack{S_1,S_2\\ S_1\cap S_2=\varnothing}} c_1(|S_1|)c_2(|S_2|)\prod_{a\in S_1\cup S_2}\tau^a_{V_1\cup V_2}\prod_{a\in\overline{S_1\cup S_2}} f^a_{V_1\cup V_2} \quad (11.64)$$
$$= \sum_S \Big(\sum_{\substack{S_1,S_2\\ S_1\cap S_2=\varnothing\\ S_1\cup S_2=S}} c_1(|S_1|)c_2(|S_2|)\Big)\prod_{a\in S}\tau^a_{V_1\cup V_2}\prod_{a\in\overline{S}} f^a_{V_1\cup V_2}\,.$$

We just have to prove that the numerical coefficients satisfy the product rule on the generating functions, as described in the lemma. Any set of cardinality k can be partitioned in two sets of cardinality h and $k-h$ in $\binom{k}{h}$ ways. So we can write for the new coefficients $c_{12}(k)$

$$c_{12}(k) = \sum_{h=0}^{k}\binom{k}{h}c_1(h)c_2(k-h)\,, \quad (11.65)$$

i.e., in generating function,

$$g_{12}(x) = \sum_k \frac{x^k}{k!}c_{12}(k) = \sum_k \frac{x^k}{k!}\sum_{h=0}^{k}\binom{k}{h}c_1(h)c_2(k-h)$$
$$= \sum_h \frac{x^h}{h!}c_1(h)\;\sum_\ell \frac{x^\ell}{\ell!}c_2(\ell) = g_1(x)g_2(x)\,, \quad (11.66)$$

as was to be proven. □

Now we prove together the Lemmas 11.1 and 11.2. The reasoning goes through a "strange" induction, which mixes the power k of $(1-\mathbf{n}_i\cdot\mathbf{n}_j)^k$ with the dimension n of the complex Grassmann Algebra. We write

$$F(\mathbf{n})|_{(1|2n)} \tag{11.67}$$

for an expression of vector fields $\mathbf{n}$ on the $\mathbb{R}^{1|2n}$ supersphere. For example, we can define

$$X(n,\kappa) := \left.\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)^{n+\kappa}}{(n+\kappa)!}\right|_{(1|2n)}, \tag{11.68}$$

and the two statements in the lemmas are $X(n,1)=0$ and $X(n,0)=Q_{ij}(e^x\sqrt{1-2x})$ respectively. The step zero of the induction is the already studied case of OSP(1|2), where $X(1,0)=f_{ij}$ (in accord with the the lemma, as $c(0)=1$ and $c(1)=0$), and $X(1,\kappa)=0$ for $\kappa\geq 1$, while the relevant object in the induction step is the quantity

$$A := (1-\mathbf{n}_i\cdot\mathbf{n}_j)|_{(1|2n+2)} - (1-\mathbf{n}_i\cdot\mathbf{n}_j)|_{(1|2n)} \tag{11.69}$$

An explicit calculation gives

$$A = \left(\frac{\phi_j}{\phi_i}-1\right)\tau_i^{n+1} + \left(\frac{\phi_i}{\phi_j}-1\right)\tau_j^{n+1} + \left(1-\frac{1}{\phi_i\phi_j}\right)\tau_{ij}^{n+1} + f_{ij}^{n+1} \tag{11.70}$$

where $\phi=\sqrt{1-2\sum_{a=1}^n\bar\psi^a\psi^a}$ is the value of the bosonic field in the $(1|2n)$ theory.

Remarkably, A is nilpotent of degree 2, instead of the naïve degree 3, as

$$A^2 = 2\tau_{ij}^{n+1}\left[\left(\frac{\phi_j}{\phi_i}-1\right)+\left(\frac{\phi_i}{\phi_j}-1\right)+\left(\frac{\phi_j}{\phi_i}-1\right)\cdot\left(\frac{\phi_i}{\phi_j}-1\right)\right] = 0\,. \tag{11.71}$$

This shows that

$$\begin{aligned} X(n+1,\kappa) &= \left.\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)^{n+1+\kappa}}{(n+1+\kappa)!}\right|_{(1|2n+2)} = \left.\frac{\big((1-\mathbf{n}_i\cdot\mathbf{n}_j)+A\big)^{n+1+\kappa}}{(n+1+\kappa)!}\right|_{(1|2n)} \\ &= X(n,\kappa+1) + A\,X(n,\kappa) = X(n,\kappa)\left(A+\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)|_{(1|2n)}}{n+\kappa+1}\right). \end{aligned} \tag{11.72}$$

This proves immediately by induction in n Lemma 11.1 ($\kappa=1$), and will prove also Lemma 11.2 ($\kappa=0$) after some extra effort. First notice that, by using Lemma 11.1, we can simply write for the $\kappa=0$ case of equation (11.72)

$$X(n+1,0) = A\,X(n,0)\,. \tag{11.73}$$

This already gives some striking constraints in the direction of the lemma. Indeed, the most general expression for $X(n,0)$ compatible with the symmetry was

$$X(n,0) = \sum_{\substack{(S_0,S_1,S_2,S_3,S_4)\\ \text{partitions of } [n]}} c(|S_0|,|S_1|,|S_2|,|S_3|,|S_4|)\prod_{a\in S_0}\tau_{ij}^a\prod_{a\in S_1}\tau_i^a\prod_{a\in S_2}\tau_j^a\prod_{a\in S_3}f_{ij}^a\,, \tag{11.74}$$

while the lemma states that $c(|S_0|,|S_1|,|S_2|,|S_3|,|S_4|)$ is zero unless $S_1=S_2=S_4=\varnothing$, does not depend explicitly on $|S_3|$, and gives a generating function for the coefficients, in the single parameter left $|S_0|$. We now justify these claims.

Equation (11.73) states that $c(|S_0|,|S_1|,|S_2|,|S_3|,|S_4|)$ is zero unless $S_4 = \varnothing$, because A has non-zero content of fermionic fields on the $(n+1)$-th component, and that $c(|S_0|,|S_1|,|S_2|,|S_3|,|S_4|)$ does not depend explicitly on $|S_3|$, because the coefficient of f_{ij}^{n+1} in A is 1.

Proving that S_1 (and, symmetrically, S_2) must be empty corresponds in the induction to prove that

$$\left(\frac{\phi_j}{\phi_i} - 1\right) \sum_S c(|S|) \prod_{a\in S} \tau_{ij}^a \prod_{a\in\overline{S}} f_{ij}^a = 0\,. \tag{11.75}$$

This is obvious if one uses Lemma 11.3 to write the expression above as

$$\left(Q_j(\sqrt{1-2x})Q_i\left(\frac{1}{\sqrt{1-2x}}\right) - 1\right) Q_{ij}(\hat{c}(x)) = Q_{ij}(\hat{c}(x)) - Q_{ij}(\hat{c}(x)) = 0 \tag{11.76}$$

or more directly through antisymmetry in $i \leftrightarrow j$ if one restates $Q_{ij}(\hat{c}(x))$ through (11.40a).

Finally, one has to prove that also the coefficient of A proportional to τ_{ij}^{n+1} behaves as claimed in the lemma. The thesis of the lemma states

$$X(n+1,0) = f_{ij}^{n+1}(\cdots) + \tau_{ij}^{n+1} \sum_{S\subseteq[n]} c(|S|+1) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a \tag{11.77}$$

and the coefficient of τ_{ij}^{n+1} must coincide with the one obtained through (11.73), i.e.

$$\sum_{S\subseteq[n]} c(|S|+1) \prod_{a\in S} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S} f_{ij}^a = \left(1 - \frac{1}{\phi_i\phi_j}\right) \sum_{S'\subseteq[n]} c(|S'|) \prod_{a\in S'} \tau_{ij}^a \prod_{a\in[n]\smallsetminus S'} f_{ij}^a \tag{11.78}$$

On the right side, using Lemma 11.3 and the expression for $\hat{c}(x)$ one gets

$$Q_{ij}\left(e^x\sqrt{1-2x} - \frac{e^x}{\sqrt{1-2x}}\right) \tag{11.79}$$

On the left side, the shift of the coefficient has an easy counterpart in generating function

$$\sum_k c(k+1)\frac{x^k}{k!} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_k c(k)\frac{x^k}{k!} = \frac{\mathrm{d}}{\mathrm{d}x}\hat{c}(x)\,. \tag{11.80}$$

The proof is completed by realizing that our $\hat{c}(x)$ satisfies the induced differential equation, i.e. that

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x\sqrt{1-2x} = e^x\sqrt{1-2x} - \frac{e^x}{\sqrt{1-2x}} \tag{11.81}$$

and satisfies the OSP(1|2) initial condition $\hat{c}(0) = 1$ and $\hat{c}'(0) = 0$. □

11.7 Correlation functions

The proof of the theorem performed in Section 11.5 has some underlying robustness which allows with small effort to understand combinatorially the expectation values of a product of fermionic fields. Given a monomial in the fields

$$\mathcal{O} = \prod_{a=1}^{n} \left(\bar{\psi}_{i_1(a)}^a \psi_{j_1(a)}^a \cdots \bar{\psi}_{i_{k(a)}(a)}^a \psi_{j_{k(a)}(a)}^a\right) \tag{11.82}$$

it is interesting to calculate the expectation value

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}})\, \mathcal{O}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi})\, \exp\big(\mathcal{S}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi})\big)}{\int \mathcal{D}(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}})\, \exp\big(\mathcal{S}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi})\big)} \tag{11.83}$$

where the action $\mathcal{S}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi})$ is the one described in equation (11.41). Conversely, it is not useful to evaluate expectations of monomials in a different form, as they are zero at sight because of the colour-wise "charge symmetry" in the action (i.e., the fact that fermions only enter in pairs $\bar\psi^a_i \psi^a_j$ in the action, requires that in $\mathcal{O}$ there are as many $\bar\psi^a$'s as ψ^a's, separately for each colour a).

Also in the presence of the observable $\mathcal{O}$, the factorization (11.52) and the consequences of Lemma 11.3 hold as well. Then, as $\mathcal{O}$ is factorized on the single sites, nothing prevents from factorizing the contributions of the Q's on the components of the forest: up to rearranging the fields in $\mathcal{O}$ (and in case taking a minus sign overall), one thus has a factor as in (11.53) per component, times a monomial in fermionic fields.

Now we are ready for fermionic integration. If we expand $Q_{V_\gamma}(\cdots)$ according to its definition, we have a linear combination of a product of τ^a's and f^a's, which, term by term, allow to factorize the integration on colours a. This is still valid in presence of $\mathcal{O}$, which is fully factorized. From here we already see a consequence: according to the algebra of τ's and f's, we can have non-zero integral only if, for each colour a and each forest component $T_\gamma = (V_\gamma, E_\gamma)$, we have either no fields $\bar\psi^a_i$ and ψ^a_j with $i, j \in V_\gamma$, or exactly one $\bar\psi^a_i$ and one ψ^a_j for some $i, j \in V_\gamma$. Call $R_\gamma \subseteq [n]$ the set of colours a for which the latter event occurs on component γ, and r_γ its cardinality.

Putting in formulas the arguments above, we have that, for any colour,

$$\int \mathcal{D}_V(\psi, \bar\psi) \tau_V = 1\,; \qquad \int \mathcal{D}_V(\psi, \bar\psi)\, \bar\psi_i \psi_j \tau_V = 0\,; \tag{11.84}$$

$$\int \mathcal{D}_V(\psi, \bar\psi) f_V = 1 - |V|\,; \qquad \int \mathcal{D}_V(\psi, \bar\psi)\, \bar\psi_i \psi_j f_V = 1\,; \tag{11.85}$$

so, in the expansion of Q

$$Q_{V_\gamma}\left(\frac{e^{x(|V_\gamma|-1)}}{\sqrt{1-2x}}\right) = \sum_{S \subseteq [n]} \tilde c(|S|, |V_\gamma|) \prod_{a \in S} \tau^a_{V_\gamma} \prod_{a \in [n] \smallsetminus S} f^a_{V_\gamma} \tag{11.86}$$

only sets S such that $S \cap R_\gamma = \varnothing$ can contribute (they provide a factor f, which multiplied by $\bar\psi_i \psi_j$ gives a τ, instead of a factor τ, which multiplied by $\bar\psi_i \psi_j$ makes zero). That is, for $a \in R_\gamma$, calling $i(a,\gamma)$ and $j(a,\gamma)$ the i and j indices in V_γ such that $\bar\psi^a_i$ and ψ^a_j are in $\mathcal{O}$, one has

$$\Big(\prod_{a \in R_\gamma} \bar\psi^a_{i(a,\gamma)} \psi^a_{j(a,\gamma)}\Big) Q_{V_\gamma}\left(\frac{e^{x(|V_\gamma|-1)}}{\sqrt{1-2x}}\right) = \sum_{S \subseteq [n] \smallsetminus R_\gamma} \tilde c(|S|, |V_\gamma|) \prod_{a \in S \cup R_\gamma} \tau^a_{V_\gamma} \prod_{\substack{a \in [n] \smallsetminus \\ (S \cup R_\gamma)}} f^a_{V_\gamma} \tag{11.87}$$

The integration over colours in R_γ is trivial and just gives a factor 1, while the integration over remaining colours is analogous to the case with no observables, but with n replaced by $n - r_\gamma$, and thus produces a relative factor

$$\frac{(2(n-r_\gamma)-1)!!}{(2n-1)!!}\,. \tag{11.88}$$

Such a factor did not appear explicitly in our treatment of the OSP(1|2) theory just because in that case n is 1 and r_γ is either 0 or 1, so that we were dealing with $1! = (-1)! = 1$.

A n-uple $\boldsymbol{\pi} = \{\pi^{(a)}\}$ of permutations in $\otimes_a \mathcal{S}_{k(a)}$ identifies a pairing of fermions $\bar\psi_i^a$ to ψ_j^a in $\mathcal{O}$. If $W \subseteq V(G)$ is the set of indices i, j appearing overall in $\mathcal{O}$, we will concentrate on the set $\mathit{\Pi}(W)$ of partitions of W (call P a partition in this set).

The properties of a given forest F relevant to the expectation value of our fermionic operators, (namely, the parameters r_γ and the constraints on the marking of vertices), are fully encoded in its underlying connectivity pattern $P(F)$ over W (that is, the partitition $P(F) \in \mathit{\Pi}(V)$ associated to the components of F, restricted to set $W \subseteq V$), so that forests sharing the same restricted connectivity pattern contribute with the same combinatorial factor in the expectation value of $\mathcal{O}$.

We call $\mathrm{prob}_W(P)$ the probability, w.r.t. our weighted measure for the spanning forests, that the connectivity pattern P over W occurs in a forest. Then the correlation function $\langle\mathcal{O}\rangle$ is a linear combination of these probabilities. Call also $\Theta_W(P,\mathcal{O})$ the function valued on 0 and 1 which implements the check that $\mathcal{O}$ restricted to each component of P and to each colour has either one $\bar\psi$ and one ψ or no fermionic factors. If $\Theta = 1$, then a single set of permutations, $\boldsymbol{\pi}$, exists which reorders the fields in $\mathcal{O}$ into $\prod_\gamma \Big(\prod_{a\in R_\gamma} \bar\psi^a_{i(a,\gamma)}\psi^a_{j(a,\gamma)}\Big)$ as required from our procedure. Then, with both $\boldsymbol{\pi}$ and $\{r_\gamma\}$ implicitly defined from P (and W defined from $\mathcal{O}$), we have

$$\langle\mathcal{O}\rangle = \sum_{P\in\mathit{\Pi}(W)} \mathrm{prob}_W(P)\ \Theta_W(P,\mathcal{O})\Big(\prod_a \epsilon(\pi^{(a)})\Big)\Big(\prod_\gamma \frac{(2(n-r_\gamma)-1)!!}{(2n-1)!!}\Big)\,. \tag{11.89}$$

Although this most general formula looks quite complicated, a few examples can illustrate how it is indeed effective. In particular, as a general recipe, chains of the form $\ldots\psi_i^a\bar\psi_i^b\ldots$ force large connectivities, and leave with a small number of summands in the linear combination above.

We can use a more compact notation for probabilities of connectivity patterns, e.g.

$$[xy|z|w\ldots] = \mathrm{prob}_{\{x,y,z,w,\ldots\}}\left(\{\{x,y\},\{z\},\{w,\ldots\}\ldots\}\right). \tag{11.90}$$

The first example of application of (11.89) is, for $n \geq 2$,

$$\left\langle\bar\psi_x^1\psi_y^1\bar\psi_y^2\psi_z^2\right\rangle_{(1|2n)} = \frac{[xyz]}{(2n-1)(2n-3)}\,; \tag{11.91}$$

where the factor $1/((2n-1)(2n-3))$ comes from the fact that the component with $\{x,y,z\}$ has $r=2$, and thus a factor $(2n-5)!!/(2n-1)!!$.

Another example for $n \geq 2$ is the four-point function

$$\left\langle\bar\psi_x^1\psi_y^1\bar\psi_z^2\psi_u^2\right\rangle_{(1|2n)} = \frac{[xy|zu]}{(2n-1)^2} + \frac{[xyzu]}{(2n-1)(2n-3)}\,; \tag{11.92}$$

which, combined with the customary expression available also in OSP(1|2)

$$\left\langle\bar\psi_x^1\psi_y^1\bar\psi_z^1\psi_u^1\right\rangle_{(1|2n)} = \frac{[xy|zu]}{(2n-1)^2} - \frac{[xu|zy]}{(2n-1)^2}\,; \tag{11.93}$$

gives the probabilistic identity, valid for $n \geq 2$,

$$\left\langle\bar\psi_x^1\left(\psi_y^1\bar\psi_z^2\psi_u^2 + \psi_y^2\bar\psi_z^2\psi_u^1 - \psi_y^1\bar\psi_z^1\psi_u^1\right)\right\rangle_{(1|2n)} = 0\,. \tag{11.94}$$

Given these motivational examples, the complicated expression (11.89) can be manipulated in order to prove the more general theorem

Theorem 11.6. *For any finite set W and $P \in \Pi(W)$ with $K(P)$ components, there exists a value n, a finite set of operators $\{\mathcal{O}_\xi\}$ in the Grassmann algebra on $W \times [n]$ and of coefficients $c_\xi(n')$ such that*

$$\left\langle \sum_\xi c_\xi(n')\mathcal{O}_\xi \right\rangle_{(1|2n')} = \mathrm{prob}_W(P) \tag{11.95}$$

for each $n' \geq n$. In our proof we get

$$n(P) = \left\lceil \frac{|W| - K(P) + 1}{2} \right\rceil . \tag{11.96}$$

Clearly, if P is composed of all atomic sets, $P = \{\{i_1\}, \dots, \{i_{|W|}\}\}$, already in OSP(1|2) it is sufficient to take $\mathcal{O} = \prod_{i \in W} \bar\psi_i \psi_i$, and also, given a partition $P \in \Pi(W)$ with no atomic sets, and a $P' \in \Pi(W \cup W')$ with all sites in W' atomic in P', and $P'|_W = P$, one has a proper operator for P' from the one of P by including also the factor $\big((2n-1)!!\big)^{|W'|} \prod_{i \in W'} \prod_{a=1}^n \bar\psi_i^a \psi_i^a$ to all monomials $\mathcal{O}_\xi$. Our formula (11.96), although not optimal, behaves well under this respect: it exactly predicts $n = 1$ regardless from $|W|$ in the case of all atomic sets, and the contribution of atomic sets to $n(P)$ cancels between $|W|$ and $K(P)$.

The statement of the theorem is equivalent to say that for every P as above we can exhibit an operator $\mathcal{O}(P)$ in the Grassmann algebra over $W \times [n]$ (with n given by (11.96)) such that, for $n' \geq n$,

$$\langle \mathcal{O}(P) \rangle_{(1|2n')} = \sum_{P'} c(n'; P', P)\, \mathrm{prob}_W(P)\,, \tag{11.97}$$

where $c(n'; P, P) \neq 0$ for each $n' \geq n$, and the $P' \in \Pi(W)$ appearing above are, besides P, such that $K(P') > K(P)$. Indeed, if this holds, one can build recursively a combination as in (11.95), because the related linear system is triangular, and $n(P)$ is decreasing in $K(P)$.

So here we prove this equivalent statement. We will exhibit an operator $\mathcal{O}(P)$ of degree $nK(P)$ both in $\bar\psi$ and in ψ, so that no P' can appear in the combination with $K(P') < K(P)$ and, if $c(n'; P', P) \neq 0$ for a P' with $K(P') = K(P)$, it must be that $r_\gamma = n$ for each component γ of P'. From this starting point we will prove that such a P' must coincide with P, thus concluding the proof.

Say $P = \{\{i_1^1, \dots, i_{\ell(1)}^1\}, \dots, \{i_1^K, \dots, i_{\ell(K)}^K\}\}$, with $K = K(P)$. Call $L = \sum_\gamma (\ell(\gamma) - 1) = |W| - K(P)$. Consider L vectors $\boldsymbol{V} = \{\mathbf{v}(i_\mu^\gamma)\}$ on $\{0,1\}^{2n}$ (with labels $\gamma = 1, \dots, K$ and $\mu = 1, \dots, \ell(\gamma) - 1$) such that

(a) $\mathbf{v}(i_{\ell(\gamma)}^\gamma) := (1, \dots, 1) - \sum_{\mu=1}^{\ell(\gamma)-1} \mathbf{v}(i_\mu^\gamma)$ is valued in $\{0,1\}^{2n}$ for each γ;
(b) In the set of $\boldsymbol{V} \cup \{\mathbf{v}(i_{\ell(\gamma)}^\gamma)\}$ there are no other combinations with coefficients in $\{0,1\}$, besides the K ones implicit above, such that the result is $(1, \dots, 1)$.

Such a set of vectors identify an operator $\mathcal{O}$ by putting in correspondence the $2n$ vector components with the $2n$ fermions $\bar\psi^1$, ψ^1, …, $\bar\psi^n$, ψ^n, and setting

$$\mathcal{O} = (\pm 1) \prod_{i \in W} \prod_{a=1}^n \bar\psi_i^{v_{2a-1}(i)} \psi_i^{v_{2a}(i)}\,. \tag{11.98}$$

Then, by construction (the property on point 1), $\mathcal{O}$ has $r_\gamma = n$ on each component of P. A sufficient condition to our purposes is the one of excluding that there exists a set

B not in the partition P such that the product of fermions with index in B has exactly one factor $\bar\psi^a$ and one ψ^a for each $a \in [n]$ (this condition is sufficient because, by the arguments above, we can restrict our attention to the case $r_\gamma = n$). This is exactly the statement of point 2.

Instead of requiring the property at point 2, we require the stronger but easier statement that no non-trivial linear combinations in $GF(2)$ of vectors in W give $(0,\dots,0)$ or $(1,\dots,1)$, i.e. that the matrix having all the vectors in $\boldsymbol{V}$, plus $(1,\dots,1)$, has maximal rank $L+1$. Clearly, this can be done if and only if $2n \ge L+1 = |W| - K(P) + 1$, i.e. the relation (11.96). We should prove that such a matrix $\boldsymbol{V}$ can be chosen also in order to satisfy point 1. The canonical basis on the first L components (of the $2n$) is a suitable choice. □

We should say that the result of Theorem 11.6 is quite far from optimal, and is presented only in order to give a flavour of how the OSP(1|2n) theories with increasing n asymptotically cover the whole set of probabilistic observables, in terms of connectivity patterns among k points on the lattice. An exception is the extreme case in which the partition has a single set, $P = \{W\}$: instead of having $n = |W| - 1$, as in the naïve choice $\mathcal{O} = \bar\psi_1^1\psi_2^1\bar\psi_2^2\psi_3^2\cdots\bar\psi_n^n\psi_{n+1}^n$, we have the asymptotically better behaviour $n = \lceil |W|/2 \rceil$, which is maybe not optimal only by 1, as suggested by direct investigation of the cases $n = 1$ and $n = 2$, where "short" optimal operators are

$$\mathcal{O}_{|W|=3} = (2n-1)!!\,\big(1 - (2n-1)\bar\psi_1^1\psi_1^1\big)\,\bar\psi_2^1\psi_3^1\,; \tag{11.99}$$

$$\begin{aligned}\mathcal{O}_{|W|=5} = (2n-1)!!\,\big((1-(2n-1)\bar\psi_1^1\psi_1^1)(1-(2n-1)\bar\psi_2^1\psi_2^1)\\ + (2n-1)(2n-3)\bar\psi_1^1\psi_2^1\bar\psi_1^2\psi_2^2\big)\,\bar\psi_3^1\psi_4^1\bar\psi_4^2\psi_5^2\,.\end{aligned} \tag{11.100}$$

We have a far better control than in Theorem 11.6 for partitions at the other extremum: the ones in which each component is either a dimer or a monomer. Indeed, in this case we can prove

Theorem 11.7. *In the conditions of Theorem 11.6, with P having L sets of size 2 and all other sets of size 1, and*

$$L \le 2^{n-1}(2^n - 1)\,, \tag{11.101}$$

there esists an operator $\mathcal{O}$ in the Grassmann algebra on $W \times [n]$ such that for $n' \ge n$

$$\langle \mathcal{O} \rangle_{(1|2n')} = \left(\frac{(2n'-2n-1)!!}{(2n'-1)!!}\right)^{K(P)} \mathrm{prob}_W(P)\,. \tag{11.102}$$

The proof of Theorem 11.7 is done mostly in the case in which P has no monomers, the generalization following trivially at the end. So we write $P = \{\{i_1, j_1\},\dots,\{i_L, j_L\}\}$. Consider the partitions of $[n]$ into four sets $(S_1,\dots,S_4)$, in case singular (i.e. with some empty set), but such that $1 \in S_1 \cup S_2$, and $|S_2| + |S_3| \ge 1$. The number of distinct such partitions is $2^{n-1}(2^n-1)$, as can be seen with elementary combinatorics. For a choice of L distinct such partitions $\{(S_1(\gamma),\dots,S_4(\gamma))\}_{\gamma=1,\dots,L}$, a proper operator $\mathcal{O}$ realizing (11.102) is the following

$$\mathcal{O} = \prod_{\gamma=1}^{L}\Big(\prod_{a\in S_1(\gamma)}(\bar\psi_{i_\gamma}^a\psi_{i_\gamma}^a)\prod_{a\in S_2(\gamma)}(\bar\psi_{i_\gamma}^a\psi_{j_\gamma}^a)\prod_{a\in S_3(\gamma)}(\bar\psi_{j_\gamma}^a\psi_{i_\gamma}^a)\prod_{a\in S_4(\gamma)}(\bar\psi_{j_\gamma}^a\psi_{j_\gamma}^a)\Big)\,. \tag{11.103}$$

It is easily seen that the coefficient with whom P appears in the combination (11.89) is the one claimed in (11.102), and we now go and prove that no other partitions $P \in \Pi(W)$ have $\Theta_W(P,\mathcal{O}) \neq 0$.

Indeed, $\Theta \neq 0$ only if, for each $A \in P$ and each $a \in [n]$, we have either a $\bar{\psi}_i^a \psi_j^a$ for $i, j \in A$, or no fermions with index a on sites of A, so $r_A = \deg_A(\bar{\psi}) = \deg_A(\psi) \leq n$.

But no singletons $\{i\}$ are possible in P, as either S_2 or S_3 are non-empty for each index, and thus in a singleton there would be unpaired fermions. As $|W| = 2L$, P is composed of at most L sets, and exactly of L sets only if P is composed only of pairs.

As the degree of $\mathcal{O}$ is $2Ln$, and P cannot have any set of size 1, P cannot have either any set of size larger than 2, because otherwise there would be some set A with $r_A > n$, which is impossible as $\mathcal{O}$ has only components a up to n. The only partitions that survive are the pairings of W, and such that every pair has $r_A = n$.

Only $\{i_\alpha, j_\beta\}$ can be paired, because $\bar{\psi}^1$ appears in all i_α, so that $\{i_\alpha, i_\beta\}$ is not allowed (and there are as many i's as j's).

Furthermore, at sight, no $\{i_\alpha, j_\beta\}$ can be paired unless $S_2(\alpha) = S_3(\beta)$ and $S_3(\alpha) = S_2(\beta)$, otherwise there would be unpaired fermions.

Also, in this case we get for the fermions on our pair

$$\prod_{a\in S_1(\alpha)} (\bar{\psi}^a\psi^a) \prod_{a\in S_4(\beta)} (\bar{\psi}^a\psi^a) \prod_{a\in S_2(\alpha)} \bar{\psi}^a \prod_{a\in S_3(\beta)} \bar{\psi}^a \prod_{a\in S_2(\beta)} \psi^a \prod_{a\in S_3(\alpha)} \psi^a$$
$$= (\pm 1) \prod_{a\in S_1(\alpha)} (\bar{\psi}^a\psi^a) \prod_{a\in S_4(\beta)} (\bar{\psi}^a\psi^a) \prod_{a\in S_2(\alpha)\cup S_3(\alpha)} (\bar{\psi}^a\psi^a) \quad (11.104)$$

So, as we need that the three sets $S_1(\alpha)$, $S_4(\beta)$ and $S_2(\alpha) \cup S_3(\alpha)$ above are all distinct, and that their union is $[n]$, we conclude that $S_4(\beta) = [n] \setminus (S_1(\alpha) \cup S_2(\alpha) \cup S_3(\alpha))$, and thus coincides with $S_4(\alpha)$. Similarly $S_1(\alpha) = S_1(\beta)$. So the partitions with index α and β coincide, and, as the partitions in our operator $\mathcal{O}$ are all different, we conclude that $\alpha = \beta$.

The procedure above generalizes immediately to P composed of sets of size 1 or 2. It suffices, for a singleton $\{\ell\} \in P$, to add a factor $\prod_{a=1}^n \bar{\psi}_\ell^a \psi_\ell^a$ to $\mathcal{O}$. □

We want to stress here an implication of the theorem we have just proven: in the OSP(1|2n) theory we have precise control on the expectation value of pairwise connectivity patterns, up to a number of pairs k which scales *exponentially* with n. While in the OSP(1|2) theory a single pair is accessible, through $\langle \bar{\psi}_i \psi_j \rangle$, and already $\langle \bar{\psi}_i \psi_j \bar{\psi}_{i'} \psi_{j'} \rangle \propto [ij|i'j'] - [ij'|i'j]$, in the theory OSP(1|4) one can fit up to 6 pairs, through

$$\begin{aligned}\mathcal{O} = &(\bar{\psi}^1_{i_1}\psi^1_{i_1}\bar{\psi}^2_{i_1}\psi^2_{j_1})(\bar{\psi}^1_{i_2}\psi^1_{i_2}\bar{\psi}^2_{j_2}\psi^2_{i_2})(\bar{\psi}^1_{i_3}\psi^1_{j_3}\bar{\psi}^2_{i_3}\psi^2_{i_3}) \\ &\cdot (\bar{\psi}^1_{i_4}\psi^1_{j_4}\bar{\psi}^2_{i_4}\psi^2_{j_4})(\bar{\psi}^1_{i_5}\psi^1_{j_5}\bar{\psi}^2_{j_5}\psi^2_{i_5})(\bar{\psi}^1_{i_6}\psi^1_{j_6}\bar{\psi}^2_{j_6}\psi^2_{j_6}) \,,\end{aligned} \quad (11.105)$$

and in OSP(1|6) up to 28 pairs.

It is conceivable that the function $n(L)$ implicit in (11.101) is optimal, but we do not try to prove it here. We do not discuss further these topics, and postpone the discussion to some more specific papers in preparation [140, 156].

11.8 On the quotient of Forest Algebra induced by OSP(1|2n) representation

Here we explain how to generalize the relation proven in Section 10.4.6 to scalar products $(1 - \mathbf{n}_i \cdot \mathbf{n}_j)/\lambda$ of vectors on the $RP^{1|2n}$ supersphere.

In particular, we need to reproduce the main step in the proof of that section, by showing the following proposition:

Proposition 11.8. *$\big((1-\mathbf{n}_i\cdot\mathbf{n}_j)-(1-\mathbf{n}_i\cdot\mathbf{n}_k)\big)/\lambda$ is linear in the $2n$ fermionic fields $\bar\psi^a_j-\bar\psi^a_k$ and $\psi^a_j-\psi^a_k$, for $1\le a\le n$.*

Given this statement, define again $R_{a|b_1b_2\cdots b_k|c}$ as

$$R_{a|b_1b_2\cdots b_k|c}=\prod_{i=1}^{k}\left(\frac{1-\mathbf{n}_a\cdot\mathbf{n}_{b_i}}{\lambda}-\frac{1-\mathbf{n}_{b_i}\cdot\mathbf{n}_c}{\lambda}\right). \tag{11.106}$$

Then we would get as a corollary that

$$R_{a|b_1b_2\cdots b_k|c}=0\qquad\text{for } k\ge 2n+1. \tag{11.107}$$

Proof of Proposition 11.8. Start by recalling that

$$\frac{1-\mathbf{n}_i\cdot\mathbf{n}_j}{\lambda}=\frac{1}{\lambda}(1-\phi_i\phi_j)-\sum_a(\bar\psi^a_i\psi^a_j+\bar\psi^a_j\psi^a_i) \tag{11.108}$$

where ϕ_i is a shortcut for $\sqrt{1-2\lambda\bar{\boldsymbol{\psi}}_i\boldsymbol{\psi}_i}$. Thus

$$-\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)-(1-\mathbf{n}_i\cdot\mathbf{n}_k)}{\lambda}=\frac{1}{\lambda}\phi_i(\phi_j-\phi_k)+\sum_a\left(\bar\psi^a_i(\psi^a_j-\psi^a_k)+(\bar\psi^a_j-\bar\psi^a_k)\psi^a_i\right). \tag{11.109}$$

The right-most summands have already the desired form described in the statement of the proposition, and we are only left with the proof that also the factor $\phi_j-\phi_k$ has this form. As $\phi_j+\phi_k$ is a (commuting) polynomial in Grassmann variables starting with 2 (i.e., it has a non-zero "body"), it is an invertible expression, and it is legitimate to write

$$\phi_j-\phi_k=\frac{\phi_j^2-\phi_k^2}{\phi_j+\phi_k} \tag{11.110}$$

(at this aim notice that ϕ_j and ϕ_k are commuting polynomials). Then, for $\phi_j^2-\phi_k^2$ we have explicitly

$$\frac{\phi_j^2-\phi_k^2}{\lambda}=-2\sum_a(\bar\psi^a_j\psi^a_j-\bar\psi^a_k\psi^a_k)=-\sum_a\big((\bar\psi^a_j-\bar\psi^a_k)(\psi^a_j+\psi^a_k)+(\bar\psi^a_j+\bar\psi^a_k)(\psi^a_j-\psi^a_k)\big) \tag{11.111}$$

This already proves our statement. However, it is interesting to recognize, in the expressions above, a structure similar to the one emerging in equation (10.69b), that is

$$\begin{aligned}\frac{(1-\mathbf{n}_i\cdot\mathbf{n}_j)-(1-\mathbf{n}_i\cdot\mathbf{n}_k)}{\lambda}&=\frac{1}{\phi_j+\phi_k}\\&\times\sum_a\big((\bar\psi^a_j-\bar\psi^a_k)((\psi^a_j+\psi^a_k)(\phi_j+\phi_k)-2\psi^a_i\phi_i)\\&\qquad+((\bar\psi^a_j+\bar\psi^a_k)(\phi_j+\phi_k))-2\bar\psi^a_i\phi_i)(\psi^a_j-\psi^a_k)\big).\end{aligned} \tag{11.112}$$

12

OSP(2|2) non-linear σ-model and dense polymers

It is well known, since the pionereeing work of De Gennes [157], that the $O(n)$ non-linear σ-model in the limit $n \to 0$ describes a theory of self-avoiding polymers. The combinatorial structure of this expansion is more deeply elucidated in [158].

The case of OSP$(2m|2m)$, for m integer, is expected to be analogous, because of Parisi-Sourlas mechanism [159]. Here we investigate the OSP(2|2) model on a generic weighted graph, with combinatorial techniques of Grassmann variables. The high-temperature limit is non-trivial, and describes a theory of dense polymers (also known as Hamiltonian paths, or, on a planar graph, Jordan curves).

Similarities and differences with the De Gennes theory (in the formulation of [158]), the $n \to 0$ limit of Nienhuis Loop Gas, and Spanning Trees when on a planar graph, and issues of universality, are discussed.

12.1 The model with OSP(2|2) symmetry

Now we specialize the notations above to the $RP^{2|2}$ supersphere:

$$\mathbf{n}_i = (\bar\phi_i, \phi_i, \bar\psi_i, \psi_i) = (r_i, \theta_i; \bar\psi_i, \psi_i)\,; \tag{12.1}$$

$$\mathbf{n}_i \cdot \mathbf{n}_j = r_i r_j \cos(\theta_i - \theta_j) + \bar\psi_i\psi_j + \bar\psi_j\psi_i\,; \tag{12.2}$$

$$|\mathbf{n}_i|^2 = r_i^2 + 2\bar\psi_i\psi_i\,, \tag{12.3}$$

where $\phi_j = r_j \exp(i\theta_j)$ and $\bar\phi_j = r_j \exp(-i\theta_j)$.

Denote $\theta_i - \theta_j$ as θ_{ij}. Define the formal integration measure over the supersphere

$$\mathrm{d}_{(2|2)}\mathbf{n} := \frac{\mathrm{d}\theta}{2\pi}\, r\mathrm{d}r\, \mathrm{d}\psi\mathrm{d}\bar\psi\, 2\delta(\mathbf{n}^2 - 1) = \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{d}(r^2)\, \mathrm{d}\psi\mathrm{d}\bar\psi\, \delta(r^2 - (1 - 2\bar\psi\psi))\,. \tag{12.4}$$

Solving w.r.t. the radius of the bosonic component, i.e. $r_i^2 = 1 - 2\bar\psi_i\psi_i$, gives

$$\int \mathrm{d}_{(2|2)}\mathbf{n}\, F(\mathbf{n}) = \int \frac{\mathrm{d}\theta}{2\pi}\mathrm{d}\psi\mathrm{d}\bar\psi\ F\big((r = 1 - \bar\psi\psi, \theta; \bar\psi, \psi)\big)\,. \tag{12.5}$$

Remark that there is no functional Jacobian from the integration of the delta function, but just a constant factor.[†] For $n > 1$, we do not have two determinations in the square root, as

[†]More generally, on OSP$(n|2m)$, we would have that $r = \sqrt{1 - 2\bar{\boldsymbol\psi}\boldsymbol\psi}$, and the Jacobian of a single-variable integration is

the radius must be "positive", i.e., in the expansion of fermionic variables, perturbatively connected to the branch containing $+1$ (and not the one containing -1). The role of the sign determination appearing in the OSP$(1|2m)$ theories is here played by the residual angular degree of freedom θ_i, and, more generally, by the $(n-1)$-dimensional manifold of angular degrees of freedom on a n-dimensional (real) sphere.

Consider a weighted graph $G=(V,E)$, with N vertices, and a set of (real or complex, symmetric) weights, with weight w_{ij} on edge (ij), thus $w_{ji}=w_{ij}$. The partition function for a nearest-neighbour quadratic interaction on this graph is

$$\begin{aligned} Z_G(w) &= \int \prod_{i\in V(G)} \mathrm{d}_{(2|2)}\mathbf{n}_i \exp\Big(\sum_{(ij)\in E(G)} w_{ij}\ \mathbf{n}_i\cdot\mathbf{n}_j\Big) \\ &= \int \prod_i \frac{\mathrm{d}\theta_i}{2\pi}\ e^{\sum_{(ij)} w_{ij}\cos\theta_{ij}} \int \mathcal{D}(\bar\psi,\psi) \\ &\quad\times \exp\Big(\sum_{(ij)} w_{ij}\ \big(\cos\theta_{ij}(-\bar\psi_i\psi_i-\bar\psi_j\psi_j+\bar\psi_i\psi_i\bar\psi_j\psi_j)+\bar\psi_i\psi_j+\bar\psi_j\psi_i\big)\Big)\,. \end{aligned} \tag{12.7}$$

We avoid certain tedious signs coming out, making the change of variables $\bar\psi_i\to-\bar\psi_i$, and $\psi_i\to\psi_i$, with Jacobian $(-1)^N$. We also forget about this overall sign, and reintroduce it only at the end.

$$\begin{aligned} Z_G(w) &= \int \prod_i \frac{\mathrm{d}\theta_i}{2\pi}\ e^{\sum_{(ij)} w_{ij}\cos\theta_{ij}} \int \mathcal{D}(\bar\psi,\psi) \\ &\quad\times \exp\Big(\sum_{(ij)} w_{ij}\ \big(\cos\theta_{ij}(\bar\psi_i\psi_i+\bar\psi_j\psi_j+\bar\psi_i\psi_i\bar\psi_j\psi_j)-\bar\psi_i\psi_j-\bar\psi_j\psi_i\big)\Big)\,. \end{aligned} \tag{12.8}$$

The angular integration is complicated by the fact that the variables of integration are $\{\theta_i\}_{i\in V}$, while the integrand actually depends on $\{\theta_{ij}\}_{(ij)\in E}$. As customary, we introduce the representation of a delta for angular variables,

$$\delta(\theta)=\sum_{m\in\mathbb{Z}}\exp(im\theta)\,, \tag{12.9}$$

which leaves us with "flow" configurations m

$$\begin{aligned} \int\prod_{i\in V}\frac{\mathrm{d}\theta_i}{2\pi}\ F\big(\{\theta_{ij}\}\big) &= \int\prod_{i\in V}\frac{\mathrm{d}\theta_i}{2\pi}\int\prod_{(ij)\in E}\frac{\mathrm{d}\theta_{ij}}{2\pi}\ \prod_{(ij)}\delta(\theta_{ij}-\theta_i+\theta_j)F\big(\{\theta_{ij}\}\big) \\ &= \int\prod_{i\in V}\frac{\mathrm{d}\theta_i}{2\pi}\sum_{m\in\mathbb{Z}^E}\int\prod_{(ij)\in E}\frac{\mathrm{d}\theta_{ij}}{2\pi}\ e^{\sum_{(ij)} im_{ij}(\theta_{ij}-\theta_i+\theta_j)}F\big(\{\theta_{ij}\}\big) \\ &= \sum_{\substack{m\in\mathbb{Z}^E\\ \nabla\cdot m=0}}\int\prod_{(ij)\in E}\frac{\mathrm{d}\theta_{ij}}{2\pi}\ e^{\sum_{(ij)} im_{ij}\theta_{ij}}F\big(\{\theta_{ij}\}\big) \end{aligned} \tag{12.10}$$

Summarizing up, the new variables $m=\{m_{ij}\}_{(ij)\in E}$, are an integer-valued vector field (and $m_{ij}=-m_{ji}$). It has zero divergence, i.e. $(\nabla\cdot m)_i=\sum_{j:(ij)\in E} m_{ij}=0$ for all vertices i. It can be represented by an integer number of "arrows" on the edges, such that at each vertex the flow is conserved. An example is given in Figure 12.1.

$$\int \mathrm{d}r\ r^{n-1}\,2\delta(r^2-(1-2\bar\psi\psi))=\frac{1}{n}\int \mathrm{d}(r^n)\ 2\delta((r^n)^{\frac{2}{n}}-(1-2\bar\psi\psi))=(1-2\bar\psi\psi)^{\frac{n}{2}-1}\,. \tag{12.6}$$

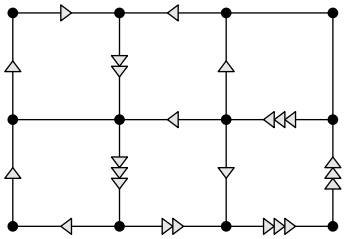


Fig. 12.1. An example of (divergence-free) flow configuration $\{m_e\}$ with zero divergence on a small graph. Although the graph is planar for simplicity, planarity plays no role in this paper.

So we have for expression (12.8)

$$\begin{aligned} Z_G(w) = \sum_{\substack{m\in\mathbb{Z}^E \\ \nabla\cdot m=0}} \int \prod_{(ij)\in E} \frac{\mathrm{d}\theta_{ij}}{2\pi}\, e^{\sum_{(ij)}(im_{ij}\theta_{ij}+w_{ij}\cos\theta_{ij})} \int \mathcal{D}(\bar\psi,\psi) \\ \times \exp\Big(\sum_{(ij)} w_{ij}\,\big(\cos\theta_{ij}(\bar\psi_i\psi_i+\bar\psi_j\psi_j+\bar\psi_i\psi_i\bar\psi_j\psi_j)-\bar\psi_i\psi_j-\bar\psi_j\psi_i\big)\Big)\,. \end{aligned} \tag{12.11}$$

At a given set of $\{\theta_{ij}, m_{ij}\}$, the integral in fermionic variables is of the form discussed in the Appendix of [136] (specialized to graphs), and in Section 8.2, and in the following we use results and notations from there.

We have an expansion over spanning rooted oriented subgraphs, in which each component is either a tree rooted on an edge, or an unicyclic rooted on its cycle.[‡]

This expansion is convenient in approaching (12.11), because, at any fixed value of $\{m_{ij}\}$ and of subgraph in the expansion, the dependence from θ_{ij} variables is factorized on the edges. So, as we see in the following, integrals over θ_{ij} are performed immediately in terms of some Bessel functions, and the combinatorial expansion trivializes integrations over *both* Grassmann and angular variables.

W.r.t. definitions in [136], the parameters in this case are

$$w^*_{ij} = w_{ij}\cos\theta_{ij}\,; \qquad w_{ij;i} = w_{ij;j} = w_{ij}\cos\theta_{ij}\,; \tag{12.12}$$

$$\overline{w}^*_{ij} = w_{ij}\cos\theta_{ij} + w^2_{ij}(\cos^2\theta_{ij}-1)\,; \qquad w_{ij;i,j} = w_{ij;j,i} = w_{ij}\,. \tag{12.13}$$

We recall the definition of the *modified Bessel function of the first kind*, $I_m(x)$, for $m\in\mathbb{Z}$ and $x\in\mathbb{C}$,

$$I_m(x) := \int \frac{\mathrm{d}\theta}{2\pi}\, e^{im\theta+x\cos\theta}\,. \tag{12.14}$$

We will use three properties. The first one is simply parity w.r.t. the index: $I_m(x) = I_{-m}(x)$. The second one is

$$\frac{\mathrm{d}}{\mathrm{d}x} I_m(x) = \int \frac{\mathrm{d}\theta}{2\pi}\,\cos\theta\, e^{im\theta+x\cos\theta} = \frac{1}{2}\big(I_{m+1}(x)+I_{m-1}(x)\big)\,; \tag{12.15}$$

and the third one comes from an integration by parts

[‡]No trees are rooted on vertices, as implicitly all mass parameters t_i are zero, this being a consequence of the fact that, as we have two bosonic components, radial integration does not produce any functional Jacobian.

$$\begin{aligned} 0 &= \int \frac{\mathrm{d}\theta}{2\pi}\, \frac{\mathrm{d}}{\mathrm{d}\theta} e^{im\theta + x\cos\theta} = i \int \frac{\mathrm{d}\theta}{2\pi} \left(m + x \frac{e^{i\theta} - e^{-i\theta}}{2} \right) e^{im\theta + x\cos\theta} \\ &= i\big(m I_m(x) + \tfrac{x}{2} I_{m+1}(x) - \tfrac{x}{2} I_{m-1}(x) \big)\,, \end{aligned} \tag{12.16}$$

which gives

$$m I_m(x) = -\frac{x}{2}\big(I_{m+1}(x) - I_{m-1}(x) \big)\,. \tag{12.17}$$

In the combinatorial expansion of the fermionic fields, still following the names in [136], we have edges of four kinds: the ones not chosen at all in the expansion ("unmarked edges"), the ones corresponding to "pointing edges", to "dashed edges", and to "rooted edges". Integration of variable θ_{ij} for the four corresponding factors gives

$$\int \frac{\mathrm{d}\theta}{2\pi}\, e^{im\theta + x\cos\theta} = I_m(x)\,; \qquad \text{(unmarked edge)} \tag{12.18a}$$

$$\int \frac{\mathrm{d}\theta}{2\pi}\, x\cos\theta\; e^{im\theta + x\cos\theta} = \frac{x}{2}\big(I_{m+1}(x) + I_{m-1}(x) \big)\,; \qquad \text{(pointing edge)} \tag{12.18b}$$

$$\int \frac{\mathrm{d}\theta}{2\pi}\, x\; e^{im\theta + x\cos\theta} = x I_m(x)\,; \qquad \text{(dashed edge)} \tag{12.18c}$$

$$\int \frac{\mathrm{d}\theta}{2\pi}\, \big(x\cos\theta + x^2(\cos^2\theta - 1) \big) e^{im\theta + x\cos\theta} = m^2 I_m(x)\,; \qquad \text{(rooted edge)} \tag{12.18d}$$

The case of the rooted edge has made use of some small algebra

$$\begin{aligned} \int \frac{\mathrm{d}\theta}{2\pi}\, x^2(\cos^2\theta - 1) e^{im\theta + x\cos\theta} &= \frac{x^2}{4}\big(I_{m+2}(x) - 2I_m(x) + I_{m-2}(x) \big) \\ &= -\frac{x}{2}\big((m+1) I_{m+1}(x) - (m-1) I_{m-1}(x) \big) \\ &= -\frac{x}{2}\big(I_{m+1}(x) + I_{m-1}(x) \big) + m^2 I_m(x)\,. \end{aligned} \tag{12.19}$$

We thus introduce a function $A_e : E(G) \to \{0, b, f, R\}$, describing the "decoration" of a valid subgraph $S \subseteq G$, such that $A_e = 0, b, f, R$ respectively if e is an edge not in S, or a pointed, dashed or rooted edge in S. [Letters b and f stand for "boson" and "fermion", and refer to the fact that a cycle of pointing edges has no "topological factor", while a cycle of dashed edges has a "topological minus sign".] We define the functions

$$f_{0;m}(x) = I_m(x)\,; \qquad f_{b;m}(x) = \frac{x}{2}\big(I_{m+1}(x) + I_{m-1}(x) \big)\,; \tag{12.20}$$

$$f_{f;m}(x) = x I_m(x)\,; \qquad f_{R;m}(x) = m^2 I_m(x)\,. \tag{12.21}$$

Then, the partition function is written in terms of pairs of flow-subgraph configurations $(m; (S, A))$:

$$Z_G(w) = \sum_{\substack{m \in \mathbb{Z}^E \\ \nabla \cdot m = 0}} \sum_{(S,A) \subseteq G} W(m; S, A)\,; \tag{12.22}$$

$$W(m; S, A) = \prod_{e \in E(G)} f_{A_e, m_e}(w_e)\,. \tag{12.23}$$

Remark that there is no ambiguity from the fact that m is antisymmetric ($m_{ij} = -m_{ji}$), because all four functions $f_{A,m}(x)$ are symmetric, $f_{A,m}(x) = f_{A,-m}(x)$. A typical configuration $(m; (S, A))$ is depicted in figure 12.2.

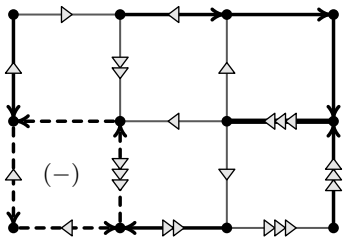


Fig. 12.2. A typical configuration $(m;(S,A))$ on a small graph. Both the graph and the flow variables are the same as in figure 12.1. If all edge weights are w, the weight of this configuration is $W(m;S,A) = -\frac{9}{16}w^{10}I_0I_1^8I_2I_3^3\big(I_0+I_2\big)^2\big(I_1+I_3\big)\big(I_2+I_4\big)$. All I_k are $I_k(w)$.

The configuration A for a given valid S is strongly constrained: any unicyclic component has everything fixed except for two independent binary choices, for the direction of the arrows on the cycle, and having the cycle dashed or not, while any tree component can be rooted on each of its edges (but the weight vanishes if the candidate root edge e has $m_e = 0$). Summation over these choices for A leaves us with an effective weight $\widehat{W}(m;S)$ for valid subgraphs S

$$Z_G(w) = \sum_{\substack{m\in\mathbb{Z}^E\\ \nabla\cdot m=0}} \sum_{S\subseteq G} \widehat{W}(m;S)\,; \qquad \widehat{W}(m;S) = \sum_{\substack{A \text{ valid}\\ \text{for } S}} W(m;S,A)\,. \tag{12.24}$$

12.2 Solution on a cycle graph

We remark that the partition function trivially factorizes on a graph G with an isthmus (i.e. an edge e such that $G\smallsetminus e$ has one more component than G), or with a loop (i.e. an edge with equal endpoints), in a way similar to the "easy" conditions for a Tutte-Grothendieck invariant, and in particular it vanishes if G has an isthmus. Indeed, if $G = G_1 \cup G_2 \cup \{e\}$ and e is an isthmus,

$$Z_G = Z_{G_1} Z_{G_2} Z_I(w_e) \tag{12.25}$$

where $Z_I(w)$ is the partition function for the two-vertex one-edge graph I

$$\begin{aligned} Z_I(w) &= \int \frac{\mathrm{d}\theta}{2\pi} \int \mathcal{D}_{1,2}(\bar\psi,\psi) e^{w\left(\cos\theta(1+\bar\psi_1\psi_1)(1+\bar\psi_2\psi_2)-\bar\psi_1\psi_2-\bar\psi_2\psi_1\right)} \\ &= w\left(I_1(w) - \frac{w}{2}(I_0(w)-I_2(w))\right)\,, \end{aligned} \tag{12.26}$$

calculated just by expanding the fermionic part of the action and keeping all the terms proportional to $\bar\psi_1\psi_1\bar\psi_2\psi_2$, then integrating over θ and using the definition of Bessel functions. Then, recalling the relation (12.17), we see that $Z_I(w) = 0$ identically.

This implies that graphs G with non-trivial partition function have no tree part, and thus, if we guess that complicancies come from the cycles in the graph, the first interesting family is the one of cycle-graphs, i.e. graphs C_ℓ with ℓ vertices and edges making a regular polygon.

We use the expansion in (12.22), where now the set of flow variables is trivial: we have the same flow m on all the edges. For all values of m we can have a "fermionic cycle"

covering the whole graph, in one of the two orientations, so we have a first contribution $-2\sum_{m\in\mathbb{Z}}\prod_e(w_eI_m(w_e))$. The other possibilities are a cycle of pointing edges, or a spanning forest with each component rooted on an edge. These contributions are collected together in a compact way by a simple transfer-matrix technique: consider all the combinations of valid occupancies on edges $1,\dots,i$, and collect in the 2-component vector $\mathbf{v}_i$ the contributions such that the last edge is respectively pointing forward or unoccupied (in the first component), or pointing backwards or rooted (in the second component). Then one has

$$\mathbf{v}_i = T_m(w_i)\mathbf{v}_{i-1}\,; \qquad T_m(w_i) = \begin{pmatrix} f_{b,m}(w_i) & f_{R,m}(w_i) \\ f_{0,m}(w_i) & f_{b,m}(w_i)\end{pmatrix}\,; \tag{12.27}$$

and the contribution of forests and pointing cycles is given by the trace of the product of these matrices. So one finally gets

$$Z_{C_\ell}(w) = -2\sum_{m\in\mathbb{Z}}\prod_e(w_eI_m(w_e)) + \sum_{m\in\mathbb{Z}}\operatorname{tr}\prod_{i=1}^{\ell}T_m(w_i)\,. \tag{12.28}$$

In the case in which all weights are equal one simply gets

$$Z_{C_\ell}(w) = -2\sum_{m\in\mathbb{Z}}(wI_m(w))^\ell + \sum_{m\in\mathbb{Z}}\operatorname{tr}T_m(w)^\ell\,, \tag{12.29}$$

and it is easy to extract a thermodynamic limit. Indeed, the two eigenvalues of T_m are found to be

$$\lambda_\pm^{(m)} = \pm\sqrt{f_{R,m}f_{0,m}} + f_{b,m} = \pm mI_m + \frac{w}{2}(I_{m-1}+I_{m+1}) = wI_{m\mp1}(w)\,, \tag{12.30}$$

again by using the property (12.17). So we get

$$Z_{C_\ell}(w) = 2w^\ell\Big[-I_0^\ell - 2\sum_{m\geq1}I_m^\ell + I_1^\ell + \sum_{m\geq1}\big(I_{m-1}^\ell + I_{m+1}^\ell\big)\Big] = 0\,. \tag{12.31}$$

12.3 High-temperature expansion

Now we consider the partition function in a limit in which the coupling parameters w_{ij} are scaled to zero. It is interesting that the leading terms in the expansion are not a trivial ensemble, and, with the insertion of a local operator, each valid configuration has a combinatorial interpretation as a single dense polymer on G.

In particular we prove

Theorem 12.1. *For G a weighted graph with N vertices and weights w_{ij}, and $Z_G(w)$ as in (12.7), we have*

$$\lim_{\epsilon\to0}\epsilon^{-N}Z_G(\epsilon w) = 0 \tag{12.32}$$

and

$$\lim_{\epsilon\to0}\epsilon^{-N+1}Z_G(\epsilon w)\,\langle\bar\psi_i\psi_j\rangle = (-1)^N\sum_{\gamma:i\to j}\prod_{e\in\gamma}w_e \tag{12.33}$$

where the sum runs over the Hamiltonian paths on G from i to j, i.e. the simple paths with endpoints i and j which visit all vertices of G.

Proof. The proofs of equations (12.32) and (12.33) are done on similar grounds: we go through (12.32) first, then we sketch the required modifications for (12.33).

Going from equation (12.7) to (12.8) we left apart a factor $(-1)^N$. For what we say in the Appendix of [136], specialized to graphs, locally in each vertex we have an arbitrary number of incident unmarked edges, an arbitrary number of incident incoming pointing edges, and one of the three possibilities: one outcoming pointing edge; one incoming and one outcoming dashed edge; one rooted edge.

We recall that, for $m \geq 0$,

$$I_m(x) = \frac{x^m}{m!\, 2^m} \left(1 + \frac{x^2}{4(m+1)} + \mathcal{O}(x^4)\right) \tag{12.34}$$

and that $I_m(x) = I_{-m}(x)$. Specializing $f_{A,m}(x)$ to the first non-vanishing order, (so that everything is intended multiplied by $(1 + \mathcal{O}(x^2))$), we get

A	0	b	f	R
$m = 0$	1	$\frac{1}{2}x^2$	x	0
$m = 1$	$\frac{1}{2}x$	$\frac{1}{2}x$	$\frac{1}{2}x^2$	$\frac{1}{2}x$
$m \geq 1$	$\frac{1}{m!2^m}x^m$	$\frac{1}{(m-1)!2^m}x^m$	$\frac{1}{m!2^m}x^{m+1}$	$\frac{m}{(m-1)!2^m}x^m$

(the case $m = 1$ is a special case of the last row, but we wrote it explicitly in order to have it at hand in the following, when we get rid of the cases $|m| \geq 2$ and work out the remaining combinatorics). The weight $W(m; S, A)$ of each pair of flow-subgraph configurations can be factorized into "quasi-local" factors over the vertices, at a price of a double counting for each edge, so that we will take "square roots" (remark that, as the overall partition function is polynomial in the weights, these square roots must come in pair, and, if the weights are formal indeterminates, must be understood as useful shortcuts for counting the degree of the polynomial in each variable)

$$\begin{aligned} W(m; S, A) &= \prod_{e \in E(G)} f_{A_e, m_e}(w_e) = \Big(\prod_{i \in V(G)} \prod_{e \sim i} f_{A_e, m_e}(w_e) \Big)^{\frac{1}{2}} \\ &=: \Big(\prod_{i \in V(G)} \mathcal{W}_i(m; S, A) \Big)^{\frac{1}{2}} . \end{aligned} \tag{12.35}$$

As we want to perform an expansion of small w's, we start by investigating the degree of the factors $\mathcal{W}_i(m; S, A)$, according to the set of pairs $\{(A_e, m_e)\}$ for edges e adjacent on i. If we have an incident rooted edge e with $m_e = 0$, the weight is exactly zero. Otherwise, say $\mathcal{W}_i(m; S, A) \sim w^{d(i)}$ (and call $d(i)$ the *degree* of i), then we have

$$d(i) = \sum_e \left(|m_e| + \delta_{A_e, f} + 2\delta_{A_e, b}\delta_{m_e, 0} \right) . \tag{12.36}$$

It is easy to show that $d(i)$ is even for every site i in every valid configuration. Indeed, the two potentially odd summands, $\sum_e |m_e|$ and $\sum_e \delta_{A_e, f}$, are always even in avalid configuration, respectively because m is divergence-free, and thenumber of dashed edges incident on a vertex is either zero or two. Furthermore, as S is spanning on G without atomic components (because we have no "mass term" in the action), the lowest possible degree is 2.

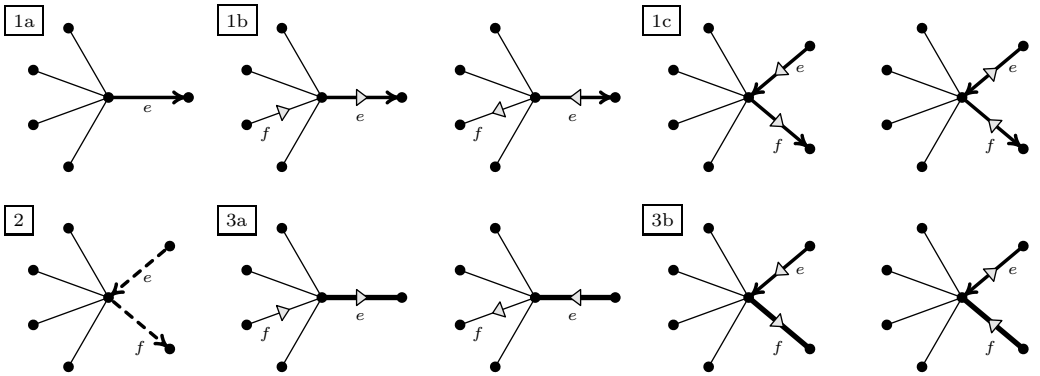


Fig. 12.3. Allowed local configurations of $(m;(S,A))$ near a vertex, having a weight $\mathcal{W}_i(m;S,A)$ of degree 2. The lowest-order contribution is, with the labels as in figure, $w_e w_f/4$ for all of the diagrams, except for (2), for which is $w_e w_f$, and (1a), for which is $w_e^2/2$. Configuration (1a) is however globally incompatible, as no other configurations have an incoming pointing edge with $m=0$.

This proves the claim that the quantity defined in (12.32) as a potentially singular limit, is finite. And in particular, in order to have a finite contribution in the limit and contraddict the theorem, we should have a term of order 2 in each vertex. We follow this assumption up to get an absurd and thus prove the first statement of the theorem. We have degree 2 in the following cases:

1a:

A_e	b	0	0	$\cdots$
m_e	0	0	0	$\cdots$

1b:

A_e	b	0	0	$\cdots$
m_e	±1	∓1	0	$\cdots$

1c:

A_e	b	b	0	$\cdots$
m_e	±1	∓1	0	$\cdots$

2:

A_e	f	f	0	$\cdots$
m_e	0	0	0	$\cdots$

3a:

A_e	R	0	0	$\cdots$
m_e	±1	∓1	0	$\cdots$

3b:

A_e	R	b	0	$\cdots$
m_e	±1	∓1	0	$\cdots$

In figure 12.3 we report the ten corresponding allowed local configurations (six, but four of them come with a $\pm$ choice for the direction of the flow), where values of $m=\pm1$ are encoded in a small triangular arrow in the middle of the edge (as already in figures 12.1 and 12.2), while arrows in dashed and pointing edges are situated on the tip of the corresponding edge (as already in figure 12.2). Of these cases, the case (1a) has to be rejected for global reasons: it contains an outgoing pointing edge with $m=0$, but there are no lowest-degree configurations with ingoing pointing edges having $m=0$.

Say that an edge is "used" if it is not both unmarked and with $m=0$. As, in all the nine diagrams in figure 12.3 (excluded (1a)), locally we have exactly two used edges incident on each vertex, this globally identifies spanning configurations of cycles. In order to prove the first part of the theorem, it suffices to prove that a cancellation occurs separately within each cycle, and in the following we concentrate on a single cycle γ of length ℓ.

Actually, because of the results of Section 12.2, we would have already proven (12.32), but it is instructive, and useful in the following, to proceed in the present formalism.

The cycle γ is produced in one of three possible circumstances:

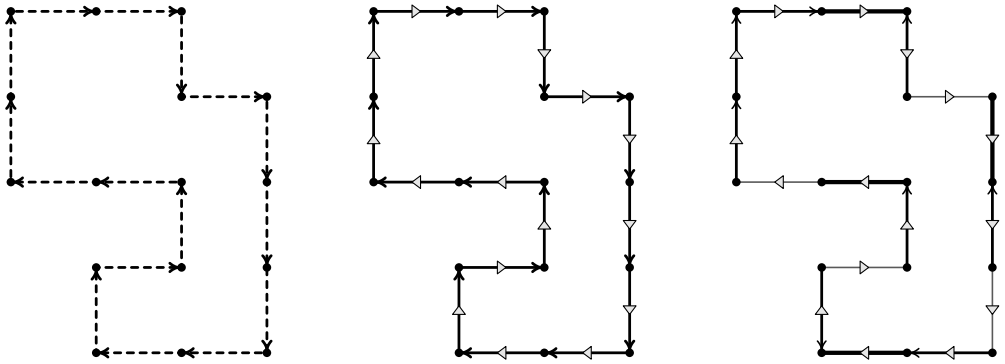


Fig. 12.4. A cycle of marked edges, in the three possible situations of lowest order in w's described in the text: a cycle of dashed edges with zero flow, a cycle of pointing edges with flow ± 1, and a sequence of edge-rooted chains with flow ± 1.

- it is a cycle of dashed edges, with zero flow everywhere;
- it is a cycle of pointing edges, and also a cycle of ± 1 flow;
- it is a cycle of ± 1 flow, composed of a certain number of "chains". Each chain is a special case of edge-rooted tree, with all vertices of degree at most 2, and there is exactly one unmarked edge with ± 1 flow between any two consecutive chains.

These three cases are described in figure 12.4. The first combination, at leading order, gives a factor

$$-2 \prod_{(ij)\in\gamma} w_{ij} \,, \tag{12.37}$$

where the minus sign comes from the reordering of the fermionic fields, and the 2 comes from the two possible orientations of the dashed arrows.

The second combination, again at leading order, gives a factor

$$4 \prod_{(ij)\in\gamma} \frac{w_{ij}}{2} = 2^{2-\ell} \prod_{(ij)\in\gamma} w_{ij} \,, \tag{12.38}$$

where the factor 4 comes from the two possible orientations of the pointing edges, *and* the two possible orientations of the flow.

The third combination is more intricated. Say that we have k chains. We must have $1 \le k \le \ell/2$, then we can choose the $2k$ locations for the roots and unmarked edges along the cycles, getting a combinatorial factor $\binom{\ell}{2k}$, a factor 2 for the choice of the first of these edge being unmarked or rooted (then rooted and unmarked edges are alternating, and pointing edges are oriented towards the nearby root), a factor 2 for the orientation of the flow, and a factor $2^{-\ell}\prod_e w_e$ from the leading terms in factors $f_{A,m}(x)$. So the full expression is

$$\Big(2^{2-\ell} \prod_{(ij)\in\gamma} w_{ij}\Big) \sum_{k\ge 1} \binom{\ell}{2k} \tag{12.39}$$

but the series of binomial factors is easily summed:

$$\sum_{k\ge 1} \binom{\ell}{2k} = \frac{1}{2}\Big(\sum_k \binom{\ell}{k} + \sum_k \binom{\ell}{k}(-1)^k - 2\Big) = \frac{1}{2}\big(2^\ell + 0^\ell - 2\big) = 2^{\ell-1} - 1 \tag{12.40}$$

so we get complexively for the expression (12.39)

$$(2 - 2^{2-\ell}) \prod_{(ij)\in\gamma} w_{ij} \,, \tag{12.41}$$

which combined with (12.37) and (12.38) finally gives the claimed cancellation at leading order, thus implying the first part of the theorem. We stress again that this had to be expected from the supersymmetry of the two bosonic and two fermionic degrees of freedom, and one gets a flavour of this mechanism from the cancellation of the two "clean" contributions, of (12.37) (fermionic) and (12.38) and (12.39) summed together (bosonic).

Things go similarly for the second part of the theorem, where we have the insertion of the operator $\bar{\psi}_i\psi_j$. Now we have two special vertices: there is locally a dashed edge with tip on i, and a dashed edge with tail on j, an no other edges incident on i or j, dashed or rooted or pointing outward (but only unmarked or pointing inward, again, this is discussed in [136]). In such a situation, if and only if both these vertices have no other incident incoming pointing edges, and all zero flow variables, they achieve $d(i) = d(j) = 1$, and give configurations with a contribution in the limit expression (12.33). So, in a subgraph S with the minimal degree, i and j are endpoints of an oriented open path of dashed edges, and the rest of S is composed of cycles. But cycles suffer from cancellations for the same mechanism as before, so we see that the open path must visit all the vertices of G, as claimed in the theorem. Checking that also the weight is as claimed is also easy. First realize that factors $f_{f,0}(x) = x + \mathcal{O}(x^3)$, without further numerical factors, then realize that the direction on the path is fixed, finally check that no minus signs come from the reordering of the fermions. This concludes the proof. □

As a final remark, going back to the general n-vector notations, the invariance under OSP(2|2) fixes that

$$\lim_{\epsilon\to 0} \epsilon^{-N+1} Z_G(\epsilon w) \langle n_i^\alpha n_j^\beta \rangle = (-1)^N g^{\alpha\beta} \sum_{\gamma: i\to j} \prod_{e\in\gamma} w_e \,, \tag{12.42}$$

which is more satisfactory than (12.33), as it is an intrinsic, base-independent statement.

A further point we want to stress is the following: our high-temperature limit is *not* related in any way to a high-radius limit in the $\mathbb{R}^{2|2}$ supersphere, as, by comparison, was the case for the spanning-tree limit within the full $\mathbb{R}^{1|2}$ setting for spanning forests.

This difference is very structural: while the radius of the supersphere is a *global* parameter, the high-temperature limit involve the *local* thermodynamic parameters w_{ij}, and could be done, for example, only in a subset of the edges of the graph (e.g. all edges adjacent to a subset V' of the vertices), leading to statements similar to Theorem 12.1 (e.g., restricted to the induced subgraph of V').

A

Grassmann-Berezin calculus

In a few words, (complex) Grassmann variables are scalar fermionic variables, i.e. they represent the exact counterpart of the Fock construction for a set of (bosonic) oscillators, originated from the assumption of *anticommutation* among variables, instead of commutation.

Beyond the strong similarities of the two procedures for bosonic and fermionic fields, the fermionic case has the pedagogical drawback of requiring from the very beginning an abstraction to formal variables and formal rules, especially for integration, while the bosonic counterpart corresponds to the traditional real- and complex-calculus.

On the other side, if one accepts this level of abstraction, and takes confidence with the combinatorial implications of Grassmann calculus, one gets also unexpected facilitations, coming from the bound on occupation numbers: the Hilbert space of functions is *finite-dimensional*, the finite set of multilinear Taylor coefficients univocally identifying a function, and the Stokes formula does not receive boundary contributions.

Also, various facts that, within commuting variables, are proven in an analytical framework and using certain hypothesis, have Grassmann counterparts that are perfectly rigorous at a purely combinatorial level, without need of addressing analysis. The master example of this is the Gaussian integral formula, that we anticipate: for "complex bosons" $(\bar{\phi}, \phi)$ and "complex fermions" $(\bar{\psi}, \psi)$ we have respectively

$$(2\pi)^{-n} \int \mathcal{D}_n(\bar{\phi}, \phi) e^{-\bar{\phi} A \phi} = (\det A)^{-1} \qquad \int \mathcal{D}_n(\psi, \bar{\psi}) e^{\bar{\psi} A \psi} = \det A$$

giving a first taste of a "duality" between bosonic and fermionic degrees of freedom, which in physics goes under the name of *supersymmetry*, and of *Parisi-Sourlas dimensional reduction*. However, the two formulas above are valid on very different grounds: while, as everybody knows, the "bosonic" one on the left is valid only if A is positive definite (and thus, *a fortiori*, valued on a mathematical structure for which this makes sense), the "fermionic" one is just an application of the formal integration rule that we discuss in a while, and simple combinatorics, and holds for an arbitrary matrix A, valued on an arbitary commutative ring (not necessarily a field, and, despite the apparences, not even necessarily containing the rationals).

For what concerns physics, of course, the Spin and Statistics theorem [160] would force us to make use of Grassmann variables only into multiplets of half-integer spin, in order to ensure unitarity. Nonetheless, there are two good reasons to consider also Grassmann variables with integer spin, and in particular, no spin at all, i.e. scalar fermions.

The first one, not much addressed in this work, is that scalar fermions arise in various "fundamental" field theories (for particle physics) in the form of Faddeev-Popov ghosts,

whose non-unitary contribution produces the exact cancellation of the "non-physical" longitudinal modes of massless vector fields (a sophisticated version of the observation above on Gaussian integration), and makes the theory unitary. The second reason, that we discuss all along this work, is that among the partition functions of models with scalar fermions on a lattice we annoverate the generating functions of various interesting combinatorial problem, with application in mathematics, or as effective theories for statistical mechanics.

In this pages, we review some basic definitions and useful formulas, which are employed in the body of the work.

A.1 Grassmann and Clifford Algebras

Consider the set of N symbols $\{\theta_i\}_{i=1,\dots,N}$ (*Grassmann* variables) satisfying the anticommutation relations

$$\theta_i\theta_j + \theta_j\theta_i = 0\,. \tag{A.1}$$

As in particular $\theta_i^2 = 0$,[†] the set of functions in these variables is a vector space of dimension 2^N, spanned by all monomials of degree at most 1 in each variable, i.e., calling $[N] = \{1, \dots, N\}$,

$$f(\theta) = \sum_{I\subseteq[N]} f_I\, \theta_{i_1}\cdots\theta_{i_k}\,, \tag{A.2}$$

where it is understood that $i_1, \dots, i_k$ are the elements of I *in ordered sequence*. The product of two functions is obtained by iterated application of relation (A.1), with the result

$$\begin{aligned} f(\theta)g(\theta) &= \sum_{I,J\subseteq[N]} \delta(I\cap J,\varnothing)\; f_I g_J\; \theta_{i_1}\cdots\theta_{i_k}\theta_{j_1}\cdots\theta_{j_h} \\ &= \sum_{I\subseteq[N]}\sum_{I'\subseteq I} f_{I'}\; g_{I\smallsetminus I'}\; \epsilon_I(I')\; \theta_{i_1}\cdots\theta_{i_k}\,. \end{aligned} \tag{A.3}$$

The sign involved in the reordering of the variables is given by the expression $\epsilon_I(I')$, defined as

$$\epsilon_I(I') = (-1)^{\frac{h(h+1)}{2}+\alpha_1+\dots+\alpha_h}\,; \qquad \begin{array}{l} I = \{i_1,\dots,i_k\}\,; \\ I' = \{i_{\alpha_1},\dots,i_{\alpha_h}\}\subseteq I\,; \end{array} \tag{A.4}$$

A small calculation (or direct commutation) gives the simple relation

$$\epsilon_I(I')\;\epsilon_I(I\smallsetminus I') = (-1)^{h(k-h)}\,, \tag{A.5}$$

which implies that in $[f,g]$ we have contribution only from pairs $f_I g_J$ with both I and J of odd cardinality, and that in $\{f,g\}$ we have contribution from the complementary set of pairs. In particular, if two functions $f(\theta)$ and $g(\theta)$ are both odd, they anticommute, while if $f(\theta)$ is an even function, it commutes with the whole algebra of functions. The set of even functions is a subalgebra of dimension 2^{N-1}. We call *Grassmann Algebra* the construction above. We can extend this framework including also formal derivatives w.r.t. the formal

[†] If we are in a ring of characteristic two, we may impose separately that $\theta_i^2 = 0$, as (A.1) only implies $2\theta_i^2 = 0$, which is an empty statement in this case. Also, in a ring of charateristic two, commutation and anticommutation are the same concept.

symbols θ_i, which have a behaviour analogue to the natural one for traditional derivatives, up to a sign, such that also derivatives are anticommuting variables

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}+\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}=0\,; \qquad\qquad \frac{\partial}{\partial\theta_i}\theta_j+\theta_j\frac{\partial}{\partial\theta_i}=\delta_{ij}\,. \tag{A.6}$$

This formulation with "variables" and "derivatives" is the natural anticommuting construction which parallels a Weyl algebra, and we name it *Clifford Algebra*, with a slight abuse of notation, as we now explain.

Indeed, the relations (A.6) allow to recognize a structure of fermionic oscillators,

$$\{a_i,a_j\}=\{a_i^\dagger,a_j^\dagger\}=0\,; \qquad\qquad \{a_i,a_j^\dagger\}=\delta_{ij}\,; \tag{A.7}$$

with the identifications

$$a_i^\dagger=\theta_i\,; \qquad\qquad a_i=\frac{\partial}{\partial\theta_i}\,. \tag{A.8}$$

As commonly done for (bosonic) oscillators, we can also consider sums and differences of creators and annihilators, which lead to the $2N$ symbols Γ_j

$$\Gamma_j=a_j+a_j^\dagger\,; \qquad \Gamma_{N+j}=i(a_j-a_j^\dagger)\,; \qquad j=1,\dots,N\,; \tag{A.9}$$

satisfying the relation $\{\Gamma_i,\Gamma_j\}=2\delta_{ij}$. These are the generators of the Clifford Algebra in the formulation which occurs more often in the literature.

A.2 Grassmann-Berezin integration

After having defined formal derivatives, we can define formal integration. We can start from the one-variable algebra

$$\int\mathrm{d}\theta_1\,f(\theta_1)=\int\mathrm{d}\theta_1\,(f_\varnothing+f_{[1]}\theta_1)=\mathcal{I}(f_\varnothing,f_{[1]})\,, \tag{A.10}$$

where the formal rule defining $\mathcal{I}$ is to be determined. We have two fundamental requirements: linearity, that is, for any commuting quantity λ,

$$\int\mathrm{d}\theta\,\big(\lambda f(\theta)\big)=\lambda\int\mathrm{d}\theta\,f(\theta)\,, \tag{A.11}$$

and invariance under translation,[‡] that is, for any anticommuting quantity ψ,

$$\int\mathrm{d}\theta\,f(\theta)=\int\mathrm{d}\theta\,f(\theta+\psi)\,. \tag{A.12}$$

Up to an irrelevant multiplicative constant, these two requirements give a single consistent choice:

$$\mathcal{I}(f_\varnothing,f_{[1]})=f_{[1]}\,. \tag{A.13}$$

As a consequence, and analogously to what happens for Riemann integration on a domain without boundary, the integral of a gradient is zero

[‡]This is the analogue of the invariance of Riemann integration over a domain with no boundary and some translational invariance, such as $\mathbb{R}$ or $[0,2\pi]$ with periodic conditions.

$$\int \mathrm{d}\theta_1 \, \frac{\partial}{\partial \theta_1} f(\theta_1) = \int \mathrm{d}\theta_1 \, \frac{\partial}{\partial \theta_1} (f_\varnothing + f_{[1]} \theta_1) = \int \mathrm{d}\theta_1 \, f_{[1]} = 0 \,. \tag{A.14}$$

The case of more variables is dealt through iteration of the procedure, with an *inverted* order of the indices

$$\int \mathcal{D}(\theta) \, f(\theta) = \int \mathrm{d}\theta_N \cdots \int \mathrm{d}\theta_1 \, f(\theta) = f_{[N]} \,. \tag{A.15}$$

Making use of nilpotence of the variables, it is easy to extract any other coefficient of the function f, in the form of an integral of a product (or, in a notation referring to statistical mechanics, as an "expectation value" w.r.t. the measure f)

$$\langle \theta_{i_1} \cdots \theta_{i_k} \rangle_f = \int \mathcal{D}(\theta) \, \theta_{i_1} \cdots \theta_{i_k} f(\theta) = \epsilon_{[N]}(I) \; f_{[N] \smallsetminus I} \,. \tag{A.16}$$

Moreover, it is easy to extract the result of a linear change of variables. First, for the one-variable case

$$\int \mathrm{d}\theta_1 \, f(J\theta_1) = J f_{[1]} = J \int \mathrm{d}\theta_1 \, f(\theta_1) \,. \tag{A.17}$$

We stress here a discrepance with real integration, for which $\int \mathrm{d}x \, f(Jx) = J^{-1} \int \mathrm{d}x \, f(x)$. Then, for the general case

$$\int \mathcal{D}(\theta) \, f(J\theta) = (\det J) \int \mathcal{D}(\theta) \, f(\theta) \,, \tag{A.18}$$

as could be easily proven through (A.17) and the translation formula (A.12), realizing that these are the basic ingredients in a Gauss Algorithm decomposition of the linear transformation.

For scalar bosons, in many cases it is useful to combine the real degrees of freedom in pairs, related by some (charge) conjugation operation when this shows to be a natural feature of the theory, i.e. when charge is "conserved" w.r.t. the application of some relevant operators in the theory. This leads to the definition of complex bosons.

We want to describe a similar construction in Grassmann Algebra. So, consider a Grassmann Algebra over a set of $2N$ variables named $\psi_1, \ldots, \psi_N, \bar{\psi}_1, \ldots, \bar{\psi}_N$. The most general function is of the form

$$f(\bar{\psi}, \psi) = \sum_{I, I' \subseteq [N]} f_{I,I'} \; (-1)^{\frac{k(k-1)}{2}} \; \bar{\psi}_{i_1} \cdots \bar{\psi}_{i_k} \psi_{i'_1} \cdots \psi_{i'_h} \,, \tag{A.19}$$

where the sign is chosen in order to allow reordering in pairs $\bar{\psi}_i \psi_j$ when $k = h$. The notation

$$\bar{\psi}\psi = \sum_i \bar{\psi}_i \psi_i \tag{A.20}$$

for scalar products will be understood, and more generally

$$\bar{\psi} A \psi = \sum_{i,j} \bar{\psi}_i A_{ij} \psi_j \tag{A.21}$$

for any quadratic form, and the measure of integration is given by the ordering convention

$$\mathcal{D}(\psi, \bar{\psi}) \; = \mathrm{d}\psi_N \mathrm{d}\bar{\psi}_N \cdots \mathrm{d}\psi_1 \mathrm{d}\bar{\psi}_1 \,. \tag{A.22}$$

The full algebra has dimension 2^{2N}, the subalgebra of even functions, of dimension 2^{2N-1}, contains the smaller subalgebra of functions with "globally neutral charge", where only monomials with $k = h$ have non-zero coefficients, whose dimension is $\sum_{k=0}^{N} \binom{N}{k}^2$. A still smaller subalgebra is the one with "locally neutral charge", where $I = I'$, whose dimension is 2^N.

There are at least two good reasons for studying complex Grassmann variables. The first one is that the general combinatorics implicit in equations (A.15, A.16), although powerful, easily drives crazy because of signs jumping all over, while the less general one deriving from complex variables is more handly, while still including a large number of interesting cases. In particular, we will see in a moment how the Pfaffian formula, analogue to the real-Gaussian integral, leads as a special case to a Determinant formula, analogue to the complex-Gaussian integral. A second reason, that we explain in the following, is that a natural scalar product has an integral formulation under a *doubling* of the algebra, or, equivalently, if the two functions in the product are "purely holomorphic". This second application has a large extension to the representation of linear operators as integral kernels, that plays an important role in the solution of the two-dimensional Ising model as is performed, for example, in [46], and is briefly discussed in the next section.

We can start with the study of some special integral formulas. A first case is integration with a Grassmann Gaussian, which gives a projection on the local neutral charge subalgebra

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{m\bar{\psi}\psi} f(\bar{\psi}, \psi) = \sum_I m^{N-|I|} f_{I,I}\,. \tag{A.23}$$

This formula has many combinatorial applications: if the function $f(\bar{\psi}, \psi)$ is such that its formal monomial expansion gives a sum over certain diagrams, with marks of two species over the elementary constituents, integration with the Gaussian measure restricts the sum to diagrams in which every constituent is either saturated by both marks (with a factor 1), or empty (with a factor m). Remark that there is no ambiguity between $e^{m\bar{\psi}\psi}$ and

$$\prod_i e^{m\bar{\psi}_i\psi_i} = \prod_i (1 + m\bar{\psi}_i\psi_i)\,, \tag{A.24}$$

where the RHS product is taken in an arbitrary order, because all the factors are in the even subalgebra (actually, in the local neutral charge subalgebra). For example, this would *not* hold for

$$e^{m\sum_i \theta_i} = 1 + m\sum_i \theta_i \neq \prod_i (1 + m\theta_i)\,, \tag{A.25}$$

while it would hold for any antisymmetric matrix A, and the expressions

$$\begin{aligned} e^{\frac{1}{2}\theta A\theta} &= e^{\sum_{i<j}\theta_i\theta_j A_{ij}} = \prod_{i<j}(1 + A_{ij}\theta_i\theta_j) \\ &= \prod_i \Big(1 + \theta_i\Big(\sum_{j>i} A_{ij}\theta_j\Big)\Big) = \prod_j \Big(1 + \Big(\sum_{i<j}\theta_i A_{ij}\Big)\theta_j\Big)\,, \end{aligned} \tag{A.26}$$

of which the complex case is a special case with A the symplectic identity matrix J, i.e. a diagonal sequence of matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. At this point, making use of the formula (A.18) for the change of variables, and some basic algebraic facts, as the general formula $\mathrm{Pf}\, A = \det B\, \mathrm{Pf}\, A'$ for $A = B^T A' B$, A and A' being antisymmetric matrices, and the fact that any antisymmetric matrix A can be decomposed into $A = B^T J B$, we would be able to immediately derive the Pfaffian and Determinant formulas. The first, for A of even dimension $2N$ reads

$$\int \mathcal{D}(\theta)\, e^{\frac{1}{2}\theta A\theta} = \det B \int \mathcal{D}(\theta)\, e^{\frac{1}{2}\theta J\theta} = \mathrm{Pf}\, A \int \mathcal{D}(\theta) \prod_{i=1}^{N} (1+\theta_{2i-1}\theta_{2i}) = \mathrm{Pf}\, A\,; \quad \text{(A.27)}$$

while the second reads

$$\int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi Q\psi} = \det Q \int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi} = \det Q \int \mathcal{D}(\psi,\bar\psi) \prod_{i=1}^{N} (1+\bar\psi_i\psi_i) = \det Q\,; \quad \text{(A.28)}$$

and more generically, if Q is invertible, and $\bar\eta$, η are some Grassmann-odd elements,

$$\begin{aligned}\int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi Q\psi+\bar\psi\eta+\bar\eta\psi} &= \int \mathcal{D}(\psi,\bar\psi)\, e^{(\bar\psi+\bar\eta Q^{-1})Q(\psi+\eta)-\bar\eta Q^{-1}\eta} \\ &= e^{-\bar\eta Q^{-1}\eta} \int \mathcal{D}(\psi,\bar\psi)\, e^{(\bar\psi+\bar\eta Q^{-1})Q(\psi+\eta)} = \det Q\; e^{-\bar\eta Q^{-1}\eta}\,. \quad \text{(A.29)}\end{aligned}$$

Nonetheless, we find instructive to derive all these formulas also directly with the combinatorial identification of the two expressions, as a first example of Grassmann variables manipulation for combinatorial applications. As a result, these alternate proofs will hold directly without any analytic assumption (decomposition of A, and invertibility of Q).

The Pfaffian of an antisymmetric matrix of even dimension N is defined as a sum over all arrangement of integers $\{1,\dots,2N\}$ in pairings, $\mathcal{M} = \{(i_1,i_2),(i_3,i_4),\dots,(i_{2N-1},i_{2N})\}$ (say, for definiteness, $i_{2k-1} < i_{2k}$ and $i_{2k-1} < i_{2k'-1}$ for $k < k'$):

$$\mathrm{Pf}\, A = \sum_{\mathcal{M}} \epsilon(\mathcal{M}) \prod_{k=1}^{N} A_{i_{2k-1}i_{2k}}\,, \quad \text{(A.30)}$$

where the sign $\epsilon(\mathcal{M})$ is the sign of the permutation σ such that $\sigma(k) = i_k$. Remark that, because of the antisymmetry of matrix A, the definition is not sensible to the convention above on the ordering of the pairings: indeed, if one transpose a pair, gets a minus sign both in the corresponding factor A_{ij} and in ϵ, while if one exchanges two pairs no minus signs are involved. A redundant but more symmetric notation can be chosen

$$\mathrm{Pf}\, A = \frac{1}{2^N N!} \sum_{\sigma\in\mathfrak{S}_{2N}} \epsilon(\sigma) \prod_{k=1}^{N} A_{\sigma(2k-1)\sigma(2k)}\,. \quad \text{(A.31)}$$

On the other side the expansion of the Gaussian form gives

$$e^{\frac{1}{2}\theta A\theta} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2}\theta A\theta\right)^n\,, \quad \text{(A.32)}$$

but, at the aim of Grassmann integration, only the term $n = N$ could give a non-vanishing contribution, thus we have

$$\frac{1}{N!}\left(\frac{1}{2}\theta A\theta\right)^N = \frac{1}{N!}\left(\sum_{i,j}\left(\frac{1}{2}A_{ij}\theta_i\theta_j\right)\right)^N = \frac{1}{2^N N!} \sum_{i_1,j_1,\dots,i_N,j_N} \prod_{k=1}^{N} \left(A_{i_kj_k}\theta_{i_k}\theta_{j_k}\right)\,. \quad \text{(A.33)}$$

Again nilpotency of Grassmann variables forces the indices i_1, j_1, …, i_N, j_N to be all different, and thus can be identified with the sequence $\sigma(1)$, …, $\sigma(2N)$ of a permutation of $[2N]$. The reordering of the product of Grassmann variables gives the desired sign

$$\theta_{i_1}\theta_{j_1}\cdots\theta_{i_N}\theta_{j_N} = \theta_{\sigma(1)}\theta_{\sigma(2)}\cdots\theta_{\sigma(2N-1)}\theta_{\sigma(2N)} = \epsilon(\sigma)\theta_1\cdots\theta_{2N}\,,$$

and the presence of proper factors A_{ij} allows us to recover the identity (A.27).

Similar (easier) reasonings apply to the case of the determinant formula (A.28). On one side

$$\det Q = \sum_{\sigma\in\mathfrak{S}_N} \epsilon(\sigma)\prod_{i=1}^{N} Q_{i\sigma(i)}\,, \tag{A.34}$$

while on the other side, again in the Taylor expansion of the exponential only the term $n = N$ can give contribution, and produces

$$\frac{1}{N!}\sum_{i_1,j_1,\ldots,i_N,j_N}\prod_{k=1}^{N}\left(Q_{i_k j_k}\bar{\psi}_{i_k}\psi_{j_k}\right)\,. \tag{A.35}$$

All indices i_k must be different. We can imagine to reorder them into the sequence $1,\ldots,N$, at the price of the combinatorial factor $N!$, which cancels out in (A.35). The sequence of indices j_k, which are all different, is fixed by this procedure, and can be rewritten as the sequence $\sigma(k)$ for a certain permutation $\sigma\in\mathfrak{S}_N$. This gives

$$\sum_{\sigma\in\mathfrak{S}_N}\prod_{k=1}^{N} Q_{k\sigma(k)}\bar{\psi}_k\psi_{\sigma(k)}\,. \tag{A.36}$$

The reordering of Grassmann variables produces the desired sign

$$\prod_{k=1}^{N}\bar{\psi}_k\psi_{\sigma(k)} = \epsilon(\sigma)\prod_{k=1}^{N}\bar{\psi}_k\psi_k\,; \tag{A.37}$$

(it suffices to check the relation $\epsilon(\sigma)\prod_{k=1}^{N}\bar{\psi}_k\psi_{\sigma(k)} = \epsilon(\tau)\prod_{k=1}^{N}\bar{\psi}_k\psi_{\tau(k)}$ for σ and τ differing by a transposition).

A.3 Integral kernels in Clifford Algebra

Consider a complex Grassmann Algebra in variables $\bar{\psi}_i$ and ψ_i, and a function g depending on the ψ_i's only (we name it an *holomorphic* function). We introduce the following conjugation operation

$$g(\psi) = \sum_I g_I\psi_{i_1}\cdots\psi_{i_k} \qquad\longleftrightarrow\qquad \overline{g}(\bar{\psi}) = \sum_I \overline{g}_I\bar{\psi}_{i_k}\cdots\bar{\psi}_{i_1}\,. \tag{A.38}$$

I.e., as in every non-commutative ring (think for example to an algebra of matrices), conjugation is defined through both conjugation of the complex-valued elements, and inversion of the multiplication order.

Then, we introduce a quadratic form among pairs of holomorphic functions

$$(g,f) = \sum_{I\subseteq[N]} f_I\overline{g}_I \tag{A.39}$$

Now we can easily show that the quadratic form (A.39) has an integral formulation, analogous to an average with the Gaussian Grassmann measure as in the expression (A.23):

$$\begin{aligned}\int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi}\overline{g}(\bar\psi)f(\psi) &= \int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi}\sum_{I,J}\overline{g}_I\ \bar\psi_{i_k}\cdots\bar\psi_{i_1}\ f_J\ \psi_{j_1}\cdots\psi_{j_h}\\ &= \int \mathcal{D}(\psi,\bar\psi)\,\Big(\sum_I \overline{g}_I\ f_I\ e^{\bar\psi\psi}\bar\psi_{i_k}\cdots\bar\psi_{i_1}\psi_{i_1}\cdots\psi_{i_k}+\text{V.T.}\Big)\\ &= \int \mathcal{D}(\psi,\bar\psi)\,\Big(\sum_I \overline{g}_I\ f_I\ (\bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}})\bar\psi_{i_k}\cdots\bar\psi_{i_1}\psi_{i_1}\cdots\psi_{i_k}+\text{V.T.}\Big)\end{aligned} \tag{A.40}$$

Here and in the following it is understood that, if $i_1, \ldots, i_k$ are the elements of I in ordered sequence, $i'_1, \ldots, i'_{N-k}$ are the elements of the complementary set $\overline{I} = [N] \setminus I$ in ordered sequence, and that the expression "+V.T.", standing for "plus vanishing terms", means that other Grassmann variable monomials, whose contribution to the integral is zero at sight, are omitted. We are almost done, we just need to check the proper sign

$$\bar\psi_{i_k}\cdots\bar\psi_{i_1}\psi_{i_1}\cdots\psi_{i_k} = \bar\psi_{i_1}\psi_{i_1}\bar\psi_{i_k}\cdots\bar\psi_{i_2}\psi_{i_2}\cdots\psi_{i_k} = \cdots = \bar\psi_{i_1}\psi_{i_1}\cdots\bar\psi_{i_k}\psi_{i_k}$$

in order to recognize

$$\int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi}\overline{g}(\bar\psi)f(\psi) = \int \mathcal{D}(\psi,\bar\psi)\,\Big(\sum_I \overline{g}_I\ f_I\ \bar\psi_1\psi_1\cdots\bar\psi_N\psi_N+\text{V.T.}\Big) = (g,f)\,. \tag{A.41}$$

An important fact is that not only the bilinear form (g,f) has an integral expression, but also any linear operator A acting on the functions, such that $(Af)(\psi)$ can be written as a convolution of function $f(\psi)$ with an integral kernel $\mathcal{A}(\eta,\bar\eta)$

$$Af(\eta) = \int \mathcal{D}(\psi,\bar\psi)\,\mathcal{A}(\eta,\bar\psi)e^{\bar\psi\psi}f(\psi)\,. \tag{A.42}$$

In particular, the kernel corresponding to the identity operator is $\mathcal{I}(\eta,\bar\eta) = e^{-\bar\eta\eta}$. Indeed

$$\begin{aligned}\int \mathcal{D}(\psi,\bar\psi)\,\mathcal{I}(\eta,\bar\psi)e^{\bar\psi\psi}f(\psi) &= \int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi(\psi-\eta)}f(\psi)\\ &= \int \mathcal{D}(\psi,\bar\psi)\ \sum_I f_I\psi_{i_1}\cdots\psi_{i_k}\Big(\bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}e^{-\bar\psi\eta}+\text{V.T.}\Big)\\ &= \int \mathcal{D}(\psi,\bar\psi)\ \sum_I f_I\psi_{i_1}\cdots\psi_{i_k}\Big(\bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}(-\bar\psi_{i_1}\eta_{i_1})\cdots(-\bar\psi_{i_k}\eta_{i_k})+\text{V.T.}\Big)\,,\end{aligned} \tag{A.43}$$

the first restriction is due to the only possible saturation of variables ψ, and the second to an analogous reasoning on $\bar\psi$. A reordering produces

$$\begin{aligned}&\psi_{i_1}\cdots\psi_{i_k}\bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}(-\bar\psi_{i_1}\eta_{i_1})\cdots(-\bar\psi_{i_k}\eta_{i_k})\\ &\quad= \bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}(-\psi_{i_1}\bar\psi_{i_1}\eta_{i_1})\cdots(-\psi_{i_k}\bar\psi_{i_k}\eta_{i_k})\\ &\quad= \bar\psi_{i'_1}\psi_{i'_1}\cdots\bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}(\eta_{i_1}\bar\psi_{i_1}\psi_{i_1})\cdots(\eta_{i_k}\bar\psi_{i_k}\psi_{i_k})\\ &\quad= \eta_{i_1}\cdots\eta_{i_k}\bar\psi_1\psi_1\cdots\bar\psi_N\psi_N\,,\end{aligned} \tag{A.44}$$

and after integration we are left with $f(\eta)$, as claimed. The explicit expression for the integral kernel, together with the expression for the action of creators and annihilators

which we go to derive, allow for the expression of any finite string of Fock operators into an integral kernel. Recalling that $a_i^\dagger = \psi_i$ and $a_i = \frac{\partial}{\partial \psi_i}$, we simply obtain for the action on the left

$$a_i^\dagger A f(\eta) = \int \mathcal{D}(\psi,\bar\psi)\, \eta_i \mathcal{A}(\eta,\bar\psi) e^{\bar\psi\psi} f(\psi)\,; \tag{A.45}$$

$$a_i A f(\eta) = \int \mathcal{D}(\psi,\bar\psi)\, \frac{\partial}{\partial \eta_i} \mathcal{A}(\eta,\bar\psi) e^{\bar\psi\psi} f(\psi)\,; \tag{A.46}$$

while the action on the right requires some more care

$$\begin{aligned} A a_i^\dagger f(\eta) &= \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{A}(\eta,\bar\psi) e^{\bar\psi\psi} \psi_i f(\psi) = \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{A}(\eta,\bar\psi) \left(\frac{\partial}{\partial \bar\psi_i} e^{\bar\psi\psi}\right) f(\psi) \\ &= -\int \mathcal{D}(\psi,\bar\psi) \left(\mathcal{A}(\eta,\bar\psi) \overleftarrow{\frac{\partial}{\partial \bar\psi_i}}\right) e^{\bar\psi\psi} f(\psi)\,; \end{aligned} \tag{A.47}$$

$$\begin{aligned} A a_i f(\eta) &= \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{A}(\eta,\bar\psi) e^{\bar\psi\psi} \left(\frac{\partial}{\partial \psi_i} f(\psi)\right) \\ &= -\int \mathcal{D}(\psi,\bar\psi)\, \mathcal{A}(\eta,\bar\psi) \left(e^{\bar\psi\psi} \overleftarrow{\frac{\partial}{\partial \psi_i}}\right) f(\psi) = -\int \mathcal{D}(\psi,\bar\psi)\, \mathcal{A}(\eta,\bar\psi) \bar\psi_i e^{\bar\psi\psi} f(\psi)\,; \end{aligned} \tag{A.48}$$

from which the rules

$$a_i A \to \frac{\partial}{\partial \psi_i} \mathcal{A}(\psi,\bar\psi)\,; \qquad A a_i \to -\mathcal{A}(\psi,\bar\psi)\bar\psi_i\,; \tag{A.49}$$

$$a_i^\dagger A \to \psi_i \mathcal{A}(\psi,\bar\psi)\,; \qquad A a_i^\dagger \to -\mathcal{A}(\psi,\bar\psi) \overleftarrow{\frac{\partial}{\partial \bar\psi_i}}\,; \tag{A.50}$$

In particular, to a sequence of Fock operators $a_{i_1}^\dagger \cdots a_{i_k}^\dagger a_{j_h} \cdots a_{j_1}$ corresponds the integral kernel

$$a_{i_1}^\dagger \cdots a_{i_k}^\dagger a_{j_h} \cdots a_{j_1} \quad \longrightarrow \quad \psi_{i_1} \cdots \psi_{i_k} e^{-\bar\psi\psi} (-\bar\psi_{j_h}) \cdots (-\bar\psi_{j_1})\,. \tag{A.51}$$

Collecting Fock operators is not the only way to cook up integral kernels from linear operators. It is also possible to deduce the product of two operators, from a convolution formula. If $C = AB$, we have

$$C f(\eta) = \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{C}(\eta,\bar\psi) e^{\bar\psi\psi} f(\psi)\,; \tag{A.52}$$

$$\begin{aligned} AB f(\eta) &= A \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{B}(\eta,\bar\psi) e^{\bar\psi\psi} f(\psi) \\ &= \int \mathcal{D}(\xi,\bar\xi)\, \mathcal{A}(\eta,\bar\xi) e^{\bar\xi\xi} \int \mathcal{D}(\psi,\bar\psi)\, \mathcal{B}(\xi,\bar\psi) e^{\bar\psi\psi} f(\psi) \\ &= \int \mathcal{D}(\psi,\bar\psi) \left(\int \mathcal{D}(\xi,\bar\xi)\, \mathcal{A}(\eta,\bar\xi) e^{\bar\xi\xi} \mathcal{B}(\xi,\bar\psi)\right) e^{\bar\psi\psi} f(\psi)\,; \end{aligned} \tag{A.53}$$

from which

$$\mathcal{C}(\eta,\bar\psi) = \int \mathcal{D}(\xi,\bar\xi)\, \mathcal{A}(\eta,\bar\xi) e^{\bar\xi\xi} \mathcal{B}(\xi,\bar\psi)\,. \tag{A.54}$$

Also the trace of an operator turns out to have a simple expression. Indeed, by definition, if we consider the canonical basis

$$\mathbf{e}_I = \psi_{i_1} \cdots \psi_{i_k} \qquad I \subseteq [N]\,, \tag{A.55}$$

the trace is given by

$$\begin{aligned}
\operatorname{tr} A &= \sum_I (\mathbf{e}_I, A\mathbf{e}_I) = \sum_I \int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi} \bar\psi_{i_k} \cdots \bar\psi_{i_1} \int \mathcal{D}(\eta,\bar\eta)\, \mathcal{A}(\psi,\bar\eta) e^{\bar\eta\eta} \eta_{i_1} \cdots \eta_{i_k} \\
&= \sum_I \int \mathcal{D}(\psi,\bar\psi)\, \left(\bar\psi_{i'_1}\psi_{i'_1} \cdots \bar\psi_{i'_{N-k}}\psi_{i'_{N-k}} + \text{V.T.}\right) \bar\psi_{i_k} \cdots \bar\psi_{i_1} \\
&\qquad \times \int \mathcal{D}(\eta,\bar\eta)\, \mathcal{A}(\psi,\bar\eta) \left(\bar\eta_{i'_1}\eta_{i'_1} \cdots \bar\eta_{i'_{N-k}}\eta_{i'_{N-k}} + \text{V.T.}\right) \eta_{i_1} \cdots \eta_{i_k}\,.
\end{aligned} \tag{A.56}$$

As the integrand, except for the generic $\mathcal{A}$, is an even function of Grassmann variables, only the even part of $\mathcal{A}$ can contribute to the integral, and thus we can commute also the factor $\eta_{i_1} \cdots \eta_{i_k}$. So we can write

$$\begin{aligned}
\operatorname{tr} A &= \sum_I \int \mathcal{D}(\psi,\bar\psi)\mathcal{D}(\eta,\bar\eta)\, \left(\bar\psi_{i'_1}\psi_{i'_1}\bar\eta_{i'_1}\eta_{i'_1} \cdots \bar\psi_{i'_{N-k}}\psi_{i'_{N-k}}\bar\eta_{i'_{N-k}}\eta_{i'_{N-k}} + \text{V.T.}\right) \\
&\qquad \bar\psi_{i_k} \cdots \bar\psi_{i_1}\eta_{i_1} \cdots \eta_{i_k}\mathcal{A}(\psi,\bar\eta) \\
&= \sum_I \int \mathcal{D}(\psi,\bar\psi)\mathcal{D}(\eta,\bar\eta)\, \left(\bar\psi_{i'_1}\eta_{i'_1}\bar\eta_{i'_1}\psi_{i'_1} \cdots \bar\psi_{i'_{N-k}}\eta_{i'_{N-k}}\bar\eta_{i'_{N-k}}\psi_{i'_{N-k}} + \text{V.T.}\right) \\
&\qquad (-1)^{N-k}\bar\psi_{i_1}\eta_{i_1} \cdots \bar\psi_{i_k}\eta_{i_k}\mathcal{A}(\psi,\bar\eta)
\end{aligned} \tag{A.57}$$

then use here the fact that $\mathrm{d}\psi\mathrm{d}\bar\psi\mathrm{d}\eta\mathrm{d}\bar\eta = -\mathrm{d}\eta\mathrm{d}\bar\psi\mathrm{d}\psi\mathrm{d}\bar\eta$, to integrate over one set of variables

$$\begin{aligned}
&= \sum_I \int \mathcal{D}(\eta,\bar\psi)\mathcal{D}(\psi,\bar\eta)\ (-1)^k\ \bar\psi_1\eta_1 \cdots \bar\psi_N\eta_N\ \bar\eta_{i'_1}\psi_{i'_1} \cdots \bar\eta_{i'_{N-k}}\psi_{i'_{N-k}}\ \mathcal{A}(\psi,\bar\eta) \\
&= \int \mathcal{D}(\psi,\bar\eta)\ \sum_I (-1)^k \bar\eta_{i'_1}\psi_{i'_1} \cdots \bar\eta_{i'_{N-k}}\psi_{i'_{N-k}}\mathcal{A}(\psi,\bar\eta) \\
&= \int \mathcal{D}(\psi,\bar\eta)\,(-1)^N e^{-\bar\eta\psi}\mathcal{A}(\psi,\bar\eta) = \int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi\psi}\mathcal{A}(\psi,-\bar\psi)\,.
\end{aligned} \tag{A.58}$$

We can combine this formula with (A.54), to write a formula for the trace of a power

$$\begin{aligned}
\operatorname{tr} A^n = &\int \mathcal{D}(\eta^{(1)},\bar\eta^{(1)}) \cdots \mathcal{D}(\eta^{(n)},\bar\eta^{(n)}) \\
&\mathcal{A}(\eta^{(n)},\bar\eta^{(1)}) e^{\bar\eta^{(1)}\eta^{(1)}} \mathcal{A}(\eta^{(1)},\bar\eta^{(2)}) e^{\bar\eta^{(2)}\eta^{(2)}} \cdots \mathcal{A}(\eta^{(n-1)},-\bar\eta^{(n)}) e^{\bar\eta^{(n)}\eta^{(n)}}\,.
\end{aligned} \tag{A.59}$$

Remark the minus sign in the last element of the product, for this and the previous formulas, This is important, and characteristic of fermions. As an example of its effect, this formula can be applied to the calculation of the free energy of the Ising Model on a rectangular strip, via a method of transfer matrix, and this sign appears in the boundary conditions in the direction parallel to the growth line, (while an analogous sign appears, apparently from other sources, for the boundary conditions in the other direction). Other methods of solution show up how these signs have the same origin in the non-planarity induced by periodic boundary conditions "exchanging information" from side to side, e.g. the "four pfaffians" of Kasteleyn formula on a surface of genus 1.

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