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The Euclidean Matching Problem

A Thesis for the degree of Philosophiæ Doctor
in Theoretical Physics

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Ssd Fis/02

Stilicidi casus lapidem cavat.

LUCRETIUS
De rerum natura, I.313

To my family
and to Elena.

Abstract

In the present work we discuss some connections between Combinatorial optimization and Statistical physics. In particular, we analyze the so-called *Euclidean bipartite matching problem*, i.e., the matching problem between two different sets of points on an Euclidean domain. We consider the random version of the problem, where the points are independently and identically distributed according to a given probability distribution density. The presence of both randomness and Euclidean constraints makes the study of the average properties of the solution highly non trivial. We first summarize some known results about both matching problems in general and Euclidean matching problems in particular. We provide a complete and general solution for the one dimensional problem in the case of convex cost functional. Moreover, we propose an ansatz for the average optimal matching cost in the quadratic case, obtaining both an analytical expression for the finite size corrections in any dimension $d \geq 3$, and the correlation functions in the thermodynamical limit. Finally, we provide, using a functional approach, a general recipe for the computation of the correlation function of the optimal matching in any dimension and on a generic domain.

Contents

Chapter 1

Supervisor's foreword

The Euclidean Matching Problem is a particular combinatorial optimization problem traditionally considered in the realm of computer science and mathematics.

When, instead of a given instance of an optimization problem, a whole class of problems is considered, according to a suitable probability distribution, methods, ideas and powerful mathematical tools that physicists have developed in the context of statistical mechanics of systems with frustration and disorder can be applied and have been shown to be very effective also in this area.

For example, the Random Assignment Problem, in which the costs associated to each couple are uncorrelated and identically distributed random variables, has been deeply understood in this way. Its solution represents a sort of mean field approximation for the case, as the one studied here, where correlations are present, for example, because of the geometry of the underlying ambient space.

In this thesis, among other results, a new elegant method is introduced to study the effects of these correlations. It is amusing to discover that a field theory in the continuum is of help to study the asymptotic properties of a discrete number of points in the limit in which this number becomes very large. And the relevant continuum field theory is similar to a reduced version of electromagnetism in which the role of the Gauss law as a constraint is replaced by a transport condition.

Such a similarity allows to study not only the average optimal cost but also the correlation function, under very general conditions, for any distribution of points in Euclidean space (of any dimension). Deep relations among discrete optimization problems, variational methods in the continuum, probability theory, statistical mechanics of disordered systems, classical field theory are put into evidence in this work. So that readers from different background can find useful inspirations to enlarge their view-point.

Sergio Caracciolo
Università degli Studi di Milano
May 2016, Milan, Italy

Chapter 2

Introduction

In 1781 Gaspard Monge (1746–1818), published his *Mémoire sur la théorie des déblais at des remblais* [14], where he discussed the following, very simple, problem.

> "Let us suppose that we have a certain number of mines and the same number of deposits. We want to associate each mine to one deposit only (where the production of the considered mine will be transported and stored). How can we perform this matching in such a way that the total transport cost is minimum?"

Monge made the quite natural assumption that the transport cost from a certain mine to a certain deposit is a given function of the distance between the mine and the deposit themselves. The positions of the mines and of the deposits are supposed assigned and therefore the problem is fixed in all its details. However the problem of finding an *optimal matching* between mines and deposits, e.g. given their positions on a chart, is simple in its formulation, but quite difficult to solve. Indeed, if the number of mines is N, we have $N!$ ways to match mines and deposits and we have to select the cheapest one among them. It is evident that, if N is quite large, a brute force approach is not feasible. Even if we are able to compute the cost of a matching in, let us say, 10^{-3} seconds, for $N = 20$ the computation of the cost of all configurations requires $7.7 \cdot 10^7$ years. It is clear therefore that a smarter approach is needed. Only in 1955[1] Harold Kuhn proposed an algorithm, called *Hungarian algorithm*, that is able to solve the problem in a computation time that scales as N^3 in the size N of the original problem [9]. The Hungarian algorithm proves that the problem is in the so called P computational complexity class, but still the required computation time grows quite rapidly as N increases.

At the time of Monge, the idea of an algorithmic solution for the matching problem was interesting by itself, but of no practical use, due to the lack of computational resources. Monge therefore reformulated the problem in a "continuum" version, in which the matching between points of different types was replaced by a transport map between two different measures on the same domain. This map had to minimize a certain cost functional. In measure theory this problem is called *optimal transport problem*. The original combinatorial problem took therefore the new form of an interesting problem, at the edge between measure theory and geometry, and it started to be extensively analyzed by mathematicians working on these topics. In 1938, Leonid V. Kantorovič (1912–1986) proposed a new (dual) reformulation of the transport problem, that is now of paramount importance in measure theory, economics and linear programming and led him to the Nobel prize in Economics in 1975. During the last decades the interest in the theory of optimal transport has increased exponentially in the mathematical community, due to the results of Luigi Ambrosio, Luis Caffarelli, Alessio Figalli, Cédric Villani and others on the existence and the properties of optimal transport maps (see the recent review paper of Bogachev and Kolesnikov [1] and the monograph by Villani [15]).

Both the previous approaches (the continuum one and the combinatorial one) assume that no disorder or randomness is present in the problem (the point positions or the measures are supposed given). When we find the optimal matching between the two sets of points or the optimal map between the two measures, the problem is solved. But we can consider the problem under a different point of view. Let us suppose, for example, that we have two sets of *random* points on a certain domain and that we ask for the optimal matching between them in such a way that a

[1] Remarkably, in 2006 it has been discovered that the problem had been solved by Carl Gustav Jacobi [8]. His work was published in 1890 in Latin, but was ignored at the time.

certain functional is minimized. The specific solution in this case is not of great interest. More interestingly, we may ask for the *average* properties of the optimal matching. This new problem, called in this version *random (or stochastic) Euclidean matching problem*, is not trivial at all. Indeed, *correlation* among the distances is present, due to the Euclidean constraints. Marc Mézard and Giorgio Parisi treated the Euclidean matching problem in a set of papers published between 1985 and 1988 [10–13]. They considered the so called *random assignment problem* as first order approximation. In the random assignment problem, the distances between points are supposed totally uncorrelated. The random assignment problem is therefore a sort of mean field model of the Euclidean problem. Subsequently, they introduced correlations as corrections to the mean field results, sketching the complete computation, that requires an infinite number of steps. Their results are remarkable for different important reasons. First, they were able to give a complete solution of the purely random case, obtaining the correct average optimal cost and its distribution. Second, their results were obtained using statistical physics techniques developed for the study of disordered systems. Their approach was therefore not rigorous, but the results were completely confirmed sixteen years later with rigorous probability arguments, inspired again by the theory of disorderd systems. They showed therefore that the statistical physics methods are extremely powerful to treat combinatorial problems in the presence of randomness.

In the present thesis we overview the main results obtained on the Euclidean matching problem in the last fifty years and we present the results of our investigation on this subject. The material is organized as follows.

In Chapter 3 and Chapter 4 we briefly review optimization theory and some basic results of physics of disordered systems. From one hand, the availability, in the last decades, of powerful computational resources encouraged the development of efficient algorithms to solve difficult optimization problems on graphs. On the other hand, dating back to the work of Edwards and Anderson [7] on spin glasses, the physics of disordered systems developed a pletora of powerful techniques with an impressive impact in many scientific fields and in particular in combinatorial optimization.

In Chapter 5 we deal with the main subject of the thesis, the Euclidean matching problem. We follow very different approaches, each one of them able to provide an insight on the problem and its peculiarities. In particular:

- we solve the one dimensional matching problem with convex cost using measure theory arguments and showing that, in the thermodynamical limit, the solution of the problem is equivalent to a Brownian bridge process [3], as first pointed out by Boniolo, Caracciolo, and Sportiello [2];
- we propose an ansatz for the optimal cost in any dimension, obtaining a precise analytical prediction for its scaling respect to the system size and for the finite size corrections to the average optimal cost [6];
- we compute the correlation function in any dimension for the optimal transport map in the thermodynamical limit, using both a generalization of the previous ansatz and a functional argument that provides a recipe for a very general case [4, 5];
- we develope the computation of Mézard and Parisi for the corrections to the mean field approximation of the Euclidean matching problem.

The previous results have been partially obtained in collaboration with Carlo Lucibello and Giorgio Parisi, from University “Sapienza” in Rome.

January 2015, Pisa, Italy

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Chapter 3

Graphs and optimization

3.1. Graph Theory

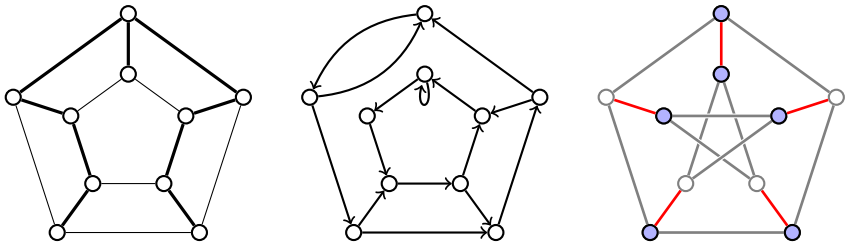
The matching problem is an important combinatorial problem defined on a *graph*. Graphs provide very often a pictorial representation of the mathematical structure underlying combinatorial optimization problems. On the other hand, graph theory is by itself rich of elegant results that can give us useful insights on many combinatorial and physical problems. For these reasons, we present here a very short introduction to the basic definitions and results of graph theory. We will refer mostly to the standard textbook of Diestel [3].

3.1.1. Fundamental definitions

A graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ is a pair of sets $(\mathscr{V}, \mathscr{E})$ such that $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. The elements of $\mathscr{V}$ are called *vertices* (or *nodes*), whilst the elements of $\mathscr{E}$ are usually called *edges*. We will denote by[1] $V = |\mathscr{V}|$ the number of elements of $\mathscr{V}$ (V is sometimes called *order* of $\mathtt{G}$) and we will suppose always that $V \in \mathbb{N}$ is finite, unless otherwise specified. Moreover, given a vertex v and an edge e, we say that v is *incident* with e if $v \in e$. In this case e is an edge *at* v and we will write $e \to v$. We will call $\mathscr{E}(v)$ the set of edges at v and $|\mathscr{E}(v)|$ the *degree* of v. We say that $u, v \in \mathscr{V}$ are *adjacent* if $(u, v) \in \mathscr{E}$; we will denote by ∂v the set of adjacent vertices to v. We define the *complete graph* $\mathtt{K}_V$ as the graph with V vertices in which each vertex is adjacent to all the others.

Two graphs $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ and $\mathtt{G}' = \mathrm{Graph}(\mathscr{V}'; \mathscr{E}')$ are *isomorphic*, or $\mathtt{G} \sim \mathtt{G}'$, if a bijection $\varphi \colon \mathscr{V} \to \mathscr{V}'$ exists such that $(u, v) \in \mathscr{E} \Leftrightarrow (\varphi(u), \varphi(v)) \in \mathscr{E}'$. Finally, given two graphs $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ and $\mathtt{G}' = \mathrm{Graph}(\mathscr{V}'; \mathscr{E}')$, if $\mathscr{V} \subseteq \mathscr{V}'$ and $\mathscr{E} \subseteq \mathscr{E}'$, than we say that $\mathtt{G}$ is a *subgraph* of $\mathtt{G}'$ and $\mathtt{G}'$ is a *supergraph* of $\mathtt{G}$: in symbols, $\mathtt{G} \subseteq \mathtt{G}'$. We say that $\mathtt{G}$ is a *spanning subgraph* of $\mathtt{G}'$ if and only if $\mathscr{V} = \mathscr{V}'$, see Fig. 3.1.1a. A *directed graph* (or *digraph*) is a graph in which we assign an *initial vertex* and a *terminal vertex* to each edge in the edge set, see Fig. 3.1.1b. In digraphs edges are *ordered* couples of vertices. In particular, if the vertex u and

[1] In the present work, given a set A of $N \in \mathbb{N}$ elements, we will use the notation $|A| = N$ for the cardinality of the set.



(a) Example, in bold, of spanning subgraph (in this case, a tree) of a given graph.

(b) Directed graph with a loop.

(c) The Petersen's graph with a vertex cover (in blue) and an edge cover (in red).

Figure 3.1.1.: Three examples of graphs.

the vertex v are respectively the initial vertex and the terminal vertex of the edge (u, v), we write

$$\vec{e} = \overrightarrow{(u, v)}.$$

In a digraph, an edge in which the initial vertex and terminal vertex coincide is called *loop*.

A *vertex cover* of $\mathtt{G}$ is a subset of $\mathscr{V}$ such that any edge of $\mathtt{G}$ has at least one endpoint in it. The *vertex covering number* $c_V(\mathtt{G})$ is the smallest possible size of a vertex cover of $\mathtt{G}$. Similarly, an *edge cover* of $\mathtt{G}$ is a subset of $\mathscr{E}$ such that any vertex of $\mathtt{G}$ is the endpoint of at least one element in it. The *edge covering number* $c_E(\mathtt{G})$ is the smallest possible size of an edge cover of $\mathtt{G}$ (see Fig. 3.1.1c).

Given two graphs $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ and $\mathtt{G}' = \mathrm{Graph}(\mathscr{V}; \mathscr{E}')$ with the same vertex set $\mathscr{V}$, we define

$$\mathtt{G} \triangle \mathtt{G}' \coloneqq \mathrm{Graph}(\mathscr{V}_{\mathtt{G}\triangle\mathtt{G}'}; \mathscr{E} \triangle \mathscr{E}'), \tag{3.1.1}$$

where

$$\mathscr{E} \triangle \mathscr{E}' \coloneqq (\mathscr{E} \cup \mathscr{E}') \setminus (\mathscr{E} \cap \mathscr{E}')$$

is the symmetric difference between the two edge sets and $\mathscr{V}_{\mathtt{G}\triangle\mathtt{G}'}$ is the set of the vertices that are ends for the edges in $\mathscr{E} \triangle \mathscr{E}'$.

A certain graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ can be represented also by a $V \times E$ matrix, called *incidence matrix*, $\mathsf{B} \coloneqq (b_{ij})_{ij}$ in such a way that

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Similarly, we introduce the more commonly used $V \times V$ *adjacency matrix* $\mathsf{A} \coloneqq (a_{ij})_{ij}$, such that

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in \mathscr{E} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

An undirected graph has a symmetric adjacency matrix, and, therefore, A has a real spectrum, called *spectrum of* $\mathtt{G}$. A *weighted* graph is a graph in which we associate a certain function $w \colon \mathscr{E} \to \mathbb{R}$ to the graph itself. Given an edge $e \in \mathscr{E}$, we say that $w(e)$ is the *weight* of e. For a given weighted graph $\mathtt{G}$, we can introduce the *weighted adjacency matrix* as $\mathsf{W} \coloneqq (w(e_{ij})a_{ij})_{ij}$.

3.1.2. Paths, forests and multipartite graphs

A *path* $\mathtt{P} = \mathrm{Graph}(\mathscr{V}_\mathtt{P}; \mathscr{E}_\mathtt{P})$ in a graph $\mathtt{G}$ is a particular subgraph $\mathtt{P} \subseteq \mathtt{G}$ such that $\mathscr{V}_\mathtt{P} = \{v_0, \dots, v_k\}$ is a set of distinct vertices and the edge set is given by

$$\mathscr{E}_\mathtt{P} = \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)\}.$$

We say that the path links v_0 and v_k and has length k. A graph $\mathtt{G}$ is *connected* if, for any couple of vertices $v, u \in \mathscr{V}$, there exists a path in $\mathtt{G}$ linking them. The length $\delta(v, u)$ of the shortest path linking two vertices u and v of $\mathtt{G}$ is called *distance* of u and v on $\mathtt{G}$. The diameter of a graph $\mathrm{diam}(\mathtt{G})$ is given by

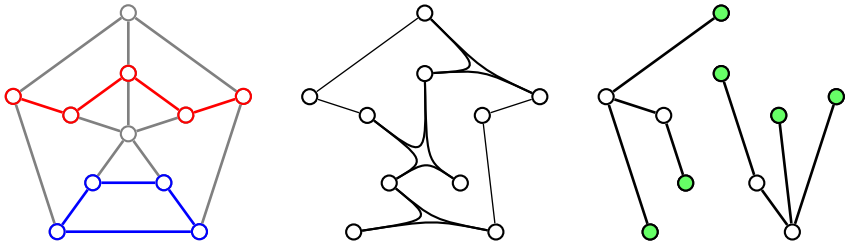
$$\mathrm{diam}(\mathtt{G}) = \max_{u,v} \delta(u, v). \tag{3.1.4}$$

If the graph $\mathtt{G}$ is not connected then, by definition, $\mathrm{diam}(\mathtt{G}) = +\infty$. Any graph can be expressed as union of maximal connected subgraphs, called *components*, and a connected graph is a graph with only one component. Given a connected graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$, the subset $\mathfrak{X} \subset \mathscr{V} \cup \mathscr{E}$ is said to be a *separating set* if $\mathtt{G}' = \mathrm{Graph}(\mathscr{V} \setminus \mathfrak{X}; \mathscr{E} \setminus \mathfrak{X})$ is not connected. If $\mathfrak{X}$ contains only a single vertex, the vertex is said a *cutvertex*. Similarly if $\mathfrak{X}$ contains one edge, we say that the selected edge is a *bridge*. A graph is called *separable* if it is not connected or has a cutvertex. In particular, we will call *biconnected* a connected graph with no cutvertices.

For $k \geq 3$, a *cycle* $\mathtt{C} = \mathrm{Graph}(\mathscr{V}_\mathtt{C}; \mathscr{E}_\mathtt{C})$ in a graph $\mathtt{G}$ is a subgraph $\mathtt{C} \subseteq \mathtt{G}$ such that $\mathscr{V}_\mathtt{C} = \{v_0, \dots, v_{k-1}\}$ is a set of distinct vertices and

$$\mathscr{E}_\mathtt{C} = \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-2}, v_{k-1}), (v_{k-1}, v_0)\}, \tag{3.1.5}$$

see Fig. 3.1.2a. We say that such a cycle has length k. The minimum length of a cycle contained in a certain graph $\mathtt{G}$ is called *girth* of $\mathtt{G}$, whilst the maximum length is called *circumference* of



(A) Path (in red) and cycle (in blue) in a graph.

(B) Example of hypergraph with two 3-hyperedges and one 4-hyperedge.

(C) Forest composed by two trees. Leaves are colored in green.

FIGURE 3.1.2.: Examples of paths, cycles, separable graphs, trees and forests.

$\mathtt{G}$. A *Hamiltonian path* in a graph is a path traversing all vertices of the graph exactly once. A Hamiltonian path that is a cycle is called *Hamiltonian cycle*. Similarly, an *Eulerian path* in a graph is a path traversing all edges of the graph exactly once, whereas an Eulerian path that is a cycle is called *Eulerian cycle*

Clearly, not all graphs contain cycles: an acyclic graph is called *forest*, a connected forest is called *tree*. A non-trivial forest (i.e., a forest with $|\mathscr{E}| \neq 0$) has always 1-degree vertices, called *leaves*, see Fig. 3.1.2c. Given a tree, sometimes a specific vertex is considered special and called *root* of the tree. With reference to the root, we define the *height* of a vertex as the distance of the vertex itself from the root.

A graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ is called *q-partite* (or, less precisely, multipartite) if we can partition $\mathscr{V}$ into q subsets (or *classes*),

$$\mathscr{V} = \bigcup_{i=1}^{q} \mathscr{V}_i, \quad \mathscr{V}_i \cap \mathscr{V}_j = \emptyset \text{ for } i \neq j,$$

in such a way that every edge in $\mathscr{E}$ connects vertices in different classes. We will denote such a graph as $\mathtt{G} = \mathrm{Graph}(\mathscr{V}_1, \ldots, \mathscr{V}_q; \mathscr{E})$. A q-partite graph is called *complete* if, given two vertices in two different classes, there exists an edge connecting them. We will denote the complete q-partite graph by $\mathtt{K}_{V_1,\ldots,V_q}$, $V_i := |\mathscr{V}_i|$. If $q = 2$ a multipartite graph is called *bipartite*. Bipartite graphs have the characterizing property of having no odd cycles.

3.1.3. Euler's formula and planar graphs

Let us consider now the set $\mathcal{S}_\mathtt{G}$ of the *spanning* subgraphs of a given graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$. The set $\mathcal{S}_\mathtt{G}$ contains the set $\mathcal{S}_\mathtt{G}^\mathrm{E}$ of the *Eulerian subgraphs*, i.e., the subgraphs containing a cycle such that each edge of the subgraph is used only once. This space has the peculiar property of being closed under the symmetric difference operation $\triangle$. In other words, if $\mathtt{G}_1 \in \mathcal{S}_\mathtt{G}^\mathrm{E}$ and $\mathtt{G}_2 \in \mathcal{S}_\mathtt{G}^\mathrm{E}$, then $\mathtt{G}_1 \triangle \mathtt{G}_2 \in \mathcal{S}_\mathtt{G}^\mathrm{E}$. The dimension of $\mathcal{S}_\mathtt{G}^\mathrm{E}$ respect to the operation $\triangle$ is called *cyclomatic number* L of the graph $\mathtt{G}$. Indeed, L is the number of cycles in $\mathtt{G}$ that cannot be obtained by other subgraphs through symmetric difference. These cycles are called *independent cycles* and play a role of a "basis" in the space of Eulerian subgraphs. Let us call $\mathscr{L}(\mathtt{G})$ the set of independent cycles. If a graph $\mathtt{G}$ has κ components, then the following general Euler's formula can be proved:

$$V + L = E + \kappa. \tag{3.1.6}$$

The relation above is particularly easy to apply for *planar graphs*. Planar graphs are graph that can be embedded (i.e., drawn) in the surface of a sphere in such a way that no edge crossing appears. If such embedding is considered, we can immediately recognize F cycles on the sphere, called *faces*. Each face is characterized by a simple property, i.e., it divides the sphere into two regions in such a way that in one of them there is no paths having both endpoints on the face itself. The cyclomatic number is then recovered as $L = F - 1$.

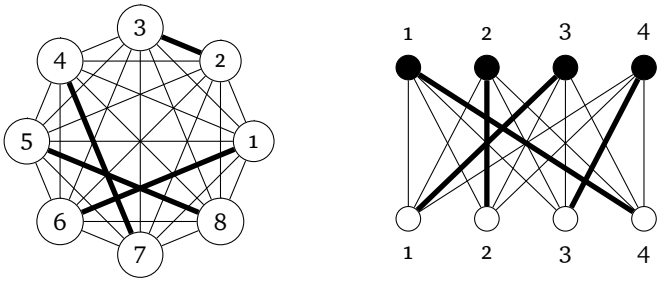


FIGURE 3.1.3.: On the left, complete graph $\mathtt{K}_8$ and an example of perfect matching on it. On the right, complete bipartite graph $\mathtt{K}_{4,4}$ and an example of perfect bipartite matching on it.

3.1.4. HYPERGRAPHS

In an *hypergraph* $\mathtt{G} = \mathrm{HyperGr}(\mathscr{V}; \mathscr{E})$ with vertex set $\mathscr{V}$, an edge $e \in \mathscr{E}$ can connect more than two vertices, see Fig. 3.1.2b. If the edge e connects k vertices, we say that e is a k-hyperedge and we write $|e| = k$. We say that a *walk* on the hypergraph is an ordered sequence $(v_0, e_1, v_1, \ldots, e_p, v_p)$, $\{v_i\}_{i=0,\ldots,p} \in \mathscr{V}$, $\{e_i\}_{i=1,\ldots,p} \in \mathscr{E}$. A *path* is a walk in which all vertices and all edges are distinct respectively. A *cycle* is a path having $p \geq 2$ and $v_0 = v_p$. An hypergraph is *connected* if, given two distinct vertices $v, u \in \mathscr{V}$, there exists a walk connecting them. Remarkably, the following generalization of the Euler's formula holds for an hypergraph with κ connected components [1]

$$\sum_{e \in \mathscr{E}} |e| - E - V + \kappa \geq 0, \tag{3.1.7}$$

where, as usual, $E = |\mathscr{E}|$ and $V = |\mathscr{V}|$.

3.1.5. MATCHINGS ON GRAPHS

We can finally introduce the main concept of this dissertation. We say that $\mathtt{M} = \mathrm{Graph}(\mathscr{V}_\mathtt{M}; \mathscr{E}_\mathtt{M}) \subseteq \mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ is a *matching* of *size* $|\mathscr{E}_\mathtt{M}|$ in $\mathtt{G}$ if, given two edges in $\mathtt{M}$, they have no vertex in common, see Fig. 3.1.3. The size of the largest matching (*maximum matching*) in $\mathtt{G}$, $m(\mathtt{G})$, is called *matching number* of $\mathtt{G}$. If $\mathscr{V}_\mathtt{M} = \mathscr{V}$ we say that $\mathtt{M}$ is *perfect*. The following fundamental result, proved in 1931 by Dénes Kőnig, holds.

Theorem 3.1.1 (Kőnig's minimax theorem)**.** *Let* $\mathtt{G}$ *be a bipartite graph. Then*

$$c_V(\mathtt{G}) = m(\mathtt{G}). \tag{3.1.8}$$

In 1935, Philip Hall proposed an equivalent formulation of the theorem above.

Theorem 3.1.2 (Hall's theorem)**.** *A bipartite graph* $\mathtt{G}$ *has a matching if and only if*

$$\left| \bigcup_{x \in \mathfrak{X}} \partial x \right| \geq |\mathfrak{X}|, \quad \forall \mathfrak{X} \subseteq \mathscr{V}_1. \tag{3.1.9}$$

In the following we will denote by $\partial_\mathtt{G}\mathfrak{X} := \bigcup_{x \in \mathfrak{X}} \partial x$, where the subscript $\mathtt{G}$ stress the fact that we refer to the topology of $\mathtt{G}$.

Corollary 3.1.3 (Marriage theorem)**.** *Let* $\mathtt{G} = \mathrm{Graph}(\mathscr{V}_1, \mathscr{V}_2; \mathscr{E})$ *be a bipartite graph. Then* $\mathtt{G}$ *admits a perfect matching if and only if the property in Eq.* (3.1.9) *holds and moreover*

$$|\mathscr{V}_1| = |\mathscr{V}_2|. \tag{3.1.10}$$

If $|\mathscr{V}_1| = |\mathscr{V}_2| = V$, numbering separately the vertices of each class in the graph, a perfect matching can be expressed as a permutation of V elements. For a proof of the previous statements, see [9]. Given a matching $\mathtt{M}$ on $\mathtt{G}$ and a path $\mathtt{P} \subset \mathtt{G}$, we say that the path $\mathtt{P}$ is $\mathtt{M}$-*alternating* if the edges of $\mathtt{P}$ are alternately in and not in $\mathtt{M}$.

3.2. Optimization problems

In the previous Section we have discussed some fundamental definitions about graphs. Graphs provides very often a mathematical representation of combinatorial problems and, in particular, many combinatorial problems can be described in terms of concepts introduced in graph theory. To proceed further, let us first introduce the definition of *optimization problem* [13].

An *instance of an optimization problem* is a couple of mathematical elements, i.e., a *space of feasible solutions* $\mathscr{F} \neq \emptyset$ and a *cost function*

$$\mathscr{C} : \mathscr{F} \to \mathbb{R}. \tag{3.2.1}$$

The target is to find the *globally optimal solution*, i.e. an element $x_o \in \mathscr{F}$ such that

$$\mathscr{C}[x_o] = \min_{x \in \mathscr{F}} \mathscr{C}[x]. \tag{3.2.2}$$

The set of all instances of a certain optimization problem is, by definition, the optimization problem itself. Observe that the cardinality of $\mathscr{F}$ was not specified and the existence of x_o is, *a priori*, not guaranteed.

Linear optimization problems To exemplify the given definitions, let us briefly discuss here the important class of linear optimization problems [13]. A *linear optimization problem* can be stated as follows: let us consider an $n \times m$ matrix of integers $\mathsf{A} = (a_{ij})_{ij}$, a vector of m integers $\mathbf{b} \in \mathbb{Z}^m$ and a vector of n integers $\mathbf{c} \in \mathbb{Z}^n$. We want to find a vector $\mathbf{X} = (X_i)_{i=1,\dots,n} \in (\mathbb{R}^+)^n$ such that

$$z \coloneqq \mathbf{c}^\intercal \cdot \mathbf{X} = \min_{\mathbf{x} \in \mathscr{F}} \mathbf{c}^\intercal \cdot \mathbf{x} \tag{3.2.3}$$

on a certain non-empty space $\mathscr{F}$ of feasible solutions, defined as follows

$$\mathscr{F} \coloneqq \left\{ \mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^+)^n : \mathsf{A} \cdot \mathbf{x} = \mathbf{b} \right\} \neq \emptyset. \tag{3.2.4}$$

The linear optimisation problem, when stated in the previous form, is said to be in a *standard form*. To have a solution of our optimization problem it is necessary that $n - \operatorname{rank}[\mathsf{A}] > 0$ (otherwise no point satisfies all the constraints). We require therefore for simplicity that $\operatorname{rank}[\mathsf{A}] = m < n$. In the space $\mathbb{R}^{n-m}$, the constraint condition $\mathsf{A} \cdot \mathbf{x} = \mathbf{b}$, with the additional constraints $x_i \geq 0 \ \forall i = 1, \dots, n$, delimits an $(n-m)$-dimensional convex polytope.

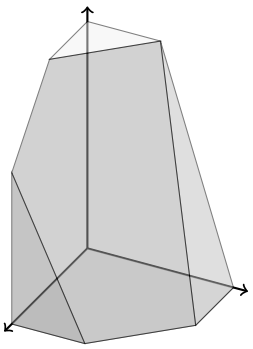


Figure 3.2.1.: An example of 3-dimensional convex polytope.

3.2.1. Combinatorial optimization

Combinatorial optimization deals with optimization problems in which the cardinality of $\mathscr{F}$ is *finite* for all instances, $|\mathscr{F}| \in \mathbb{N}$. In this case the problem has *always* at least one solution. However, in many cases the number of feasible solution is extremely large, and a brute-force approach is computationally unfeasible. To better exemplify these aspects, let us discuss some relevant (classical) combinatorial optimization problems in more details.

The Travelling Salesman Problem In the *Travelling Salesman Problem* (Tsp) a complete graph $\mathtt{K}_N = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ is given with a weight function $w : \mathscr{E} \to \mathbb{R}^+$, in such a way that a weight $w(e) \in \mathbb{R}^+$ is associated to each edge e of the graph. In the Tsp, the space $\mathscr{F}$ is given by all possible closed paths $\mathtt{h} = \mathrm{Graph}(\mathscr{V}_\mathtt{h}; \mathscr{E}_\mathtt{h})$ passing only once through each vertex. In other words, the the space of feasible solution in the Tsp is the set of Hamiltonian cycle on $\mathtt{K}_N$. The cost function is given by

$$\mathscr{C}^{\mathrm{Tsp}}[\mathtt{h}] \coloneqq \sum_{e \in \mathscr{E}_\mathtt{h}} w(e). \tag{3.2.5}$$

A similar formulation of the problem can be given on a generic (connected) graph $\mathtt{G}$. Observe that, working on $\mathtt{K}_N$, $|\mathscr{F}| = \frac{(N-1)!}{2}$. Therefore the direct inspection of the solution by computing all possible values of the cost function requires a huge amount of steps even for relatively small values of N. In a variation of this problem, the *Chinese Postman Problem*, the set $\mathscr{F}$ of feasible solutions is given by the set of all Eulerian cycles of the considered graph.

THE GRAPH q-COLORING PROBLEM The graph q-coloring problem (q-COL) is a problem defined on a graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$, $V := |\mathscr{V}|$. We want to assign to each vertex $v \in \mathscr{V}$ a number (a "color") $q_v \in \{1, \dots, q\}$, $q \in \mathbb{N}$, in such a way that the cost function

$$\mathscr{C}^{q\text{-COL}}[\mathbf{q}] := \sum_{(v,u)\in\mathscr{E}} \delta_{q_v, q_u} \tag{3.2.6}$$

is minimized. The set $\mathscr{F}$ is therefore given by all sets $\mathbf{q} = \{q_v\}_{v\in\mathscr{V}}$ such that $q_v \in \{1, \dots, q\}$. The number of feasible solutions is therefore q^V.

THE k-SAT PROBLEM The k-SAT problem is defined on an hypergraph $\mathtt{G} = \mathrm{HyperGr}(\mathscr{V}; \mathscr{E})$, $|\mathscr{V}| = V$, such that $|e| = k \ \forall e \in \mathscr{E}$. We give a quantity $J_e^v \in \{-1, 1\}$ for each edge e at $v \in \mathscr{V}$, depending on the edge and on the vertex. We search for the set $\boldsymbol{\sigma} = \{\sigma_v\}_{v\in\mathscr{V}} \in \mathscr{F}$, $\sigma_v \in \{-1, 1\}^V$ such that

$$\mathscr{C}^{k\text{-SAT}}[\boldsymbol{\sigma}] = \sum_{e\in\mathscr{E}} \prod_{v\in e} \frac{1 - J_e^v \sigma_v}{2} \tag{3.2.7}$$

is minimized. Again, in this case $|\mathscr{F}| = 2^V$ is exponential in the size V of the problem. Observe also that the cost function above, Eq. (3.2.7), reminds immediately a Hamiltonian function for a spin system on a graph. This analogy will become extremely relevant in the next Chapter.

MATCHING PROBLEMS Let us now consider a complete weighted graph $\mathtt{K}_{2N} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ with weight function $w\colon \mathscr{E} \to \mathbb{R}^+$. In the (monopartite) matching problem we want to find a perfect matching $\mathtt{M} \subseteq \mathtt{G}$ such that the cost functional

$$\mathscr{C}^{\mathrm{M}}[\mathtt{M}] := \frac{1}{|\mathscr{E}_{\mathtt{M}}|} \sum_{e\in\mathscr{E}_{\mathtt{M}}} w(e) \tag{3.2.8}$$

is minimized. If $w(e) = 1 \ \forall e \in \mathscr{E}$ the problem is sometimes called *cardinality matching problem*. The *k-assignment problem* is formulated on the complete weighted bipartite graph $\mathtt{K}_{N,M}$. In this problem we ask for an optimal matching in the graph $\mathtt{K}_{N,M}$ of cardinality $k \leq \min\{N, M\}$.

The *bipartite matching problem*, or simply *assignment problem*, is the N-assignment problem on the complete bipartite graph $\mathtt{K}_{N,N}$. Observe that, in assignment problems, the matching can be represented as a permutation of N elements $\sigma \in \mathscr{P}_N$, $\mathscr{P}_N$ set of all permutations of N elements. Indeed, given the complete bipartite graph $\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E})$, we can number the vertices as $\mathscr{V} = \{v_1, \dots, v_N\}$ and $\mathscr{U} = \{u_1, \dots, u_N\}$, and assume that $w\colon (v_i, u_j) \mapsto w_{ij}$. The optimal cost associated to the optimal matching $\mathtt{M}_o$ can be expressed therefore as

$$\mathscr{C}^{\mathrm{M}}[\mathtt{M}_o] = \min_{\sigma\in\mathscr{P}_N} \frac{1}{N} \sum_{i=1}^{N} w_{i\,\sigma(i)}. \tag{3.2.9}$$

In the assignment problem there are $N!$ possible solutions. However, we will show that, from the algorithmic point of view, this problem belongs to the class of "simple" combinatorial problems and can be solved with quite fast algorithms.

POLYNOMIAL AND NON-DETERMINISTIC POLYNOMIAL ALGORITHMIC CLASSES

Given an instance of an optimization problem, the main target is often to find the optimal solution. However many different kinds of questions can be asked about an optimization problem. For example, we may wonder if, for a given constant c, the set

$$S_c = \{x \in \mathscr{F}\colon \mathscr{C}[x] \leq c\} \tag{3.2.10}$$

is empty or not. This type of problem is called *decision problem*. In the theory of computational complexity [10, 12], each optimization problem is classified according to the running time (number of computational operations) and memory required to evaluate the decision problem or to find its solution. In particular, the class of *non-deterministic polynomial problems* NP is the set of problems such that, given a feasible solution x and a constant c, it is easy to evaluate if $x \in S_c$ or not. Here "easy" means that the check can be performed by a certain algorithm in a number of

computational operations that is polynomial in the size of the input. An algorithm is *polynomial* if the running time is bounded from above by a certain polynomial in the size of the input and *superpolynomial* if such a bound does not exist. We say that a certain optimization problem belongs to the class $\mathtt{P} \subseteq \mathtt{NP}$, or that it is a *polynomial-time problem*, if there exists a polynomial algorithm that solves it. It is still unknown whether $\mathtt{P} = \mathtt{NP}$ or $\mathtt{P} \neq \mathtt{NP}$. In NP it is possible to identify a special set of problems, called NP-complete problems. Every problem in NP can be mapped in an NP-complete problem with, at most, an additional polynomial computational overhead. It follows that, if a NP-complete problem is found to be in P, it would follow that $\mathtt{P} = \mathtt{NP}$.

Among the problems discussed above, TSP, q-COL with $q > 2$ and k-SAT with $k > 2$, are NP-complete problems. The assignment problem, instead, belongs to the P computational complexity class. Indeed, we will show below that a fast algorithm is available for its solution.

3.2.2. Algorithms for assignment

We present now two algorithms for the solution of the assignment problem. *Dantzig's algorithm*, called also *simplex method*, is a general algorithm for the solution of linear optimization problems. The assignment problem can be indeed seen as a linear optimization problem, as we will show below, and therefore the simplex method can be applied to it. The *Hungarian algorithm* is the classical algorithm for the solution of the assignment problem: its computational complexity is polynomial and therefore the assignment problem is in P. Another very important algorithm, derived from the *cavity method*, will be discussed in the next Chapter.

Here we consider the assignment problem on a weighted complete bipartite graph

$$\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E}), \quad \mathscr{V} = \{v_i\}_{i=1,\dots,N}, \quad \mathscr{U} = \{u_i\}_{i=1,\dots,N}. \tag{3.2.11}$$

The weight function is such that

$$w\colon (v_i, u_j) \mapsto w_{ij} \in \mathbb{R}^+. \tag{3.2.12}$$

A matching $\mathtt{M} = \mathrm{Graph}(\mathscr{V}_{\mathtt{M}}; \mathscr{E}_{\mathtt{M}}) \subset \mathtt{G}$ on the graph can be represented by a $N \times N$ matrix $\mathsf{M} = (m_{ij})_{ij}$ such that

$$m_{ij} = \begin{cases} 1 & \text{if } (v_i, u_j) \in \mathscr{E}_{\mathtt{M}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.13}$$

We can therefore identify the set of matchings on the graph $\mathtt{K}_{N,N}$ with the space of $N \times N$ matrices

$$\mathscr{F} = \left\{ \mathsf{M} = (m_{ij})_{\substack{i=1,\dots,N \\ j=1,\dots,N}} \,\middle|\, m_{ij} \in \{0,1\} \text{ and } \sum_{i=1}^{N} m_{ij} = \sum_{j=1}^{N} m_{ij} = 1 \ \forall i, j \right\}. \tag{3.2.14}$$

The matching cost for a given $\mathsf{M} \in \mathscr{F}$ is

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}] \coloneqq \frac{1}{N} \sum_{i,j} w_{ij} m_{ij}. \tag{3.2.15}$$

It is evident that this is completely equivalent to Eq. (3.2.8). M_o is the optimal matching if and only if

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o] \coloneqq \min_{\mathsf{M} \in \mathscr{F}} \frac{1}{N} \sum_{i,j} w_{ij} m_{ij}. \tag{3.2.16}$$

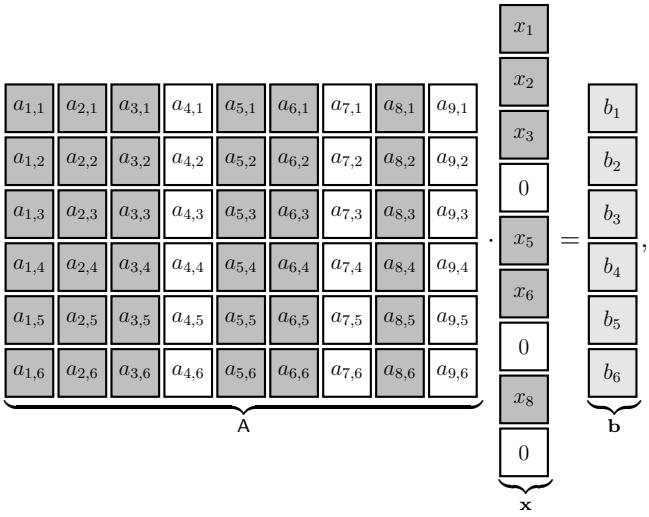
The simplex method

The algorithmic solution of linear optimization problems is due to G. B. Dantzig that formulated the celebrated simplex algorithm [2, 13]. Here we will sketch the approach, skipping some details that the reader can find properly treated in the cited references. As anticipated above, in a linear optimization problem we search for a vector $\mathbf{X} = (X_i)_{i=1,\dots,n} \in (\mathbb{R}^+)^n$ such that $\mathbf{c}^\intercal \cdot \mathbf{X} = \min_{\mathbf{x} \in \mathscr{F}} \mathbf{c}^\intercal \cdot \mathbf{x}$, where $\mathscr{F}$ is the space of vectors $\mathbf{x}$ of m real positive elements satisfying the constraint $\mathsf{A} \cdot \mathbf{x} = \mathbf{b}$. Here A is a matrix of $n \times m$ integers, $n > m$, $\mathbf{b}$ is a vector of m integers and $\mathbf{c}$ is a vector

of n integers. We suppose that $\operatorname{rank}[\mathsf{A}] = m$. We can select therefore m linearly independent columns in A, let us call them $\mathsf{B} := \{\mathbf{a}_{i_k}\}_{k=1,\dots,m}$, such that the system

$$\mathsf{A} \cdot \mathbf{x} = \mathbf{b}, \qquad x_j = 0 \text{ if } j \neq i_k \, \forall k = 1, \dots m \tag{3.2.17}$$

has a unique solution. The problem is now in the form



(3.2.18)

where we have highlighted the submatrix B. If the solution $\mathbf{x}$ of the previous problem has $x_i \geq 0$, it is called *basic feasible solution* (BFS). Remarkably, it can be proved that the optimal solution that we are searching for, *is* a BFS. BFSs have an easy geometrical meaning. Indeed, let us consider the polytope associated to the matrix A. It can be proved that

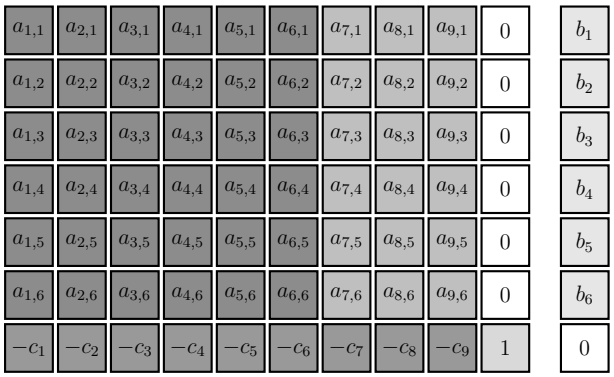
$$\mathbf{x} \in \mathscr{F} \text{ and } \mathbf{x} \text{ is BFS} \Leftrightarrow \mathbf{x} \text{ is a vertex of the polytope.} \tag{3.2.19}$$

For any instance of a linear programming problem there is an optimal BFS. By the previous operation, let us suppose that a certain BFS $\mathbf{x}^*$ is known and let us suppose also that this BFS corresponds for the sake of simplicity to the set of columns $\{\mathbf{a}_i\}_{i=1,\dots,m}$ of A, in such a way that $\mathbf{x}^* = (x_1^*, \dots, x_m^*, 0, \dots, 0)$. This solution can be the starting pointing of our search for the optimal solution. From what we said, indeed, an optimal solution can be found among the vertices of the polytope. The simplex method therefore is such that *we move from one vertex to another* through proper pivoting operations, until the optimal vertex is found.

The first step is to write down the *simplex tableaux*:

$$\begin{cases} \mathsf{A} \cdot \mathbf{x} = \mathbf{b} \\ -\mathbf{c}^\intercal \cdot \mathbf{x} + z = 0 \end{cases} \Rightarrow \left(\begin{array}{cc|c} \mathsf{A} & 0 & \mathbf{b} \\ -\mathbf{c}^\intercal & 1 & 0. \end{array} \right) \tag{3.2.20}$$

The structure of the matrix is therefore (in the pictures, $m = 6$ and $n = 9$)



(3.2.21)

The first m columns of A correspond to the submatrix B associated to the solution $\mathbf{x}^*$ discussed above, that is our starting point. After a sequence of row operation we can transform the simplex tableaux in the following way

$$\left(\begin{array}{cccccc|ccc|c|c}
1 & 0 & 0 & 0 & 0 & 0 & \tilde{a}_{7,1} & \tilde{a}_{8,1} & \tilde{a}_{9,1} & 0 & x_1^* \\
0 & 1 & 0 & 0 & 0 & 0 & \tilde{a}_{7,2} & \tilde{a}_{8,2} & \tilde{a}_{9,2} & 0 & x_2^* \\
0 & 0 & 1 & 0 & 0 & 0 & \tilde{a}_{7,3} & \tilde{a}_{8,3} & \tilde{a}_{9,3} & 0 & x_3^* \\
0 & 0 & 0 & 1 & 0 & 0 & \tilde{a}_{7,4} & \tilde{a}_{8,4} & \tilde{a}_{9,4} & 0 & x_4^* \\
0 & 0 & 0 & 0 & 1 & 0 & \tilde{a}_{7,5} & \tilde{a}_{8,5} & \tilde{a}_{9,5} & 0 & x_5^* \\
0 & 0 & 0 & 0 & 0 & 1 & \tilde{a}_{7,6} & \tilde{a}_{8,6} & \tilde{a}_{9,6} & 0 & x_6^* \\
0 & 0 & 0 & 0 & 0 & 0 & -\tilde{c}_7 & -\tilde{c}_8 & -\tilde{c}_9 & 1 & z_{\mathsf{B}}
\end{array}\right) \tag{3.2.22}$$

where z_{B} is the value of z on the current BFS. The solution $\mathbf{x}^*$ appears in the last column, being x_i^* for $i = 1, \dots, m$. If all the entries of $\{\tilde{c}_i\}_{i=m+1,\dots,n}$ are positive, then the current BFS is optimal and z_{B} is the optimal cost. Indeed, if we consider a solution in the form $\mathbf{x}^{**} = (x_1^*, \dots, x_m^*, x_{m+1}, 0, \dots, 0)$, we will have for it $z = z_{\mathsf{B}} + \tilde{c}_{m+1} x_{m+1}^* > z_{\mathsf{B}}$.

If this is not the case, we have to proceed further. We choose a non-zero pivot element $\tilde{a}_{rc} \neq 0$ in the simplex tableaux, and we multiply the corresponding row for $\tilde{a}_{rc}^{-1}$. Proper multiples of the new row are added to the remaining rows of $\tilde{\mathsf{A}}$ in such a way that the c-th column of the new matrix has 1 in correspondence of the position (r, c) and zero otherwise. The chosen variable is a *new* basic variable and it is called *entering variable*. It substitute the old r-th basic variable, called now *leaving variable*. We switch the r-th column with the current c-th column to obtain a new simplex tableaux in the form (3.2.22). Due to the fact that the value of z must be minimized, the entering variable is chosen in a column c in such a way that $\tilde{c}_c < 0$ (non-zero values in the direction of the new selected component decrease z). The condition that the new solution must be *feasible* determines a criterion for the row: it can be shown that this condition implies that, being c the chosen pivot column, the row r must be such that $\tilde{a}_{rc}^{-1} x_r^*$ is minimum among all rows r. The iteration of this sequence of steps leads to the optimal solution exploring the BFSs. Note that the method requires as starting point a BFS.

Finally, observe that if we have constraints expressed in terms of inequalities, e.g.,

$$\sum_{j=1}^{n} a_{ij} x_j < b_i, \tag{3.2.23}$$

we can introduce a new *slack variable* $s_i \geq 0$ for each inequality and write it as $\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$. Obviously we can introduce a "dependence" of z from the new variables in a trivial way as $z = \sum_{i=1}^{n} c_i x_i + 0 \cdot \sum_j s_j$.

The simplex method is, practically, very efficient. However, Klee and Minty [7] proved that there exist linear problems such that, in the worst case, the convergence time of the simplex algorithm is exponentially large in the size of the input.

Application to the assignment problem The simplex algorithm can be applied to the assignment problem quite straightforwardly [2]. Given a matching problem with a cost function as in Eq. (3.2.15), let us consider the following space of feasible solutions,

$$\mathscr{F} = \left\{ \mathbf{x} \in \left(\mathbb{R}^+\right)^{N^2} | \mathsf{A} \cdot \mathbf{x} = \mathbf{1} \right\}. \tag{3.2.24}$$

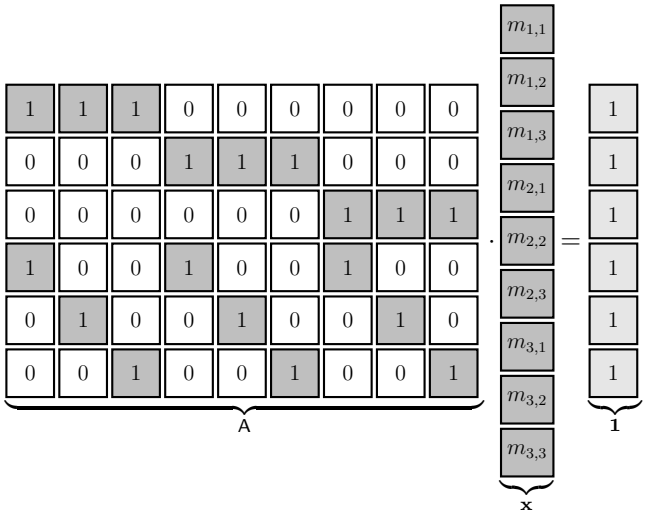
Here $\mathbf{1}$ is a vector of $2N$ elements all equal to 1 and $\mathsf{A} = (a_{ij})_{ij}$ is a $2N \times N^2$ matrix such that

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq i \leq N \text{ and } (i-1)N < j \leq iN, \\ 1 & \text{if } N+1 \leq i \leq 2N \text{ and } j - i + N \mod N = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.25}$$

We search for the vector $\mathbf{x}_o$ such that

$$\mathbf{c}^{\intercal} \cdot \mathbf{x}_o = \min_{\mathbf{x} \in \mathscr{F}} \mathbf{c}^{\intercal} \cdot \mathbf{x}. \tag{3.2.26}$$

Here $\mathbf{c} = (c_i)_i$ is a N^2 dimensional column vector such that $w_{ij} = c_{(i-1)N+j}$ for $i = 1, \dots, N$ and $j = 1, \dots, N$. The vector $\mathbf{x} = (x_i)_i$ can be easily identified with a $N \times N$ matching matrix $\mathsf{M} = (m_{ij})_{ij}$, putting $m_{ij} \equiv x_{(i-1)N+j}$ and Eq. (3.2.18) recovers the constraints in Eq. (3.2.14). For $N = 3$, Eq. (3.2.18) has therefore the form



(3.2.27)

The Hungarian algorithm

We analyze now in some details the *Hungarian algorithm* [6] for the solution of the assignment problem on the complete graph[2] $\mathtt{K}_{N,N}$. This algorithm was proposed by Harold W. Kuhn in 1955 [8] and it has polynomial computational complexity, proving that the assignment problem belongs to the P computational complexity class. Indeed, the Hungarian algorithm, in the version of Munkres [11], has time complexity $O(N^4)$ in the size N of the input. Dinic and Kronrod [4] and Edmonds and Karp [5] were later able to decrease the computational complexity to $O(N^3)$. Remarkably, the Hungarian algorithm is deeply connected with the more general theory of the cavity method. The algorithm is named in honor of two Hungarian mathematicians, Dénes Kőnig and Jenő Egerváry, that proved the fundamental theoretical results behind the algorithm elaborated by Kuhn.

As above, we consider the complete bipartite graph $\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E})$, $\mathscr{V} = \{v_i\}_{i=1,\dots,N}$, $\mathscr{U} = \{u_j\}_{j=1,\dots,N}$. The target is to find the matching matrix M_0 such that the matching cost

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}] \coloneqq \frac{1}{N} \sum_{i,j} w_{ij} m_{ij}, \quad w_{ij} \geq 0 \ \forall i, j, \tag{3.2.28}$$

is minimized. The matrix M_o minimizing the cost above is also a solution for the matching problem associated with the shifted cost

$$\mathscr{C}^{\mathrm{M}}_{h_0}[\mathsf{M}] \coloneqq \frac{1}{N} \sum_{i,j} w_{ij} m_{ij} + h_0. \tag{3.2.29}$$

In particular, the cost function in Eq. (3.2.29) is invariant under the *gauge transformation*

$$w_{ij} \mapsto w_{ij} - \lambda_i - \mu_j, \quad h_0 \mapsto h_0 + \frac{1}{N} \sum_{i=1}^{N} \lambda_i + \frac{1}{N} \sum_{j=1}^{N} \mu_j. \tag{3.2.30}$$

[2] More general polynomial algorithms are available to solve the matching problem on weighted graph $\mathtt{K}_{N,M}$, $N \neq M$.

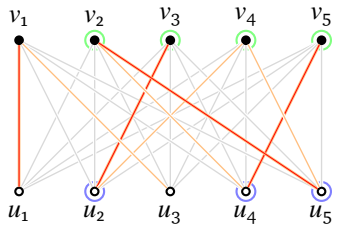


FIGURE 3.2.2.: The complete $\mathtt{K}_{5,5}$ with an equality graph $\mathtt{Z}$ (in orange) and the corresponding maximum matching $\mathtt{M}$ (in red) on it. An example of set $\mathscr{V}^*$ (green) with corresponding $\partial_{\mathtt{Z}}\mathscr{V}^*$ (blue) is also depicted.

Gauge transformations work directly on the $N \times N$ weight matrix $\mathsf{W} = (w_{ij})_{ij}$, that becomes the only object to work on for the solution. The two column vectors of real values $\boldsymbol{\lambda} = (\lambda_i)_{i=1,\dots,N}$ and $\boldsymbol{\mu} = (\mu_i)_{i=1,\dots,N}$ identify the gauge transformation. We say that the gauge is *proper* if

$$w_{ij} \mapsto w_{ij} - \lambda_i - \mu_j \geq 0, \quad \forall i, j. \tag{3.2.31}$$

It is said in this case that the set $\{\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N\}$ is a *feasible node-weighting*. The following theorem about proper gauges holds [9].

Theorem 3.2.1 (Egerváry's theorem)**.** *There exists a proper gauge* $(\boldsymbol{\lambda}^o, \boldsymbol{\mu}^o)$ *such that the cost of the optimal assignment* M_o *is given by*

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o] = \frac{\sum_{i=1}^N \lambda_i^o + \sum_{j=1}^N \mu_j^o}{N}. \tag{3.2.32}$$

Moreover, this value is maximal among all possible proper gauges, i.e.,

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o] = \max_{(\boldsymbol{\lambda},\boldsymbol{\mu})\ proper} \frac{\sum_{i=1}^N \lambda_i + \sum_{j=1}^N \mu_j}{N}. \tag{3.2.33}$$

Let us suppose that a proper gauge transformation has been performed on our matrix

$$\mathsf{W} \xrightarrow[\boldsymbol{\mu}]{\boldsymbol{\lambda}} \tilde{\mathsf{W}} = \mathsf{W} - \boldsymbol{\mu} \otimes \mathbf{1}^\intercal - \mathbf{1} \otimes \boldsymbol{\lambda}^\intercal, \tag{3.2.34}$$

where $\mathbf{1} = (1)_{i=1,\dots,N}$ is an N-dimensional column vector. We can construct, on the basis of the new weight matrix $\tilde{\mathsf{W}}$, the *equality subgraph* $\mathtt{Z} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E}_{\mathtt{Z}}) \subseteq \mathtt{K}_{N,N}$ such that $e \in \mathscr{E}_{\mathtt{Z}} \Leftrightarrow \tilde{w}(e) = 0$, being $\tilde{w}(e)$ the weight associated to the edge e by the new weight matrix $\tilde{\mathsf{W}}$. If $\mathtt{Z}$ contains a perfect matching $\mathtt{M}_o \subseteq \mathtt{Z} \subseteq \mathtt{K}_{N,N}$, then $\mathtt{M}_o$ is the optimal matching[3], having a matching cost given by the Eq. (3.2.32).

If $\mathtt{Z}$ does not contain a perfect matching, we have to perform a new gauge transformation (see Fig. 3.2.2). We search now in $\mathscr{V}$ for a set of vertices $\mathscr{V}^*$ such that $|\partial_{\mathtt{Z}}\mathscr{V}^*| < |\mathscr{V}^*|$. The existence of such subset is guaranteed by the Kőnig's theorem (see the formulation of Hall). We apply then the following gauge $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$:

$$\boldsymbol{\lambda}^* = (\lambda_i^*)_i \colon \lambda_i^* = \begin{cases} \gamma & \text{if } v_i \in \mathscr{V}^*, \\ 0 & \text{if } v_i \notin \mathscr{V}^*, \end{cases} \tag{3.2.35}$$

$$\boldsymbol{\mu}^* = (\mu_j^*)_j \colon \mu_j^* = \begin{cases} -\gamma & \text{if } u_j \in \partial_{\mathtt{Z}}\mathscr{V}^*, \\ 0 & \text{if } u_j \notin \partial_{\mathtt{Z}}\mathscr{V}^*, \end{cases} \tag{3.2.36}$$

where

$$\gamma = \min_{\substack{v_i \in \mathscr{V}^* \\ u_j \notin \partial_{\mathtt{Z}}\mathscr{V}^*}} \tilde{w}_{ij}. \tag{3.2.37}$$

The obtained weight matrix can be associated again to a new equality subgraph, in which we search again for a perfect matching. The algorithm proceeds repeating the last steps (see Fig. 3.2.3) until a perfect matching is found. The convergence of the algorithm in polynomial time is proved, e.g., in [6].

[3] Observe that different matching solutions are in general possible, but corresponding to the same cost.

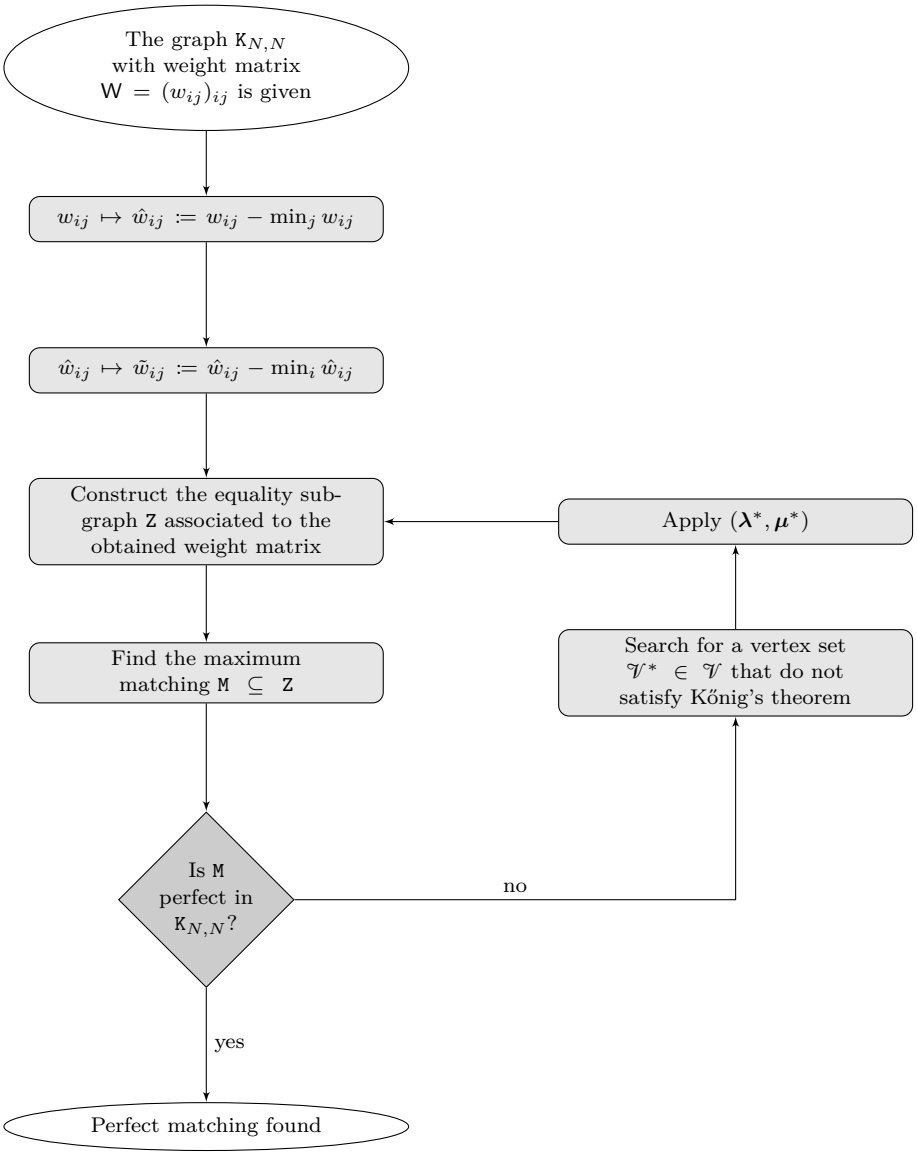


FIGURE 3.2.3.: The Hungarian algorithm, with reference to the description given in the main text.

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Chapter 4

Random optimization problems and statistical mechanics

4.1. Random optimization problems

In the previous Chapter we discussed optimization problems defined on graphs, and in particular matching problems. For each of these problems we supposed that the parameters of the problem (e.g., the weight matrix for the considered graph) were given once and for all. For a given instance of an optimization problem, the solution can be found running specific algorithms available in the literature.

However, to study some general properties of a given optimization problem, it is often useful to consider *random instances* and study both the complexity of the problem and its solution *in average*, for large sizes of the input. This probabilistic approach to combinatorial optimization problems shed new light on their mathematical properties. Many results have been obtained in computer science, combinatorics and probability. Remarkably, since a seminal work by Mézard and Parisi [15], statistical physics has played a central role in this kind of investigations. Indeed, many techniques developed by physicists for the study of disordered systems and phase transitions, are tremendously effective in the investigation of random optimization problems and, for many important problems, some phase-like transitions were identified respect to the parameters of the problems. These phase transitions are related to changes in the structure of the set of feasible solutions [3, 13].

The satisfiability transition A first kind of randomization for a combinatorial optimization problem defined on a graph can be performed on the graph itself. A typical ensemble of *random graphs* is the Erdős–Rényi random graph $\mathtt{G}_{V,E}$. The generic graph of this ensemble has V vertices and it is obtained selecting uniformly at random E edges among all the $\binom{V}{2}$ possible edges. The limit $V \to \infty$ is performed assuming $E = \alpha V$ with α is fixed. We can then consider an optimization problem, e.g., q-Col, on this graph, adopting the cost function in Eq. (3.2.6). In the case of k-Sat, we can similarly define an ensemble of random hypergraphs with V vertices and $M = \alpha V$ k-edges, assuming that each k-edge is chosen uniformly at random among the $\binom{V}{k}$ possible k-edges. In the random k-Sat, a second kind of randomness is typically present. Indeed, the cost function in Eq. (3.2.7) depends on a set of parameters $\{J_v^e\}$: these parameters are usually chosen to be ± 1 with equal probability. Obviously, in this random optimization problems the solution for a given instance is not relevant, but some average properties are of a certain importance. For example, both in the random q-Col and in the random k-Sat, we may ask what is the probability $P(\alpha, V)$ that an instance with V vertices has a feasible state with zero energy, i.e., if the problem is *satisfiable*. This probability depends on the only parameter of the ensemble, α. Since the seminal works of Cheeseman, Kanefsky, and Taylor [6] and Mitchell, Selman, and Levesque [18], it became evident that there is, for both the random q-Col and the random k-Sat, a sharp transition between two regimes, i.e.,

$$\lim_{V\to\infty} P(\alpha, V) = \theta(\alpha_s - \alpha), \tag{4.1.1}$$

where α_s is a critical value depending on the problem and $\theta(x)$ is the Heaviside function. Eq. (4.1.1) is now called *satisfiability conjecture* and many results have been obtained both about upper and

lower bounds on α_s and about its specific value.

In this thesis we will not discuss further the random k-SAT and the random q-COL. Many extremely interesting results about these problems, as anticipated, were obtained by means of statistical physics techniques [25]. The interested reader can find more information in the book of Mézard and Montanari [13].

4.1.1. Random matching problems

Given a certain instance of the matching problem on a weighted graph, we can find a solution of the problem using, for example, the algorithms discussed in the previous Chapter. Here however we are interested in the study of the average properties of the *random* matching problem. In the random matching problem on the weighted graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$, the weights $\{w(e)\}_{e\in\mathscr{E}}$ are supposed to be *random variables* with a certain distribution, both in the case $\mathtt{G} \equiv \mathtt{K}_{2N}$ and in the case $\mathtt{G} \equiv \mathtt{K}_{N,N}$. In the simplest case, all weights are *independently and identically distributed (i.i.d.) random variables* with a distribution density $\rho(w)$. The distribution itself defines therefore an ensemble of random instances and we ask for the typical properties (average optimal cost, optimal cost distribution...) for a given ensemble in the thermodynamical limit. The simplest case of independent random variables is already quite difficult to tackle. In the following we will discuss some results for this particular case. In Chapter 5 we will include Euclidean correlations among the weights of the graph.

We call random *monopartite* matching problem (RMMP) the matching problem defined on a complete graph $\mathtt{K}_{2N}$, whilst in the *random bipartite matching problem* (RBMP) the problem is supposed formulated on the complete bipartite graph $\mathtt{K}_{N,N}$. Both the RMMP and the RBMP have been investigated using purely combinatorial and probabilistic arguments, but also through statistical physics approaches.

The RBMP: Aldous' solution and Parisi's formula

In 2001 Aldous [1] provided a rigorous treatment of the RBMP, or random assignment problem, on the complete bipartite graph $\mathtt{K}_{N,N}$ in the $N \to \infty$ limit. Aldous assumed i.i.d. weights and he chose a probability distribution density for the weights given by

$$\rho(w) = \theta(w)\, \mathrm{e}^{-w}\,. \tag{4.1.2}$$

Due to the importance of his solution, we will give a brief sketch of his results. We want to stress here that Aldous' solution was deeply inspired by the so-called *cavity method*, first introduced in the study of glassy systems. We will discuss this method later.

Let us consider our weighted complete bipartite graph

$$\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E}), \quad \mathscr{V} = \{v_1, \dots, v_N\}, \quad \mathscr{U} = \{u_1, \dots, u_N\}, \quad w\colon (v_i, u_j) \mapsto w_{ij}. \tag{4.1.3}$$

The quantities w_{ij} for $i, j = 1, \dots, N$, as anticipated, are supposed i.i.d. random variables with probability distribution density given by Eq. (4.1.2).

Aldous performed an *unfolding procedure* to construct, from the weighted complete graph $\mathtt{K}_{N,N}$, an infinite ordered tree in which each vertex has a fixed degree[1] N

$$\mathtt{T} = \mathrm{Graph}(\mathscr{V}_\mathtt{T}; \mathscr{E}_\mathtt{T}). \tag{4.1.4}$$

The construction procedure is the following. Let us select a vertex of $\mathtt{K}_{N,N}$ at random and let us call it ϕ. For the sake of simplicity, let us suppose that $\phi \equiv v_N$: we will write $\tau(\phi) = v_N$, where τ is a "folding map". This vertex will be the root of our tree. We connect now ϕ to N new vertices, and we denote them by $t_1, \dots, t_N$. Moreover, we associate to each edge (ϕ, t_i) a weight $\omega_{\phi i}$, in such a way that

$$\omega_{\phi i} = i\mathrm{th}\left[\{w((\tau(\phi), u_l)\}_{l=1,\dots,N}\right]. \tag{4.1.5}$$

In the notation above, $i\mathrm{th}\left[\{a_l\}_{l=1,\dots,K}\right]$ denotes the i-th element, in increasing order, of the set $\{a_l\}_{l=1,\dots,K}$. If the i-th weight in the set is, e.g., w_{Nk}, we say that $\tau(t_i) = u_k$.

[1] The tree is therefore an infinite N-Cayley tree.

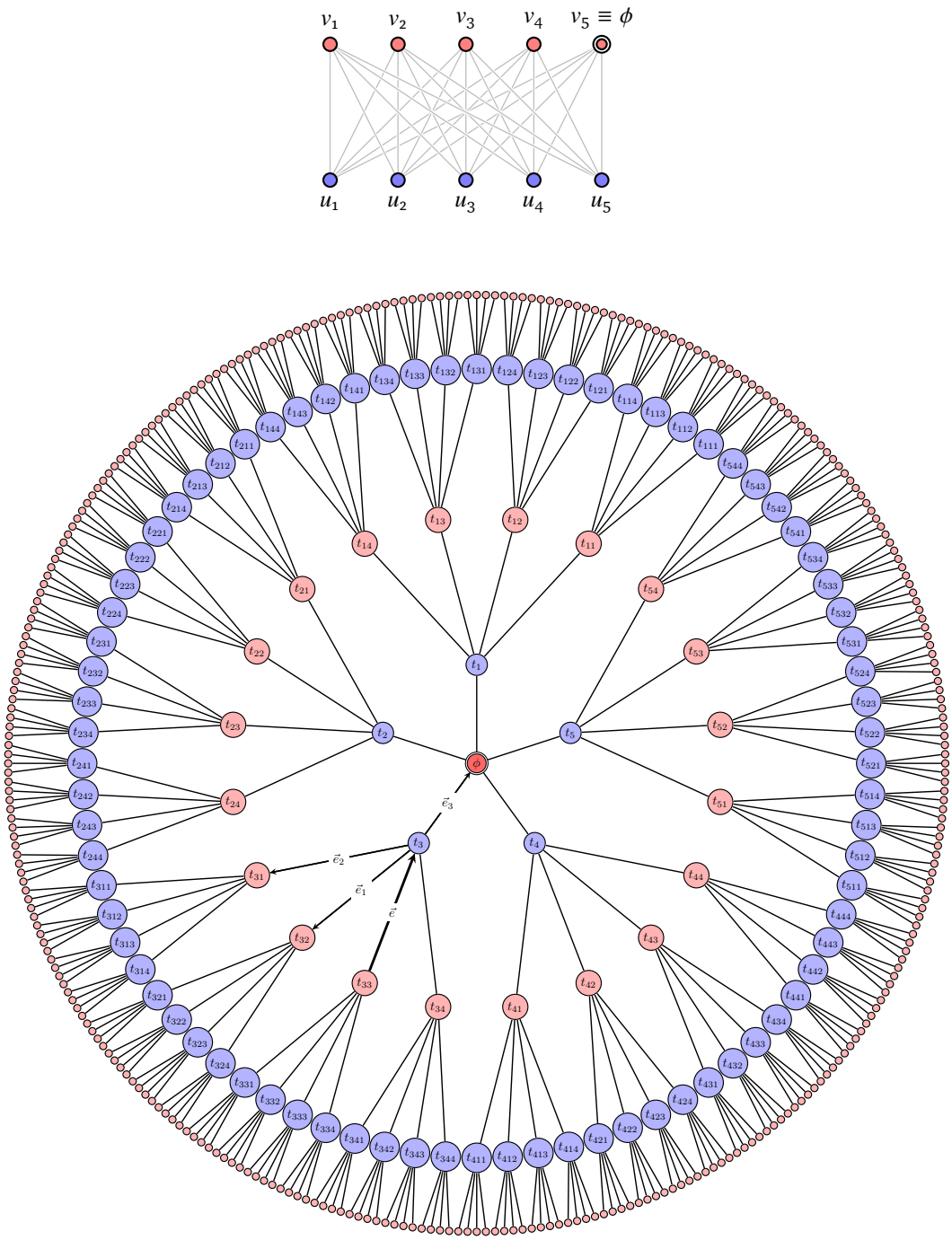


FIGURE 4.1.1.: Structure of the unfolding map for a $K_{5,5}$ graph The directed edge $\vec{e}$ is represented in bold with its children $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$.

Let us take now one of the N vertices generated, let us say t_k. We attach to t_k $N-1$ new vertices, let us call them t_{ki}, $i = 1, \dots, N-1$. Again, we associate to each edge (t_k, t_{ki}) a weight $\omega_{k\,ki}$, in such a way that

$$\omega_{k\,ki} = i\text{th}\left[\{w((v_j, \tau(t_k))\}_{j=1,\dots,N-1}\right]. \tag{4.1.6}$$

Observe that the weight of the edge $(v_N, \tau(t_k))$ in $\mathtt{K}_{N,N}$ has been excluded: this happens because the vertex t_k is already attached to a vertex (here ϕ) corresponding to v_N. We repeat this construction for all vertices $\{t_l\}_{l=1,\dots,N}$.

We proceed in this way constructing an infinite tree like in Fig. 4.1.1. Each vertex of the tree has $(N-1)$ "children" and each "child" is associated to one and only one vertex in the original graph $\mathtt{K}_{N,N}$. The different "levels" of the tree correspond alternately to vertices in $\mathcal{V}$ or in $\mathcal{U}$ of the original graph. Taking the limit $N \to \infty$, an element of the *infinite tree* at distance $d > 0$ from the root ϕ is identified by a sequence of integers number, like

$$t_k, \text{ with } k = k_1 k_2 \cdots k_d, \quad k_i \in \{1, \dots, N-1\} \tag{4.1.7}$$

Denoting by $e = (t_k, t_{k'}) \in \mathscr{E}_{\mathtt{T}}$ an edge of the tree, there is only one corresponding edge $(\tau(t_k), \tau(t_{k'})) \in \mathscr{E}$. The constructed tree is called *Poisson–weighted infinite tree*, due to the fact that the weights on the tree are distributed according to Eq. (4.1.2).

An optimal *matching* on this tree is given by a matching $\mathtt{M}_o$ such that

$$\mathscr{C}_{\mathtt{T}}[\mathtt{M}_o] = \min_{\mathtt{M}} \sum_{e \in \mathtt{M}} w(e), \tag{4.1.8}$$

where the minimum is taken over all possible matchings $\mathtt{M} \subseteq \mathtt{T}$ on the tree. Observe that the previous quantity is formally divergent in the $N \to \infty$ limit. However, the search and the study of an optimal matching on this graph will be easier than on the original graph, due to the absence of cycles.

The edges of the graph were not considered, up to now, directed. We specify the direction of an edge writing

$$\vec{e} = \overrightarrow{(t_k, t_{k'})}, \tag{4.1.9}$$

meaning that the tail is in t_k and the head in $t_{k'}$. We call the $N-1$ oriented edges having tail in $t_{k'}$ the *children* of $\vec{e} = \overrightarrow{(t_k, t_{k'})}$. "Cutting" the edge $(t_k, t_{k'})$ we obtain two trees: we say that the one containing $t_{k'}$ is the three of the *descendants* of $\vec{e}$ and we will denote it by $\mathtt{T}^{\vec{e}}$.We call $\mathtt{T}^{\vec{e}}_{\text{nc}}$ the forest of $N-1$ trees obtained removing from $\mathtt{T}^{\vec{e}}$ the $N-1$ edges incident at $t_{k'}$.

Due to the tree-like structure of the graph, we can write a self-consistent equation for the cost $\mathscr{C}_{\mathtt{T}^{\vec{e}}}$ of the optimal matching on the tree of descendants $\mathtt{T}^{\vec{e}}$ as

$$\mathscr{C}_{\mathtt{T}^{\vec{e}}} = \min_{\vec{e}_i \text{ child of } \vec{e}} \left[w(\vec{e}_i) + \mathscr{C}_{\mathtt{T}^{\vec{e}_i}_{\text{nc}}} + \sum_{j \neq i} \mathscr{C}_{\mathtt{T}^{\vec{e}_j}} \right]. \tag{4.1.10}$$

This recursive equation must hold due to the fact that we are considering trees. If we introduce

$$X(\vec{e}) \coloneqq \mathscr{C}_{\mathtt{T}^{\vec{e}}} - \mathscr{C}_{\mathtt{T}^{\vec{e}}_{\text{nc}}}, \tag{4.1.11}$$

difference between the cost of the optimal matching on the descendants' tree with and without the edges incident in $t_{k'}$, we have

$$\begin{aligned} X(\vec{e}) &= \min_{\vec{e}_i \text{ child of } \vec{e}} \left[w(\vec{e}_i) + \mathscr{C}_{\mathtt{T}^{\vec{e}_i}_{\text{nc}}} + \sum_{j \neq i} \mathscr{C}_{\mathtt{T}^{\vec{e}_j}} \right] - \sum_j \mathscr{C}_{\mathtt{T}^{\vec{e}_j}} \\ &= \min_{\vec{e}_i \text{ child of } \vec{e}} \left[w(\vec{e}_i) - X(\vec{e}_i) \right]. \end{aligned} \tag{4.1.12}$$

We expect that $X(\vec{e})$ and $X(\vec{e}_i)$ are identically distributed. In particular, Aldous proved the following result.

Theorem 4.1.1. *A given edge $e \in \mathscr{E}_{\mathrm{T}}$ is an edge of the optimal matching of the original problem if and only if the following inequality holds*

$$w(e) < X(\vec{e}) + X(\overleftarrow{e}). \tag{4.1.13}$$

Assuming that all weights are i.i.d. random variables, distributed according to Eq. (4.1.2), it can be proved that $X(\vec{e})$ has a logistic probability distribution density:

$$p_X(x) = \frac{1}{4\cosh^2\frac{x}{2}}. \tag{4.1.14}$$

From this result and from Eq. (4.1.13), the distribution of the weights *on the edges of the optimal matching* is

$$\rho_o(w) \coloneqq \Pr\left[w(e) < X(\vec{e}) + X(\overleftarrow{e})\right] = \theta(w)\frac{\mathrm{e}^{-w} + w - 1}{4\sinh^2\frac{w}{2}}. \tag{4.1.15}$$

The average optimal cost of the RBMP follows straightforwardly,

$$\mathscr{C}^{\mathrm{RBMP}} \coloneqq \overline{\mathscr{C}^{\mathrm{M}}} \xrightarrow{N\to\infty} \int_0^\infty w\rho_o(w)\,\mathrm{d}\,w = \frac{\pi^2}{6}, \tag{4.1.16}$$

where we denoted by $\overline{\bullet}$ the average over all instances.

Both Eq. (4.1.15) and Eq. (4.1.16) had been already obtained by Mézard and Parisi [14, 15] in 1985 using the replica method. In the entire construction of Aldous there are, moreover, evident resemblances with the so called *cavity method* (see Chapter 5).

Aldous' results are valid in the $N \to \infty$ limit but no information about finite size effects can be obtained from the arguments above. This information is given by the following theorem, proposed as a conjecture by Coppersmith and Sorkin [7] in 1999 for the random k-assignment problem and later independently proved by Linusson and Wästlund [11] and Nair, Prabhakar, and Sharma [20].

Theorem 4.1.2 (Coppersmith–Sorkin formula)**.** *Let us consider a complete bipartite graph $\mathtt{K}_{N,M} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E})$ and let us associate to each edge $e \in \mathscr{E}$ a weight $w(e)$, in such a way that the costs $\{w(e)\}_{e\in\mathscr{E}}$ are a set of i.i.d. random variables with density distribution given by Eq. (4.1.2). Then the average optimal cost of the k-assignment problem on it is*

$$\overline{\mathscr{C}^{k\mathrm{M}}} = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\frac{\mathbb{I}_{[0,k)}(i+j)}{(M-i)(N-j)} \tag{4.1.17}$$

where $\mathbb{I}_A(x)$ is the indicator function:

$$\mathbb{I}_A(x) \coloneqq \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.18}$$

From the previous theorem, the following corollary follows for the assignment problem, conjectured by Parisi [22] in 1998

Corollary 4.1.3 (Parisi's formula)**.** *In the assignment problem, we have*

$$\mathscr{C}_N^{\mathrm{RBMP}} \coloneqq \overline{\mathscr{C}^{\mathrm{M}}} = \sum_{i=1}^{N}\frac{1}{i^2}. \tag{4.1.19}$$

Apart from the original works, a complete discussion of the previous results can be found in the monograph by Mézard and Montanari [13].

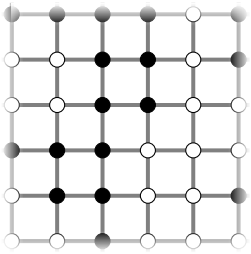


FIGURE 4.2.1.: Ising model on a two-dimensional lattice.

4.2. Disordered systems, spin glasses and assignment

As anticipated in the previous Section, when dealing with random optimization problems we are not interested in the properties of a specific instance, but in the average properties of the problem, possibly depending on some parameters (e.g., the size or some fixed parameters in the cost function) and on the way we introduce randomness in the problem itself.

Statistical physics has developed a plethora of techniques to obtain the average properties of systems with a huge number of degrees of freedom, even in presence of disorder. In particular, in *disordered systems*, disorder is typically introduced in the Hamiltonian function,

$$H = H(\boldsymbol{\sigma}, \mathbf{J}), \tag{4.2.1}$$

as a set of parameters $\mathbf{J} \coloneqq \{J_k\}_k$ randomly extracted from a certain probability distribution. These parameters usually couple with the degrees of freedom $\boldsymbol{\sigma} \coloneqq \{\sigma_i\}_i$ of the system itself. Different techniques have been developed to treat properly these systems depending on the fact that the disorder is considered fixed on the time scale over which the degrees of freedom of the system fluctuate (*quenched* disorder) or otherwise (*annealed* disorder).

Spin glasses play the role of reference frame in which physicists analyze the peculiar effects of disorder on the behavior of systems with a large number of degrees of freedom. The importance of spin glasses goes beyond the application to physical systems. In a famous set of seminal papers, Mézard and Parisi [15] discussed the application of spin glasses methods to some optimization problems. In particular they studied the TSP and the RMMP, showing the power of these techniques in the analysis of random optimization problems. Remarkably, the results on random matching problems, published in 1985, were rigorously proved by Aldous for the RBMP only in 2001, as explained in Section 4.1.1. To understand their results, however, some concepts of the general theory are necessary. We will present here a brief survey on spin glass theory [5, 12, 21] and, subsequently, we will reproduce their argument for the RMMP, neglecting the first order finite size corrections obtained by Parisi and Ratiéville [24] by means of similar techniques.

4.2.1. Preliminaries: the Ising model

Let us consider a graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$, $V = |\mathscr{V}|$, and suppose that we assign to each node $v_i \in \mathscr{V}$ of $\mathtt{G}$ an *Ising spin variable* $\sigma_i \in \{-1, 1\}$ and to each edge $e \equiv (v_i, v_j) \in \mathscr{E}$ an interaction energy $-J\sigma_i\sigma_j$, $J \in \mathbb{R}$, depending on the value of the Ising spins on the end points of the edge. We introduce also an *Hamiltonian* functional for the graph in the following form:

$$H_{\mathtt{G}}[\boldsymbol{\sigma}; J, h] \coloneqq -J \sum_{\langle ij \rangle} \sigma_i\sigma_j - h \sum_{i=1}^{V} \sigma_i. \tag{4.2.2}$$

In the previous Hamiltonian, $h \in \mathbb{R}$ is a fixed real quantity. Moreover, we denoted by $\sum_{\langle ij \rangle} \equiv \sum_{(v_i,v_j)\in\mathscr{E}}$. The model defined above is an *Ising model* on the graph $\mathtt{G}$. If $\mathtt{G}$ is a complete graph, the model is sometimes called *Curie–Weiss model*, or *infinite range Ising model*. In the problem of magnetism, the Ising spins represent microscopic magnetic moments on a certain lattice, coupled in a ferromagnetic ($J > 0$) or antiferromagnetic ($J < 0$) way and in presence of a certain external uniform magnetic field h. In Eq. (4.2.2) we denoted by $\boldsymbol{\sigma} \coloneqq \{\sigma_i\}_{i=1,\dots,V}$ the generic configuration of the system.

In the spirit of Boltzmann–Gibbs statistical mechanics, we can associate a Boltzmann–Gibbs weight to each configuration $\boldsymbol{\sigma}$,

$$\mu_{\mathtt{G}}[\boldsymbol{\sigma};\beta,J,h] \coloneqq \frac{1}{Z_{\mathtt{G}}(\beta,J,h)}\,\mathrm{e}^{-\beta H_{\mathtt{G}}[\boldsymbol{\sigma};J,h]}, \tag{4.2.3}$$

where the normalization is given by the partition function of the system

$$Z_{\mathtt{G}}(\beta,J,h) \coloneqq \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H_{\mathtt{G}}[\boldsymbol{\sigma};J,h]}, \tag{4.2.4}$$

in such a way that $\sum_{\boldsymbol{\sigma}} \mu_{\mathtt{G}}[\boldsymbol{\sigma};\beta,J,h] = 1$. The parameter $\beta^{-1} > 0$ is the *temperature* of the system. Given a function $f \coloneqq f(\boldsymbol{\sigma})$, we introduce the notation

$$\langle f \rangle \coloneqq \sum_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma})\mu_{\mathtt{G}}[\boldsymbol{\sigma};\beta,J,h]. \tag{4.2.5}$$

It is well known that the partition function plays a central role in the computation of many physical quantities of interest. For example, the *magnetization* is given by

$$m_{\mathtt{G}}(\beta,J,h) \coloneqq \frac{1}{V}\sum_{i=1}^{V}\langle\sigma_i\rangle = -\frac{1}{V}\frac{\partial \ln Z_{\mathtt{G}}(\beta,J,h)}{\partial h} \in [-1,1]. \tag{4.2.6}$$

If $N \coloneqq V$ is finite, $Z_{\mathtt{G}}(\beta,J,h)$ is an analytic function of its arguments β, J and h. However, in the limit $N \to \infty$, some singularities may appear in the (β,J,h) space in some points (*critical points*) or surfaces (*critical surfaces*). The existence and location of these singularities is of paramount interest. Indeed, a lack of analiticity in the partition function the fingerprint of *phase transitions*, and critical surfaces identify the interfaces between different macroscopic behaviors of the system. This modification of the macroscopic behavior can be, in general, characterized by the sharp change in the value of a certain quantity, that is called *order parameter*. Unfortunately, there are no general criteria to identify the proper order parameter for a given system.

To more precise in our exposition, let us consider a ferromagnetic Ising model on the hypercubic lattice in d dimensions of side L. On this lattice, there are $L^d \equiv N$ vertices and each of them is connected to its $2d$ nearest neighbors. If we denote by H_N the Hamiltonian (4.2.2) of the system, the partition function $Z_N(\beta,J,h)$ and the Boltzmann–Gibbs measure $\mu_N[\boldsymbol{\sigma};\beta,J,h]$ can be defined as above. The magnetization $m_N(\beta,J,h)$, defined as in Eq. (4.2.6), gives a measure of how much the system is "ordered". In fact, $|m_N| \approx 1$ is indicative of the fact that, on average, almost all spins variables have the same value, whilst $|m_N| \approx 0$ suggests that the spins are randomly oriented. Not surprisingly, in the $N \to \infty$ limit the magnetization m_N is an order parameter in the sense specified above. Indeed, for $d > 1$ and fixed $J > 0$, the function

$$m(\beta,J,h) \coloneqq \lim_{N} m_N(\beta,J,h) \tag{4.2.7}$$

has a *branch cut* in the (β,h) plane on the axis $h = 0$ and for $\beta > \beta_c(d,J)$. Here $\beta_c(d,J)$ is the *critical inverse temperature* and it depends on dimensionality. Formally, for $d = 1$ $\beta_c(1,J) = +\infty$, whilst for $d = 2$ a famous duality argument by Kramers and Wannier [10] shows that

$$\beta_c(2,J) = \frac{\ln\left(1+\sqrt{2}\right)}{2J}. \tag{4.2.8}$$

For $d > 2$ only numerical estimations of β_c are available. The branch cut of $m(\beta,J,h)$ shows that $Z_N(\beta,J,h)$ is not analytic in the $N \to \infty$ limit. Indeed, for $\beta > \beta_c(d,J)$, we have that

$$\lim_{h\to 0^+} m(\beta,J,h) \neq \lim_{h\to 0^-} m(\beta,J,h) \tag{4.2.9}$$

and therefore

$$\lim_{h\to 0^+}\lim_{N}\mu_N[\boldsymbol{\sigma};\beta,J,h] \neq \lim_{h\to 0^-}\lim_{N}\mu_N[\boldsymbol{\sigma};\beta,J,h]. \tag{4.2.10}$$

On the critical line two different equilibrium measures coexist and we have a *spontaneous symmetry breaking* of the $\mathbb{Z}_2$ invariance $\{\sigma_i\} \mapsto \{-\sigma_i\}$ satisfied by the Hamiltonian functional (4.2.2) for $h = 0$. We say that for $\beta > \beta_c(d,J)$ and $h = 0$ we have *two different phases*, each of them

identified by the value of the order parameter m. The intuitive concept of pure phase can be more rigorously formulated in the context of statistical field theory [26]: here we simply state that an equilibrium measure μ describes a *pure phase* if, and only if, given two local observables $A(x)$ and $B(x)$ (where x is a graph site) then in the thermodynamical limit

$$\lim_{\delta(x,y)\to+\infty} [\langle A(x)B(y)\rangle - \langle A(x)\rangle\langle B(y)\rangle] = 0, \tag{4.2.11}$$

where $\delta(x,y)$ is the distance between the node x and the node y in the graph. The previous condition is called *cluster property*. If, however, for a system described by a Hamiltonian $H(\boldsymbol{\sigma})$, ergodicity breaking happens, the phase space is partitioned in Ω pure phases $\{\omega_\alpha\}_{\alpha=1,\dots,\Omega}$. Each phase has its own Gibbs measure

$$\mu_\alpha(\boldsymbol{\sigma}) = \frac{\mathbb{I}_{\omega_\alpha}(\boldsymbol{\sigma})}{Z_\alpha}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}, \quad Z_\alpha = \sum_{\boldsymbol{\sigma}} \mathbb{I}_{\omega_\alpha}(\boldsymbol{\sigma})\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\,. \tag{4.2.12}$$

We can also define a free energy $F_\alpha = -\beta^{-1}\ln Z_\alpha$ corresponding to the phase α.

The Ising model is easily solvable on a one dimensional lattice. In 1944 Lars Onsager announced his celebrated solution for the Ising model with $h = 0$ in $d = 2$, later reformulated in different ways [19]. Unfortunately, no exact solution is available for $d > 2$, even in absence of external magnetic field. The discussion of the different, inspiring solutions proposed for the Ising model in $d = 2$ [4, 19] is outside the purposes of the present work. However, to give a qualitative description of the phase transition in the Ising model, we can adopt a very simple and common approximation, called *mean field approach*. We assume that $\sigma_i = m+\delta\sigma_i$, where $m \equiv m(\beta, J, h)$ is the magnetization in the $N \to \infty$ limit. We can insert this approximate form in the generic Hamiltonian in Eq. (4.2.2) on the graph $\mathtt{G}$ and neglect fluctuations in the non-linear terms. For the Ising model on a d-dimensional lattice, we have

$$\begin{aligned} H_N[\boldsymbol{\sigma}; J, h] &= -J\sum_{\langle ij\rangle} (m+\delta\sigma_i)\,(m+\delta\sigma_j) - h\sum_{i=1}^{N}\sigma_i \\ &\approx -2dJNm^2 - (2dJm+h)\sum_{i=1}^{N}\sigma_i. \end{aligned} \tag{4.2.13}$$

In this approximation it is easily obtained

$$Z_N(\beta, J, h) = \mathrm{e}^{\beta 2dNJm^2}\,[2\cosh\beta\,(Jmd+h)]^N\,. \tag{4.2.14}$$

Therefore the magnetization satisfies the equation

$$m = \tanh\beta(2dJm+h). \tag{4.2.15}$$

From the last equation, we have that a non-vanishing magnetisation can be obtained for $h = 0$ only if $2d\beta J > 1$, i.e. for $\beta > \frac{1}{2dJ}$. The critical value obtained from this mean field approximation is then

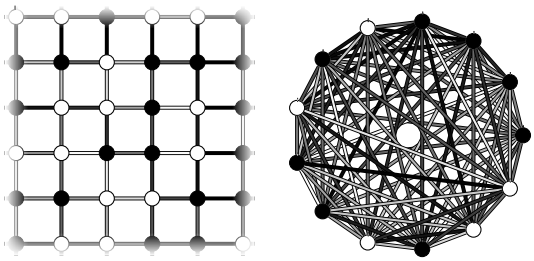
$$\beta_c^{\mathrm{mf}} = \frac{1}{2dJ}. \tag{4.2.16}$$

This value *is not* correct: for example, for $d = 2$ does not recover the solution in Eq. (4.2.8) obtained through an exact computation. However, the mean field theory qualitatively reproduces the critical behavior of the original model and correctly predict the *existence* of a critical temperature.

The mean field theory is able also to give us a hint about what happens at the critical temperature. If we compute the free energy for the Ising model in this approximation, we have

$$\begin{aligned} F_N(\beta, J, h) &= -\frac{1}{\beta}\ln Z_N(\beta, J, h) \\ &\approx -\frac{N}{\beta}\ln 2 + \frac{N}{2\beta_c^{\mathrm{mf}}}\left(1-\frac{\beta}{\beta_c^{\mathrm{mf}}}\right)m^2 + \frac{N}{12}\left(\frac{m}{\beta_c^{\mathrm{mf}}}\right)^4\beta^3 + o\left(m^4\right) \end{aligned} \tag{4.2.17}$$

Note that the functional above has a global minimum in $m = 0$ for $\beta < \beta_c^{\mathrm{mf}}$. This value becomes a local maximum for $\beta > \beta_c^{\mathrm{mf}}$ and, in the same regime, two new minima appear, symmetric



(A) EA-model. (B) SK-model.

FIGURE 4.2.2.: Pictorial representation of 2-dimensional Edwards–Anderson model and of the Sherrington-Kirkpatrick model.

respect to $m = 0$. This fact suggests that, above the critical temperature, the system is in a zero–magnetization phase. Below the critical temperature, the system "falls" in one of the two valleys corresponding to nonzero magnetization. This simple observation, proposed for the first time by Landau, gives a qualitative explanation of the spontaneous symmetry breaking.

4.2.2. Spin glasses

Let us consider an Ising-type model on a graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$ with the following Hamiltonian,

$$H_{\mathtt{G}}[\boldsymbol{\sigma}; \mathbf{J}, h] \coloneqq -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i. \tag{4.2.18}$$

This time, each J_{ij} is supposed to be a *random quantity*. In particular, each J_{ij} is extracted, independently from all the others, from a given probability distribution density $\rho(J)$, identical for all $(v_i, v_j) \in \mathscr{E}$. The set of values $\mathbf{J} = \{J_{ij}\}$ is supposed to be extracted once and for all for each instance, and therefore we say that the disorder is *quenched*. Usually both positive and negative value of J_{ij} are admitted, so we have both ferromagnetic and paramagnetic interactions. If we consider our system on the hypercubic lattice in d dimensions, the obtained model is called *Edwards–Anderson model* (EA-model) in d dimensions. The EA-model is an example of *disordered system*, and, in particular, of *spin glass*.

A disordered system is a system characterized by the presence of two elements. First, in disordered systems some *randomness* appears: in the case of the EA-model, the randomness is obviously in the fact that the coupling constants are random variables extracted from a certain distribution function. Second, in a disordered system there is *frustration*. This concept is slightly more subtle. A lattice model is *frustrated* whenever there exists a set of local constraints in conflict each other. Let us consider for example a cycle of length four in the EA-model in $d = 2$, in which the product of the coupling constants J along the cycle is negative. For example, denoting by a black line an interaction with $J < 0$ and by a white line an interaction with $J > 0$, such a cycle can have the form

It is evident that no configuration of Ising spins is capable of minimizing simultaneously the energy contribution of all edges separately. The energy landscape of frustrated systems is therefore often nontrivial and it is not obvious what is the structure of the "minimum energy configuration".

In disordered systems we are not interested in the properties of a precise realization, but we have to average, in some sense, on the disorder to extract the useful information. In the case of the EA-model, we need therefore to average over all possible values of $\mathbf{J} = \{J_{ij}\}_{ij}$. We will denote by $\bar{\bullet}$ the average of a function respect to the possible values of $\mathbf{J}$. How this average must be performed on thermodynamical functionals, however, is a delicate matter. In fact, it turns out that the free energy $F_{\mathtt{G}}$ for a disordered system on a generic graph $\mathtt{G}$ is *self averaging* [5, 21], i.e.,

$$\lim_{|\mathscr{V}|\to\infty} \frac{\overline{F_{\mathtt{G}}^2}}{\overline{F_{\mathtt{G}}}^2} = 1, \tag{4.2.19}$$

but the partition function is *not*. The property of being self-averaging is crucial, because we want that physical observables *do not* depend from the specific realization. On the other hand, free energy is fundamental for the computation of all physical quantities.

It follows that we have to compute in general the following *configurational average*:

$$\overline{F_{\mathtt{G}}[\mathbf{J};\beta]} \coloneqq -\frac{\overline{\ln Z_{\mathtt{G}}[\mathbf{J};\beta]}}{\beta} = -\frac{1}{\beta}\left(\prod_{\langle ij\rangle}\int \rho(J_{ij})\,\mathrm{d}\,J_{ij}\right)\ln Z_{\mathtt{G}}[\mathbf{J};\beta]. \tag{4.2.20}$$

Parisi's solution of the SK model

To perform the integral (4.2.20), a famous and powerful trick, called *replica trick*, is usually adopted. The manipulation is based on the following identity:

$$\ln x = \lim_{n\to 0}\frac{x^n-1}{n}, \qquad x>0. \tag{4.2.21}$$

To proceed with our computation, let us assume that the graph on which the Hamiltonian (4.2.18) is defined is a *complete graph*, $\mathtt{G}\equiv \mathtt{K}_N$. The obtained model is called *Sherrington–Kirkpatrick model* (SK-model), and it is an mean field version of the EA-model. In this model the probability distribution density for the set of parameters $\{J_{ij}\}_{ij}$ is given by

$$\rho(J) = \frac{\sqrt{N}}{\Gamma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{NJ^2}{2\Gamma^2}}, \quad \Gamma>0. \tag{4.2.22}$$

We want to compute the average free energy density using the replica trick in Eq. (4.2.22) and in the thermodynamical limit,

$$\beta\bar{f} \coloneqq \lim_{\substack{n\to 0\\ N\to\infty}}\frac{1-\overline{Z_N^n}}{nN}. \tag{4.2.23}$$

It follows that

$$\begin{aligned}\overline{Z^n} &= \left(\prod_{\langle ij\rangle}\int \rho(J_{ij})\,\mathrm{d}\,J_{ij}\right)\exp\left(\beta\sum_{i<j}J_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^\alpha\right)\\ &= \exp\left(\frac{N\beta^2\Gamma^2 n}{4}\right)\left[\prod_{\alpha}\sum_{\{\sigma^\alpha\}}\right]\exp\left[\frac{\beta^2\Gamma^2}{2n}\sum_{\alpha<\beta}\left(\sum_i\sigma_i^\alpha\sigma_i^\beta\right)^2+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^\alpha\right]\\ &= \exp\left(\frac{N\beta^2\Gamma^2 n}{4}\right)\left[\prod_{\alpha<\beta}\int \mathrm{d}\,q_{\alpha\beta}\right]\exp\left[\frac{N\beta^2\Gamma^2}{2n}\sum_{\alpha<\beta}q_{\alpha\beta}^2+N\ln z[\mathsf{q}]\right],\end{aligned} \tag{4.2.24}$$

where we have introduced

$$z[\mathsf{q}] \coloneqq \sum_{\sigma^1,\dots,\sigma^n}\exp\left(\beta^2\Gamma^2\sum_{\alpha<\beta}q_{\alpha\beta}\sigma^\alpha\sigma^\beta+\beta h\sum_\alpha\sigma^\alpha\right), \quad \mathsf{q}\coloneqq(q_{\alpha\beta})_{\alpha\beta}. \tag{4.2.25}$$

Observe that z appears as a sort of partition function for a set of n coupled spins, each one associated to one of the n replica indexes. Being the exponent of the integrand proportional to N, we can use the steepest descent method to evaluate the expression. In particular, the extremality condition respect to the generic element $q_{\alpha\beta}$ gives

$$q_{\alpha\beta} = \frac{\partial\ln z[\mathsf{q}]}{\partial q_{\alpha\beta}} \equiv \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z. \tag{4.2.26}$$

In Eq. (4.2.26) we have adopted the notation

$$\left\langle O\left(\sigma^1,\dots,\sigma^n\right)\right\rangle_z \coloneqq \frac{1}{z}\sum_{\sigma^1,\dots,\sigma^n}O\left(\sigma^1,\dots,\sigma^n\right)\exp\left(\beta^2\Gamma^2\sum_{\alpha<\beta}q_{\alpha\beta}\sigma^\alpha\sigma^\beta+\beta h\sum_\alpha\sigma^\alpha\right). \tag{4.2.27}$$

Let $\mathsf{Q} \coloneqq (Q_{\alpha\beta})_{\alpha\beta}$ be the solution of the saddle point condition in Eq. (4.2.26). Calling

$$S[\mathsf{Q}] \coloneqq -\frac{\beta^2\Gamma^2}{2}\sum_{\alpha\neq\beta} Q_{\alpha\beta}^2 + \frac{1}{4}\beta^2\Gamma^2 + \frac{1}{n}\ln z[\mathsf{Q}], \tag{4.2.28}$$

$$H_{(\alpha\beta)(\gamma\delta)}(\mathsf{Q}) \coloneqq \left.\frac{\partial^2 S[\mathsf{q}]}{\partial q_{\alpha\beta}\partial q_{\gamma\delta}}\right|_{\mathsf{q}=\mathsf{Q}}, \tag{4.2.29}$$

it follows that

$$\overline{Z^n} \sim \frac{1}{\sqrt{\det \mathsf{H}}}\left[1 + nN\left(-\frac{\beta^2\Gamma^2}{4}\sum_{\alpha\neq\beta} Q_{\alpha\beta}^2 + \frac{1}{4}\beta^2\Gamma^2 + \frac{1}{n}\ln z[\mathsf{Q}]\right)\right]. \tag{4.2.30}$$

Notably, $Q_{\alpha\beta}$ are not merely a set of variables. Indeed they express a sort of "average superposition" between replicas, being (as can be directly checked)

$$Q_{\alpha\beta} = \overline{\langle \sigma_i^\alpha \sigma_i^\beta \rangle_{\mathsf{R}}}, \tag{4.2.31}$$

where $\langle \bullet \rangle_{\mathsf{R}}$ is the average respect to the replicated system whose Hamiltonian function is

$$H^{\mathsf{R}}[\{\boldsymbol{\sigma}^\alpha\}_\alpha; \mathbf{J}, h] \coloneqq \sum_{\alpha=1}^{n}\left[-\sum_{i<j} J_{ij}\sigma_i^\alpha\sigma_j^\alpha + h\sum_{i=1}^{N}\sigma_i^\alpha\right]. \tag{4.2.32}$$

The variables $Q_{\alpha\beta}$ play the role of *spin glass order parameters*, as we will see. If we suppose that all replicas are equivalent (*replica symmetric hypothesis*)

$$Q_{\alpha\beta} = \overline{\langle \sigma_i^\alpha \sigma_i^\beta \rangle_{\mathsf{R}}} = \overline{\langle \sigma_i^\alpha \rangle_{\mathsf{R}} \langle \sigma_i^\beta \rangle_{\mathsf{R}}} = \overline{\langle \sigma_i \rangle}^2 = q =: q_{\mathrm{EA}}, \quad \alpha \neq \beta. \tag{4.2.33}$$

The quantity q_{EA} is called *Edward–Anderson order parameter*. We expect that for $\beta \to 0$ the spins are randomly oriented, and therefore $q_{\mathrm{EA}} = 0$, whilst in the $\beta \to +\infty$ limit $q_{\mathrm{EA}} > 0$, having $\langle \sigma_i \rangle \neq 0$ for each realization.

A careful computation (see Section A.1) shows that

$$\frac{1}{\sqrt{\det \mathsf{H}}} \xrightarrow{n\to 0} 1. \tag{4.2.34}$$

In our replica symmetric hypothesis we get [21]

$$-\beta\bar{f} = \frac{\beta^2 J^2}{4}(1-q)^2 + \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{-\frac{w^2}{2}} \ln\left[2\cosh\left(\beta J\sqrt{q}w + \beta h\right)\right] \mathrm{d}\, w, \tag{4.2.35}$$

from which we get

$$m = \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{-\frac{w^2}{2}} \tanh\left(\beta J\sqrt{q}w + \beta h\right) \mathrm{d}\, w, \tag{4.2.36}$$

whilst the value of q to be inserted in the previous equation can be obtained from the extremization condition on Eq. (4.2.35):

$$q = \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{-\frac{w^2}{2}} \tanh^2\left(\beta J\sqrt{q}w + \beta h\right) \mathrm{d}\, w. \tag{4.2.37}$$

The replica approach for the Sherrington–Kirkpatrick model seems to give a complete solution. However, our solution presents an *anomalous behavior in the low temperature limit*. For example, a direct computation of the entropy density s gives

$$\lim_{\beta\to+\infty} s = \lim_{\beta\to+\infty} \beta^2 \frac{\partial \bar{f}}{\partial \beta} = -\frac{1}{2\pi}. \tag{4.2.38}$$

The origin of this pathological behavior, and of the instability of the solution analyzed by Almeida and Thouless [2] (see Subsection A.1), was identified by Parisi [23] in 1983. Parisi showed in particular that a breaking of the replica symmetry is needed.

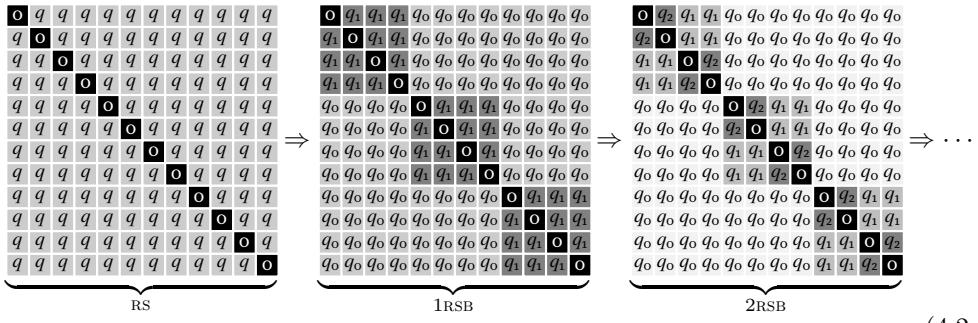
REPLICA SYMMETRY BREAKING AND PARISI'S SOLUTION In the replica symmetric hypothesis, Eq. (4.2.33), the matrix $\mathsf{Q} = (Q_{\alpha\beta})_{\alpha\beta}$ can be represented in the following way (in the example, $n = 12$):

$$\mathsf{Q} = \begin{pmatrix} 0 & q & q & q & q & q & q & q & q & q & q & q \\ q & 0 & q & q & q & q & q & q & q & q & q & q \\ q & q & 0 & q & q & q & q & q & q & q & q & q \\ q & q & q & 0 & q & q & q & q & q & q & q & q \\ q & q & q & q & 0 & q & q & q & q & q & q & q \\ q & q & q & q & q & 0 & q & q & q & q & q & q \\ q & q & q & q & q & q & 0 & q & q & q & q & q \\ q & q & q & q & q & q & q & 0 & q & q & q & q \\ q & q & q & q & q & q & q & q & 0 & q & q & q \\ q & q & q & q & q & q & q & q & q & 0 & q & q \\ q & q & q & q & q & q & q & q & q & q & 0 & q \\ q & q & q & q & q & q & q & q & q & q & q & 0 \end{pmatrix}. \tag{4.2.39}$$

We impose $Q_{\alpha\alpha} = 0\ \forall\alpha$. Parisi suggested to *break* this symmetry in a set of steps. In the first step we choose an integer m_1 such that all replicas are divided into $\frac{n}{m_1}$ groups. If α and β belong to the same group, then $Q_{\alpha\beta} = q_1$, otherwise $Q_{\alpha\beta} = q_0$. The matrix Q has, under this hypothesis, the following form (in the example, $n = 12$ and $m_1 = 4$):

$$\mathsf{Q} = \begin{pmatrix} 0 & q_1 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & 0 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & q_1 & 0 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & q_1 & q_1 & 0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & 0 & q_1 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_1 & 0 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & 0 & q_1 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & q_1 & 0 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_1 & q_1 & q_1 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & 0 & q_1 & q_1 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & 0 & q_1 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & q_1 & 0 \end{pmatrix}. \tag{4.2.40}$$

We can proceed further, subdividing each one of the m_1 groups of replicas in m_2 subgroups, and so on. We finally obtain a set of integer numbers $n \geq m_1 \geq m_2 \cdots \geq 1$ and a gradually finer subdivision of the matrix Q, in which at the k-th step of symmetry breaking we admit $k + 1$ possible different q values. Pictorially, (in the example below, $n = 12$, $m_1 = 4$, $m_2 = 2$)



(4.2.41)

The idea of evaluating the free energy taking into account different levels of replica symmetry breaking is not justified by a general principle, or a rigorous argument, but only by a deep intuition. The obtained results are, however, in excellent agreement with the data. For the SK-model, numerical evidences show that a full RSB is necessary, i.e. we have to proceed in the breaking of replica symmetry *ad infinitum*. Note however that a *full* RSB is not always needed, and some disordered systems requires only a finite number of steps. It might seem, being $n \to 0$, that an infinite number of steps is simply impossible. However, we can still proceed in a nonrigorous way

and note that, at the k-th level of RSB,

$$\frac{1}{n}\sum_{\alpha\beta} Q^l_{\alpha\beta} = \sum_{j=0}^{k}(m_j - m_{j+1})q_j^l, \qquad m_0 \equiv n,\ m_{k+1} \equiv 1,\ l \in \mathbb{N}. \tag{4.2.42}$$

In the previous equation, $Q^l_{\alpha\beta}$ is simply the l-th power of the element $Q_{\alpha\beta}$ of the matrix Q. We can define a function

$$q^l(x) \coloneqq \sum_{i=1}^{k} q_i^l \mathbb{I}_{[m_{i+1},m_i]}(x) \tag{4.2.43}$$

where $\mathbb{I}_{[a,b]}(x)$ is the indicator function. In the limit $n \to 0$ we have that

$$n \geq m_1 \geq m_2 \cdots \geq 1 \longrightarrow 0 \leq m_1 \leq m_2 \leq \cdots \leq 1, \tag{4.2.44}$$

and therefore $q(x)$ is defined in the interval $[0,1]$. At the same time, $m_j - m_{j+1} \to -\,\mathrm{d}\,x$ and we can write

$$\frac{1}{n}\sum_{\alpha\beta} Q^l_{\alpha\beta} = \sum_{j=0}^{k}(m_j - m_{j+1})q_j^l \xrightarrow{k\to+\infty} -\int_0^1 q^l(x)\,\mathrm{d}\,x. \tag{4.2.45}$$

From Eq. (4.2.30) and Eq. (4.2.34), we obtain

$$-\beta\bar{f} \sim -\frac{\beta^2\Gamma^2}{4}\sum_{\alpha\neq\beta} Q^2_{\alpha\beta} + \frac{1}{4}\beta^2\Gamma^2 + \frac{1}{n}\ln z[\mathsf{Q}]. \tag{4.2.46}$$

Therefore we get for the energy and the susceptibility

$$\bar{\epsilon} \coloneqq \partial_\beta \bar{f} = -\tfrac{\beta\Gamma^2}{2}\left(1 + \tfrac{1}{n}\textstyle\sum_{\alpha\neq\beta} Q^2_{\alpha\beta}\right) \longrightarrow -\tfrac{\beta\Gamma^2}{2}\left(1 - \textstyle\int_0^1 q^2(x)\,\mathrm{d}\,x\right), \tag{4.2.47a}$$

$$\bar{\chi} \coloneqq \partial_h \bar{f} = \beta\left(1 + \tfrac{1}{n}\textstyle\sum_{\alpha\neq\beta} Q_{\alpha\beta}\right) \longrightarrow \beta\left(1 - \textstyle\int_0^1 q(x)\,\mathrm{d}\,x\right). \tag{4.2.47b}$$

The complete expression for $\bar{f}$ in the full RSB scheme can be obtained after some computations [21]:

$$\beta\bar{f} = -\frac{\beta^2\Gamma^2}{4}\left(1 + \int_0^1 q^2(x)\,\mathrm{d}\,x - 2q(1)\right) - \frac{1}{\sqrt{2\pi}}\int_0^1 P\left(0, \sqrt{q(0)}w\right)\mathrm{e}^{-\frac{w^2}{2}}\,\mathrm{d}\,w, \tag{4.2.48}$$

where the function $P(x,h)$ satisfies the so called *Parisi equation*:

$$\frac{\partial P(x,h)}{\partial x} = -\frac{\Gamma^2}{2}\frac{\mathrm{d}q(x)}{\mathrm{d}x}\left[\frac{\partial^2 P(x,h)}{\partial h^2} + x\left(\frac{\partial P(x,h)}{\partial h}\right)\right], \qquad P(1,h) = \ln\left(2\cosh\beta h\right). \tag{4.2.49}$$

The function $q(x)$ must be found extremizing the functional (4.2.48). This is a quite difficult task but numerical simulations are in perfect agreement with Parisi's solution when the limit $\beta \to +\infty$ is taken.

The reason of the success of Parisi's approach is that, in the thermodynamical limit, the free energy of a spin glass has a *multivalley structure* in which many minima separated by infinitely high barriers appear. The RSB is necessary to probe all these different valleys simultaneously and this is the intuition behind the procedure. It is said that each valley corresponds to a phase, or *state*, and ergodicity breaking occurs. To better understand this concept, let us introduce, for a certain realization $\mathbf{J} = \{J_{ij}\}$, an overlap function between states of the system in the form

$$\Omega_{ab} \coloneqq \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i\rangle_a\langle\sigma_i\rangle_b, \tag{4.2.50}$$

where $\langle\bullet\rangle_a$ denote the average restricted to the state a. The distribution of the overlaps is

$$\varrho_{\mathbf{J}}(\Omega) \coloneqq \sum_{ab} w_a w_b \delta(\Omega - \Omega_{ab}), \tag{4.2.51}$$

where w_a is the probability of being in the state a. In the Ising model, $J_{ij} \equiv J > 0$, for $\beta \to +\infty$ and $h = 0$ only two phases are possible and therefore

$$\varrho(\Omega) = \frac{\delta(\Omega - 1)}{2} + \frac{\delta(\Omega + 1)}{2}. \tag{4.2.52}$$

In the SK-model we are interested, as usual, to the average over the disorder of $\varrho_{\mathbf{J}}$, distribution of the overlaps for a given set of parameters $\mathbf{J}$,

$$\varrho(\Omega) \coloneqq \overline{\varrho_{\mathbf{J}}(\Omega)}. \tag{4.2.53}$$

In the full RSB scheme, it can be proved [23] that the momenta of $\varrho(\Omega)$ coincide with the average of the powers of $Q_{\alpha\beta}$, solutions of (4.2.26). In other words,

$$\int \Omega^k \varrho(\Omega)\, \mathrm{d}\,\Omega = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\alpha\neq\beta} Q_{\alpha\beta}^k = \int_0^1 q^k(x)\, \mathrm{d}\, x, \tag{4.2.54}$$

In particular,

$$\int \Omega\varrho(\Omega)\, \mathrm{d}\,\Omega = \int_0^1 q(x)\, \mathrm{d}\, x = \int_{-1}^1 q \frac{\mathrm{d}x}{\mathrm{d}q}\, \mathrm{d}\, q \Rightarrow x(q) = \int_{-\infty}^q \varrho(\Omega)\, \mathrm{d}\,\Omega. \tag{4.2.55}$$

where we denoted by $x(q)$ the inverse function of $q(x)$. This equation stresses the strong relation between the matrix of overlaps of replicas $\mathsf{Q} = (Q_{\alpha\beta})_{\alpha\beta}$ and the matrix of overlaps of states $\Omega = (\Omega_{ab})_{ab}$.

4.2.3. The solution of the random matching problem by replicas

Let us now turn back to random matching problems, and in particular to the RMMP, both to give an example of analysis of random optimization problems by means of statistical physics techniques, and to complement the results on the RBMP that will be presented in Subsection 4.1.1. We will follow, with slight modifications, the original paper by Mézard and Parisi [15].

As explained before, the RMMP is formulated on the complete graph K_{2N}. To each edge e of the graph, we associate a weight $w(e)$ extracted from a certain probability density function $\rho(w)$, independently from all other edges. Let us suppose that this distribution is exactly the same given in Eq. (4.1.2), i.e.,

$$\rho(w) = \theta(w)\, \mathrm{e}^{-w}\,. \tag{4.2.56}$$

Labeling by $\{v_i\}_{i=1,\dots 2N}$ the $2N$ vertices of the graph, we denote the weight of the edge $e = (v_i, v_j)$ as w_{ij}. The weighted adjacency matrix $\mathsf{W} = (w_{ij})$, $w_{ij} = w_{ji}$, identifies a realization of our problem. The cost of a matching on this graph can be expressed as

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}] = \frac{1}{N} \sum_{1\leq i<j\leq 2N} m_{ij} w_{ij}, \tag{4.2.57}$$

whereas the optimal cost is given by

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o] \coloneqq \min_{\mathsf{M}} \mathscr{C}^{\mathrm{M}}[\mathsf{M}] = \min_{\sigma\in\mathscr{P}_N} \frac{1}{N} \sum_{i=1}^N w_{i\,\sigma(i)}, \tag{4.2.58}$$

where $\mathscr{P}_N$ is the set of all permutations of N elements. As before, the symmetric matrix $\mathsf{M} = (m_{ij})_{ij}$ is such that $m_{ij} = 1$ if the vertices v_i and v_j are connected, zero otherwise. We assume $m_{ii} = 0\ \forall i$. In a perfect matching on K_{2N}, we have

$$\sum_{j=1}^{2N} m_{ij} = 1, \quad \forall i = 1, \dots, 2N. \tag{4.2.59}$$

Let us start writing down a "partition function" for our "system", assuming $\mathscr{C}^{\mathrm{M}}[\mathsf{M}]$ as an "energy" density functional of the "configuration" M, in the form

$$\begin{aligned} Z(\beta) &\coloneqq \sum_{\mathsf{M}} \left[\prod_{i=1}^{2N} \delta\left(1, \sum_{j=1}^{2N} m_{ij}\right) \right] \mathrm{e}^{-\beta \mathscr{C}^{\mathrm{M}}[\mathsf{M}]} \\ &= \left(\prod_{i=1}^{2N} \int_0^{2\pi} \frac{\mathrm{e}^{i\lambda_i}\, \mathrm{d}\,\lambda_i}{2\pi} \right) \prod_{j<k} \left[1 + \mathrm{e}^{-\beta N^{-1} w_{jk} - i(\lambda_j + \lambda_k)} \right], \end{aligned} \tag{4.2.60}$$

where we introduced the Kronecker symbol $\delta(a,b) \equiv \delta_{a,b}$. Observe that the average value of w_{ij} is $\overline{w_{ij}} = 1$, but the *minimum* weight in a set of N weights $\{w_i\}_{i=1,\dots,N}$ is distributed according to

$$\Pr\left[x \le \min_i\{w_i\}\right] = \prod_{i=1}^{N} \Pr[x \le w_i] = \mathrm{e}^{-Nx}\,. \tag{4.2.61}$$

and therefore, $\overline{\min_i\{w_i\}} = \frac{1}{N}$. The minimum weight appearing in the optimal solution gives a contribution to the optimal cost scaling as $\frac{1}{N^2}$, due to the factor $\frac{1}{N}$ in front of the expression of the cost. The "energy scale" of our interest corresponds therefore to "temperature" regimes given by $\beta^{-1} \sim \frac{1}{N^2}$. For this reason, it is convenient to substitute

$$\beta \mapsto \beta N^2. \tag{4.2.62}$$

We proceed using the *replica approach*. For n replicas, we need to compute

$$\overline{\prod_{\alpha=1}^{n} \prod_{j<k} \left[1 + \mathrm{e}^{-\beta N w_{jk} - i(\lambda_j^\alpha + \lambda_k^\alpha)} \right]} = \prod_{j<k} \overline{\prod_{\alpha=1}^{n} \left[1 + \mathrm{e}^{-\beta N w_{jk} - i(\lambda_j^\alpha + \lambda_k^\alpha)} \right]}. \tag{4.2.63}$$

As in the SK-model, we can now proceed easily with the average over the disorder. In particular, for $N \gg 1$,

$$\overline{\prod_{\alpha=1}^{n} \left[1 + \mathrm{e}^{-\beta N w_{jk} - i(\lambda_j^\alpha + \lambda_k^\alpha)} \right]} = 1 + \frac{1}{\beta N} \sum_{p=1}^{n} \frac{1}{p} \sum_{1 \le \alpha_1 < \cdots < \alpha_p \le n} \mathrm{e}^{-i \sum_{r=1}^{p} (\lambda_j^{\alpha_r} + \lambda_k^{\alpha_r})}\,. \tag{4.2.64}$$

After some computations [15], we get in the large N limit

$$\overline{Z^n} = \int \prod_{p=1}^{n} \prod_{\alpha_1 < \cdots < \alpha_p} \frac{\sqrt{\beta p}\, \mathrm{d}\, q_{\alpha_1 \dots \alpha_p}}{\sqrt{2\pi N}}\, \mathrm{e}^{-\frac{N\beta}{2} \sum_{p=1}^{n} p \sum_{\alpha_1 < \cdots < \alpha_p} q_{\alpha_1 \dots \alpha_p}^2 + 2N \ln z[\mathsf{q}]}, \tag{4.2.65}$$

where

$$z[\mathsf{q}] \coloneqq \left(\prod_{\alpha=1}^{n} \int_0^{2\pi} \frac{\mathrm{e}^{i\lambda^\alpha}\, \mathrm{d}\,\lambda^\alpha}{2\pi} \right) \exp\left[i \sum_{\alpha=1}^{n} \lambda^\alpha + \sum_{p=1}^{n} \sum_{\alpha_1 < \cdots < \alpha_p} q_{\alpha_1 \dots \alpha_p}\, \mathrm{e}^{-i(\lambda^{\alpha_1} + \cdots + \lambda^{\alpha_p})} \right]. \tag{4.2.66}$$

The saddle point equation gives

$$q_{\alpha_1 \dots \alpha_p} = \frac{2}{p\beta} \left\langle \mathrm{e}^{-i(\lambda^{\alpha_1} + \cdots + \lambda^{\alpha_p})} \right\rangle_z, \tag{4.2.67}$$

where $\langle \bullet \rangle_z$ is the average computed respect to the measure defined by the partition function $z[\mathsf{q}]$. In the following we will denote by $Q_{\alpha_1 \dots \alpha_p}$ the solution of the saddle point equation.

In the *replica symmetric hypothesis* $Q_{\alpha_1 \dots \alpha_p} \equiv Q_p$,

$$-\beta \bar{f} = -\frac{\beta}{2} \sum_{p \to 0}^{\infty} (-1)^{p-1} Q_p^2 + 2 \int_{-\infty}^{\infty} \left(\mathrm{e}^{-\mathrm{e}^w} - \mathrm{e}^{-G(w)} \right) \mathrm{d}\, w, \qquad G(x) \coloneqq \sum_{p=1}^{\infty} (-1)^{p-1} \frac{Q_p\, \mathrm{e}^{xp}}{p!}. \tag{4.2.68}$$

Observe that Q_p plays the role of an order parameter in this computation. Here, even in the replica symmetric hypothesis, an *infinite* number of order parameters is necessary, contrary to what happens for the SK-model.

The saddle point equation can be rewritten in terms of $G(w)$ as

$$G(w) = \frac{2}{\beta} \int_{-\infty}^{+\infty} \left[1 - J_0\left(2\,\mathrm{e}^{\frac{w+y}{2}}\right)\right] \mathrm{e}^{-G(y)}\,\mathrm{d}\,y. \tag{4.2.69}$$

where we have introduced the Bessel function of the first kind

$$J_0(x) \coloneqq \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{2^{2p}(p!)^2}. \tag{4.2.70}$$

From the expression of $\bar{f}$, the average cost at finite temperature $\overline{\langle \mathscr{C}^{\mathrm{M}}(\beta)\rangle}$ can be easily obtained:

$$\overline{\langle \mathscr{C}^{\mathrm{M}}(\beta)\rangle} = \frac{1}{\beta} \int_{-\infty}^{+\infty} G(w)\,\mathrm{e}^{-G(w)}\,\mathrm{d}\,w. \tag{4.2.71}$$

Remarkably, defining $\hat{G}(x) \coloneqq G(\beta x)$, in the limit $\beta \to +\infty$ we can still write an expression for the average cost of the ground state, i.e., the average optimal cost of the problem

$$\mathscr{C}^{\mathrm{RMMP}} = \lim_{N\to\infty} \overline{\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o]} = \int_{-\infty}^{+\infty} \tilde{G}(w)\,\mathrm{e}^{-\tilde{G}(w)}\,\mathrm{d}\,w = \frac{\pi^2}{12}, \qquad \tilde{G}(w) = \ln\left(1 + \mathrm{e}^{2w}\right). \tag{4.2.72}$$

Moreover, Mézard and Parisi [15] were able to obtain the distribution $\rho_o(w)$ of the optimal weight appearing in the matching as

$$\rho_o(w) = \theta(w) \frac{\mathrm{e}^{-2w} + w - 1}{2\sinh^2 w}. \tag{4.2.73}$$

Observe that this distribution is extremely similar to the one appearing in Eq. (4.1.15) (obtained for the RBMP). By similar arguments, they also showed that in the RBMP [14]

$$\mathscr{C}^{\mathrm{RBMP}} = \lim_{N\to\infty} \overline{\mathscr{C}^{\mathrm{M}}[\mathsf{M}_o]} = \frac{\pi^2}{6}. \tag{4.2.74}$$

We stress again that all the results above were derived *for the first time* using the previous techniques and, remarkably, through a pretty straightforward approach.

4.3. The cavity method

In the previous Section we have shown, by an example, the power of the replica trick in the analysis of the properties of the ground state (i.e., the optimal solution) in a random optimization problems. In the approach above, however, there is no hint about the algorithmic procedure to find a solution of a given instance of the problem. We might wonder if there is any technique in the theory of disordered systems that can help us to solve a specific instance, besides the investigation of the average properties. The *cavity method* [16, 17], introduced in the context of the theory of spin glasses, provides an answer to this question.

4.3.1. General theory of the cavity method and belief propagation

The cavity method is a quite general algorithm that successfully treated many random optimization problems, including the assignment problem. In the present Subsection we introduce this method, following the pedagogical expositions in [8, 9, 13].

We suppose that our system is characterized by N spin-type variables $\sigma_i \in \{-1, 1\}$, $i = 1, \dots, N$, in such a way that $\boldsymbol{\sigma} = \{\sigma_i\}$ fixes a configuration of the system itself. We assume also that a Hamiltonian function for the system is given, in the form

$$H(\boldsymbol{\sigma}) = \sum_{a=1}^{M} E_a(\boldsymbol{\sigma}_{\partial a}) + \sum_{i=1}^{N} W_i(\sigma_i). \tag{4.3.1}$$

In the previous expression E_a is a function of a subset $\boldsymbol{\sigma}_{\partial a} \subseteq \boldsymbol{\sigma}$, whilst the functions W_i depends on one variable σ_i only (one body term). We can associate to the previous Hamiltonian a *factor graph*, that pictorially represents the involved variables and the interactions. The factor graph is a bipartite graph

$$\mathtt{F}_H = \mathrm{Graph}(\mathscr{V}_\sigma, \mathscr{V}_E; \mathscr{E}), \quad |\mathscr{V}_\sigma| = N, \quad |\mathscr{V}_E| = M, \quad \mathscr{E} \subseteq \mathscr{V}_\sigma \times \mathscr{V}_E, \tag{4.3.2}$$

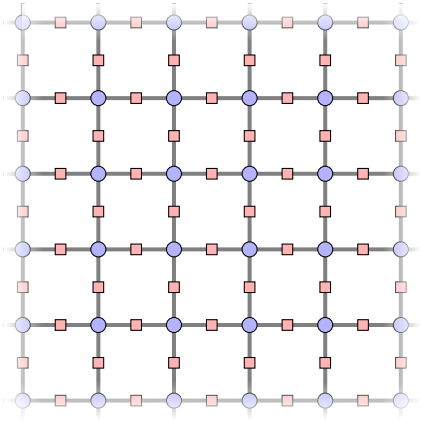
constructed as follows:

- we associate N "variable" vertices $\mathscr{V}_\sigma = \{i\}_i$ to the N variables $\boldsymbol{\sigma}$, in such a way that $i \leftrightarrow \sigma_i$;
- we associate M "function" vertices $\mathscr{V}_E = \{a\}_a$ to the M terms $\{E_a\}_a$ in the Hamiltonian (4.3.2);
- $(i, a) \in \mathscr{E}$ if and only if $\sigma_i \in \boldsymbol{\sigma}_{\partial a}$.

Observe that no edge exists connecting two vertices of the same type. Moreover, for the moment we consider implicit in the vertex i the representation the possible contribution $W_i(\sigma_i)$ to the Hamiltonian. For example, the Ising model on the two-dimensional lattice with zero magnetic field

$$H = \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j, \tag{4.3.3}$$

has the following factor graph



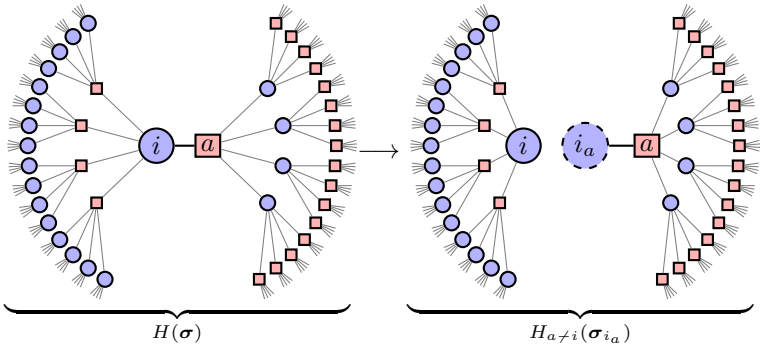
where the square nodes are factor nodes and the circle nodes are variables nodes. The factor graph is a good starting point to understand the cavity method. Indeed, the cavity method is based on local modifications of this graph and on the study of the obtained new system with reference to the old one. This is typically preformed in three ways.

Let us consider a Hamiltonian in the form (4.3.2), and suppose that $(i, a) \in \mathscr{E}$ in the factor graph. We can therefore define the following new Hamiltonian from Eq. (4.3.2)

$$H_{a \neq i}(\boldsymbol{\sigma}_{i_a}) = \sum_{b \neq a} E_b(\boldsymbol{\sigma}_{\partial b}) + E_a(\boldsymbol{\sigma}_{\partial a}^{i \leftrightarrow i_a}) + \sum_{i=1}^{N} W_i(\sigma_i). \tag{4.3.4}$$

This new Hamiltonian has $N + 1$ variables, $\boldsymbol{\sigma}_{i_a} = \boldsymbol{\sigma} \cup \{\sigma_{i_a}\}$. In particular, a new variable, σ_{i_a}, appears. The second term in the Hamiltonian above needs some explanation. The notation

$E_a(\boldsymbol{\sigma}_{\partial a}^{i\leftrightarrow i_a})$ means that E_a is evaluated on its usual variables except for the variable σ_i, that must be replaced with the variable σ_{i_a}. This is the only term where the new variable appears. Observe, for example, that no one body term is associated to σ_{i_a}. Pictorially, this means that, in the factor graph, we are "detaching" the node i and the node a, inserting a new node, let us call it i_a, connected to a:

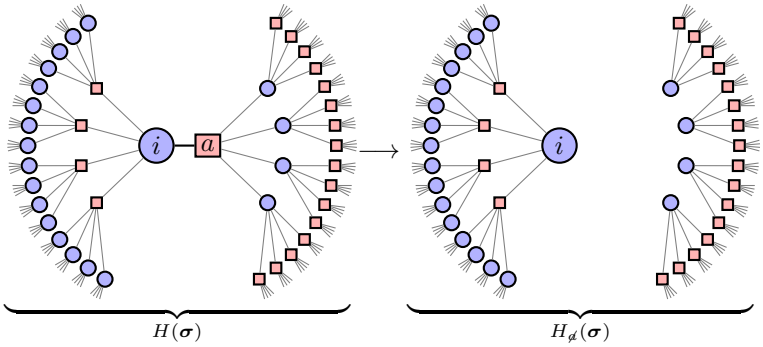


(4.3.5)

Similarly, we can remove a function vertex, let it be a, obtaining the Hamiltonian

$$H_{\not a}(\boldsymbol{\sigma}) = \sum_{b\neq a} E_b(\boldsymbol{\sigma}_{\partial b}) + \sum_{i=1}^{N} W_i(\sigma_i). \tag{4.3.6}$$

The new factor graph has $M-1$ function vertices. Pictorially,

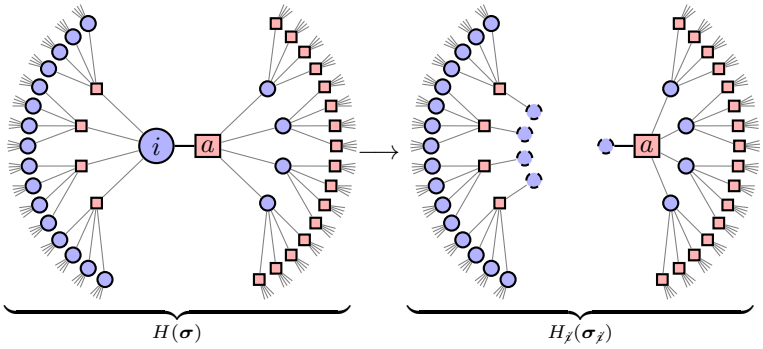


(4.3.7)

Finally, we can remove from the original factor graph a variable vertex i, writing the Hamiltonian with $N-1+|\partial i|$ spin variables,

$$H_{\not i}(\boldsymbol{\sigma}_{\not i}) = \sum_{b\colon \sigma_i\notin\boldsymbol{\sigma}_{\partial b}} E_b(\boldsymbol{\sigma}_{\partial b}) + \sum_{b\colon \sigma_i\in\boldsymbol{\sigma}_{\partial b}} E_b(\boldsymbol{\sigma}_{\partial b}^{i\leftrightarrow i_b}) + \sum_{j\neq i} W_j(\sigma_j). \tag{4.3.8}$$

In the previous Hamiltonian, we have introduced $|\partial i|$ variables σ_{i_b}, one for each value of b such that $\sigma_i \in \boldsymbol{\sigma}_{\partial b}$. We denoted by $\boldsymbol{\sigma}_{\not i} \coloneqq (\boldsymbol{\sigma}\setminus\{\sigma_i\}) \cup \bigcup_{b\colon \sigma_i\in\boldsymbol{\sigma}_{\partial b}} \{\sigma_{i_b}\}$. Pictorially, we have



(4.3.9)

The dashed vertices represents the new variables σ_{i_b}. Remember that we do not insert the one body contribution for these variables.

The introduced three types of alteration of the original factor graph create, in all cases, a local "cavity" and are indeed a sort of "local perturbation" of the original graph. For each of them, we can write down the usual thermodynamical functionals. More importantly, we can define three types of *fields* on these graphs that will play an important role. Let us first consider the system described by the Hamiltonian in Eq. (4.3.2). Then we can always write

$$\Pr\left[\sigma_i = \sigma\right] \equiv \langle \delta_{\sigma_i \sigma} \rangle = \frac{\mathrm{e}^{-\beta h_i \sigma}}{2\cosh(\beta h_i)} \tag{4.3.10a}$$

Formally, h_i plays the role of parameter of the distribution in Eq. (4.3.10a) over the site i, but it is also a sort of local *magnetic field*, acting on the spin σ_i. Similarly, let us consider the system described by Eq. (4.3.4). Then we define the *cavity field* $h_{i\to a}$ such that

$$\Pr\left[\sigma_i = \sigma\right] \equiv \langle \delta_{\sigma_i \sigma} \rangle_{i \neq a} = \frac{\mathrm{e}^{-\beta h_{i\to a} \sigma}}{2\cosh(\beta h_{i\to a})}. \tag{4.3.10b}$$

In the same system, we define the *cavity bias* $u_{a\to i}$ such that

$$\Pr\left[\sigma_{i_a} = \sigma\right] \equiv \left\langle \delta_{\sigma_{i_a} \sigma} \right\rangle_{i \neq a} = \frac{\mathrm{e}^{-\beta u_{a\to i} \sigma}}{2\cosh(\beta u_{a\to i})}. \tag{4.3.10c}$$

Up to now, we have simply discussed deformed models of the original one with no reference to the possible presence of disorder or to the specific structure of the factor graph. To proceed further, some hypotheses are needed to relate the fields introduced above. In particular, we require that the so called *cavity ansatz* holds.

Ansatz (Cavity ansatz)**.** *We assume that, in the $N \to \infty$ limit, in a pure state the approximations*

$$\left\langle \prod_{b \in \partial i} \delta_{\sigma_{i_b} \sigma} \right\rangle_{\not i} \simeq \prod_{b \in \partial i} \left\langle \delta_{\sigma_{i_b} \sigma} \right\rangle_{\not i}, \tag{4.3.11a}$$

$$\left\langle \prod_{j \in \partial a} \delta_{\sigma_j \sigma} \right\rangle_{\not a} \simeq \prod_{j \in \partial a} \left\langle \delta_{\sigma_j \sigma} \right\rangle_{\not a}, \tag{4.3.11b}$$

hold for any variable node i and any factor node a. Moreover,

$$\left\langle \delta_{\sigma_{i_a} \sigma} \right\rangle_{\not i} \simeq \left\langle \delta_{\sigma_{i_a} \sigma} \right\rangle_{i \neq a}, \tag{4.3.11c}$$

$$\langle \delta_{\sigma_i \sigma} \rangle_{\not a} \simeq \langle \delta_{\sigma_i \sigma} \rangle_{i \neq a}. \tag{4.3.11d}$$

As usual, in the previous expression we denoted by ∂v the set of neighbors of the vertex v in the factor graph. The cavity ansatz is sometimes called *replica symmetric assumption*. In other words, the cavity ansatz requires that, in the cavity factor graphs, the correlations between the two "sides" of the cavity are almost negligible. This happens exactly when the factor graph is a tree (in this case, the cavity separates the tree in two, or more, new trees completely uncorrelated) or if the factor graph has very long cycles and correlations decay fast enough on these long cycles. With reference to Eqs. (4.3.10), Eqs. (4.3.11) are immediately rewritten as

$$\frac{\mathrm{e}^{-\beta h_{i\to a} \sigma}}{2\cosh(\beta h_{i\to a})} \propto \mathrm{e}^{-\beta W_i(\sigma_i)} \prod_{\substack{b \in \partial i \\ b \neq a}} \frac{\mathrm{e}^{-\beta u_{b\to i} \sigma}}{2\cosh(\beta u_{b\to i})}, \tag{4.3.12a}$$

$$\frac{\mathrm{e}^{-\beta u_{a\to i} \sigma}}{2\cosh(\beta u_{a\to i})} \propto \sum_{\boldsymbol{\sigma}_{\partial a}} \delta_{\sigma_i \sigma}\, \mathrm{e}^{-\beta E_a(\boldsymbol{\sigma}_{\partial a})} \prod_{\substack{j \in \partial a \\ j \neq i}} \frac{\mathrm{e}^{-\beta h_{j\to a} \sigma_j}}{2\cosh(\beta h_{j\to a})}. \tag{4.3.12b}$$

The previous equations are called *cavity equations*. If $W_i(\sigma_i) \equiv w_i\sigma_i$, then they can be rewritten more simply as

$$h_{i\to a} = w_i + \sum_{\substack{b\in\partial i\\ b\neq a}} u_{b\to i}, \tag{4.3.13a}$$

$$u_{a\to i} = \frac{1}{2\beta}\ln \frac{\displaystyle\sum_{\substack{\boldsymbol{\sigma}_{\partial a}\\ \sigma_i=-1}} \exp\left[-\beta E_a(\boldsymbol{\sigma}_{\partial a}) - \beta\sum_{\substack{j\in\partial a\\ j\neq i}} h_{j\to a}\right]}{\displaystyle\sum_{\substack{\boldsymbol{\sigma}_{\partial a}\\ \sigma_i=+1}} \exp\left[-\beta E_a(\boldsymbol{\sigma}_{\partial a}) - \beta\sum_{\substack{j\in\partial a\\ j\neq i}} h_{j\to a}\right]}. \tag{4.3.13b}$$

Once that the previous values are found, it is easily seen that the free energy F of the original system can be written in terms of the free energy $F_{i\neq a}$, $F_{\not a}$ and $F_{\not i}$ of the system described by Eq. (4.3.4), Eq. (4.3.6) and Eq. (4.3.8) respectively as

$$F \simeq F_{i\neq a} - \frac{1}{\beta}\ln\sum_{\sigma\in\{-1,1\}} \frac{\mathrm{e}^{-\beta(h_{i\to a}+u_{a\to i})\sigma}}{4\cosh(\beta h_{i\to a})\cosh(\beta u_{a\to i})}, \tag{4.3.14a}$$

$$F \simeq F_{\not a} - \frac{1}{\beta}\ln\sum_{\boldsymbol{\sigma}_{\partial a}} \mathrm{e}^{-\beta E_a(\boldsymbol{\sigma}_{\partial a})}\prod_{i\in\partial a}\frac{\mathrm{e}^{-\beta h_{i\to a}\sigma_i}}{2\cosh(\beta u_{a\to i})}, \tag{4.3.14b}$$

$$F \simeq F_{\not i} - \frac{1}{\beta}\ln\sum_{\sigma\in\{-1,1\}} \mathrm{e}^{-\beta W_i(\sigma)}\prod_{a\in\partial i}\frac{\mathrm{e}^{-\beta u_{a\to i}\sigma}}{2\cosh(\beta u_{a\to i})}. \tag{4.3.14c}$$

From the last equation in particular, and from the cavity equations in Eqs. (4.3.12), we see that, for a given $a \in \partial i$

$$\Pr[\sigma_i = \sigma] \propto \exp\left(-\beta W(\sigma_i) - \beta\sigma\sum_{b\in\partial_i} u_{b\to i}\right) = \exp\left[-\beta\sigma(u_{a\to i} + h_{i\to a})\right]. \tag{4.3.15}$$

Belief propagation So far we have discussed the cavity method thinking to a *spin system* and we introduced the fields $\{h_{i\to a}, u_{a\to i}\}_{(i,a)}$ as a sort of local magnetic fields on the edges. However, the spirit of the cavity method is quite general. We can indeed formulate an algorithm, called *belief propagation*, that can be used to solve combinatorial optimization problem. Suppose that we have a model in N variables $\mathbf{x} = \{x_i\}_{i=1,\dots,N}$, taking their values in a finite set, and suppose also that their joint probability has the form

$$\mu(\mathbf{x}) = \frac{1}{Z}\prod_{a=1}^{M}\psi_a(\mathbf{x}_{\partial a}), \tag{4.3.16}$$

where by $\mathbf{x}_{\partial a} \subseteq \mathbf{x}$ we denote a certain subset of the set of variables, depending on a, and by Z a proper normalization constant. As in the case of our spin system, a probability measure of the form (4.3.16) can be graphically represented by a factor graph,

$$\mathtt{F}_\mu = \mathrm{Graph}(\mathscr{V}_x, \mathscr{V}_\psi; \mathscr{E}), \quad |\mathscr{V}_x| = N, \quad |\mathscr{V}_\psi| = M, \quad \mathscr{E} \subseteq \mathscr{V}_x \times \mathscr{V}_\psi. \tag{4.3.17}$$

Each function vertex $a \in \mathscr{V}_\psi$ is associated to the function $\psi_a(\mathbf{x}_{\partial a})$ and each vertex $i \in \mathscr{V}_x$ is associated to the variable x_i. We have that $(i,a) \in \mathscr{E}$ if and only if $x_i \in \mathbf{x}_{\partial a}$. Again, we can define on each edge of this bipartite graph two functions, one for each direction in which the edge can be crossed. In particular, for an edge $e = (i,a)$, we define the *messages*

$$\upsilon_{i\to a}(x), \quad \nu_{a\to i}(x). \tag{4.3.18}$$

The messages are elements in the space of probability distribution functions, subject to the normalization conditions

$$\sum_{i\in\partial a} \upsilon_{i\to a} = 1 \quad \forall a, \qquad \sum_{a\in\partial i} \nu_{a\to i} = 1 \quad \forall i. \tag{4.3.19}$$

Proceeding exactly as in the case of the cavity method, we can write down an iterative "message–passing" algorithm in which we update a message on an edge on the basis of the values of the incoming messages on the tail of the directed edge. This means that we can write iterative equations that are completely equivalent to the cavity equations above. These equations are called *belief propagation equations* and they provide the recipe to update the messages. After t steps, we have that

$$\upsilon^{t+1}_{i\to a}(x) \simeq \prod_{b\in\partial i\setminus a} \nu^{t}_{b\to i}(x), \tag{4.3.20a}$$

$$\nu^{t+1}_{a\to i}(x) \simeq \sum_{\substack{\mathbf{x}_{\partial a}\\ x_i\equiv x}} \psi_a(\mathbf{x}_{\partial a}) \prod_{\substack{j\in\partial a\\ j\neq i}} \upsilon^{t}_{j\to a}(x_j). \tag{4.3.20b}$$

After T iterations, we can write

$$\upsilon^{T}_{i}(x_i) \simeq \prod_{a\in\partial i} \nu^{T-1}_{a\to i}(x_i), \tag{4.3.21}$$

It can be proved that, if the factor graph $\mathtt{F}_\mu$ is a tree, Eqs. (4.3.20) converge irrespectively of the initial conditions, after at most

$$T^* = \operatorname{diam}(\mathtt{F}_\mu) \tag{4.3.22}$$

steps. Let $\upsilon^*_i(x)$, $\nu^*_i(x)$ be the limiting distributions. We have also that υ^*_i and ν^*_i are related to the marginal distribution corresponding to the variable on the site i. Indeed,

$$\upsilon^*_i(x_i) \equiv \sum_{x_j\neq x_i} \mu(\mathbf{x}). \tag{4.3.23}$$

The cavity method presented above is based on the idea that the structure of the factor graph is tree-like (i.e., there are no small cycles) and variables are weakly correlated at large distances. If these hypotheses are not satisfied, we cannot expect that the cavity method works. The existence of strong long-range correlation among the variables suggest the existence of a highly correlated phase and, therefore, the breaking of ergodicity in the phase space. To solve this fundamental issue, Mézard and Parisi [16, 17] introduced the *one-step replica symmetry breaking* (1RSB) cavity method. The 1RSB cavity method is *not* a rigorous method and it is still a very active research topic. We will not discuss it here, despite the fact that it played a fundamental role in the analysis of important optimization problems. The interested reader can find a modern presentation of this technique in [13].

4.3.2. Belief propagation for the assignment problem

Let us consider again our main problem, the RBMP, or assignment problem, on the weighted complete bipartite graph $\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E})$. As usual, we associate to each edge a weight, $w\colon (v_i, u_j) \mapsto w_{ij} \in \mathbb{R}^+$ and the cost function is

$$\mathscr{C}^{\mathrm{M}}[\mathsf{M}] = \frac{1}{N}\sum_{ij} m_{ij} w_{ij}, \tag{4.3.24}$$

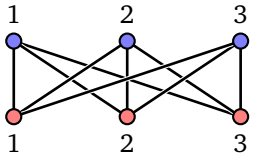
where $\mathsf{M} = (m_{ij})_{ij}$ is a matching matrix, such that $m_{ij} = 1$ if the edge (v_i, u_j) is an edge of the matching, 0 otherwise. As known, the matching matrix must satisfy the conditions

$$\sum_{i=1}^{N} m_{ik} = \sum_{j=1}^{N} m_{kj} = 1, \quad k = 1, \dots, N. \tag{4.3.25}$$

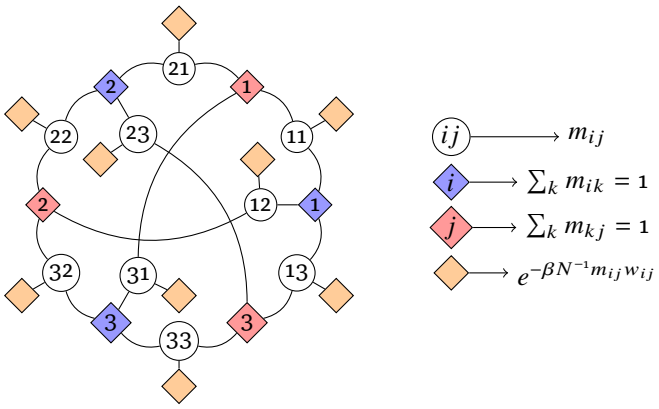
We can write a joint probability in the form of Eq. (4.3.16) for a given matching matrix M,

$$\mu(\mathsf{M}) = \prod_{ij} \delta\left(\sum_{i=1}^{N} m_{ij}, 1\right) \delta\left(\sum_{j=1}^{N} m_{ij}, 1\right) \frac{\mathrm{e}^{-\beta N^{-1} m_{ij} w_{ij}}}{Z}, \quad \beta \in \mathbb{R}^{+}. \tag{4.3.26}$$

A factor graph associated to this measure is easily constructed, the variables being the elements m_{ij} of the matrix M. For example, for $N = 3$ the graph $\mathtt{K}_{3,3}$ is



The corresponding factor graph for can be depicted as follows:



As the reader can see, the graph has many quite small cycles and this suggests that we cannot apply the cavity approach, at least in its replica symmetric version. We will make however the assumption that the cavity ansatz still holds. We will justify this strong (and counterintuitive) statement later. Under this hypothesis and with reference to Eq. (4.3.26) and Eqs. (4.3.20), the belief propagation equations on the fixed point are

$$\upsilon_{(ij)\to i}(m) \simeq \nu_{j\to(ij)}(m)\, \mathrm{e}^{-\beta N^{-1} m w_{ij}}, \tag{4.3.27a}$$

$$\nu_{i\to(ij)}(m) \simeq \sum_{\{m_{kj}\}_{k\neq i}} \delta\left(m + \sum_{k\neq i} m_{kj}, 1\right) \prod_{k\neq i} \upsilon_{(kj)\to j}(m_{kj}). \tag{4.3.27b}$$

To take the $\beta \to \infty$ limit, let us introduce, for each oriented edge $\vec{e}_{ij} = \overrightarrow{(v_i, u_j)}$ of the original complete bipartite graph,

$$X\left(\vec{e}_{ij}\right) := \frac{1}{\beta} \ln \frac{\nu_{i\to(ij)}(1)}{\nu_{i\to(ij)}(0)}. \tag{4.3.28}$$

For the same edge $e \in \mathscr{E}$ we define also

$$X\left(\overleftarrow{e}_{ij}\right) := \frac{1}{\beta} \ln \frac{\nu_{j\to(ij)}(1)}{\nu_{j\to(ij)}(0)}. \tag{4.3.29}$$

Through this change of variable the belief propagation equations become

$$X^{t+1}\left(\vec{e}_{ij}\right) = -\frac{1}{\beta} \ln \sum_{k\neq i} \exp\left[-\frac{\beta w_{kj}}{N} + \beta X^{t}\left(\overleftarrow{e}_{kj}\right)\right], \tag{4.3.30a}$$

$$X^{t+1}\left(\overleftarrow{e}_{ij}\right) = -\frac{1}{\beta} \ln \sum_{k\neq j} \exp\left[-\frac{\beta w_{ik}}{N} + \beta X^{t}\left(\overleftarrow{e}_{ik}\right)\right]. \tag{4.3.30b}$$

Using belief propagation, we are eventually back to solve our problem on the original complete bipartite graph. Being interested in the ground state, we have to take the $\beta \to \infty$ limit. We obtain the so called MIN-SUM algorithm for the assignment problem:

$$X^{t+1}(\vec{e}_{ij}) = \min_{k \neq i} \left[w_{kj} N^{-1} - X^t(\overleftarrow{e}_{kj}) \right], \tag{4.3.31a}$$

$$X^{t+1}(\overleftarrow{e}_{ij}) = \min_{k \neq j} \left[w_{ik} N^{-1} - X^t(\vec{e}_{ik}) \right]. \tag{4.3.31b}$$

For a given edge $e \in \mathscr{E}$ we have two message functions, i.e. $X(\vec{e})$ and $X(\overleftarrow{e})$ and we have to update both simultaneously. The criterion of occupancy of an edge can be easily identified observing that, if (ij) is an edge of the optimal graph, $\nu_{i\to(ij)}(1) \sim \nu_{j\to(ij)}(1) \sim 1$ and, by consequence, $\nu_{i\to(ij)}(0) \sim \nu_{j\to(ij)}(1) \sim 0$, and therefore in this case

$$X(\vec{e}_{ij}) + X(\overleftarrow{e}_{ij}) \geq 0. \tag{4.3.32}$$

On the other hand, if e_{ij} is not an edge of the optimal matching, at the fixed point $X(\vec{e}_{ij}) + X(\overleftarrow{e}_{ij}) \leq 0$. Our belief propagation equations suggest that the problem of finding an optimal matching on the bipartite complete graph is equivalent to search for an optimal matching on an *infinite tree* in the $N \to \infty$ limit: it is indeed enough to write down the belief propagation equations for the tree introduced by Aldous, see Fig. 4.1.1. Eqs. (4.3.31) are perfectly identical to Eqs. (4.1.12), and the criterion in Eq. (4.3.32) is equivalent to the one in Eq. 4.1.13 observing that in our case we have to replace $w \mapsto wN^{-1} \xrightarrow{N\to\infty} 0$. This fact partially justifies the success of the replica symmetric cavity method for the solution of the assignment problem for $N \to \infty$ and the cavity method is indeed a rigorous approach for the investigation, as Aldous showed. However the convergence for finite N is not guaranteed. Indeed, Fichera [8] showed that, for finite N, after a certain time t_c the message functions enter in a quasi-periodic regime that however still allows us to identify the correct solution. Moreover, if there is a nonoptimal matching $\mathsf{M} = (m_{ij})_{ij}$ differing for one cycle from the optimal one $\mathsf{M}_o = (m^o_{ij})_{ij}$ and having

$$\frac{\mathscr{C}^{\mathrm{M}}[\mathsf{M}] - \mathscr{C}^{\mathrm{M}}[\mathsf{M}^o]}{N - \sum_{ij} m_{ij} m^o_{ij}} \ll 1, \tag{4.3.33}$$

the algorithm converges very slowly.

Up to now, we did not introduce any randomness in the problem. Suppose then that $\{w_{ij}\}_{ij}$ are i.i.d. random variables, distributed accordingly to the density distribution in Eq. (4.1.2). Comparing Eq. (4.3.31) with Eq. (4.1.12), it is easily seen that the fixed point solution of the previous equation $X(\vec{e})$ is, up to a rescaling, the same function introduced by Aldous on the infinite tree that we constructed in Subsection 4.1.1. It is clear that we may reproduce here exactly the same computations, obtaining the same results. The cavity method is therefore able to provide both an algorithm for the solution of a given instance of the random assignment, and the correct equations for its probabilistic analysis.

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Chapter 5

Euclidean Matching Problems

5.1. Euclidean optimization problems

In the previous Chapter we presented some random optimization problems on weighted graphs and some useful tools for their solution. However, we did not discuss the effect of possible correlations among the weights associated to the edges of the considered graphs, or we supposed explicitly that no correlation at all was present.

In the present Chapter we will introduce *Euclidean correlations* in our problems. Let us suppose that the problem is defined on a graph $\mathtt{G} = \mathrm{Graph}(\mathscr{V}; \mathscr{E})$. We will assume also that a connected convex domain $\Omega \subset \mathbb{R}^d$ is given and we will consider a map $\boldsymbol{\Phi}$,

$$\boldsymbol{\Phi} \colon \mathscr{V} \to \Omega \text{ such that } v_i \in \mathscr{V} \mapsto \boldsymbol{\Phi}(v_i) \equiv \boldsymbol{\xi}_i \in \Omega. \tag{5.1.1}$$

The map $\boldsymbol{\Phi}$ can be a random point process. We assume that a weight function $w \colon \mathscr{E} \to \mathbb{R}^+$ is given in such a way that

$$w(e_{ij}) \equiv w\left(\left\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\right\|\right), \quad e_{ij} \equiv (v_i, v_j). \tag{5.1.2}$$

In other words, the weight associated to the edge (v_i, v_j) is a function of the Euclidean distance $\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\|$. Apart from this difference, the Euclidean versions of optimization problems on graphs are formulated exactly as in the previous Chapter. In this thesis, we will analyze a particular form of weight function (5.1.2), in particular we will assume

$$w(e_{ij}) \equiv w^{(p)}(e_{ij}) \equiv w_{ij}^{(p)} \coloneqq \left\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\right\|^p, \quad p \in \mathbb{R}^+. \tag{5.1.3}$$

The asymptotic analysis of the optimal cost with weight functions in the form (5.1.3) has been widely studied in the hypothesis that the points $\{\boldsymbol{\xi}_i\}_i$ are randomly generated on Ω [31]. In particular, the asymptotic of the Tsp problem was firstly analyzed by Beardwood, Halton, and Hammersley [6], and subsequently by Redmond and Yukich [28]. In particular, let us assume that we are searching for a subgraph $\mathtt{O} \subseteq \mathtt{G}$ in a set of suitable subgraph $\mathcal{S}$ such that the functional

$$\mathscr{C}[\mathtt{S}] = \frac{1}{|\mathscr{E}_\mathtt{S}|} \sum_{e \in \mathscr{E}_\mathtt{S}} w^{(p)}(e), \quad \mathtt{S} = \mathrm{Graph}(\mathscr{V}_\mathtt{S}; \mathscr{E}_\mathtt{S}) \in \mathcal{S}, \tag{5.1.4}$$

is minimized, i.e.,

$$\mathscr{C}[\mathtt{O}] = \min_{\mathtt{S} \in \mathcal{S}} \mathscr{C}[\mathtt{S}]. \tag{5.1.5}$$

The defined functional is *homogeneous* and *translationally invariant*, i.e.,

$$\mathscr{C}[\mathtt{S}] \xrightarrow{\boldsymbol{\xi}_i \mapsto \lambda \boldsymbol{\xi}_i + \mathbf{r}} \lambda^p \mathscr{C}[\mathtt{S}], \qquad \lambda \in \mathbb{R}^+, \quad \mathbf{r} \in \mathbb{R}^d. \tag{5.1.6}$$

Redmond and Yukich [28] proved that the optimal cost (5.1.5) for the Tsp and for the monopartite matching problem on the complete graph $\mathtt{K}_{2N}$, embedded in the unit hypercube in d dimensions, scales as $N^{-\frac{p}{d}}$ for $p \geq 1$ for $N \gg 1$.

Euclidean optimization problems are of great interest, due to the fact that some optimization problems are *actually* defined in the geometric space (consider, for example, the Tsp to be solved in a certain geographical area). The Euclidean origin of the weights in the graph is not relevant

for the solution of a given instance and the algorithms available in the literature obviously work perfectly. However, the study in presence of randomness is more complicated. In random Euclidean optimization problems, randomness is typically introduced in the embedding process of the graph in Ω. In this case a probability distribution density is given on Ω,

$$\rho\colon \Omega \to \mathbb{R}^+, \quad \int_\Omega \rho(\mathbf{x})\, \mathrm{d}^d\, x = 1. \tag{5.1.7}$$

We suppose then that $N := |\mathcal{V}|$ points, $\mathfrak{X} := \{\boldsymbol{\xi}_i\}_{i=1,\dots,N} \in \Omega$, are independently randomly generated on Ω. We associate therefore to each vertex $v_i \in \mathcal{V}$ a point $\boldsymbol{\xi}_i$ at random. In this way all quantities $w(e)$, generated according to Eq. (5.1.2), are random, but in general an *Euclidean correlation* appears, due to the correlation among the distances of the points. The average procedure is therefore more subtle than in the purely uncorrelated case, and the point generation procedure becomes of paramount importance. Once the points are generated, we can introduce the *empirical measure*

$$\rho_{\mathfrak{X}}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N} \delta^{(d)}\left(\mathbf{x} - \boldsymbol{\xi}_i\right). \tag{5.1.8}$$

It can be proved [14] that for $N \to \infty$ the empirical measures $\rho_{\mathfrak{X}}$ converges to ρ almost surely.

5.2. Euclidean matching problems

Random Euclidean matching problems on a domain $\Omega \subset \mathbb{R}^d$ will be the main topic of the following Sections. In the present Section we first fix the notation and the terminology that will be used throughout this Chapter.

Euclidean monopartite matching problem In the *random Euclidean monopartite matching problem* (rEm), we consider a matching problem on the complete graph $\mathtt{K}_{2N}$, assuming that a point $\boldsymbol{\xi}_i \in \Omega$ is associated to each vertex $v_i \in \mathcal{V}$. The points $\{\boldsymbol{\xi}_i\}_i$ are supposed randomly generated on Ω, according to a given probability density function ρ, as in Eq. (5.1.7). We are interested in the perfect matching $\mathtt{M}_o \subseteq \mathtt{K}_{2N}$ that minimizes the functional (5.1.4),

$$\mathscr{C}^{(p,\mathrm{Em})}[\mathtt{M}] := \frac{1}{N} \sum_{e \in \mathscr{E}_{\mathtt{M}}} w^{(p)}(e), \qquad \mathtt{M} = \mathrm{Graph}(\mathcal{V}_{\mathtt{M}}; \mathscr{E}_{\mathtt{M}}) \in \mathcal{S}, \tag{5.2.1}$$

where $w^{(p)}(e)$ is defined in Eq. (5.1.3) and

$$\mathcal{S} := \{\mathtt{M} \mid \mathtt{M} \subset \mathtt{K}_{2N} \text{ perfect matching}\}. \tag{5.2.2}$$

In the following we will denote the optimal matching cost by

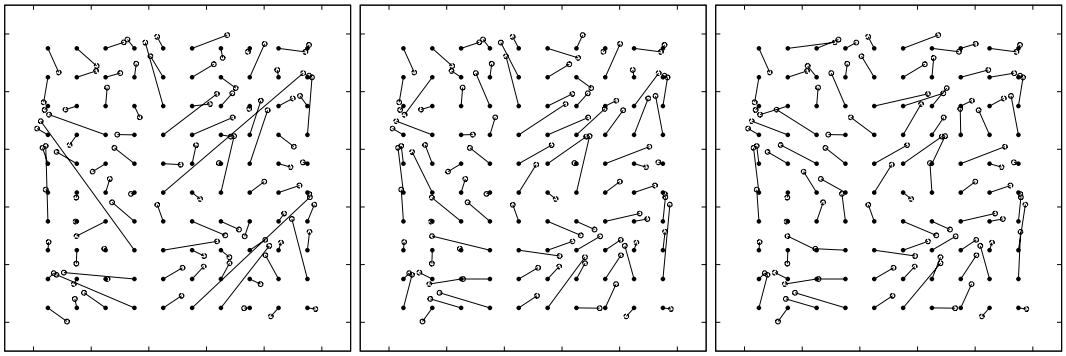
$$\mathscr{C}_N^{(p,\mathrm{Em})} := \mathscr{C}^{(p,\mathrm{Em})}[\mathtt{M}_o] = \min_{\mathtt{M} \in \mathcal{S}} \mathscr{C}^{(p,\mathrm{Em})}[\mathtt{M}]. \tag{5.2.3}$$

The *average* optimal cost will be denoted by

$$\mathscr{C}_{N,d}^{(p,\mathrm{rEm})} := \overline{\mathscr{C}_N^{(p,\mathrm{Em})}}, \tag{5.2.4}$$

where the average $\overline{\bullet}$ is on the positions of the points. Observe that the previous quantity strongly depends on the considered domain Ω, on the number of points N and on the distribution ρ.

Euclidean bipartite matching problem In the *random Euclidean bipartite matching problem* (rEb), we consider a matching problem on the complete bipartite graph $\mathtt{K}_{N,N} = \mathrm{Graph}(\mathcal{V}, \mathcal{U}; \mathscr{E})$, $N := |\mathcal{V}| = |\mathcal{U}| \in \mathbb{N}$. Randomness can be introduced in two different ways. We can assume that both vertices in $\mathcal{V}$ and vertices in $\mathcal{U}$ are associated to points in Ω, randomly and independently generated according to a distribution ρ: in this case we call the problem *random–random Euclidean bipartite matching problem* (rr–Eb). If $\rho(\mathbf{x}) = \frac{1}{|\Omega|}$, where $|\Omega|$ is the Lebesgue measure of Ω, the problem is called *Poisson–Poisson Euclidean bipartite matching problem*. Another possibility is to associate, e.g., the vertices of $\mathcal{V}$ to random points on Ω



(A) $p = 0.5$. (B) $p = 1$. (C) $p = 1.5$.

FIGURE 5.2.1.: Optimal grid–random Euclidean bipartite matching with $N = 100$ on the square with open boundary conditions for the same instance. The random points are generated with uniform distribution on the square: in this case, the matching problem is called also grid–Poisson Euclidean bipartite matching problem. Observe that for $p = 1$ there are no intersecting links.

and the vertices of $\mathcal{U}$ to a fixed lattice on Ω. If a square lattice is adopted, the problem is called *grid–random Euclidean bipartite matching problem* (GR–EB, see for example Fig. (5.2.1)). If the points associated to the vertices of $\mathcal{V}$ are uniformly distributed on Ω, then we say that the problem is a *grid–Poisson Euclidean bipartite matching problem.*

As in the assignment problem, in the EB we are interested on the perfect matching $\mathtt{M}_o$ such that the functional

$$\mathscr{C}^{(p,\mathrm{EB})}[\mathtt{M}] := \frac{1}{N}\sum_{e\in\mathscr{E}_{\mathtt{M}}} w^{(p)}(e), \qquad \mathtt{M} = \mathrm{Graph}(\mathscr{V}_{\mathtt{M}};\mathscr{E}_{\mathtt{M}}) \in \mathcal{S}. \tag{5.2.5}$$

is minimized. As in the monopartite case, $w^{(p)}(e)$ is given by Eq. (5.1.3) and

$$\mathcal{S} := \{\mathtt{M} \mid \mathtt{M} \subset \mathtt{K}_{N,N} \text{ perfect matching}\}. \tag{5.2.6}$$

Let us denote by $\mathcal{V} := \{v_i\}_{i=1,\dots,N}$ and by $\mathcal{U} := \{u_i\}_{i=1,\dots,N}$. We can write also $w^{(p)}(v_i, u_j) = w^{(p)}_{ij}$. A given matching can be uniquely associated to an element $\sigma \in \mathcal{P}_N$ of the set of permutations of N elements, in such a way that

$$\sigma \leftrightarrow \mathtt{M} = \mathrm{Graph}(\mathscr{V}_{\mathtt{M}};\mathscr{E}_{\mathtt{M}}) \in \mathcal{S} \iff (v_i, u_{\sigma(i)}) \in \mathscr{E}_{\mathtt{M}}. \tag{5.2.7}$$

In the following we will use quite often this correspondence and the previous functional (5.2.5) will be written as

$$\mathscr{C}^{(p,\mathrm{EB})}[\sigma] := \frac{1}{N}\sum_{i=1}^{N} w^{(p)}_{i\,\sigma(i)}. \tag{5.2.8}$$

We will denote the optimal matching cost by

$$\mathscr{C}^{(p,\mathrm{EB})}_N := \min_{\mathtt{M}\in\mathcal{S}} \mathscr{C}^{(p,\mathrm{EB})}[\mathtt{M}] = \min_{\sigma\in\mathcal{P}_N} \mathscr{C}^{(p,\mathrm{EB})}[\sigma]. \tag{5.2.9}$$

The result of the average on the point positions depends on N, Ω, ρ but also if we are considering the RR–EB or the GR–EB. In the former case we write

$$\mathscr{C}^{(p,\mathrm{RR})}_{N,d} := \overline{\mathscr{C}^{(p,\mathrm{EB})}_N} \tag{5.2.10}$$

where the average is performed over both sets of points. In the latter case we have instead

$$\mathscr{C}^{(p,\mathrm{GR})}_{N,d} := \overline{\mathscr{C}^{(p,\mathrm{EB})}_N}\Big|_{\mathcal{U} \text{ on the grid}}, \tag{5.2.11}$$

where, in the average procedure, one set of vertices is supposed fixed. Observe finally that, although the EB appears as a slight variation of the EM, a crucial complication appears. Let us consider

for example an Euclidean matching problem on Ω and a certain partition $\bigcup_i \omega_i \subset \Omega$. In a greedy approach, in the monopartite case, we can always perform an optimal matching procedure inside each ω_i leaving at worst one point unmatched for each element of the partition. In the bipartite case, instead, it may happen that a relevant fraction of points remains unmatched for each ω_i and therefore they have to be connected with points outside ω_i. This suggests that local fluctuations in the number of points of different type in the EB can be relevant in the scaling properties of the average optimal cost in a non trivial way.

SCALING OF THE AVERAGE OPTIMAL COST The scaling of the average optimal cost for the EM can be directly obtained in the context of the general theory of Euclidean optimization problems [31]. In particular, let us consider the unit hypercube in d dimensions,

$$\Omega_d \coloneqq [0,1]^d \subset \mathbb{R}^d, \tag{5.2.12}$$

and let us fix the $\rho(\mathbf{x}) = 1$ on this domain. It turns out that, for $0 < p < d$, in the EM on Ω_d we have that

$$\mathscr{C}_{N,d}^{(p,\mathrm{rEM})} \sim N^{-\frac{p}{d}}. \tag{5.2.13}$$

From a mathematical point of view, the treatment of the scaling of the EB appears more difficult, in particular in low dimensions. We will treat exactly this problem for $d = 1$ later. For $d = 2$, Ajtai, Komlós, and Tusnády [2] proved that the average optimal cost scales as

$$\mathscr{C}_{N,2}^{(p,\mathrm{RR})} \sim \left(\frac{\ln N}{N}\right)^{\frac{p}{2}}. \tag{5.2.14}$$

Observe that a logarithmic correction appears respect to the monopartite case: this correction is indeed due to the relevance of local fluctuations in the number of points of different types. For $d \geq 3$ and $p = 1$ Dobrić and Yukich [13] proved rigorously that

$$\mathscr{C}_{N,d}^{(1,\mathrm{RR})} \sim N^{-\frac{1}{d}}. \tag{5.2.15}$$

The previous result can be justified by an heuristic argument. Given a certain point in Ω_d, its nearest neighbor of different type is at a distance $\sim \frac{1}{\sqrt[d]{N}}$, and therefore we expect

$$\mathscr{C}_{N,d}^{(p,\mathrm{RR})} \sim N^{-\frac{p}{d}}. \tag{5.2.16}$$

This result is indeed confirmed by numerical simulations for $d \geq 3$. To our knowledge, however, no further scaling properties of the average optimal costs for Euclidean matching problems are known from a rigorous point of view. In the following, we will show that, using a proper ansatz, we will be able not only to derive the correct scaling properties for the EB in any dimension, but also to obtain the finite size corrections to the leading terms for $d \geq 2$.

5.3. Random Euclidean monopartite matching problem by replicas

The replica approach is, once again, powerful enough to help us in the study of the random EM. The computation follows directly the method used for the RMMP but, as anticipated, in this context a correlation between the different weights appears, due to the presence of an Euclidean structure [21]. The idea, introduced once again by Mézard and Parisi [25], is to consider these correlations as corrections to the purely random case, namely the RMMP. Mézard and Parisi computed the very first order corrections, due to correlations among *three* different weights (two weights are always uncorrelated). We try here to expand their results. We consider the problem on the complete graph $\mathtt{K}_{2N}$ embedded in the unit hypercube Ω_d. We assume also that the points are independently generated on Ω_d with *uniform* distribution. For a perfect matching $\mathtt{M} \subset \mathtt{K}_{2N}$, the cost function is given by Eq. (5.2.3). In particular, let us introduce the adjacency matrix M for the matching $\mathtt{M}$ in the usual way,

$$\mathsf{M} = (m_{ij})_{1 \leq i,j \leq 2N}, \tag{5.3.1}$$

with the constraints

$$m_{ij} \in \{0,1\}, \quad m_{ij} = m_{ji}, \quad \sum_{k=1}^{2N} m_{kj} = \sum_{k=1}^{2N} m_{ik} = 1, \quad \forall i,j. \tag{5.3.2}$$

The functional in Eq. (5.2.3) can be written as

$$\mathscr{C}^{(p,\mathrm{EM})}[\mathsf{M}] \equiv \mathscr{C}^{(p,\mathrm{EM})}[\mathsf{M}] = \frac{1}{N}\sum_{i<j} m_{ij} w_{ij}^{(p)}. \tag{5.3.3}$$

In the following we will denote for simplicity by $w_{ij} \equiv w_{ij}^{(p)} \equiv w_e$ the weight corresponding to the edge $e = (i,j)$. As in the RMMP, we start from the following partition function

$$Z_N(\beta) \coloneqq \sum_{\mathsf{M}} \mathrm{e}^{-\beta N^{\frac{p}{d}+1}\mathscr{C}^{(p,\mathrm{EM})}[\mathsf{M}]} = \left(\prod_{i=1}^{2N}\int_0^{2\pi}\frac{\mathrm{e}^{i\lambda_i}\,\mathrm{d}\,\lambda_i}{2\pi}\right)\prod_{i<j}\left[1+\exp\left(-\beta N^{\frac{p}{d}} w_{ij} - i\lambda_i - i\lambda_j\right)\right]. \tag{5.3.4}$$

In the previous expression we have properly rescaled the temperature $\beta \to \beta N^{\frac{p}{d}+1}$ to obtain a finite average cost in the large N limit (note that the average optimal cost scales as $N^{-\frac{p}{d}}$ in the EM). As in the purely random case, we have to properly average over the disorder, and therefore we proceed via a replica trick. The replicated partition function is

$$\begin{aligned} Z_N^n(\beta) &= \left(\prod_{a=1}^{n}\prod_{i=1}^{2N}\int_0^{2\pi}\frac{\mathrm{e}^{i\lambda_i^a}\,\mathrm{d}\,\lambda_i^a}{2\pi}\right)\prod_{a=1}^{n}\prod_{i<j}\left[1+\exp\left(-\beta N^{\frac{p}{d}} w_{ij} - i\lambda_i^a - i\lambda_j^a\right)\right] \\ &= \left(\prod_{a=1}^{n}\prod_{i=1}^{2N}\int_0^{2\pi}\frac{\mathrm{e}^{i\lambda_i^a}\,\mathrm{d}\,\lambda_i^a}{2\pi}\right)\prod_{i<j}\left[1+u_{ij}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)\right], \end{aligned} \tag{5.3.5}$$

where we have introduced the function

$$u_{ij}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j) \coloneqq \sum_{r=1}^{\infty}\exp\left(-r\beta N^{\frac{p}{d}} w_{ij}\right)\sum_{a_1<\cdots<a_r}\exp\left[-i\sum_{m=1}^{r}\left(\lambda_i^{a_m}+\lambda_j^{a_m}\right)\right]. \tag{5.3.6}$$

At this point, in the computation for the RMMP the average is performed easily and immediately, using the fact that the weights are all independent random variables. In this case, however, this is not true any more, due to the Euclidean constraint. The average is more complicated and we have

$$\overline{\prod_{i<j}(1+u_{ij})} = 1+\sum_{E=1}^{\infty}\sum_{\mathsf{S}\in\mathcal{S}_E}\overline{\prod_{e\in\mathscr{C}_{\mathsf{S}}}u_e} = \exp\left[\sum_{E=1}^{\infty}\sum_{\mathsf{S}\in\mathcal{S}_E^{\mathrm{B}}}\overline{\prod_{e\in\mathscr{C}_{\mathsf{S}}}u_e}\right]. \tag{5.3.7}$$

We have introduced the sets

$$\mathcal{S}_E = \{\mathsf{S} = \mathrm{Graph}(\mathscr{V}_{\mathsf{S}};\mathscr{C}_{\mathsf{S}}) \mid \mathsf{S}\subseteq\mathfrak{K}_{2N} \text{ and } |\mathscr{C}_{\mathsf{S}}| = E\}, \quad |\mathcal{S}_E| = \binom{2N^2-N}{E}, \tag{5.3.8}$$

$$\mathcal{S}_E^{\mathrm{B}} = \{\mathsf{S}\in\mathcal{S}_E \mid \mathsf{S} \text{ is biconnected}\}\subset\mathcal{S}_E. \tag{5.3.9}$$

In other words, $\mathcal{S}_E$ contains all the subsets of different E edges in the set of edges of the complete graph $\mathtt{K}_{2N}$. The set $\mathcal{S}_E^{\mathrm{B}}$ contains, instead, all distinct biconnected subgraphs of $\mathcal{S}_E$. For a given $\mathsf{S}\in\mathcal{S}_E^{\mathrm{B}}$, the average must be performed using the following joint distribution

$$\rho_{\mathsf{S}}(\{w_e\}_{e\in\mathscr{C}_{\mathsf{S}}}) = \left(\prod_{e\in\mathscr{C}_{\mathsf{S}}}\int \mathrm{d}^d z_e\right)\prod_{e=1}^{E}\delta\left(w_e - \|\mathbf{z}_e\|^p\right)\prod_{\mathsf{c}\in\mathscr{L}(\mathsf{S})}\delta^{(d)}\left(\sum_{e\in\mathscr{C}_{\mathsf{c}}}\mathbf{z}_e\right), \tag{5.3.10}$$

where $\mathscr{L}(\mathsf{S})$ is the set of independent cycles $\mathsf{c} = \mathrm{Graph}(\mathscr{V}_{\mathsf{c}};\mathscr{C}_{\mathsf{c}})$ in S. To evaluate $\overline{\prod_{e\in\mathscr{C}_{\mathsf{S}}}u_e}$ we have therefore to compute an integral in the form

$$\left(\prod_{e\in\mathscr{C}_{\mathsf{S}}}\int \mathrm{d}\,w_e\,\mathrm{e}^{-r_e\beta N^{\frac{p}{d}}w_e}\right)\rho_{\mathsf{S}}(\{w_e\}_{e\in\mathscr{C}_{\mathsf{S}}}), \quad \{r_e\}_e\in\mathbb{N}. \tag{5.3.11}$$

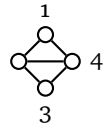
In the relevant limit of short links[1], the previous quantity can be rewritten as

$$\frac{1}{N^{E-L}}\left(\prod_{e\in\mathscr{C}_{\mathtt{S}}}\int \mathrm{d}\,w_e\,\mathrm{e}^{-r_e\beta w_e}\right)\rho_{\mathtt{S}}(\{w_e\}_{e\in\mathscr{C}_{\mathtt{S}}}) = N^{L-E}\overline{\prod_{e=1}^{E}\mathrm{e}^{-r_e\beta w_e}}, \tag{5.3.12}$$

where $L \coloneqq |\mathscr{L}(\mathtt{S})|$ is the cyclomatic number. Using the Euler formula, $E-L=V-\kappa$, where κ is the number of components of the graph and $V=|\mathscr{C}_{\mathtt{S}}|$, we have that the contribution of the subgraph $\mathtt{S}$ scales as N^{1-V}. Moreover, the set of subgraphs $\mathcal{S}_E^{\mathrm{B}}$ can be partitioned using the equivalence relation of isomorphism between graphs. Let us call $\mathtt{g}_E$ the generic equivalence class containing all distinct biconnected subgraphs of $\mathtt{K}_{2N}$ with E edges with the same topology with E edges and V vertices. Importantly, the average contribution of each element of the class is the same. The number of elements in $\mathtt{g}_E$ is[2]

$$\frac{(2N)!}{(2N-V)!}\sim(2N)^V \text{ for } N\gg 1. \tag{5.3.13}$$

In the following, we identify $\mathtt{g}_E$ with its topology. If $\mathtt{g}_E$ has some additional symmetry under the exchange of vertices, Eq. (5.3.13) overcounts the number of elements of the equivalence class. A certain correcting factor taking into account this symmetry, let us call it $\sigma(\mathtt{g}_E)$, must be introduced. Let us consider for example the graph topology

$\mathtt{g}=$ 

For a given set $\{\boldsymbol{\xi}_i\}_{i=1,\dots,N}$ of points, we can arrange them in 4! ways but only 3! are really different, due to the symmetry under the exchanges $1\leftrightarrow 3$ and $2\leftrightarrow 4$. For this graph $\sigma(\mathtt{g})=\frac{1}{4}$. Note that the factor $\sigma(\mathtt{g})$ does not depend on N. Finally, we can conclude that a given topology gives a contribution of the order N.

We can therefore rewrite Eq. (5.3.7), up to $o\left(\frac{1}{N}\right)$ terms, as a sum over different biconnected topologies, each one of order N, with proper symmetry factors

$$\overline{\prod_{i<j}(1+u_{ij})} = \exp\left[\frac{1}{2}\,\raisebox{-0.3em}{} + \frac{1}{6}\,\raisebox{-0.3em}{} + \frac{1}{8}\,\raisebox{-0.3em}{  } + \frac{1}{10}\,\raisebox{-0.3em}{  } + \frac{1}{4}\,\raisebox{-0.3em}{  } + \dots\right]. \tag{5.3.14}$$

The previous result generalizes an analogue one appearing in [25] in which $E\leq 3$ was considered. Following Mézard and Parisi [25], let us introduce now, using a proper set of Lagrange multipliers $\hat{q}_{a_1,\dots,a_k}$, the order parameter,

$$q_{a_1\dots a_k} \coloneqq \frac{1}{2N}\sum_{i=1}^{2N}\exp\left(-i\sum_{j=1}^{k}\lambda_i^{a_j}\right), \quad 1\leq k\leq n, \tag{5.3.15}$$

symmetric under permutation of replica indices. For a given topology $\mathtt{g}=\mathrm{Graph}(\mathscr{V};\mathscr{E})$ with E edges and V vertices, we have in the notation above

$$\sum_{\mathtt{S}\in\mathtt{g}}\overline{\prod_{e\in\mathscr{C}_{\mathtt{S}}}u_e} = 2^V N\sum_{\mathbf{r}}\overline{\prod_{e=1}^{E}\frac{\mathrm{e}^{-r_e\beta w_e}}{r_e!}}\sum_{\{a_i^1\}_i}\cdots\sum_{\{a_i^E\}_i}\prod_{v\in\mathscr{V}}q_{a(v)}. \tag{5.3.16}$$

We have denoted by $\mathbf{r}\coloneqq\{r_e\}_e$ a set of E integers, each one associated to an edge of the graph. The quantity

$$\sum_{\{a_i^1\}_i}\cdots\sum_{\{a_i^E\}_i}\prod_{v\in\mathscr{V}}q_{a(v)} \tag{5.3.17}$$

[1] We take into account that the occupied links are of the order $N^{-\frac{p}{d}}$.

[2] Indeed, there are

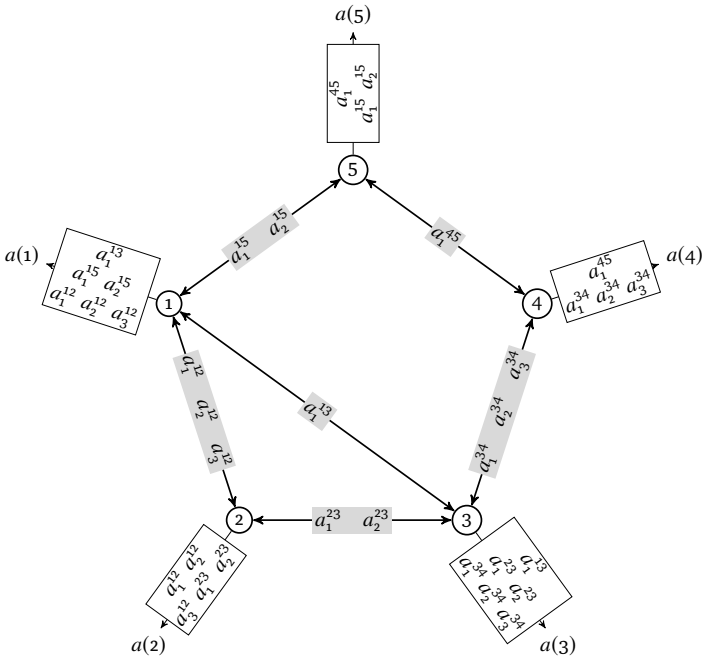
$$\frac{(2N)!}{(2N-V)!}$$

different ordered subsets of V vertices in $\mathtt{K}_{2N}$.

must be read as follows. We associate r_e replica indices $\{a_i^e\}_i = \{a_1^e, \dots, a_{r_e}^e\}$ to each edge e and we construct the set of indices

$$a(v) \coloneqq \bigcup_{e\colon v\in e} \{a_i^e\}_i, \quad r(v) \coloneqq |a(v)| = \sum_{e\colon v\in e} r_e. \tag{5.3.18}$$

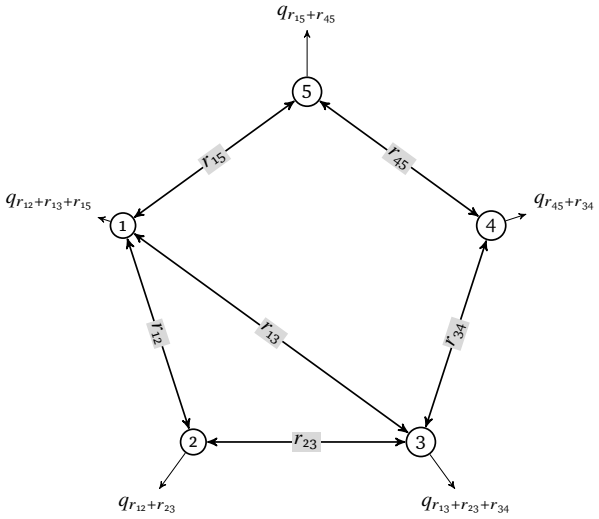
Pictorially, we have for example



The sum then runs over all different sets of replica indices. We proceed however adopting an important, simplifying hypothesis. We assume that the solution is *replica symmetric*, i.e., that the replicas introduced above are indistinguishable, and therefore

$$q_{a_1\dots a_r} \equiv q_r \Rightarrow \sum_{\mathsf{S}\in\mathsf{g}} \overline{\prod_{e\in\mathscr{E}_\mathsf{S}} u_e} = 2^V N \sum_{\mathbf{r}} \hat{\sigma}_n(\mathsf{g};\mathbf{r}) \overline{\prod_{e\in\mathscr{E}} \frac{\mathrm{e}^{-r_e\beta w_e}}{r_e!}} \prod_{v\in\mathscr{V}} q_{r(v)}. \tag{5.3.19}$$

Pictorially, we have



In the previous expression we have introduced the combinatorial coefficient $\hat{\sigma}_n(\mathsf{g};\mathbf{r})$, number of ways to select $\sum_e r_e$ (not necessarily distinct) replica indices, such that we can choose E sets R_e of them, e edge of the graph, satisfying the property

$$e \cap e' \neq \emptyset \Rightarrow R_e \cap R_{e'} = \emptyset, \quad |R_e| = r_e. \tag{5.3.20}$$

Note that $\hat{\sigma}_0 = 0$. We denote by

$$\hat{\sigma}(\mathsf{g};\mathbf{r}) \coloneqq \lim_{n\to 0} \frac{\hat{\sigma}_n(\mathsf{g};\mathbf{r})}{n}. \tag{5.3.21}$$

The averaged replicated partition function is therefore

$$\overline{Z^n} = \left[\prod_{k=1}^{n} \int_{-\infty}^{+\infty} \mathrm{d}\, q_k \int_{-i\infty}^{+i\infty} \mathrm{d}\, \hat{q}_k\right] \mathrm{e}^{-N\beta S[\beta;q,\hat{q}]}, \tag{5.3.22a}$$

$$-\beta S[\beta;q,\hat{q}] \coloneqq \Sigma(q) + 2\ln z(\hat{q}) - 2\sum_{k=1}^{n} \binom{n}{k} \hat{q}_k q_k, \tag{5.3.22b}$$

with

$$\begin{aligned}\Sigma(q) &\coloneqq \sum_{\mathsf{g}} 2^V \sigma(\mathsf{g}) \sum_{\mathbf{r}} \hat{\sigma}_n(\mathsf{g};\mathbf{r}) \overline{\prod_{e\in\mathscr{E}} \frac{\mathrm{e}^{-r_e\beta w_e}}{r_e!}} \prod_{v\in\mathscr{V}} q_{r(v)} \\ &\xrightarrow{n\to 0} n \sum_{\mathsf{g}} 2^V \sigma(\mathsf{g}) \sum_{\mathbf{r}} \hat{\sigma}(\mathsf{g};\mathbf{r}) \overline{\prod_{e\in\mathscr{E}} \frac{\mathrm{e}^{-r_e\beta w_e}}{r_e!}} \prod_{v\in\mathscr{V}} q_{r(v)}.\end{aligned} \tag{5.3.22c}$$

The sum runs over all biconnected graphs $\mathsf{g} = \mathrm{Graph}(\mathscr{V};\mathscr{E})$. The purely random link contribution is related to the $E = 1$ contribution and omitting all other terms we recover the RMMP as mean field approximation. Following [24],

$$\begin{aligned}2\ln z(\hat{q}) &\coloneqq 2\ln\left[\left(\prod_{a=1}^{n} \int_0^{2\pi} \frac{\mathrm{e}^{i\lambda^a}\,\mathrm{d}\,\lambda^a}{2\pi}\right) \exp\left(\sum_{r=1}^{n} \binom{n}{r} \hat{q}_r\, \mathrm{e}^{-i\sum_{j=1}^{r}\lambda^{a_j}}\right)\right] \\ &\xrightarrow{n\to 0} 2n\beta \int_{-\infty}^{+\infty} \left(\mathrm{e}^{-\,\mathrm{e}^{\beta x}} - \mathrm{e}^{-G(x)}\right) \mathrm{d}\, x,\end{aligned} \tag{5.3.22d}$$

where we have introduced the function

$$G(x) \coloneqq \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\hat{q}_r\, \mathrm{e}^{\beta r x}}{r!} \tag{5.3.23}$$

and we have used the fact that

$$\binom{n}{k} \xrightarrow{n\to 0} n\frac{(-1)^{k-1}}{k}. \tag{5.3.24}$$

The saddle point equations can be written as

$$Q_r = \beta \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\beta r x - G(x)}}{(r-1)!}\, \mathrm{d}\, x, \tag{5.3.25a}$$

and

$$\begin{aligned}G(x) = \sum_{\mathsf{g}} 2^{V-1}\sigma(\mathsf{g}) \Bigg[\sum_{v\in\mathscr{V}} \left(\prod_{e\in\mathscr{E}} \int \mathrm{d}\, w_e \prod_{\tilde{v}\neq v} \int \mathrm{d}\, x_{\tilde{v}}\, G'(x_{\tilde{v}})\, \mathrm{e}^{-G(x_{\tilde{v}})}\right) \frac{\rho_{\mathsf{g}}(\{w_e\}_e)}{\beta} \cdot \\ \cdot \frac{\partial}{\partial x_v} K_{\mathsf{g}}[\{\beta(x_{v_i} + x_{v_j} - w_{ij})\}]\Bigg]\Bigg|_{x_v\equiv x}.\end{aligned} \tag{5.3.25b}$$

In the previous expression, we introduced the following auxiliary function, defined on the graph $\mathtt{g}$:

$$K_{\mathtt{g}}(\{x_e\}_e) \coloneqq \sum_{\mathbf{r}} \left[\frac{\hat{\sigma}(\mathtt{g};\mathbf{r})}{\prod_v r(v)!} \prod_e \frac{\mathrm{e}^{r_e x_e}}{r_e!} \right]. \tag{5.3.26}$$

We can finally write down a general expression for the free energy at finite temperature as

$$\bar{f}(\beta) \coloneqq \lim_{\substack{N\to\infty \\ n\to 0}} \frac{1-\overline{Z^n}}{nN\beta} = -2\int_{-\infty}^{+\infty} \left(\mathrm{e}^{-\mathrm{e}^{\beta x}} - \mathrm{e}^{-G(x)}\right) \mathrm{d}\, x + \int_{-\infty}^{+\infty} G(x)\, \mathrm{e}^{-G(x)}\, \mathrm{d}\, x$$
$$-\sum_{\mathtt{g}} 2^V \sigma(\mathtt{g}) \left[\left(\prod_e \int \mathrm{d}\, w_e \prod_v \int \mathrm{d}\, x_v\, G'(x_v)\, \mathrm{e}^{-G(x_v)} \right) \frac{\rho_{\mathtt{g}}(\{w_e\}_e)}{\beta} K_{\mathtt{g}}[\{\beta(x_a + x_b - w_{ab})\}] \right]. \tag{5.3.27}$$

From the previous expression we have that

$$\lim_N N^{\frac{p}{d}} \mathscr{C}_{N,d}^{(p,\mathrm{REM})} \coloneqq \lim_N N^{\frac{p}{d}} \overline{\min_{\mathtt{M}} \mathscr{C}^{(p,\mathrm{EM})}[\mathtt{M}]} = \lim_{\beta\to\infty} \bar{f}(\beta). \tag{5.3.28}$$

5.3.1. MEAN FIELD APPROXIMATION

In the mean field approximation only the $E = 1$ contribution is considered (i.e., different weights in the complete graph are considered identically and independently distributed). The $E = 1$ contribution is given by the only graph

$$\mathtt{g}_1 = \quad , \quad \sigma(\mathtt{g}_1) = \frac{1}{2},$$

for which, in Eq. (5.3.25b), we get

$$2\sigma(\mathtt{g}_1) \left[\int \frac{\mathrm{d}\, w \rho_{\mathtt{g}_1}(w)}{\beta} \sum_{i=1}^{2} \left(\int \mathrm{d}\, x_i\, G'(x_i)\, \mathrm{e}^{-G(x_i)} \right) \frac{\partial}{\partial x} K_{\mathtt{g}_1}[\beta(x_i + x - w)] \right]$$
$$= 2 \iint \frac{e^{-G(y)} \rho_{\mathtt{g}_1}(w)}{\beta} \frac{\partial^2}{\partial x^2} K_{\mathtt{g}_1}[\beta(y + x - w)]\, \mathrm{d}\, y\, \mathrm{d}\, w. \tag{5.3.29}$$

On the other hand, we have that[3]

$$\hat{\sigma}_n(\mathtt{g}_1; r) = r! \binom{n}{r} \xrightarrow{n\to 0} n(-1)^{r-1}(r-1)!. \tag{5.3.30}$$

and therefore

$$\frac{\partial K_{\mathtt{g}_1}(x)}{\partial x} \coloneqq -\sum_{r=1}^{\infty} (-1)^r \frac{\mathrm{e}^{rx}}{(r!)^2} = 1 - J_0\left(2\,\mathrm{e}^{\frac{x}{2}}\right), \tag{5.3.31}$$

where $J_0(x)$ is a Bessel function of the first kind. The saddle point equations become

$$Q_r = \beta \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\beta r x - G(x)}}{(r-1)!}\, \mathrm{d}\, x, \tag{5.3.32a}$$

$$G(x) = -2 \iint \rho_{\mathtt{g}_1}(w)\, \mathrm{e}^{-G(y)} \frac{\partial}{\partial x} J_0\left(2 \exp\left(\frac{\beta(x+y-w)}{2}\right)\right) \mathrm{d}\, w\, \mathrm{d}\, y. \tag{5.3.32b}$$

In the short-link limit

$$\rho_{\mathtt{g}_1}(w) \sim \frac{\Sigma_d}{p} w^{\frac{d}{p}-1}, \quad \Sigma_d \coloneqq \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{5.3.33}$$

[3] Indeed, we have $\binom{n}{r}$ ways of selecting r replica indices out of n and $r!$ ways of ordering them.

In the large β limit, we have that

$$1 - J_0\left(2\exp\left(\frac{\beta x}{2}\right)\right) \xrightarrow{\beta\to\infty} \theta(x) \tag{5.3.34}$$

and therefore eq. (5.3.32b) becomes

$$G(x) = \frac{4\pi^{\frac{d}{2}}}{p\Gamma\left(\frac{d}{2}\right)} \int_0^\infty w^{\frac{d}{p}-1}\, \mathrm{e}^{-G(w-x)}\, \mathrm{d}\, w, \tag{5.3.35}$$

mean field equation for the function G. It can be numerically solved for given values p, d. The mean field contribution to the average optimal length is therefore obtained from Eq. (5.3.27) in the $\beta \to \infty$ limit:

$$\lim_N N^{\frac{p}{d}} \mathscr{C}^{(p,\mathrm{rEM})}_{N,d,\mathrm{mf}} = \int_{-\infty}^{\infty} G(x)\, \mathrm{e}^{-G(x)}\, \mathrm{d}\, x - 2\int_{-\infty}^{+\infty} \left(\theta(x) - \mathrm{e}^{-G(x)}\right) \mathrm{d}\, x. \tag{5.3.36}$$

The previous equation and Eq. (5.3.35) show that, up to a rescaling factor, the mean field solution for the EM with parameters p, d corresponds to the mean field solution of a RMMP whose weight distribution has the form $\rho(w) \sim w^{\frac{d}{p}-1}$ for $w \to 0$. Observing that, for $\alpha > 0$,

$$f(x) = \alpha \int_0^\infty \mathrm{e}^{-f(y-x)}\, \mathrm{d}\, y \Rightarrow f(x) = \ln\left(1 + \mathrm{e}^{\alpha x}\right), \tag{5.3.37}$$

then for $p = d$ we can write down the explicit solution

$$G(x) = \ln\left[1 + \exp\left(\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} x\right)\right] \tag{5.3.38}$$

and the average optimal length is

$$\lim_N N \mathscr{C}^{(d,\mathrm{rEM})}_{N,d,\mathrm{mf}} = \frac{d}{\Sigma_d}\frac{\pi^2}{12}. \tag{5.3.39}$$

5.3.2. Polygonal corrections and zero temperature limit

We have shown that the leading contribution to the optimal cost is given by all biconnected graphs. An analytic treatment for the entire set of these graphs is a formidable task. However, there is a class of graphs with a special symmetry, i.e., the class of polygons, for which we can attempt a computation that helps us to obtain a numerical recipe for the polygonal corrections [21]. Observe that the triangular corrections have been already evaluated by Mézard and Parisi [25] in their seminal paper on the random EM. The generic polygon $\mathtt{P}_E$ with E edges has E vertices and

$$\sigma(\mathtt{P}_E) = \frac{1}{2E}. \tag{5.3.40}$$

Moreover, the distribution $\rho_{\mathtt{P}_E}(\{w_e\}_e)$ is given by

$$\rho_{\mathtt{P}_E}(\{w_e\}_e) \equiv \rho_E(\{w_e\}_e) := \left[\prod_{e=1}^E \int \mathrm{d}^d\, z_e\, \delta\left(w_e - \|\mathbf{z}_e\|^p\right)\right] \delta^{(d)}\left(\sum_{e=1}^E \mathbf{z}_e\right). \tag{5.3.41}$$

Here and in the following we will denote by e, $e = 1, \ldots, E$, the e-th edge according to a given orientation of the polygon. Therefore e and $e+1$ correspond to consecutive edges. The contribution of $\mathtt{P}_E$ to the action is

$$\begin{aligned} -\beta S_E[\beta; Q, \hat{Q}] &:= 2^E \sum_{\mathbf{r}} \left[\hat{\sigma}_{\mathtt{P}_E}(\mathbf{r}) \overline{\prod_{e=1}^E e^{-r_e \beta w_e}} \prod_{e=1}^E \frac{Q_{r_e + r_{e+1}}}{r_e!}\right] \\ &= \frac{2^E \Sigma_d}{(2\pi)^d} \sum_{\mathbf{r}} \hat{\sigma}_{\mathtt{P}_E}(\mathbf{r}) \int_0^\infty k^{d-1} \prod_{e=1}^E \frac{Q_{r_e + r_{e+1}} g_{r_e}(k)}{r_e!}\, \mathrm{d}\, k, \quad \Sigma_d := \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \end{aligned} \tag{5.3.42}$$

In the previous we have introduced

$$g_r(k) \coloneqq \Sigma_d \int_0^\infty z^{d-1} e^{-r\beta z^p} {}_0F_1\left[\begin{matrix} - \\ \frac{d}{2} \end{matrix}; -\frac{k^2 z^2}{4} \right] \mathrm{d}\, z. \tag{5.3.43}$$

As anticipated, we can proceed in analogy with the computation performed by Parisi and Ratiéville [27] for the finite size corrections to the RMMP. We introduce the $(2^n - 1) \times (2^n - 1)$ matrix $\mathsf{T}(k)$ whose elements are given by

$$T_{\alpha\alpha'}(k) \coloneqq \delta_{\alpha\cap\alpha'=\emptyset} Q_{|\alpha|+|\alpha'|} \sqrt{g_{|\alpha|}(k) g_{|\alpha'|}(k)}. \tag{5.3.44}$$

Here α is an element of the power set of the replica indices with cardinality $|\alpha|$ and

$$\delta_{\alpha\cap\alpha'=\emptyset} = \begin{cases} 1 & \text{if } \alpha \cap \alpha' = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.45}$$

Therefore the contribution of the polygon P_E can be written as

$$-\beta S_E[\beta; Q, \hat{Q}] = \frac{2^E \Sigma_d}{(2\pi)^d} \int_0^\infty k^{d-1} \operatorname{tr}\left[\mathsf{T}^E(k)\right] \mathrm{d}\, k. \tag{5.3.46}$$

The diagonalization of the matrix $\mathsf{T}(k)$ can be performed through the classical strategy of Almeida and Thouless [3], already adopted by [27]. In particular, we analyze eigenvectors with q distinguished replica indices. It is possible to show that the contribution of the sectors $q \geq 2$ is formally analogous to the finite size correction in the random monopartite matching problem. The sectors $q = 0$ and $q = 1$ give instead a non-zero contribution, unlike in the purely random case where their contribution is absent. We present below the proof of the final result.

DERIVATION OF THE POLYGONAL CONTRIBUTION An eigenvector $\mathsf{c} = (c_\gamma)_\gamma$ of the matrix T must satisfy the equation

$$\sum_\gamma T_{\alpha\gamma} c_\gamma = \sum_{\gamma\colon\, \alpha\cap\gamma=\emptyset} Q_{|\alpha|+|\gamma|} \sqrt{g_{|\alpha|}(k) g_{|\gamma|}(k)} c_\gamma = \lambda c_\alpha. \tag{5.3.47}$$

In the spirit of the classical approach of Almeida and Thouless [3], we search for eigenvectors c^q with q distinguished replicas, in the form

$$c^q_\alpha = \begin{cases} 0 & \text{if } |\alpha| < q, \\ d^i_{|\alpha|} & \text{if } \alpha \text{ contains } q - i + 1 \text{ different indices, } i = 1, \dots, q+1. \end{cases} \tag{5.3.48}$$

For $q \geq 2$, if we consider $q - 1$ distinguished replicas, it can be proved [23] that the following orthogonality condition holds:

$$\sum_{k=0}^{q-j} \binom{k}{q-j} \binom{|\alpha| - (k+j)}{n-q} d^{q+1-(k+j)}_{|\alpha|} = 0. \tag{5.3.49}$$

The orthogonality condition provides a relation between all the different values $d^i_{|\alpha|}$, showing that we can keep only one value, say $d^1_{|\alpha|}$, as independent. Using this assumption, the eigenvalues of the original $\mathsf{T}(k)$ matrix can be evaluated diagonalizing the infinite dimensional matrices $\mathsf{N}^{(q)}(k)$ [27] whose elements, in the $n \to 0$ limit, are given by

$$N^{(q)}_{ab}(k) = (-1)^b \frac{\Gamma(a+b)\Gamma(b) Q_{a+b} \sqrt{g_a(k) g_b(k)}}{\Gamma(a)\Gamma(b-q+1)\Gamma(b+q)}. \tag{5.3.50}$$

In particular, for $q = 0$ a direct computation gives

$$N^{(0)}_{ab}(k) = \binom{n-a}{b} Q_{a+b} g_b(k) \xrightarrow{n\to 0} (-1)^b \frac{\Gamma(a+b)}{\Gamma(a) b!} Q_{a+b} \sqrt{g_a(k) g_b(k)} \tag{5.3.51}$$

whereas for $q = 1$ we obtain

$$N^{(1)}_{ab}(k) = \binom{n-a}{b} \frac{b}{b-n} Q_{a+b} \sqrt{g_a(k) g_b(k)} \xrightarrow{n\to 0} N^{(0)}_{ab}(k) + \frac{n}{b} N^{(0)}_{ab} + o(n). \tag{5.3.52}$$

We have that

$$\operatorname{tr}\left[\mathsf{T}^E(k)\right] = \sum_{q=0}^\infty \left[\binom{n}{q} - \binom{n}{q-1} \right] \operatorname{tr}\left[\left(\mathsf{N}^{(q)}(k) \right)^E \right] \tag{5.3.53}$$

We proceed distinguishing the case $q \geq 2$ from the case $q < 2$. For $q \geq 2$, computing the spectrum of the matrix $\mathsf{N}^{(q)}$ for $q \geq 2$ is equivalent to the computation of the spectrum of $\mathsf{M}^{(q)}(k)$, that has elements

$$M^{(q)}_{ab}(k) \coloneqq (-1)^{a+b}\sqrt{\frac{g_{b+q}(k)}{g_{a+q}(k)}}\frac{\Gamma(a+1)\Gamma(b+q)}{\Gamma(b+1)\Gamma(a+q)}N^{(q)}_{b+q\ a+q}(k) = (-1)^{a+q}\frac{\Gamma(a+b+2q)}{\Gamma(a+2q)b!}Q_{a+b+2q}g_{b+q}(k). \tag{5.3.54}$$

The eigenvalue equation for $\mathsf{M}^{(q)}(k)$ has the form

$$\lambda c^{(q)}_a = \sum_{b=1}^{\infty} M^{(q)}_{ab}(k)c^{(q)}_b = \beta(-1)^q \int \frac{(-1)^a e^{(a+q)\beta u}}{\Gamma(a+2q)}\phi^{(q)}(u;k)\,\mathrm{d}\,u, \tag{5.3.55}$$

where we have introduced

$$\phi^{(q)}(u;k) \coloneqq \sum_{b=1}^{\infty}\frac{e^{(b+q)\beta u - \frac{G(u)}{2}}}{b!}c^{(q)}_b g_{b+q}(k). \tag{5.3.56}$$

The eigenvalue equation in Eq. (5.3.55) can be written as

$$\lambda\phi^{(q)}(u;k) = (-1)^q \int A^{(q)}(u,v;k)\phi^{(q)}(v;k)\,\mathrm{d}\,v, \tag{5.3.57}$$

where $\mathsf{A}^{(q)}(k) = (A^{(q)}(u,v;k))_{u,v}$ is

$$A^{(q)}(u,v;k) \coloneqq \beta e^{-\frac{G(u)+G(v)}{2}+q\beta(u+v)}\sum_{a=1}^{\infty}\frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a+2q)a!}g_{a+q}(k). \tag{5.3.58}$$

In the $n \to 0$ limit, from Eq. (5.3.53) we have therefore

$$\begin{aligned}\sum_{q=2}^{\infty}\left[\binom{n}{q} - \binom{n}{q-1}\right]\operatorname{tr}\left[\left(\mathsf{N}^{(q)}(k)\right)^E\right] &= \sum_{q=2}^{\infty}(-1)^{qE}\left[\binom{n}{q} - \binom{n}{q-1}\right]\operatorname{tr}\left[\left(\mathsf{A}^{(q)}(k)\right)^E\right] \\ &\xrightarrow{n\to 0} n\sum_{q=2}^{\infty}(-1)^{q(E+1)}\frac{2q-1}{q(1-q)}\operatorname{tr}\left[\left(\mathsf{A}^{(q)}(k)\right)^E\right] \\ = n\sum_{q=1}^{\infty}\frac{4q-1}{2q(1-2q)}\operatorname{tr}\left[\left(\mathsf{A}^{(2q)}(k)\right)^E\right] &+ (-1)^E n\sum_{q=1}^{\infty}\frac{4q+1}{2q(2q+1)}\operatorname{tr}\left[\left(\mathsf{A}^{(2q+1)}(k)\right)^E\right].\end{aligned} \tag{5.3.59}$$

Here we used the fact that

$$\binom{n}{q} - \binom{n}{q-1} \xrightarrow{n\to 0} \begin{cases} 1 & \text{if } q = 0, \\ -1+n & \text{if } q = 1, \\ n(-1)^q\frac{2q-1}{q(1-q)} & \text{if } q > 1. \end{cases} \tag{5.3.60}$$

Let us now evaluate the contributions of the sectors $q = 0$ and $q = 1$. We have

$$\sum_{q=0}^{1}\left[\binom{n}{q} - \binom{n}{q-1}\right]\operatorname{tr}\left[\left(\mathsf{N}^{(q)}(k)\right)^E\right] = \operatorname{tr}\left[\left(\mathsf{N}^{(0)}(k)\right)^E\right] + (n-1)\operatorname{tr}\left[\left(\mathsf{N}^{(1)}(k)\right)^E\right] + o(n). \tag{5.3.61}$$

We define the matrices $\mathsf{M}^{(0)}(k)$ and $\tilde{\mathsf{M}}^{(1)}(k)$ as follows

$$M^{(0)}_{ab}(k) \coloneqq (-1)^{a+b}\sqrt{\frac{g_b(k)}{g_a(k)}}\frac{a}{b}N^{(0)}_{ba}(k) = (-1)^a\frac{\Gamma(a+b)}{\Gamma(a)b!}Q_{a+b}g_b(k), \tag{5.3.62a}$$

$$\tilde{M}^{(1)}_{ab}(k) \coloneqq (-1)^{a+b}\sqrt{\frac{g_b(k)}{g_a(k)}}\frac{a}{b}N^{(1)}_{ba} = (-1)^a\frac{\Gamma(a+b)}{\Gamma(a)b!}Q_{a+b}g_b(k) + n(-1)^a\frac{\Gamma(a+b)}{\Gamma(a+1)b!}Q_{a+b}g_b(k). \tag{5.3.62b}$$

As in the $q \geq 2$ case, we can introduce the operator $\mathsf{A}^{(0)}(k)$ as follows

$$\begin{aligned}A^{(0)}(u,v;k) &= \beta e^{-\frac{G(u)+G(v)}{2}}\sum_{a=0}^{\infty}\frac{(-1)^a e^{a\beta(u+v)}g_a(k)}{\Gamma(a)a!} \\ &= \Sigma_d e^{-\frac{G(u)+G(v)}{2}}\int_0^{\infty} z^{d-1}{}_0F_1\left[\begin{matrix}-\\ \frac{d}{2}\end{matrix};-\frac{k^2z^2}{4}\right]\frac{\partial}{\partial y}J_0\left(2e^{\beta\frac{y}{2}}\right)\Big|_{y=u+v-z^p}\mathrm{d}\,z.\end{aligned} \tag{5.3.63}$$

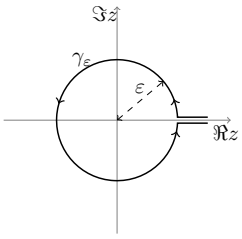


FIGURE 5.3.1.: Hankel path in the complex plane.

having the same eigenvalues of $\mathsf{M}^{(0)}(k)$. Therefore

$$\operatorname{tr}\left[\left(\mathsf{N}^{(0)}(k)\right)^E\right] = \operatorname{tr}\left[\left(\mathsf{M}^{(0)}(k)\right)^E\right] = \operatorname{tr}\left[\left(\mathsf{A}^{(0)}(k)\right)^E\right]. \tag{5.3.64}$$

Similarly, we have that the eigenvalues of $\tilde{\mathsf{M}}^{(1)}(k)$ are obtained from

$$\lambda\tilde{c}_a = \sum_b \tilde{M}^{(1)}_{ab}(k)\tilde{c}_b = \sum_{r'}(-1)^a \frac{\Gamma(a+b)Q_{a+b}g_b(k)}{\Gamma(a)b!}\left(1+\frac{n}{a}\right)\tilde{c}_b = \int \frac{e^{au\beta-\frac{G(u)}{2}}}{\Gamma(a)}\left(1+\frac{n}{a}\right)\tilde{\phi}(u;k)\,\mathrm{d}\,u, \tag{5.3.65}$$

where $\tilde{\phi}(u;k)$ is given by

$$\tilde{\phi}(u;k) \coloneqq \sum_{b=1}^{\infty} \frac{e^{b\beta u-\frac{G(u)}{2}}}{b!}\tilde{c}_b g_b(k). \tag{5.3.66}$$

It is natural, therefore, to introduce once again the operator $\tilde{\mathsf{A}}^{(1)}(k)$ defined as follows

$$\tilde{A}^{(1)}(u,v;k) \coloneqq \beta e^{-\frac{G(u)+G(v)}{2}} \sum_{a=1}^{\infty} \frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a)a!} g_a(k)\left(1+\frac{n}{a}\right) = A^{(0)}(u,v;k) + nB(u,v;k). \tag{5.3.67}$$

The operator $\mathsf{B}(k)$ introduced above is

$$\begin{aligned} B(u,v;k) &\coloneqq \beta e^{-\frac{G(u)+G(v)}{2}} \sum_{a=1}^{\infty} \frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a+1)a!} g_a(k) \\ &= \Sigma_d \beta e^{-\frac{G(u)+G(v)}{2}} \int_0^{\infty} z^{d-1}\,{}_0F_1\left[\begin{matrix}-\\ \frac{d}{2}\end{matrix}; -\frac{k^2z^2}{4}\right]\left[J_0\left(2e^{\beta\frac{u+v-z^p}{2}}\right)-1\right]\mathrm{d}\,z. \end{aligned} \tag{5.3.68}$$

We have then, up to higher orders in n,

$$\begin{aligned} \operatorname{tr}\left[\left(\mathsf{N}^{(0)}(k)\right)^E\right] + (n-1)\operatorname{tr}\left[\left(\mathsf{N}^{(1)}(k)\right)^E\right] &= \operatorname{tr}\left[\left(\mathsf{A}^{(0)}(k)\right)^E\right] + (n-1)\operatorname{tr}\left[\left(\mathsf{A}^{(0)}(k)+n\mathsf{B}(k)\right)^E\right] \\ &= n\operatorname{tr}\left[\left(\mathsf{A}^{(0)}(k)\right)^E\right] + nE\operatorname{tr}\left[\left(\mathsf{A}^{(0)}(k)\right)^{E-1}\mathsf{B}(k)\right] \end{aligned} \tag{5.3.69}$$

We need now to calculate the $\beta\to\infty$ limit, being interested in the optimal cost configuration. Let us consider the $q\geq 2$ contribution. First, we introduce the identity

$$\sum_{r=1}^{\infty}\frac{(-x)^r}{\Gamma(r+2q)r!} = \frac{i}{2\pi}\oint_{\gamma_\epsilon} e^{-\zeta-2q\ln(-\zeta)+\frac{x}{\zeta}}\,\mathrm{d}\,\zeta. \tag{5.3.70}$$

The path γ_ϵ, in the complex plane, is the Hankel path, represented in Fig. 5.3.1. This identity can be proved starting from the Hankel representation for the reciprocal gamma function [1]

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\oint_{\gamma_\epsilon} e^{-\zeta-z\ln(-\zeta)}\,\mathrm{d}\,\zeta. \tag{5.3.71}$$

Using the identity in Eq. (5.3.70), we can rewrite Eq. (5.3.58) for $q\geq 2$ as

$$\begin{aligned} &A^{(q)}(u,v;k) = \\ &= \frac{i\beta\Sigma_d}{2\pi}e^{-\frac{G(u)+G(v)}{2}}\int_0^{+\infty}\mathrm{d}\,w\oint_{\gamma_\epsilon}\mathrm{d}\,\zeta\,\frac{w^{\frac{d}{p}-1}}{p}\,{}_0F_1\left[\begin{matrix}-\\ \frac{d}{2}\end{matrix}; -\frac{k^2w^{\frac{2}{p}}}{4}\right]\mathrm{e}^{\beta q(u+v-w)-w-2q\ln(-\zeta)+\frac{e^{\beta(u+v-w)}}{\zeta}}. \end{aligned} \tag{5.3.72}$$

To compute the $\beta \to \infty$ limit, we perform a saddle point approximation, obtaining

$$\begin{cases} \zeta_{\mathrm{sp}} = -q, \\ w_{\mathrm{sp}} = u + v - \frac{2 \ln q}{\beta}. \end{cases} \tag{5.3.73}$$

The saddle point has fixed position assuming that $\ln q = t\beta$ for some t. Taking instead q fixed and $\beta \to \infty$, it is easily seen from Eq. (5.3.58) that

$$\lim_{\beta \to \infty} A^{(q)}(u, v; k) = \begin{cases} \infty & \text{for } u + v > 0, \\ 0 & \text{for } u + v < 0. \end{cases} \tag{5.3.74}$$

Indeed, only for $u + v - 2t > 0$ the saddle point is inside the range of integration. For this reason, we take $\frac{\ln q}{\beta} = t$ fixed, obtaining the limit operator $\mathsf{H}(t, k)$,

$$H(u, v; t, k) \coloneqq \lim_{\substack{\beta \to \infty,\ q \to \infty \\ \beta^{-1} \ln q = t}} A^{(q)}(u, v; k) \approx \frac{\Sigma_d}{p} e^{-\frac{G(u)+G(v)}{2}} \left. x^{\frac{d}{p}-1} {}_0F_1\left[\begin{matrix} - \\ \frac{d}{2} \end{matrix}; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \right|_{x=u+v-2t}. \tag{5.3.75}$$

Observing that $\sum_{q=2}^{\infty} \frac{1}{\beta q} \to \int_0^{+\infty} \mathrm{d}\, t$ the contribution to the (rescaled) average optimal cost from the $q \geq 2$ sectors is

$$\epsilon_E^{(2)} \coloneqq \begin{cases} \frac{2^E \Sigma_d}{E (2\pi)^d} \iint_0^{\infty} k^{d-1} \operatorname{tr}\left[\mathsf{H}^E(t, k)\right] \mathrm{d}\, t\, \mathrm{d}\, k & E \text{ odd}, \\ 0 & E \text{ even}. \end{cases} \tag{5.3.76}$$

For the sectors $q = 0$ and $q = 1$ the $\beta \to \infty$ limit can be performed quite straightforwardly. In particular, using Eq. (5.3.34), we obtain the two limit operators $\mathsf{H}(0, k)$ and $\mathsf{K}(k)$,

$$A^{(0)}(u, v; k) \xrightarrow{\beta \to \infty} -\Sigma_d e^{-\frac{G(u)+G(v)}{2}} \left. \frac{x^{\frac{d}{p}-1}}{p} {}_0F_1\left[\begin{matrix} - \\ \frac{d}{2} \end{matrix}; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \right|_{x=u+v} \equiv -H(u, v; 0, k), \tag{5.3.77a}$$

$$B(u, v; k) \xrightarrow{\beta \to \infty} -\Sigma_d \beta e^{-\frac{G(u)+G(v)}{2}} \left. \frac{x^{\frac{d}{p}}}{d} {}_0F_1\left[\begin{matrix} - \\ \frac{d}{2}+1 \end{matrix}; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \right|_{x=u+v} =: -\beta K(u, v; k). \tag{5.3.77b}$$

The contribution to the (rescaled) average optimal cost from the sectors $q = 0$ and $q = 1$ is

$$\epsilon_E^{(1)} \coloneqq (-1)^E \frac{2^{E-1} \Sigma_d}{(2\pi)^d} \int_0^{\infty} k^{d-1} \operatorname{tr}\left[\mathsf{H}^{E-1}(0, k) \mathsf{K}(k)\right] \mathrm{d}\, k. \tag{5.3.78}$$

Summarizing, the average optimal cost is given by

$$\lim_{N \to \infty} N^{\frac{p}{d}} \mathscr{C}_N = \lim_{N \to \infty} N^{\frac{p}{d}} \mathscr{C}_N^{(p,\mathrm{mf})}(d) + \sum_{E=3}^{\infty} \left(\epsilon_E^{(1)} + \epsilon_E^{(2)} \right) + \text{non-polygonal terms}, \tag{5.3.79}$$

As anticipated, the contribution $\epsilon_E^{(2)}$ has an expression that is totally analogue to the one for the finite size corrections computed in [27] for the RMMP, whilst $\epsilon_E^{(1)}$ has no equivalent contribution in that computation. In both expressions above, the function G appears. The saddle point equation for G is given by Eq. (5.3.25b). However, keeping the polygonal contribution only, we can similarly write an equation for G in terms of the matrices $\mathsf{H}(t, k)$ and $\mathsf{K}(k)$ as

$$G(u) = \int \rho(w) e^{-G(w-u)} \mathrm{d}\, w - \frac{e^{G(u)}}{2} \sum_{E=3}^{\infty} \left[\frac{\delta \epsilon_E^{(1)}}{\delta G(u)} + \frac{\delta \epsilon_E^{(2)}}{\delta G(u)} \right] = \int \rho(w) e^{-G(w-u)} \mathrm{d}\, w$$
$$- \frac{2^{E-1} e^{G(u)} \Sigma_d}{(2\pi)^d} \sum_{E=3}^{\infty} \int_0^{\infty} k^{d-1} \left[\frac{(-1)^{E-1}}{2} \sum_{m=0}^{E-1} \mathsf{H}^{E-1-m}(0, k) \mathsf{K}(k) \mathsf{H}^m(0, k) - \int_0^{\infty} \mathsf{H}^E(t, k) \mathrm{d}\, t \right]_{uu} \mathrm{d}\, k. \tag{5.3.80}$$

To proceed further, a numerical evaluation of the previous quantities must be performed. This numerical investigation has been carried on in [21]. We refer to this work for the numerical details and additional comments.

5.4. The Monge–Kantorovič transport problem

In the previous Section we discussed the REM *via* a replica approach: similar calculations can be reproduced for the REB. The replica method is quite straightforward but sometimes a little bit involved in the calculation. However, for the bipartite case a different approach is possible, and moreover it is able to provide us the desired results with a little amount of work. This approach is not rigorous but very effective, and it is inspired by the theory of Monge and Kantorovič for the transport problem. The *Monge–Kantorovič problem*, or *optimal transport problem*, is a kind of "continuum version" of the EB. It is a well studied problem in measure theory and a large amount of results were obtained in recent years [30].

Let us suppose that two non-negative measures $\rho_{\mathscr{R}}(\mathbf{x})$ and $\rho_{\mathscr{B}}(\mathbf{y})$ are given on $\mathbb{R}^d$, such that the overall *mass balance condition* is satisfied

$$\int_{\mathbb{R}^d} \mathrm{d}\,\rho_{\mathscr{R}}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{d}\,\rho_{\mathscr{B}}(\mathbf{y}). \tag{5.4.1}$$

Let us define also the set of maps

$$\mathscr{M}(\rho_{\mathscr{R}}, \rho_{\mathscr{B}}) \coloneqq \left\{ \mathbf{T}\colon \mathbb{R}^d \to \mathbb{R}^d \,\middle|\, \int_{\mathbb{R}^d} h(\mathbf{T}(\mathbf{x}))\,\mathrm{d}\,\rho_{\mathscr{R}}(\mathbf{x}) = \int_{\mathbb{R}^d} h(\mathbf{y})\,\mathrm{d}\,\rho_{\mathscr{B}}(\mathbf{y}) \quad \forall h \in \mathsf{C}^0(\mathbb{R}^d) \right\}. \tag{5.4.2}$$

We suppose also that a *cost function*

$$\omega\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+ \tag{5.4.3}$$

is given. In the optimal transport problem, we ask for the map $\mathbf{M} \in \mathscr{M}(\rho_{\mathscr{R}}, \rho_{\mathscr{B}})$ such that, given the cost functional

$$\mathscr{C}[\mathbf{T}; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] \coloneqq \int_{\mathbb{R}^n} \omega(\mathbf{x}, \mathbf{T}(\mathbf{x}))\,\mathrm{d}\,\rho_{\mathscr{R}}(\mathbf{x}), \tag{5.4.4}$$

we have that

$$\mathscr{C}[\rho_{\mathscr{R}}, \rho_{\mathscr{B}}] \coloneqq \mathscr{C}[\mathbf{M}; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] = \inf_{\mathbf{T} \in \mathscr{M}(\rho_{\mathscr{R}}, \rho_{\mathscr{B}})} \mathscr{C}[\mathbf{T}; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}]. \tag{5.4.5}$$

This problem is called *transport problem* and it was introduced in the 18th century by the French mathematician Gaspard Monge. In the original problem, Monge [26] considered a finite set of piles of soil and a finite set of excavation sites, both sets of the same cardinality. He wanted to find a proper matching to minimize the transportation cost. Monge's problem was, therefore, an EB. The similarity between the continuous problem above and our problem is striking. In particular, the transport problem appears as a generalization of the EB, in which empirical measures are replaced with generic measures on a certain support.

The analogies, however, go beyond the formulation above. In the 1940s Kantorovič [18, 19] suggested a slightly different approach. He considered the set of *transfer plans*

$$\mathscr{M}_K(\rho_{\mathscr{R}}, \rho_{\mathscr{B}}) \coloneqq \left\{ \pi\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+ \,\middle|\, \int_{\mathbf{y}\in\mathbb{R}^d} \mathrm{d}\,\pi(\mathbf{x},\mathbf{y}) = \mathrm{d}\,\rho_{\mathscr{R}}(\mathbf{x}),\ \int_{\mathbf{x}\in\mathbb{R}^d} \mathrm{d}\,\pi(\mathbf{x},\mathbf{y}) = \mathrm{d}\,\rho_{\mathscr{B}}(\mathbf{y}) \right\}. \tag{5.4.6}$$

and the relaxed cost functional

$$\mathscr{C}_K[\pi; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] \coloneqq \int_{\mathbb{R}^d\times\mathbb{R}^d} \omega(\mathbf{x},\mathbf{y})\,\mathrm{d}\,\pi(\mathbf{x},\mathbf{y}). \tag{5.4.7}$$

He asked therefore for the optimal transfer plan π_o such that

$$\mathscr{C}_K[\pi_o; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] \coloneqq \inf_{\pi \in \mathscr{M}_K(\rho_{\mathscr{R}}, \rho_{\mathscr{B}})} \mathscr{C}_K[\pi; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}]. \tag{5.4.8}$$

As mentioned before, this problem is a weak formulation of the original Monge transport problem. The formulation of Kantorovič opens interesting and nontrivial questions, e.g. about the the existence of an optimal transport plan [30]. Moreover, the existence of an optimal transport plan π_o

does not guarantee that an optimal map $\mathbf{M}$ in the Monge sense exists. In some sense, the optimal plans of Kantorovič remind us the adjacency matrix M for a matching, but without the constraint $m_{ij} \in \{0, 1\}$. Quite brutally, we will assume that an optimal map always exists. It can be proved indeed that, if ω is a continuous function on its support and some additional requirements[4] on $\rho_{\mathscr{R}}(\mathbf{x})$, both an optimal transport plan and a corresponding optimal map always exist [4]. In the following we will consider only cost functions in the form $\omega(\mathbf{x}, \mathbf{y}) \equiv \omega(\mathbf{x} - \mathbf{y})$.

The formulation of Kantorovič is interesting not only because it offers an alternative view on the problem, but also because it suggests a road for its solution. He introduced indeed a dual formulation, defining the following space:

$$\mathscr{K}(\omega) \coloneqq \left\{(u, v) | u, v \colon \mathbb{R}^d \to \mathbb{R},\ u, v \in \mathsf{C}^0(\mathbb{R}^d),\ u(\mathbf{x}) + v(\mathbf{y}) \leq \omega(\mathbf{x} - \mathbf{y}),\ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d\right\}. \tag{5.4.9}$$

He asked for the couple $(u_o, v_o) \in \mathscr{K}(w)$ that maximizes the following dual cost

$$\hat{\mathscr{C}}_K[u, v; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] \coloneqq \int_{\mathbb{R}^d} u(\mathbf{x}) \,\mathrm{d}\, \rho_{\mathscr{R}}(\mathbf{x}) + \int_{\mathbb{R}^d} v(\mathbf{y}) \,\mathrm{d}\, \rho_{\mathscr{B}}(\mathbf{y}). \tag{5.4.10}$$

It can be proved [15] that, if ω is uniformly continuous, such a couple of functions exists and, moreover, they satisfy the following relations

$$\begin{cases} u_o(\mathbf{x}) = \inf_{\mathbf{y} \in \mathbb{R}^d} \left[w(\mathbf{x} - \mathbf{y}) - v_o(\mathbf{y})\right] \\ v_o(\mathbf{y}) = \inf_{\mathbf{x} \in \mathbb{R}^d} \left[w(\mathbf{x} - \mathbf{y}) - u_o(\mathbf{x})\right]. \end{cases} \tag{5.4.11}$$

The correspondence between the previous set of equations and the belief propagation equations (4.3.31) is remarkable. The relation between the couple of functions (u_o, v_o) and the optimal map of the original problem $\mathbf{M}$, is clarified by the following theorem [15].

Theorem 5.4.1 (Strictly convex cost)**.** *Let $\omega \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ a strictly convex function, i.e.,*

$$\omega(\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2) < \alpha\omega(\mathbf{x}_1) + (1 - \alpha)\omega(\mathbf{x}_2), \quad \alpha \in (0, 1). \tag{5.4.12}$$

Then an optimal map $\mathbf{M}$ for the original plan exists almost everywhere and it satisfies the following differential equation

$$\nabla\omega\left(\mathbf{x} - \mathbf{M}(\mathbf{x})\right) = \nabla u_o(\mathbf{x}). \tag{5.4.13}$$

Moreover,

$$\hat{\mathscr{C}}_K[u_o, v_o; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}] = \mathscr{C}[\mathbf{M}; \rho_{\mathscr{R}}, \rho_{\mathscr{B}}]. \tag{5.4.14}$$

If the cost function ω is not strictly convex (e.g., $\omega(\mathbf{x}) = |\mathbf{x}|$), then the uniqueness of the map is not guaranteed.

If the cost function ω is strictly convex, we can also write a differential equation for $\mathbf{M}$ in the form of a change-of-variable formula (sometimes called *Jacobian equation*)

$$\mathrm{d}\, \rho_{\mathscr{R}}(\mathbf{x}) = \mathrm{d}\, \rho_{\mathscr{B}}(\mathbf{M}(\mathbf{x})) \det \mathsf{J}_{\mathbf{M}}(\mathbf{x}), \tag{5.4.15}$$

where $\mathsf{J}_{\mathbf{M}}(\mathbf{x})$ is the Jacobian matrix for the map $\mathbf{M}$. This formula does not hold in general for non convex costs. Note that the optimal map is not usually "smooth" and the meaning of the Jacobian matrix itself must be clarified. The validity conditions of the Jacobian equation (5.4.15) are pointed out in the following theorem[5]

Theorem 5.4.2 (Jacobian equation)**.** *Let be $\rho_{\mathscr{R}}, \rho_{\mathscr{B}}$ two non-negative measures on $\mathbb{R}^d$. Let us assume that*

- $\mathrm{d}\, \rho_{\mathscr{R}}(\mathbf{x}) = \rho_{\mathscr{R}}(\mathbf{x})\, \mathrm{d}^d x$, *where* $\rho_{\mathscr{R}}(\mathbf{x}) \in \mathsf{L}^1(\mathbb{R}^d)$;

[4] In particular, it is required that $\rho_{\mathscr{R}}(\mathbf{x})$ has no atoms. In measure theory, given a measurable space (m, Σ) with m measure on the σ-algebra Σ, $A \subset \Sigma$ is called *atom* if $m(A) > 0$ and $\forall B \subset A$ measurable such that $m(B) < m(A)$, $m(B) = 0$.

[5] A more general statement can be found in the monograph by Villani [30, Chapter 11].

- $\mathbf{M} \in \mathscr{M}(\rho_{\mathscr{R}}, \rho_{\mathscr{B}})$;
- $\exists \Sigma \subset \mathbb{R}^d$ *such that* $\rho_{\mathscr{R}}(\mathbf{x}) = 0$ *almost everywhere outside* Σ *and* $\mathbf{M}$ *is injective on* Σ;
- $\mathbf{M}$ *is approximately differentiable*[6] *on* Σ *almost everywhere.*

If $\nabla\mathbf{M}(\mathbf{x})$ *is the approximate gradient of* $\mathbf{M}$*, then by definition* $\mathsf{J}_{\mathbf{M}}(\mathbf{x}) \coloneqq \det|\nabla\mathbf{M}(\mathbf{x})|$ *and Eq.* (5.4.15) *holds.*

Working with strictly convex potentials and compact submanifolds of $\mathbb{R}^d$, the hypotheses of the previous theorem are satisfied. As an important particular case, let us consider the following cost function:

$$\omega(\mathbf{x}-\mathbf{y}) \coloneqq \|\mathbf{x}-\mathbf{y}\|^2, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, \tag{5.4.16}$$

where $\|\bullet\|$ is the Euclidean norm. The cost is strictly convex. We suppose also that two measures are given on $\mathbb{R}^d$ such that

$$\mathrm{d}\,\rho_{\mathscr{R}}(\mathbf{x}) \coloneqq \rho_{\mathscr{R}}(\mathbf{x})\,\mathrm{d}^d x, \quad \mathrm{d}\,\rho_{\mathscr{B}}(\mathbf{y}) \coloneqq \rho_{\mathscr{B}}(\mathbf{y})\,\mathrm{d}^d y, \tag{5.4.17}$$

such that $\rho_{\mathscr{R}}(\mathbf{x}), \rho_{\mathscr{B}}(\mathbf{x}) \in \mathsf{L}^1(\mathbb{R}^d)$. The optimal map is a minimum of the following functional

$$\mathscr{L}[\mathbf{M},\lambda] \coloneqq \int\limits_{\mathbb{R}^d} \Big[\|\mathbf{x}-\mathbf{M}(\mathbf{x})\|^2 + \lambda(\mathbf{x})\left(\rho_{\mathscr{B}}(\mathbf{M}(\mathbf{x}))\det \mathsf{J}_{\mathbf{M}}(\mathbf{x}) - \rho_{\mathscr{R}}(\mathbf{x})\right)\Big]\,\mathrm{d}^d x \tag{5.4.18}$$

where $\lambda(\mathbf{x})$ is a Lagrange multiplier. A direct computation of the Euler–Lagrange equations gives that

$$\mathbf{M}(\mathbf{x}) = \nabla\phi(\mathbf{x}) \tag{5.4.19}$$

for a certain (convex) potential ϕ. In other words, *in the quadratic case, the optimal map can be expressed as a gradient of a convex potential.* The Jacobian equation assumes the well known form of a *Monge–Ampère equation*,

$$\rho_{\mathscr{R}}(\mathbf{x}) = \rho_{\mathscr{B}}(\nabla\phi(\mathbf{x}))\det \mathsf{H}[\phi](\mathbf{x}), \tag{5.4.20}$$

where $\mathsf{H}[\phi]$ is the Hessian matrix of ϕ. This equation will be extremely useful in the what follows.

5.5. Ansatz for the Euclidean matching problem

Inspired by the Monge–Kantorovič theory of optimal transport, and in particular by the Monge–Ampère equation, we try to obtain useful information on the discrete problem for any dimension, through a proper regularization procedures, under the assumption of a quadratic cost. In particular, let us consider the random EB on the unit hypercube,

$$\Omega \equiv \Omega_d = [0,1]^d \subset \mathbb{R}^d. \tag{5.5.1}$$

We suppose that two sets of N points, let us call them $\mathscr{R} = \{\mathbf{r}_i\}_{i=1,\dots N}$ and $\mathscr{B} = \{\mathbf{b}_i\}_{i=1,\dots N}$, are generated with *uniform* distribution on the hypercube, $\rho(\mathbf{x}) = 1$ (Poisson–Poisson matching problem). As usual, we randomly associate the vertices of the complete bipartite graph $\mathsf{K}_{N,N} \coloneqq \mathrm{Graph}(\mathscr{V},\mathscr{U};\mathscr{E})$ to the points on Ω_d, let be

$$v_i \in \mathscr{V} \quad \mapsto \quad \mathbf{r}_i \in \mathscr{R}, \quad i = 1,\dots N, \tag{5.5.2a}$$
$$u_i \in \mathscr{U} \quad \mapsto \quad \mathbf{b}_i \in \mathscr{B}, \quad i = 1,\dots N. \tag{5.5.2b}$$

We want to find the optimal map $\mathbf{M}_o\colon \Omega_d \to \Omega_d$ that minimizes the following functional:

$$\mathscr{C}^{(2,\mathrm{EB})}[\mathbf{M}] \coloneqq \frac{1}{N}\sum_{i=1}^{N} \|\mathbf{M}(\mathbf{r}_i) - \mathbf{r}_i\|^2. \tag{5.5.3}$$

[6] See, for example, [30] for a rigorous definition of approximate differentiability and approximate gradient.

The distance $\|\bullet\|$ is the Euclidean distance on the domain Ω evaluated *with periodic boundary conditions*. This means that, to be precise, we are considering the matching problem on the d–dimensional flat hypertorus, T^d. The map $\mathbf{M}_o$ must be searched in the set of suitable maps, given by

$$\mathcal{S} = \{\mathbf{M}\colon \Omega_d \to \Omega_d | \exists \sigma \in \mathcal{P}_N \text{ such that } \mathbf{M}(\mathbf{r}_i) = \mathbf{b}_{\sigma(i)}\ \forall i\}. \tag{5.5.4}$$

We can associate to the set $\mathcal{R}$ and to the set $\mathcal{B}$ two empirical measures respectively

$$\rho_{\mathcal{R}}(\mathbf{x}) \coloneqq \frac{1}{N}\sum_{i=1}^{N} \delta\left(\mathbf{x} - \mathbf{r}_i\right), \tag{5.5.5a}$$

$$\rho_{\mathcal{B}}(\mathbf{x}) \coloneqq \frac{1}{N}\sum_{i=1}^{N} \delta\left(\mathbf{x} - \mathbf{b}_i\right), \tag{5.5.5b}$$

such that $\rho_{\mathcal{R}}(\mathbf{x}), \rho_{\mathcal{B}}(\mathbf{x}) \xrightarrow{N\to\infty} \rho(\mathbf{x}) = 1$ weakly. The difference between the two densities

$$\varrho(\mathbf{x}) \coloneqq \rho_{\mathcal{R}}(\mathbf{x}) - \rho_{\mathcal{B}}(\mathbf{x}) \tag{5.5.6}$$

has corresponding Fourier modes given by

$$\hat{\varrho}(\mathbf{k}) \coloneqq \sum_{i=1}^{N} \frac{\mathrm{e}^{-2\pi i \mathbf{k}\cdot\mathbf{r}_i} - \mathrm{e}^{-2\pi i \mathbf{k}\cdot\mathbf{b}_i}}{N}, \qquad \mathbf{k} \in \mathbb{Z}^d. \tag{5.5.7}$$

A suitable map $\mathbf{M} \in \mathcal{S}$ is therefore a map between the two atomic densities (5.5.5) that minimizes the following transport functional

$$\mathscr{C}[\mathbf{M}; \rho_{\mathcal{R}}, \rho_{\mathcal{B}}] \coloneqq \int_{\Omega_d} \|\mathbf{x} - \mathbf{M}(\mathbf{x})\|^2 \rho_{\mathcal{R}}(\mathbf{x})\, \mathrm{d}^d x. \tag{5.5.8}$$

The functional in Eq. (5.5.8) is identical to the the cost in Eq. (5.5.3). An optimal map $\mathbf{M}_o$ is such that

$$\mathscr{C}[\mathbf{M}_o; \rho_{\mathcal{R}}, \rho_{\mathcal{B}}] = \min_{\mathbf{M}\in\mathcal{S}} \mathscr{C}[\mathbf{M}; \rho_{\mathcal{R}}, \rho_{\mathcal{B}}]. \tag{5.5.9}$$

Neglecting for a moment the high nonregularity of our measures, we have that, for a quadratic cost as in Eq. (5.5.8), the optimal map can be expressed as a gradient of a scalar function $\varphi(\mathbf{x})$, see Eq. (5.4.19). We write therefore

$$\mathbf{M}(\mathbf{x}) \coloneqq \mathbf{x} + \boldsymbol{\mu}(\mathbf{x}), \quad \boldsymbol{\mu}(\mathbf{x}) = \nabla\varphi(\mathbf{x}). \tag{5.5.10}$$

We impose also the Monge–Ampère equation (5.4.20),

$$\rho_{\mathcal{R}}(\mathbf{x}) = \rho_{\mathcal{B}}\left(\mathbf{x} + \nabla\varphi(\mathbf{x})\right) \det\left(\mathbb{I} + \frac{\partial^2\varphi(\mathbf{x})}{\partial x_i \partial x_j}\right), \tag{5.5.11}$$

where $\mathbb{I} = (\delta_{ij})_{ij}$. In the limit $N \to \infty$ we expect that the optimal map has $|\boldsymbol{\mu}(\mathbf{x})| \ll 1$. Indeed, the two empirical measures converge weakly to the same uniform measure and therefore $\mathbf{M}_o(\mathbf{x}) \to \mathbf{x}$. Under this working hypothesis, the Monge–Ampère equation can be linearized as a simple Poisson equation,

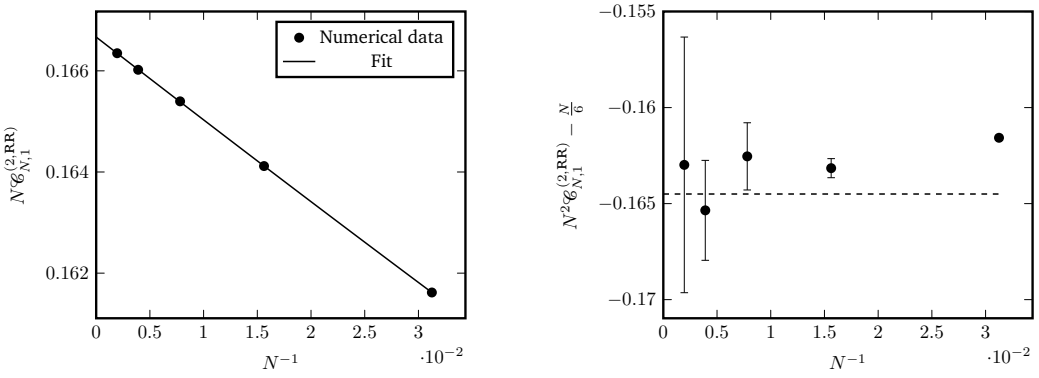
$$\Delta\varphi(\mathbf{x}) = \varrho(\mathbf{x}). \tag{5.5.12}$$

The previous equation has only *one* non trivial solution on the d-dimensional hypertorus and therefore identifies uniquely the optimal map. Substituting in the equation (5.5.8) the solution, we have that for the optimal map $\mathbf{M}_o$

$$\mathscr{C}^{(2,\mathrm{EB})}[\mathbf{M}_0] \simeq \mathscr{C}[\mathbf{M}_o; \rho_{\mathcal{R}}, \rho_{\mathcal{B}}] = \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{|\hat{\varrho}(\mathbf{k})|^2}{4\pi^2\|\mathbf{k}\|^2}. \tag{5.5.13}$$

Our hypothesis is that the previous quantity, for $N \gg 1$, captures the leading terms of the exact optimal cost of our original matching problem. In particular, observing that

$$\overline{|\hat{\varrho}(\mathbf{k})|^2} = \frac{2}{N} \tag{5.5.14}$$



(A) Numerical results for $\mathscr{C}_{N,1}^{(2,\mathrm{RR})}$.

(B) Numerical results for $N\mathscr{C}_{N,1}^{(2,\mathrm{RR})} - \frac{1}{6}$.

FIGURE 5.5.1.: Numerical simulations for $d = 1$. We averaged the optimal cost of the EB given by an exact algorithm [12] for system sizes up to $N = 2048$. The fit was performed using a fitting function in the form of Eq. (5.5.17).

we obtain our expression for the average optimal cost [11]

$$\mathscr{C}_{N,d}^{(2,\mathrm{RR})} \simeq \frac{1}{2\pi^2 N} \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{k}\|^2}. \tag{5.5.15}$$

ONE DIMENSIONAL CASE Let us consider the quadratic random EB in dimension $d = 1$. In low dimensions, that is for $d = 1, 2$, fluctuations of the density of points are dominant and the scaling of the cost is expected to be anomalous. Moreover, for $d = 1$ the optimal cost is also not self-averaging [17]. The one-dimensional case is the simplest application of our formula and we obtain straightforwardly

$$\mathscr{C}_{N,1}^{(2,\mathrm{RR})} = \frac{1}{6N} + o\left(\frac{1}{N}\right). \tag{5.5.16}$$

This result is *exact*. In the next Section we will prove the equivalent result for the GR–EB using a different approach. We numerically checked the validity of our prediction. In Fig. 5.5.1 we show the perfect agreement of numerical results with our formula. In particular, the fit for $N\mathscr{C}_{N,1}^{(2,\mathrm{RR})}$ was performed using the three parameters fit function

$$F_1(N) = c_1 + \frac{c_1^{(1)}}{N} + \frac{c_1^{(2)}}{N^2}. \tag{5.5.17}$$

From a least square fit we obtained the coefficient $c_1 = 0.166668(3)$, in perfect agreement with our analytical prediction. Once verified the validity our theoretical result, we used it to extrapolate the subleading coefficient $c_1^{(1)}$, fixing $c_1 \equiv \frac{1}{6}$ and using the fitting function $F_1(N)$ with two free parameters (see Table 5.1).

HIGHER DIMENSIONS For $d \geq 2$ the sum in Eq. (5.5.15) diverges. To regularize it, let us introduce a proper cut-off in the momentum space, i.e., let us consider only the element of the sum with

$$\|\mathbf{k}\| \leq \frac{2\pi}{\ell}, \quad \ell := \frac{1}{\sqrt[d]{N}}. \tag{5.5.18}$$

The cut-off above has a clear meaning. Indeed, for finite N, ℓ is the characteristic length of the system, being of the order of the distance between two points of different type. Clearly $2\pi\ell^{-1} \to +\infty$ for $N \to +\infty$. We introduce a regularizing smooth function $\Phi(x)$ such that $\Phi(0) = 1$ and $\lim_{x\to+\infty} \Phi(x) = 0$. The function has to decrease rapidly enough to make the series

$$\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{n}\|^2} \Phi\left(\frac{2\pi\|\mathbf{n}\|}{2\pi\ell^{-1}}\right) \tag{5.5.19}$$

converge. Let us denote by

$$\mathcal{N}_d(r) \coloneqq \left|\{\mathbf{x} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} | \|\mathbf{x}\| < r\}\right|, \tag{5.5.20}$$

the number of lattice points (excluded the origin) in a ball of radius r centred in the origin in dimension d. Then, for arbitrary $a \in (0,1)$, we can write

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{k}\|^2}\Phi\left(\frac{\|\mathbf{k}\|}{\sqrt[d]{N}}\right) &= \lim_{R\to\infty}\int_a^R \frac{1}{k^2}\,\Phi\left(\frac{k}{\sqrt{N}}\right)\left[\frac{\partial\mathcal{N}_d(k)}{\partial k} - \Sigma_d k^{d-1}\right]\mathrm{d}\,k \\ &+ N^{\frac{d-2}{d}}\Sigma_d \int_{\frac{a}{\sqrt[d]{N}}}^{\infty} \Phi\left(k\right) k^{d-3}\,\mathrm{d}\,k \simeq \lim_{R\to\infty}\int_a^R \frac{1}{k^2}\left[\frac{\partial\mathcal{N}_d(k)}{\partial k} - \Sigma_d k^{d-1}\right]\mathrm{d}\,k \\ &+ N^{\frac{d-2}{d}}\Sigma_d \int_{\frac{a}{\sqrt[d]{N}}}^{\infty} \Phi\left(k\right) k^{d-3}\,\mathrm{d}\,k. \end{aligned} \tag{5.5.21}$$

To proceed further, we have to distinguish the $d = 2$ case from the $d > 2$ case. Let us start from $d = 2$. In Eq. (5.5.21) we have

$$\int_a^R \left[\frac{\partial\mathcal{N}_2(k)}{\partial k} - 2\pi k\right]\frac{\mathrm{d}\,k}{k^2} = \left.\frac{\mathcal{N}_2(k) - \pi k^2}{k^2}\right|_a^R + \int_a^R \frac{\mathcal{N}_2(k) - \pi k^2}{2k^3}\,\mathrm{d}\,k. \tag{5.5.22}$$

Both the first and the second term are finite in the $R \to \infty$ limit due to the result of Hardy and Ramanujan [16]

$$\mathcal{N}_2(k) - \pi k^2 \leq 1 + 2\sqrt{2}\pi k. \tag{5.5.23}$$

Therefore we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{k}\|^2}\Phi\left(\frac{\|\mathbf{k}\|}{\sqrt{N}}\right) \simeq \int_a^{+\infty} \frac{\mathcal{N}_2(k) - \pi k^2}{2k^3}\,\mathrm{d}\,k + 2\pi\log\frac{\sqrt{N}}{a} + 2\pi\int_1^{\infty}\frac{\Phi(k)}{k}\,\mathrm{d}\,k. \tag{5.5.24}$$

Eq. (5.5.15) for the case $d = 2$ can then be rewritten as

$$\mathscr{C}_{N,2}^{(2,\mathrm{RR})} \simeq \frac{\ln N}{2\pi N} + \frac{c_2^{(1)}}{N}, \tag{5.5.25}$$

where $c_2^{(1)}$ is some constant. To our knowledge the result

$$\lim_{N\to\infty}\frac{N\mathscr{C}_{N,2}^{(2,\mathrm{RR})}}{\ln N} = \frac{1}{2\pi} \tag{5.5.26}$$

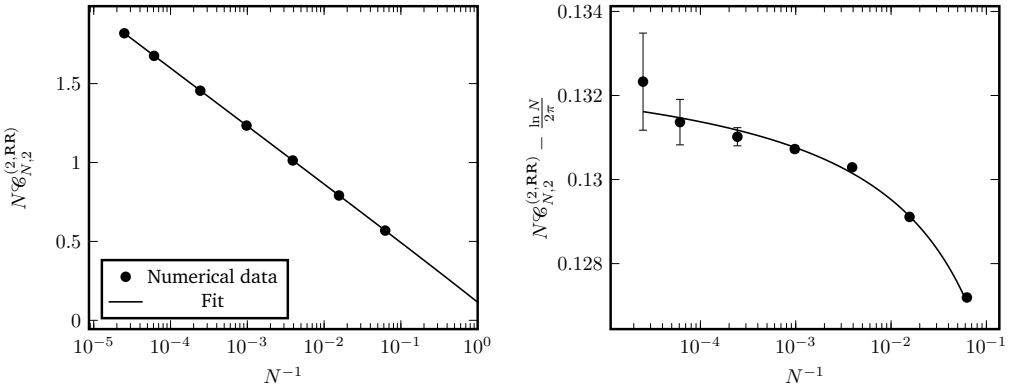
is new to the literature. The validity of Eq. (5.5.25) has been confirmed by numerical simulations. In Fig. 5.5.2 we fitted our numerical data for $N\mathscr{C}_{N,2}^{(2,\mathrm{RR})}$ using the function

$$F_2(N) = c_2 \ln N + c_2^{(1)} + \frac{c_2^{(2)}}{\ln N}. \tag{5.5.27}$$

The $\frac{1}{\ln N}$ correction was suggested by the plot in Fig. 5.5.2b. From a least square fit we obtained

$$2\pi c_2 = 1.0004(6), \tag{5.5.28}$$

in perfect agreement with our analytical prediction. Once verified the theoretical prediction for c_2, we used it to extrapolate the subleading coefficient $c_2^{(1)}$, fixing $c_2 = \frac{1}{2\pi}$ and fitting the other two parameters (see Table 5.1).



(A) Numerical data for $N\mathscr{C}_{N,2}^{(2,\mathrm{RR})}$.

(B) Numerical results for $N\mathscr{C}_{N,2}^{(2,\mathrm{RR})} - \frac{\ln N}{2\pi}$.

FIGURE 5.5.2.: Numerical simulations for $d = 2$.

	Dimension				
	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
$\frac{1}{2\pi^2}\zeta_d(1)$	$\frac{1}{6}$	$\frac{1}{2\pi}$	$-0.45157\ldots$	$-0.28091\ldots$	$-0.21423\ldots$
c_d	0.166668(3)	0.15921(6)	0.66251(2)	0.571284(6)	0.584786(2)
$c_d^{(1)}$	−0.1645(13)	0.1332(5)	−0.4489(16)	−0.282(4)	−0.2139(13)

TABLE 5.1.: Results of numerical simulations for $p = 2$. In the first line the analytical predictions for c_1, c_2 and $c_d^{(1)}$ for $d > 2$ are presented.

For $d \geq 3$, the last term in Eq. (5.5.21) is finite but cannot be explicitly computed since it depends on the choice of the regularizing function Φ. Unfortunately, it corresponds exactly to the leading term in the cost. We write therefore

$$c_d = \Sigma_d \int_0^\infty \Phi\left(k\right) k^{d-3} \,\mathrm{d}\, k. \tag{5.5.29}$$

We name instead

$$\Xi_d \coloneqq \int_0^{+\infty} \left[\frac{\partial \mathcal{N}_d(k)}{\partial k} - \Sigma_d\, k^{d-1}\right] \frac{\mathrm{d}\, k}{k^2}. \tag{5.5.30}$$

It can be shown that $\Xi_d = \zeta_d(1)$, where $\zeta_d(1)$ is the analytic continuation to the point $s = 1$ of the *Epstein zeta function*

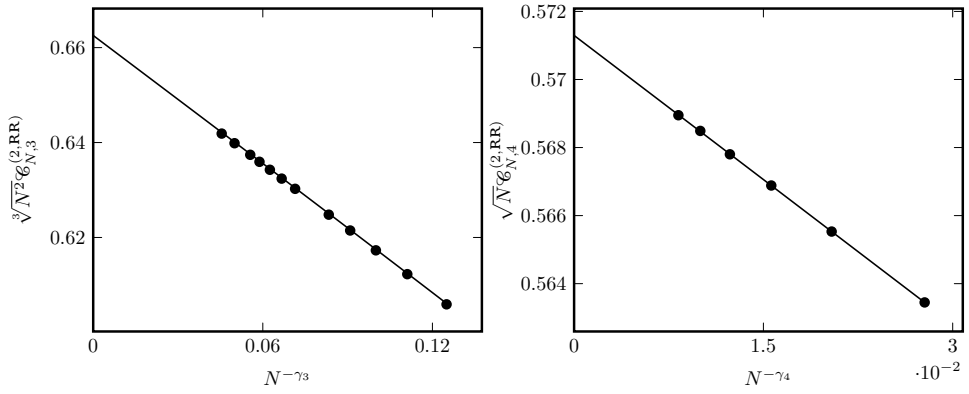
$$\zeta_d(s) \coloneqq \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{k}\|^{2s}} \qquad \text{for } \Re s > \frac{d}{2}. \tag{5.5.31}$$

After some computations [11] we obtained the result

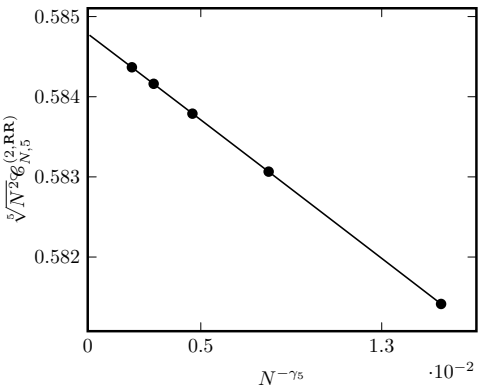
$$\Xi_d = \zeta_d(1) = \pi\left[\frac{2}{2-d} - 1 + \int_1^{+\infty} \left(1 + z^{\frac{d}{2}-2}\right)\left(\Theta^d(z) - 1\right) \mathrm{d}\, z\right]. \tag{5.5.32}$$

In the previous expression, $\Theta(z)$ is defined by

$$\Theta(z) \coloneqq \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\pi n^2 z} \equiv \vartheta_3(0; iz), \tag{5.5.33}$$



(A) $d = 3$. (B) $d = 4$.



(C) $d = 5$.

FIGURE 5.5.3.: Numerical simulations for $d > 2$. We verified the validity of Eq. (5.5.35) with numerical simulations on systems with sizes up to $N = 10648$ in dimension $d = 3$, $N = 14641$ in dimension $d = 4$ and $N = 32768$ in dimension $d = 5$. The scaling exponents γ_d are readily confirmed to be the exact ones.

where $\vartheta_3(\tau; z)$ is the third Jacobi theta function. Therefore we finally have that for $d > 2$,

$$\mathscr{C}^{(2,\mathrm{RR})}_{N,d} \simeq c_d N^{-\frac{2}{d}} + \frac{\zeta_d(1)}{2\pi^2 N} + o\left(\frac{1}{N}\right). \tag{5.5.34}$$

The expression for $\zeta_d(1)$ is given by Eq. (5.5.32), while c_d has to be determined numerically. Note that for $d \to +\infty$ we recover the correct mean field scaling behavior already analyzed by Houdayer, Boutet de Monvel, and Martin [17] for the average optimal cost $\mathscr{C}^{(\mathrm{mf})}_N$ of the random assignment problem. Observe indeed that

$$\sqrt[d]{N^2}\mathscr{C}^{(2,\mathrm{RR})}_N(d) \simeq c_d + \frac{\zeta_d(1)}{2\pi^2 N^{\gamma_d}}, \qquad \gamma_d \coloneqq 1 - \frac{2}{d} \xrightarrow{d\to\infty} 1. \tag{5.5.35}$$

However, for finite d, the scaling behavior can be very different from the mean field one [27] that Houdayer, Boutet de Monvel, and Martin [17] used to fit the Euclidean optimal cost. The predictions above were confirmed by proper numerical simulations (see Fig. 5.5.3). We fitted our numerical data for $\sqrt[d]{N^2}\mathscr{C}^{(2,\mathrm{RR})}_{N,d}$ using the function

$$F_d(N) = c_d + c_d^{(1)} N^{-\gamma_d} + \frac{c_d^{(2)}}{N}. \tag{5.5.36}$$

5.5.1. Correlation functions for the rEB

Let us now discuss a different aspect of the rEB. Let us consider again the optimal matching field

$$\boldsymbol{\mu}_o(\mathbf{r}_i) \coloneqq \mathbf{b}_{\sigma_o(i)} - \mathbf{r}_i, \tag{5.5.37}$$

with $\sigma_o \in \mathcal{P}_N$ optimal permutation that minimizes the functional in Eq. (5.5.3). Eq. (5.5.3) can be written, in particular, as

$$\mathscr{C}^{(2,\mathrm{EB})}[\boldsymbol{\mu}] \coloneqq \frac{1}{N}\sum_{i=1}^{N} \|\boldsymbol{\mu}(\mathbf{r}_i)\|^2 \equiv \mathscr{C}^{(2,\mathrm{EB})}[\sigma], \quad \boldsymbol{\mu} = \mathbf{b}_{\sigma(i)} - \mathbf{r}_i \ \sigma \in \mathcal{P}_N. \tag{5.5.38}$$

Let us analyze the *correlation function* of the optimal matching field in the EB with quadratic cost, defined as

$$C_d(\mathbf{x}) \coloneqq \overline{\boldsymbol{\mu}_o(\mathbf{r}_i)\cdot\boldsymbol{\mu}_o(\mathbf{r}_j)}\Big|_{\mathbf{r}_i-\mathbf{r}_j=\mathbf{x}}. \tag{5.5.39}$$

To avoid boundary effects, we will work on the unit hypertorus T^d (i.e., we will consider periodic boundary conditions) and we will assume that the random points are generated with uniform distribution on the hypertorus.

To obtain the quantity in Eq. (5.5.39), we will generalize our ansatz, following [10]. The generalization is quite straightforward. Using the Poisson equation (5.5.12), obtained by a proper linearization of the Monge–Ampére equation in the large N limit, we can write down an expression for the correlation function assuming that

$$\boldsymbol{\mu}_o(\mathbf{x}) = \nabla\varphi(\mathbf{x}), \quad \Delta\varphi(\mathbf{x}) = \varrho(\mathbf{x}). \tag{5.5.40}$$

Here ϱ is given by Eq. (5.5.6). As mentioned above, being $\int_{\Omega_d}\varrho(\mathbf{x})\,\mathrm{d}^d x = 0$, Eq. (5.5.12) has a unique solution on the compact manifold T^d, given by

$$\varphi(\mathbf{x}) = \int_{\mathsf{T}^d} \varrho(\mathbf{y}) G_d(\mathbf{y},\mathbf{x})\,\mathrm{d}^d y, \tag{5.5.41}$$

where G_d is the Green's function for the Laplace operator Δ on the flat hypertorus T^d, i.e.,

$$\Delta_{\mathbf{y}} G_d(\mathbf{x},\mathbf{y}) = \delta^{(d)}(\mathbf{x}-\mathbf{y}) - 1. \tag{5.5.42}$$

In a Fourier mode expansion, we can write

$$G_d(\mathbf{x},\mathbf{y}) \equiv G_d(\mathbf{x}-\mathbf{y}) = -\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{\mathrm{e}^{2\pi i\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}}{4\pi^2\|\mathbf{n}\|^2}. \tag{5.5.43}$$

We have finally that

$$\begin{aligned} C_d(\mathbf{x}-\mathbf{y}) \coloneqq& \overline{\nabla\phi(\mathbf{x})\cdot\nabla\phi(\mathbf{y})} \\ =& \iint \nabla_{\mathbf{z}} G_d(\mathbf{z}-\mathbf{x})\cdot\nabla_{\mathbf{w}} G_d(\mathbf{w}-\mathbf{y})\,\overline{\varrho(\mathbf{z})\varrho(\mathbf{w})}\,\mathrm{d}^d z\,\mathrm{d}^d w \end{aligned} \tag{5.5.44}$$

where we used the fact that periodic boundary conditions are assumed and Eq. (5.5.41) holds.

We distinguish now two different cases. In the RR–EB (here Poisson–Poisson EB) both the points of $\mathcal{R}$ and the points of $\mathcal{B}$ are random points uniformly distributed on the considered domain. In this case

$$\overline{\varrho(\mathbf{x})\varrho(\mathbf{y})} = 2\frac{\delta^{(d)}(\mathbf{x}-\mathbf{y})-1}{N}, \tag{5.5.45}$$

and therefore the correlation function is

$$C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}) = -\frac{2}{N}G_d(\mathbf{x}-\mathbf{y}). \tag{5.5.46}$$

In the GR–EB (here grid–Poisson EB) we suppose that $N = L^d$, for some natural $L \in \mathbb{N}$, and that one set of points, e.g. the set $\mathcal{R} = \{\mathbf{r}_i\}_{i=1,\dots,N}$, is fixed on the vertices of an hypercubic lattice,

in such a way that $\{\mathbf{r}_i\}_{i=1,\dots,N} = \left\{\frac{\boldsymbol{k}}{L} | \boldsymbol{k} \in [0,L]^d \cap \mathbb{N}^d\right\}$. The set $\mathscr{B} = \{\mathbf{b}_i\}_{i=1,\dots,N} \subset \Omega_d$ is instead obtained as before, considering the points uniformly randomly generated. In this case we have

$$\overline{\varrho(\mathbf{x})\varrho(\mathbf{y})} = \frac{1}{N}\delta^{(d)}(\mathbf{x}-\mathbf{y}) + \frac{N^2-N}{N^2} + \sum_{ij} \frac{\delta^{(d)}(\mathbf{x}-\mathbf{r}_i)\,\delta^{(d)}(\mathbf{y}-\mathbf{r}_j)}{N^2} - \sum_i \frac{\delta^{(d)}(\mathbf{x}-\mathbf{r}_i) + \delta^{(d)}(\mathbf{y}-\mathbf{r}_i)}{N}. \tag{5.5.47}$$

The correlation function is therefore

$$C_d^{\mathrm{GR}}(\mathbf{x}-\mathbf{y}) = -\frac{1}{N} G_d(\mathbf{x}-\mathbf{y}). \tag{5.5.48}$$

We will consider also the correlation function for the normalized transport field, i.e., the following quantity:

$$c_d(\mathbf{x}-\mathbf{y}) = \overline{\frac{\boldsymbol{\mu}_o(\mathbf{x})}{\|\boldsymbol{\mu}_o(\mathbf{x})\|} \cdot \frac{\boldsymbol{\mu}_o(\mathbf{y})}{\|\boldsymbol{\mu}_o(\mathbf{y})\|}}, \tag{5.5.49}$$

Note that $\boldsymbol{\mu}_o(\mathbf{x})\|\boldsymbol{\mu}_o(\mathbf{x})\|^{-1}$ lives on the d-dimensional unit sphere. To compute the correlation function (5.5.49) for the normalized field in the RR case, we assume a Gaussian behavior for the joint probability distribution of two values of the optimal transport field, and therefore we have

$$c_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}) = \iint \frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{\|\boldsymbol{\mu}_1\|\|\boldsymbol{\mu}_2\|} \frac{\mathrm{e}^{-\frac{1}{2}\boldsymbol{\mu}^T \cdot \boldsymbol{\Sigma}^{-1}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\mu}}}{\left(2\pi\sqrt{\det \boldsymbol{\Sigma}}\right)^d} \,\mathrm{d}^d\mu_1\,\mathrm{d}^d\mu_2, \tag{5.5.50}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \tag{5.5.51}$$

and $\boldsymbol{\Sigma}(\mathbf{x},\mathbf{y})$ is the covariance matrix,

$$\boldsymbol{\Sigma}(\mathbf{x},\mathbf{y}) \coloneqq \begin{pmatrix} \overline{\boldsymbol{\mu}_o(\mathbf{x})\cdot\boldsymbol{\mu}_o(\mathbf{x})} & \overline{\boldsymbol{\mu}_o(\mathbf{x})\cdot\boldsymbol{\mu}_o(\mathbf{y})} \\ \overline{\boldsymbol{\mu}_o(\mathbf{y})\cdot\boldsymbol{\mu}_o(\mathbf{x})} & \overline{\boldsymbol{\mu}_o(\mathbf{y})\cdot\boldsymbol{\mu}_o(\mathbf{y})} \end{pmatrix} \equiv \begin{pmatrix} C_d^{\mathrm{RR}}(\mathbf{0}) & C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}) \\ C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}) & C_d^{\mathrm{RR}}(\mathbf{0}) \end{pmatrix}. \tag{5.5.52}$$

The case $d = 1$ is peculiar and was first solved by Boniolo, Caracciolo, and Sportiello [7]. Let us consider $d \geq 2$ and introduce

$$A \coloneqq \frac{C_d^{\mathrm{RR}}(\mathbf{0})}{\det \boldsymbol{\Sigma}(\mathbf{x},\mathbf{y})}, \tag{5.5.53a}$$

$$B \coloneqq \frac{C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y})}{\det \boldsymbol{\Sigma}(\mathbf{x},\mathbf{y})}. \tag{5.5.53b}$$

Observe that

$$\lim_{N\to\infty} \frac{B}{A} = \lim_{N\to\infty} \frac{C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y})}{C_d^{\mathrm{RR}}(\mathbf{0})} = 0. \tag{5.5.54}$$

Indeed, $NC_d^{\mathrm{RR}}(\mathbf{x})$ is finite for $\mathbf{x} \neq 0$, whereas $NC_d^{\mathrm{RR}}(\mathbf{0}) \sim N^{1-\frac{2}{d}}$ for $d > 2$, $NC_d^{\mathrm{RR}}(\mathbf{0}) \sim \ln N$ for $d = 2$. We have therefore that, in the notation above,

$$\det \boldsymbol{\Sigma} = \frac{1}{A^2 - B^2} \tag{5.5.55}$$

and

$$c_d^{\mathrm{RR}}(\mathbf{x},\mathbf{y}) = \frac{B}{A}\frac{2\Gamma^2\left(\frac{d+1}{2}\right)}{d\Gamma^2\left(\frac{d}{2}\right)}\left(1-\frac{B^2}{A^2}\right)^{\frac{d}{2}} {}_2F_1\left[\begin{matrix}\frac{d+1}{2} & \frac{d+1}{2} \\ \frac{d}{2}+1\end{matrix}; \frac{B^2}{A^2}\right] \xrightarrow[\frac{B}{A}\to 0]{N\to\infty} \frac{2}{d}\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^2 \frac{C_d^{\mathrm{RR}}(\mathbf{x}-\mathbf{y})}{\mathscr{C}_{N,d}^{(2,\mathrm{RR})}}. \tag{5.5.56}$$

In the previous expression, we have introduced the hypergeometric function

$$ {}_2F_1\left[\begin{matrix} a & b \\ c \end{matrix}; z\right] \coloneqq \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (a)_n \coloneqq \frac{\Gamma(a+1)}{\Gamma(a-n+1)}. \tag{5.5.57} $$

In the GR case, the correlation function for the normalized field has the same expression, i.e.,

$$ c_d^{\mathrm{GR}}(\mathbf{x}-\mathbf{y}) = \frac{2}{d} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \right)^2 \frac{C_d^{\mathrm{GR}}(\mathbf{x}-\mathbf{y})}{\mathscr{C}_{N,d}^{(2,\mathrm{GR})}}. \tag{5.5.58} $$

Finally, for $d \geq 2$ we can compute also the so called *wall-to-wall correlation function*. In the RR case it is given by

$$ W_d^{\mathrm{RR}}(r) \coloneqq \prod_{i=2}^{d} \left(\int_0^1 \mathrm{d}\, x_i \right) c_d^{\mathrm{RR}}(r, x_2, \dots, x_d) = -\frac{4}{d} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \right)^2 \frac{G_1(r)}{\mathscr{C}_{N,d}^{(2,\mathrm{RR})}}. \tag{5.5.59} $$

Similarly, the computation for the GR case gives

$$ W_d^{\mathrm{GR}}(r) \coloneqq \prod_{i=2}^{d} \left(\int_0^1 \mathrm{d}\, x_i \right) c_d^{\mathrm{GP}}(r, x_2, \dots, x_d) = -\frac{2}{d} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \right)^2 \frac{G_1(r)}{\mathscr{C}_{N,d}^{(2,\mathrm{GR})}}. \tag{5.5.60} $$

Let us now analyze in details the solutions for different values of d.

One dimensional case In Section 5.6 we discuss an exact solution of the one dimensional matching problem in the GR case. Let us consider now the RR case, remembering that similar considerations can be carried on for the GR case. We have that

$$ G_1(r) = -\sum_{n \neq 0} \frac{\mathrm{e}^{2\pi i n r}}{4\pi^2 n^2} = -\frac{1}{12} + \frac{|r|}{2}\left(1 - |r|\right). \tag{5.5.61} $$

It follows from Eq. (5.5.46) that

$$ C_1^{\mathrm{RR}}(x-y) = \frac{1 - 6|x-y|\left(1-|x-y|\right)}{6N}. \tag{5.5.62} $$

Note that the average optimal cost is

$$ C_1^{\mathrm{RR}}(0) = \mathscr{C}_{N,1}^{(2,\mathrm{RR})} = \tfrac{1}{6N}. \tag{5.5.63} $$

In the GR case the results of Section 5.6 are easily recovered. Boniolo, Caracciolo, and Sportiello [7] obtained the correlation function for the normalized matching field as

$$ c_1^{\mathrm{RR}}(x) = c_1^{\mathrm{GR}}(x) = \frac{2}{\pi} \arctan\left[\frac{1 - 6x(1-x)}{\sqrt{12x(1-x)\left(1-3x(1-x)\right)}} \right]. \tag{5.5.64} $$

Two dimensional case For $d = 2$, we have that

$$ G_2(\mathbf{r}) = -\sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{4\pi^2 \|\mathbf{n}\|^2} \mathrm{e}^{2\pi i \mathbf{n}\cdot\mathbf{r}}. \tag{5.5.65} $$

The Green's function G_2 can be expressed in terms of first Jacobi theta functions [20]. From Eq. (5.5.46) we have simply

$$ C_2^{\mathrm{RR}}(\mathbf{x}) = -\frac{2G_2(\mathbf{x})}{N}. \tag{5.5.66} $$

In the GR case, we have as usual

$$ C_2^{\mathrm{GR}}(\mathbf{x}-\mathbf{y}) = \frac{1}{2} C_2^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}). \tag{5.5.67} $$

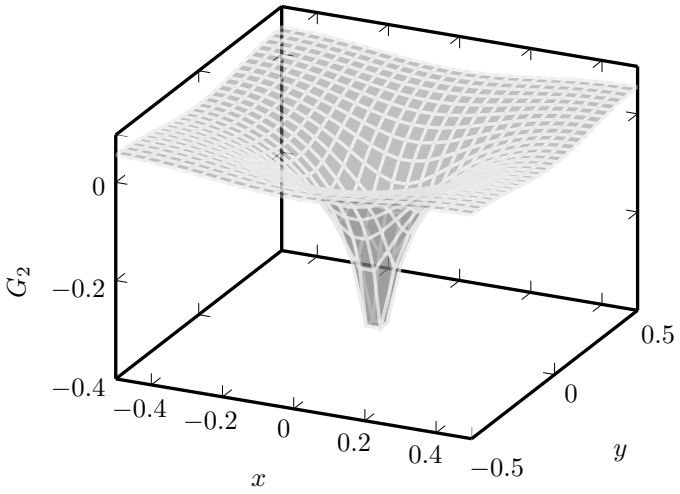


FIGURE 5.5.4.: Laplacian Green's function $G_2(\mathbf{r})$, $\mathbf{r} = (x, y)$, on T^2.

Observe that the previous expressions contains no free parameters and therefore a direct comparison with numerical data is possible. We present our numerical results both from the GR case and the RR case in Fig. 5.5.5a. As explained in Section 5.5, a proper regularization led to the correct scaling for the average optimal cost,

$$\mathscr{C}_{N,2}^{(2,\mathrm{RR})} = C_2^{\mathrm{RR}}(\mathbf{0}) = \frac{1}{N}\left(\frac{\ln N}{2\pi} + c_2^{(1,\mathrm{RR})}\right) + o\left(\frac{1}{N}\right), \quad c_2^{(1,\mathrm{RR})} \equiv c_2^{(1)} = 0.1332(5). \tag{5.5.68}$$

A numerical fit of the optimal costs for $d = 2$ for the GR EBMP gives

$$\mathscr{C}_{N,2}^{(2,\mathrm{GR})} = C_2^{\mathrm{GR}}(\mathbf{0}) = \frac{1}{2N}\left(\frac{\ln N}{2\pi} + c_2^{(1,\mathrm{GR})}\right) + o\left(\frac{1}{N}\right), \; c_2^{(1,\mathrm{GR})} = 0.3758(5). \tag{5.5.69}$$

The correlation function (5.5.49) for the normalised matching field in the RR case has the expression (5.5.56),

$$c_2^{\mathrm{RR}}(\mathbf{x} - \mathbf{y}) = \frac{\pi}{4}\frac{C_2^{\mathrm{RR}}(\mathbf{x} - \mathbf{y})}{\mathscr{C}_{N,2}^{(2,\mathrm{RR})}}. \tag{5.5.70}$$

The only free parameter in this quantity is $\mathscr{C}_{N,2}^{(2,\mathrm{RR})}$. Inserting the value obtained in Section 5.5, Eq. (5.5.68), we have the theoretical prediction in Fig. 5.5.5b. Similarly, we can write down the form of the correlation function for the normalized transport field in the GR case, $c_2^{\mathrm{GR}}(\mathbf{x} - \mathbf{y})$. In particular, using Eq. (5.5.69) in Eq. (5.5.58) for $d = 2$, we obtain the theoretical curve depicted in Fig. (5.5.5b), where, once again, an excellent agreement is found with numerical data.

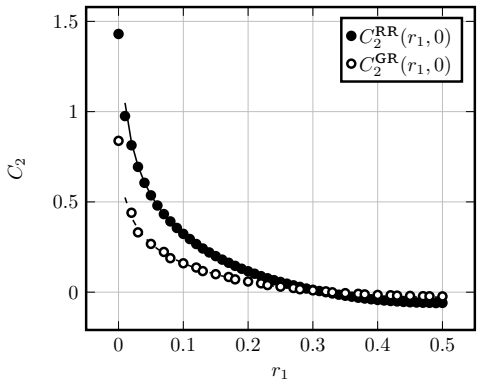
Finally, let us compute the wall-to-wall correlation function. The theoretical prediction in the RR case is given by Eq. (5.5.59),

$$W_2^{\mathrm{RR}}(r) = -\frac{\pi G_1(r)}{2N\mathscr{C}_{N,2}^{(2,\mathrm{RR})}}. \tag{5.5.71}$$

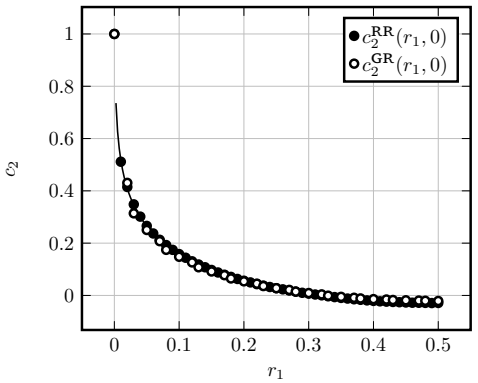
In the GR case, instead, we have

$$W_2^{\mathrm{GR}}(r) = -\frac{\pi G_1(r)}{4N\mathscr{C}_{N,2}^{(2,\mathrm{GR})}}. \tag{5.5.72}$$

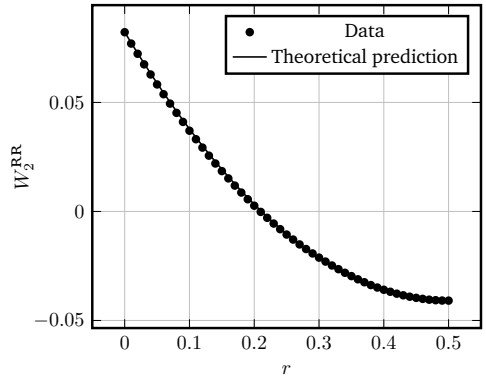
Numerical results both for the RR case and for the GR case are presented in Fig. 5.5.5c and Fig. 5.5.5d. The values of the average optimal cost in the corresponding cases, Eq. (5.5.68) and Eq. (5.5.68), fix completely the expression of the wall-to-wall correlation function.



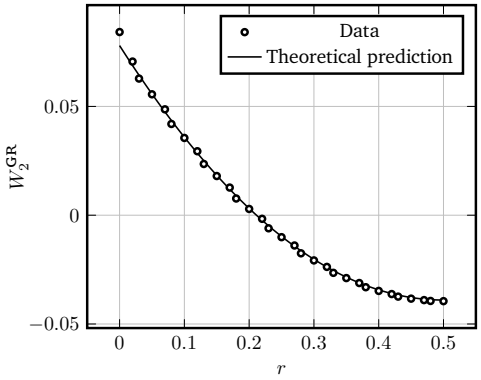
(A) Section $C_2^{\mathrm{RR}}(r_1,0)$ and $C_2^{\mathrm{GR}}(r_1,0)$ of the correlation function for $N = 10^4$ in the RR case and for $N = 3600$ in the GR case and corresponding theoretical predictions.



(B) Section $c_2^{\mathrm{RR}}(r_1,0)$ for $N = 10^4$ and $c_2^{\mathrm{GP}}(r_1,0)$ for $N = 3600$ and theoretical predictions. Note that the theoretical curves almost overlap.



(C) Wall-to-wall correlation function in two dimensions for the RR matching with $N = 10^4$ on the unit flat torus. The continuous line corresponds to the analytical prediction.



(D) Wall-to-wall correlation function in two dimensions for the GR matching with $N = 3600$ on the unit flat torus. The continuous line corresponds to the analytical prediction.

FIGURE 5.5.5.: Correlation functions for the Euclidean bipartite matching problem in two dimensions and numerical results.

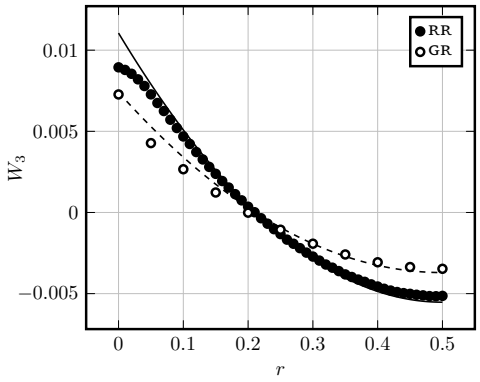


FIGURE 5.5.6.: Wall-to-wall correlation function in three dimensions for the RR matching and the GR matching with $d = 3$ and $N = 8000$ on the unit flat hypertorus.

THREE-DIMENSIONAL CASE For $d = 3$ Eq. (5.5.43) and Eq. (5.5.46) give

$$C_3^{\mathrm{RR}}(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi^2 N} \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} \frac{\mathrm{e}^{2\pi i \mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}}{\|\mathbf{n}\|^2}. \tag{5.5.73}$$

From the correlation function $C_3^{\mathrm{RR}}(\mathbf{x})$, the wall to wall correlation function can be obtained as before in the form

$$W_3^{\mathrm{RR}}(r) = -\frac{16 G_1(\mathbf{r})}{3\pi N \mathscr{C}_{N,3}^{(2,\mathrm{RR})}}. \tag{5.5.74}$$

As in the previous cases, $\mathscr{C}_{N,3}^{(2,\mathrm{RR})} \equiv C_3^{\mathrm{RR}}(\mathbf{0})$ can be evaluated from the cost fit and it is equal to

$$\mathscr{C}_{N,3}^{(2,\mathrm{RR})} = 0.66251(2) N^{-\frac{2}{3}} - \frac{0.45157\ldots}{N} + o\left(\frac{1}{N}\right) \tag{5.5.75}$$

(see Table 5.1). Following the same procedure of the RR case, we can compute the wall-to-wall correlation function for on the unit hypercube in the three-dimensional case for the GR matching problem. Reproducing the computations of the $d = 2$ case we have

$$W_3^{\mathrm{GP}}(r) = -\frac{8 G_1(r)}{3\pi N \mathscr{C}_{N,3}^{(2,\mathrm{GR})}}. \tag{5.5.76}$$

From a direct cost evaluation, we obtain

$$\mathscr{C}_{N,3}^{(2,\mathrm{GR})} = 0.4893(4) N^{-\frac{2}{3}} - \frac{0.23(5)}{N} + o\left(\frac{1}{N}\right). \tag{5.5.77}$$

The prediction obtained and the numerical data are presented in Fig. 5.5.6.

5.6. ONE DIMENSIONAL GR–REB: EXACT SOLUTION FOR CONVEX WEIGHTS

5.6.1. PRELIMINARIES

In this Section we give special attention to the REB on the line and on the circumference. The low dimensionality allows us to give an exact solution to the problem in the case of *convex* cost functions. In particular, for the sake of simplicity, we will consider the GR–EB in one dimension both with open boundary conditions (OBC) and with periodic boundary conditions (PBC) on the interval $\Omega_1 = [0, 1]$. A similar treatment of the RR–EB can be performed. In the GR–EB we suppose that we work on the complete graph $\mathtt{K}_{N,N} = \mathrm{Graph}(\mathscr{V}, \mathscr{U}; \mathscr{E})$, in which the first set of vertices is associated to a set of fixed points on the interval. In particular

$$v_i \in \mathscr{V} \mapsto r_i \equiv \frac{2i-1}{2N}, \quad i = 1, \ldots, N, \tag{5.6.1}$$

whilst the set of vertices $\mathcal{U}$ is associated to a set of N points, $\{b_i\}_{i=1,\dots,N}$, randomly generated in the interval, such that $u_i \in \mathcal{U} \mapsto b_i$. We will suppose the $\mathcal{U}$-vertices indexed in such a way that $i \le j \Rightarrow b_i \le b_j$. Finally, along with the functional $\mathscr{C}^{(p,\mathrm{EB})}$,

$$\mathscr{C}^{(p,\mathrm{EB})}[\sigma] = \frac{1}{N}\sum_{i=1}^{N} w^{(p)}_{i\,\sigma(i)}, \quad \mathscr{C}^{(p,\mathrm{GR})}_{N,1} := \overline{\min_{\sigma\in\mathscr{P}_N} \mathscr{C}^{(p,\mathrm{EB})}[\sigma]}, \tag{5.6.2a}$$

we will consider also

$$\tilde{\mathscr{C}}^{(p,\mathrm{EB})}[\sigma] = \sqrt[p]{\frac{1}{N}\sum_{i=1}^{N} w^{(p)}_{i\,\sigma(i)}}, \quad \tilde{\mathscr{C}}^{(p,\mathrm{GR})}_{N,1} := \overline{\min_{\sigma\in\mathscr{P}_N} \tilde{\mathscr{C}}^{(p,\mathrm{EB})}[\sigma]}, \tag{5.6.2b}$$

in which $p > 1$ and the function $w^{(p)} \colon [0,1] \to [0,1]$ is given by:

$$w^{(p)}(x) = \begin{cases} |x|^p & \text{for OBC,} \\ |x|^p\,\theta\left(\frac{1}{2} - |x|\right) + (1-|x|)^p\,\theta\left(|x| - \frac{1}{2}\right) & \text{for PBC.} \end{cases} \tag{5.6.3}$$

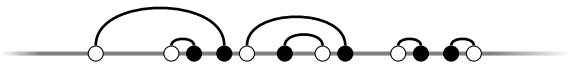
Before any analytical consideration on the optimal cost, let us discuss the structure of the optimal matching. The point configurations in our problem are of this type:

Let us consider for a moment the cost functional (5.6.2) assuming $p \in (0, +\infty)$. It can be easily seen that, in the hypothesis $p > 1$, the optimal matching, let us say the optimal permutation σ_o, is always *ordered*, i.e.

$$\sigma_o(i) = \begin{cases} i & \text{for OBC,} \\ i + \lambda \mod N & \text{for PBC, for a certain } \lambda \in \{0, 1, \dots, N-1\}. \end{cases} \tag{5.6.4}$$

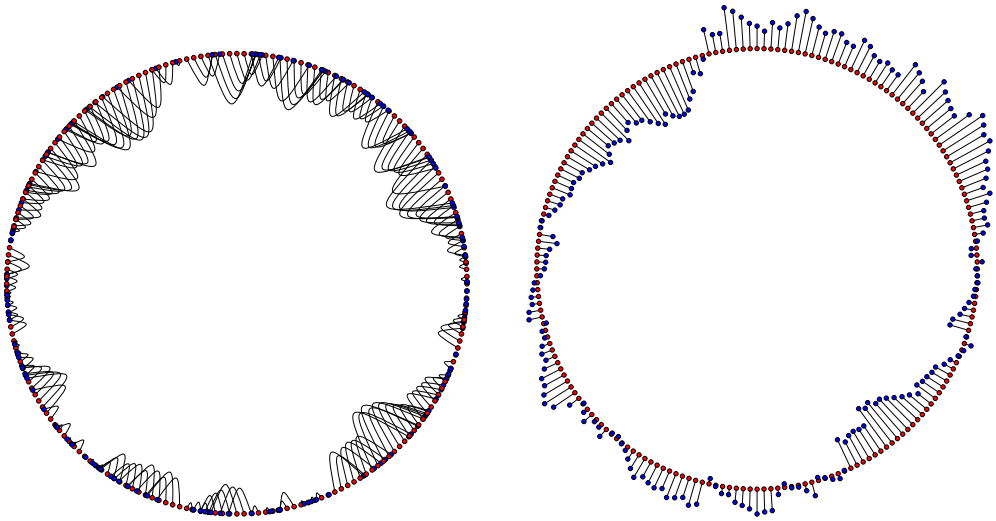
The optimal matching can be pictorially represented as

This simple fact follows directly from the convexity of the weight function $w^{(p)}$ for $p > 1$. Indeed, it can be proved by direct inspection in the $N = 2$ case that the ordered solution minimizes the cost functional among all possibilities. If instead $0 < p < 1$, the weight function $w^{(p)}$ is concave and the ordered solution is not necessarily the optimal one. The optimal matching σ_o has to satisfy a different requirement, i.e., it must be *uncrossing*, or *nested* [7]: given two intervals $[r_i, b_{\sigma_o(i)}]$ and $[r_j, b_{\sigma_o(j)}]$, $i \neq j$, either $[r_i, b_{\sigma_o(i)}] \cap [r_j, b_{\sigma_o(j)}] = \emptyset$ or one of the two intervals is a subset of the other one. Pictorially, drawing arcs connecting matched pairs above the line, they must be uncrossing, as in the picture below:



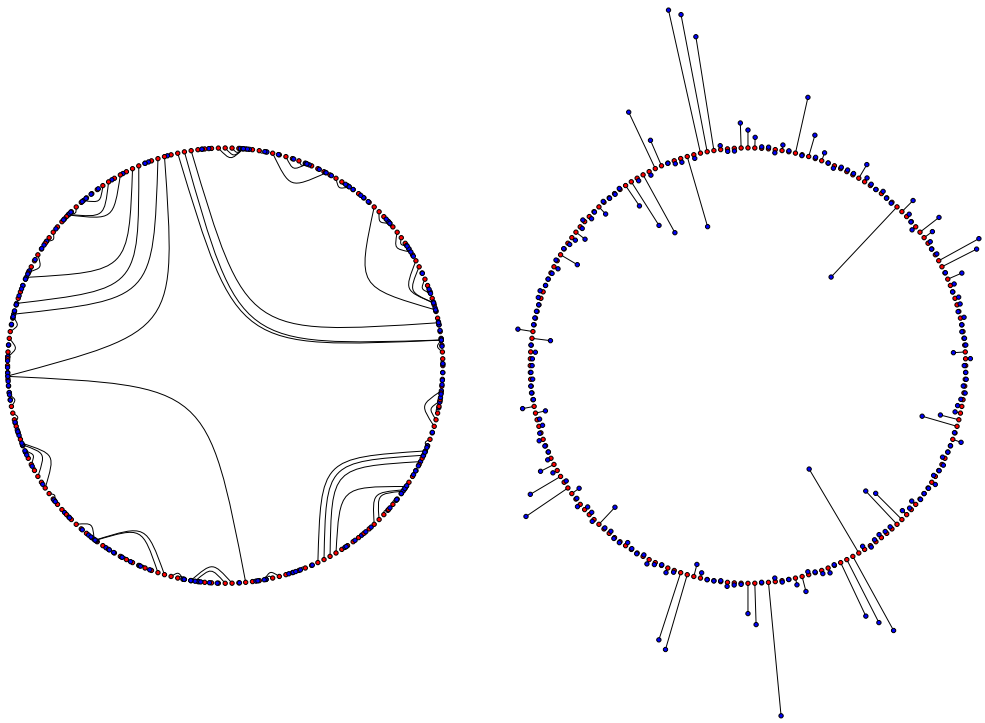
Clearly many nested solutions are possible: the uncrossing condition is necessary for the optimal solution in the $p \in (0,1)$ case but not sufficient to identify it. This simple fact in the discrete case, appears as a *no crossing rule* for the optimal map in the one dimensional Monge–Kantorovič transport problem in presence of concave cost functions [22]. Remarkably, for $p = 1$ it is possible to have instances in which the same optimal cost is obtained by different matching solutions.

The observation above on the properties of the solution for $p > 1$ solves completely the problem. Indeed, we can perform a simple probabilistic argument to obtain the correct average optimal cost and its distribution in the large N limit both on the line and on the circumference [7]. Here we will perform a different calculation following our paper [8], again inspired by the Monge–Kantorovič theory.



(A) Optimal matching for $p = 2$.

(B) Optimal matching field for $p = 2$.



(C) Optimal matching for $p = \frac{1}{2}$.

(D) Optimal matching field for $p = \frac{1}{2}$.

FIGURE 5.6.1.: Grid–Poisson Euclidean matching problem on the *same set* of $N = 200$ points on the circumference. In figures 5.6.1a and 5.6.1c, arcs join each red point to the corresponding blue point in the optimal matching. In figures 5.6.1b and 5.6.1d we represented, for each red point, the corresponding optimal matching field on it as a radial segment. The matching field is supposed as a field from a red point to a blue point. The lenght is proportional to the lenght of the arc joining the red point to the matched blue point, whilst the segment is directed outwards if the matching field is oriented clockwise, inwords otherwise. It is evident that the structure of the optimal matching for $p > 1$ and $p \in (0, 1)$ is very different.

5.6.2. General solution

Let us consider the interval $\Omega_1 := [0,1] \subset \mathbb{R}$ and let us suppose also that two different measures are given on it, i.e., the uniform (Lebesgue) measure

$$\mathrm{d}\,\rho_0(x) := \mathrm{d}\,x, \tag{5.6.5}$$

and a non uniform measure $\mathrm{d}\,n(x)$ with measure density $\nu(x)$,

$$\mathrm{d}\,\rho(x) := \rho(x)\,\mathrm{d}\,x = \mathrm{d}\,x + \mathrm{d}\,x \sum_{k=1}^{\infty} \left[\hat{\rho}^{(1)}(k)\cos(2\pi k x) + \hat{\rho}^{(2)}(k)\sin(2\pi k x)\right]. \tag{5.6.6}$$

We ask for the optimal map $M_o\colon \Omega_1 \to \Omega_1$ in the set

$$\mathcal{S} := \left\{ M\colon \Omega_1 \to \Omega_1 \,\middle|\, \int_A \mathrm{d}\,x = \int_{M^{-1}(A)} \mathrm{d}\,\rho(x) \quad \forall A \subset \Omega \text{ measurable} \right\} \tag{5.6.7}$$

that minimizes the following functional

$$\mathscr{C}^{(p)}[M;\rho] := \int_0^1 w^{(p)}\left(x - M(x)\right)\mathrm{d}\,\rho(x), \quad p \in \mathbb{R}^+. \tag{5.6.8}$$

or the functional

$$\tilde{\mathscr{C}}^{(p)}[M;\rho] := \sqrt[p]{\int_0^1 w^{(p)}\left(x - M(x)\right)\mathrm{d}\,\rho(x)}, \quad p \in \mathbb{R}^+. \tag{5.6.9}$$

In the hypothesis $p > 1$ the Jacobian equations (5.4.15) can be rewritten as a change-of-variable formula:

$$\mathrm{d}\,\rho(x) = \mathrm{d}\,M(x). \tag{5.6.10}$$

Adopting PBC,

$$M(0) = M(1) - 1, \tag{5.6.11}$$

and the solution of (5.6.10) determines the optimal map up to a constant as

$$M_o(x) = x + M_o(0) + \varphi(x). \tag{5.6.12}$$

In the previous equation we have introduced

$$\varphi(x) := \sum_{k=1}^{\infty} \hat{\rho}^{(1)}(k)\frac{\sin(k\pi x)\cos(k\pi x)}{\pi k} + \sum_{k=1}^{\infty} \rho^{(2)}(k)\frac{\sin^2(k\pi x)}{\pi k}. \tag{5.6.13}$$

Note that $\varphi(0) = \varphi(1) = 0$. The value of $M(0)$ must be determined requiring that the functional (5.6.8) is minimum, i.e., imposing

$$p\int_0^1 \mathrm{sign}\left(M(0) + \varphi(x)\right)\left|M(0) + \varphi(x)\right|^{p-1}\mathrm{d}\,\rho(x) = 0. \tag{5.6.14}$$

If instead OBC are considered, then $M(0) = 0$ and the solution is obtained explicitly $\forall p > 1$.

Let us now suppose that the measure $\mathrm{d}\,\rho(x)$ is obtained as a limit measure of a random atomic measure of the form

$$\mathrm{d}\,\rho(x) := \frac{\mathrm{d}\,x}{N}\sum_{i=1}^{N}\delta\left(x - b_i\right) = \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N}\theta\left(x - b_i\right)\right), \tag{5.6.15}$$

where $\mathcal{B} = \{b_i\}_{i=1,\dots,N}$ is a set of N points uniformly randomly distributed in Ω_1. With this hypothesis, we expect that

$$\mathscr{C}_{N,1}^{(p,\mathrm{GR})} = \overline{\min_{M\in\mathcal{S}} \mathscr{C}^{(p)}[M;\rho]} \text{ and } \tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})} = \overline{\min_{M\in\mathcal{S}} \tilde{\mathscr{C}}^{(p)}[M;\rho]}, \tag{5.6.16}$$

where $\overline{\bullet}$ denote the average over the positions of the points. The measure in Eq. (5.6.15) can be written as

$$\rho(x) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{N}} \frac{Z_k}{\pi k} \sin(\pi k x) + x + \frac{1}{N}\sum_{i=1}^{N} b_i - 1. \tag{5.6.17}$$

where we have introduced

$$Z_k \equiv Z_k(x) \coloneqq \frac{1}{N}\sum_{i=1}^{N} z_i(x), \quad z_i(x) \coloneqq -\sqrt{2N}\cos(2\pi k b_i + \pi k x). \tag{5.6.18}$$

Observe now that $Z_k(x)$ is a sum of independent identically distributed random variables. Applying the central limit theorem, we have that $Z_k(x)$ is normally distributed as

$$Z_k \sim \mathcal{N}(0,1) \quad \forall k \in \mathbb{N}. \tag{5.6.19}$$

Remarkably the previous distribution does not depend on x. Moreover, the Z_k and Z_l are independent random variables for $k \neq l$, being Gaussian distributed and $\langle Z_l Z_k\rangle = 0$, where the average $\langle\bullet\rangle$ is intended over the possible values $\{b_i\}_i$. In Eq. (5.6.17) the *Karhunen–Loève expansion for the Brownian bridge* [5] on the interval $[0,1]$ appears:

$$\mathsf{B}(x) = \sum_{k=1}^{\infty} \sqrt{2}\frac{Z_k}{\pi k}\sin(\pi k x), \quad Z_k \sim \mathcal{N}(0,1) \quad \forall k \in \mathbb{N}. \tag{5.6.20}$$

The Brownian bridge is continuous time stochastic process defined as

$$\mathsf{B}(t) \coloneqq \mathsf{W}(t) - t\mathsf{W}(1), \quad t \in [0,1], \tag{5.6.21}$$

where $\mathsf{W}(t)$ is a Wiener process (see Appendix B). It follows that $\rho(x)$ can be written, for large N and up to irrelevant additive constants, as

$$\mathrm{d}\,\rho(x) \simeq \mathrm{d}\left(\frac{\mathsf{B}(x)}{\sqrt{N}} + x\right), \tag{5.6.22}$$

and therefore we cannot associate a density measure to it, due to the fact that the Brownian bridge is not differentiable.

Periodic boundary conditions Considering PBC, we have

$$M_o(x) \equiv M_o^{(p)}(x) = M_o^{(p)}(0) + x + \frac{\mathsf{B}(x)}{\sqrt{N}}, \quad x \in [0,1]. \tag{5.6.23}$$

Denoting by

$$\mu_o^{(p)}(x) \coloneqq M_o^{(p)}(x) - x, \tag{5.6.24}$$

it follows that $\forall p > 1$ [8]

$$\overline{\mu_o^{(p)}(x)\mu_o^{(p)}(y)} = \frac{c_1^{(p,\mathrm{PBC})}}{N} - \frac{\phi(x-y)}{N}. \tag{5.6.25}$$

where $c_1^{(p,\mathrm{PBC})}$ is a constant depending on p and

$$\phi(x) \coloneqq |x|\frac{1-|x|}{2}. \tag{5.6.26}$$

This implies also that

$$\min_{M\in\mathcal{S}} \mathscr{C}^{(p)}[M;\rho] = O\left(N^{-\frac{p}{2}}\right), \tag{5.6.27}$$

For $p = 2$, Eq. (5.6.14) becomes

$$M^{(2)}(0) = -\frac{1}{\sqrt{N}}\int_0^1 \mathsf{B}(x) \circ \mathrm{d}\,\mathsf{B}(x) - \frac{1}{\sqrt{N}}\int_0^1 \mathsf{B}(x)\,\mathrm{d}\,x = -\frac{1}{\sqrt{N}}\int_0^1 \mathsf{B}(x)\,\mathrm{d}\,x \tag{5.6.28}$$

where the first integral is intended in the Stratonovič sense. We can therefore directly compute

$$\mathscr{C}_{N,1}^{(2,\mathrm{GR})} = \frac{c_1^{(2,\mathrm{PBC})}}{N} = \frac{1}{12N}. \tag{5.6.29}$$

If we consider the transport cost functional

$$\tilde{\mathscr{C}}^{(p)}[M;\rho] \coloneqq \sqrt[p]{\int_0^1 |x - M(x)|^p \,\mathrm{d}\,\rho(x)} \tag{5.6.30}$$

the matching problem has the same solution $\tilde{M}_o^{(p)} \equiv M_o^{(p)}$ obtained for the cost (5.6.8) for all values of p, due to the fact that the function $f(x) = \sqrt[p]{x}$ is monotone. However, for the functional cost (5.6.30), in the $p \to \infty$ limit, we have

$$\lim_{p\to\infty} \tilde{\mathscr{C}}^{(p)}[M_o^{(p)};\rho] = \lim_{p\to\infty}\left(\int_0^1 \left|\frac{\mathsf{B}(x)}{\sqrt{N}} + \tilde{M}_o^{(p)}(0)\right|^p \mathrm{d}\,x\right)^{\frac{1}{p}} = \sup_{x\in[0,1]}\left|\frac{\mathsf{B}(x)}{\sqrt{N}} + \tilde{M}_o^{(\infty)}(0)\right| \tag{5.6.31}$$

obtaining

$$\lim_{p\to\infty} \tilde{M}_o^{(\infty)}(0) = -\frac{\sup_{x\in[0,1]}\mathsf{B}(x) + \inf_{x\in[0,1]}\mathsf{B}(x)}{2\sqrt{N}}. \tag{5.6.32}$$

The correlation function can be directly found using the known joint distributions for the Brownian bridge and its sup and for the sup and inf of a Brownian bridge (see Appendix B). After some calculations we obtain, for $p = \infty$,

$$\overline{\mu_o^{(\infty)}(x)\mu_o^{(\infty)}(x+r)} = \frac{12-\pi^2}{24N} - \frac{\phi(r)}{N}. \tag{5.6.33}$$

The value $c_1^{(\infty,\mathrm{PBC})} \coloneqq \frac{12-\pi^2}{24} = 0.0887665\ldots$ is very close to the value obtained for $p = 2$, $c_1^{(2,\mathrm{PBC})} = \frac{1}{12} = 0.08\bar{3}$. In figure 5.6.2b we plot the values of $c_1^{(p,\mathrm{PBC})}$ as function of p.

The optimal cost in the $p \to \infty$ limit can be instead evaluated as the average spread of the Brownian bridge. Denoting

$$\xi \coloneqq \sup_{x\in[0,1]} \mathsf{B}(x) - \inf_{x\in[0,1]} \mathsf{B}(x), \tag{5.6.34}$$

the distribution of the spread ξ is given by [14]

$$\Pr(\xi < u) = \vartheta\left(\mathrm{e}^{-2u^2}\right) + u\frac{\mathrm{d}}{\mathrm{d}\,u}\vartheta\left(\mathrm{e}^{-2u^2}\right), \tag{5.6.35}$$

where

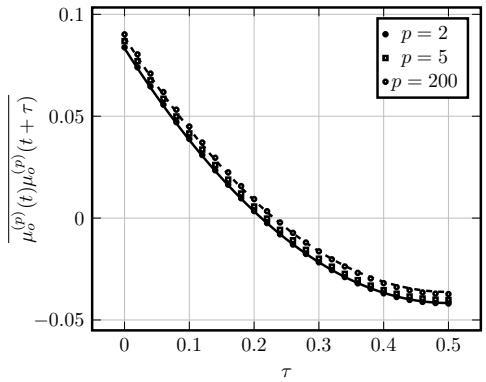
$$\vartheta(x) = 1 + 2\sum_{n=1}^{\infty} x^{n^2}. \tag{5.6.36}$$

From Eq. (5.6.35) the distribution of the optimal cost in the $p \to \infty$ limit is easily derived. Moreover,

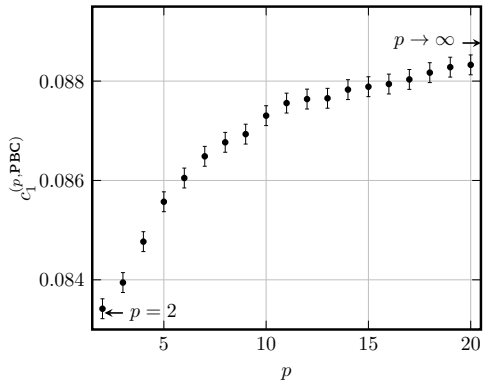
$$\sqrt{N}\tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})} \xrightarrow[p\to\infty]{N\to\infty} \frac{1}{2}\sqrt{\frac{\pi}{2}}. \tag{5.6.37}$$

In Fig. 5.6.2c we plotted

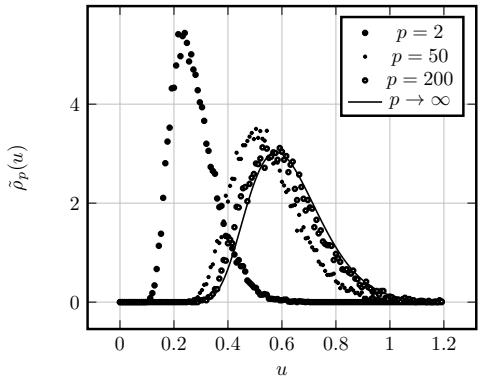
$$\tilde{\rho}_p(u) \coloneqq \frac{\mathrm{d}}{\mathrm{d}u}\Pr\left(\sqrt{N}\tilde{\mathscr{C}}^{(p)} < u\right) \tag{5.6.38}$$



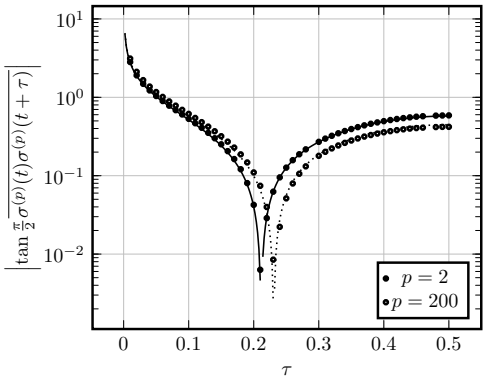
(A) Correlation function in the PBC case for $N = 1000$ obtained averaging over 10^4 realisations (for clarity, not all data are represented). The continuous line is the theoretical prediction for $p = 2$; the dashed line corresponds to the $p \to \infty$ case.



(B) Value of $c^{(p,\mathrm{PBC})}$ in the PBC case for $N = 1000$ obtained from the fit of the correlation function, averaging over 5000 realisations, for different values of p.



(C) Density distribution $\tilde{\rho}_p(u)$ of the rescaled cost for different values of p computed with $2 \cdot 10^4$ iterations and $N = 1000$ for the PBC case.



(D) Plot for the normalised map correlation function (5.6.40) obtained with 10^4 iterations and $N = 1000$ for the PBC case. The continuous line is the theoretical prediction for $p = 2$, whilst the dotted line is obtained from the theoretical prediction for $p \to \infty$. The pole is at $\tau(p) = \frac{1-\sqrt{1-8c_1^{(p,\mathrm{PBC})}}}{2}$.

FIGURE 5.6.2.: One dimensional GR–EB on the circumference.

for different values of p, showing that the distribution approaches the predicted one for $p \to \infty$.

Let us now introduce the normalized matching field

$$\sigma^{(p)}(x) \coloneqq \frac{\mu_o^{(p)}(x)}{\left|\mu_o^{(p)}(x)\right|} = \operatorname{sign}(\mu_o^{(p)}(x)). \tag{5.6.39}$$

The correlation function $\overline{\sigma^{(p)}(x)\sigma^{(p)}(y)}$ can be computed from the covariance matrix observing that the process is Gaussian and it is given by

$$\overline{\sigma^{(p)}(x)\sigma^{(p)}(x+r)} = \frac{2}{\pi}\arctan\frac{c_1^{(p,\mathrm{PBC})} - \phi(r)}{\sqrt{\phi(r)\left(2c_1^{(p,\mathrm{PBC})} - \phi(r)\right)}}. \tag{5.6.40}$$

Open boundary conditions If OBC are considered, then $M^{(p)}(0) = 0\ \forall p \in (1, +\infty)$ and we have simply,

$$M_o(x) = x + \frac{1}{\sqrt{N}}\mathsf{B}(x), \quad x \in [0,1]. \tag{5.6.41}$$

Defining as above

$$\mu_o(x) \coloneqq M_o(x) - x = \frac{1}{\sqrt{N}}\mathsf{B}(x), \quad x \in [0,1], \tag{5.6.42}$$

it can be easily seen that

$$\overline{\langle \mu_o(x)\mu_o(x+r)\rangle} = \int_0^1 \overline{\mu_o(x)\mu_o(x+r)}\,\mathrm{d}\,x = \frac{1}{6N} + \frac{\phi\,(r)}{N}, \tag{5.6.43}$$

where the average $\langle\bullet\rangle$ is intended on the position x, whilst we denoted by $\overline{\bullet}$ the average over different realizations of the problem as usual. Here the function ϕ is the same function defined in Eq. (5.6.26). The distribution of $\mu_o(x)$ is the one of a Brownian bridge process, but it can be also computed directly from the discrete configuration [7], using the fact that the optimal matching is ordered. Indeed, considering that the optimal permutation in this case is simply $\sigma_o(i) = i$, the probability density distribution for the position of the i-th b-point is:

$$\Pr\left(b_i \in \mathrm{d}\,b\right) = \binom{N}{i} y^i (1-b)^{N-i}\frac{i}{b}\,\mathrm{d}\,b, \tag{5.6.44}$$

where we used the short-hand notation $x \in \mathrm{d}\,z \Leftrightarrow x \in [z, z + \mathrm{d}\,z]$. In the $N \to \infty$ limit, a nontrivial result is obtained introducing the variable $\mu(x)$

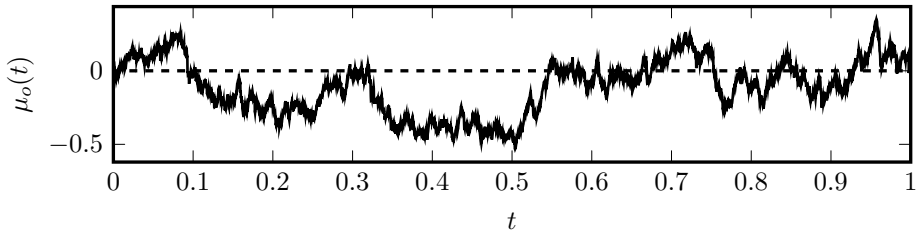
$$\mu(x) \coloneqq x - \frac{i}{N}, \tag{5.6.45}$$

expressing the rescaled (signed) matching field between the b-point in $[b, b + \mathrm{d}\,b]$ and its corresponding r-point in the optimal matching. After some calculations, we obtain a distribution for the variable $\mu(x)$ depending on the position on the interval $x \in [0,1]$:

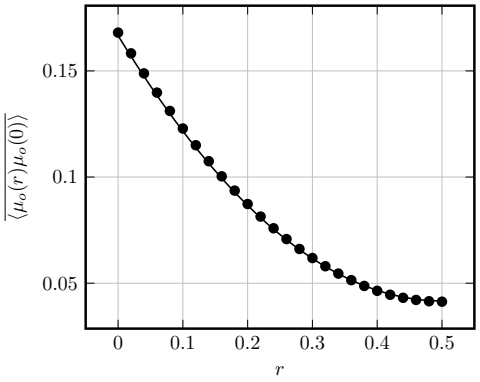
$$\Pr\left(\mu(x) \in \mathrm{d}\,\mu\right) = \frac{\mathrm{e}^{-\frac{\mu^2}{2Nx(1-x)}}}{\sqrt{2\pi N x(1-x)}}\,\mathrm{d}\,\mu. \tag{5.6.46}$$

The distribution (5.6.46) is the one of a (rescaled) Brownian bridge on the interval $[0,1]$. The joint distribution of the process can be derived similarly. In particular, the covariance matrix for the 2-points joint distribution has the form, for $x, y \in [0,1]$,

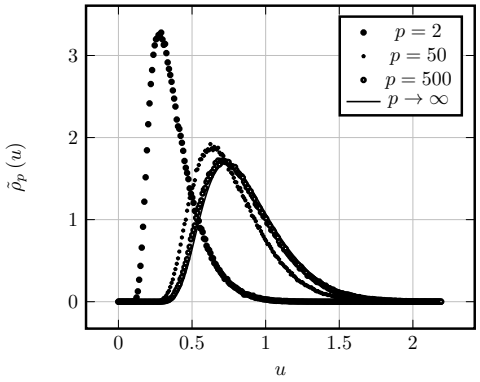
$$\Sigma_2 = \frac{1}{N}\begin{pmatrix} 2\phi(x) & \phi(x) + \phi(y) - \phi(y-x) \\ \phi(x) + \phi(y) - \phi(y-x) & 2\phi(y) \end{pmatrix}. \tag{5.6.47}$$



(A) Plot of $\mu_o(t)$ for a certain realisation of the problem with $N = 4096$.



(B) Correlation function $\forall p > 1$ in the OBC case for $N = 1000$ obtained averaging over 10^4 realisations (for clarity, not all data are represented).



(C) Distribution density of the rescaled cost $\tilde{\rho}_p$ for different values of p computed with $5 \cdot 10^5$ iterations and $N = 5000$ for the OBC case.

FIGURE 5.6.3.: One dimensional GR–EB on the interval.

Averaging over the positions x, y and fixing the distance $r := |y - x|$, we have

$$\langle \Sigma_2(r) \rangle = \frac{1}{N} \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - \phi(r) \\ \frac{1}{6} - \phi(r) & \frac{1}{6} \end{pmatrix}. \tag{5.6.48}$$

Eq. (5.6.43) follows immediately.

Introducing the normalized variable

$$\sigma(x) := \frac{\mu_o(x)}{|\mu_o(x)|} = \operatorname{sign}(\mu_o(x)), \tag{5.6.49}$$

Boniolo, Caracciolo, and Sportiello [7] computed also the correlation function

$$\overline{\sigma(x)\sigma(y)} = \frac{2}{\pi} \arctan \sqrt{\frac{\min(x,y)(1 - \max(x,y))}{|y - x|}} \Rightarrow \int_0^{1-x} \overline{\sigma(y)\sigma(x+y)}\, \mathrm{d}\, y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}. \tag{5.6.50}$$

Both formulas were confirmed numerically. The average cost of the matching is

$$N^{\frac{p}{2}} \mathscr{C}_N^{(p,\mathrm{GR})} \xrightarrow{N \to \infty} \int_0^1 \overline{|\mathsf{B}(x)|^p}\, \mathrm{d}\, x = \frac{1}{2^{\frac{p}{2}}(p+1)} \Gamma\left(\frac{p}{2} + 1\right). \tag{5.6.51}$$

The optimal cost $\sqrt{N}\tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})}$ in the $N \to \infty$ limit can be written as

$$\sqrt{N}\tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})} \xrightarrow{N \to \infty} \overline{\sqrt[p]{\int_0^1 |\mathsf{B}(x)|^p\, \mathrm{d}\, x}}. \tag{5.6.52}$$

Although the previous expression is difficult to evaluate exactly for finite p [29], the calculation can be easily performed in the relevant limit $p \to \infty$, being

$$\sqrt{N}\tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})} \xrightarrow{p\to\infty} \overline{\sup_{x\in[0,1]} |\mathsf{B}(x)|}. \tag{5.6.53}$$

The distribution of the sup of the absolute value of the Brownian bridge is the well known Kolmogorov distribution [14]

$$\Pr\left(\sup_{x\in[0,1]} |\mathsf{B}(x)| < u\right) = \sum_{k=-\infty}^{+\infty} (-1)^k \, \mathrm{e}^{-2k^2u^2} \tag{5.6.54}$$

and therefore

$$\sqrt{N}\tilde{\mathscr{C}}_{N,1}^{(p,\mathrm{GR})} \xrightarrow{N\to\infty} \sqrt{\frac{\pi}{2}} \ln 2. \tag{5.6.55}$$

In Fig. 5.6.3c we plotted

$$\tilde{\rho}_p(u) \coloneqq \frac{\mathrm{d}}{\mathrm{d}\,u}\left[\Pr\left(\sqrt{N}\tilde{\mathscr{C}}_N^{(p,\mathrm{GR})} < u\right)\right] \tag{5.6.56}$$

for different values of p. Observe that $\tilde{\rho}_p$ approaches the Kolmogorov distribution in the large p limit.

Universality If we consider the problem of a matching of N random b-points to N lattice r-points on the interval $[0, N]$, the correlation function assumes the form

$$C(\tau) = \overline{\left\langle \mu_o^{(p)}(x)\mu_o^{(p)}(x+r)\right\rangle} = c^{(p)}N - \frac{|r|}{2}\left(1 - \frac{|r|}{N}\right), \tag{5.6.57}$$

both in the case of OBC and in the case of PBC. It follows that for $N \to \infty$ the correlation function has a divergent part, $C(0) = c^{(p)}N$, depending through $c^{(p)}$ on the specific details of the problem (e.g., the boundary conditions adopted or the value of p), a universal finite part $-\frac{|r|}{2}$ and a (universal) finite size correction $\frac{r^2}{2N}$. This fact suggests that all Euclidean matching problems in one dimension with strictly convex cost functionals belong to the same universality class and that the specific details of the model determine only the value of the constant $c^{(p)}$ in the divergent contribution $C(0)$.

5.7. Functional approach to the quadratic rEb

The approaches presented in Section 5.5 and Section 5.6 can be further generalized. We can indeed recover all the results above adopting a general approach proposed in [9]. In this general framework, two sets of N points, let us call them $\mathscr{R} \coloneqq \{\mathbf{r}_i\}_{i=1,\dots,N}$ and $\mathscr{B} \coloneqq \{\mathbf{b}_i\}_{i=1,\dots,N}$, are considered on a domain $\Omega \subseteq \mathbb{R}^d$ in d dimensions. The domain is supposed to be connected and bounded. As usual, we want to find the permutation σ_o that minimizes the functional

$$\mathscr{C}^{(2,\mathrm{Eb})}[\sigma] \coloneqq \frac{1}{N}\sum_{i=1}^{N} w_{i\,\sigma(i)}^{(2)}, \quad w_{ij}^{(2)} \coloneqq \|\mathbf{r}_i - \mathbf{b}_j\|^2, \tag{5.7.1}$$

in the set of all permutations of N elements $\mathscr{P}_N$. The $2N$ points are extracted independently with a given probability distribution density $\rho(\mathbf{x})$ on the domain Ω (not necessarily the uniform one) and as usual we are interested in the quantity

$$\mathscr{C}_N^{(2,\mathrm{RR})} \coloneqq \overline{\min_{\sigma\in\mathscr{P}_N} \mathscr{C}^{(2,\mathrm{Eb})}[\sigma]}. \tag{5.7.2}$$

We associate both to the set $\mathscr{R}$ and to the set $\mathscr{B}$ an empirical measure, let us call them $\rho_{\mathscr{R}}$ and $\rho_{\mathscr{B}}$ respectively, as in Eqs. (5.5.5). We define the functional

$$\mathscr{C}^{(2,\mathrm{Eb})}[\boldsymbol{\mu}] \coloneqq \int_\Omega \|\boldsymbol{\mu}(\mathbf{x})\|^2 \rho_{\mathscr{R}}(\mathbf{x})\,\mathrm{d}^d x \tag{5.7.3}$$

for a map $\boldsymbol{\mu}\colon \Omega \to \mathbb{R}^d$. The previous functional provides a correct matching cost, Eq. (5.7.1), if, and only if, a permutation $\sigma \in \mathcal{P}_N$ exists such that

$$\boldsymbol{\mu}(\mathbf{r}_i) = \mathbf{b}_{\sigma(i)} - \mathbf{r}_i, \quad \forall i = 1, \dots, N. \tag{5.7.4}$$

This additional constraint implies

$$\int_\Omega \delta^{(d)}\left(\mathbf{x} - \mathbf{y} - \boldsymbol{\mu}(\mathbf{y})\right) \rho_{\mathscr{R}}(\mathbf{y})\, \mathrm{d}^d y = \rho_{\mathscr{B}}(\mathbf{x}). \tag{5.7.5}$$

We can write down a *partition function* for our problem introducing a proper Lagrange multiplier $\varphi(\mathbf{x})$ to impose the constraint in Eq. (5.7.5):

$$Z(\beta) \propto \int [\mathscr{D}\boldsymbol{\mu}] \int_{-i\infty}^{+i\infty} [\mathscr{D}\varphi]\, \mathrm{e}^{-\beta S[\boldsymbol{\mu},\varphi]}, \tag{5.7.6}$$

the optimal solution being recovered for $\beta \to +\infty$. The exponent in the functional integral is

$$\begin{aligned} S[\boldsymbol{\mu},\varphi] &\coloneqq \frac{1}{2}\mathscr{C}^{(2,\mathrm{EB})}[\boldsymbol{\mu}] + \int_\Omega \left[\varphi(\mathbf{x})\rho_{\mathscr{B}}(\mathbf{x}) - \varphi(\mathbf{x}+\boldsymbol{\mu}(\mathbf{x}))\rho_{\mathscr{R}}(\mathbf{x})\right] \mathrm{d}^d x \\ &= \frac{1}{2}\mathscr{C}^{(2,\mathrm{EB})}[\boldsymbol{\mu}] - \int_\Omega \left[\varphi(\mathbf{x})\varrho(\mathbf{x}) + \rho_{\mathscr{R}}(\mathbf{x})\boldsymbol{\mu}(\mathbf{x})\cdot\nabla\varphi(\mathbf{x})\right] \mathrm{d}^d x + s[\boldsymbol{\mu},\varphi], \end{aligned} \tag{5.7.7}$$

where $s[\boldsymbol{\mu},\varphi] = O\left(\|\boldsymbol{\mu}\|^2\varphi\right)$ are higher order nonlinear terms in the fields obtained from the Taylor series expansion of $\varphi(\mathbf{x}+\boldsymbol{\mu})$ around $\boldsymbol{\mu} = \mathbf{0}$. We have introduced also

$$\varrho(\mathbf{x}) \coloneqq \rho_{\mathscr{R}}(\mathbf{x}) - \rho_{\mathscr{B}}(\mathbf{x}). \tag{5.7.8}$$

Observing that $\rho_{\mathscr{R}}(\mathbf{x})$ is almost surely zero everywhere on the boundary, the Euler–Lagrange equations are

$$\varrho(\mathbf{x}) = \nabla\cdot\left(\rho_{\mathscr{R}}(\mathbf{x})\boldsymbol{\mu}(\mathbf{x})\right) - \frac{\delta s[\boldsymbol{\mu},\varphi]}{\delta\varphi(\mathbf{x})}, \tag{5.7.9a}$$

$$\rho_{\mathscr{R}}(\mathbf{x})\boldsymbol{\mu}(\mathbf{x}) = \rho_{\mathscr{R}}(\mathbf{x})\nabla\varphi(\mathbf{x}) - \frac{1}{2}\frac{\delta s[\boldsymbol{\mu},\varphi]}{\delta\boldsymbol{\mu}(\mathbf{x})}. \tag{5.7.9b}$$

In the limit $N \to \infty$ both $\rho_{\mathscr{R}}(\mathbf{x})$ and $\rho_{\mathscr{B}}(\mathbf{x})$ both converge (in weak sense) to $\rho(\mathbf{x})$, and the optimal field $\boldsymbol{\mu}_o$ is trivially $\boldsymbol{\mu}_o(\mathbf{x}) \equiv \mathbf{0}\ \forall \mathbf{x} \in \Omega$. For $N \gg 1$ we expect that the relevant contribution is given by small values of $\|\boldsymbol{\mu}_o\|$ and the nonlinear terms in s are higher order corrections to the leading quadratic terms. The saddle point equations simplify as

$$\varrho(\mathbf{x}) = \nabla\cdot\left(\rho(\mathbf{x})\boldsymbol{\mu}(\mathbf{x})\right), \tag{5.7.10a}$$

$$\boldsymbol{\mu}(\mathbf{x}) = \nabla\varphi(\mathbf{x}). \tag{5.7.10b}$$

It is remarkable that Eq. (5.7.10b) reproduces the known result in measure theory that the transport field is a gradient but, in our approach, this is specified as the gradient of the introduced Lagrange multiplier. We impose Neumann boundary conditions

$$\nabla_{\mathbf{n}(\mathbf{x})}\varphi(\mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} \equiv \nabla\varphi(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} = 0, \tag{5.7.11}$$

where $\mathbf{n}(\mathbf{x})$ is the normal unit vector to the boundary in $\mathbf{x} \in \partial\Omega$. Indeed, this condition guarantees that the shape of the boundary is not modified in the $N \to \infty$ limit. The potential φ is therefore the solution of the following problem

$$\nabla\cdot\left[\rho(\mathbf{x})\nabla\varphi(\mathbf{x})\right] = \varrho(\mathbf{x}) \tag{5.7.12}$$

on the domain Ω with boundary conditions given by Eq. (5.7.11). To solve Eq. (5.7.12), we introduce the *modified* Green's function $G_\rho(\mathbf{x},\mathbf{y})$ of the operator $\nabla\cdot[\rho(\mathbf{x})\nabla\bullet]$ on Ω,

$$\nabla_{\mathbf{x}}\cdot[\rho(\mathbf{x})\nabla_{\mathbf{x}}G_\rho(\mathbf{x},\mathbf{y})]=\delta^{(d)}(\mathbf{x}-\mathbf{y})-\frac{1}{|\Omega|},\qquad\text{with}\quad\left.\frac{\partial G_\rho(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})}\right|_{\mathbf{x}\in\partial\Omega}=0. \tag{5.7.13}$$

In Eq. (5.7.13), $|\Omega|$ is the Lebesgue measure of Ω.

The expression for $\boldsymbol{\mu}_o(\mathbf{x})$, solution of the saddle point equations, becomes

$$\boldsymbol{\mu}_o(\mathbf{x})=\int_\Omega\nabla_{\mathbf{x}}G_\rho(\mathbf{x},\mathbf{y})\varrho(\mathbf{y})\,\mathrm{d}^d y. \tag{5.7.14}$$

Averaging over the disorder, we obtain easily the following *two-point correlation function*

$$\begin{aligned}C(\mathbf{x},\mathbf{y})\coloneqq\overline{\boldsymbol{\mu}_o(\mathbf{x})\cdot\boldsymbol{\mu}_o(\mathbf{y})}&=\iint_{\Omega_N(\mathbf{x})\times\Omega_N(\mathbf{y})}\left[\overline{\varrho(\mathbf{z})\varrho(\mathbf{w})}\nabla_{\mathbf{x}}G_\rho(\mathbf{x},\mathbf{z})\cdot\nabla_{\mathbf{y}}G_\rho(\mathbf{y},\mathbf{w})\right]\mathrm{d}^d z\,\mathrm{d}^d w\\&=\frac{2}{N}\int_{\Omega_N(\mathbf{x},\mathbf{y})}[\rho(\mathbf{z})\nabla_{\mathbf{x}}G_\rho(\mathbf{x},\mathbf{z})\cdot\nabla_{\mathbf{y}}G_\rho(\mathbf{y},\mathbf{z})]\,\mathrm{d}^d z\\&-\frac{2}{N}\iint_{\Omega_N(\mathbf{x})\times\Omega_N(\mathbf{y})}[\rho(\mathbf{z})\rho(\mathbf{w})\nabla_{\mathbf{x}}G_\rho(\mathbf{x},\mathbf{z})\cdot\nabla_{\mathbf{y}}G_\rho(\mathbf{y},\mathbf{w})]\,\mathrm{d}^d z\,\mathrm{d}^d w,\end{aligned} \tag{5.7.15}$$

We used the following general result

$$\overline{\varrho(\mathbf{z})\varrho(\mathbf{w})}=2\tfrac{\rho(\mathbf{z})}{N}\left[\delta^{(d)}(\mathbf{z}-\mathbf{w})-\rho(\mathbf{w})\right]. \tag{5.7.16}$$

We introduced also the following sets

$$\Omega_N(\mathbf{x})\coloneqq\{\mathbf{y}\in\Omega\colon\|\mathbf{x}-\mathbf{y}\|>\alpha\delta_N\}, \tag{5.7.17a}$$

$$\Omega_N(\mathbf{x},\mathbf{y})\coloneqq\{\mathbf{z}\in\Omega\colon\|\mathbf{x}-\mathbf{z}\|>\alpha\delta_N\text{ and }\|\mathbf{y}-\mathbf{z}\|>\alpha\delta_N\},\quad\alpha\in\mathbb{R}^+. \tag{5.7.17b}$$

Here δ_N is the scaling law in N of the average distance between two *nearest neighbor* points randomly generated on Ω accordingly to $\rho(\mathbf{x})$. In other words, we introduced a cut-off to take into account the discrete original nature of the problem and avoid divergences. Observe that $\delta_N\to 0$ as $N\to\infty$. The results of the computation may depend upon the regularizing parameter α.

Eq. (5.7.15) provides a recipe for the calculation of the average optimal cost and for the correlation function. In particular, in our approximation we have that

$$\mathscr{C}_N^{(2,\mathrm{RR})}\simeq\int_\Omega C(\mathbf{x},\mathbf{x})\rho(\mathbf{x})\,\mathrm{d}^d x. \tag{5.7.18}$$

If no regularization is required ($\alpha=0$) the previous expression simplifies and we obtain

$$\mathscr{C}_{N,d}^{(2,\mathrm{RR})}\simeq\frac{2}{N}\iint_{\Omega\times\Omega}\rho(\mathbf{x})\left[\rho(\mathbf{y})G_\rho(\mathbf{x},\mathbf{y})-\frac{G_\rho(\mathbf{x},\mathbf{x})}{|\Omega|}\right]\mathrm{d}^d x\,\mathrm{d}^d y. \tag{5.7.19}$$

For $d=1$ the previous expression have a simpler form. Suppose that $\Omega=[a,b]\subset\mathbb{R}$, and a certain probability density distribution $\rho(x)$ on Ω is given. In this case Eq. (5.7.15) and Eq. (5.7.18) have the form

$$C(x,y)=\frac{2}{N}\frac{\Phi_\rho(\min\{x,y\})-\Phi_\rho(x)\Phi_\rho(y)}{\rho(x)\rho(y)}, \tag{5.7.20a}$$

$$\mathscr{C}_{N,1}^{(2,\mathrm{RR})}=\frac{2}{N}\int_a^b\frac{\Phi_\rho(x)(1-\Phi_\rho(x))}{\rho(x)}\,\mathrm{d}\,x, \tag{5.7.20b}$$

where we have introduced the cumulative function

$$\Phi_\rho(x)\coloneqq\int_a^x\rho(\xi)\,\mathrm{d}\,\xi. \tag{5.7.21}$$

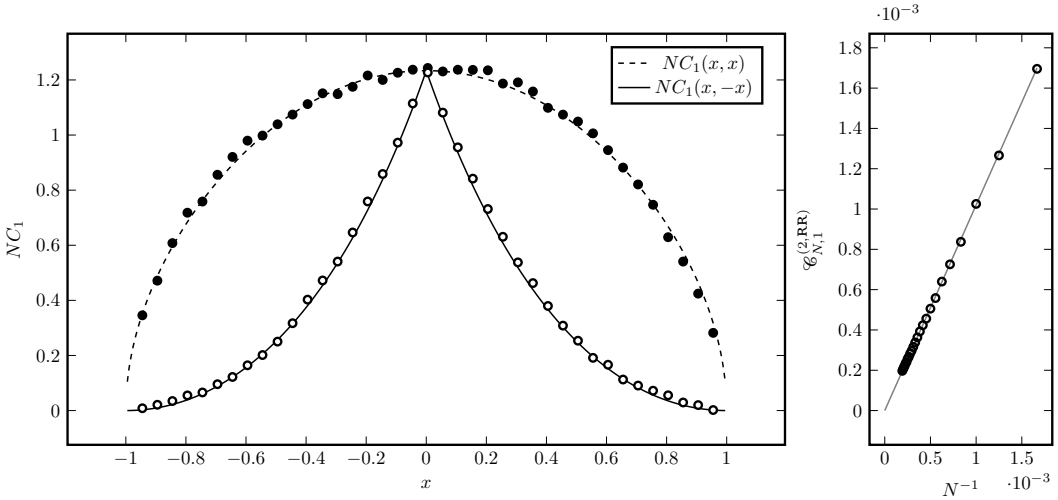


FIGURE 5.7.1.: REB on the real line with points generated using a semi-circle distribution, Eq. (5.7.23). On the left, correlation function $C(x,x)$ and $C(x,-x)$ for $N = 3000$, obtained averaging over 5000 instances of the problem. We compare with the theoretical predictions obtained from Eq. (5.7.20a). On the right, the average optimal cost obtained averaging over 5000 instances. We compare with the theoretical prediction obtained from Eq. (5.7.20b), presented in Eq. (5.7.24).

MATCHING PROBLEM ON THE INTERVAL As application of Eqs. (5.7.20), let us assume, for example,

$$\Omega \equiv [-1,1] \tag{5.7.22}$$

and a semicircle distribution on it,

$$\rho(x) = \frac{2\sqrt{1-x^2}}{\pi} \qquad x \in [-1,1], \tag{5.7.23a}$$

$$\Phi_\rho(x) = 1 + \frac{x\sqrt{1-x^2} - \arccos x}{\pi}. \tag{5.7.23b}$$

Applying Eqs. (5.7.20) we obtain immediately both the correlation function and the average optimal cost. In particular,

$$\mathscr{C}_{N,1}^{(2,\mathrm{RR})} = \frac{1}{N}\left(\frac{\pi^2}{6} - \frac{5}{8}\right) + o\left(\frac{1}{N}\right). \tag{5.7.24}$$

In Fig. 5.7.1 we compare the numerical results with the analytical predictions.

Observe also that Eq. (5.7.20a) provides the correct correlation function for the REB on $\Omega \equiv [0,1]$ with $\rho(x) = \theta(x)\theta(1-x)$. We have

$$C_1(x,y) = \begin{cases} 2\frac{\min\{x,y\}-xy}{N} & (x,y) \in [0,1]^2 \\ 0 & \text{otherwise,} \end{cases} \tag{5.7.25a}$$

$$\mathscr{C}_{N,1}^{(2,\mathrm{RR})} = \frac{1}{3N} + o\left(\frac{1}{N}\right). \tag{5.7.25b}$$

MATCHING PROBLEM ON THE UNIT SQUARE Let us now consider the unit square,

$$\Omega \equiv \Omega_2 = [0,1]^2, \tag{5.7.26}$$

with uniform distribution, $\rho(\mathbf{x}) = 1$. Using Eq. (5.7.15) we can compute $C(\mathbf{x},\mathbf{y})$ as function of the modified Green's function of the Laplacian on the square with Neumann boundary condition $G_s(\mathbf{x},\mathbf{y})$. However, it can be seen that $NC(\mathbf{x},\mathbf{x}) \to \infty$ for $N \to \infty$ and we need to regularize the correlation function proceeding as in the case of the computation on the torus [9]. We eventually obtain

$$\mathscr{C}_{N,2}^{(2,\mathrm{RR})} = \frac{\ln N}{2\pi N} + \frac{c_2^{(1)}}{N} + o\left(\frac{1}{N}\right). \tag{5.7.27}$$

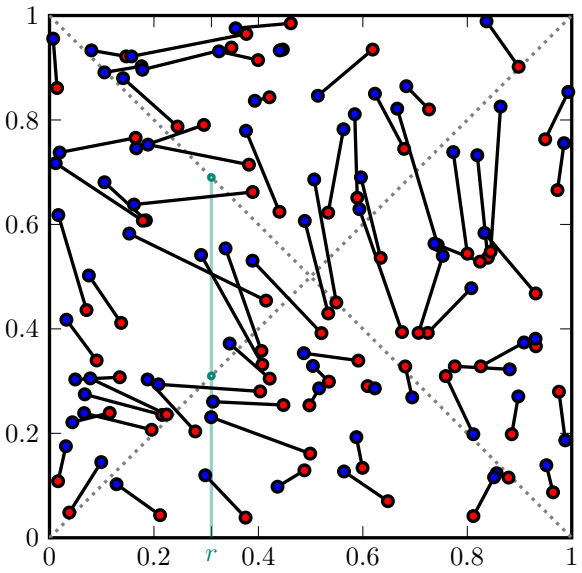


Figure 5.7.2.: Eb on the square. The geometrical meaning of the variable r in Eq. (5.7.28) is also depicted.

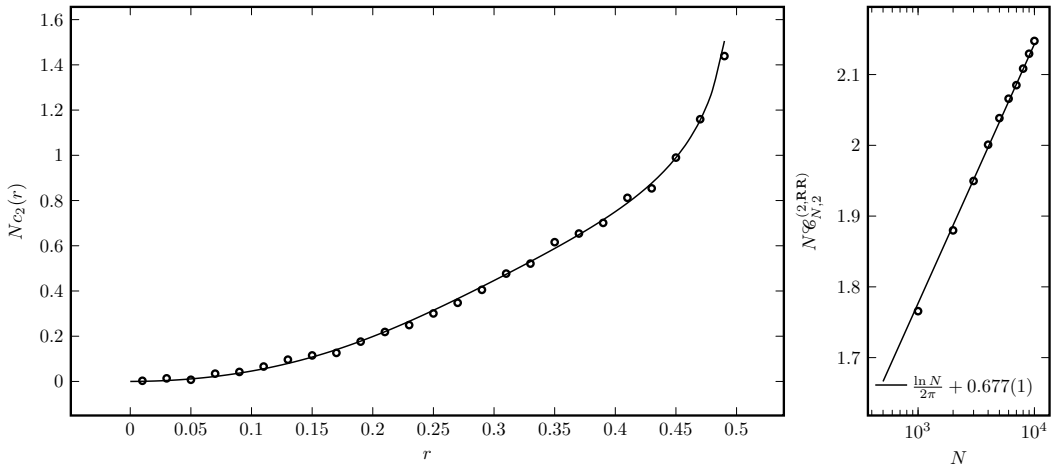


Figure 5.7.3.: Matching problem on the square. On the left, correlation function between points on the diagonals of the square, see Eq. (5.7.28), obtained for $N = 3000$ and averaging over $2 \cdot 10^4$ instances. We compare with our analytical prediction. On the right, we compare our theoretical prediction for the averagge optimal cost, Eq. (5.7.27), with numerical results obtained averaging over $2 \cdot 10^4$ instances. In particular, the value of $c_2^{(1)}$ is obtained by a fit procedure, obtaining $c_2^{(1)} = 0.677(1)$.

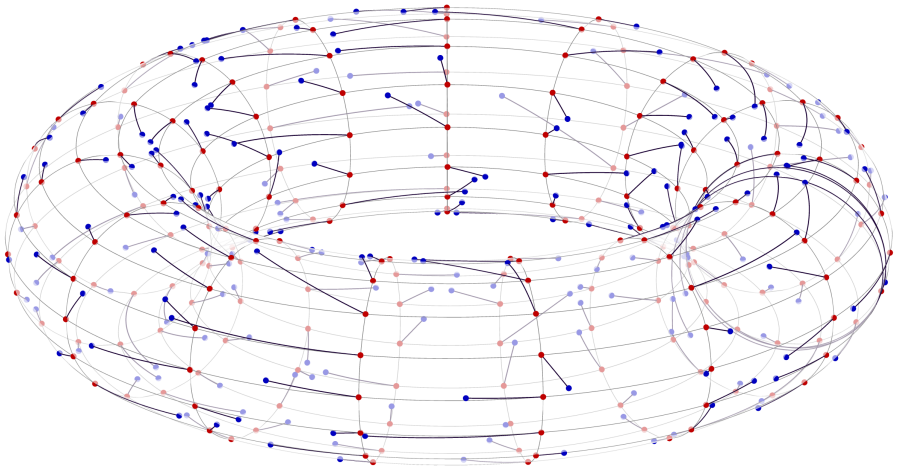


FIGURE 5.7.4.: Optimal GR Euclidean bipartite matching with $N = 225$ and $p = 2$ on the torus.

The leading term is exactly the same obtained for the REB on the 2-dimensional torus. In Fig. 5.7.3 we plotted the numerical results for the average optimal cost, comparing with the prediction in Eq. (5.7.27). Moreover, we compare also our numerical results with the theoretical prediction for $c_2(r)$, defined as follows (see also Fig. 5.7.2)

$$\mathbf{x}_r \coloneqq (r, r), \ \mathbf{y}_r \coloneqq (r, 1-r), \quad c_2(r) \coloneqq NC_2(\mathbf{x}_r, \mathbf{y}_r). \tag{5.7.28}$$

MATCHING PROBLEM ON THE FLAT HYPERTORUS Finally, let us consider $\Omega \equiv \mathsf{T}^d \coloneqq \mathbb{R}^d/\mathbb{Z}^d$. We can restate the results above for this case simply substituting the Neumann boundary conditions in Eq. (5.7.12) and Eq. (5.7.13) with periodic boundary conditions. The Euclidean distance between the points $\mathbf{x} = (x_i)_{i=1,\dots,d}$ and $\mathbf{y} = (y_i)_{i=1,\dots,d}$ in Ω must be understood as the geodetic distance on the flat hypertorus. Assuming uniform distribution, $\rho(\mathbf{x}) = 1$, and $\delta_N = 0$, we obtain our previous results

$$C(\mathbf{x}, \mathbf{y}) = -\frac{2}{N} G_d(\mathbf{x} - \mathbf{y}), \quad \mathscr{C}_{N,d}^{(2,\mathrm{RR})} = -\frac{2}{N} G_d(\mathbf{0}), \tag{5.7.29}$$

where, as above, $G_d(\mathbf{x})$ is the Green's function of the Laplacian on T^d. As we know, the quantity $G_d(\mathbf{0})$ is divergent for $d \geq 2$, but a proper regularization can be performed, as already explained. We therefore completely recover our results in Section 5.5.

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Chapter 6

Conclusions and perspectives

In this thesis we discussed many variants of the random (or stochastic) Euclidean matching problem, a matching problem between points randomly generated on a certain Euclidean domain. We supposed the cost of the matching depending on the Euclidean distances of the matched points only. In particular, we tried to evaluate the average optimal cost of the optimal matching. We introduced also the concept of *correlation function* for the optimal matching, a quantity that, in this optimization problem, is meaningful because of the underlying Euclidean support. We investigated both these quantities (average optimal cost and correlation function) with different methods, inspired by very different research area of mathematics and physics. We stressed the inspiring analogy between some currently hot research topics in measure theory and recent results in statistical physics and theory of optimization problems. In some specific cases, we were able to obtain closed formulas for the average optimal cost or its finite size corrections, and we gave a general recipe for the evaluation of the cost and the correlation function in full generality in the quadratic case. Finally, we were able to find a very interesting connection between the solution of the problem with convex costs in $d = 1$ and the Brownian bridge process. In this sense, the theory of stochastic processes plays also an important role in the study of the random Euclidean bipartite matching problem.

There are still many open problems related to the random Euclidean matching problem that deserve a careful investigation.

A first observation is that many obtained results are not, strictly speaking, rigorous from the mathematical point of view, despite the fact that an excellent agreement with the numerical simulations was found. A rigorous derivation of our results, therefore, can be a quite important and challenging project for further research on this topic.

Another important task is a better evaluation of the corrections to the mean field contributions in the replica approach sketched in Section 5.3. Indeed, we presented the computation for the polygonal contribution, but a more general solution is required, in particular taking care of the delicate zero temperature limit, to obtain a better approximation of the average optimal cost.

Finally, the study performed in the Chapter 5 concerns the random Euclidean matching problem with *convex* cost functions. The concave case is an elusive problem, and, even in one dimension, our approaches fail. The properties of the optimal matching in the concave problem, and the possible correspondence with a different stochastic process in one dimension, deserve, without any doubt, a deeper investigation.

Appendix A

Additional classical results on glasses

A.1. Instability of the replica symmetric solution

A question that arises about the replica symmetric solution for the SK-model is if it is *stable* respect to fluctuations. From a mathematical point of view, the solution is stable if the Hessian matrix $\mathsf{H}(\mathsf{q})$ in Eq. (4.2.29) is *positive definite* in the replica symmetric solution in Eq. (4.2.33), $Q_{\alpha\beta} = q$. The simplest way to verify the stability is therefore by solving the eigenvalue problem

$$\mathsf{H}(\mathsf{Q})\boldsymbol{\eta} = \lambda\boldsymbol{\eta}, \tag{A.1.1}$$

and verify that all eigenvalues are positive. This analysis was performed for the first time by Almeida and Thouless [1]. We will not reproduce here all the steps of their computation, but we will present only the main results. By direct computation we can obtain the following expression for the Hessian matrix

$$H_{(\alpha\beta)(\gamma\delta)} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \beta^2\Gamma^2\left(\left\langle\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\left\langle\sigma^\gamma\sigma^\delta\right\rangle_z\right). \tag{A.1.2}$$

In other words, the structure of the Hessian matrix is the following

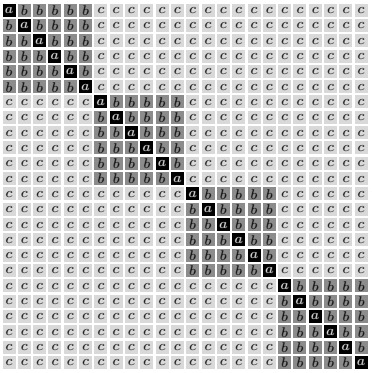
$$H_{(\alpha\beta)(\alpha\beta)} = 1 - \beta^2\Gamma^2\left(1 - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z^2\right) \equiv a, \tag{A.1.3a}$$

$$H_{(\alpha\beta)(\alpha\gamma)} = -\beta^2\Gamma^2\left(\left\langle\sigma^\beta\sigma^\gamma\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\left\langle\sigma^\alpha\sigma^\gamma\right\rangle_z\right) \equiv b, \tag{A.1.3b}$$

$$H_{(\alpha\beta)(\gamma\delta)} = -\beta^2\Gamma^2\left(\left\langle\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\left\langle\sigma^\gamma\sigma^\delta\right\rangle_z\right) \equiv c, \tag{A.1.3c}$$

The Hessiam matrix has therefore a particular symmetric form of the type (here $n = 4$)

$$\mathsf{H} = \tag{A.1.4}$$



In the *paramagnetic phase*, being $q = 0$, a direct computation shows that $b = c = 0$ and $a = 1$, so the solution is stable.

In the *ferromagnetic phase*, Almeida and Thouless [1] first assumed an eigenvector symmetric under the exchange of replica indexes, i.e., in the form

$$\eta^{\alpha\beta} = \eta, \quad \forall\alpha, \beta. \tag{A.1.5}$$

Under this assumption, they found the following corresponding eigenvalue:

$$\lambda_1 = \frac{1 - a - 4b + 3c \pm |1 - a + 4b - 3c|}{2}. \tag{A.1.6}$$

They proved also that the eigenvalue λ_2 obtained for the $n-1$ eigenvectors in the form

$$\eta^{\alpha\beta} \equiv \eta_0(\delta_{\alpha\theta} + \delta_{\beta\theta}) + \eta_1 \tag{A.1.7}$$

for a certain θ (i.e., in each eigenvector one replica is "special"), is such that $\lambda_2 \xrightarrow{n\to 0} \lambda_1$. The third eigenvalue λ_3 is obtained assuming that there are two replicas, of indices ν, θ, such that

$$\eta^{\theta\nu} \equiv \eta_1, \quad \eta^{\theta\alpha} = \eta^{\nu\alpha} \equiv \eta_2, \quad \eta^{\alpha\beta} \equiv \eta_3, \quad \forall \alpha, \beta. \tag{A.1.8}$$

There are $\frac{n(n-3)}{2}$ eigenvectors in this form. Again, they proved that the corresponding eigenvalue is

$$\lambda_3 = a - 2b + c. \tag{A.1.9}$$

It can be seen that, for $Q_{\alpha\beta} \equiv q$, $\lambda_1 = \lim_{n\to 0} \lambda_2$ is always positive. However

$$\lambda_3 = a - 2b + c > 0 \Rightarrow \frac{1}{\beta^2\Gamma^2} > \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\frac{w^2}{2}} \operatorname{sech}^2 (\beta\Gamma\sqrt{q}w + \beta h)\, \mathrm{d}\, w. \tag{A.1.10}$$

The previous inequality defines a region in the (β, h) space in which the replica symmetric solution is *unstable*, corresponding to values (β, h) that violate the inequality. The curve obtained imposing the equality is called *Almeida–Thouless line* (AT-line).

A.2. The TAP approach for glasses

A different but instructive approach to the study of the SK-model was proposed in 1977 by Thouless, Anderson, and Palmer [5]. They obtained a set of equations for the local magnetisation in the SK-model. Given a certain realisation of the SK-model, the free energy can be written as

$$-\beta F[\beta; \mathbf{h}; \mathbf{J}] \coloneqq \ln \left[\sum_{\{\sigma_i\}} \exp \left(\beta \sum_{i \neq j} J_{ij}\sigma_i\sigma_j + \beta \sum_i h_i\sigma_i \right) \right]. \tag{A.2.1}$$

In the previous equation, we denoted by $\mathbf{h} \coloneqq \{h_i\}_{i=1,\dots,N}$ a set of local magnetic field, and by $\mathbf{J} \coloneqq \{J_{ij}\}_{ij}$ the set of coupling constants. Note that here we suppose that the external magnetic field depends on the site position. As known from the general theory, the local magnetisation is given by $m_i \coloneqq \langle \sigma_i \rangle = -\partial_{h_i} F$, F free energy. Let us now perform a Legendre transform, introducing

$$G[\beta; \mathbf{m}; \mathbf{J}] = \max_{h_1,\dots,h_N} \left[F[\beta; \mathbf{h}; \mathbf{J}] + \sum_i m_i h_i \right], \tag{A.2.2}$$

where $\mathbf{m} \coloneqq \{m_i\}_i$. The new functional is a *constrained free energy*, in the form

$$-\beta G[\beta; \mathbf{m}; \mathbf{J}] = \ln \left[\sum_{\{\sigma_i\}} \exp \left(\beta \sum_{i \neq j} J_{ij}\sigma_i\sigma_j + \beta \sum_i h_i(m_i)(\sigma_i - m_i) \right) \right], \tag{A.2.3}$$

where $h_i(m_i)$ are such that $\langle \sigma_i \rangle = m_i$. Let us now expand the previous functional around $\beta = 0$ (infinite temperature). We obtain

$$\begin{aligned} -\beta G[\beta; \mathbf{m}; \mathbf{J}] = \sum_{i=1}^{N} \left[\frac{1 - m_i}{2} \ln \left(\frac{1 - m_i}{2} \right) - \frac{1 + m_i}{2} \ln \left(\frac{1 + m_i}{2} \right) \right] \\ + \beta \sum_{ij} J_{ij} m_i m_j + \frac{\beta^2}{2} \sum_{ij} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) + o(\beta^2). \end{aligned} \tag{A.2.4}$$

Plefka [4] showed that, for the SK-model, the additional terms in $o(\beta^2)$ can be neglected in the $N \to +\infty$. The minimum condition respect to m_i, $\partial_{m_i} G = 0$, gives the celebrated *Thouless–Anderson–Palmer equations* (TAP-equations):

$$m_i = \tanh\left[\beta\left(\sum_{j\neq i} J_{ij} m_j + h_i - m_i \sum_{j\neq i} J_{ij}^2 \beta(1-m_j)^2\right)\right] \tag{A.2.5}$$

In the previous expression an additional term appears respect to the equation for the mean field magnetization in the Ising model (4.2.14), i.e. the so called *Onsager reaction*

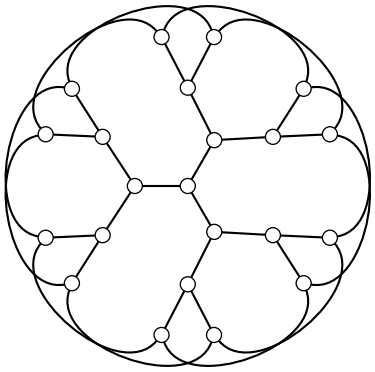
$$h_i^O \coloneqq -m_i \sum_{j\neq i} J_{ij}^2 \beta(1-m_j)^2. \tag{A.2.6}$$

This additional term is not negligible in the spin glass case due to the fact that $J_{ij} \sim \frac{1}{\sqrt{N}}$ and *not* $J_{ij} \sim \frac{1}{N}$ as in the ferromagnetic case.

A.3. The cavity method for spin glasses on a Bethe lattice

The cavity method was originally conceived for the analysis of the ground state of spin glasses on particular lattices. In this Section, therefore, we start from this problem, to identify properly the physics behind the algorithm that the method inspired, following the seminal paper of Mézard and Parisi [3].

Let us first introduce the so called *Bethe lattice*. The Bethe lattice $\mathtt{B}_N^k = \mathrm{Graph}(\mathscr{V};\mathscr{E})$ is a random graph of $N \gg 0$ vertices with fixed connectivity $k+1$. For example, $\mathtt{B}_N^2$ has the structure



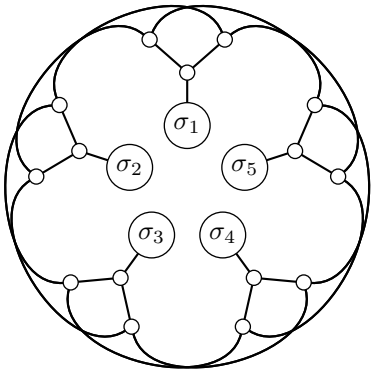
The Bethe lattice $\mathtt{B}_N^k$ has cycles[1] of typical length $\ln N$, and therefore we expect that we can consider the $(k+1)$-Cayley tree as the limiting graph of a Bethe lattice $\mathtt{B}_N^k$ having $N \to \infty$.

We define now a spin model on the Bethe lattice, associating to each vertex $v_i \in \mathscr{V}$ a spin variable $\sigma_i \in \{-1, 1\}$, and to each edge (v_i, v_j) a random coupling J_{ij}. In particular, we suppose that $\{J_{ij}\}_{ij}$ are identically and independently distributed random variables with a certain probability distribution density $\rho(J)$. The spin glass Hamiltonian is simply

$$H_{\mathtt{B}_N^k}[\boldsymbol{\sigma};\mathbf{J}] = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j. \tag{A.3.1}$$

To find the global ground state (GGS) energy $\mathscr{E}_N$ of the system, we benefit of the tree-like structure of the graph. Indeed, let us consider a new graph $\mathtt{G}_{N,q}^k \subseteq \mathtt{B}_N^k$ obtained from $\mathtt{B}_N^k$ decreasing the degree of $q > k$ spins $\{\sigma_1, \dots, \sigma_q\}$ from $k+1$ to k. This spins are called *cavity spins* and surround a "cavity" in the graph. For example, $\mathtt{G}_{N,5}^2$ has the shape

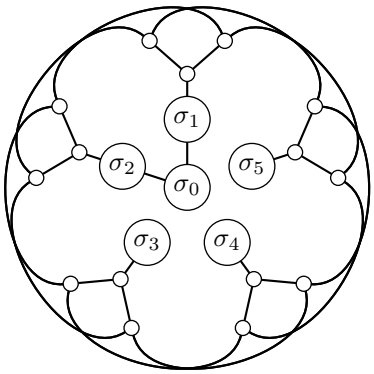
[1] Note that in the literature it is common to identify a Bethe lattice with a Cayley tree, that indeed it is a tree with no cycles.



We suppose now the values of these spins $\sigma_1, \ldots, \sigma_q$ *fixed* and that the GGS energy of this new system, let us call it $\mathscr{C}_N(\sigma_1, \ldots, \sigma_q)$, is related to the old one in the following way

$$\mathscr{C}_N(\sigma_1, \ldots, \sigma_q) - \mathscr{C}_N = \sum_{i=1}^{q} h_i \sigma_i, \tag{A.3.2}$$

for a proper set $\{h_i\}_{i=1,\ldots,q}$ of *local cavity fields*. Due to the random nature of the system, we assume that $\{h_i\}_i$ are i.i.d. random variables as well, with distribution $\varrho(h)$. Now suppose that a new spin σ_0 is added to the lattice and coupled to k cavity spins $\sigma_1, \ldots, \sigma_k$ by $J_1, \ldots, J_k$ respectively,



We fix the value of σ_0 and change the value of $\sigma_1, \ldots, \sigma_k$ in such a way that the new GGS energy is minimized. The energy ϵ_i of each link (σ_i, σ_0) is minimized taking

$$\epsilon_i = \min_{\sigma_i} \left[(-h_i - J_i\sigma_0)\sigma_i\right] =: -a(J_i, h_i) - \sigma_0 b(J_i, h_i). \tag{A.3.3}$$

In this notation, it is clear that the final cavity field on σ_0 is

$$h_0 = \sum_{i=1}^{k} b(J_i, h_i). \tag{A.3.4}$$

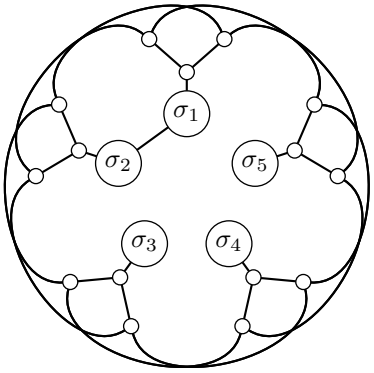
It follows that the recursion relation

$$\varrho(h) = \overline{\prod_{i=1}^{k} \left(\int \varrho(h_i)\, \mathrm{d}\, h_i \right) \delta\left(h - \sum_{i=1}^{k} b(J_i, h_i) \right)} \tag{A.3.5}$$

holds. Suppose now that we are able to solve the previous equation for ϱ. Then, denoting by $\delta\epsilon^1$ the energy shift of the GGS due to a site addition, we have

$$\delta\epsilon_0^1 = -\overline{\prod_{i=1}^{k+1}\left(\int \varrho(h_i)\,\mathrm{d}\,h_i\right)\left(\sum_{j=1}^{k+1} a(J_j,h_j) + \left|\sum_{j=1}^{k+1} b(J_j,h_j)\right|\right)}. \tag{A.3.6}$$

Let us consider back the graph $\mathtt{G}_{N,q}^k$ and let us add, this time, an edge, e.g. connecting σ_1 and σ_2,



Again, the energy shift of the GGS $\delta\epsilon^2$ due to the addition of an edge can be written as

$$\delta\epsilon^2 = -\overline{\prod_{i=1}^{2}\left(\int \varrho(h_i)\,\mathrm{d}\,h_i\right)\max_{\sigma_1,\sigma_2}(h_1\sigma_1 + h_2\sigma_2 + J_{12}\sigma_1\sigma_2)}. \tag{A.3.7}$$

We have now two expression for the shifts in energy due to local alterations of the graph, a vertex addition and an edge addition. The importance of these to quantities is due to the fact that Mézard and Parisi [2] proved that

$$\lim_{N\to\infty}\frac{\mathscr{C}_N}{N} = \delta\epsilon^1 - \frac{k+1}{2}\delta\epsilon^2. \tag{A.3.8}$$

Note that the tree-like structure of the Bethe lattice plays a fundamental role in the derivation of the equations above, allowing us to write recursive relations. The cavity method suggests that, at least on a tree-like graph, we can obtain information about the ground state evaluating the shift in energy after a local modification of the graph and writing down proper self-consistent equations for the distributions of the random quantities. The method, in the formulation presented here, works however only in the replica symmetric hypothesis. In other words, we assume that there are *no local ground states*, i.e., local minima in energy whose corresponding configurations can be obtained from the global minimum ones only by an infinite number of spin flips. This eventuality is not so rare and, in this case, it might be necessary to break the replica symmetry to correctly reproduce the GGS energy.

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Appendix B

The Wiener process and the Brownian bridge process

In the present Appendix we will introduce some fundamental properties of the Wiener process and of the Brownian bridge process. Excellent books on the subject are available in the literature (see, for example, the book of Rogers and Williams [5] or the monograph of Mörters and Peres [3]). Here we summarize the main properties, presenting also some useful probability distributions for the Wiener process and the Brownian bridge process.

B.1. Wiener process and Brownian bridge process

Let us suppose that Ω is the sample space of the process and that $\mathscr{F}$ corresponds to the set of events, σ-algebra over Ω. Recall that a σ-algebra on a certain set A is a collection Σ of subset of A such that $A \in \Sigma$, $a \in \Sigma \Rightarrow A \setminus a \in \Sigma$ and $a, b \in \Sigma \Rightarrow a \cup b \in \Sigma$. If $\mathscr{C}$ is a set of subsets of A, then $\sigma(\mathscr{C})$ is the smallest σ-algebra on A containing $\mathscr{C}$. Let us suppose also that a probability $\mathbb{P}\colon \mathscr{F} \to \mathbb{R}^+$ is given on the space of events. In this space, we denote by $\mathbb{E}(\bullet)$ the expected value of a certain quantity $f\colon \Omega \to \mathbb{R}$,

$$\mathbb{E}(f) \coloneqq \int_\Omega \mathbb{P}(\omega) f(\omega)\, \mathrm{d}\,\omega. \tag{B.1.1}$$

Similarly, if $\mathscr{G} \subset \mathscr{F}$ is a sub σ-algebra of $\mathscr{F}$, $\mathbb{E}(f|\mathscr{G}) \coloneqq \int_\Omega f(\omega)\mathbb{P}(\omega|\mathscr{G})\, \mathrm{d}\,\omega$.

A *Wiener process* W, or *standard Brownian process* $\mathsf{W}\colon \mathbb{R}^+ \to \mathbb{R}$ on this probability space is such that

- $\forall \omega \in \Omega\ \mathsf{W}(0) = 0$, i.e., any realization of the process starts in the origin;
- the map $t \to \mathsf{W}(t)$ is continuous $\forall t \in \mathbb{R}^+$ and $\forall \omega \in \Omega$;
- $\forall t, \tau \in \mathbb{R}^+$ the increment $\mathsf{W}(t+\tau) - \mathsf{W}(t)$ is independent from $\mathsf{W}(u)$, $u \in [0, t]$;
- the increment $\mathsf{W}(t+\tau) - \mathsf{W}(t)$, $\forall t, \tau \in \mathbb{R}^+$, is normally distributed with zero mean and variance τ.

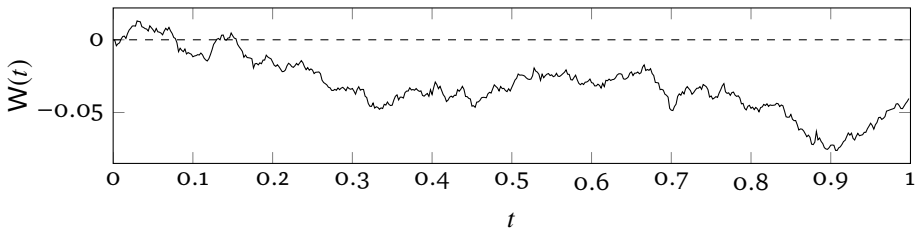

Figure B.1.1.: A realization of a Wiener process.

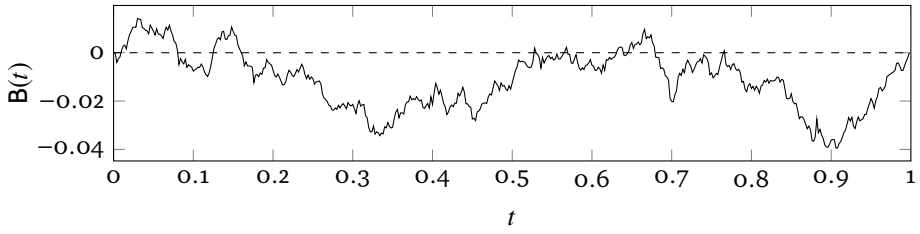

FIGURE B.1.2.: A realization of a Brownian bridge process.

This set of properties uniquely identifies the Wiener process: it can be proved that such a process *exists*. A Wiener process can be introduced also using different probabilistic terminologies connected to its main properties that follow directly from the definition.

Definition B.1.1 (Gaussian process)**.** A *Gaussian process* is a certain stochastic process $\mathsf{X}(t)\colon T \subseteq \mathbb{R} \to \mathbb{R}$ such that, given a set of n element $\{t_1, \dots, t_n\} \in T$, the joint distribution of $(\mathsf{X}(t_1), \dots, \mathsf{X}(t_n))$ is a multivariate Gaussian distribution. It follows that we have to specify only the mean $\mu(t) \coloneqq \mathbb{E}(\mathsf{X}(t))$ and the covariance $\mathbb{E}\left(\mathsf{X}(t)\mathsf{X}(t')\right)$.

The Wiener process is a Gaussian process with mean $\mathbb{E}(\mathsf{W}(t)) = 0$ and covariance $\mathbb{E}\left(\mathsf{W}(t)\mathsf{W}(t')\right) = \min\{t, t'\}$. Most importantly, if a continuous process satisfies the previous relations, it is a Wiener process.

Definition B.1.2 (Martingale)**.** Let us consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as above. We say that a family $\{\mathscr{F}_t\colon t \in \mathbb{R}^+\}$, sub σ-algebras of $\mathscr{F}$, is a *filtration* if, for $s < t$,

$$\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \sigma\left(\bigcup_\tau \mathscr{F}_\tau\right) \subseteq \mathscr{F}, \tag{B.1.2}$$

A certain process $\mathsf{X}(t)$, $t \in \mathbb{R}^+$ on our probability space is said to be *adapted* to the filtration if $\mathsf{X}(t)$ is measurable on $\mathscr{F}_t$. Given an adapted process such that $\mathbb{E}\left(\mathsf{X}(t)\right) < +\infty\ \forall t$ and $\mathbb{E}\left(\mathsf{X}(t)|\mathscr{F}_s\right) = \mathsf{X}(s)$ for $s < t$ almost surely, then the process is called *martingale*.

The Wiener process is a martingale, due to the fact that, denoting by $\mathscr{W}_s \coloneqq \sigma\left(\{\mathsf{W}(\tau)\colon \tau \leq s\}\right)$ the filtration of the probability space, $\mathbb{E}\left(\mathsf{W}(t) - \mathsf{W}(s)|\mathscr{W}_s\right) = 0 \Rightarrow \mathbb{E}\left(\mathsf{X}(t)|\mathscr{W}_s\right) = \mathsf{X}(s)$ directly from the defining properties of the Wiener process.

Definition B.1.3 (Markov process)**.** An adapted process $\mathsf{X}(t)$, $t \in \mathbb{R}^+$, with filtration $\{\mathscr{F}_s\}_{s\in\mathbb{R}^+}$, is *Markov process* if there exists a Markov kernel $p_{\mathsf{X}}(\tau, \mathsf{X}(s), A)$ for A open subset of $\mathbb{R}$, such that

$$\Pr\left(\mathsf{X}(t) \in A|\mathscr{F}_s\right) = p_{\mathsf{X}}(t - s, \mathsf{X}(s), A). \tag{B.1.3}$$

The Wiener process is a Markov process, having

$$p_{\mathsf{W}}(\tau, y, (x, x + \mathrm{d}\,x)) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(x-y)^2}{2\tau}\right) \mathrm{d}\,x =: \rho(x - y; \tau)\,\mathrm{d}\,x. \tag{B.1.4}$$

It is well known that the probability density $\rho(x, \tau)$ is the fundamental solution of the heat equation

$$\partial_t \rho(x, t) = \frac{1}{2}\partial_x^2 \rho(x, t), \tag{B.1.5}$$

obtained by Einstein in his celebrated work on Brownian diffusion.

A stochastic process strictly related to the Wiener process is the *Brownian bridge process*. A Brownian bridge $\mathsf{B}(t)$ is a Gaussian process on the unit interval $[0, 1]$ such that $\mathbb{E}(\mathsf{B}(t)) = 0$ and $\mathbb{E}(\mathsf{B}(s)\mathsf{B}(t)) = \min\{s, t\} - st$. It can be written in terms of a Wiener process on the unit interval as

$$\mathsf{B}(t) \coloneqq \mathsf{W}(t) - t\mathsf{W}(1), \quad t \in [0, 1]. \tag{B.1.6}$$

It can be proved that the previous process can be obtained simply conditioning a Wiener process to be equal to zero for $t = 1$. We can introduce also the Gaussian stochastic process

$$\mathsf{B}^{(-1)}(t) \coloneqq \int_0^t \mathsf{B}(s)\,\mathrm{d}\,s, \quad t \in [0,1]. \tag{B.1.7}$$

It is easily proved that

$$\mathbb{E}\left(\mathsf{B}^{(-1)}(t)\right) = 0, \quad \mathbb{E}\left(\left[\mathsf{B}^{(-1)}(1)\right]^2\right) = \frac{1}{12}. \tag{B.1.8}$$

Finally, the *zero-area Brownian bridge* can be defined as

$$\mathsf{B}^0(t) = \mathsf{B}(t) - \mathsf{B}^{(-1)}(1). \tag{B.1.9}$$

The previous process has

$$\mathbb{E}\left(\mathsf{B}^0(t)\right) = 0, \quad \mathbb{E}\left(\mathsf{B}^0(s)\mathsf{B}^0(t)\right) = \frac{1}{2}\left(|s-t| - \frac{1}{2}\right)^2 - \frac{1}{24}. \tag{B.1.10}$$

B.2. Probability distributions

Many mathematical results are available for the probability distributions of some interesting quantities related to the Wiener process or the Brownian bridge process on the unit interval. We collect here some of these results, without proving them (for further details and more general results, see the detailed paper of Beghin and Orsingher [1]). The probability analysis performed is usually carried on through the so-called *reflection principle*. Using the reflection principle, it can be proved that

$$\Pr\left(\sup_{\tau\in(0,t)} \mathsf{W}(\tau) > W\right) = 2\int_W^\infty \frac{\mathrm{e}^{-\frac{z^2}{2t}}}{\sqrt{2\pi t}}\,\mathrm{d}\,z. \tag{B.2.1}$$

Similarly we have

$$\Pr\left(\sup_{\tau\in(0,t)} \mathsf{W}(\tau) > W | \mathsf{W}(t) = w\right) = \begin{cases} \exp\left(-\frac{2W(W-w)}{t}\right) & W > w, \\ 1 & W < w. \end{cases} \tag{B.2.2}$$

It follows that, if $w = 0$ and $t = 1$ we have the probability distribution for the sup of the Brownian bridge process,

$$\Pr\left(\sup_{\tau\in(0,1)} \mathsf{B}(\tau) > B\right) = \mathrm{e}^{-2B^2}. \tag{B.2.3}$$

For the absolute value of the Wiener process we have that

$$\Pr\left(\sup_{\tau\in(0,t)} |\mathsf{W}(\tau)| < W | \mathsf{W}(t) = w\right) = \sum_{k\in\mathbb{Z}} (-1)^k \exp\left(-\frac{2kW(kW-w)}{t}\right). \tag{B.2.4}$$

For $w = 0$ and $t = 1$ we have

$$\Pr\left(\sup_{\tau\in(0,1)} |\mathsf{B}(\tau)| < B\right) = \sum_{k\in\mathbb{Z}} (-1)^k \exp\left(-2k^2B^2\right). \tag{B.2.5}$$

Darling [2] proved that for the zero-area Brownian bridge the following distribution holds:

$$\Pr\left(\mathsf{B}^0(t) < B\right) = \frac{4\sqrt{\pi}}{3}\sum_{n=1}^\infty \frac{1}{\alpha_n}\psi\left(\frac{\sqrt{8}B}{3\alpha_n}\right), \quad B > 0, \tag{B.2.6}$$

where $0 < \alpha_1 < \cdots < \alpha_n < \ldots$ are the zeros of the combination of Bessel functions

$$f(\alpha) \coloneqq J_{\frac{1}{3}}(\alpha) + J_{-\frac{1}{3}}(\alpha), \tag{B.2.7}$$

whilst $\psi(x)$ is the solution of the following integral equation:

$$\int_0^\infty \mathrm{e}^{-\lambda x}\,\psi(x)\,\mathrm{d}\,x = \mathrm{e}^{-\lambda^{\frac{2}{3}}}\,. \tag{B.2.8}$$

The theorem of Darling is not trivial at all, both in the derivation and in the final result. The explicit treatment of the probability distribution density is indeed quite involved. More implicit results on the Laplace transform of the probability distribution of $\mathsf{B}^{(-1)}$ have been obtained by Perman and Wellner [4], but we do not present them here.

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Education and employment

- **Postdoctoral fellow** (since February 2015), Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro (Brasil).
- **Ph.D. degree** in Theoretical Physics (2012–2015), University of Pisa, Pisa (Italy). Supervisor: Sergio Caracciolo.
- **Master degree** in Theoretical Physics (2009–2011), University of Salento, Lecce (Italy). Graduated with honors. Supervisors: Rosario Antonio Leo and Piergiulio Tempesta.
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School, visits and conferences

- *Conference on Entanglement and Non–Equilibrium Physics of Pure and Disordered Systems* at ICTP (2016), Trieste (Italy).
- *26th IUPAP International conference on Statistical Physics – Statphys26* (2016), Lyon (France).
- *31st International Colloquium on Group Theoretical Methods in Physics* (2016), Rio de Janeiro (Brazil).
- *International workshop of foundations of complexity* (2015), Rio de Janeiro (Brazil).
- *New Trends in Statistical Mechanical Foundations of Complexity* (2015), Erice (Italy).
- *Statistical Field Theories Lectures* at GGI (2014), Florence (Italy).
- *Beg Rohu Summer School* (2013), St Pierre Quiberon (France).
- *Physics and Mathematics of Nonlinear Phenomena* (2013), Gallipoli (Italy).
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Publications

In refereed journals

1. R.A. Leo, G. Sicuro and P. Tempesta, *A theorem on the existence of symmetries of fractional PDEs*, Comptes rendus mathématique, 354(3), p. 219–222 (2014).
2. S. Caracciolo, C. Lucibello, G. Parisi and G. Sicuro, *Scaling hypothesis for the Euclidean bipartite matching problem*, Physical Review E, 90(1), 012118 (2014).
3. S. Caracciolo and G. Sicuro, *One-dimensional Euclidean matching problem: Exact solutions, correlation functions, and universality*, Physical Review E, 90(4), 042112 (2014).
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5. G. Sicuro, P. Tempesta, A. Rodríguez, C. Tsallis, *On the robustness of the q–Gaussian family*, Annals of Physics, 363, p. 316–336 (2015).

6. S. Caracciolo and G. Sicuro, *Quadratic stochastic Euclidean bipartite matching problem*, Physical Review Letters, 115, 230601 (2015).

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8. G. Sicuro, D. Bagchi and C. Tsallis, *On the connection between linear combination of entropies and linear combination of extremizing distributions*, Physics Letters A, 380, 2025–2030 (2016).

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1. R.A. Leo, G. Sicuro and P. Tempesta, *A foundational approach to the Lie theory for fractional order partial differential equations*, ArXiv preprint, 1405.1017 (2016).

Others

Referee for *Annalen der Physik*, *Physical Review E* and *Theoretical and Mathematical Physics*.