Tesi di dottorato

Settembre 2008

Integrals Of Motion:
a Minimal and a Logarithmic CFT

Alessandro Nigro

Matr. ???

Codice P.A.C.S.: 11.25.Hf

Tutore:

Prof. Sergio Caracciolo

A.A. 2007/2008

Integrals Of Motion:
a Minimal and a Logarithmic CFT

Dedicato ai miei genitori, for my parents.

Riassunto

Contents

Chapter 1

Conformal Invariance in Field Theory

In this chapter we will present a brief overview of Conformal Field Theory fundamentals, with particular attention to its connection to Statistical Mechanics and Quantum Field Theory, indeed CFT is a powerful tool capable in many situations of shedding light on connections between SM and QFT.

We will be particularly interested in unitary CFTs, and more specifically in Minimal Models since they have been proved to be in correspondence with a wide class of solvable models in SM, the simplest of which is the Ising Model.

Infact at the end of the chapter we will show how all we have said about CFT perfectly applies to the Ising Model, which will be presented both as a Statistical Mechanical model and as a Fermionic Field Theory (we will not talk about its bosonic representation which is obtained as a particular limit of the $\lambda\phi^4$ theory).

Throughout this chapter the main reference (unless otherwise stated) will be [1].

1.1 The Conformal Group and Its Algebra

The conformal group is the group of coordinate transformations under which the metric tensor gets rescaled by a position-dependent Ω factor. That is, if $x \to x'$, then $g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$, where $g'_{\mu\nu} = \Omega(x) g_{\mu\nu}(x)$.

Now, if we define $v \cdot w = g_{\mu\nu} v^\mu w^\nu$ we are led to conclude that the angle between two vectors is preserved:

$$\cos\theta = \frac{v \cdot w}{\sqrt{v^2 w^2}} \to \frac{\Omega v \cdot w}{\Omega \sqrt{v^2 w^2}} \ .$$

1

Some properties of such transformations are more easily derived through their infinitesimal form:

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \qquad , \qquad |\epsilon(x)| \ll 1 \tag{1.1.1}$$

Now,

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\epsilon^{\mu} \tag{1.1.2}$$

so that to first order in ϵ:

$$\delta g_{\mu\nu} = -2\partial_{(\mu}\epsilon_{\nu)} \tag{1.1.3}$$

The requirement that the transformation be conformal is readily seen to sound as:

$$2\partial_{(\beta}\epsilon_{\alpha)} = \Psi g_{\alpha\beta} \tag{1.1.4}$$

which is known as the Conformal Killing Equation. By taking the trace of both sides of (1.1.4) we see that $\Psi = \frac{2}{D}\partial \cdot \epsilon$ (D is the underlying space dimension) and then

$$\Omega = 1 + \Psi \tag{1.1.5}$$

By taking the 4-divergence of (1.1.4) it follows that:

$$\Box\epsilon_{\beta} + \frac{(D-2)}{2}\partial_{\beta}\Psi = 0 \quad , \text{ where } \Box = g_{\mu\nu}\partial^{\mu}\partial^{\nu} \tag{1.1.6}$$

and repeating the same trick we conclude:

$$(D-1)\Box\Psi = 0 \qquad \rightarrow \qquad \Box\Psi = 0 \quad , \quad D > 1 \tag{1.1.7}$$

Now, applying ∂_{λ} to (1.1.6) and simmetrizing we obtain, after using (1.1.7):

$$(D-2)\partial_{\lambda}\partial_{\beta}\Psi = 0 \tag{1.1.8}$$

This, for $D > 2$ implies that $\Psi \sim \partial \cdot \epsilon$ is at most quadratic in x so we have only a finite set of generators for conformal transformations.

For $D = 2$ the last equation is not necessarily true so that we can define a Global Conformal Group (and algebra) like in higher dimensions, but beside this structure we have a Local Conformal Algebra which happens to be ∞-dimensional (a property which makes 2 dimensional conformal field theories exactly solvable as we will see).

In the case $D > 2$ we can write:

$$\epsilon^{\mu} = \begin{cases} a^{\mu} & \text{(traslations)} \\ \omega^{\mu}{}_{\nu}x^{\nu} & \text{(rotations, } \omega \text{ skew symmetric)} \\ \lambda x^{\mu} & \text{(dilatations)} \\ b^{\mu}x^2 - 2x^{\mu}b \cdot x & \text{(special conformal)} \end{cases} \tag{1.1.9}$$

So that by counting the parameters we have that the dimension of the $D > 2$ conformal group is $\frac{(D+1)(D+2)}{2}$.

The finite transformations are all obvious but the special conformal ones for which:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu \quad \rightarrow \quad \delta\left(\frac{x^\mu}{x^2}\right) = b^\mu \tag{1.1.10}$$

so that by integrating the equation still holds, and keeping higher order terms we find:

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + b^2 x^2 + 2b \cdot x} \tag{1.1.11}$$

1.1.1 $D = 2$ Conformal Algebra

In this case we adopt complex coordinates $z, \overline{z}$ in the $\mathbb{R}^2$ plane:

$$\begin{cases} z = x + iy \\ \overline{z} = x - iy \end{cases} \tag{1.1.12}$$

In these coordinates the Conformal Killing Equation (1.1.4) takes the form of Cauchy-Riemann Equations so that $\epsilon(z)$ is holomorphic.

Under an holomorphic transformation $w = f(z)$ we have:

$$ds^2 = dz\,d\overline{z} \quad \rightarrow \quad \left|\frac{\partial f}{\partial z}\right|^2 dz\,d\overline{z} \quad, \quad \Omega = \left|\frac{\partial f}{\partial z}\right|^2 \tag{1.1.13}$$

If we write the infinitesimal version of this transformation $z \rightarrow z + \epsilon(z)$, and expand $f(z)$ as:

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n \tag{1.1.14}$$

It makes sense to take as a basis for such transformations $\epsilon_n = -z^{n+1}$, so that:

$$\delta z = [l_n, z] = -z^{n+1} \qquad n \in \mathbb{Z} \tag{1.1.15}$$

where the l_n are the generators of the infinitesimal conformal transformations $z \rightarrow z + \epsilon_n(z)$, satisfying:

$$[l_n, l_m] = (n - m)l_{n+m} \tag{1.1.16}$$

It is readily understood that we have analogous relations for antiholomorphic transformations with their generators satisfying

$$[l_m, \overline{l}_n] = 0 \qquad \forall \quad n, m \in \mathbb{Z} \tag{1.1.17}$$

This is the classical local conformal algebra, which is, as we will soon see, the $c = 0$ case of a Virasoro Algebra.

If we furthermore impose a regularity requirement on the Local Conformal Algebra so that changes of coordinates are well defined as $z \to 0, \infty$, that is on the whole Riemann Sphere $\mathbf{S}^2 \cup \infty$, we can easily realize that only the $\mathbf{SL}(2, \mathbb{C})/\mathbb{Z}_2$ subgroup, generated by l_0, l_1, l_{-1} and their antiholomorphic counterparts, survives as a global transformation generator. Specifically we have that $l_{-1}, \bar{l}_{-1}$ generate left and right traslations, $l_1, \bar{l}_1$ special conformal, whereas $l_0 + \bar{l}_0$ generates the scalings and $i(l_0 - \bar{l}_0)$ the rotations.

We also notice that since the 2D conformal algebra is a direct sum of two disjoint algebras we have that conformal transformations are factorized into holomorphic and antiholomorphic parts.

1.2 Stress Tensor, Symmetries and Conservation Laws

Let us consider now a classical field theory with an action

$$S = \int d^D x \sqrt{g} \mathcal{L}(\phi, \nabla^c \phi)$$

built on some fields and their derivatives respecting principles such as locality, causality and invariance under some group of transformations. We know from Nöether's theorem that to each invariance of the lagrangian we can associate a conserved current J.

We want to produce an expression for J, restricting our consideration to consequences of conformal invariance. As it might be noticed we are momentarily releasing the requirement of working in a flat space for the sake of generality.

We define the Stress Tensor variationally through:

$$\delta S = -\frac{1}{2} \int d^D x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{1.2.1}$$

now, under a conformal transformation $x \to x + \epsilon$ we have, using (1.1.4):

$$\delta g_{ab} = -2 \nabla_{(a} \epsilon_{b)} = -\frac{2}{D} \nabla^d \epsilon_d g_{ab} \tag{1.2.2}$$

so that

$$\delta S = \int d^D x \sqrt{g} T^{ab} \nabla_{(a} \epsilon_{b)} \tag{1.2.3}$$

This tells us that T^{ab} is a symmetric tensor. Now if we pose $J^a = T^{ab} \epsilon_b$ we have:

$$\delta S = \int d^D x \sqrt{g} (\nabla^a J_a - \epsilon_b \nabla_a T^{ab}) \tag{1.2.4}$$

and if $J \to 0$ quickly enough as $|x| \to \infty$ we conclude from traslation invariance that the Stress Tensor is conserved.

$$\nabla_a T^{ab} = 0 \tag{1.2.5}$$

Now, if we consider (1.2.3) and use (1.1.4) dilatation invariance tells us that:

$$T^a_a = T = 0 \tag{1.2.6}$$

so that T^{ab} is traceless as a consequence of scaling invariance.

Special Conformal invariance tells us nothing more about properties of T^{ab}.

Moreover, the properties so far estabilished are sufficient to conclude that the special conformal current is already conserved as a consequence of scaling plus traslation invariance, so that the special conformal symmetry comes as a gift of scaling invariance. This is not a coincidence but is a well known property of statistical mechanical models near a second order phase transition.

The interpretation of T^{ab} itself as a current and the possibility to build conformal currents from it gives us a natural way to implement conformal symmetry at the quantum level.

1.3 Conformal Field Theories

We will now define a conformal field theory as a field theory satisfying the following properties:

1. There exists a set of fields $\{A_i\}$,which is usually infinite, and $\partial_\mu A_i \in \{A_i\}$

2. There exists another set of fields $\{\phi_j\} \subset \{A_i\}$, called *Quasi Primary* which transform as tensor densities under conformal transformations, that is:

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/D} \phi_j(x') \tag{1.3.1}$$

   where Δ_j is the dimension of ϕ_j, and $\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-D/2}$.

   As a consequence we have a covariance property of the correlation functions under conformal transformations.In the sense that:

$$\langle \phi_1(x_1)\ldots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/D}_{x=x_1} \ldots \left|\frac{\partial x'}{\partial x}\right|^{\Delta_n/D}_{x=x_n} \langle \phi_1(x'_1)\ldots\phi_n(x'_n)\rangle \tag{1.3.2}$$

3. All the remaining fields in the family $\{A_i\}$ are expressible as linear combinations of quasi primary fields and their derivatives.

4. There exists a vacuum invariant under global conformal transformations (in $D = 2$ this means $\mathbf{SL}(2, \mathbb{C})$ invariance).

This definition is valid only for scalar fields, we will see how in 2 dimensions, due to the decoupling of holomorphic and antiholomorphic parts we will be able to include also fields with a spin s.

1.3.1   Form of the Correlators in D Dimensions

The second property exposed above has the virtue of completely determining 2 and 3 point functions for quasi primary fields in a conformal field theory, while higher order correlators will be completely fixed for $D = 2$, as we will see, by the requirement of Minimality for the representation of the conformal group acting on the Hilbert space of the theory.

Before going any further it will be convenient to spend some words about the conformal invariants upon which a correlator may depend; so we fix N points $x_1 \ldots x_N$ in our D-dimensional space and look for a traslational invariant to find out that the allowed dependence must be of the form $(x_i - x_j)$, if we now look for a rotational and traslational invariant we are forced to choose $r_{ij} = |x_i - x_j|$, again adding scaling invariance to the other two requirements we are left with the invariant $\frac{r_{ij}}{r_{kl}}$.

Special conformal invariance is a more complicated matter but it can be settled by noting that the following relation holds:

$$|x_i' - x_j'|^2 = \frac{|x_i - x_j|^2}{(1 + b^2 x_i^2 + 2b \cdot x_i)(1 + b^2 x_j^2 + 2b \cdot x_j)} \tag{1.3.3}$$

so that the quantity:

$$\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}} \tag{1.3.4}$$

is invariant under all the global conformal group, such a quantity is called a *cross-ratio*; the number of indipendent cross ratios is $N(N - 3)/2$.

By using the covariance properties of the correlators under conformal transformations it is straightforward to show that 2 and 3 point functions are of the form:

$$\langle \phi(x_1)\phi(x_2) \rangle = \begin{cases} \frac{C_{12}}{r_{12}^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{1.3.5}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.3.6}$$

Higher order correlators begin to have a dependence on arbitrary functions of the indipendent cross-ratios, these functions, as we will see, can be determined for $D = 2$ as solutions of differential equations due to the existence of so-called null states in the Hilbert space. For example, by considering that for $N = 4$ we have only 2 independent cross-ratios, and imposing all the constraints coming from global conformal invariance we can show that:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}}\right) \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\sum_{i=1}^{4}\Delta_i/3} \tag{1.3.7}$$

1.4 $D = 2$ Conformal Field Theories

The $D = 2$ situation shows some little difference from higher dimensional cases because of the factoring of holomorphic and antiholomorphic transformations that takes place in the conformal group.

For this reason it's more convenient to choose two indipendent conformal dimensions h and $\overline{h}$ such that under a transformation of the group $z \to f(z)$ a primary field changes as:

$$\phi(z, \overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\overline{\partial f}}{\overline{\partial z}}\right)^{\overline{h}} \phi(f, \overline{f}) \tag{1.4.1}$$

We recover the previous definitions in the special case in which $h = \overline{h} = \Delta_\phi/2$, this situation corresponds to a spinless field since we will be quite soon able to show that $s = h - \overline{h}$ (this implies some more knowledge about the algebraic structure of the Hilber Space, which we will soon provide).

Now, under an infinitesimal transformation $z \to z + \epsilon(z)$ it is simple to show that:

$$\phi(z, \overline{z}) \to \phi(z, \overline{x}) + ((h\partial\epsilon + \epsilon\partial) + (\overline{h}\partial\overline{\epsilon} + \overline{\epsilon}\overline{\partial}))\phi(z, \overline{z}) \tag{1.4.2}$$

that is

$$\delta_{\epsilon\overline{\epsilon}}\phi(z, \overline{z}) = ((h\partial\epsilon + \epsilon\partial) + (\overline{h}\partial\overline{\epsilon} + \overline{\epsilon}\overline{\partial}))\phi(z, \overline{z}) \tag{1.4.3}$$

If we now consider that

$$\delta_\epsilon\phi := [Q, \phi] \tag{1.4.4}$$

where Q is some conformal charge yet to be explicitly identified (but already introduced at a classical level), we can with ease derive a set of differential equations for correlation functions by setting $\epsilon = z^n$, which for $n = 0, 1, 2$ is enough to completely determine 2 and 3 point functions as above. For example the 2 point function satisfies:

$$((h_1\partial_1\epsilon(z_1) + \epsilon(z_1)\partial_1) + (\overline{h}_1\overline{\partial}_1\overline{\epsilon}(\overline{z}_1) + \overline{\epsilon}(\overline{z}_1)\overline{\partial}_1) + (h_2\partial_2\epsilon(z_2) +$$
$$+ \epsilon(z_2)\partial_2) + (\overline{h}_2\overline{\partial}_2\overline{\epsilon}(\overline{z}_2) + \overline{\epsilon}(\overline{z}_2)\overline{\partial}_2))\langle\phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)\rangle = 0 \tag{1.4.5}$$

without further comments we shall write the 2 and 3 points correlators as:

$$\langle\phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)\rangle = \begin{cases} \frac{C_{12}}{z_{12}^{2h}\overline{z}_{12}^{2\overline{h}}} & h_1 = h_2 = h,\ \overline{h}_1 = \overline{h}_2 = \overline{h} \\ 0 & \text{otherwise} \end{cases} \tag{1.4.6}$$

$$\langle\phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)\phi_3(z_3, \overline{z}_3)\rangle =$$
$$= \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3+h_1} z_{13}^{h_3+h_1-h_2}\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3}\overline{z}_{23}^{\overline{h}_2+\overline{h}_3+\overline{h}_1}\overline{z}_{13}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}} \tag{1.4.7}$$

1.4.1 Quantization for $D = 2$ and the OPE

To truly quantize our field theory we need operator charges to use as generators of continuous transformations for the system, such as traslations, rotations and time evolution. We may want to start from an underlying manifold that is somehow more familiar such as the infinite cylinder, with a spatial periodic coordinate $\sigma_1 \in [0, 2\pi]$, and a time $\sigma_0 \in \mathbb{R}$ (this is the case of String Theory's world sheet, or more simply of a field theory where we have compactified a dimension in order to mitigate the possible infrared divergences); we can now map this manifold (which has a trivial Euclidean metric) to the complex plane through the exponential map:

$$z = e^{\sigma_0 + i\sigma_1} \tag{1.4.8}$$

Now the remote past is represented by the origin of the complex plane, and the future is the infinity point; equal time sections become circles of constant radius about the origin and so time flows radially outward from $z = 0$.

What suggests us that we are going in the right way is the fact that now the generator of time evolution for the system is indeed the dilatation generator of the complex plane; spatial traslations on the cylinder are generated in the plane by the rotation generator.

In this context it is natural to look at the stress tensor as the fundamental object of our theory since as we showed previously a theory which is invariant under conformal transformations possesses a current J^a, built from the stress tensor, whose divergence is closely related to the stress tensor's divergence and its trace, as it can be seen by combining (1.2.3) and (1.2.4), so that since T^{ab} is conserved and traceless, J^a is also conserved. In complex coordinates (1.1.12) the conservation law for the stress tensor reads:

$$\overline{\partial} T_{zz} = 0 \quad \partial T_{\overline{z}\overline{z}} = 0 \tag{1.4.9}$$

since the tracelessness condition translates into:

$$T_{z\overline{z}} = T_{\overline{z}z} = 0 \tag{1.4.10}$$

It is then natural to define the charges as:

$$Q_{\epsilon\overline{\epsilon}} = \frac{1}{2\pi i} \oint (T(z)\epsilon(z)dz + \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})d\overline{z}) \tag{1.4.11}$$

where our contour of integration is an equal time slice and is counterclockwise oriented for both z and $\overline{z}$.

Now that we have a charge, we can look at it as originating (1.4.3) through the commutator (1.4.4). The problem now is to define what we mean by commutator in this context; since our charge is given as a slice integral of a locally defined expression which we may want to make sense as an insertion inside a Path Integral beside other fields evaluated at other

points, we should remember that Green Functions are generally guaranteed to converge for time ordered products of operators so that we are led to introduce a Radial Ordering:

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases} \tag{1.4.12}$$

In this way we will define the equal time commutator as:

$$[\oint_{E.T.slice} dz A(z), B(w)] := \oint dz R(A(z)B(w)) \tag{1.4.13}$$

where in the righthand side the contour of integration is a small circle around w obtained as a difference of two contours centered about the origin and avoiding w so that in one case $|z| < |w|$ and in the other case $|z| > |w|$. From now on the radial ordering symbol will be omitted.

With our new definition of $Q_{\epsilon\bar{\epsilon}}$ (1.4.11) and the explicit expression for the variation of a primary field $\phi(z,\bar{z})$ (1.4.3) it is immediate to infere that the product between $T(z)$ and $\phi(w,\bar{w})$ must have the following short distance singular behaviour, which we will call OPE (Operator Product Expansion) between T and ϕ:

$$T(z)\phi(w,\bar{w}) = \frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)} + \dots \tag{1.4.14}$$

with a similar expression for the antiholomorphic part.

The OPE is not a peculiarity of Conformal Field Theories, it is quite common to find short distance singularities as two operators approach one another as insertions of a path integral, in general if we have two operators A,B approaching one another and a complete set of local operators O_i, we can write:

$$A(x)B(y) = \sum_i C_i(x-y)O_i(y) \tag{1.4.15}$$

and if all the operators in this expression have definite scaling properties, the functions C_i are constrained to behave as:

$$C_i \sim \frac{1}{|x-y|^{\Delta_A+\Delta_B-\Delta_{O_i}}} \tag{1.4.16}$$

This is infact the case of CFTs where in addition the OPE defines an associative algebra that is fully characterized by the algebraic structure of the Hilbert space that will allow us to exactly express the OPE between two primary fields as a sum over conformal families whose coefficients will be related to the 3 point function's coefficients C_{ijk} in a very simple way.

1.4.2 Conformal Ward Identities

We have already met some forms of Ward Identities, although we did not mention it when we produced differential equations for correlation functions as a consequence of (1.4.3), now we know that the natural way to obtain infinitesimal transfromations is to commute the fields with the right charges so that:

$$\delta_\epsilon(\phi_1(x_1,)\ldots\phi_n(x_n)) = [Q_\epsilon, \phi_1(x_1)\ldots\phi_n(x_n)] = \sum_{k=1}^{n}(\phi_1(x_1))\ldots[Q_\epsilon, \phi_k(x_k)]\ldots\phi_n(x_n))$$

(1.4.17)

When we defined the charge in (1.4.11), we could as well have pointed out that $Q_{\epsilon\overline{\epsilon}}$ can be decomposed as a sum of two separate holomorphic and antiholomorphic charges (obviously defined):

$$Q_{\epsilon\overline{\epsilon}} = Q_\epsilon + Q_{\overline{\epsilon}}$$

(1.4.18)

So that now the meaning of our equations is made clear and we can proceed to put (1.4.17) inside an expectation value and to substitute the espression for Q_ϵ, and then use the OPE (1.4.14) to obtain:

$$\begin{aligned}
\Big\langle \oint \frac{dz}{2\pi i}\epsilon(z)T(z)\phi_1(w_1,\overline{w}_1)\ldots\phi_n(w_n,\overline{w}_n)\Big\rangle = \\
= \sum_{k=1}^{n}\oint \frac{dz}{2\pi i}\epsilon(z)\Big(\frac{h_k}{(z-w_k)^2}+\frac{\partial_{w_k}}{(z-w_k)}\Big)\big\langle\phi_1(w_1,\overline{w}_1)\ldots\phi_n(w_n,\overline{w}_n)\big\rangle
\end{aligned}$$

(1.4.19)

or, in a non integrated version:

$$\big\langle T(z)\phi_1(w_1,\overline{w}_1)\ldots\phi_n(w_n,\overline{w}_n)\big\rangle = \sum_{k=1}^{n}\Big(\frac{h_k}{(z-w_k)^2}+\frac{\partial_{w_k}}{(z-w_k)}\Big)\big\langle\phi_1(w_1,\overline{w}_1)\ldots\phi_n(w_n,\overline{w}_n)\big\rangle$$

(1.4.20)

These equations can be a very powerful tool to compute the form of correlators.

1.4.3 TT OPE and Central Charge

In general primary fields always have transformation laws of the type shown in (1.4.1) and (1.4.3), from the first of these two equations we can notice that the field ϕ has a transformation law that can be interpreted as a tensor's, with the field having h lower z indexes and $\overline{h}$ lower $\overline{z}$ indexes, so that its infinitesimal variation is such that it could be derived as the most general expression, linear in ϵ , with $(h+1)$ lower z indexes and $\overline{h}+1$ lower $\overline{z}$ indexes. The right coefficients can be then chosen to agree with the OPE. If we now want to exctract information about the $T(z)T(w)$ OPE, we must proceed in a way that is quite similar. First of all we notice that $T(z)$ is an object with two lower z

indexes, so that in analogy with the above situation we may risk to pose $h = 2$; the most general form for the variation of $T(z)$ is then:

$$\delta_\epsilon T(z) = \alpha\epsilon(z)\partial T(z) + \beta\partial\epsilon(z)T(z) + \gamma\partial^3\epsilon(z) \tag{1.4.21}$$

Which implies an OPE of the form:

$$T(z)T(w) = \frac{6\gamma}{(z-w)^4} + \frac{\beta T(w)}{(z-w)^2} + \frac{\alpha\partial T(w)}{(z-w)} \tag{1.4.22}$$

now if we require that $\beta = h = 2$, $\alpha = 1$ (in analogy with the primary case) and that the 2 point function be normalized as:

$$\langle T(z)T(w) \rangle = \frac{c}{2}\frac{1}{(z-w)^4} \tag{1.4.23}$$

so that $\gamma = \frac{c}{12}$, we obtain the following infinitesimal transformation law:

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\partial^3\epsilon(z) \tag{1.4.24}$$

and and OPE of the form:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1.4.25}$$

With a twin equation for the antiholomorphic part.

The choice of the γ constant has been done simply for convenience, since it will influence the structure of the Virasoro Algebra (soon to be introduced) which in turn will fix the normalization of the stress tensor's two point function. The constant c will be called *Central Charge* and we will see that the requirement of a unitary theory will constrain c to be greater or equal to 0. We finally want to say something about the finite transformation law for $T(z)$ under $z \to f(z)$, which takes the form:

$$T(z) \to (\partial f)^2 T(f) + \frac{c}{12}S(f, z) \tag{1.4.26}$$

where

$$S(f, z) = \frac{\partial f\partial^3 f - 3/2(\partial^2 f)^2}{(\partial f)^2} \tag{1.4.27}$$

is called the Swartzian derivative. It is interesting that the Swartzian derivative of a global conformal transformation vanishes, implying an infinitesimal transformation law identical to (1.4.3), this could also have been inferred from the fact that $\epsilon(z)$ is at most quadratic in z for $\mathbf{SL}(2, \mathbb{C})$ mappings. This fact is the statement that $T(z)$ is a *Quasi Primary Field*.

$S(f, z)$ will be also very useful in measuring the shift of the vacuum energy due to the change of geometry of the background manifold for the theory.

1.5   Algebraic Structure of the Hilbert Space

Our goal is to build the Hilbert space of a conformal field theory starting from the tools already in our hands, that is the globally invariant vacuum $|0\rangle$, the stress tensor, the primary fields of the theory and the OPE.

1.5.1   Mode Expansions and the Virasoro Algebra

We begin by defining a mode expansion for a generic scaling operator $A(z)$ of dimension h through:

$$A(z) = \sum_{n \in Z-h} \frac{A_n}{z^{n+h}} \qquad (1.5.1)$$

so that the modes A_n have scaling dimension n. The inverse relation is given by:

$$A_n = \oint \frac{dz}{2\pi i} z^{h+n-1} A(z) \qquad (1.5.2)$$

In particular for the stress tensor we have $h = 2$ and we conventionally call the modes L_n. Using (1.5.2) and (1.4.25), we now want to try to evaluate the commutator of two modes; a problem arises about how to define the commutator of two contour integrations$[\oint dz, \oint dw]$, this is done by fixing w and performing the z integration on a small circle around w, the w contour is then taken to be a circle around $w = 0$. So that we readily find the commutation relations of the *Virasoro Algebra*:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,\,0} \qquad (1.5.3)$$

and the analogous antiholomorphic relation; we point out that the full algebra is a direct sum of the two holomorphic and antiholomorphic algebras so that

$$[L_m, \overline{L}_n] = 0 \qquad (1.5.4)$$

The first remarkable property of this algebra is that it admits a subalgebra generated by L_0, L_1, L_{-1} that does not involve the central charge, so that the $\mathbf{SL}(2, \mathbb{C})$ subalgebra of the local conformal algebra still determines an exact symmetry for the theory even if we have a central charge term.

We will now provide the concept of adjoint, which is essential to our algebraic structure. If we consider an in state created by the insertion of a primary operator A at $z = 0$ and call it $A(0)|0\rangle = |h\rangle$, we are naturally led to consider a similar out state which has an insertion at the ∞ point; we want such a state to be considered as the adjoint of $A(0)|0\rangle$. The right thing to do is to consider the $\mathbf{SL}(2, \mathbb{C})$ mapping $z \rightarrow 1/z$ which takes ∞ to

the origin and to proceed by defining the adjoint as the transformed field under such a mapping:

$$A^\dagger(z,\overline{z}) = \frac{1}{z^{2h}\overline{z}^{2\overline{h}}} A(\frac{1}{z},\frac{1}{\overline{z}}) \tag{1.5.5}$$

So that in the limit $z \to \infty$ we can write:

$$\langle h| = \lim_{z \to \infty} \langle 0| z^{2h}\overline{z}^{2\overline{h}} A(z,\overline{z}) \tag{1.5.6}$$

As a consequence of this definition it can be easily shown that the stress tensor's modes of expansion satisfy:

$$L_n^\dagger = L_{-n} \tag{1.5.7}$$

So that we also have that $\mathbf{SL}(2,\mathbb{C})$ is a subalgebra of the Virasoro Algebra stable under the adjoint operation.

Let us now consider the effects of regularity requirements imposed on the state representation of the stress tensor. If we require regularity as $z \to 0$ for the state $T(z)|0\rangle$ we obtain the condition:

$$L_n|0\rangle = 0 \text{ for } n \geq -1 \tag{1.5.8}$$

and, taking the adjoint:

$$\langle 0|L_n \text{ for } n \leq 1 \tag{1.5.9}$$

So that we find out once more that the vacuum must be $\mathbf{SL}(2,\mathbb{C})$ invariant. Now by inserting the mode expansions for $T(z)$ into $\langle 0|T(z)T(w)|0\rangle$ we could easily show that the two point function really has the form of (1.4.23), and we could also compute mechanically although in a laborious way all higher order correlators.

1.5.2   Highest Weight States and Descendants

If we consider once more the state $\phi(z)|0\rangle$ (where ϕ is a primary field) as $z \to 0$ and call it $|h\rangle$, the usual problem arises to impose regularity conditions on it, which tells us:

$$\phi_n|0\rangle = 0 \text{ for } n \geq -h + 1 \tag{1.5.10}$$

Now, if we consider the commutator:

$$[L_n, \phi(z)] = h(n+1)z^n\phi(z) + z^{n+1}\partial\phi(z) \tag{1.5.11}$$

we discover that it is 0 for $n > 0$ and $z = 0$, while for $n = z = 0$ we find the very useful relation:

$$[L_0, \phi(0)]|0\rangle = h\phi(0)|0\rangle \tag{1.5.12}$$

so we discover that:

$$L_0|h\rangle = h|h\rangle \tag{1.5.13}$$

And still another gift comes from (1.5.11) and (1.5.2):

$$[L_n, \phi_m] = (n(h-1) - m)\phi_{m+n,} \tag{1.5.14}$$

so that for $n = 0$ we find:

$$[L_0, \phi_m] = -m\phi_m \tag{1.5.15}$$

This is interesting because it tells us that we can also identify $|h\rangle$ as:

$$|h\rangle = \phi_{-h}|0\rangle \tag{1.5.16}$$

We will now define the notion of *Descendant* at level N as a state generated by a primary operator (called a *Highest Weight State*), on which we let act a string of operators $L_{-n_1} \ldots L_{-n_k}$ ($n_i > 0$) with $\sum_i n_i = N$. Such a state has a conformal weight $h + N$, and the number of these states amounts to $P(N)$, that is the number of partitions of N into positive integer parts.

The action of the L_n is made more manifest by the commutation relation:

$$[L_n, L_0] = nL_n \tag{1.5.17}$$

That implies:

$$L_0 L_n |h\rangle = (h - n)L_n|h\rangle \tag{1.5.18}$$

So we have learned that the L_n act as rising operators for the conformal weight for $n < 0$, and in the case $n > 0$ using (1.5.10), (1.5.14) and (1.5.16) we deduce:

$$L_n|h\rangle = 0 \text{ for } n > 0 \tag{1.5.19}$$

Some constraints due to unitarity come from the relation:

$$\langle h|L_{-n}^\dagger L_{-n}|h\rangle = (2nh + c/12(n^3 - n))\langle h|h\rangle \tag{1.5.20}$$

Which for n large, united to the requirement of a positive definite norm requires $c > 0$, while for $n = 1$ it implies $h \geq 0$.

Summing up we have learned that a unitary representation of the Virasoro Algebra must have:

$$h \geq 0 \text{ and } c > 0 \tag{1.5.21}$$

In a short while we will see how an even more restricted range of values for c and h can be picked out from a more detailed analisys, these will characterize the Minimal Unitary Models.

Now, as an interesting example of descendant state it would be worth to consider $T(0)|0\rangle$. If we notice that:

$$L_{-2} = \oint \frac{dz}{2\pi i} \frac{T(z)}{z} \sim T(0) \tag{1.5.22}$$

we immediately arrive at the relation:

$$T(0)|0\rangle = L_{-2}|0\rangle \tag{1.5.23}$$

so that the stress tensor is a level 2 descendant of the identity operator, incidentally this gives us still another explanation of why T is not a primary field, and also gives us a criterion to recognize a quasi primary operator: it is not a level 1 descendant of the previous level.

1.5.3   Correlators of Descendants and Fusion Rules

It is interesting to consider more complicated correlators formed not just by primary fields, but also from descendants. Let us consider for simplicity a correlator involving only one descendant of the form:

$$\langle \phi_1(w_1,\overline{w}_1)\ldots\phi_{n-1}(w_{n-1},\overline{w}_{n-1})L_{-k_1}\ldots L_{-k_l}\phi_n(z,\overline{z})\rangle \tag{1.5.24}$$

By using the commutator (1.5.11) to annihilate L_{-k_s} on the out vacuum we can show it to take the form:

$$\langle \phi_1(w_1,\overline{w}_1)\ldots\phi_{n-1}(w_{n-1},\overline{w}_{n-1})L_{-k_1}\ldots L_{-k_l}\phi_n(z,\overline{z})\rangle =$$
$$\mathcal{L}_{-k_1}\ldots\mathcal{L}_{-k_l}\langle \phi_1(w_1,\overline{w}_1)\ldots\phi_{n-1}(w_{n-1},\overline{w}_{n-1})\phi_n(z,\overline{z})\rangle \tag{1.5.25}$$

where:

$$\mathcal{L}_{-k} = -\sum_{i=1}^{n-1}\left(\frac{(1-k)h_i}{(w_i-z)^k} + \frac{\partial_{w_i}}{(w_i-z)^{k-1}}\right) \tag{1.5.26}$$

So that the above correlator is completely determined in terms of primary fields correlators.

The same could in principle be carried out for more complicated correlators so that the only fundamental correlators for the theory are those of primary fields.

Now, if we turn our attention to the OPE structure we can use the primary-descendant structure of the Hilbert space to group it as (you can check that the scaling dimensions in the sum terms are right):

$$\phi_i(z,\overline{z})\phi_j(w,\overline{w}) = \sum_{p(k\overline{k})} C_{ijp}^{(k\overline{k})} z^{h_p-h_i-h_j+\sum_l k_l} \overline{z}^{\overline{h}_p-\overline{h}_i-\overline{h}_j+\sum_l \overline{k}_l} \phi_p^{(k\overline{k})}(w,\overline{w}) \tag{1.5.27}$$

where by $\phi_p^{(k\bar{k})}(w,\overline{w})$ we mean the descendant at level $(k\bar{k})$ of ϕ_p with respect to the two Virasoro algebras of the theory. If we now consider an arbitrary 3 point function, we have shown that it can be computed in terms of the three point function for the primaries so that:

$$\langle\phi_i\phi_j\phi_k^{(l,0)}\rangle = C_{ijk}\mathcal{L}_{-l}\frac{1}{z_{ij}^{h_i+h_j-h_k}z_{jk}^{h_j+h_k+h_i}z_{ik}^{h_k+h_i-h_j}\overline{z}_{ij}^{\overline{h}_i+\overline{h}_j-\overline{h}_k}\overline{z}_{jk}^{\overline{h}_j+\overline{h}_k+\overline{h}_i}\overline{z}_{ik}^{\overline{h}_k+\overline{h}_i-\overline{h}_j}} \qquad (1.5.28)$$

If on the other hand we use the OPE we find:

$$\langle\phi_i\phi_j\phi_k^{(l,0)}\rangle = \sum_{p(s\overline{s})} C_{ijp}^{(s\overline{s})} z^{h_p-h_i-h_j+\sum_l s_l}\overline{z}^{\overline{h}_p-\overline{h}_i-\overline{h}_j+\sum_l \overline{s}_l}\langle\phi_p^{(s\overline{s})}(w,\overline{w})\phi_k^{(l,0)}\rangle \qquad (1.5.29)$$

And remembering that 2 point functions of operators of different conformal weight must vanish:

$$\langle\phi_i\phi_j\phi_k^{(l,0)}\rangle = \sum_{(s\overline{s})} C_{ijk}^{(s\overline{s})} z^{h_k-h_i-h_j+\sum_l s_l}\overline{z}^{\overline{h}_k-\overline{h}_i-\overline{h}_j+\sum_l \overline{s}_l}\langle\phi_k^{(s\overline{s})}(w,\overline{w})\phi_k^{(l,0)}\rangle \qquad (1.5.30)$$

Now, equating the two expressions we see that both of them must be null when C_{ijk} is null so that also the second must be proportional to it and we are led to write:

$$C_{ijk}^{(s\overline{s})} = C_{ijk}\beta_{ij}^{k(s)}\overline{\beta}_{ij}^{k(\overline{s})} \qquad (1.5.31)$$

The functions $\beta_{ij}^{k(s)}$ are in principle determinable, but to our goals it suffices to notice that the 3 point functions' non null coefficients determine which conformal families are allowed in the OPE between two fields; most often the null coefficients are determinable on the basis of some symmetry principle for the field theory under consideration, as is the case for example in the Ising Model, where the symmetries are, as we will see, Duality and Spin Reversal. So we will say that the null C_{ijk} determine the so called *Fusion Rules* for the model which we can simbolically write as:

$$[\phi_i] \times [\phi_j] = \sum_p C_{ijp}[\phi_p] \qquad (1.5.32)$$

1.5.4   Verma Modules, Kac Determinant and Unitarity

We have seen how a highest weight state $|h\rangle$ has descendants which can be characterized by their eigenvalue N under the L_0 operator, these states can also be organized into an hierarchy as the level of descendance N increases. In this way we could hope to build a state representation of the Virasoro Algebra, but a problem arises about wether at a given level of descendance all states are linearly independent or not. In the latter case there must exist a linear combination of states which we must quotient out of the Hilbert Space

by requiring it to be equivalent to the null vector. This procedure is called dimensional reduction; before dimensional reduction the set composed of a highest weight state and its descendants $\mathcal{V}(c, h)$ is called a *Verma Module*, and after dimensional reduction it constitutes an irreducible representation of the Virasoro Algebra. The Hilbert Space is then built as a direct sum over Verma Modules:

$$\mathcal{H} = \bigoplus_{(h,\overline{h})} \mathcal{V}(c, h) \otimes \mathcal{V}(c, \overline{h}) \tag{1.5.33}$$

Let us now spend some words about how to determine the existence of null states in a Verma Module.

If, for a fixed N, we consider the Gram Matrix (i.e. the matrix of all possible inner products) for the $P(N)$ states of the form $L_{-n_1} \ldots L_{-n_k} |h\rangle$ with $\sum_i n_i = N$, it happens that such a matrix has a vanishing determinant (The *Kac Determinant*) if the vectors are not all linearly independent, and its null eigenvectors expanded on the $P(N)$ states at a given level of descendance N give the linear combination that must vanish.

Let us give 2 instructive examples, for $N = 1$, normalizing $\langle h|h \rangle = 1$ (i.e. the 2 point functions are normalized to 1), we have:

$$\langle h|L_1 L_{-1}|h\rangle = 2h \tag{1.5.34}$$

which for $h = 0$ simply states what we already know, that is $L_{-1}|0\rangle = 0$. For $N = 2$, taking as a basis $\{L_{-2}, L_{-1}^2\}$, we easily build the gram matrix:

$$\begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1 + 2h) \end{pmatrix} \tag{1.5.35}$$

and taking the determinant we find:

$$2h(16h^2 + (2c - 10)h + c) \tag{1.5.36}$$

Which can be trivially null for $h = 0$, corresponding to the fact that $L_{-1}^2|0\rangle = 0$, or it could have nontrivial zeroes for special values of h that can be determined obviously as a function of c.

The corresponding null vector can be shown to be:

$$\left(L_{-2} - \frac{3}{4h + 2}L_{-1}^2\right)|h\rangle = 0 \tag{1.5.37}$$

In general the occurence of a null state at level n will imply that at a level $N > n$ there will be $P(N - n)$ null states.

This will prove very useful because it will provide us with more constraints for the N

point correlators, since for example they will be annihilated by a differential operator as a consequence of (1.5.37):

$$(\mathcal{L}_{-2} - \frac{3}{4h+2}\mathcal{L}_{-1}^2)\langle\phi(z,\overline{z})\ldots\rangle = 0 \tag{1.5.38}$$

Now let us spend some words about unitarity. In the last section we have learned that unitary representations of the Virasoro Algebra can occur only for values of h and c in the range given by (1.5.21), this is surely true, but we can say more. A detailed analisys of the Kac Determinant, whose goal is to dermine the existence of imaginary norm states, can show that although for $h \geq 0$, $c \geq 1$ there is nothing that prevents us form having unitary representations, in the region $h \geq 0$, $0 \leq c < 1$ unitary prepresentations of the Virasoro Algebra may occur only at discrete values of the central charge indicized by an integer $m \geq 2$ and for a set of fields depending on two more integers $1 \leq p \leq m-1$, $1 \leq q \leq p$. Explicitly we have:

$$c = 1 - \frac{6}{m(m+1)} \quad m = 3, 4, \ldots \tag{1.5.39}$$

$$h_{p,\,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad 1 \leq p \leq m-1, \ 1 \leq q \leq p \tag{1.5.40}$$

These are called *Minimal Unitary Models* $\mathcal{M}_{m,m+1}$ and it has been shown, by comparison of critical exponents, that the operator content of each one falls within the universality class of a critical statistical mechanical model; for example $m = 3$ is the Ising Model, $m = 4$ the Tricritical Ising Model, m=5 the 3-States Potts Model and $m = 6$ the Tricritical 3-States Potts Model.

If we release the requirement of unitarity we can still find finite operator content theories as above Indicized by two integers, these are just the *Minimal Models* $\mathcal{M}_{m,\,n}$ (without "unitary"), and their central charge satisfies:

$$c = 1 - \frac{6(m-n)^2}{mn} \tag{1.5.41}$$

where m and n are coprime integers.

Going back to unitary models we can notice that the conformal weights (1.5.40) possess the symmetry $p \rightarrow m-p$, $q \rightarrow m+1-q$ so that we can extend the range of q to $1 \leq q \leq m$ so that we obtain $m(m-1)/2$ couples of primary fields with equal conformal weight. The model $\mathcal{M}_{3,4}$ for examples possesses 3 couples of primary fields of weight $h_{1,1} = h_{2,3} = 0$ (identity operator), $h_{1,2} = h_{2,2} = 1/16$ and $h_{2,1} = h_{1,3} = 1/2$.

1.5.5   CFT on the Torus and Modular Invariance

We are now interested in finding a way to define a CFT on a manifold of higher genus such as the torus. Such a manifold can be obtained by twisting a finite length cylinder so

that we get periodic boundary conditions in both directions (peculiar mixes of periodic and antiperiodic boundary conditions produce non orientable underlying manifolds such as Klein bottles).

If we adopt complex coordinates on the cylinder $z = x + iy$, and take x to be periodic under $x \to x + 1$, we are still left with the need to introduce a new direction (i.e. a complex number τ) which has to be periodically identified, so that $z = z + \tau$.

Another thing we cannot do without are the two generators of space and time traslations i.e. the Hamiltonian H and the momentum P, these are obtained considering the exponential mapping $w = exp(2\pi i z)$ from the complex plane, where the Hamiltonian is simply the generator of dilatations $L_0 + \overline{L}_0$ and the momentum the generator of rotations $i(L_0 - \overline{L}_0)$.

In order to obtain $(L_0)_{Cyl}$ we need to consider the Swartzian derivative (1.4.27) of the exponential mapping which gives the following transformation law for the Stress Tensor:

$$T_{Cyl}(z) = -4\pi^2 \left(w^2 T(w) - \frac{c}{24} \right) \tag{1.5.42}$$

where w is the coordinate in the complex plane.

This tells us that only L_0 is changed by the mapping so that:

$$(L_0)_{Cyl} = L_0 - \frac{c}{24} \tag{1.5.43}$$

This gives us the following expressions for the generators on the cylinder:

$$H = L_0 + \overline{L}_0 - \frac{c}{24} - \frac{\overline{c}}{24} \tag{1.5.44}$$

$$P = i \left(L_0 - \overline{L}_0 - \frac{c - \overline{c}}{24} \right) \tag{1.5.45}$$

It is very important to notice that the mapping between complex plane and cylinder has caused a shift in the vacuum energy of the system, called *Casimir Energy*, this has simply the effect of changing the normalization of the functional integral.

Furthermore combining the two periodic conditions tells us that $z = z + \tau + 1$ so that it would have been the same to choose $\tau + 1$ (more generally $\tau + k$ with $k \in \mathbb{Z}$) or τ as the periodic direction, this is the hint of a more rich underlying structure that has been identified with the modular group $SL(2, \mathbb{Z})$.

The theory under consideration must be invariant under redefinitions of the modular parameter τ of the form:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \frac{SL(2, \mathbb{Z})}{\mathbb{Z}_2} \tag{1.5.46}$$

Such a group of transformations is generated by:

$$\begin{cases} \tau \to \tau + 1 \\ \tau \to -\frac{1}{\tau} \end{cases} \tag{1.5.47}$$

If we now define the *Virasoro Characters* as:

$$\chi_{c,h}(q) = Tr_{\mathcal{V}(c,h)}q^{L_0 - \frac{c}{24}} = \sum_{N=0}^{\infty} d_N q^{h+N-\frac{c}{24}} \quad q = e^{2\pi i \tau}, \tag{1.5.48}$$

where d_N is the degeneracy of the Nth level of descendance inside the Verma module $\mathcal{V}(c,h)$, we have that as a consequence of modular invariance, under modular transformations, the Virasoro Characters transform among themselves in a unitary representation of the modular group, so that we get:

$$\chi_{c,h}(\tau + 1) = \sum_{h'} \mathcal{T}_{h,h'} \chi_{c,h'}(\tau) \tag{1.5.49}$$

$$\chi_{c,h}\left(-\frac{1}{\tau}\right) = \sum_{h'} \mathcal{S}_{h,h'} \chi_{c,h'}(\tau) \tag{1.5.50}$$

With all these concepts in hand we are now ready to build a partition function starting form the expression:

$$Z(\tau) = Tr e^{2\pi(i\mathfrak{Re}\tau P - \mathfrak{Im}\tau H)} \tag{1.5.51}$$

which, introduced the modular parameter $q = exp(2\pi i \tau)$, and for $\bar{c} = c$ takes the form:

$$Z(q) = (q\bar{q})^{-\frac{c}{24}} Tr q^{L_0} \bar{q}^{\bar{L}_0} \tag{1.5.52}$$

Where the trace is taken over all the Hilbert Space.
It is now possible (this is due to the Hilbert Space structure) to decompose the partition function into a bilinear form in the Virasoro Characters:

$$Z(q) = \sum_{h,\bar{h}} \mathcal{N}_{h,\bar{h}} \chi_{c,h}(q) \chi_{c,\bar{h}}(\bar{q}) \tag{1.5.53}$$

In this last expression $\mathcal{N}_{h,\bar{h}}$ is an integer that numbers the multiplicity of occurrence of $\mathcal{V}(c,h) \otimes \mathcal{V}(c,\bar{h})$ in the Hilbert Space; modular invariance of the partition function is warranted by the unitarity of the representation of the modular group carried by the Virasoro characters.
In non chiral (i.e. whose content is only of spinless fields) theories only tensor products of Verma Modules with $h = \bar{h}$ may occur, this situation corresponds to a *diagonal theory* whose partition function is simply:

$$Z(q) = \sum_{h} \left|\chi_{c,h}(q)\right|^2 \tag{1.5.54}$$

An example of such a theory is $\mathcal{M}_{3,4}$ which thanks to its diagonal form can be identified with the Critical Ising Model. There are also examples of non diagonal chiral theories as is the case for example of $\mathcal{M}_{5,6}$ which corresponds to the 3 States Potts Model.

1.6 Some Identifications

1.6.1 $\mathcal{M}_{3,4}$ as the Ising Model

Let us first of all recall some common lore about the Ising Model.
This model is defined on a square lattice trough the Hamiltonian:

$$H = J \sum_{<i,j>} \sigma_i \sigma_j + h \sum_i \sigma_i \tag{1.6.1}$$

where $< i, j >$ denotes the sum over nearest neighbor sites, and the variables σ_i must belong to the set $\{-1, 1\}$; h represents an external magnetic field, which to our purposes will be considered equal to 0 (the $h \neq 0$ situation will be achievable in the context of CFT perturbation theory).
This theory presents a 2nd order phase transition separating an ordered phase ($\langle \sum_i \sigma_i \rangle \neq 0$) from a disordered phase ($\langle \sum_i \sigma_i \rangle = 0$).
In the disordered phase we have a finite correlation length ξ and 2 point functions fall off exponentially as $\langle \sigma_n \sigma_0 \rangle \sim exp(-n/\xi)$, while at the critical point ξ diverges and correlators fall of with power law (a manifest signal of scaling invariance):

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{n^{d-2+\eta}} \tag{1.6.2}$$

We have also a lattice interaction energy $\epsilon_i = \frac{1}{2d} \sum_k \sigma_i \sigma_{i+k}$ (here k denotes the unity displacement in one of the $2d$ directions) which at criticality behaves as:

$$\langle \epsilon_n \epsilon_0 \rangle \sim \frac{1}{n^{2(d-1/\nu)}} \tag{1.6.3}$$

where ν is related to the correlation length's divergence (as $T \to T_c$) expressed in terms of the reduced temperature $t = (T - T_c)/T_c$ so that $\xi \sim t^{-\nu}$.
Now, for $d = 2$ the Ising Model has been solved by Onsager and Kaufmann and its critical exponents have been shown to be $\eta = 1/4$ and $\nu = 1$, therefore if we consider $n \sim r$ (i.e. the continuum limit) we readily see from correlation functions that at criticality σ and ϵ behave as *spinless* fields of conformal dimensions $\overline{h}_\sigma = h_\sigma = 1/16$ and $\overline{h}_\epsilon = h_\epsilon = 1/2$.
These conformal weights tell us that we must consider the $\mathcal{M}_{3,4}$ model as the correct critical continuum version of the Ising Model and the presence of spinless fields confirms

what we already knew form the requirement of modular invariance, i.e. that we must consider the diagonal theory with partition function:

$$Z(q) = \left|\chi_{1/2,0}(q)\right|^2 + \left|\chi_{1/2,1/16}(q)\right|^2 + \left|\chi_{1/2,1/2}(q)\right|^2 \tag{1.6.4}$$

We finally want to spend some words about symmetries in the Ising Model, from the Hamiltonian (1.6.1) is is manifest that the *spin reversal* is a symmetry of the model, furthermore Kramers and Wannier have shown [?] that the model possesses a *duality* symmetry which exchanges order and disorder parameters ($\sigma \leftrightarrow \mu$) and reverses energy ($\epsilon \rightarrow -\epsilon$).

These considerations tell us that 3 point correlation functions involving an odd number of σ or an odd number of ϵ (but no σ or μ together with ϵ) must vanish so that we are left with the following fusion rules for the model:

$$\begin{cases} \epsilon \times \epsilon = 1 \\ \sigma \times \sigma = 1 + \epsilon \\ \sigma \times \epsilon = \sigma \end{cases}$$

Another very useful consequence of duality symmetry for the Ising Model is that by comparison of the partition function written in terms of the order and disorder parameter it is possible to find an equation that fixes the temperature for the phase transition:

$$e^{-2\beta_c} = \tanh(\beta_c) \tag{1.6.5}$$

where $\beta_c = 1/T_c$.

1.6.2 Massless Fermion as $\mathcal{M}_{3,4}$

Let us consider now a massless free fermion with an action

$$S = \frac{1}{8\pi} \int d^2x \, \overline{\Psi} \slashed{\partial} \, \Psi \tag{1.6.6}$$

with

$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \tag{1.6.7}$$

$$\slashed{\partial} = \sigma_x \partial_x + \sigma_y \partial_y = \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix} \tag{1.6.8}$$

$$\overline{\Psi} = \Psi^\dagger \sigma_x = \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix} \tag{1.6.9}$$

where σ_i are Pauli's sigma matrices. Using this considerations we can write the action as a sum of an holomorphic plus an antiholomorphic part:

$$S = \frac{1}{8\pi} \int d^2z(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}) \tag{1.6.10}$$

This action yields the following equations of motion:

$$\begin{cases} \overline{\partial}\psi = 0 \\ \partial\overline{\psi} = 0 \end{cases} \tag{1.6.11}$$

which can be used to compute the Stress Tensor using the formula (which holds for a lagrangian depending only on a field and its derivative):

$$T^{\mu\nu} = \frac{\delta\mathcal{L}(\phi, \partial\phi)}{\delta(\partial_\mu\phi)}\phi^\nu - g^{\mu\nu}\mathcal{L} \tag{1.6.12}$$

so that we find, after normal ordering the expression:

$$\begin{cases} T(z) = \frac{1}{2} : \psi(z)\partial\psi(z) : \\ \overline{T}(\overline{z}) = \frac{1}{2} : \overline{\psi}(\overline{z})\overline{\partial}\overline{\psi}(\overline{z}) : \end{cases} \tag{1.6.13}$$

The propagator can be computed by first putting the theory on a cylinder, decomposing into positive and negative frequency parts and then by considering the mapping (1.4.8), it is then just a matter of summing up a geometric series to reach the result:

$$\begin{cases} \langle\psi(z)\psi(w)\rangle = -\frac{1}{z-w} \\ \langle\overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle = -\frac{1}{\overline{z}-\overline{w}} \end{cases} \tag{1.6.14}$$

This result could also be quickly reached if we had used the identity:

$$\overline{\partial}\frac{1}{z} = \delta_2(z, \overline{z}) \tag{1.6.15}$$

We now want to compute the central charge for this theory by analizing the TT OPE, all we have to do is to remember we are dealing with Grassmann Variables so that for example $\psi^2(z) = 0$, $\psi(z)\psi(w) = -\psi(w)\psi(z)$, and then use Wick's Theorem for Fermionic Fields.

We readily get dropping regular terms:

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{4} : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) := \\
&= \frac{1}{4}\left[\frac{\partial\psi(z)\partial\psi(w)}{z-w} - \frac{\partial\psi(z)\partial\psi(w)}{(z-w)^2} - \frac{\psi(z)\partial\psi(w)}{(z-w)^2} - \frac{2\psi(z)\psi(w)}{(z-w)^3} + \right. \\
&\quad + \frac{2}{(z-w)^4} - \frac{1}{(z-w)^4}\Bigg] = \frac{1}{4}\left[\frac{1}{(z-w)^4} + \frac{(\partial\psi(w))^2}{z-w} - \right. \\
&\quad - \frac{\partial\psi(w)\psi(w) + (z-w)\partial^2\psi(w)\psi(w)}{(z-w)^2} + \frac{\psi(w)\partial\psi(w) + (z-w)(\partial\psi(w))^2}{(z-w)^2} - \\
&\quad - 2\frac{\psi^2(w) + \partial\psi(w)\psi(w)(z-w) + 1/2(z-w)^2\partial^2\psi(w)\psi(w)}{(z-w)^3}\Bigg] = \\
&= \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}(1/2\psi(w)\partial\psi(w)) + \frac{1}{z-w}\left[\partial(1/2\psi(w)\partial\psi(w))\right]
\end{aligned}
$$

(1.6.16)

This tells us, by comparison with (1.4.25), that we have a central charge $c = 1/2$, repeating the same calculation for the antiholomorphic part we easily discover that $c = \bar{c}$.

In close analogy we determine the conformal weight of the ψ operators by the $T\psi$ OPE:

$$
\frac{1}{2} : \psi(z)\partial\psi(z) : \psi(w) = \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}
$$

(1.6.17)

So that ψ and $\overline{\psi}$ are respectively $(1/2, 0)$ and $(0, 1/2)$ Primary Fields with a spin $s = h - \overline{h} = 1/2$, in close analogy to the operator content of the $\mathcal{M}_{3,4}$ model.

To proceed further we need to introduce the mode expansion for the field ψ as:

$$
i\psi(z) = \sum \frac{\psi_n}{z^{n+1/2}}
$$

(1.6.18)

where the i factor was put for mere convenience, and the summation set will depend on the boundary conditions we will consider as we shall shortly see. The anticommutation relations for the modes are constrained by the $\psi\psi$ short distance expansion (i.e. the propagator) to be:

$$
\{\psi_n, \psi_m\} = \delta_{n+m,0}
$$

(1.6.19)

We can now conceive to impose 2 different kinds of boundary conditions as $z \to e^{2\pi i}z$, *Periodic* and *Antiperiodic*, and these will impose respectively $n \in \mathbb{Z} + 1/2$ and $n \in \mathbb{Z}$. The change of BC will be achieved by the insertion of Primary Operators (called *Twist Fields*) on the incoming and outgoing vacuum.

Such operators will have the following short distance expansion with ψ:

$$
\psi(z)\sigma(w) = (z-w)^{-1/2}\mu(w) + \dots
$$

(1.6.20)

From dimensional analisys σ and μ will have the same conformal weight.

The in-state $\sigma(0)|0\rangle = |h_\sigma\rangle$ will be possibly annihilated by the ψ_n for $n > 0$, this is insured as long as $h_\sigma < 1$; since we do not know h_σ we could take this as an ansatz and hope to find results that are consistent.

With this anstatz we can calculate the 2 point function in the antiperiodic sector:

$$\langle h_\sigma|\psi(z)\psi(w)|h_\sigma\rangle = -\frac{1}{2}\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z - w} \tag{1.6.21}$$

This result can be used to determine the expectation value of the stress tensor in the antiperiodic sector.

Now, if $z = w + \epsilon$ we can easily show that:

$$\langle h_\sigma|\psi(z)\partial\psi(w)|h_\sigma\rangle = -\frac{1}{\epsilon^2} + \frac{1}{8w^2} \tag{1.6.22}$$

which is simply the statement that

$$\langle h_\sigma|T(z)|h_\sigma\rangle = \frac{1}{16}\frac{1}{w^2} \tag{1.6.23}$$

Let us now consider the $T\sigma$ OPE:

$$T(z)\sigma(0)|0\rangle = \frac{h_\sigma\sigma(0)}{z^2}|0\rangle + \dots \tag{1.6.24}$$

which, normalizing $\langle h_\sigma|h_\sigma\rangle$ to 1, enables us to conclude:

$$\langle h_\sigma|T(z)|h_\sigma\rangle = \frac{h_\sigma}{z^2} \tag{1.6.25}$$

so that we have the impressive result

$$h_\sigma = h_\mu = \frac{1}{16} \tag{1.6.26}$$

This tells us that we have found the $h = 1/16$ Primary Field corresponding to the other Primary Field present in $\mathcal{M}_{3,4}$, so that the identification of the two theories is complete. We want to point out that since $\mathcal{M}_{3,4}$ falls into the universality class (i.e. critical exponents are exactly the same) of the *Ising Model* we have also discovered that such a model (at the critical point and at zero external field) has a field content of a free massless Fermion; later on we will see how the off critical and zero external field situation (what we will call the thermic perturbation) will have a description in terms of a massive free Fermion, this in turn will mean breakdown of conformal symmetry.

Furthermore the presence of a *Dual* Twist Field μ is compatible with the description of the Ising Model which infact possesses a *Duality* Symmetry ($\sigma \leftrightarrow \mu$).

Chapter 2

Boundary and Integrability in $D = 2$

In this chapter we will deal with CFT defined on manifolds with boundary, in order to discover how the existence of a boundary modifies the structure of the Hilbert Space of the theory, and as a consequence of the partition function itself; furthermore we will be led to identify a set of scaling operators that naturally lives on the boundary, which will be in 1 on 1 correspondence with all the possible choices of boundary conditions for the underlying Statistical Mechanical model.

We will also introduce CFT Perturbation Theory, and we will content ourselves with providing an example of relevant perturbation. This in order to introduce the concept of CFT breaking and of Integrable Deformation of a CFT.

Integrable Deformations of CFT will thus provide a particular example of a wider class of Field Theories, which are infact the Integrable Field Theories, such theories are very interesting since they provide highly nontrivial examples of completely solvable QFT, and suggest a different approach to scattering theory, based on considering the S matrix itself as the fundamental object of interest.

2.1   CFTs with Boundary

The problem of considering CFTs on manifolds with boundary arises naturally in Statistical Mechanics when we consider a theory with some definite boundary conditions and consider its critical behaviour; another important connection (but anyway far from our goals) is open String Theory.

References for this section can be found in Cardy's works [3][4][5][6].

2.1.1   Boundary Conditions and Correlators

Two dimensional manifolds with boundary are classified by their topology, and in partic-
ular simply connected manifolds can be mapped to the complex upper half plane.

This is a good reason to consider the upper half plane from the start as our main exam-
ple. In order to preserve some sort of conformal invariance we are forced to require that
under infinitesimal transfromations $z \to z + \epsilon(z)$ the real axis be mapped into itself, this
is because the distance of the image of a point lying on the boundary from the boundary
itself would define a local scale for the system and the shortest of such local scales would
thus define a global scale which would force correlators to decay exponentially as their
distance from the boundary grows.

This requirement is easily seen to mean that ϵ must be real:

$$\epsilon(\overline{z}) = \overline{\epsilon}(z) \tag{2.1.1}$$

Another important requirement is that boundary conditions on fields must be conformally
invariant so that the scaling properties of primary operators tell us that we must consider
homogeneous B.C. such as:

$$\phi|_B = 0 \tag{2.1.2}$$

If we now want to go further we are forced to consider the antiholomorphic dependence
of the fields to be constrained to $\overline{z} = z^*$.

And following this line of thought we constrain also the stress tensor to satisfy:

$$T(z^*) = \overline{T}(z) \tag{2.1.3}$$

which in turn implies that $T = \overline{T}$ on the real axis so that there is no energy or momentum
flux across the boundary.

This procedure can be interpreted conversion of the antiholomorphic degrees of freedom
into holomorphic degrees of freedom on the lower half plane.

We are now ready to consider the conformal ward identities for this system, since $\delta_{\epsilon,\overline{\epsilon}} = \delta_\epsilon + \delta_{\overline{\epsilon}}$, considering a closed contour C (and its complex coniugated $\overline{C}$) lying in the upper
half plane we have:

$$\begin{aligned}
\delta_{\epsilon,\overline{\epsilon}}\big\langle\phi_1(z_1.\overline{z}_1),\ldots,\phi_1(z_k,\overline{z}_k)\big\rangle = \\
-\oint_C \frac{dw}{2\pi i}\epsilon(w)\big\langle T(w)\phi_1(z_1,z_1^*),\ldots,\phi_1(z_k,z_k^*)\big\rangle+ \\
-\oint_{\overline{C}} \frac{dw^*}{2\pi i}\overline{\epsilon}(w^*)\big\langle \overline{T}(w^*)\phi_1(z_1,z_1^*),\ldots,\phi_1(z_k,z_k^*)\big\rangle
\end{aligned} \tag{2.1.4}$$

Now we can deform C and $\overline{C}$ to follow very closely the real axis so that the two pieces of
contours along the real axis cancel each other and we are left with a single contour C' no

longer contained in the upper half plane encircling the insertions at $\{z_1, \ldots, z_k, z_1^*, \ldots, z_k^*\}$, so that:

$$\delta_{\epsilon,\epsilon^*}\langle\phi_1(z_1, z_1^*), \ldots, \phi_1(z_k, z_k^*)\rangle = -\oint_{C'} \frac{dw}{2\pi i}\epsilon(w)\langle T(w)\phi_1(z_1.z_1^*), \ldots, \phi_1(z_k, z_k^*)\rangle \quad (2.1.5)$$

which for $\phi(z, \overline{z}) = \phi(z)\overline{\phi}(\overline{z})$ means that a k points correlator on the complex upper half plane satisfies the same differential equation of a $2k$ points correlator on the whole plane, modulus the fact that we must impose the constraint $\overline{z} = z^*$.

As an example we can consider the 1 point function of a primary field of conformal weights $h = \overline{h}$; assuming that in the bulk ($|z| \to \infty$) the expression vanishes we are readily led to ($y = \Im z$):

$$\langle\phi(z, \overline{z})\rangle_\beta = \frac{A_\phi^\beta}{2y^{2h}} \quad (2.1.6)$$

where A_ϕ^β is an amplitude depending on the field ϕ and on the boundary condition labeled by β.

2.1.2 Boundary States and Operators

In the framework of the previous section arises the problem to consider the limit of an operator insertion at z as the point z tends to the boundary, and even more to consider the correlation of insertions living on the boundary (we are interested mainly in the first situation).

This is done by considering a Primary field as a product of holomorphic and antiholomorphic parts, constrained so that $z = z^*$ $h = \overline{h}$, and then introducing the following short distance expansion (OPE):

$$\phi(z, \overline{z}) = \phi(z)\phi(z^*) \sim \sum_i (2iy)^{h_i - 2h}C_{\phi\psi_i}^\beta\psi_i(x) \quad (2.1.7)$$

Where again β labels a boundary condition, x, y are respectively the real and imaginary parts of z and finally the $\{\psi_i(x)\}$ are a family of boundary fields of scaling dimension h_i which we normalize as:

$$\langle\psi_i(x_1)\psi_j(x_2)\rangle_\beta = \delta_{i,j}(x_1 - x_2)^{-2h_i} \quad (2.1.8)$$

In particular taking the expectation value of (2.1.7) and considering (2.1.6) we discover:

$$C_{\phi,\mathbf{1}}^\beta = A_\phi^\beta \quad (2.1.9)$$

and

$$C_{\phi,\psi_i}^\beta\langle\psi_i(x)\rangle_\beta = 0 \ , \ \psi_i \neq 1 \quad (2.1.10)$$

Further information about the state representation of boundary operators can be obtained by putting the theory on a finite length cylinder of dimensions L, T (which can be mapped through the exponential map to an anulus in the complex plane).

After compactifying one dimension we first of all need to remember that in analogy with the previous section on the boundary we must have $T(z^*) = \overline{T}(z)$ so that:

$$\sum_{n \in \mathbb{Z}} \frac{L_n}{z^{*n+2}} |\alpha\rangle = \sum_{m \in \mathbb{Z}} \frac{\overline{L}_m}{z^{m+2}} |\alpha\rangle \tag{2.1.11}$$

which taking as inner boundary S^1, so that $z^* = 1/z$, implies after renaming the summation index m:

$$(z^2 L_n - z^{-2} \overline{L}_{-n}) |\alpha\rangle = 0 \tag{2.1.12}$$

This equation must hold for all $z \in S^1$ and in particular for $z = 1$, so that we get the constraint:

$$(L_n - \overline{L}_{-n}) |\alpha\rangle = 0 \tag{2.1.13}$$

This constraint has the immediate effect of reducing the Hilbert Space of States of the theory to (remember that only half of the conformal generators survive):

$$\mathcal{H}_{Boundary} = \bigoplus_h \mathcal{V}(c, h) \tag{2.1.14}$$

which is of course embedded into the bigger Hilbert Space (1.5.33), and infact we can solve the above constraint inside this wider space as:

$$|j\rangle\rangle = \sum_N |j, N\rangle \otimes U \overline{|j, N\rangle} \tag{2.1.15}$$

where $|j\rangle\rangle$ are the so called Ishibashi States, $|j, N\rangle$ is a state belonging to the dimensionally reduced verma module $\mathcal{V}(c, j)$, N labels a state inside a given level of descendance, and finally U is a antiunitary operator satisfying:

$$U \overline{|j, 0\rangle} = \overline{|j, 0\rangle}^* \tag{2.1.16}$$

$$[\overline{L}_n, U] = 0 \tag{2.1.17}$$

the second equation in particular tells us that U can be expressed as a function of the L_n. We want to point out that the states $|j\rangle\rangle$ defined as above are orthogonal but have an infinite norm:

$$\langle\langle j | j \rangle\rangle = \sum_{NM} \langle j, N | \otimes \overline{\langle i, N |} U^\dagger U \overline{|i, M\rangle} \otimes |j, M\rangle = \sum_{NM} \delta_{NM} = \infty \tag{2.1.18}$$

This problem can be fixed by changing the normalization of $|j, N\rangle$ for example to $1/N$. With the help of these Isibashi states we can now generate all the possible boundary states by linear combination.

2.1.3 Partition Function and Modular Invariance

Now that we have shed some light on the underlying Hilbert Space structure we can observe that the choice of two different quantization schemes allows us to express the partition function in two important ways.

Taking time as the periodic direction we find out that the hamiltonian H depends on the boundary condidtions $H = H_{\alpha\beta}$, and local conformal invariance implies that its spectrum falls into irreducible representations of the Virasoro algebra so that recalling the Hilbert Space structure we are led to write:

$$Z_{\alpha\beta}(q) = \sum_h n_{\alpha\beta}^h \chi_{c,h}(q) \tag{2.1.19}$$

Where $n_{\alpha\beta}^h$ is the number of times the representation h occurs in the spectrum of the Hamiltonian, and the modular parameter q has been identified as:

$$q = e^{2\pi i \tau} \ , \ \tau = iT/2L \tag{2.1.20}$$

Now, from the previous chapter we know that under a modular transformation $\tau \to -1/\tau$ the characters transform according to a unitary representation of the modular group, furthermore such a transformation formally exchanges the roles of the compactified dimensions T, L, so that calling $\tilde{q}$ the transformed of q under the modular inversion we can write:

$$Z_{\alpha\beta}(q) = \sum_i n_{\alpha\beta}^i S_{ij} \chi_{c,j}(\tilde{q}) \tag{2.1.21}$$

On the other hand taking space as the compactified dimension the Hamiltonian does not depend on the boundary condidtions and has the simple expression already introduced in the previous chapter, so that the partition function takes the form:

$$Z_{\alpha\beta}(q) = \langle \alpha | \tilde{q}^{L_0 - \frac{c}{24}} | \beta \rangle = \sum_{ij} \langle \alpha | i \rangle\rangle \langle\langle i | (\tilde{q}^{1/2})^{L_0 + \overline{L}_0 - \frac{c}{12}} | j \rangle\rangle \langle\langle j | \beta \rangle =$$
$$= \sum_j \langle \alpha | i \rangle\rangle \langle\langle i | \beta \rangle \chi_{c,j}(\tilde{q}) \tag{2.1.22}$$

Comparison of the two expressions yields:

$$\sum_i S_{ij} n_{\alpha\beta}^i = \langle \alpha | j \rangle\rangle \langle\langle j | \beta \rangle \tag{2.1.23}$$

If we now consider a boundary state $|\tilde{0}\rangle$ such that the only representation that occurs in the Hamiltonian $H_{\tilde{0}\tilde{0}}$ is the identity, that is to say $n_{\tilde{0}\tilde{0}}^i = \delta_{i,0}$, it immediately follows from (2.1.23) that $S_{0,j} = \left| \langle\langle j | \tilde{0} \rangle \right|^2$, so that since $S_{0,j} > 0$ because of unitarity we have:

$$|\tilde{0}\rangle = \sum_j \sqrt{S_{0,j}} | j \rangle\rangle \tag{2.1.24}$$

and similarly requiring that only the representation l propagates in $H_{\tilde{0}\tilde{l}}$ (i.e. $n^i_{\tilde{0}\tilde{l}} = \delta_{i,l}$), we get:

$$|\tilde{l}\rangle = \sum_j \frac{S_{l,j}}{\sqrt{S_{0,j}}}|j\rangle\rangle \tag{2.1.25}$$

This is an important result because using the Ishibashi states and modulare invariance of the partition function we have been able to build a complete correspondence between bulk and boundary states.

In particular by taking for $|\alpha\rangle$ and $|\beta\rangle$ two states as above we easily get the result:

$$\sum_i S_{i,j} n^i_{\tilde{k}\tilde{l}} = \frac{S_{k,i} S_{l,j}}{S_{0,j}} \tag{2.1.26}$$

which, once we know the modular S matrix tells us all we need to know about the operator content of theories with certain boundary conditions.

2.2   Perturbation Theory

So far we have dealt only with critical theories, but it's also interesting to approach off critical theories by studying perturbations to the conformally invariant action inside a Path Integral formalism.

This will lead us to distinguish a new class of field theories which shares an important property with the Conformal Field Theories, that is the existence of an infinite set of conserved currents which will make the theory in principle completely solvable (*Integrable*). Further details and rigorous proofs of what follows can be found in [6][1][7][2].

2.2.1   Breakdown of Conformal Symmetry

If we now consider some CFT with a conformally invariant action S_{CFT} and perturb this critical fixed point with some linear combination of relevant operators so that:

$$S = S_{CFT} + \sum_i \lambda_i \int d^2 z \phi_i(z, \overline{z}) \tag{2.2.1}$$

we have that in general the perturbed action loses scaling invariance so that the stress tensor stops being a traceless object, but anyway since the action is given as a sum of an invariant plus a symmetry breaking term, we are still able to analyze the pattern of such a breaking.

The perturbed action may then flow under the Renormalization Group transformations to another fixed point, which might be another CFT (for example the $\phi_{1,3}$ perturbation of $\mathcal{M}_{p,p-1}$ flows to $\mathcal{M}_{p-1,p-2}$), or simply a (noncritical) massive field theory.

Let us now consider for the sake of simplicity the effect of a perturbation with a single relevant primary field ϕ, recalling (1.2.3) and (1.2.2), under $z \to z + \epsilon(z)$ we find the following expression for the variation of the action:

$$\delta S = \frac{1}{2} \int d^2 z (\partial \cdot \epsilon) \Theta(z, \overline{z}) \tag{2.2.2}$$

where Θ denotes the trace of the stress tensor which happens to be also its spin 0 part. On the other hand since $\delta S_{CFT} = 0$ using (1.4.3) we get:

$$\delta S = \lambda \int d^2 z \delta \phi(z, \overline{z}) = \lambda \int d^2 z (h-1)(\partial \cdot \epsilon) \phi(z, \overline{z}) \tag{2.2.3}$$

so that we find the following expression for the trace component of the stress tensor:

$$\Theta(z, \overline{z}) = 2\lambda(h-1)\phi(z, \overline{z}) \tag{2.2.4}$$

This example concretely shows how a perturbation may have the effect of breaking conformal invariance, since as we already know the tracelessness of the stress tensor is strictly related to scaling invariance of the theory.

In this context it is also possible to compute correlation functions perturbatively as a series involving only the conformal correlators, this is done by expanding the interaction term as a power series:

$$\langle X \rangle = \int \mathcal{D}[\phi] X e^{S_{CFT} + \lambda \int d^2 w \phi(w, \overline{w})} =$$
$$= \langle X \rangle_{CFT} + \sum_{k=1}^{\infty} \lambda^k \int d^2 w_1 \ldots \int d^2 w_k \langle X \phi(w_1, \overline{w}_1) \ldots \phi(w_k, \overline{w}_k) \rangle_{CFT} \tag{2.2.5}$$

where X denotes an insertion in the Path Integral. The integrals appearing in the last term above are not always finite, and may need an appropriate regularization procedure to make sense.

2.2.2 Deformations of CFT and Integrability

A Conformal Field Theory has an infinite set of conserved currents , for example every expression built purely on the holomorphic or antiholomorphic part of the stress tensor and its derivatives does define a conserved current since the dependence of such objects is purely on the z or $\overline{z}$ variables.

If we wanted to be quantistically rigorous we had to normal order such objects (when there happen to be products of operators that diverge as their points of insertion approach each other), following either the usual normal ordering prescription or the Conformal Normal Ordering prescription, which is simply obtained by arranging all the operator modes in

increasing order after substituting their mode expansions.

It is possible to show that the two prescriptions are equivalent since they both reproduce Wick's theorem, and that it's possible to pass from one prescription from the other.

The existence of this infinite set of conserved currents is the reason that makes a CFT a solvable theory, that is we can in principle determine everything about it, all the theories sharing this property are called *Integrable*.

It can happen that a perturbation of a CFT defines an integrable theory, in this case the deformed theory's conservation laws can be interpreted as deformations of conformal conservation laws.

The first trivial example is the stress tensor itself, which as long as we do not lose traslational invariance remains conserved:

$$\overline{\partial} T(z, \overline{z}) = -\frac{1}{4} \partial \Theta(z, \overline{z}) \qquad (2.2.6)$$

where from (2.2.4) we see what is already ovious, i.e. the fact that as $\lambda \to 0$ the conservation law reduces to the statement that T is purely holomorphic. The corresponding integral of motion is the momentum:

$$P = \oint (dz T + d\overline{z} \Theta) \qquad (2.2.7)$$

In general the study of the deformation of a conservation law is carried out as follows.

Let $J(z, \overline{z})$ be a conserved current for the Conformal Action of dimension $(s, 0)$, the statement that J is conserved must be interpreted as an operator statement, that is to say it holds weighted on the conformal measure:

$$\left\langle X \overline{\partial} J_{z,\dots} \right\rangle_{CFT} + \left\langle X \partial J_{\overline{z},\dots} \right\rangle_{CFT} = 0 \qquad (2.2.8)$$

Let ϕ be the perturbing field, we define the OPE of J and ϕ as:

$$J(z) \phi(w, \overline{w}) = \sum_k \frac{A^{(k)}(w, \overline{w})}{(z - w)^k} \qquad (2.2.9)$$

where the modes $A^{(k)}(z, \overline{z})$ have scaling dimension $(s + h - k, \overline{h})$, and only a finite number of $A^{(k)}(z, \overline{z})$ with $k > 0$ can exist because otherwise for k greater than some $\tilde{k}$ they would have a negative scaling dimension.

We will now substitute this OPE inside the path integral, and carry the calculation out only to first order in λ, although we could in principle carry it out completely since only a finite number of terms are involved.

Now, to first order in λ we have:

$$\left\langle X J(z, \overline{z}) \right\rangle = \left\langle X J(z) \right\rangle_{CFT} + \lambda \int d^2 w \left\langle X J(z) \phi(w, \overline{w}) \right\rangle_{CFT} \qquad (2.2.10)$$

And applying $\overline{\partial}$ to (2.2.10) we finally get:

$$\overline{\partial}\langle XJ(z,\overline{z})\rangle = \lambda\overline{\partial}\int d^2w\langle XJ(z)\phi(w,\overline{w})\rangle_{CFT} \tag{2.2.11}$$

and regulating the integral term with a step function cut off we get:

$$\overline{\partial}\int d^2w\langle XJ(z)\phi(w,\overline{w})\rangle_{CFT} = \overline{\partial}\lim_{a\to 0}\int d^2w H(|z-w|^2-a^2)\langle XJ(z)\phi(w,\overline{w})\rangle_{CFT} =$$

$$= \overline{\partial}\lim_{a\to 0}\sum_k\int d^2w\frac{H(|z-w|^2-a^2)}{(z-w)^k}\langle XA^{(k)}(w,\overline{w})\rangle_{CFT} =$$

$$= \lim_{a\to 0}\sum_k\int d^2w(z-w)\frac{\delta(|z-w|^2-a^2)}{(z-w)^k}\langle XA^{(k)}(w,\overline{w})\rangle_{CFT} =$$

$$= \lim_{a\to 0}\sum_k\int_0^\infty d\rho\rho\int_0^{2\pi}d\theta e^{(1-k)\theta}\rho^{1-k}\delta(\rho^2-a^2)\langle XA^{(k)}(z-\rho e^{i\theta},\overline{z}-\rho e^{-i\theta})\rangle_{CFT} =$$

$$= \lim_{a\to 0}\sum_k\int_0^\infty d\rho\int_0^{2\pi}d\theta e^{(1-k)\theta}\rho^{2-k}\frac{\delta(\rho-a)}{2a}\langle XA^{(k)}(z-\rho e^{i\theta},\overline{z}-\rho e^{-i\theta})\rangle_{CFT} =$$

$$= \sum_k\lim_{a\to 0}\frac{1}{2}\int_0^{2\pi}d\theta e^{(1-k)\theta}a^{1-k}\langle XA^{(k)}(z-ae^{i\theta},\overline{z}-ae^{-i\theta})\rangle_{CFT} =$$

$$= \sum_k\pi\delta_{k,1}\langle XA^{(k)}(z,\overline{z})\rangle_{CFT} = \pi\langle XA^{(1)}(z,\overline{z})\rangle_{CFT}$$

$$\tag{2.2.12}$$

This equation tells us that the Conformal conservation law is spoiled by the perturbation already at the first order unless $A^{(1)}(z,\overline{z})$ is a total z derivative.

This first order result allows us to achieve useful information about the $\phi_{1,3}$ (Thermal) perturbation of the Ising Model, if we consider infact the holomorphic fermion (of dimensions $(1/2,0)$) itself as a current which is conserved in virtue of the equations of motion, considering that $A^{(1)}(z,\overline{z})$ has scaling dimensions $(0,1/2)$ we see that the only possible choice is:

$$A^{(1)} = \overline{\psi} \tag{2.2.13}$$

and all the other expressions for $k>1$ must vanish since they have negative scaling dimensions, so that our first order calculation is an exact calculation that tells us (we give also the antiholomorphic twin equation):

$$\begin{cases} \overline{\partial}\psi = \pi\lambda\overline{\psi} \\ \partial\overline{\psi} = \pi\lambda\psi \end{cases} \tag{2.2.14}$$

and, using (1.6.8) we readily obtain:

$$(\slashed{\partial} - \pi\lambda)\Psi(z,\overline{z}) = 0 \tag{2.2.15}$$

Which is simply the massive Dirac Equation, so that the off critical Ising Model in zero magnetic field is described by a massive fermion theory. This situation provides also an example of situation where the conformal conservation law is broken by the perturbation. A non trivial example of deformable conservarion law is that of $J(z) =: T^2(z) :$, under a $\phi_{1,3}$ perturbation, whose conservation is ensured by the existence of a third level null vector in $\mathcal{V}(1/2, h_{1,3})$. As a consequence we have a spin 3 conserved charge.

Indeed A.B. Zamolodchikov has shown [6] that the $\phi_{1,3}$ perturbation Ising Model possesses an infinite set of conserved currents of the form:

$$X_{2n+k}^{(k+1)} = zX_{2n+k+1}^{(k)} + \overline{z}X_{2n+k-1}^{(k)}$$
$$n \in \mathbb{Z}, \ \ k = -1, 0, 1, 2, \ldots \tag{2.2.16}$$

satisfying:

$$\overline{\partial}X_{2n+k+1}^{(k)} = \partial X_{2n+k-1}^{(k)} \tag{2.2.17}$$

with the corresponding $s = 2n + k$ conserved charges:

$$P_{2n+k}^{(k)} = \oint (X_{2n+k+1}^{(k)}dz + X_{2n+k-1}^{(k)}d\overline{z}) \tag{2.2.18}$$

Where the first term in the recurrence relation (2.2.16) is given by:

$$X_{2n+1}^{(0)} = zT_{2n+2} + \overline{z}T_{2n} \tag{2.2.19}$$

with:

$$\begin{aligned}
&T_{2n} = \lambda^{1-2n} : \partial^{n-1}\psi\partial^n\psi : \ \ , \ n = 1, 2, 3, \ldots \\
&T_0 = \lambda : \overline{\psi}\psi : \sim \Theta \\
&T_2 \sim T \\
&T_{-2n} = \overline{T}_{2n}
\end{aligned} \tag{2.2.20}$$

We notice that $P_{2n-1}^{(-1)}$ corresponds to the integrals of motion of the conformal family of the holomorphic part of the stress tensor.

2.3 Consequences of Integrabiliy on Scattering Theory

If we consider an Integrable field theory with an infinite set of integrals of motion P_s labelled by their spin s, we discover that Intergability constrains the $n-$particle S matrix to be factorized into $n(n - 1)/2$ $2-$particle amplitudes and to satisfy the Yang Baxter Equation.

Such an equation can be considered as a starting point of view in the search for Integrable Models, indeed in an axiomatic approach one looks for self consistent solutions (i.e. S matrices) of the equations describing Integrable Theories, and then tries to gain insight on the integrals of motion. It is in this framework that it has been shown that the magnetic perturbation of the Ising Model is described by a highly nontrivial scattering theory containing 8 particles.

Detailed information about the subject can be found in [7][2][6].

2.3.1 Unitary Equations and Crossing Symmetry

Let us consider a scattering process described in momentum space $A_i(p_i)A_j(p_j) \rightarrow A_k(p_k)A_l(p_l)$. Lorentz invariance constrains the S matrix to be a function of the 3 Mandelstam variables s, t, u which are infact Lorentz scalars:

$$
\begin{aligned}
s &= (p_i + p_j)^2 \\
t &= (p_i - p_k)^2 \\
u &= (p_i - p_l)^2 \\
s + t + u &= \sum_{b=1}^{4} m_b^2
\end{aligned}
\tag{2.3.1}
$$

The constraint satisfied by these variables tells us at a first glance that we need only consider 2 of them, furthermore momentum conservation forces us to discard another one so that we are left with only one independent variable.

Let us now introduce the rapidity variable θ which has the virtue of parametrizing the on mass-shell condition:

$$
p_i = m_i \begin{pmatrix} \cosh \theta_i \\ \sinh \theta_i \end{pmatrix}
\tag{2.3.2}
$$

in this parametrization the s variable can be written as:

$$
s = m_i^2 + m_j^2 + 2m_i m_j \cosh \theta_{ij} \quad \theta_{ij} = \theta_i - \theta_j
\tag{2.3.3}
$$

this tells us that in scattering processes only the relative rapidity θ_{ij} is important, and that the function $s(\theta_{ij})$ is periodic of period $2\pi i$ so that the complex plane (in the variable θ_{ij}) is foliated into strips and as a consequence the S matrix will be characterized by its analytic structure inside one of these strips.

We will now define the S matrix elements through:

$$
\left| A_i(\theta_1) A_j(\theta_2) \right\rangle_{in} = S_{ij}^{kl}(\theta_{12}) \left| A_k(\theta_2) A_l(\theta_1) \right\rangle_{out}
\tag{2.3.4}
$$

Unitarity requirement can be translated into:

$$\sum_{n,m} S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l \qquad (2.3.5)$$

If we want also a crossing symmetry that allows us to equate the S matrix elements of the two processes:

$$A_i(p_i) A_j(p_j) \to A_k(p_k) A_l(p_l)$$
$$A_i(p_i) A_k(p_k) \to A_j(p_j) A_l(p_l) \qquad (2.3.6)$$

the result can be heuristically achieved by observing that the Mandlestam variables $s(\theta)$ and $t(\theta)$ differ as functions of θ only in the sign of the term $2m_i m_j \cosh \theta$, so that we can write $s(\theta) = t(i\pi - \theta)$ (this is true if $m_j = m_k$ so this justifies the exchange $j \leftrightarrow k$ in the S matrix) which suggests us:

$$S_{ik}^{lj}(\theta) = S_{ij}^{kl}(i\pi - \theta) \qquad (2.3.7)$$

More constraints could be imposed on the S matrix if we required it to be invariant under Parity and Time Reversal symmetries:

$$S_{ij}^{kl}(\theta) = S_{ji}^{kl}(\theta) \quad P$$
$$S_{ij}^{kl}(\theta) = S_{kl}^{ij}(\theta) \quad T \qquad (2.3.8)$$

2.3.2 Consequences of Integrability

It is common lore in quantum field theory that to a conserved current is associated a charge that generates a group of symmetries for the system, so that Integrability tells us that the theory under consideration has an $\infty-$dimensional symmetry.

The presence of such an infinite number of constraints on scattering processes has the effect of forbidding particle production, so that only elastic scattering processes may occur, and furthermore the sets of initial and final momenta are forced to coincide. If we label the charges by their spin s and call them P_s we can consider the evolutor:

$$U_s(a) = e^{iaP_s} \qquad (2.3.9)$$

where we observe that U_1 produces a traslation of a on the fields in configuration space, while in general higher s evolutors shift plane waves by a momentum dependent phase, so that by acting appropriately with a combination of such operators we can arbitrarily shift the points of interaction in a generic process without altering the scattering amplitude (let's recall that the P_s generate symmetries of the action, and that the S matrix is a functional of the action).

As a consequence only $2-$particle scatterings are fundamental, and the $n-$particles S

matrix must be factorized into $n(n-1)/2$ $2-$particle amplitudes.

Another consequence comes from equating 2 different ways of factoring $3-$particles processes, which yields the Yang-Baxter Equation:

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 k_3}^{j_1 j_3}(\theta_{13}) S_{k_2 k_3}^{j_2 k_3}(\theta_{23}) = S_{i_1 i_3}^{k_1 k_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) S_{i_1 k_3}^{k_2 j_3}(\theta_{23}) \qquad (2.3.10)$$

where summation over repeated indexes is understood.

2.3.3 Analytic Structure of S and Bootstrap Principle

As we already mentioned when we introduced it, the S matrix has an analytic structure which is completely specified once we have knowledge of its poles in an analitycity strip of width $2\pi i$, the S matrix becomes then a meromorphic function in the complex plane. Let us consider the S matrix in the neighborhood of one of those singularities:

$$S_{ij}^{kl}(\theta) \sim \frac{i R_{ij}^n}{\theta - i u_{ij}^n} \qquad (2.3.11)$$

Such poles represent resonances in scattering processes, which can typically be identified with bound states (n labels the bound state).

The bootstrap principle consists in identifying the bound states themselves as asymptotic states so that (2.3.3) tells us:

$$m_n^2 = m_i^2 + m_j^2 + 2 m_i m_j \cos u_{ij}^n \qquad (2.3.12)$$

This equation provides a constraint on the location of the poles since the cyclic permutations of u_{ij}^n must be the angles of a triangle of sides m_i, m_j, m_n, so that:

$$u_{ij}^n + u_{in}^j + u_{jn}^i = 2\pi \qquad (2.3.13)$$

Now, if we consider a theory with a non degenerate mass spectrum, or anyway where mass degeneracy is resolved by the different eigenvalues under the P_s, we have that the S matrix is diagonal (i.e. the upper indexes are redundant) and that the bootstrap principle can be expressed as:

$$S_{i\bar{l}}(\theta) = S_{ij}(\theta + i\overline{u}_{j\ l}^k) S_{ik}(\theta - i\overline{u}_{l\ k}^j) \qquad (2.3.14)$$

with $\overline{u}_{j\ l}^k = \pi - u_{j\ l}^k$. In this situation the general solution of the unitarity and crossing symmetry equations (2.3.5) and (2.3.7) can be expressed as an arbitrary product of functions $s_x(\theta)$ [7], where:

$$s_x(\theta) = \frac{\sinh((\theta + i\pi x)/2)}{\sinh((\theta - i\pi x)/2)} \qquad (2.3.15)$$

with the $s_x(\theta)$ satisfying the following properties:

$$s_x(\theta) = s_x(\theta + 2\pi i)$$
$$s_x(\theta)s_x(-\theta) = s_{-x}(\theta) = 1$$
$$s_x(\theta) = s_{x+2}(\theta) = s_{-x}(-\theta) \qquad (2.3.16)$$
$$s_0(\theta) = -s_1(\theta) = 1$$
$$s_x(i\pi - \theta) = -s_{1-x}(\theta)$$

Thanks to the periodicity property it is possible to choose the range for the x variable to be the interval $[-1, 1]$. We notice that the s_x have a simple pole at $\theta = i\pi x$ and a simple zero at $\theta = -i\pi x$.

We still have to solve the boostrap equation, this is a very delicate job, since we must look for solutions that are consistent with Integrability and with a set of Bootstrap Fusion Rules we have to choose as an ansatz (characterizing the model).

If we consider an asymptotic state $|A_a(\theta)\rangle$ that is an eigenstate of the P_s with eigenvalue $\omega_s^a(\theta)$ we have that Lorentz invariance constrains ω to have the form:

$$\omega_s^a(\theta) = \chi_s^a e^{s\theta} \qquad (2.3.17)$$

where $\chi_1^a = m_a$; furthermore locality imposes for a many particle state:

$$P_s|A_{a_1}(\theta_1) \ldots A_{a_k}(\theta_k)\rangle = (\omega_s^{a_1}(\theta_1) + \ldots + \omega_s^{a_k}(\theta_k))|A_{a_1}(\theta_1) \ldots A_{a_k}(\theta_k)\rangle \qquad (2.3.18)$$

Now, for some resonant values of the rapidity 2$-$particle states fuse together to give another asymptotic state:

$$\lim_{\epsilon \to 0} \epsilon|A_a(\theta + i\overline{u}_{ac}^b - \frac{\epsilon}{2})A_b(\theta - i\overline{u}_{bc}^a + \frac{\epsilon}{2})\rangle = |A_{\overline{c}}(\theta)\rangle \qquad (2.3.19)$$

and, applying the P_s we get the following set of equations for the χ:

$$\chi_s^a e^{is\overline{u}_{ac}^b} + \chi_s^b e^{is\overline{u}_{bc}^a} = \chi_s^{\overline{c}} \qquad (2.3.20)$$

such equation can be solved only after we have chosen some Bootstrap Fusion Rule:

$$A_i \times A_j = \sum_k N_{ij}^k A_k \qquad (2.3.21)$$

where $N_{ij}^k \in \{0, 1\}$.

A rather simple example of solution of these equations is given by a system containing only one particle with the fusion rule:

$$A \times A \to A \qquad (2.3.22)$$

so that $\overline{u}_{aa}^a = \pi/3$, and (2.3.20) becomes:

$$\cos\left(\frac{s\pi}{3}\right) = \frac{1}{2} \tag{2.3.23}$$

which implies consistency with an infinite set of charges of spin:

$$s = 1, 5, 7, 12, 13, 18, \ldots \tag{2.3.24}$$

This is a rather curios situation since the particle A appears as a bound state of itself.

Bibliography

[1] P.Ginsparg, in: Champs, Cordes et Phenomenes Critiques/Field, String and Critical Phenomena, eds E. Brezin and J. Zinn-Justin (Elsevier,Amsterdam, 1989).

[2] J.L. Cardy, in: Champs, Cordes et Phenomenes Critiques/Field, String and Critical Phenomena, eds E. Brezin and J. Zinn-Justin (Elsevier,Amsterdam, 1989); Phase Trans. Critical Phenom. 11(1987) 55.

[3] J.L. Cardy, Boundary Conditions in Conformal Field Theory, Advanced Studies In Pure Mathematics 19,1989,pp.127

[4] J.L. Cardy and D.C.Lewellen, Bulk and boundary operators in conformal field theory, Physics Letters B 259,3,1991, pp.274.

[5] J.L.Cardy, Conformal Invariance and Surface Critical Behaviour, Nuclear Physics B240[FS12](1984) pp.514 .

[6] J.L.Cardy , Boundary Conditions, Fusion Rules nd the Verlinde Formula, Nuclear Physics B324 (1989) pp.581 .

[7] G.Mussardo, Off-Critical statistical models: factorized scattering theories and bootstrap program, Physics Reports 218, Nos. 5 and 6 (1992) pp.215, North-Holland .

[8] A.B.Zamolodchikov, Integrable Field Theory from conformal Field Theory, Advanced Studies In Pure Mathematics 19, 1989, pp. 641 .

[9] VV. Bazanov, S.L. Lukyanov, A.B. Zamolodchikov, arXiv:hep-th/9412229 (1996), arXiv:hep-th/9604044 (1997), arXiv:hep-th/9805008 (1999) .

[10] R.Sasaki, I.Yamanaka, Adv. Stud. Pure Math. 16(1988) 271 .

Chapter 3

Integrals Of Motion for the A$_3$ Lattice Model

Starting from the lattice A_3 realization of the Ising model defined on a strip with integrable boundary conditions, the exact spectrum (including excited states) of all the local integrals of motion is derived in the continuum limit by means of TBA techniques. It is also possible to follow the massive flow of this spectrum between the UV $c = 1/2$ conformal fixed point and the massive IR theory. The UV expression of the eigenstates of such integrals of motion in terms of Virasoro modes is found to have only rational coefficients and their fermionic representation turns out to be simply related to the quantum numbers describing the spectrum.

PACS: 11.25.Hf

3.1 Introduction

It is well known that a deep connection exists between integrable models of statistical mechanics and integrable quantum field theories. In particular in quantum field theory the Yang Baxter equation (YBE) plays an important role as a constraint on the 2-particles $S-$matrix. On the other hand in statistical mechanics the same equation appears as an equation satisfied by the Boltzmann Weights. Boltzmann weights satisfying the YBE are then used to build families of commuting transfer matrices, which is another quite general feature of integrability for lattice models.

If we take as a prototype the $\mathbf{A}_n$ RSOS models one finds that the Boltzmann Weights can be understood as elliptic solutions of the YBE, and actually it is possible to recognize that the solutions one finds are connected with $S-$matrices in the Sine-Gordon theory [1].

Furthermore it is well known that upon a suitable restriction of the couplings the Sine-Gordon model is equivalent to minimal conformal field theories [2][3][4], and actually also the $\mathbf{A}_n$ RSOS have been shown to be in the universality class of minimal CFTs [22].

Nonetheless the $\mathbf{A}_n$ are in correspondence with CFTs only in an appropriate continuum limit called UV scaling limit. In general there will be a continuum scaling limit depending on a mass parameter μ which will generate a RG flow to a massive IR theory, where the relevant processes will essentially be the scattering of kinks.

The continuum field theory corresponding to $\mathbf{A}_n$ models can be interpreted as a $\phi_{1,3}$ thermal perturbation of the $\mathcal{M}_{n,n+1}$ minimal conformal field theory [4] . Such a perturbation is known to be integrable [6][7], this means that there exists an infinite number of commuting currents which remain conserved in the perturbed theory.

In particular the first conserved quantity is the energy, if one considers its value on the vacuum state it is well known that this is proportional to the central charge of the underlying CFT in the UV limit. The flow to the IR of such a quantity represents an example of the famous $c-$theorem.

A powerful tool for having access to the vacuum energy is the Thermodynamic Bethe Ansatz (TBA)(see for example [8]), which in some cases has been generalized to excited states [22][9][10][11].

In this work we derive the excited TBA equations for the $\mathbf{A}_3$ model on a strip with integrable boundary conditions by diagonalizing the transfer matrix. We then proceed to define the continuum scaling limit of the transfer matrix eigenvalues which we then use as generating functions for some quantities which we eventually identify with the conserved quantities of the thermally perturbed conformal field theory.

It is then possible to analytically follow all the conserved quantities along the massive flow to the IR theory. Comparison of the results which are obtained in the UV limit with the spectrum of the BLZ local integrals of motion provides an exact identification of the conserved quantities and allows to put the lattice boundary conditions in correspondence with the CFT operator content of the theory.

The eigenstates of the BLZ integrals of motion are computed in their Virasoro form, and once expressed in terms of the fermion field turn out to be labelled by the same quantum numbers which label the exact formula for their eigenvalues which has been derived independently through TBA.

3.2   The A_3 Model

The $\mathbf{A}_3$ model is a lattice model which provides a convenient realization of the Ising model. It is built on a square lattice where to each site j is assigned a height variable

$a_j \in \{1, 2, 3\}$. The local height variables a_j are constrained to satisfy an adjacency rule which holds for i, j nearest neighbours:

$$|a_i - a_j| = 1 \qquad (3.2.1)$$

The model is characterized by its Boltzmann Weights [16], which can be of two different types: bulk weights and boundary weights. The only nonvanishing bulk weights are:

$$W \left(\begin{matrix} a \pm 1 & a \\ a & a \mp 1 \end{matrix} \middle| u \right) = \frac{\theta_1(\lambda - u, q)}{\theta_1(\lambda, q)}$$

$$W \left(\begin{matrix} a & a \pm 1 \\ a \mp 1 & a \end{matrix} \middle| u \right) = \frac{\sqrt{\theta_1((a - 1)\lambda, q)\theta_1((a + 1)\lambda, q)}}{\theta_1(a\lambda, q)} \frac{\theta_1(u, q)}{\theta_1(\lambda, q)} \qquad (3.2.2)$$

$$W \left(\begin{matrix} a & a \pm 1 \\ a \pm 1 & a \end{matrix} \middle| u \right) = \frac{\theta_1(a\lambda \pm u, q)}{\theta_1(a\lambda, q))}$$

while the non vanishing boundary weights are:

$$K_L \left(\begin{matrix} a \\ & a \mp 1 \\ a \end{matrix} \middle| u \right) = \sqrt{\frac{\theta_1((a \mp 1)\lambda, q)}{\theta_1(a\lambda, q)}} \frac{\theta_4(u \mp \xi_L(a), q)\theta_4(u \pm a\lambda \pm \xi_L(a), q)}{\theta_4^2(\lambda, q)} \qquad (3.2.3)$$

$$K_R \left(\begin{matrix} & a \\ a \mp 1 & \\ & a \end{matrix} \middle| u \right) = \sqrt{\frac{\theta_1((a \mp 1)\lambda, q)}{\theta_1(a\lambda, q)}} \frac{\theta_4(u \mp \xi_R(a), q)\theta_4(u \pm a\lambda \pm \xi_R(a), q)}{\theta_4^2(\lambda, q)} \qquad (3.2.4)$$

where $\lambda = \pi/4$ is the so called crossing parameter, u is the spectral parameter and the ξ are related to the choice of boundary condition.

We also define the elliptic theta functions of nome q as:

$$\begin{cases} \theta_1(u, q) = 2q^{1/4} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \sin((2k + 1)u) & |q| < 1 \\ \theta_2(u, q) = 2q^{1/4} \sum_{k=0}^{\infty} q^{k(k+1)} \cos((2k + 1)u) & |q| < 1 \\ \theta_4(u, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2ku) & |q| < 1 \end{cases} \qquad (3.2.5)$$

this is the so called q-series, which will prove more useful to our goal, more typical definitions of these functions are given in terms of infinite products.

It is important to remark that the role of the nome q is to control the criticality of the model, which becomes critical as $q \to 0$. In what will follow we will focus on the region $0 < q < 1$ which is the so called regime III of [22], actually this regime corresponds to

a low temperature phase, but because of the duality symmetry of the Ising model this is the same as a high temperature phase.

Now, in terms of the above objects it is known that the model admits a transfer matrix description (on a lattice of width N):

$$
\langle a_1 \ldots a_{N+1} | \mathbf{T}(u) | b_1 \ldots b_{N+1} \rangle = \sum_{c_1 \ldots c_{N+1}} K_L \left(\begin{array}{c|c} b_1 \\ c_1 & \lambda - u \\ a_1 \end{array} \right) \cdot
$$

$$
\cdot \left[\prod_{j=1}^{N} W \left(\begin{array}{cc} c_j & c_{j+1} \\ a_j & a_{j+1} \end{array} \middle| u \right) W \left(\begin{array}{cc} b_j & b_{j+1} \\ c_j & c_{j+1} \end{array} \middle| \lambda - u \right) \right] K_R \left(\begin{array}{c|c} b_{N+1} \\ c_{N+1} & u \\ a_{N+1} \end{array} \right) \tag{3.2.6}
$$

such a transfer matrices form a one parameter commuting family with respect to the spectral parameter u, and it is well known that this property makes the model integrable. The transfer matrix $\mathbf{T}(u)$ satisfies the following functional equation:

$$
\mathbf{T}(u)\mathbf{T}(u + \lambda) = \left(\mathbf{1} + \mathbf{d}(u) \right) F_N(u) \, S(u, \xi_L, \xi_R) = \mathcal{F}(u, q) \tag{3.2.7}
$$

With

$$
F_N(u, q) = \left[\frac{\theta_1(u - \lambda)\theta_1(u + \lambda)}{\theta_1(\lambda)^2} \right]^{2N} \tag{3.2.8}
$$

$$
S(u, \xi_L, \xi_R) = \frac{\theta_1(2u - 2\lambda)\theta_1(2u + 2\lambda)}{\theta_1(2u - \lambda)\theta_1(2u + \lambda)} \, \mathcal{A}_L(u, q, \xi_L, a_L)\mathcal{A}_R(u, q, \xi_R, a_R) \tag{3.2.9}
$$

where $\mathbf{d}$ is a matrix proportional to the identity that takes the form:

$$
\mathbf{d}(u, q) = \mathbf{1}(-1)^N \left[\frac{\theta_1(u)\theta_1(u - 2\lambda)}{\theta_1(u - \lambda)\theta_1(u + \lambda)} \right]^{2N} \left\{ \frac{\theta_1(2u)^2}{\theta_1(2u - 2\lambda)\theta_1(2u + 2\lambda)} \right\} \left(\mathcal{B}_L(u, \xi_L, a_L)\mathcal{B}_R(u, \xi_R, a_R) \right) \tag{3.2.10}
$$

being

$$
\mathcal{B}_L = e^{i\pi a_L} \frac{\theta_4(u + \pi/4 - \xi_L)\theta_4(u - \pi/4 - \xi_L)\theta_4(u + \pi/4 + \xi_L)\theta_4(u - \pi/4 + \xi_L)}{\theta_4(u - a_L\pi/4 - \xi_L)\theta_4(u + a_L\pi/4 + \xi_L)\theta_4(u - (a_L + 2)\pi/4 - \xi_L)\theta_4(u + (a_L - 2)\pi/4 + \xi_L)} \tag{3.2.11}
$$

$$
\mathcal{B}_R = e^{-i\pi a_R} \frac{\theta_4(u + \pi/4 - \xi_R)\theta_4(u - \pi/4 - \xi_R)\theta_4(u + \pi/4 + \xi_R)\theta_4(u - \pi/4 + \xi_R)}{\theta_4(u - a_R\pi/4 - \xi_R)\theta_4(u + a_R\pi/4 + \xi_R)\theta_4(u - (a_R + 2)\pi/4 - \xi_R)\theta_4(u + (a_R - 2)\pi/4 + \xi_R)} \tag{3.2.12}
$$

$$
\mathcal{A}_L(u, q, \xi_L, a_L) = \frac{\theta_4(u - \xi_L)\theta_4(u + \xi_L)\theta_4(u + a_L\pi/4 + \xi_L)\theta_4(u - a_L\pi/4 - \xi_L)}{\theta_4(\lambda)^4} \tag{3.2.13}
$$

$$
\mathcal{A}_R(u, q, \xi_R, a_R) = \frac{\theta_4(u - \xi_R)\theta_4(u + \xi_R)\theta_4(u + a_R\pi/4 + \xi_R)\theta_4(u - a_R\pi/4 - \xi_R)}{\theta_4(\lambda)^4} \tag{3.2.14}
$$

The phases in the $\mathcal{B}$ terms may seem strange, but they turn out to be necessary. This fact has been observed also in [13] where the TBA equations for the $\mathbf{A}_3$ model were derived. Such a matrix satisfies a functional equation which for obvious reasons is called the Inversion Equation:

$$\mathbf{d}(u)\mathbf{d}(u+\lambda) = \mathbf{1} \tag{3.2.15}$$

As a consequence of the simple form of the $\mathbf{d}$ matrix (which for more complicated models is not diagonal but is expressed in terms of $\mathbf{T}$ itself), we have that the Functional equation written in terms of the eigenvalues T of $\mathbf{T}$ is independent of the eigenvalue under consideration.

Before moving on to discuss the TBA equations, it is useful to spend some word to comment on the periodicities of the transfer matrix $\mathbf{T}(u)$. Such periodicities come directly from the properties of the elliptic θ functions and read:

$$\mathbf{T}(u+\pi) = \mathbf{T}(u) \tag{3.2.16}$$

$$\mathbf{T}(u-i\log q) = \mathbf{T}(u) \tag{3.2.17}$$

As a consequence we have that $\mathbf{T}$ is a doubly periodic function which is completely defined by its analytic properties inside a rectangle that we may take as:

$$(-\frac{\lambda}{2}, \frac{7}{2}\lambda) \times i(\frac{1}{2}\log(q), -\frac{1}{2}\log(q)) \tag{3.2.18}$$

If we now consider the functional equation (3.2.7) it is clear that D's periodicities are inherited by the righthand side $\mathcal{F}(u)$ so that the object of our interest will be the zeroes of $\mathcal{F}$ inside the periodicity rectangle (3.2.18).

Such zeroes can be shown by numerical analisys (and indeed analytically in the critical limit [18]) to be organized on lines parallel to $u = \lambda/2 + ix$ with periodicity λ. It can be argued (see again [18] for the critical case) that as a consequence of the periodicities and of the structure of the functional equation (3.2.7) the eigenvalues $T(u)$ must have zeroes which are organized in structures called $1 - strings$ and $2 - strings$.

1-strings are just single zeroes of real part $\lambda/2$ and imaginary part $0 < v_k < -1/2\log q$ such that:

$$T(\lambda/2 \pm iv_k) = 0 \quad k = 1, \ldots, m \tag{3.2.19}$$

where m denotes the number of 1-strings, while 2 strings are couples of zeroes sharing the same imaginary part, while their real part takes the values $\lambda/2 \pm \lambda$, and we shall call their number n.

It is clear that a 1-string is a zero both for $T(u)$ and for $\mathcal{F}(u)$, therefore is will be convenient in our case to look for such zeroes in the expression for $\mathcal{F}$. Such zeroes coul in
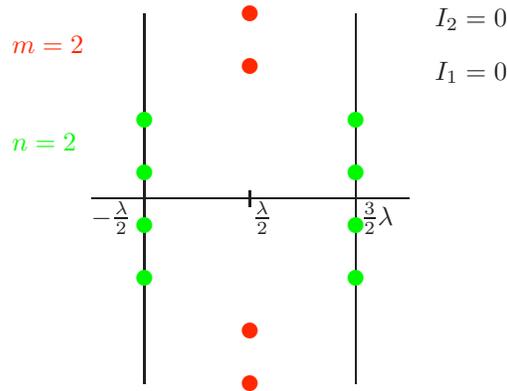
Figure 3.1. Example of the structure of zeroes labeled by the topological number {0,0}

principle be found in either one of the three factors of which $\mathcal{F}$ is composed, but since (as we shall shortly see) F_N and S are going to be eliminated in the scaling limit 1-strings will essentially be zeroes of the $(1 + d(u))$ term. It is finally worth mentioning that for finite N it is possible to give a characterization of the states (i.e. transfer matrix eigenvalues) in terms of a non increasing sequence of numbers $\{I_1, I_2, \ldots, I_m\}$ called *quantum numbers* which express the position of 1-strings related to the position of 2-strings.

Each I_k tells us how many 2-strings the k-th 1-string has to go through in order to reach its position in the pattern of zeroes starting from the configuration where all the 2-strings are heaped on the bottom, so that ordering the imaginary part of the 1-strings v_k into an increasing sequence $\{v_k\}_{k=1}^m$ we have that the quantum numbers $\{I_k\}$ must necessarily arrange into a non increasing succession.

Clearly, the I_k have to satisfy the following constraint:

$$I_k \leq n \ \forall k \tag{3.2.20}$$

Such a characterization of the eigenvalues in terms of 1-strings and 2-strings also happens to give us a natural criterion for ordering the states, first of all we order the states by their increasing m value, the ordering between equal m states is done so that the state with all the 2-strings at the bottom of the tower comes first, and then each time a 2-string is pushed over a 1-string the "energy" increases by one "unit".

We shall see that in the continuum limit a more natural set of quantum numbers will arise to describe the pattern of zeroes.

3.3 Excited TBA Equations

In this section we are going to derive the excited state TBA equations for the A_3 model by solving (3.2.7). Considerable work has been done in the past on the excited states TBA, here we will essentially follow the work of [22].

First of all let us recall the form of the Functional equation (3.2.7), we then define an x coordinate in the following way:

$$u = \frac{\lambda}{2} + i\frac{x}{4}, \qquad T_1(x) \overset{\text{def}}{=} T(u) \tag{3.3.1}$$

where we are going to solve (3.2.7) for the following values of x:

$$x \in (2\log q, -2\log q) \tag{3.3.2}$$

for convenience we will rewrite (3.2.7) after applying a traslation:

$$u \to u - \frac{\lambda}{2}: \qquad T(u - \frac{\lambda}{2})\, T(u + \frac{\lambda}{2}) = \mathcal{F}(u - \frac{\lambda}{2}) \tag{3.3.3}$$

we then use (3.3.1) to write (3.2.7) in the following form:

$$u = \frac{\lambda}{2} + i\frac{x}{4}, \quad T_1(x + i\frac{\pi}{2})\, T_1(x - i\frac{\pi}{2}) = \mathcal{F}(u - \frac{\lambda}{2}) = \mathcal{F}(i\frac{x}{4}) \overset{\text{def}}{=} \mathcal{F}_1(x) \tag{3.3.4}$$

At this point we could be tempted to follow the solution method used in [22] and try to Fourier-expand the logaritmic derivative of (3.3.4), anyway before being allowed to do so, we have to remove the zeroes of $T_1(x)$ in order to deal with an analytic function for which a Fourier expansion does make sense.

Now, if we consider what has been said in the previous section about the position of the zeroes, one observes that for real x, $|x| < -2\log q$, the function $T_1(x)$ which are due to the presence of 1-strings.

In order to reach our result we define the function

$$p(x, v_k) = i\frac{\theta_1(\frac{i}{2}(x - 4v_k), q^2)}{\theta_2(\frac{i}{2}(x - 4v_k), q^2)} \tag{3.3.5}$$

We observe that the p function satisfies an equation which is similar to $T_1(x)$:

$$p(x + i\frac{\pi}{2}, v_k)\, p(x - i\frac{\pi}{2}, v_k) = 1 \tag{3.3.6}$$

furthermore we observe that p can be used to collect all the zeroes (for real x) of T_1 through the product:

$$\prod_{k=1}^{m} p(x, v_k)p(x, -v_k) \tag{3.3.7}$$

so that we can assert that the function

$$T_{\mathrm{ANZ}}(x) \overset{\text{def}}{=} \frac{T_1(x)}{\displaystyle\prod_{k=1}^{m} p(x, v_k) p(x, -v_k)} = T_1(x) \prod_{k=1}^{m} \frac{\theta_2(\frac{i}{2}(x - 4v_k), q^2)}{\theta_1(\frac{i}{2}(x - 4v_k), q^2)} \frac{\theta_2(\frac{i}{2}(x + 4v_k), q^2)}{\theta1(\frac{i}{2}(x + 4v_k), q^2)}$$

(3.3.8)

does not have zeroes for real x $|x| < -2 \log q$ (ANZ stands for analytic and not zero).
We then observe that as a consequence of (3.3.6)T_{ANZ} still satisfies the Functional equation:

$$T_{\mathrm{ANZ}}(x + i\frac{\pi}{2}) \, T_{\mathrm{ANZ}}(x - i\frac{\pi}{2}) = \mathcal{F}_1(x) \tag{3.3.9}$$

so that now one is authorized to fourier-expand the logarithmic derivative of the above equation.Bymeans of some algebra one can determine $T_{ANZ}(x)$, and thus $T_1(x)$ to be:

$$\log T_1(x) = \sum_{k=1}^{m} \log[p(x, v_k) p(x, -v_k)] + k * \log \mathcal{F}_1 + D \tag{3.3.10}$$

Where $k(x)$ is a convolution kernel defined as:

$$k(x - y) = -\frac{1}{4 \log q} \sum_{k=-\infty}^{\infty} \frac{e^{\frac{ik\pi(x-y)}{2 \log q}}}{e^{-\frac{k\pi^2}{4 \log q}} + e^{\frac{k\pi^2}{4 \log q}}} \tag{3.3.11}$$

The convolution kernel $k(x)$ can be computed in terms of Elliptic θ functions. It has been computed in [22] to have the following form:

$$k(x, q) = \frac{\theta_2(0, q^4) \theta_3(0, q^4) \theta_3(ix, q^4)}{2\pi \theta_2(ix, q^4)} \tag{3.3.12}$$

Finally if we recall $\mathcal{F}(u)$'s definition we can write:

$$\log \mathcal{F}_1(x) = \log \mathcal{F}(i\frac{x}{4}) = \log\left(1 + d(i\frac{x}{4})\right) + \log F_N(i\frac{x}{4}) + \log S(i\frac{x}{4}, \xi_L, \xi_R) \tag{3.3.13}$$

so that remembering the expression for $F_N(i\frac{x}{4})$ we observe that its logarithm gives a contribution proportional to N and thus can be identified with a *Bulk Energy* term (which diverges in the thermodynamic limit $N \to \infty$), while $S(i\frac{x}{4}, \xi_L, \xi_R)$ gives a contribution independent of N which anyway diverges as we approach the critical regime ($q \to 0$) and can thus be identified with a *Surface Energy*.
Now, since in the next section we are going to deal with the continuum limit of $\log T(u)$ (which consists both of $N \to \infty$ and $q \to 0$), it is natural to conclude by defining a subtracted Energy so as to give rise to meaningful quantities in the continuum limit:

$$\log T_{\mathrm{finite}}(x) \overset{\text{def}}{=} \log T_1(x) - k * \log\left[F_N(i\frac{x}{4}) S(i\frac{x}{4}, \xi_L, \xi_R)\right] \tag{3.3.14}$$

and, more explicitly:

$$\log T_{\text{finite}}(x) = \sum_{k=1}^{m} \log[p(x, v_k)p(x, -v_k)] + k * \log\left(1 + d(i\frac{x}{4})\right) \tag{3.3.15}$$

Where we observe that the constant of integration D has finally disappeared in the subtraction, corresponding to a shift in the vacuum energy.

3.4   Scaling Limit

Taking a scaling limit of a lattice model essentially means considering its critical behaviour in the thermodynamical limit.

such a double limit ($N \to \infty$, $q \to 0$) can in principle be computed along infinite paths, however it can be shown [22] that there exists a prescription which allows us to obtain a meaningful limit for $\log D_{finite}$.

such a prescription has the form:

$$q = t^{\frac{1}{2}} \tag{3.4.1}$$

$$u = \frac{i}{4}(x + \log N) \tag{3.4.2}$$

where the size N and the *reduced temperature t* satisfy:

$$N \to \infty \tag{3.4.3}$$

$$Nt^{\nu} = \mu \tag{3.4.4}$$

ν is understood as the critical exponent of the correlation length, which for the $\mathbf{A}_n$ models in regime III of [22] is known to be:

$$\nu = \frac{n+1}{4} \tag{3.4.5}$$

The μ parameter plays the role a regulator for the continuum system, and it will be used to generate a massive RG flow connecting the UV ($\mu = 0$) and IR ($\mu = +\infty$) fixed points. Such a regulator can be thought of as arising from the product of a mass m and a length R:

$$\mu = \frac{1}{4}mR \tag{3.4.6}$$

being

$$R = \lim_{N \to \infty, l \to 0} Nl \tag{3.4.7}$$

$$m = \lim_{t \to 0, l \to 0} \frac{t^{\nu}}{l} \tag{3.4.8}$$

and l is understood as the lattice spacing.

Our first goal is to build a continuum version of $\mathbf{d}(u,q)$ which we will use to expand the continuum transfer matrix as a series whose coefficients will be the integrals of motion, we will then use the information we will gain to dicuss the full continuum expression for $\log T_{\text{finite}}(x)$.

The limit we are essentially interested in computing is the following:

$$\hat{d}(x) := \lim_{N \to \infty} d\Big(\frac{i}{4}(x + \log N), \Big(\frac{\mu}{N}\Big)^{\frac{1}{2}}\Big) \tag{3.4.9}$$

From now on we shall fix the right boundary to $a_R = 1$ so that already before going into the scaling limit the R boundary term is fixed to:

$$\mathcal{B}_R = -1 \tag{3.4.10}$$

Therefore from now on the only surviving boundary height a_L will be simply called a and we shall realize in what will follow that $a = s$ where s is the Kac label of $\Delta_{r,s}$, and of course in this case $r = a_R = 1$.

In order to achieve the correct scaling behaviour for the boundary term one has to postulate the following scaling behaviour for the boundary parameter ξ (we are dropping the subscript L):

$$\xi \sim \xi^* + \frac{1}{4}\log N \tag{3.4.11}$$

It is also useful to split the boundary term as:

$$\mathcal{B} = e^{i\pi a}\mathcal{B}^+\mathcal{B}^- \tag{3.4.12}$$

so that $\mathcal{B}^+$ depends only on $u + \xi$ and similarly $\mathcal{B}^-$ depends only on $u - \xi$. Proceeding further one finds:

$$\hat{\mathcal{B}}^+ = \frac{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{-2i(\xi^* + \frac{\pi}{4})}}{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{-2i(\xi^* + a\frac{\pi}{4})}}\frac{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{-2i(\xi^* - \frac{\pi}{4})}}{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{-2i(\xi^* - (a+2)\frac{\pi}{4})}} \tag{3.4.13}$$

$$\hat{\mathcal{B}}^- = \frac{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{-2i(\frac{\pi}{4} - \xi^*)}}{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{2i(a\frac{\pi}{4} + \xi^*)}}\frac{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{2i(\frac{\pi}{4} + \xi^*)}}{1 - \mu^{\frac{1}{2}}e^{\frac{x}{2}}e^{2i((a+2)\frac{\pi}{4} + \xi^*)}} \tag{3.4.14}$$

One then notices that in the UV limit $\mu \to 0$ the boundary $\hat{\mathcal{B}}$ flows into

$$\hat{\mathcal{B}} \sim e^{i\pi a} \tag{3.4.15}$$

and takes the value -1 for $a = 1, 3$ and $+1$ for $a = 2$.

After carrying out the calculation one finds the following continuum version of $\mathbf{d}$:

$$\hat{d}(x,\mu) = e^{-8\mu\cosh(x+\log\mu)}\tanh^2\Big(\frac{x + \log\mu}{2}\Big)(\hat{\mathcal{B}}_L\hat{\mathcal{B}}_R) \tag{3.4.16}$$

One should at this point discuss the scaling limit of the convolution term in $\log T_{\text{finite}}(x)$, such a calculation turns out to yield:

$$\lim_{N \to \infty, q \to 0} k * \log\left(1 + d\left(\frac{i}{4}(x + \log N)\right)\right) = \int_{-\infty}^{+\infty} \frac{1}{2\pi \cosh(x - y)} \log(1 + \hat{d}(y)) dy \quad (3.4.17)$$

We can now deal with the scaling limit of the excitations.
As we approach the scaling limit the 1-strings will have the following asymptotic behaviour (y_k is the finite part):

$$4v_k \sim y_k + \log N \quad (3.4.18)$$

The 1-string term will then become:

$$\sum_{k=1}^{m} \log(p(x, v_k)p(x, -v_k)) \sim \sum_{k=1}^{m} \log\left(\tanh\left(\frac{x - y_k + \log \mu}{2}\right) \tanh\left(\frac{x + y_k + \log \mu}{2}\right)\right)$$

$$(3.4.19)$$

3.5 Expansion

3.5.1 Study of the Ground State

We first of all begin by studying the behaviour of the continuum ground state eigenvalue of the transfer matrix; in the previous section we have shown it takes the following form:

$$\log \hat{D}(x) = \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \frac{\log(1 + \hat{d}(y))}{\cosh(x - y)} \quad (3.5.1)$$

such an expression, following the spirit of [20] has to be expanded in the following series:

$$\log \hat{D}(x) = -\sum_{n=1}^{\infty} C_n I_{2n-1}(\mu) e^{(2n-1)x} \quad (3.5.2)$$

which in our case yields the following expression for the vacuum integrals of motion:

$$C_n I_{2n-1}^{vac}(\mu) = \frac{(-1)^n}{\pi} \int_{-\infty}^{+\infty} dy e^{-(2n-1)y} \log(1 + \hat{d}(y)) \quad (3.5.3)$$

we're now going to study this expression in the UV and IR limits.
We start by observing that the above expression can be manipulated into the form:

$$C_n I_{2n-1}^{vac} = \frac{(-1)^n \mu^{2(2n-1)}}{\pi 4^{(2n-1)}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n}} \int_0^{\infty} \frac{dt}{t} t^{(2n-1)} e^{-(t + \frac{16k^2 \mu^2}{t})} \left(\frac{t - 4k\mu}{t + 4k\mu}\right)^{2k} \left(\hat{\mathcal{B}}_R \hat{\mathcal{B}}_L\right)^k$$

$$(3.5.4)$$

Where we once we fix the right boundary to $a_R = 1$ $\hat{\mathcal{B}}_R$ disappears from the equations. One at this point decides to get rid of ξ^* fixing it to zero. Actually one could keep it, and let it scale once again as $\mu \to 0$ so that, for a suitable Λ:

$$\mu^\Lambda e^{4i\xi^*} = \tau \tag{3.5.5}$$

The parameter τ would then generate a flow between different conformal boundary conditions, see for example [16].

Now, by means of standard techniques one can prove that the following inequality holds for all values of a:

$$|C_n I_{2n-1}^{vac}(\mu)| \leq \frac{2\mu^{(2n-1)}}{\pi} \sum_{k=1}^\infty \frac{1}{k} \mathrm{K}_{1-2n}(8k\mu) \tag{3.5.6}$$

where the $K_l(z)$ are the modified Bessel functions of the second kind. A study of the large μ asymptotics of the above series allows one to conclude that the ground states of the integrals of motion decay exponentially in the IR limit.

We are now ready to move our attention to the UV asymptotic behaviour.

In the limit $\mu \to 0$ it is not difficult to show that:

$$C_n I_{2n-1}^{vac}(\mu) \sim \frac{(-1)^{n+1}}{\pi 4^{(2n-1)}} \Gamma(2n - 1) \mathrm{Li}_{2n}(e^{i\pi a}) \tag{3.5.7}$$

where

$$\mathrm{Li}_\nu(z) = \sum_{k=1}^\infty \frac{z^k}{k^\nu} \tag{3.5.8}$$

is the Polylogarithm function. It is worth spending a word to observe that considering the energy I_1 one finds a dilogarithm of a phase, actually it is well known that the central charge is usually proportional to a dilogarithm. Furthermore one notices that all the vacuum expectation values of the integrals of motion are proportional to polylogarithms, this seems to be a rather general structure.One could ask himself if such polylogarithms satisfy sum rules similar to those holding for dilogarithms.

3.5.2   Excited States

We now ask ourselves what is the behaviour of the excitation terms in the UV and IR and compare it with the ground states.

Now we want to expand the 1-string term.

The expansion is readily obtained if one considers the following identity:

$$\log\left(\frac{1-t}{1+t}\right) = -2 \sum_{k=1}^\infty \frac{t^{2k-1}}{2k-1} \tag{3.5.9}$$

and for example writes:

$$\log\tanh\frac{x - y_k + \log\mu}{2} = i\pi - 2\sum_{n=1}^{\infty} e^{(2n-1)x}\frac{e^{-(2n-1)(y_k-\log\mu)}}{2n-1} \tag{3.5.10}$$

Clearly one obtains the following result:

$$\left(1 - string\right) = -\sum_{n=1}^{\infty}\frac{2e^{(2n-1)x}}{2n-1}\left(\sum_{k=1}^{m}\left(e^{-(2n-1)y_k} + \mu^{2(2n-1)}e^{(2n-1)y_k}\right)\right) \tag{3.5.11}$$

In order to proceed further it is necessary to have a closer look at the asymptotic behaviour of the y_k.

Let us recall the equation satisfied by the y_k:

$$\hat{d}\left(y_k - i\frac{\pi}{2}\right) = -1 \tag{3.5.12}$$

so that by taking the logarithm of both sides, using the expression for $\hat{d}$ given previously we have:

$$-4\mu(e^{-2i\lambda}e^{y_k}\mu - e^{2i\lambda}e^{-y_k}\mu^{-1}) + 2\log\left(\frac{1 - e^{2i\lambda}\mu^{-1}e^{-y_k}}{1 + e^{2i\lambda}\mu^{-1}e^{-y_k}}\right) + \log(\hat{\mathcal{B}}_L\hat{\mathcal{B}}_R) = i\pi n_k \tag{3.5.13}$$

where the n_k are odd numbers.

We now introduce the following function:

$$g_k(\mu) := \mu e^{y_k} \tag{3.5.14}$$

we are now interested in expressing the y_k equation in terms of this new function g_k, in order to do so we first rewrite the boundary term as:

$$\hat{\mathcal{B}}\left(y_k - i\frac{\pi}{2}\right) = e^{i\pi a}\frac{(g_k + i)^2}{(g_k + i)^2 - 2ig_k(1 + \cos(\pi a))} \tag{3.5.15}$$

so that the y_k equation becomes simpler and reads:

$$4i\mu\frac{(g_k^2 - 1)}{g_k} + \log\left(\frac{(g_k - i)^2}{(g_k + i)^2 - 2ig_k(1 + \cos(\pi a))}\right) = i\pi(n_k + 1 - a) \tag{3.5.16}$$

this equation gives us the inverse function $\mu(g_k)$, and we are interested in its behaviour as $\mu \to 0$ and $\mu \to +\infty$. In order to reach our goal it suffices to pick the branch of the $\mu(g_k)$ function which passes through the origin, $g_k \to 0$ corresponds to the UV limit, whereas $g_k \to 1$ is the IR limit.

Expanding μ around $g_k = 0$ we obtain:

$$\mu \sim \frac{\pi}{4}(a - 1 - n_k)g_k + O(g_k^2) \tag{3.5.17}$$

If we now decide to rewrite the 1-string term as follows:

$$(1 - string) = -\sum_{n=1}^{\infty} \frac{2e^{(2n-1)x}}{2n-1} \mu^{\eta(2n-1)} \sum_{k=1}^{m} \left(g_k^{(2n-1)} + \frac{1}{g_k^{(2n-1)}} \right) \tag{3.5.18}$$

we conclude by applying the above UV expansion that the expression

$$\mu^{(2n-1)} \left(g_k^{(2n-1)} + \frac{1}{g_k^{(2n-1)}} \right) \tag{3.5.19}$$

is UV limited, so that in this limit the excitations have the same scaling behaviour as the ground state.

On the other hand the IR excitations cannot avoid to grow faster than the ground state term, this observation united to the fact that $g_k \to 1$ dictates the particular structure of the IR spectrum.

3.5.3   Full Expansion

We finally have arrived at the point of writing the analytic expression for all the integrals of motion of the model, such expression reads:

$$\begin{aligned}
C_n I_{2n-1}(\mu) =& \frac{2\mu^{(2n-1)}}{2n-1} \sum_{k=1}^{m} \left(g_k^{(2n-1)} + \frac{1}{g_k^{(2n-1)}} \right) + \\
& + \frac{(-1)^n}{\pi} \int_0^{+\infty} dy\, e^{-(2n-1)y} \log(1 + \hat{d}(y,\mu))
\end{aligned} \tag{3.5.20}$$

In the limit $\mu \to \infty$, $g_k \to 1$ the ground state drops off exponentially, so that considering only the excitations it is immediate to realize that:

$$C_n I_{2n-1}(\mu) \sim \frac{4m\mu^{(2n-1)}}{2n-1} \tag{3.5.21}$$

this spectrum happens to be integrally spaced, and looses all memory of the quantum numbers aside from the length of the sequence of the n_k.

It is worth remarking that this result should not surprise us very much since it is very similar to what has been achieved in [17] for the $\mathbf{A}_4$ model corresponding to the tricritical Ising Model universality class.

In the UV limit it is clear from what we said so far that the integrals of motion have the following behaviour:

$$\begin{aligned}
C_n I_{2n-1}(\mu) \sim& \left(\frac{\pi}{4} \right)^{(2n-1)} \left(\frac{2}{2n-1} \sum_{k=1}^{m} \left(a - 1 - n_k \right)^{(2n-1)} + \right. \\
& \left. + (-1)^n e^{i\pi a} ((1 - e^{i\pi a}) 2^{-2n} - 1) \Gamma(2n-1) \frac{\zeta(2n)}{\pi^{2n}} \right)
\end{aligned} \tag{3.5.22}$$

For $a = 1$ the energy takes the following form

$$C_1 I_1(\mu) \sim \pi \left(\frac{1}{2} \sum_{k=1}^{m} (-n_k) - \frac{1}{48} \right) \tag{3.5.23}$$

For $a = 2$ we have:

$$C_1 I_1(\mu) \sim \pi \left(\frac{1}{2} \sum_{k=1}^{m} (1 - n_k) + \frac{1}{24} \right) \tag{3.5.24}$$

For $a = 3$ we have:

$$C_1 I_1(\mu) \sim \pi \left(\frac{1}{2} \sum_{k=1}^{m} (2 - n_k) - \frac{1}{48} \right) \tag{3.5.25}$$

It is readily recognized that these formulae are in agreement respectively with the $h = 0$, $h = 1/16$ and $h = 1/2$ sectors of the minimal model $\mathcal{M}_{3,4}$ if we choose:

$$C_1 = \pi \tag{3.5.26}$$

One notices that in the vacuum sector m must be even whereas in the $1/2$ sector m must be odd for trivial reasons. In the $1/16$ sector m must be odd but this fact is less trivial to understand from the formula, let us simply say that m−parity is fixed in the sector and is odd because for the highest weight state the only possible quantum number must satisfy $1 - n_1 = 0$.

We will understand better the structure of the quantum numbers in the next section.

If we now want to compute the constants C_2, C_3 in the vacuum sector of the model this can be done by using the explicit expressions for the integrals of motion, which can be found in [1]:

$$\mathbf{I}_1 = L_0 - \frac{c}{24} \tag{3.5.27}$$

$$\mathbf{I}_3 = 2 \sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c + 22)}{2880} \tag{3.5.28}$$

$$\mathbf{I}_5 = \sum_{m,n,p \in \mathbb{Z}} \delta_{m+n+p,\, 0} : L_m L_n L_p : + \frac{3}{2} \sum_{n=1}^{\infty} L_{1-2n} L_{2n-1} +$$

$$+ \sum_{n=1}^{\infty} \left(\frac{11+c}{6} n^2 - \frac{c}{4} - 1 \right) L_n L_{-n} - \frac{c+4}{8} L_0^2 + \frac{(c+2)(3c+20)}{576} L_0 + \tag{3.5.29}$$

$$- \frac{c(3c+14)(7c+68)}{290304}$$

where the $:\ :$ denotes *Conformal Normal Ordering* which can be obtained by arranging all the L_n in an increasing sequence with respect to n.

So that we have the following vacuum expectation values:

$$\langle 0 | \mathbf{I}_3 | 0 \rangle = \frac{49}{11520} \tag{3.5.30}$$

$$\langle 0|\mathbf{I}_5|0\rangle = -\frac{4433}{2322432} \tag{3.5.31}$$

which gives us the following equations:

$$C_2\langle 0|\mathbf{I}_3|0\rangle = \frac{\pi^3\Gamma(3)(1-2^{-3})}{4^3 90} \tag{3.5.32}$$

$$C_3\langle 0|\mathbf{I}_3|0\rangle = -\frac{\pi^5\Gamma(5)(1-2^{-5})}{4^5 945} \tag{3.5.33}$$

so that we readily get:

$$C_2 = \frac{\pi^3}{14} \tag{3.5.34}$$

$$C_3 = \frac{9}{715}\pi^5 \tag{3.5.35}$$

This values for the vacuum constants C_n actually can be extracted from [19], so that in general we have for the vacuum sector the following expression:

$$C_n = \frac{3^n 4^{2-3n}\pi^{-\frac{1}{2}+2n}\Gamma(4n-2)}{n!\Gamma(3n-\frac{1}{2})} \tag{3.5.36}$$

Actually these values of the C_n are computed from the vacuum sector, but direct calculation allows one to verify that they are independent of the sector.

We stress that the above formulas describe exactly the conformal spectrum of the minimal model $\mathcal{M}_{3,4}$.

3.6   Fermionic modes and TBA quantum numbers

Actually the quantum numbers n_k themselves have a very simple interpretation in terms of fermionic modes.

To understand this one needs only to remember that the stress energy tensor for the Ising model is built out of the fermion field as:

$$T(z) := \frac{1}{2} : \psi(z)\partial\psi(z) : \tag{3.6.1}$$

So that by introducing the well known mode expansion

$$i\psi(z) = \sum_n \frac{\psi_n}{z^{n+\frac{1}{2}}} \tag{3.6.2}$$

one gets

$$L_n = \frac{1}{2}\sum_k (k+\frac{1}{2}) : \psi_{n-k}\psi_k : \tag{3.6.3}$$

CFT State	Fermionic State	TBA State
$\lvert 0\rangle$	$\lvert 0\rangle$	$()$
$2L_{-2}\lvert 0\rangle$	$\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(-1,-3)$
$L_{-3}\lvert 0\rangle$	$\psi_{-\frac{5}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(-1,-5)$
$\frac{5}{7}L_{-4}\lvert 0\rangle - \frac{6}{7}L^2_{-2}\lvert 0\rangle$	$\psi_{-\frac{5}{2}}\psi_{-\frac{3}{2}}\lvert 0\rangle$	$(-3,-5)$
$\frac{3}{7}L_{-4}\lvert 0\rangle + \frac{2}{7}L^2_{-2}\lvert 0\rangle$	$\psi_{-\frac{7}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(-1,-7)$
$\frac{3}{7}L_{-5}\lvert 0\rangle - \frac{4}{7}L_{-3}L_{-2}\lvert 0\rangle$	$\psi_{-\frac{7}{2}}\psi_{-\frac{3}{2}}\lvert 0\rangle$	$(-3,-7)$
$\frac{2}{7}L_{-5}\lvert 0\rangle + \frac{2}{7}L_{-3}L_{-2}\lvert 0\rangle$	$\psi_{-\frac{9}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(-1,-9)$
$\frac{5}{14}L_{-6}\lvert 0\rangle + \frac{3}{7}L_{-4}L_{-2}\lvert 0\rangle - \frac{23}{56}L^2_{-3}\lvert 0\rangle$	$\psi_{-\frac{7}{2}}\psi_{-\frac{5}{2}}\lvert 0\rangle$	$(-5,-7)$
$\frac{1}{4}L_{-6}\lvert 0\rangle - \frac{1}{2}L_{-4}L_{-2}\lvert 0\rangle + \frac{1}{16}L^2_{-3}\lvert 0\rangle$	$\psi_{-\frac{9}{2}}\psi_{-\frac{3}{2}}\lvert 0\rangle$	$(-3,-9)$
$\frac{5}{28}L_{-6}\lvert 0\rangle + \frac{3}{14}L_{-4}L_{-2}\lvert 0\rangle + \frac{5}{112}L^2_{-3}\lvert 0\rangle$	$\psi_{-\frac{11}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(-1,-11)$

Table 3.1. U.V. state correspondence CFT$\longrightarrow$TBA for the $h=0$ sector

CFT State	Fermionic State	TBA State
$\lvert 1/2\rangle$	$\lvert 0\rangle$	(1)
$L_{-1}\lvert 1/2\rangle$	$\psi_{-\frac{3}{2}}\lvert 0\rangle$	(-1)
$\frac{2}{3}L_{-2}\lvert 1/2\rangle$	$\psi_{-\frac{5}{2}}\lvert 0\rangle$	(-3)
$\frac{1}{2}L_{-3}\lvert 1/2\rangle$	$\psi_{-\frac{7}{2}}\lvert 0\rangle$	(-5)
$\frac{1}{4}L_{-4}\lvert 1/2\rangle + \frac{1}{8}L_{-3}L_{-1}\lvert 1/2\rangle$	$\psi_{-\frac{9}{2}}\lvert 0\rangle$	(-7)
$\frac{3}{4}L_{-4}\lvert 1/2\rangle - \frac{5}{8}L_{-3}L_{-1}\lvert 1/2\rangle$	$\psi_{-\frac{5}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(1,-1,-3)$
$\frac{3}{16}L_{-5}\lvert 1/2\rangle + \frac{1}{8}L_{-4}L_{-1}\lvert 1/2\rangle$	$\psi_{-\frac{11}{2}}\lvert 0\rangle$	(-9)
$\frac{7}{16}L_{-5}\lvert 1/2\rangle - \frac{3}{8}L_{-4}L_{-1}\lvert 1/2\rangle$	$\psi_{-\frac{7}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(1,-1,-5)$
$\frac{1}{8}L_{-6}\lvert 1/2\rangle + \frac{3}{32}L_{-5}L_{-1}\lvert 1/2\rangle + \frac{1}{36}L_{-4}L_{-2}\lvert 1/2\rangle$	$\psi_{-\frac{13}{2}}\lvert 0\rangle$	(-11)
$\frac{1}{4}L_{-6}\lvert 1/2\rangle - \frac{5}{16}L_{-5}L_{-1}\lvert 1/2\rangle + \frac{1}{18}L_{-4}L_{-2}\lvert 1/2\rangle$	$\psi_{-\frac{9}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(1,-1,-7)$
$\frac{3}{8}L_{-6}\lvert 1/2\rangle + \frac{9}{32}L_{-5}L_{-1}\lvert 1/2\rangle - \frac{13}{36}L_{-4}L_{-2}\lvert 1/2\rangle$	$\psi_{-\frac{7}{2}}\psi_{-\frac{5}{2}}\psi_{-\frac{1}{2}}\lvert 0\rangle$	$(1,-3,-5)$

Table 3.2. U.V. state correspondence CFT$\longrightarrow$TBA for the $h=1/2$ sector

CFT State	Fermionic State	TBA State
$\|1/16\rangle_-$	$\sqrt{2}\psi_0\|1/16\rangle_+$	(1)
$2\sqrt{2}L_{-1}\|1/16\rangle_-$	$\psi_{-1}\|1/16\rangle_+$	(-1)
$\sqrt{2}L_{-2}\|1/16\rangle_-$	$\psi_{-2}\|1/16\rangle_+$	(-3)
$\frac{8}{7}L_{-3}\|1/16\rangle_- - \frac{12}{7}L_{-2}L_{-1}\|1/16\rangle_-$	$\sqrt{2}\psi_{-2}\psi_{-1}\psi_0\|1/16\rangle_+$	(1,-1,-3)
$\frac{2\sqrt{2}}{7}L_{-3}\|1/16\rangle_- + \frac{4\sqrt{2}}{7}L_{-2}L_{-1}\|1/16\rangle_-$	$\psi_{-3}\|1/16\rangle_+$	(-5)
$\frac{5}{8}L_{-4}\|1/16\rangle_- - L_{-3}L_{-1}\|1/16\rangle_-$	$\sqrt{2}\psi_{-3}\psi_{-1}\psi_0\|1/16\rangle_+$	(1,-1,-5)
$\frac{3}{8\sqrt{2}}L_{-4}\|1/16\rangle_- + \frac{1}{\sqrt{2}}L_{-3}L_{-1}\|1/16\rangle_-$	$\psi_{-4}\|1/16\rangle_+$	(-7)
$\frac{1}{2}L_{-5}\|1/16\rangle_- + \frac{3}{2}L_{-4}L_{-1}\|1/16\rangle_- - L_{-3}L_{-2}\|1/16\rangle_-$	$\sqrt{2}\psi_{-3}\psi_{-2}\psi_0\|1/16\rangle_+$	(1,-3,-5)
$\frac{9}{28}L_{-5}\|1/16\rangle_- - \frac{29}{28}L_{-4}L_{-1}\|1/16\rangle_- + \frac{3}{14}L_{-3}L_{-2}\|1/16\rangle_-$	$\sqrt{2}\psi_{-4}\psi_{-1}\psi_0\|1/16\rangle_+$	(1,-1,-7)
$\frac{3}{14\sqrt{2}}L_{-5}\|1/16\rangle_- + \frac{9}{14\sqrt{2}}L_{-4}L_{-1}\|1/16\rangle_- + \frac{1}{7\sqrt{2}}L_{-3}L_{-2}\|1/16\rangle_-$	$\psi_{-5}\|1/16\rangle_+$	(-9)
$\frac{19}{64\sqrt{2}}L_{-6}\|1/16\rangle_- - \frac{5}{8\sqrt{2}}L_{-5}L_{-1}\|1/16\rangle_- + \frac{1}{\sqrt{2}}L_{-4}L_{-2}\|1/16\rangle_- - \frac{9}{16\sqrt{2}}L_{-3}^2\|1/16\rangle_-$	$\psi_{-3}\psi_{-2}\psi_{-1}\|1/16\rangle_+$	(-1,-3,-5)
$\frac{409}{896}L_{-6}\|1/16\rangle_- + \frac{81}{112}L_{-5}L_{-1}\|1/16\rangle_- - \frac{3}{14}L_{-4}L_{-2}\|1/16\rangle_- - \frac{59}{224}L_{-3}^2\|1/16\rangle_-$	$\sqrt{2}\psi_{-4}\psi_{-2}\psi_0\|1/16\rangle_+$	(1,-3,-7)
$\frac{5}{32}L_{-6}\|1/16\rangle_- - \frac{3}{4}L_{-5}L_{-1}\|1/16\rangle_- + \frac{1}{8}L_{-3}^2\|1/16\rangle_-$	$\sqrt{2}\psi_{-5}\psi_{-1}\psi_0\|1/16\rangle_+$	(1,-1,-9)
$\frac{69}{448\sqrt{2}}L_{-6}\|1/16\rangle_- + \frac{29}{56\sqrt{2}}L_{-5}L_{-1}\|1/16\rangle_- + \frac{1}{7\sqrt{2}}L_{-4}L_{-2}\|1/16\rangle_- + \frac{1}{112\sqrt{2}}L_{-3}^2\|1/16\rangle_-$	$\psi_{-6}\|1/16\rangle_+$	(-11)

Table 3.3. U.V. state correspondence CFT⟶TBA for the $h = 1/16$ sector

TBA State	I_3 Eigenvalue	I_5 Eigenvalue
()	$\frac{49}{11520}$	$\frac{-4433}{2322432}$
$(-1,-3)$	$\frac{47089}{11520}$	$\frac{17581135}{2322432}$
$(-1,-5)$	$\frac{211729}{11520}$	$\frac{225292639}{2322432}$
$(-3,-5)$	$\frac{255409}{11520}$	$\frac{242734063}{2322432}$
$(-1,-7)$	$\frac{577969}{11520}$	$\frac{1211381743}{2322432}$
$(-1,-9)$	$\frac{1226449}{11520}$	$\frac{4255847167}{2322432}$
$(-3,-7)$	$\frac{621649}{11520}$	$\frac{1228823167}{2322432}$
$(-5,-7)$	$\frac{786289}{11520}$	$\frac{1436534671}{2322432}$
$(-3,-9)$	$\frac{1270129}{11520}$	$\frac{4273288591}{2322432}$
$(-1,-11)$	$\frac{2237809}{11520}$	$\frac{11607335311}{2322432}$

Table 3.4. Table of eigenvalues in the vacuum sector

TBA State	I_3 Eigenvalue	I_5 Eigenvalue
(1)	$\frac{1729}{11520}$	$\frac{67639}{2322432}$
(-1)	$\frac{45409}{11520}$	$\frac{17509063}{2322432}$
(-3)	$\frac{210049}{11520}$	$\frac{225220567}{2322432}$
(-5)	$\frac{576289}{11520}$	$\frac{1211309671}{2322432}$
(-7)	$\frac{1224769}{11520}$	$\frac{4255775095}{2322432}$
$(1,-1,-3)$	$\frac{257089}{11520}$	$\frac{242806135}{2322432}$
(-9)	$\frac{2236129}{11520}$	$\frac{11607263239}{2322432}$
$(1,-1,-5)$	$\frac{623329}{11520}$	$\frac{1228895239}{2322432}$
(-11)	$\frac{3691009}{11520}$	$\frac{26759824663}{2322432}$
$(1,-1,-7)$	$\frac{1271809}{11520}$	$\frac{4273360663}{2322432}$
$(1,-3,-5)$	$\frac{787969}{11520}$	$\frac{1436606743}{2322432}$

Table 3.5. Table of eigenvalues in the 1/2 sector

TBA State	I_3 Eigenvalue	I_5 Eigenvalue
(1)	$\frac{-7}{1440}$	$\frac{143}{72576}$
(-1)	$\frac{1673}{1440}$	$\frac{72215}{72576}$
(-3)	$\frac{13433}{1440}$	$\frac{2306447}{72576}$
$(1,-1,-3)$	$\frac{15113}{1440}$	$\frac{2378519}{72576}$
(-5)	$\frac{45353}{1440}$	$\frac{17513639}{72576}$
$(1,-1,-5)$	$\frac{47033}{1440}$	$\frac{17585711}{72576}$
(-7)	$\frac{107513}{1440}$	$\frac{73801871}{72576}$
$(1,-3,-5)$	$\frac{58793}{1440}$	$\frac{19819943}{72576}$
$(1,-1,-7)$	$\frac{109193}{1440}$	$\frac{73873943}{72576}$
(-9)	$\frac{209993}{1440}$	$\frac{225225143}{72576}$
$(-1,-3,-5)$	$\frac{60473}{1440}$	$\frac{19892015}{72576}$
$(1,-3,-7)$	$\frac{120953}{1440}$	$\frac{76108175}{72576}$
$(1,-1,-9)$	$\frac{211673}{1440}$	$\frac{225297215}{72576}$
(-11)	$\frac{362873}{1440}$	$\frac{560432015}{72576}$

Table 3.6. Table of eigenvalues in the 1/16 sector

Where the fermionic modes will have a half integer index when we will be working in the $0, 1/2$ sectors, whereas the index will be integer in the twisted $1/16$ sector.

It is then just a matter of unraveling the normal ordering and using the fermionic algebra $\{\psi_n, \psi_m\} = \delta_{n+m,0}$ to work out the fermionic expression for the eigenvectors of the integrals of motion.

If we consider for example the sixth level of descendance in the vacuum sector we have:

$$20L_{-6}|0\rangle + 24L_{-4}L_{-2}|0\rangle + 5L_{-3}^2|0\rangle = 112\psi_{-\frac{11}{2}}\psi_{-\frac{1}{2}}|0\rangle \tag{3.6.4}$$

$$4L_{-6}|0\rangle - 8L_{-4}L_{-2}|0\rangle + L_{-3}^2|0\rangle = 16\psi_{-\frac{9}{2}}\psi_{-\frac{3}{2}}|0\rangle \tag{3.6.5}$$

$$20L_{-6}|0\rangle + 24L_{-4}L_{-2}|0\rangle - 23L_{-3}^2|0\rangle = 56\psi_{-\frac{7}{2}}\psi_{-\frac{5}{2}}|0\rangle \tag{3.6.6}$$

So that comparing with table 3.1 we see that $\frac{n_k}{2}$ are simply the labels of the fermionic modes, and we can easily either guess or explicitly work out straightforwardly the form of all the other eigenstates which is obvious aside from a normalization.

Similarly in the $1/2$ sector one has at level 6:

$$36L_{-6}|1/2\rangle + 27L_{-5}L_{-1}|1/2\rangle + 8L_{-4}L_{-2}|1/2\rangle = 288\psi_{-\frac{13}{2}}|0\rangle \tag{3.6.7}$$

$$36L_{-6}|1/2\rangle - 45L_{-5}L_{-1}|1/2\rangle + 8L_{-4}L_{-2}|1/2\rangle = 144\psi_{-\frac{9}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle \tag{3.6.8}$$

$$108L_{-6}|1/2\rangle + 81L_{-5}L_{-1}|1/2\rangle - 26L_{-4}L_{-2}|1/2\rangle = 72\psi_{-\frac{7}{2}}\psi_{-\frac{5}{2}}\psi_{-\frac{1}{2}}|0\rangle \tag{3.6.9}$$

so that the fermionic modes have indexes which are simply $\frac{n_k-2}{2}$ and one understands how the fermionic represetation of table 3.2 should be.

The $1/16$ sector has to be worked out expanding the Virasoro modes in integer fermionic modes, this is not very different from the previous situation, except for the presence of the zero-mode ψ_0 which generates the zero mode algebra:

$$\psi_0|1/16\rangle_- = \frac{1}{\sqrt{2}}|1/16\rangle_+ \tag{3.6.10}$$

$$\psi_0|1/16\rangle_+ = \frac{1}{\sqrt{2}}|1/16\rangle_- \tag{3.6.11}$$

So that actually there are 2 $1/16$ vacua, one has no 1-strings ($|1/16\rangle_+$) and the other has 1 1-string ($|1/16\rangle_-$). The fermionic modes have indexes which are simply $\frac{n_k-1}{2}$, so that the (1) quantum number is recognized as coming from the insertion of ψ_0.

For example one has:

$$|1/16\rangle_- = \sqrt{2}\psi_0|1/16\rangle_+ \tag{3.6.12}$$

$$L_{-1}|1/16\rangle_- = \frac{1}{2\sqrt{2}}\psi_{-1}|1/16\rangle_+ \tag{3.6.13}$$

$$L_{-2}\big|1/16\big\rangle_{-} = \frac{1}{\sqrt{2}}\psi_{-2}\big|1/16\big\rangle_{+} \qquad (3.6.14)$$

So that comparing with table 3.3 we easily understand the how things work out and, once again, aside from the normalizations one knows perfectly from the beginning which results he will find upon expressing eigenstates in terms of fermionic modes.

It has to be remarked that the fermionic description of the eigenstates of the RSOS transfer matrix fits naturally into the description of [20], where it is shown that the TBA quantum numbers can be described also in terms of "fermionic paths".

Finally one notices that the TBA quantum numbers appearing in the 1/16 and in the 1/2 sector are actually the same, this can be understood from the existence of a boundary flow connecting the 2 sectors, see for example [21].

3.7 Conclusion

In this work we have derived the excited TBA equations for the A_3 model defined on a strip with integrable boundary conditions. The continuum limit of the TBA equations was then derived and an axpansion defined which led to integrals of motion. Such integrals of motion were identified to be the BLZ local integrals of motion. The identification was carried out by comparing the exact diagonalization of the Virasoro expressions with the spectrum derived form TBA. The eigenvalues were found to be exactly the same and the eigenvectors, once expressed in terms of fermionic modes, turned out to be labelled by the same quantum numbers as the eigenvalues.

It has to be remarked that this is the first time that an exact diagonalization of all the BLZ local integrals of motion has been carried out in a particular case including all the excited states.

A more detailed analisys of the spectra obtained would surely be interesting and will be the subject of successive work.

3.8 Acknowledgments

Bibliography

[1] A.N. Kirillov and N.Yu. Reshetikhin, Representations of the Algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, LOMI preprint E-9-88

[2] D.Bernard, A.Leclair, Residual Quantum Symmetries Of The Restricted Sine-Gordon Theories, Nucl.Phys.B340 (1990) 721-751

[3] A.Leclair, Restricted Sine-Gordon Theory and the Minimal Conformal Series, Physics Letters B 230 1,2 1989 pp.103

[4] N.Reshtikhin, F.Smirnov, Hidden Quantum Group Symmetry and Integrable Perturbations of Conformal Field Theories, Commun.Math.Phys. 157-177 (1990)pp.157

[5] P. A. Pearce and B. Nienhuis, "Scaling Limit of RSOS Models and TBA Equations", Nucl. Phys. B **519** (1998) 579 [arXiv:hep-th/9711185].

[6] A.B.Zamolodchikov, Integrable Field Theory from conformal Field Theory, Advanced Studies In Pure Mathematics 19, 1989, pp. 641 .

[7] Subir Ghoshal, Alexander Zamolodchikov, Bounday S-Matrix and Boundary State in Two-Dimensional Integrable Quantum field Theory, Int.J.Mod.Phys. A9 (1994) 3841-3886; (hep-th/9306002)

[8] G.Mussardo, Off-Critical statistical models: factorized scattering theories and bootstrap program, Physics Reports 218, Nos. 5 and 6 (1992) pp.215, North-Holland .

[9] M.J. Martins, Complex Excitations in the thermodynamic Bethe-Ansatz Approach, Phys.Rev.Lett. 67 (1991) 419

[10] Patrick Dorey, Roberto Tateo, Excited States by Analytic Continuation of TBA Equations, Nucl.Phys. B482 (1996) 639-659 (hep-th/9607167)

[11] Paul Fendley, Excited States Thermodynamics,Nucl.Phys. B374 (1992) 667-691 (hep-th/9109021)

[12] R. E. Behrend, P. A. Pearce and D. L. O'Brien, "Interaction - round - a - face models with fixed boundary conditions: The ABF fusion hierarchy", J. Stat. Phys.84 (1996) 1-48, arXiv:hep-th/9507118.

[13] L. Chim, P. Pearce, TBA equations for a strip with integrable boundary conditions I: $\mathbf{A}_3$ -UNPUBLISHED

[14] D.O'Brien, P.Pearce, S.O.Waarnar, Finitized Conformal Spectrum of the Ising Model on the Cylinder and Torus, Physica A 228 (1996) pp.63 .

[15] P.Grinza, G.Feverati, Integrals of Motion from TBA and lattice-conformal dictionary, Nucl.Phys. B702 (2004) 495-515, (arXiv:hep-th/0405110) .

[16] C. Ahn, C. Rim, Boundary flows in general coset theories, J.Phys A: Math. Gen. 32 (1999) 2509-2525

[17] Paul.A.Pearce, Leung Chim, Changrim Ahn, Excited TBA Equations I: Massive Tricritical Ising Model,Nucl.Phys. B601 (2001) 539-568 (arXiv:hep-th/0012223).

[18] VV. Bazanov, S.L. Lukyanov, A.B. Zamolodchikov, Commun.Math.Phys. 177 (1996) 381-398 (arXiv:hep-th/9412229)

[19] VV. Bazanov, S.L. Lukyanov, A.B. Zamolodchikov, Nucl.Phys. B489 (1997) 487-531, (arXiv:hep-th/960709)

[20] G.Feverati and P.A.Pearce, "Critical RSOS and Minimal models I: Paths, Fermionic Algebras and Virasoro Modules",Nucl.Phys. B663 (2003) 409-442 (hep-th/0211185)

[21] F.Lesage, H.Saleur, P.Simonetti, "Boundary flows in minimal models",Phys.Lett. B427 (1998) 85-92 (hep-th/980206)

Chapter 4

Integrals Of Motion for $\mathcal{LM}_{1,2}$

Abstract

We consider critical dense polymers $\mathcal{LM}(1,2)$. This model is exactly integrable on the square lattice and in the continuum sclaing limit yields a logarithmic conformal field theory.

IOM, TBA, Fermionic States, Symplectic Fermions.

4.1 Introduction

It is well known [4] that the CFT corresponding to critical dense polymers has central charge $c = -2$. Such a conformal field theory is known to be logarithmic, these theories, in contrast with rational CFTs, can be realized by different models for the same value of the central charge and conformal weights. For example Hamiltonian walks on a Manhattan lattice [5, 6], the rational triplet theory[7, 8, 9], symplectic fermions [10, 11], the Abelian sanpile model [12], dimers [13], the traveling salesman problem [14] and branching polymers [15] all share the same value of the central charge, which is -2.
In particular we shall learn something along the way of the relation between critical dense polymers and symplectic fermions.
The layout of the paper proceeds by reviewing some common lore about the CFT correponding to critical dense polymers. In section 2 the lattice model is introduced, the transfer matrix is explicitly built from the Boltzmann weights and the inversion identities and selection rules are also discussed.
In section 3 we derive the TBA equations for the model and deal with its continuum limit. The eigenvalues of the BLZ involutive charges are obtained by expanding the eigenvalues

of the continuum scaled transfer matrix. In section 3.3 a new result is obtained, that is after having identified the involutive charges we are able to perform a $1/N$ expansion for the eigenvalues in which the conserved charges explicitly appear. Such an expansion is then manipulated to obtain an alternative form that provides the eigenvalues for the lattice involutive charges. In this new framework the eigenvalues of the transfer matrix are expressed in terms of Bell polynomials, and the ivnversion identity itself is expressed in terms of these polynomials.

In section 4 these results are extended to the transfer matrix itself and the $N-$tangles corresponding to the lattice involutive charges are explicitly built in terms of the Temperley Lieb algebra, thus providing a reason for the long calculation of section 3.3.

In section 5 we describe the relation of the model with symplectic fermions. We give a description of selection rules for (r, s) boundary conditions which is completely analogous to the lattice one, we decompose all the characters in terms of characters of certain fermionic modules built over the Virasoro algebra. And finally we describe the Jordan decomposition of the continuum tranfer matrix corresponding to modules with the same conformal weight but different (r, s). It is shown how $(1, s)$ modules correspond to diagonalizable tranfer matrices in agreement with the lattice behaviour of the model, on the other hand for $r \neq 1$ it can happen that the tranfer matrix ehibits a nontrivial jordan canonical form. The resaons underlying this result are then discussed in view of the results of the paper.

4.1.1   CFT

The CFT corresponding to critical dense polymers has central charge $c = -2$ and is a logarithmic CFT. It is the first member $\mathcal{L}(1, 2)$ of the logarithmic minimal models $\mathcal{L}(p, p')$ [3] with central charges

$$c = 1 - \frac{6(p - p')^2}{pp'} \tag{4.1.1}$$

With respect to the Virasoro conformal symmetry, it admits an infinite number of representations. In general, these representations are not irreducible — some are reducible yet indecomposable. The so-called Kac representations, labelled by $r, s = 1, 2, 3, \ldots$, can be arranged in an infinitely extended Kac table as in Figure 4.1. The scaling dimensions are given by the usual minimal Kac formula with $p = 1$ and $p' = 2$

$$\Delta_{r,s} = \frac{(2r - s)^2 - 1}{8} \tag{4.1.2}$$

Actually, the first column with $r = 1$ (shown shaded in Figure 4.1) contains all of the distinct values of the conformal weights.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
6	3	1	0	0	1	$\cdots$
$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
3	1	0	0	1	3	$\cdots$
$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
1	0	0	1	3	6	$\cdots$
$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
0	0	1	3	6	10	$\cdots$
$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
0	1	3	6	10	15	$\cdots$

Figure 4.1. Kac table of critical dense polymers.

The characters of the Kac representations corresponding to $\Delta_{r,s}$ are obtained by modding out the submodule generated by the null vector at level rs and are given by

$$\chi_{r,s}(q) = q^{-\frac{c}{24}} \frac{q^{\Delta_{r,s}}(1 - q^{rs})}{\prod_{n=1}^{\infty}(1 - q^n)} \tag{4.1.3}$$

These characters are obtained in the limit as $N \to \infty$ from finitized characters

$$\chi_{r,s}^{(N)}(q) = q^{-c/24 + \Delta_{r,s}} \left(\begin{bmatrix} N \\ (N-s+r)/2 \end{bmatrix}_q - q^{rs} \begin{bmatrix} N \\ (N-s-r)/2 \end{bmatrix}_q \right) \tag{4.1.4}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is a q-binomial or Gaussian polynomial.

4.2 Critical Dense Polymers

We will consider in this paper an exactly solvable model of critical dense polymers on a square lattice [4]. The degrees of freedom are localized on elementary faces, which can be found in one of the following two configurations:
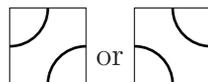
$$\boxed{} \text{ or } \boxed{} \tag{4.2.1}$$

where the arcs represent segments of the polymer. The elementary faces belong to the planar Temperley-Lieb algebra [2], and therefore satitfy the following simple equations:

$$(4.2.2)$$

where the dashed lines indicate that the corners and associated incident edges are identified.

The parameter β represents the loop fugacity which, for critical dense polymers, is set to zero. This means that the polymer is not allowed to form closed loops. Therefore it passes twice through each face of the lattice, and in the continuum scaling limit it is dense or space filling, in the sense that its fractal dimension is 2.

The transfer matrix is built out of local face operators or 2-tangles $X(u)$ and boundary 1-triangles.

The local face operators are defined diagrammatically in the planar TL algebra:

$$X(u) = \boxed{u} = \cos(u)\ \diagup + \sin(u)\ \diagdown \qquad (4.2.3)$$

which means that the weights assigned to the elementary face conficgurations are

$$W\left(\diagup\right) = \cos(u), \qquad W\left(\diagdown\right) = \sin(u) \qquad (4.2.4)$$

The local face operators satisfy the Yang-Baxter equation as well as an Inversion Identity. The $(1, s)$ boundary 1-triangles are defined as the following solutions to the boundary Yang Baxter equation [3]:

$$(4.2.5)$$

$s - 1$ columns

The YBEs, supplemented by additional local relations, are sufficient to imply commuting transfer matrices and integrability. To work on a strip with fixed boundary conditions on the right and left, we need to work with N column Double-row Transfer Matrices

represented schematically in the planar TL algebra by the N-tangle

$$\boldsymbol{D}(u) \;=\; \text{(4.2.6)}$$

This schematic representation in the *planar* TL algebra needs to be interpreted appropriately to write $\boldsymbol{D}(u)$ in terms of the generators of the *linear* TL algebra and to write down its associated matrix:

$$\boldsymbol{D}(u) =$$

$$j = \quad -1 \quad 0 \quad 1 \quad \cdots \quad N{-}1 \;\; N$$

(4.2.7)

For $(1, s)$ boundary conditions the transfer matrix acts on link states with $\ell = s - 1$ defects which have to be glued into the $(1, s)$ boundary triangle as exeplified in figure 4.2.

4.2.1 Inversion Identities

For $(1, s)$ boundary conditions the tranfer matrix satisfies an inversion identity [3], which by virtue of commutativity is satisfied also by its eigenvalues:

$$D(u)D(u + \frac{\pi}{2}) = \left(\frac{\cos^{2N}(u) - \sin^{2N}(u)}{\cos^2(u) - \sin^2(u)} \right)^2 = \mathcal{F}_N(u) \qquad (4.2.8)$$
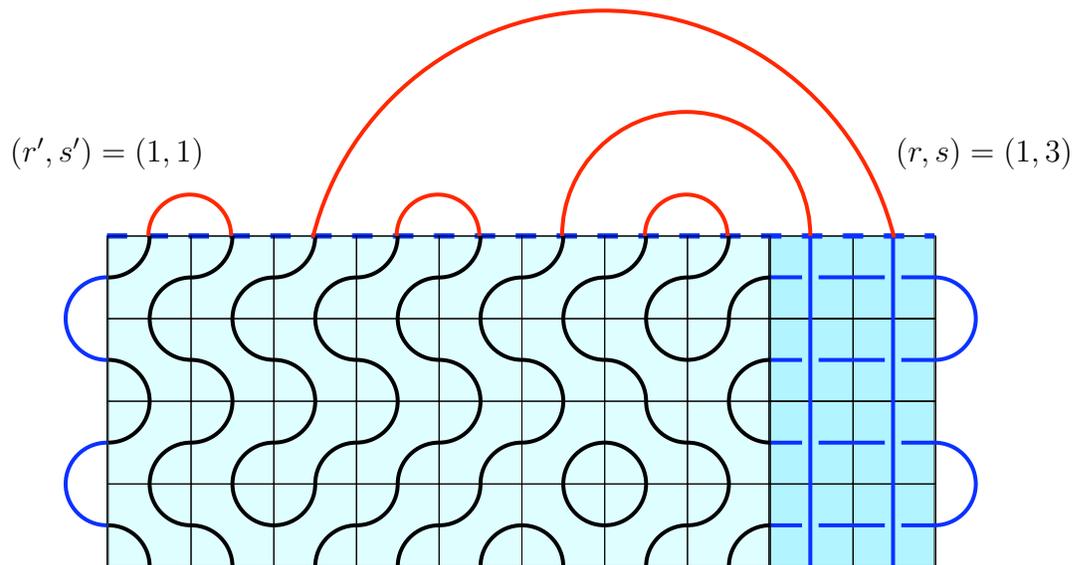
Figure 4.2. A typical configuration on the strip showing connectivities. The action on the link state is explained in the next section. The boundary condition is of type $(r', s') = (1, 1)$ on the left and type $(r, s) = (1, 3)$ on the right so there are $\ell = s-1 = 2$ defects in the bulk. The strings propagating along the right boundary are spectators connected to the defects.

Such an identity does not depend on s and it can be solved exactly for finite N, yielding a number of solutions which is larger than the size of the **D** matrix.

The idea behind the solution is the observation that $\mathcal{F}_N(u)$ is an entire function of u which can be factorized exactly. The eigenvalues $D(u)$ are determined by sharing out the zeroes of $\mathcal{F}$ between the two factors on the righthand side of (4.2.8).

The function $\mathcal{F}$, due to being a square, has only double zeroes which we can define through:

$$\mathcal{F}_N(\frac{\pi}{4} + iv_k) = 0 \qquad (4.2.9)$$

where

$$v_k = -\frac{1}{2} \log \tan(\frac{t_j}{2}) \qquad (4.2.10)$$

being $t_j = \frac{j\pi}{N}$ for even N whereas $t_j = \frac{(2j-1)\pi}{2N}$ for odd N.

It follows then that the factorized form of the eigenvalues is for even $N = 2L$:

$$D(u) = 2L2^{1-2L} \prod_{k=1}^{L-1} (\operatorname{cosec}(\frac{\pi k}{2L}) + \epsilon_k \sin(2u))(\operatorname{cosec}(\frac{\pi k}{2L}) + \mu_k \sin(2u)) \qquad (4.2.11)$$

whereas for odd $N = 2L + 1$:

$$D(u) = 2^{-2L} \prod_{k=1}^{L} (\operatorname{cosec}(\frac{\pi}{2}\frac{2k-1}{2L+1}) + \epsilon_k \sin(2u))(\operatorname{cosec}(\frac{\pi}{2}\frac{2k-1}{2L+1}) + \mu_k \sin(2u)) \qquad (4.2.12)$$

such solutions, however, are too many and one needs to impose some selection rules to pick the correct $(1, s)$ conformal boundary conditions.

The different sectors are chosen by applying selection rules to the combinatorics of zeroes.

A typical pattern of zeroes for the eigenvalues for $N = 12$ is:

$$(4.2.13)$$
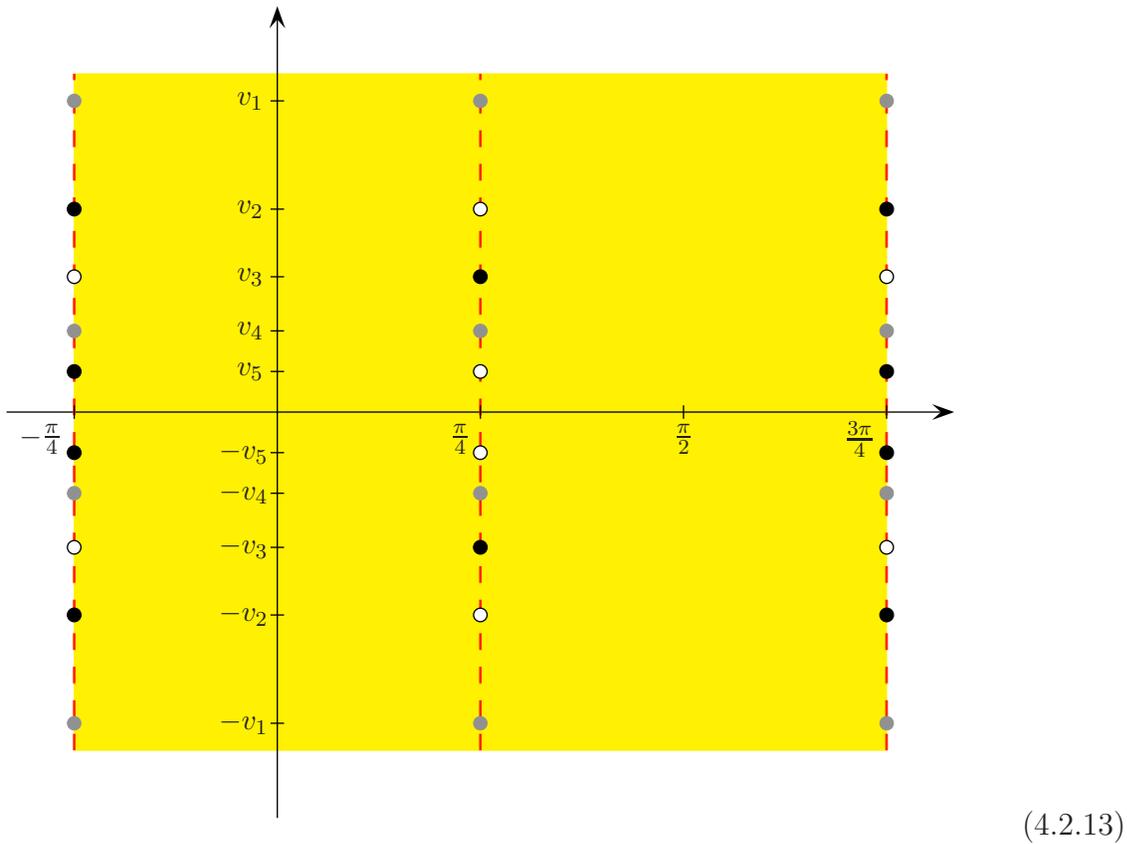
A single zero is indicated by a grey dot while a double zero is indicated by a black dot.

4.2.2   Selection Rules

A two column configuration is a couple $(\boldsymbol{l}|\boldsymbol{r})$ of vectors both of length M with integral entries arranged in decreasing order.

A two column configuration is called *admissible* if, calling m the length of $\boldsymbol{l}$ one has:

$$l_k \leq r_k, \;\; k = 1, \ldots, m \tag{4.2.14}$$

If the length of $\boldsymbol{r}$ is greater than the length of $\boldsymbol{l}$, $\boldsymbol{l}$ is understood as completed with a sequence of zeroes.

It follows then that to each zero pattern of the eigenvalues it is possible to associate only a single two-column configuration can be described as described in figure 4.2.15, where one is describing the state $(3|4, 3, 1)$.

$$(4.2.15)$$

The label k in l_k, r_k is *the same k* as in 4.2.11 (we will understand this better from the IOM) and one has:

$$\epsilon_n = -1, \text{ if } n \in \{l_1, \dots, l_M\}, \ \epsilon_n = 1 \text{ otherwise} \tag{4.2.16}$$

$$\mu_n = -1, \text{ if } n \in \{r_1, \dots, r_M\}, \ \mu_n = 1 \text{ otherwise} \tag{4.2.17}$$

We recall from [4] that the set $A_{m,n}^M$ is the set of all *admissible* two column diagrams of height M with m occupied sites on the left and n occupied sites on the right.

To each two column diagram $\mathcal{D}$ is associated a weight:

$$w(\mathcal{D}) = \sum_i l_i + \sum_j r_j \tag{4.2.18}$$

one then defines:

$$\left\langle \begin{matrix} M \\ m,n \end{matrix} \right\rangle_q = \sum_{\mathcal{D} \in A_{m,n}^M} q^{w(\mathcal{D})} \tag{4.2.19}$$

$$\left\langle \begin{matrix} M \\ m,n \end{matrix} \right\rangle_q = 0, \quad \text{if } A_{m,n}^M = \emptyset \tag{4.2.20}$$

one then has the following Fermionic formuals for the finitized characters [4].

For odd s one has:

$$\chi_{1,s}^{(N)}(q) = q^{\frac{1}{12}} \sum_{m=0}^{\frac{N-s+1}{2}} \left(\left\langle \begin{matrix} \frac{N}{2} \\ m,m+\frac{s-3}{2} \end{matrix} \right\rangle_q + \left\langle \begin{matrix} \frac{N-2}{2} \\ m,m+\frac{s-1}{2} \end{matrix} \right\rangle_q \right) \tag{4.2.21}$$

For even s, one has:

$$\chi_{1,s}^{(N)}(q) = q^{-\frac{1}{24} - \frac{s-2}{4}} \sum_{m=0}^{\frac{N-s+1}{2}} \left\langle \begin{matrix} \frac{N-1}{2} \\ m,m+\frac{s-2}{2} \end{matrix} \right\rangle_q q^{-m} \tag{4.2.22}$$

Clearly $\left\langle \begin{matrix} M \\ m,n \end{matrix} \right\rangle_q$ is the character associated to the set $A_{m,n}^M$ with respect to the weight introduced above.

From these expressions one can read off at first sight which two column diagrams are allowed to contribute to a given sector. The above characters can be reduced to the form 4.1.4 by means of the identity

$$\left\langle \begin{matrix} M \\ m,n \end{matrix} \right\rangle_q = q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)} \left(\begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M \\ n \end{bmatrix}_q - q^{n-m+1} \begin{bmatrix} M \\ n+1 \end{bmatrix}_q \begin{bmatrix} M \\ m-1 \end{bmatrix}_q \right) \tag{4.2.23}$$

4.3   TBA and Integrals of Motion

4.3.1   Derivation of TBA

The functional equation for the eigenvalues of critical dense polymers is

$$D(u)D(u + \frac{\pi}{2}) = \left(\frac{\cos^{2N}(u) - \sin^{2N}(u)}{\cos^2(u) - \sin^2(u)} \right)^2 = \mathcal{F}_N(u) \tag{4.3.1}$$

the derivation of TBA equations follows closely the work on Ising [19]. The difference being essentially that since we are at criticality one has to use fourier integrals instead of fourier series.

First of all we define:

$$u = \frac{\pi}{4} + \frac{i}{2}x \tag{4.3.2}$$

$$D_1(x) := D(u) \tag{4.3.3}$$

$$\mathcal{F}_1(x) := \mathcal{F}(i\frac{x}{2}) \tag{4.3.4}$$

one then has that the inversion identity takes the form:

$$D_1(x - i\frac{\pi}{2})D_1(x + i\frac{\pi}{2}) = \mathcal{F}_1(x) \tag{4.3.5}$$

The function $D_1(x)$ has real zeroes and we shall use auxiliary functions to remove the unwanted zeroes:

$$p(x, v_k) = i\tan(\frac{i}{2}(x - 2v_k)) \tag{4.3.6}$$

which satisfy

$$p(x + i\frac{\pi}{2}, v_k)p(x - i\frac{\pi}{2}, v_k) = 1 \tag{4.3.7}$$

one then factors the zeroes in the following way:

$$D_1(x) := D_{ANZ}(x) \prod_k p(x, v_k)p(x, -v_k) \tag{4.3.8}$$

where D_{ANZ} is analytic and non zero (ANZ). T_{ANZ} then by virtue of 4.3.7 still satisfies the same functional equation:

$$D_{ANZ}(x - i\frac{\pi}{2})D_{ANZ}(x + i\frac{\pi}{2}) = \mathcal{F}_1(x) \tag{4.3.9}$$

One then fourier transforms the logarithmic derivative of 4.3.9 and proceeds precisely as in [19] to obtain the TBA equations:

$$\log D_1(x) = \sum_{k=1}^{m} \log(p(x, v_k)p(x, -v_k)) + k * \log \mathcal{F}_1 \tag{4.3.10}$$

with k being the usual convolution kernel

$$k(x) = \frac{1}{2\pi \cosh(x)} \tag{4.3.11}$$

such a kernel arises from an integral of the type:

$$\int_{-\infty}^{+\infty} dk \frac{e^{i\alpha kx}}{e^{\beta k} + e^{-\beta k}} = \frac{\pi}{2\beta} \frac{1}{\cosh\left(\frac{\pi \alpha x}{2\beta}\right)} \tag{4.3.12}$$

which evaluated using the residues method, and actually can be used as a tool to figure out how the fourier tranform is defined.

To obtain a meaningful expression in the continuum limit one has to subtract the divergent part out of $\mathcal{F}_1$. so one has for even N

$$\mathcal{F}(u + \frac{i}{2}\log N) \sim \frac{2N^{2(N-1)e^{-4i(N-1)u}}}{4^{2N-1}}(\cos(4e^{2iu}) - 1) \tag{4.3.13}$$

whereas for odd N

$$\mathcal{F}(u + \frac{i}{2}\log N) \sim \frac{2N^{2(N-1)e^{-4i(N-1)u}}}{4^{2N-1}}(\cos(4e^{2iu}) + 1) \tag{4.3.14}$$

the scaling limit of the p functions is computed by using the exact result for the 1-strings:

$$v_k = -\frac{1}{2}\log \tan(\frac{t_j}{2}) \tag{4.3.15}$$

being $t_j = \frac{j\pi}{N}$ for even N whereas $t_j = \frac{(2j-1)\pi}{2N}$ for odd N. It turns out that for even N one has:

$$\hat{p}(x, -v_k) = \tanh(\frac{1}{2}(x + \log\left(\frac{k\pi}{2}\right))) \tag{4.3.16}$$

and

$$\hat{p}(x, v_k) = 1 \tag{4.3.17}$$

the convolution term after subtracting the explicit $N-$divergent term looks like:

$$k * \mathcal{F}_1 \sim \int_{-\infty}^{+\infty} \frac{dy}{2\pi \cosh(x - y)} \log(\cosh(4e^{-y}) \pm 1) \tag{4.3.18}$$

so that by calling $\hat{D}(x)$ the continuum scaled version of $D_1(x)$ one has the following continuum TBA:

$$\log \hat{D}(x) = \sum_{k=1}^{m} \log(\hat{p}(x, v_k)\hat{p}(x, -v_k)) + \int_{-\infty}^{+\infty} \frac{dy}{2\pi \cosh(x - y)} \log(\cosh(4e^{-y}) \pm 1) \tag{4.3.19}$$

4.3.2   Integrals of motion

We shall now deal with the expansion of $\hat{D}$ which will yield the eigenvalues of the BLZ involutive charges. Again, as in [19][20] we consider the following expansion:

$$\hat{D}(x) = -\sum_{n=1}^{\infty} U_n I_{2n-1} e^{(2n-1)x} \tag{4.3.20}$$

where the I_{2n-1} are the eigenvalues of the BLZ involutive charges.

For the auxiliary functions one uses an expansion like:

$$\log \tanh(\frac{x + 2y_k}{2}) = i\pi - 2\sum_{n=1}^{\infty} e^{(2n-1)x} \frac{\left(\frac{k\pi}{2}\right)^{(2n-1)}}{2n - 1} \tag{4.3.21}$$

We must be careful, and consider both the contributions of single and double 1-strings, in the case of double 1-strings the summation term carries an additional 2 factor coming from the log of a square.

the convolution term can also be expanded by means of:

$$\frac{1}{\cosh x} = -2\sum_{n=1}^{\infty}(-1)^n e^{(2n-1)x} \tag{4.3.22}$$

yielding:

$$U_n I_{2n-1}^{vac} = \frac{(-1)^{n+1}}{\pi 4^{2n-1}} \int_0^{\infty} dt \ t^{2(n-1)} \log(\cosh(t) \pm 1) \tag{4.3.23}$$

sadly enough after subtracting the divergent part the integral above is still ill defined, and we have to further subtract an $N-$independent part. This can be done by placing a cut off and integrating twice by parts. One then gets:

$$\int_0^{\Lambda} dt \ t^{2(n-1)} \log(1 + \cosh(t)) = \frac{1}{2n(2n-1)} \int_0^{\Lambda} ds \frac{s^{2n}}{1 + \cosh(s)} +$$
$$- \frac{1}{2n(2n-1)} \Lambda^{2n} \frac{\sinh \Lambda}{1 + \cosh \Lambda} + \frac{1}{2n - 1}\Lambda^{2n-1}\log(1 + \cosh(\Lambda)) \tag{4.3.24}$$

So that the integral on the right hand side is now convergent and the divergent part has been isolated in the surface terms.

Actually it is possible to check that (see appendix):

$$\int_0^{\infty} ds \frac{s^{2n}}{1 + \cosh(s)} = 4n(1 - 2^{1-2n})\Gamma(2n)\zeta(2n) \tag{4.3.25}$$

and also, in the other case that the useful integral is:

$$\int_0^{\infty} ds \frac{s^{2n}}{\cosh(s) - 1} = 4n\Gamma(2n)\zeta(2n) \tag{4.3.26}$$

So that one now gets in one case:

$$U_n I_{2n-1}^{vac} = \frac{(-1)^{n+1}}{\pi 4^{2n-1} 2n(2n-1)} \int_0^\infty ds \frac{s^{2n}}{1 + \cosh(s)} \qquad (4.3.27)$$

whereas on the other hand:

$$U_n I_{2n-1}^{vac} = \frac{(-1)^{n+1}}{\pi 4^{2n-1} 2n(2n-1)} \int_0^\infty ds \frac{s^{2n}}{\cosh(s) - 1} \qquad (4.3.28)$$

so that piecing up one gets

$$U_n I_{2n-1} =$$
$$2\left(\frac{\pi}{4}\right)^{(2n-1)} \left(\frac{2}{2n-1} \sum_{j \in A} \left(2j - 1\right)^{(2n-1)} + \frac{1}{2n-1} \sum_{j' \in A'} \left(2j' - 1\right)^{(2n-1)} + (-1)^n (1 - 2^{1-2n})\Gamma(2n-1)\frac{\zeta(2n)}{\pi^{2n}}\right)$$

$$(4.3.29)$$

so that one realizes that this formula describes the $h = -1/8$ sector of the $c = -2$ theory, and notices that the above expression is identical to ising, aside from a factor of 2 and contibutions of double zeroes.

The vacuum is described instead by

$$U_n I_{2n-1} =$$
$$2\left(\frac{\pi}{4}\right)^{(2n-1)} \left(\frac{2}{2n-1} \sum_{j \in A} \left(2j\right)^{(2n-1)} + \frac{1}{2n-1} \sum_{j' \in A'} \left(2j'\right)^{(2n-1)} + (-1)^{n+1}\Gamma(2n-1)\frac{\zeta(2n)}{\pi^{2n}}\right)$$

$$(4.3.30)$$

and it is also identical to ising, aside from what has been remarked above.

The constants are found by direct comparison with CFT to be:

$$U_1 = \pi \qquad (4.3.31)$$

$$U_2 = \frac{\pi^3}{12} \qquad (4.3.32)$$

$$U_3 = \frac{\pi^5}{60} \qquad (4.3.33)$$

and so on. Actually one notices that these constants are precisely those one can obtain from the ising model by requiring to describe the $c = -2$ theory instead (aside from the factor of 2 appearing in front of the formula). Actually the above expression *is* the same expression as in the Ising case, this is true of course if we consider the largest eigenvalue. The excitations are a bit different due to the presence of double zeroes.

It may be instructive to observe that the behaviour of excitations in this model is actually

encoded in some properties of the bernoulli polynomials. If we take the BLZ formula for highest weight eigenvalues in the first column of Kac's table:

$$I_{2n-1}^{vac} = 2^{-n} B_{2n}\left(\frac{s-1}{2}\right) \tag{4.3.34}$$

and use the following property of Bernoulli polynomials

$$B_{2n}\left(\frac{s-1}{2}\right) = B_{2n}\left(\frac{s-3}{2}\right) + 2n\left(\frac{s-3}{2}\right)^{2n-1} \tag{4.3.35}$$

we realize that $B_{2n}(0), B_{2n}(\frac{1}{2})$ (so $s = 1, 2$) represent the highest weights $0, -\frac{1}{8}$, and that going beyond those in Kac's table implies adding a certain number of 1-strings which is at this point trivially guessed.

Suppose for example s to be odd, one then has:

$$I_{2n-1}^{vac} = 2^{1-n} n \sum_{j=1}^{\frac{s-3}{2}} j^{2n-1} + 2^{-n} B_{2n} \tag{4.3.36}$$

Actually this shows also that the odd power behaviour of 1-string contributions is actually encoded in the BLZ formula and therefore this formula alone should be enough to suggest the structure of all the excited states.

One can be even more explicit, and resum the contribution of the quantum numbers, to get the explicit expression for $\Delta_{1,s}$ and its close relatives pertaining to the higher IOM:

$$I_1^{vac} = \frac{(s-1)(s-3)}{8} + \frac{1}{12} \tag{4.3.37}$$

$$I_3^{vac} = \frac{(s-1)^2(s-3)^2}{64} - \frac{1}{120}$$

$$I_5^{vac} = \frac{(s-1)^2(s-2)^2(s-3)^2}{512} + \frac{1}{336}$$

$$I_7^{vac} = \frac{(s-1)^2(s-3)^2(11-8s+50s^2-24s^3+3s^4)}{12288} - \frac{1}{480}$$

$$\dots$$

and, in general:

$$I_{2n-1}^{vac} = 2^{1-n} n \sum_{k=0}^{2n} \frac{(-1)^k B_{2n-k}}{2^k} \frac{(2n-1)!}{k!(2n-k)!}(s-3)^k \tag{4.3.38}$$

The constants U_n do not depend on s and can be fixed either from the highest weight $\Delta_{1,1} = 0$ or from the coefficients of 1-strings in the expressions obtained from TBA:

$$U_n 2^{-n} 2n = \frac{2\pi^{2n-1}}{2^{2n-1}(2n-1)} \tag{4.3.39}$$

This simple observation allows to obtain immediately the closed form for the coefficients U_n:

$$U_n = \frac{\pi^{2n-1}}{2^{n-1}(2n^2 - n)} \tag{4.3.40}$$

Finally, in view of the lattice selection rules explained in the previous section , it will prove useful to rearrange the 1-strings according to the double column diagram description. From this point of view there is actually no difference between lattice selection rules and continuum ones, and actually the j will be nothing but the labels of the two column diagrams.

For the $-1/8, 3/8, \ldots$ family one has

$$U_n I_{2n-1} = 2\left(\frac{\pi}{4}\right)^{(2n-1)} \left(\frac{1}{2n-1} \sum_{j_l \in A_l} \left(2j_l - 1\right)^{(2n-1)} + \frac{1}{2n-1} \sum_{j_r \in A_r} \left(2j_r - 1\right)^{(2n-1)} + \right.$$
$$\left. + (-1)^n (1 - 2^{1-2n}) \Gamma(2n-1) \frac{\zeta(2n)}{\pi^{2n}}\right)$$

$$(4.3.41)$$

whereas for the $0, 1, 3, \ldots$ family

$$U_n I_{2n-1} = 2\left(\frac{\pi}{4}\right)^{(2n-1)} \left(\frac{1}{2n-1} \sum_{j_l \in A_l} \left(2j_l\right)^{(2n-1)} + \frac{1}{2n-1} \sum_{j_r \in A_r} \left(2j_r\right)^{(2n-1)} + \right.$$
$$\left. + (-1)^{n+1} \Gamma(2n-1) \frac{\zeta(2n)}{\pi^{2n}}\right)$$

$$(4.3.42)$$

when $j_l = j_r$ this gives rise to a doouble 1-string contribution.

So that by applying the $(1, s)$ selection rules to the quantum numbers one obtains all the correct CFT characters.

4.3.3 Euler-Maclaurin and Integrals of Motion

The goal of this section is to extend the Euler Maclaurin analisys carried out for polymers in [4] and for Ising in [18] to *all* orders on $1/N$. As we already know the first order turns out by general arguments to be proportional to the eigenvalues of the energy I_1 of the underlying CFT, and is used as a tool to identify the central charge and to prove that the finitized characters yield for $N \to \infty$ the quasi rational characters of CFT, after extracting the divergent and constant parts which are proportional to the following bulk and boundary free energies:

$$f_{bulk}(u) = \log \sqrt{2} - \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\sin t} + \sin 2u\right) dt \qquad (4.3.43)$$

$$f_{bdy} = \log(1 + \sin 2u) \qquad (4.3.44)$$

The higher order corrections turn out to be related to the conserved quantities of CFT as well by the following *asymptotic* expansion:

$$\log D(u) = -2N f_{bulk}(u) - f_{bdy}(u) + \sum_{n=1}^{\infty} \frac{1}{N^{2n-1}} b_n \sin(2u) P_n(\sin(2u)) U_n I_{2n-1} \qquad (4.3.45)$$

where P_n are polynomials whose explicit form is:

$$P_n(a) = \sum_{k=1}^{n} (-1)^{(n+1)} C_{n,k} a^{2(k-1)} \tag{4.3.46}$$

being

$$C_{n,k} = \sum_{l=1}^{k} (-1)^{(l+1)} 4^{(1-k)} \frac{(2l-1)^{(2n-1)}}{l+k-1} \binom{2(k-1)}{k-l} \tag{4.3.47}$$

and the succession b_n is:

$$b_n = (-1)^{n+1} \frac{2^{(2n-1)}}{\Gamma(2n-1)} \tag{4.3.48}$$

This result is remarkably compact and independent of the parity of N, and actually up to first order it is well known to be a general feature of RSOS models and Logarithmic Minimal Models. The somewhat surprising simple result obtained here to all orders relies a lot on the factorization of the eigenvalues, and it would take a lot of additional effort just to investigate the persistence of such a property in the general case.

We will now give a brief explanation of how this calculation proceeds in the case of *even* N.

First of all introduce the auxiliary function F, defined as:

$$F(t) = \log(t\,\mathrm{cosec}(t) + t\sin(2u)) \tag{4.3.49}$$

in terms of this function one can express the logarithm of the eigenvalues in the following form:

$$\log D(u) = (1 - 2L)\log 2 + \log(2L) + 2\sum_{k=1}^{L-1} F(\frac{k\pi}{2L})) - 2\sum_{k=1}^{L-1} \log(\frac{k\pi}{2L})) +$$
$$+ \sum_{k \in A_l} \log(\frac{1 - \sin(2u)\sin(\frac{k\pi}{2L})}{1 + \sin(2u)\sin(\frac{k\pi}{2L})}) + \sum_{k \in A_r} \log(\frac{1 - \sin(2u)\sin(\frac{k\pi}{2L})}{1 + \sin(2u)\sin(\frac{k\pi}{2L})}) \tag{4.3.50}$$

the sum over F is evaluated by means of the Euler Maclaurin formula:

$$\sum_{k=1}^{L} F(\frac{k\pi}{2L}) \sim \int_{1}^{L} F(\frac{x\pi}{2L}) dx + \frac{1}{2}(F(\frac{\pi}{2L}) + F(\frac{\pi}{2})) + \sum_{k=1}^{\infty} (\frac{\pi}{2L})^{2k-1} \frac{B_{2k}}{(2k)!} (F^{2k-1}(\frac{\pi}{2}) - F^{2k-1}(\frac{\pi}{2L})) \tag{4.3.51}$$

the sum over the logarithms is evaluated by using the asymptotic of the Γ function:

$$\sum_{k=1}^{L-1} \log(\frac{k\pi}{2L}) = \frac{1}{2}\log L + L(\log\frac{\pi}{2} - 1) + \log 2 + \sum_{n=1}^{\infty} (\frac{2}{\pi})^{2n-1} \frac{B_{2n}}{2n(2n-1)} (\frac{\pi}{2L})^{2n-1} \tag{4.3.52}$$

one then uses the values of the derivatives of F

$$F(\frac{\pi}{2}) = \log(\frac{\pi}{2}) + \log(1 + \sin(2u)) \tag{4.3.53}$$

$$F^{(2k-1)}(0) = \sin(2u)P_k(\sin(2u)) \tag{4.3.54}$$

$$F^{(2k-1)}(\frac{\pi}{2}) = \frac{2^{(2k-1)}(2(k-1))!}{\pi^{(2k-1)}} \tag{4.3.55}$$

and notices that $F^{(2k-1)}(\frac{\pi}{2})$ is engineered to cancel the contribution of the Γ function, whereas the even derivatives of F drop out of the calculations regardless of their explicit form.

The excitations are included by noticing that they are generated by:

$$\log(\frac{1 + a\sin x}{1 - a\sin x}) = 2a \sum_{n=1}^{\infty} \frac{P_n(a)}{(2n-1)!} x^{(2n-1)} \tag{4.3.56}$$

And finally piecing up one arrives at 4.3.51.

If now one calls

$$G_N(u) = \sum_{n=1}^{\infty} \frac{1}{N^{2n-1}} b_n \sin(2u) P_n(\sin(2u)) U_n I_{2n-1} \tag{4.3.57}$$

it is possible to reshuffle the sum so as to collect a given power of $\sin(2u)$ as:

$$G_N(u) = \sum_{l=1}^{\infty} K_l(N) \sin^{2l-1}(2u) \tag{4.3.58}$$

being

$$K_l(N) = \sum_{r=l}^{\infty} \frac{C_{r,l} b_r U_r I_{2r-1}}{N^{(2r-1)}} \tag{4.3.59}$$

actually one can do even more, and resum the above series explicitly.

The expressions one obtains essentially depend on the parity of N, which is conveniently parametrized for even N as $N = 2D + 2$, whereas for odd N as $N = 2D + 1$.

It is also convenient to isolate the constant and divergent contributions, as well as the contribution of the excited states:

$$K_l(D) = \overline{K}_l(D) - K_l^{div} \tag{4.3.60}$$

$$\overline{K}_l(D) = K_l^{exc}(D) + K_l^{(0)}(D) \tag{4.3.61}$$

the K^{div} and K^{exc} are defined independently of the parity of N:

$$K_n^{div} = N\frac{\Gamma(n - \frac{1}{2})\Gamma(n)}{2\sqrt{\pi}\Gamma^2(n + \frac{1}{2})} - \frac{1}{2n - 1} \tag{4.3.62}$$

$$K_n^{exc}(D) = \frac{1}{(2n-1)2^{2n-3}} \sum_{j \in A_l \cup A_r} \sum_{m=0}^{n-1} (-1)^{m+1} \binom{2n-1}{m+n} \sin((2m+1)t_j) \qquad (4.3.63)$$

whereas the other pieces are, for even N:

$$t_j = \frac{j\pi}{N} = \frac{j\pi}{2(D+1)} \qquad (4.3.64)$$

$$K_n^0(D) = \frac{1}{(2n-1)2^{2n-3}} \sum_{m=0}^{n-1} \sin\left(\frac{(2m-1)\pi}{4}\right) \binom{2n-1}{m+n} \operatorname{cosec}\left((2m+1)\frac{t_1}{2}\right) \sin\left((2m+1)\frac{t_D}{2}\right) \qquad (4.3.65)$$

whereas for odd N one has:

$$t_j = \frac{(2j-1)\pi}{2N} = \frac{(2j-1)\pi}{2(2D+1)} \qquad (4.3.66)$$

$$K_n^0(D) = \frac{1}{2n-1} + \frac{1}{(2n-1)2^{2n-2}} \sum_{m=0}^{n-1} (-1)^{m+1} \binom{2n-1}{m+n} \operatorname{cosec}\left((2m+1)t_1\right) \qquad (4.3.67)$$

In the next section we will recognize the $\overline{K}_n$ as eigenvalues of suitable $N-$tangles defined in the Temperley Lieb algebra.

We now want to resum the contribution of the divergent part, for reasons that will become clear in a short time:

$$\sum_{n=1}^{\infty} K_n^{div} \sin^{2n-1}(2u) = \frac{1}{2} \log\left(\frac{1-\sin(2u)}{1+\sin(2u)}\right) + \frac{2N}{\pi}\left(\sin(2u) + \frac{2}{9} \, {}_3F_2\left((1,\frac{3}{2},2);(\frac{5}{2},\frac{5}{2}); \sin^2(2u)\right)\sin^3(2u)\right) \qquad (4.3.68)$$

where

$$_pF_q(\boldsymbol{a};\boldsymbol{b};z) = \sum_{k=0}^{\infty} \frac{\prod_i \left(\Gamma(a_i+k)/\Gamma(a_i)\right)}{\prod_j \left(\Gamma(b_j+k)/\Gamma(b_j)\right)} \frac{z^k}{k!} \qquad (4.3.69)$$

is the generalized hypergeometric function.

It is indeed remarkable that the bulk and boundary free energy produce very neat cancellations with the resummed divergent part, by means of the following identity:

$$\int_0^{\frac{\pi}{2}} dt \log(\operatorname{cosec}(t) + \sin(2u)) = \frac{\pi}{2} \log(1 + \sqrt{1 - \sin^2(2u)}) +$$

$$+ \sin(2u) + \frac{2}{9} \, {}_3F_2\left((1,\frac{3}{2},2);(\frac{5}{2},\frac{5}{2}); \sin^2(2u)\right)\sin^3(2u) \qquad (4.3.70)$$

one then uses also the following expansion

$$\log(1 + \sqrt{1 - z^2}) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi}}{2n\Gamma(\frac{1}{2} - n)\Gamma(n+1)} z^{2n} \qquad (4.3.71)$$

and ends up with the following expression for the eigenvalues D:

$$\log D(u) = \sum_{n=1}^{\infty} \frac{A_n}{n!} \sin^n(2u) \tag{4.3.72}$$

where

$$A_{2n} = (2n)! \left(\frac{1}{2n} + N(-1)^{n+1} \frac{\sqrt{\pi}}{2n\Gamma(\frac{1}{2} - n)\Gamma(n+1)} \right) \tag{4.3.73}$$

$$A_{2n-1} = (2n-1)! \overline{K}_n \tag{4.3.74}$$

one then introduces the complete Bell polynomials:

$$e^{\sum_{n=1}^{\infty} \frac{A_n}{n!} x^n} = \sum_{n=0}^{\infty} \frac{B_n(A_1, \ldots, A_n)}{n!} x^n \tag{4.3.75}$$

which are defined recursively as:

$$B_{n+1}(A_1, \ldots, A_n) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k+1} B_k(A_1, \ldots, A_k), \qquad B_0 = 1 \tag{4.3.76}$$

$$
\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_1^2 + A_2 \\
B_3 &= A_1^3 + 3A_1 A_2 + A_3 \\
B_4 &= A_1^4 + 6A_1^2 A_2 + 3A_2^2 + 4A_1 A_3 + A_4 \\
B_5 &= A_1^5 + 10A_1^3 A^2 + 15A_1 A_2^2 + 10A_2^2 A_3 + 5A_1 A_4 + A_5
\end{aligned} \tag{4.3.77}
$$

$$\cdots$$

in terms of these polynomials one has the following expansion for the eigenvalues:

$$D(u) = \sum_{n=0}^{\infty} \frac{B_n(A_1, \ldots, A_n)}{n!} \sin^n(2u) \tag{4.3.78}$$

actually it is possible to read off from the factorized form of the eigenvalues that they are polynomials in the variable $\sin(2u)$, whereas the above expansion is an infinite series. This is due to Euler Maclaurin (which was our starting point) being an asymptotic expansion. Fortunately this is not a problem. It turns out that one simply has to truncate the above expansion to get the *exact* result:

$$D(u) = 1 + \sum_{n=1}^{2D} \frac{B_n(A_1, A_2, \ldots, A_n)}{n!} \sin^n(2u) \tag{4.3.79}$$

This decomposition will be lifted from the eigenvalues to the transfer matrix itself in the next section.

It is also worth, again for the meaning it will carry in the next section, to recast the inversion identity in the following form:

$$D(u)D(u+\lambda) = \sum_{k=0}^{2D} \frac{B_k\left(2\frac{1!}{2!}A_2, \ldots, 2\frac{k!}{(2k)!}A_{2k}\right)}{k!} \sin^{2k}(2u) \tag{4.3.80}$$

while we are about it we also give the following explicit evaluation of the above Bell polynomials, which ban be obtained by explicitly expanding $\mathcal{F}(u)$:

$$B_k\left(2\frac{1!}{2!}A_2, \ldots, 2\frac{k!}{(2k)!}A_{2k}\right) = k!F_{2k} \tag{4.3.81}$$

being

$$F_{2m} = \sum_{r=0}^{2D} f_{r,m}g_{r,N} \tag{4.3.82}$$

$$f_{r,m} = \sum_{l=0}^{r} \frac{(-1)^{m+l}\Gamma(\frac{l}{2}+1)}{2^r\Gamma(m+1)\Gamma(\frac{l}{2}+1-m)} \binom{r}{l} \tag{4.3.83}$$

$$g_{r,N} = \sum_{m=0}^{r} h_{m,N}h_{r-m,N} \tag{4.3.84}$$

$$h_{m,N} = \begin{cases} \sum_{l=0}^{m}(-1)^{m-l}2^l \binom{N}{m-l} & m < N \\[2mm] ((-1)^N - 1)2^{m-N} & m \geq N \end{cases} \tag{4.3.85}$$

4.4   Integrals of Motion on the Lattice

In this section we want to put the attention on the meaning of those misterious results which we obtained from Euler Maclaurin.

What happens is that the transfer matrix admits the following expansion:

$$\mathbf{D}(u) = \mathbf{1} + \sum_{n=1}^{2D} \frac{B_n(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)}{n!} \sin^n(2u) \tag{4.4.1}$$

where, following the notation of the previous section we define

$$\mathbf{A}_{2n} = A_{2n}\mathbf{1} \tag{4.4.2}$$

$$\mathbf{A}_{2n-1} = (2n-1)!\overline{\mathbf{K}}_n \tag{4.4.3}$$

the $\overline{\mathbf{K}}_n$ rightfully deserve to be called Lattice Integrals of Motion, and they are in involution:

$$[\overline{\mathbf{K}}_l, \overline{\mathbf{K}}_m] = 0 \tag{4.4.4}$$

by construction they are diagonal in the same basis as the transfer matrix itself, so that if we label an eigenstate by the corresponding 2-column diagram $\mathcal{D}$ one has:

$$\overline{\mathbf{K}}_n|\mathcal{D}\rangle = \overline{K}_n|\mathcal{D}\rangle \tag{4.4.5}$$

$$\mathbf{D}(u)|\mathcal{D}\rangle = D(u)|\mathcal{D}\rangle \tag{4.4.6}$$

where $\overline{K}_n$ is the quantity which we computed in the previous section.

We are now going to exhibit explicitly how the lattice IOM are built from the generators of the TL algebra.

First of all we introduce the boundary symmetric $N-$tangles $\mathbf{B}_k$:

$$\mathbf{B}_k = \mathbf{e}_k + \mathbf{e}_{N-k} \tag{4.4.7}$$

and the following nested commutators, which for even $N = 2D + 2$ take the form:

$$\mathbf{H}_n = \sum_{j=1}^{2D+3-2n} [\mathbf{e}_j, [\mathbf{e}_{j+1}, [\mathbf{e}_{j+2}, [\ldots, [\mathbf{e}_{j+2n-3}, \mathbf{e}_{j+2n-2}]\ldots]]]] \tag{4.4.8}$$

while for odd $N = 2D + 1$ the bound of the summation is different:

$$\mathbf{H}_n = \sum_{j=1}^{2D+2-2n} [\mathbf{e}_j, [\mathbf{e}_{j+1}, [\mathbf{e}_{j+2}, [\ldots, [\mathbf{e}_{j+2n-3}, \mathbf{e}_{j+2n-2}]\ldots]]]] \tag{4.4.9}$$

The idea of introducing nested commutators in TL expansions is not completely new, for example it has been used in [23].

In terms of the $\mathbf{H}_n$ one has the following form for the fist few IOM:

$$\overline{\mathbf{K}}_1 = \mathbf{H}_1 \tag{4.4.10}$$

$$\overline{\mathbf{K}}_2 = \frac{1}{12}\mathbf{H}_2 + \frac{1}{6}\overline{\mathbf{K}}_1 - \frac{1}{12}\mathbf{B}_1 \tag{4.4.11}$$

$$\overline{\mathbf{K}}_3 = \frac{1}{80}\mathbf{H}_3 + \frac{1}{20}\mathbf{H}_2 - \frac{1}{80}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] + \frac{3}{40}\overline{\mathbf{K}}_1 - \frac{2}{80}\mathbf{B}_1 - \frac{3}{80}\mathbf{B}_2 \tag{4.4.12}$$

$$\overline{\mathbf{K}}_4 = \frac{1}{448}\mathbf{H}_4 + \frac{3}{224}\mathbf{H}_3 + \frac{15}{448}\mathbf{H}_2 - \frac{3}{112}[\overline{\mathbf{K}}_2, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] - \frac{1}{64}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] - \frac{1}{224}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_2]] +$$
$$+ \frac{5}{112}\overline{\mathbf{K}}_1 - \frac{1}{448}\mathbf{B}_1 - \frac{1}{28}\mathbf{B}_2 - \frac{5}{448}\mathbf{B}_3 \tag{4.4.13}$$

$$\overline{\mathbf{K}}_5 = \frac{1}{2304}\mathbf{H}_5 + \frac{1}{288}\mathbf{H}_4 + \frac{7}{576}\mathbf{H}_3 + \frac{7}{288}\mathbf{H}_2 - \frac{5}{192}[\overline{\mathbf{K}}_2, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] - \frac{1}{96}[\overline{\mathbf{K}}_2, [\overline{\mathbf{K}}_1, \mathbf{B}_2]] +$$
$$- \frac{41}{2304}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] - \frac{5}{768}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_2]] - \frac{1}{768}[\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_3]] - \frac{5}{144}[\overline{\mathbf{K}}_3, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] +$$
$$+ \frac{35}{1152}\overline{\mathbf{K}}_1 + \frac{1}{128}\mathbf{B}_1 - \frac{11}{384}\mathbf{B}_2 - \frac{19}{1152}\mathbf{B}_3 - \frac{7}{2304}\mathbf{B}_4 \tag{4.4.14}$$

We consider also the inverse relations which give the nested commutators in terms of the boundary tangles and IOM:

$$\mathbf{H}_2 = 12\overline{\mathbf{K}}_2 - 2\overline{\mathbf{K}}_1 + \mathbf{B}_1 \tag{4.4.15}$$

$$\mathbf{H}_3 = 80\overline{\mathbf{K}}_3 - 48\overline{\mathbf{K}}_2 + 2\overline{\mathbf{K}}_1 + [\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] - 2\mathbf{B}_1 + 3\mathbf{B}_2 \tag{4.4.16}$$

$$\mathbf{H}_4 = 448\overline{\mathbf{K}}_4 - 480\overline{\mathbf{K}}_3 + 108\overline{\mathbf{K}}_2 - 2\overline{\mathbf{K}}_1 + [12\overline{\mathbf{K}}_3 + \overline{\mathbf{K}}_1 + \mathbf{B}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] + [2\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_2]] - 4\mathbf{B}_1 - 2\mathbf{B}_2 + 5\mathbf{B}_3 \tag{4.4.17}$$

$$\mathbf{H}_5 = 2304\overline{\mathbf{K}}_5 - 3584\overline{\mathbf{K}}_4 + 1600\overline{\mathbf{K}}_3 - 192\overline{\mathbf{K}}_2 + 2\overline{\mathbf{K}}_1 + [-8\mathbf{B}_1 + 5\overline{\mathbf{K}}_1 - 36\overline{\mathbf{K}}_2 + 80\overline{\mathbf{K}}_3, [\overline{\mathbf{K}}_1, \mathbf{B}_1]] +$$
$$+ [-\overline{\mathbf{K}}_1 + 24\overline{\mathbf{K}}_2, [\overline{\mathbf{K}}_1, \mathbf{B}_2]] + [3\overline{\mathbf{K}}_1, [\overline{\mathbf{K}}_1, \mathbf{B}_3]] + 14\mathbf{B}_1 - 2\mathbf{B}_2 - 2\mathbf{B}_3 + 7\mathbf{B}_4 \tag{4.4.18}$$

this structure shows some remarkable regularities, indeed it is possible to suggest that the general structure should be something of the form:

$$\mathbf{H}_n = \sum_{l=1}^{n} \mathcal{C}_{l,n}\overline{\mathbf{K}}_l + \sum_{l=1}^{n-1} \mathcal{S}_{l,n}\mathbf{B}_l + \sum_{l=1}^{n-2}[P_{l,n}(\overline{\mathbf{K}}_1, \ldots, \overline{\mathbf{K}}_{n-l-1}; \mathbf{B}_1), [\overline{\mathbf{K}}_1, \mathbf{B}_l]] \tag{4.4.19}$$

where

$$P_{l,n}(\overline{\mathbf{K}}_1, \ldots, \overline{\mathbf{K}}_{n-l-1}; \mathbf{B}_1) = \sum_{h=1}^{n-l-1} p_{l,n,h}\overline{\mathbf{K}}_h + a_{l,n}\mathbf{B}_1 \tag{4.4.20}$$

in particular

$$P_{n-2,n}(\overline{\mathbf{K}}_1) = (n-2)\overline{\mathbf{K}}_1 \tag{4.4.21}$$

$$\mathcal{S}_{n-1,n} = 2n - 3 \tag{4.4.22}$$

and one has also:

$$\mathcal{C}_{l,n} = (-1)^{l+n}(2l-1)2^{2l-2}\left(\binom{n+l-3}{n-l-1} + \binom{n+l-2}{n-l} \right) \tag{4.4.23}$$

so that the task of solving the problem is reduced to identifying the $\mathcal{S}_{l,n}, p_{l,n,h}, a_{l,n}$ successions appearing in 4.4.19.

It is also worth considering the inversion identity in the Bell polynomial form:

$$\mathbf{D}(u)\mathbf{D}(u+\lambda) = \mathbf{1} + \sum_{k=1}^{2D} \frac{B_k\left(2\frac{1!}{2!}\mathbf{A}_2, \ldots, 2\frac{k!}{(2k)!}\mathbf{A}_{2k}\right)}{k!} \sin^{2k}(2u) \tag{4.4.24}$$

one notices that on the right hand side of the above equation only the even $\mathbf{A}_{2n}$ can appear. What turns this identity into an inversion identity is simply the fact that the $\mathbf{A}_{2n}$ are proportional to the identity. This seems to suggest that more general theories which do not satisfy an inversion identity will possess an additional set of non trivial IOM.

4.5 Symplectic Fermions

4.5.1 Generalities

We now want to discuss the different ways of describing states for critical dense polymers and their relation to symplectic fermion states in the continuum limit.

The CFT describing symplectic fermions is built from the following stress energy tensor [10, 11]:

$$T(z) = \frac{1}{2} : \boldsymbol{\chi}(z) \cdot \boldsymbol{\chi}(z) : \tag{4.5.1}$$

where we have introduced the notation

$$\boldsymbol{\chi} \cdot \boldsymbol{\chi} = d_{\alpha,\beta} \chi^{\alpha} \chi^{\beta} \tag{4.5.2}$$

where $d_{\alpha,\beta}$ is the antisymmetric tensor satisfying $d_{+,-} = 1$.

the field

$$\boldsymbol{\chi}(z) = \begin{pmatrix} \chi^+(z) \\ \chi^-(z) \end{pmatrix} \tag{4.5.3}$$

is a quasi primary field of scaling dimension 1, and by introducing the mode expansion

$$\boldsymbol{\chi}(z) = \sum_{n \in \mathbb{Z}} \frac{\boldsymbol{\chi}_n}{z^{n+1}} \tag{4.5.4}$$

one has that the modes satisfy the following anticommutation relations:

$$\{\chi_m^{\alpha}, \chi_n^{\beta}\} = m d^{\alpha,\beta} \delta_{m+n} \tag{4.5.5}$$

it follows that the Virasoro modes can be expanded in Symplectic Fermion modes:

$$L_n = \frac{1}{2} \sum_m : \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_{n-m} : \tag{4.5.6}$$

where the summation is over $\mathbb{Z}$ when the modes are considered to be acting on the vacuum Ω wereas the summation is over $\mathbb{Z} - \frac{1}{2}$ when the action is over the twisted vacuum μ. On notices that in the twisted sector there are no fermionic zero modes.

The energy L_0 does not have a diagonal action in the sense that there exists a logarithmic partner ω of the vacuum Ω such that:

$$L_0 \omega = \Omega \tag{4.5.7}$$

$$L_0 \Omega = 0 \tag{4.5.8}$$

By the way, is we decide to build the module over the vacuum Ω, by virtue of 4.5.8 the logarithmic partner does never appear.

In order to select the $\chi_{1,1}$ character it is necessary to require traslational invariance, which implies:

$$\boldsymbol{\chi}_{-1} \cdot \boldsymbol{\chi}_0 \Omega = 0 \tag{4.5.9}$$

this can be obtained by requiring:

$$\boldsymbol{\chi}_0 \Omega = 0 \tag{4.5.10}$$

this can be interpreted also as a condition on the fermionic states $\boldsymbol{\theta}$, defined as:

$$\boldsymbol{\chi}_0 \omega = -\boldsymbol{\theta} \tag{4.5.11}$$

$$\chi_0^\alpha \theta^\beta = d^{\alpha,\beta} \Omega \tag{4.5.12}$$

so that

$$\boldsymbol{\chi}_0 \Omega = 2 \, \chi_0^+ \chi_0^- \boldsymbol{\theta} \tag{4.5.13}$$

implies that 4.5.10 is equivalent to one of the 2 component of the zero mode annihilating the fermionic state $\boldsymbol{\theta}$.

The theory has a global $sl(2)$ isospin symmetry, and the free fermion field $\boldsymbol{\chi}$ transforms as a $j = \frac{1}{2}$ representation of $sl(2)$[10]:

$$[J^+, J^-] = 2J^0 \tag{4.5.14}$$

$$[J^\pm, J^0] = \pm J^0 \tag{4.5.15}$$

$$[J^0, \chi^\pm(z)] = \pm \frac{1}{2}\chi^\pm(z) \tag{4.5.16}$$

$$[J^\pm, \chi^\pm(z)] = 0 \tag{4.5.17}$$

$$[J^\pm, \chi^\mp(z)] = \chi^\pm(z) \tag{4.5.18}$$

$$J^0\Omega = J^\pm\Omega = 0 \tag{4.5.19}$$

by virtue of this global symmetry the highest weight states will always fall into irreducible representations of $sl(2)$ carrying isospin $j \in \frac{1}{2}\mathbb{N}$:

$$\big|j, m\big\rangle = \chi_{-2j}^{(+} \cdots \chi_{-j+m}^{+} \chi_{-j+m+1}^{-} \cdots \chi_{-1}^{-)}\Omega \tag{4.5.20}$$

where the round brackets denote symmetrization over the upper indexes.

The whole multiplet can be obtained by acting on $\big|j, -j\big\rangle$ with rising operators:

$$J^+\big|j, -j\big\rangle = J^+ \prod_{k=1}^{2j} \chi_{-k}^+\Omega = [\, J^+, \prod_{k=1}^{2j} \chi_{-k}^+]\Omega = \chi_{-2j}^-\chi_{1-2j}^+ \cdots \chi_{-1}^+\Omega + \ldots + \chi_{-2j}^+ \cdots \chi_{-2}^+\chi_{-1}^-\Omega = \big|j, 1-j\big\rangle \tag{4.5.21}$$

For example the state with weight $\Delta_{1,9} = 6$ forms a $j = \frac{3}{2}$ multiplet of $sl(2)$ which is composed by the following four states:

$$
\begin{aligned}
\left|\frac{3}{2}, \frac{3}{2}\right\rangle &= \chi^+_{-3}\chi^+_{-2}\chi^+_{-1}\Omega & , \left|\frac{3}{2}, \frac{1}{2}\right\rangle &= \chi^{(+}_{-3}\chi^+_{-2}\chi^{-)}_{-1}\Omega \\
\left|\frac{3}{2}, -\frac{1}{2}\right\rangle &= \chi^{(+}_{-3}\chi^-_{-2}\chi^{-)}_{-1}\Omega & , \left|\frac{3}{2}, -\frac{3}{2}\right\rangle &= \chi^-_{-3}\chi^-_{-2}\chi^-_{-1}\Omega
\end{aligned}
\tag{4.5.22}
$$

in general the states $|j, m\rangle$ have confomal weight $\Delta_j = j(2j + 1)$, covering all entries in Kac table with integer conformal weight.

All the other entries in Kac table can be described by introducing a twisted vacuum μ and using fermi modes labelled by half integers:

$$
|j, m\rangle = \chi^{(+}_{-2j+\frac{1}{2}} \cdots \chi^+_{-j+m+\frac{1}{2}} \chi^-_{-j+m+\frac{3}{2}} \cdots \chi^{-)}_{-\frac{1}{2}} \mu
\tag{4.5.23}
$$

in this case the multiplet has conformal weight $\Delta_j = -\frac{1}{8} + 2j^2$.

So that if one picks one of the $|j, m\rangle$, either twisted or untwisted, it is possible to build a Virasoro module over it, and all we need to remember to describe such a module in terms of fermions is the commutator between virasoro modes and fermi modes:

$$
[L_{-n}, \chi_{-l}] = l\chi_{-(n+l)}
\tag{4.5.24}
$$

To select a sector labelled by (r, s) it will be necessary to identify which combination of fermi modes corresponds to the null vector at level rs, for example both $(2, 1)$ and $(1, 5)$ have $\Delta = 1$, the difference being that by virtue of some selection rules we will be able to identify which submodules we have to throw away. These selection rules will be the same as the lattice selection rules.

4.5.2 Selection Rules and Characters

In order to deal with selections rules let us introduce some obvious notation.

Let $A^\infty_{m,n}$ be the set of all admissible two column diagrams with m occupied sides on the left and n occupied sites on the right, where each diagram has no height restriction.

One then introduces:

$$
\left\langle \begin{matrix} \infty \\ m, n \end{matrix} \right\rangle_q = \sum_{\mathcal{D} \in A^\infty_{m,n}} q^{w(\mathcal{D})}
\tag{4.5.25}
$$

and

$$
\overline{\chi}^{(2j)}(q) = q^{\frac{1}{12}} \sum_{m=0}^{\infty} \left\langle \begin{matrix} \infty \\ m, m+2j \end{matrix} \right\rangle_q
\tag{4.5.26}
$$

The object $\overline{\chi}^{(2j)}(q)$ is the character of the Virasoro module which we shall call $\mathcal{Q}^{(2j)}$ which is built on one of the highest weights $|j, m\rangle$, the choice of m is not important here,

because all such modules are isomorphic due to $sl(2)$ invariance, and can be generated by the action of the $J^{\pm}$ operators.

These characters are simply related to the $(1, 2k + 1)$ quasi rational characters by virtue of the formula:

$$\overline{\chi}^{(2j)}(q) = \sum_{k=0}^{2j}(-1)^{2j-k}\chi_{1,2k+1}(q) \tag{4.5.27}$$

which can be inverted to yield:

$$\chi_{1,4j+1}(q) = \overline{\chi}^{(2j-1)}(q) + \overline{\chi}^{(2j)}(q), \quad j \in \frac{1}{2}\mathbb{N} \tag{4.5.28}$$

notice that

$$\Delta_{1,4j+1} = j(2j - 1) \tag{4.5.29}$$

and that one defines also $\overline{\chi}^{(-1)}(q) = 0$. Since the $\overline{\chi}^{(2j)}(q)$ are well defined characters with positive coefficients, 4.5.28 can be interpreted as meaning that the Virasoro module $\mathcal{V}_{1,4j+1}$ admits the following decomposition:

$$\mathcal{V}_{1,4j+1} = \mathcal{Q}^{(2j-1)} \oplus \mathcal{Q}^{(2j)} \tag{4.5.30}$$

where, again, $\mathcal{Q}^{(-1)} = \emptyset$.

Indeed, it is possible to obtain information on generic decompositions for $r > 1$, by means of the following identity:

$$\chi_{1+k,s}(q) = \sum_{\rho=0}^{k}\chi_{1,s-2k+4\rho} \tag{4.5.31}$$

where it is understood:

$$\chi_{1,0}(q) = 0 \tag{4.5.32}$$

$$\chi_{1,-s}(q) = -\chi_{1,s}(q) \tag{4.5.33}$$

which tells us for example that:

$$\chi_{3,5} = \chi_{1,1} + \chi_{1,5} + \chi_{1,9} = \overline{\chi}^{(0)} + \overline{\chi}^{(1)} + \overline{\chi}^{(2)} + \overline{\chi}^{(3)} + \overline{\chi}^{(4)} \tag{4.5.34}$$

which implies for example that the following modules admit decompositions such as:

$$\mathcal{V}_{2,3} = \mathcal{Q}^{(0)} \oplus \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \tag{4.5.35}$$

$$\mathcal{V}_{2,5} = \mathcal{Q}^{(0)} \oplus \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \oplus \mathcal{Q}^{(3)} \tag{4.5.36}$$

$$\mathcal{V}_{3,5} = \mathcal{Q}^{(0)} \oplus \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \oplus \mathcal{Q}^{(3)} \oplus \mathcal{Q}^{(4)} \tag{4.5.37}$$

$$\mathcal{V}_{3,9} = \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \oplus \mathcal{Q}^{(3)} \oplus \mathcal{Q}^{(4)} \oplus \mathcal{Q}^{(5)} \oplus \mathcal{Q}^{(6)} \tag{4.5.38}$$

It follows from simple cancellations of characters that:

$$\chi_{n,1}(q) = \overline{\chi}^{(n-1)}(q) \tag{4.5.39}$$

which gives the identification:

$$\mathcal{V}_{n,1} = \mathcal{Q}^{(n-1)} \tag{4.5.40}$$

one the focuses on $(n,3)$:

$$\chi_{n,3} = \overline{\chi}^{(n-2)}(q) + \overline{\chi}^{(n-1)}(q) + \overline{\chi}^{(n)}(q) \tag{4.5.41}$$

so that

$$\mathcal{V}_{n,3} = \mathcal{Q}^{(n-2)} \oplus \mathcal{Q}^{(n-1)} \oplus \mathcal{Q}^{(n)} \tag{4.5.42}$$

the case of $(n,5)$ gives:

$$\chi_{n,5} = \overline{\chi}^{(n-3)}(q) + \overline{\chi}^{(n-2)}(q) + \overline{\chi}^{(n-1)}(q) + \overline{\chi}^{(n)}(q) + \overline{\chi}^{(n+1)}(q) \tag{4.5.43}$$

corresponding to:

$$\mathcal{V}_{n,5} = \mathcal{Q}^{(n-3)} \oplus \mathcal{Q}^{(n-2)} \oplus \mathcal{Q}^{(n-1)} \oplus \mathcal{Q}^{(n)} \oplus \mathcal{Q}^{(n+1)} \tag{4.5.44}$$

For odd $s = 2k + 1$, the most general situation which we can describe in terms of the modules $\mathcal{Q}^{(n)}$ is:

$$\mathcal{V}_{r,2k+1} = \bigoplus_{l=r-1-k}^{r-1+k} \mathcal{Q}^{(l)} \tag{4.5.45}$$

Where of course all the $\mathcal{Q}^{(n)}$ with negative n are empty.

We are now going to deal with the twisted case.

Following the selection rules defined on the lattice we define:

$$\overline{\chi}^{(2j)} = q^{-\frac{1}{24}-j} \sum_{m=0}^{\infty} q^{-m} \left\langle \begin{smallmatrix} \infty \\ m, m+2j \end{smallmatrix} \right\rangle_q \tag{4.5.46}$$

Notice that in order not to introduce further notation we are using for 4.5.46 the same name as 4.5.26, this should not rise any confusion, since we are working in a different sector of the theory.

One then has that, as in the previous case $\overline{\chi}^{(n)} = 0$ for $n < 0$. On the other hand, whenever the $\overline{\chi}^{(n)}$ are different from zero the following equality holds:

$$\chi_{1,4j+2} = \overline{\chi}^{(2j)} \tag{4.5.47}$$

It is also useful to notice that:

$$\Delta_{1,4j+2} = -\frac{1}{8} + 2j^2 \tag{4.5.48}$$

Again, in analogy with the previous case one introduces the Virasoro modules $\tilde{\mathcal{Q}}^{(2j)}$ built on the highest weights 4.5.23, and realizes that the $\overline{\chi}^{(2j)}$ are the characters of such modules. One then notices that by means of 4.5.31 and following it is possible to derive the following identities:

$$\chi_{r,2} = \chi_{1,2r} = \overline{\chi}^{(r-1)} \tag{4.5.49}$$

$$\chi_{r,2n} = \sum_{k=0}^{n-1} \overline{\chi}^{(r-n+2k)} \tag{4.5.50}$$

which imply that the modules $\mathcal{V}_{r,2}$ and $\mathcal{V}_{1,2r}$ are isomorphic, and that in general the following decomposition holds:

$$\mathcal{V}_{r,2n} = \bigoplus_{k=0}^{n-1} \tilde{\mathcal{Q}}^{(r-n+2k)} \tag{4.5.51}$$

So that we have finally given a description of the entire Kac table in terms of the modules $\mathcal{Q}^{(n)}$ and $\tilde{\mathcal{Q}}^{(n)}$.

In passing it is very nice to make some simple remarks about $\mathcal{W}-$modules. Notice that the $\mathcal{W}-$characters of [24] can be cast in the following form:

$$\hat{\chi}_{1,1}(q) = \sum_{j \in \frac{1}{2}\mathbb{N}} (2j+1)\overline{\chi}^{(2j)}(q) \tag{4.5.52}$$

$$\hat{\chi}_{2,1}(q) = \sum_{j \in \frac{1}{2}\mathbb{N}^+} (2j+1)\overline{\chi}^{(2j)}(q) \tag{4.5.53}$$

implying that the corresponding $\mathcal{W}-$modules have the following structure:

$$\mathcal{V}_{1,1}^{\mathcal{W}} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} (2j+1)\mathcal{Q}^{(2j)} \tag{4.5.54}$$

$$\mathcal{V}_{2,1}^{\mathcal{W}} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}^+} (2j+1)\mathcal{Q}^{(2j)} \tag{4.5.55}$$

and similarly (but remember the different meaning of $\overline{\chi}^{(2j)}$):

$$\hat{\chi}_{1,2}(q) = \sum_{j \in \frac{1}{2}\mathbb{N}} (2j+1)\overline{\chi}^{(2j)}(q) \tag{4.5.56}$$

$$\hat{\chi}_{2,2}(q) = \sum_{j \in \frac{1}{2}\mathbb{N}^+} (2j+1)\overline{\chi}^{(2j)}(q) \tag{4.5.57}$$

$$\mathcal{V}_{1,2}^{\mathcal{W}} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} (2j+1)\tilde{\mathcal{Q}}^{(2j)} \tag{4.5.58}$$

$$\mathcal{V}_{2,2}^{\mathcal{W}} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}^+} (2j+1)\tilde{\mathcal{Q}}^{(2j)} \tag{4.5.59}$$

Notice that the multiplicity $(2j + 1)$ of each module arises precisely from the multiplcity of the allowed values of m for the states $|j, m\rangle$. It would thus be more honest to label the $\mathcal{Q}, \tilde{\mathcal{Q}}$ modules as $\mathcal{Q}^{(2j,m)}, \tilde{\mathcal{Q}}^{(2j,m)}$, whenever it is necessary to keep in mind that these modules are made indeed of different states, and are only isomorphic. From these expressions it is also transparent that the $\mathcal{W}$−modules are closed under the action of $sl(2)$ rising and lowering operators, the action of the diagonal generator can be used instead to twist the monodromy of the fermion by a continuous phase, thus generating a flow between the twisted and untwisted sectors.

4.5.3 Fermionic form of the BLZ Eigenstates

We proceed now to describe the explicit relation between the selection rules and the fermionic form of the BLZ IOM.

Recalling the lattice selection rules for the vacuum sector, we introduce 2-column diagrams of infinite height $\mathcal{D} \in A_{m,m}^\infty$, labelled by $(\boldsymbol{l}, \boldsymbol{r})$ with both $\boldsymbol{l}, \boldsymbol{r}$ of length m.

In general one notices that the following state:

$$|\mathcal{D}\rangle = \prod_{i=1}^{m} \boldsymbol{\chi}_{-l_i} \cdot \boldsymbol{\chi}_{-r_i} \Omega \tag{4.5.60}$$

is such that:

$$L_0|\mathcal{D}\rangle = w(\mathcal{D})|\mathcal{D}\rangle \tag{4.5.61}$$

and one identifies $w(\mathcal{D})$ as the level of descendance.

Notice that the states $|\mathcal{D}\rangle$ can be brought to a canonical form where modes with the same label are coupled by a scalar product, to this goal the following identity proves useful:

$$\boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_n \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_l = -\frac{1}{2} \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_m \boldsymbol{\chi}_n \cdot \boldsymbol{\chi}_l \quad, m, n, l < 0 \tag{4.5.62}$$

Although the counting of states is correct one has to check that the states $|\mathcal{D}\rangle$ are always eigenstates of the BLZ IOM. It is possible to check by hand that this is indeed the case up to level 6, and it should be true at all levels.

More generally if we consider a state $|\mathcal{D}\rangle \in \mathcal{Q}^{(2j)}$ it will be of the form:

$$|\mathcal{D}\rangle = \prod_{i=1}^{2j} \chi_{-r_{m+i}}^+ \prod_{i=1}^{m} \boldsymbol{\chi}_{-l_i} \cdot \boldsymbol{\chi}_{-r_i} \Omega \tag{4.5.63}$$

and by using 4.5.62 together with:

$$\chi_m^+ \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_n = -\frac{1}{2}\chi_n^+ \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_m \tag{4.5.64}$$

$$\chi_l^+ \boldsymbol{\chi}_m \cdot \boldsymbol{\chi}_n + \chi_n^+ \boldsymbol{\chi}_l \cdot \boldsymbol{\chi}_m + \chi_m^+ \boldsymbol{\chi}_n \cdot \boldsymbol{\chi}_l = 0 \tag{4.5.65}$$

it is possible to bring all the expressions to a simple canonical form.

Indeed, it should be true that for all the modules $\mathcal{V}_{1,4j+1}$, the states $\left|\mathcal{D}\right\rangle$ are eigenstates of the IOM, leaving aside possible mixings due to degeneracy.

The situation for the modules $\tilde{\mathcal{Q}}^{(2j)}$ is slightly different, in this case the structure of the fermionic state associated to a two column diagram is:

$$\left|\mathcal{D}\right\rangle = \prod_{i=1}^{2j}\chi_{\frac{1}{2}-r_{m+i}}^+ \prod_{i=1}^{m}\boldsymbol{\chi}_{\frac{1}{2}-l_i} \cdot \boldsymbol{\chi}_{\frac{1}{2}-r_i}\mu \tag{4.5.66}$$

The singlet case $j = 0$ is understood to give rise to the analogue of 4.5.60.

In this case the action of L_0 is given by:

$$L_0\left|\mathcal{D}\right\rangle = \left(w(\mathcal{D}) - m - j - \frac{1}{8}\right)\left|\mathcal{D}\right\rangle \tag{4.5.67}$$

This difference is related to the presence of q^{-m-j} in the definition of the character 4.5.46. Aside from these differences all the considerations of the previous cases apply also here.

4.5.4   Examples

In this section we want to give a comparative description of some Verma modules corresponding to the same conformal weight. The method we shall employ is direct calculation of the matrix form of the IOM at a given level of descendance in the standard lexicographically ordered Virasoro basis, we will then compute the Jordan canonical form of such a matrix to discover that in many cases it exhibits Jordan Blocks.

In the cases of $(1, s)$ modules it is well known from the lattice theory that the action of the transfer matrix is completely diagonalizable, we shall confirm this observation for the quantum transfer matrix $\mathbf{D}$ which we can define inpiring ourselves to 4.3.20:

$$\mathbf{D}(x) = e^{\mathbf{F}(x) + \sum_{n=1}^{\infty} U_n \mathbf{I}_{2n-1}e^{(2n-1)x}} \tag{4.5.68}$$

For a suitable $\mathbf{F}(x) = F(x)\mathbf{1}$ which is introduced to resemble the structure of the expansion 4.3.72, and to take into account 4.4.2.

Notice that $\mathbf{D}$ satisfies an inversion identity:

$$\mathbf{D}\left(x - i\frac{\pi}{2}\right)\mathbf{D}\left(x + i\frac{\pi}{2}\right) = e^{F(x+i\frac{\pi}{2}) + F(x-i\frac{\pi}{2})}\mathbf{1} \tag{4.5.69}$$

In some cases, however, with $r > 1$ we shall find that the higher IOM exhibit a nontrivial Jordan structure. L_0, by the way, is always diagonalizable because we are considering modules built on the vacuum Ω by using strings of fermionic operators. What makes **D** not diagonalizable is the effect of the higher IOM. Again, this is in perfect agreement with the lattice theory for which according to 4.3.59 the eigenvalues of the lattice integrals of motion are given by a series of the continuum integrals of motion. The reason why the Hamiltonian on the lattice is not diagonalizable in some cases is that it is a superposition of continuum IOM, so that even of L_0 is diagonal, the lattice Hamiltonian receives contributions from operators which are not diagonalizable.

$\mathcal{V}_{1,5}$ **vs** $\mathcal{V}_{2,7}$

We want to give in this section an explanatory study of the module $\mathcal{V}_{1,5}$, such a module has $\Delta_{1,5} = 1$ and is known to have a null vector at level 5. We shall identify such a state as the reason preventing the matrix representations of the IOM to be indecomposable. The matrix form of the IOM is found to be diagonal up to level 4, at level 5 one finds that the Jordan canonical form of I_3 can be obtained by a similarity transformation:

$$\mathbf{U}^{-1}\mathbf{I}_3\mathbf{U} = \mathbf{J}_3 \tag{4.5.70}$$

which is explicitly realized by:

$$\mathbf{U} = \begin{pmatrix} 18 & 720 & 0 & 4 & -8 & -26 & 304 \\ \frac{18}{5} & -2160 & 234 & 8 & -12 & -12 & 258 \\ -\frac{52}{5} & -1440 & 4 & 12 & 16 & -38 & 112 \\ -5 & 1080 & -140 & -4 & -12 & 6 & 96 \\ 0 & 2880 & 0 & -4 & 4 & -14 & 46 \\ 3 & 1800 & 0 & -5 & -4 & 5 & 20 \\ -\frac{3}{10} & 180 & 3 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{4.5.71}$$

$\mathbf{I}_3$ is written in the standard basis:

$$\left\{ L_{-5}|1\rangle, L_{-4}L_{-1}|1\rangle, L_{-3}L_{-2}L_{-1}|1\rangle, L_{-3}L_{-1}^2|1\rangle, L_{-2}^2L_{-1}|1\rangle, L_{-2}L_{-1}^3|1\rangle, L_{-1}^5|1\rangle \right\} \tag{4.5.72}$$

The Jordan decomposition is found to be:

$$\mathbf{J}_3 = \begin{pmatrix} \frac{4319}{120} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4319}{120} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4319}{120} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7919}{120} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8639}{120} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15119}{120} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{25919}{120} \end{pmatrix} \qquad (4.5.73)$$

Notice that the size of the matrix is $P(5) = 7$ but the dimensionality is known from the character to be 6. $P(N)$ is the number of partitions of N into as a sum of positive integers. It may seem that a Jordan indecomposable structure is emerging for $\mathbf{I}_3$ in the module $\mathcal{V}_{1,5}$ at level 5, thus one introduces the generalized eigenvectors.

$$\rho_i \ , i = 0, \ldots, 6 \qquad (4.5.74)$$

Their virasoro form is simply found by applying the similarity transformation to the vectors $(1,0,0,0,0,0,0), (0,1,0,0,0,0,0), \ldots$.
After finding their virasoro form one can go over to the fermi modes, and one has:

$$\mathbf{I}_3 \, \rho_2 = \frac{4319}{120}\rho_2 + \rho_1 \qquad (4.5.75)$$

where

$$\rho_1 = 10(72L_{-5}-216L_{-4}L_{-1}-144L_{-3}L_{-2}L_{-1}+108L_{-3}L_{-1}^2+288L_{-2}^2L_{-1}-180L_{-2}L_{-1}^3+18L_{-1}^5)|1\rangle \qquad (4.5.76)$$

is found to be a null vector when expressed in its fermionic form, whereas:

$$\rho_0 = \frac{192}{5}\chi^+_{-1}\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-3}\Omega + 24\chi^+_{-2}\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-1}\Omega \qquad (4.5.77)$$

$$\rho_2 = 276\chi^+_{-1}\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-3}\Omega + 510\chi^+_{-2}\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-1}\Omega \qquad (4.5.78)$$

$$\rho_3 = -25\chi^+_{-4}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega \qquad (4.5.79)$$

$$\rho_6 = 2700\chi^+_{-6}\Omega \qquad (4.5.80)$$

these are the eigenstates to be found inside $\mathcal{Q}^{(1)}$, the other 2 states have to be looked for inside $\mathcal{Q}^{(2)}$

$$\rho_4 = 18\chi^+_{-4}\chi^+_{-2}\Omega \qquad (4.5.81)$$

$$\rho_5 = 90\chi^+_{-5}\chi^+_{-1}\Omega \qquad (4.5.82)$$

The discussion of this case is sufficient to show that whenever two states, are degenerate for *all* the IOM they are allowed to mix, although one can always to pick a basis within their common eigenspace for which, in this case:

$$\tilde{\rho}_0 = \chi^+_{-1}\boldsymbol{\chi}_{-2} \cdot \boldsymbol{\chi}_{-3}\Omega \tag{4.5.83}$$

$$\tilde{\rho}_2 = \chi^+_{-2}\boldsymbol{\chi}_{-3} \cdot \boldsymbol{\chi}_{-1}\Omega \tag{4.5.84}$$

The case of $\mathcal{V}_{2,7}$ is radically different. First of all one has:

$$\mathcal{V}_{2,7} = \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \oplus \mathcal{Q}^{(3)} \oplus \mathcal{Q}^{(4)} \tag{4.5.85}$$

and furthermore the submodule one wants to mod out starts at level 14, therefore the dimensionality at level 5 of $\mathcal{V}_{2,7}$ is precisely $P(5) = 7$, for this simple argument ρ_1 cannot be a null vector. The only candidate for a nonzero ρ_1 can be taken from the module $\mathcal{Q}^{(3)}$, so that it is natural to suggest (upon suitably normalizing everything):

$$\rho_1 = \chi^+_{-3}\chi^+_{-2}\chi^+_{-1}\Omega \tag{4.5.86}$$

So that in this case the action of $\boldsymbol{I}_3$ (and likewise all the higher IOM) becomes indecomposable at level 5. Such a jordan cell will propagate at successive levels of descendance. Explicitly at level 6, under the action of L_{-1} one has:

$$\eta_1 = L_{-1}\rho_1 = 3\chi^+_{-4}\chi^+_{-2}\chi^+_{-1}\Omega \tag{4.5.87}$$

which spans a Jordan cell together with some suitable linear combination:

$$\eta_2 = a_1\chi^+_{-1}\boldsymbol{\chi}_{-4} \cdot \boldsymbol{\chi}_{-2} + a_2\chi^+_{-2}\boldsymbol{\chi}_{-4} \cdot \boldsymbol{\chi}_{-1} \tag{4.5.88}$$

such that:

$$\boldsymbol{I}_3\,\eta_2 = \frac{8759}{120}\eta_2 + \eta_1 \tag{4.5.89}$$

at level 7, there are 2 Jordan cells.
The first one is spanned by:

$$\xi_1 = (L_{-2} + 3L^2_{-1})\rho_1 = 21\chi^+_{-5}\chi^+_{-2}\chi^+_{-1}\Omega \tag{4.5.90}$$

and again, some linear combination:

$$\xi_2 = b_1\chi^+_{-1}\boldsymbol{\chi}_{-5} \cdot \boldsymbol{\chi}_{-2} + b_2\chi^+_{-2}\boldsymbol{\chi}_{-5} \cdot \boldsymbol{\chi}_{-1} \tag{4.5.91}$$

such that:

$$\boldsymbol{I}_3\,\xi_2 = \frac{16079}{120}\xi_2 + \xi_1 \tag{4.5.92}$$

whereas the second Jordan cell is spanned by:

$$\alpha_1 = (-4L_{-2} + L^2_{-1})\rho_1 = 14\chi^+_{-4}\chi^+_{-3}\chi^+_{-1}\Omega \tag{4.5.93}$$

and the usual linear combination:

$$\alpha_2 = c_1\chi^+_{-1}\boldsymbol{X}_{-4} \cdot \boldsymbol{X}_{-3} + c_2\chi^+_{-3}\boldsymbol{X}_{-4} \cdot \boldsymbol{X}_{-1} \tag{4.5.94}$$

such that:

$$\boldsymbol{I}_3\,\alpha_2 = \frac{11039}{120}\alpha_2 + \alpha_1 \tag{4.5.95}$$

The Jordan cells generated by ρ_1 will initially be counted by $P(N-5)$, but for N large enough this will change due to the appearence of a rank 3 Jordan cell at level 14. In general one will notice that at levels $n(2n+3) = 5, 14, 27, 44, \ldots$ a new null vetcor will appear and in correpondence one will observe higher and higher rank Jordan cells appearing. One then will use the states available from the modules $\mathcal{Q}^{(2j)}$ to fill up the null vectors spanning the Jordan blocks.

$\mathcal{V}_{1,2}$ vs $\mathcal{V}_{2,4}$

Let us now consider a module built on a primary field of dimension $\Delta = -\frac{1}{8}$, by standard calculations with the Virasoro algebra one can start to work out the explicit form of the matrix representation of $\boldsymbol{I}_3$. Already at level 2 things start to be interesting, one finds the following Jordan decomposition:

$$\boldsymbol{I}_3 = \begin{pmatrix} \frac{3367}{960} & 1 \\ 0 & \frac{3367}{960} \end{pmatrix} \tag{4.5.96}$$

which is obtained by the similarity trasformation:

$$\boldsymbol{U} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} \\ 1 & 0 \end{pmatrix} \tag{4.5.97}$$

Indeed from the character $\chi_{1,2}$ of $\mathcal{V}_{1,2} = \tilde{\mathcal{Q}}^{(0)}$ we know that at level 2 we have a null vector, which corresponds to the combination:

$$\rho_0 = (-\frac{1}{2}L_{-2} + L^2_{-1})\mu = 0 \tag{4.5.98}$$

and a generalized eigenvector which is just an eigenvector because of the equation above:

$$\rho_1 = \frac{1}{6}L_{-1}\mu = \frac{1}{6}\boldsymbol{X}_{-\frac{3}{2}} \cdot \boldsymbol{X}_{-\frac{1}{2}}\mu \tag{4.5.99}$$

satisfying:

$$\mathbf{I}_3 \rho_1 = \frac{3367}{960} \rho_1 + \rho_0 = \frac{3367}{960} \rho_1 \tag{4.5.100}$$

The difference with the module $\mathcal{V}_{2,4} = \tilde{\mathcal{Q}}^{(0)} \oplus \tilde{\mathcal{Q}}^{(2)}$ starts early, since from the character we see that it has no null vector at level 2 and since the module $\tilde{\mathcal{Q}}^{(0)}$ has not enough states the only candidate is the highest weight of $\tilde{\mathcal{Q}}^{(2)}$:

$$\rho_0 = \chi^+_{-\frac{3}{2}} \chi^+_{-\frac{1}{2}} \mu \tag{4.5.101}$$

so that we realize that the action of $\mathbf{I}_3$ starts to be indecomposable already at level 2. The state ρ_0 will generate the whole module $\tilde{\mathcal{Q}}^{(2)}$, which will always appear inside Jordan cells having the same multiplicity at a given level of descendance as the corresponding module. At high levels however, some Jordan cells will still contain null vectors due to the fact that the next module in the sequence which is $\tilde{\mathcal{Q}}^{(4)}$ is not available for filling up those null vectors. The next null vector will be at level 8, and in general one will have that a new null vector will appear at level $2n^2$ for $n \in \mathbb{N}$.

At level 8 the Jordan decomposition of $\mathbf{I}_3$ is indeed very big, by the way the interesting part is that instead of having $P(8-2) = 11$ rank 2 Jordan cells we have 8 such cells plus one rank 3 Jordan cell. This Jordan cell clearly contains the new null vector.

We reproduce 6×6 the block of $\mathbf{I}_3$ which contains such a cell together with a rank 2 cell having the same eigenvalues:

$$\mathbf{I}_3 = \begin{pmatrix} \frac{59527}{960} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{59527}{960} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{59527}{960} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{59527}{960} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{59527}{960} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{59527}{960} \end{pmatrix} \tag{4.5.102}$$

notice that the repartition into 1 rank 3, one rank 2 and 1 spare eigenstate sums up to 6, precisely as the allowed states from the modules $\tilde{\mathcal{Q}}^{(0)}, \tilde{\mathcal{Q}}^{(2)}, \tilde{\mathcal{Q}}^{(4)}$. The states spanning the cell are:

$$(3, 1|4, 2), (2, 1|4, 3) \in \tilde{\mathcal{Q}}^{(0)} \tag{4.5.103}$$

$$(3|4, 2, 1), (2|4, 3, 1), (1|4, 3, 2) \in \tilde{\mathcal{Q}}^{(2)} \tag{4.5.104}$$

$$(|4, 3, 2, 1) \in \tilde{\mathcal{Q}}^{(4)} \tag{4.5.105}$$

At level 10, as it is natural to expect, there are $P(10-8) = 2$ rank 3 6×6 Blocks with the same structure as the one appearing for the first time at level 8. It is natural to conjecture that each time a new null vector will appear it will bring along a Jordan cell of higher rank, so that at level 18 a rank 4 Jordan cell is expected to appear for the first time.

4.6   Discussion

Symplectic fermions etc

4.7   Appendix

4.7.1   Bernoulli Numbers

The Bernoulli numbers B_n are defined as:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{4.7.1}$$

they satisfy:

$$B_{2n-1} = 0 \ , n = 2, 3, \ldots \tag{4.7.2}$$

They appear in the Euler Maclauring summation formula:

$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(b) - f^{(2k-1)}(a)) \tag{4.7.3}$$

these numbers satisfy a wide variety of identities, for example:

$$\sum_{n=0}^{m} \binom{m+1}{n} B_n = 0 \tag{4.7.4}$$

can be used for proving identities like:

$$\frac{2n-1}{2(2n+1)!} + \sum_{s=0}^{2n-1} (-1)^s \frac{B_{2n-s+1}}{(2n-s+1)!s!} = 0 \tag{4.7.5}$$

$$\frac{n}{4^n(2n+1)!} - \sum_{s=0}^{2n-1} \frac{B_{2n-s+1}}{2^s(2n-s+1)!s!} = 0 \tag{4.7.6}$$

which ensure that the even derivatives of $F(t)$ drop out of the Euler Maclaurin calculation 4.3.51.

Or they appear in the sum of powers used in 4.3.38:

$$\sum_{k=1}^{p} k^p = \sum_{k=1}^{p+1} (-1)^{p-k+1} \frac{B_{p-k+1}p!}{k!(p-k+1)!} n^k \tag{4.7.7}$$

They are also necessary to go over from the zeta functions appearing in 4.3.29 to the expressions of the highest weight BLZ IOM 4.3.34:

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n} \tag{4.7.8}$$

4.7.2 Proof of an Integral

We want to give an explanation of how 4.3.25 was obtained. The integrand is not bounded along the imaginary axis and has also double poles. Nonetheless we want to find a way to evaluate the integral by means of the residue formula. For this reason we split the double poles by inserting a regulator which we also use to introduce a dumping factor along the imaginary axis. We will evaluate this integral by using a countour running along the real axis and enclosing the poles in upper half plane with a semicircle of infinite radius.
After introducing the regulator one gets:

$$I_n(\epsilon) = \int_{-\infty}^{\infty} dx \frac{e^{i\epsilon x} x^{2n}}{1 + \frac{\epsilon^2}{2} \cosh x} \tag{4.7.9}$$

the roots of the denominator are:

$$x_n(\epsilon) = \log(-1 - \frac{\epsilon^2}{2} \pm \sqrt{1 - (1 + \frac{\epsilon^2}{2})}) \sim i\pi(2l + 1) \mp i\epsilon \tag{4.7.10}$$

one then gets by expanding in powers of ϵ and isolating the residues (the divergent part drops by virtue of the $\pm$ signs):

$$I_n(\epsilon) = -4\pi i \sum_{l=0}^{\infty} (-i)(-1)^n \pi^{2n-1} 2n(2l + 1)^{2n-1} + O(\epsilon) \tag{4.7.11}$$

one then uses in sequence:

$$\sum_{l=0}^{\infty} \frac{z^{2l+1}}{(2l + 1)^{\nu}} = \frac{1}{2}(\mathrm{Li}_{\nu}(z) - \mathrm{Li}_{\nu}(-z)) \tag{4.7.12}$$

$$\mathrm{Li}_{\nu}(-1) = (2^{1-\nu} - 1)\zeta(\nu) \tag{4.7.13}$$

$$\zeta(1 - 2n) = (-1)^{3n} 2^{1-2n} \Gamma(2n) \frac{\zeta(2n)}{\pi^{2n}} \tag{4.7.14}$$

to obtain

$$I_n(\epsilon) = 8n(2^{1-2n} - 1)\Gamma(2n)\zeta(2n) + O(\epsilon) \tag{4.7.15}$$

so that:

$$I_n(0) = \int_{-\infty}^{\infty} dx \frac{x^{2n}}{1 + \cosh x} = 8n(2^{1-2n} - 1)\Gamma(2n)\zeta(2n) \tag{4.7.16}$$

The other similar integral is evaluated precisely with the same techniques.

4.7.3   Some Virasoro-Fermi Modes Calculations

We start by analizing the module $\mathcal{Q}^{(0)}$ (being also the vacuum module), for which the most generic states up to level 6 give, after some straightforward algebra:

$$\Omega \tag{4.7.17}$$

$$L_{-2}\Omega = \frac{1}{2}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega \tag{4.7.18}$$

$$L_{-3}\Omega = \boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-1}\Omega \tag{4.7.19}$$

$$(aL_{-4}+bL_{-2}^2)\Omega = (a+b)\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-1}\Omega + \frac{a}{2}\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-2}\Omega \tag{4.7.20}$$

$$(aL_{-5}+bL_{-3}L_{-2})\Omega = (a+b)\boldsymbol{\chi}_{-4}\cdot\boldsymbol{\chi}_{-1}\Omega + a\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-2}\Omega \tag{4.7.21}$$

$$(aL_{-6}+bL_{-4}L_{-2}+cL_{-3}^2+dL_{-2}^3)\Omega = (a+b+2c+3d)\boldsymbol{\chi}_{-5}\cdot\boldsymbol{\chi}_{-1}\Omega + (\,a+c)\boldsymbol{\chi}_{-4}\cdot\boldsymbol{\chi}_{-2}\Omega+$$
$$+ (\frac{a}{2}+d)\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-3}\Omega + \frac{1}{2}(\frac{b}{2}-c)\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-2}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega \tag{4.7.22}$$

The first few generic states in the module $\mathcal{Q}^{(1)}$ up to level 5 are:

$$\chi^+_{-1}\Omega \tag{4.7.23}$$

$$L_{-1}\chi^+_{-1}\Omega = \chi^+_{-2}\Omega \tag{4.7.24}$$

$$(aL_{-2}+bL_{-1}^2)\chi^+_{-1}\Omega = (a+2b)\chi^+_{-3}\Omega \tag{4.7.25}$$

$$(aL_{-3}+bL_{-2}L_{-1}+cL_{-1}^3)\chi^+_{-1}\Omega = (\,a+2b+6c)\chi^+_{-4}\Omega + \frac{1}{2}(b-a)\chi^+_{-2}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega \tag{4.7.26}$$

$$(aL_{-4}+bL_{-3}L_{-1}+cL_{-2}^2+dL_{-2}L_{-1}^2+eL_{-1}^4)\chi^+_{-1}\Omega = (\,a+2b+3c+6d+24e)\chi^+_{-5}\Omega+$$
$$+ \frac{1}{2}(-a+c+2d)\chi^+_{-3}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega + \frac{1}{2}(a-b)\chi^+_{-1}\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-2}\Omega \tag{4.7.27}$$

$$(aL_{-5}+bL_{-4}L_{-1}+cL_{-3}L_{-2}L_{-1}+dL_{-3}L_{-1}^2+eL_{-2}^2L_{-1}+fL_{-2}L_{-1}^3+gL_{-1}^5)\chi^+_{-1}\Omega =$$
$$+ (a+2b+3c+6d+8e+24f+120g)\chi^+_{-6}\Omega + \frac{1}{2}(-a+4e+6f)\chi^+_{-4}\boldsymbol{\chi}_{-1}\cdot\boldsymbol{\chi}_{-1}\Omega+$$
$$+ (a-c-2d)\chi^+_{-1}\boldsymbol{\chi}_{-2}\cdot\boldsymbol{\chi}_{-3}\Omega + (\,b-c-2d+e)\chi^+_{-2}\boldsymbol{\chi}_{-3}\cdot\boldsymbol{\chi}_{-1}\Omega \tag{4.7.28}$$

The first few generic states in the module $\mathcal{Q}^{(2)}$ up to level 3 are:

$$\chi^+_{-2}\chi^+_{-1}\Omega \tag{4.7.29}$$

$$L_{-1}\chi^+_{-2}\chi^+_{-1}\Omega = 2\chi^+_{-3}\chi^+_{-1}\Omega \tag{4.7.30}$$

$$(aL_{-2} + bL^2_{-1})\chi^+_{-2}\chi^+_{-1}\Omega = (2b - a)\chi^+_{-3}\chi^+_{-2}\Omega + (2a + 6b)\chi^+_{-4}\chi^+_{-1}\Omega \tag{4.7.31}$$

$$(aL_{-3}+bL_{-2}L_{-1}+cL^3_{-1})\chi^+_{-2}\chi^+_{-1}\Omega =(-a+12c)\chi^+_{-4}\chi^+_{-2}\Omega+(2a+6b+24c)\chi^+_{-5}\chi^+_{-1}\Omega \tag{4.7.32}$$

We give also the first few states in the module $\tilde{\mathcal{Q}}^{(0)}$:

$$\mu \tag{4.7.33}$$

$$L_{-1}\mu = \frac{1}{2}\boldsymbol{\chi}_{-\frac{1}{2}} \cdot \boldsymbol{\chi}_{-\frac{1}{2}}\mu \tag{4.7.34}$$

$$(aL_{-2} + bL^2_{-1})\mu = (a + \frac{b}{2})\boldsymbol{\chi}_{-\frac{3}{2}} \cdot \boldsymbol{\chi}_{-\frac{1}{2}}\mu \tag{4.7.35}$$

$$(aL_{-3} + bL_{-2}L_{-1} + cL^3_{-1})\mu = (a + \frac{b}{2} + \frac{3}{4}c)\boldsymbol{\chi}_{-\frac{5}{2}} \cdot \boldsymbol{\chi}_{-\frac{1}{2}}\mu + (\frac{a}{2} + \frac{b}{4})\boldsymbol{\chi}_{-\frac{3}{2}} \cdot \boldsymbol{\chi}_{-\frac{3}{2}}\mu \tag{4.7.36}$$

$$(aL_{-4} + bL_{-3}L_{-1} + cL^2_{-2} + dL_{-2}L^2_{-1} + eL^4_{-1})\mu = (a + \frac{b}{2} + \frac{3}{2}c + \frac{3}{4}d + \frac{15}{8}e)\boldsymbol{\chi}_{-\frac{7}{2}} \cdot \boldsymbol{\chi}_{-\frac{1}{2}}\mu+$$

$$+ (a + \frac{c}{2} + \frac{d}{4} + \frac{3}{2}e)\boldsymbol{\chi}_{-\frac{5}{2}} \cdot \boldsymbol{\chi}_{-\frac{3}{2}}\mu + (\frac{b}{4} - \frac{c}{2} - \frac{d}{4} + \frac{e}{8})\boldsymbol{\chi}_{-\frac{3}{2}} \cdot \boldsymbol{\chi}_{-\frac{3}{2}}\boldsymbol{\chi}_{-\frac{1}{2}} \cdot \boldsymbol{\chi}_{-\frac{1}{2}}\mu$$

$$\tag{4.7.37}$$

These formulas, together with table 4.1 are sufficient to reconstruct the precise coefficients of the Fermionic expressions everywhere in the text.

4.7.4   Truncated action of the BLZ IOM

In this appendix we give useful truncations for the action of the BLZ IOM when acting on some descendant state at level K:

$$\mathbf{I}_3 = 2\sum_{n=1}^{K} L_{-n}L_n + L^2_0 - \frac{c + 2}{12}L_0 + \frac{c(5c + 22)}{2880} \tag{4.7.38}$$

$$\mathbf{I}_5 = 3!\left(\sum_{n=1}^{K} L_{-n}L_0L_n + \sum_{n=2}^{K}\sum_{m=1}^{n-1}(L_{-(m+n)}L_mL_n + L_{-n}L_{-m}L_{n+m}) + \frac{3}{2}\sum_{n=1}^{K} L_{1-2n}L_{2n-1}+\right.$$

$$+ \sum_{n=1}^{K}(\frac{11 + c}{6}n^2 - \frac{c}{4} - 1)L_{-n}L_n - \frac{c + 4}{8}L^2_0 + \frac{(c + 2)(3c + 20)}{576}L_0 - \frac{c(3c + 14)(7c + 68)}{290304}$$

$$\tag{4.7.39}$$

Vir	$\mathcal{Q}^{(0)}$	$\mathcal{Q}^{(1)}$	$\mathcal{Q}^{(2)}$	$\mathcal{Q}^{(3)}$	$\mathcal{D}$	$w(\mathcal{D})$
$\lvert 0\rangle$	Ω				$()$	0
$\lvert 1\rangle$		$\chi^+_{-1}\Omega$			(1)	1
$L_{-2}\lvert 0\rangle$	$\chi_{-1}\cdot\chi_{-1}\Omega$	$\chi^+_{-2}\Omega$			$(1[1])$	2
$L_{-1}\lvert 1\rangle$					$(1[2])$	
$L_{-3}\lvert 0\rangle$	$\chi_{-2}\cdot\chi_{-1}\Omega$	$\chi^+_{-3}\Omega$	$\chi^+_{-2}\chi^+_{-1}\Omega$		$(1[2])$	3
$(L_{-2}+L^2_{-1})\lvert 1\rangle$					(3)	
$\lvert 3\rangle$					$(2,1)$	
$\begin{array}{l}(L_{-4}-L^2_{-2})\lvert 0\rangle\\ L^2_{-2}\Omega\end{array}$	$\begin{array}{l}\chi_{-2}\cdot\chi_{-2}\Omega\\ \chi_{-3}\cdot\chi_{-1}\Omega\end{array}$	$\chi^+_{-4}\Omega$	$\chi^+_{-3}\chi^+_{-1}\Omega$		$\begin{array}{l}(2[2])\\ (1[3])\end{array}$	4
$(L_{-3}+L_{-2}L_{-1}+L^3_{-1})\lvert 1\rangle$					(4)	
$(-10L_{-3}+2L_{-2}L_{-1}+L^3_{-1})\lvert 1\rangle$		$\chi^+_{-2}\chi_{-1}\cdot\chi_{-1}\Omega$			$(1[2,1])$	
$L_{-1}\lvert 3\rangle$					$(1[3,1])$	
$\begin{array}{l}(L_{-5}-L_{-3}L_{-2})\Omega\\ L_{-3}L_{-2}\Omega\end{array}$	$\begin{array}{l}\chi_{-3}\cdot\chi_{-2}\Omega\\ \chi_{-4}\cdot\chi_{-1}\Omega\end{array}$	$\chi^+_{-5}\Omega$	$\chi^+_{-4}\chi^+_{-1}\Omega$	$\chi^+_{-3}\chi^+_{-2}\chi^+_{-1}\Omega$	$\begin{array}{l}(2[3])\\ (1[4])\end{array}$	5
$(26L_{-4}+26L_{-3}L_{-1}+6L^2_{-2}+10L_{-2}L^2_{-1}+L^4_{-1})\lvert 1\rangle$		$\chi^+_{-3}\chi_{-1}\cdot\chi_{-1}\Omega$	$\chi^+_{-3}\chi^+_{-2}\Omega$		$(1[5])$	
$(2L_{-4}+2L_{-3}L_{-1}+2L^2_{-2}-6L_{-2}L^2_{-1}+L^4_{-1})\lvert 1\rangle$		$\chi^+_{-1}\chi_{-2}\cdot\chi_{-2}\Omega$			$(1[3,1])$	
$(-10L_{-4}+8L_{-3}L_{-1}+6L^2_{-2}-8L_{-2}L^2_{-1}+L^4_{-1})\lvert 1\rangle$					$(2[2,1])$	
$(2L_{-2}+L^2_{-1})\lvert 3\rangle$					$(1[4,1])$	
$(-3L_{-2}+L^2_{-1})\lvert 3\rangle$					$(1[3,2])$	
$(-2L_{-6}+4L_{-4}L_{-2}+2L^2_{-3}+L^3_{-2})\Omega$	$\chi_{-5}\cdot\chi_{-1}\Omega$	$\chi^+_{-6}\Omega$	$\chi^+_{-5}\chi^+_{-1}\Omega$	$\chi^+_{-3}\chi^+_{-2}\chi^+_{-1}\Omega$	$(1[5])$	6
$(-8L_{-6}-2L_{-4}L_{-2}-L^2_{-3}+4L^3_{-2})\Omega$	$\chi_{-4}\cdot\chi_{-2}\Omega$	$\chi^+_{-4}\chi_{-1}\cdot\chi_{-1}\Omega$	$\chi^+_{-4}\chi^+_{-2}\Omega$		$(2[4])$	
$(L_{-6}-2L_{-4}L_{-2}-L^2_{-3}+L^3_{-2})\Omega$	$\chi_{-3}\cdot\chi_{-3}\Omega$	$\chi^+_{-1}\chi_{-2}\cdot\chi_{-3}\Omega$			$(3[3])$	
$(-2L_{-6}-5L_{-4}L_{-2}+2L^2_{-3}+L^3_{-2})\Omega$	$\chi_{-3}\cdot 2\chi_{-2}\chi_{-1}\cdot\chi_{-1}\Omega$	$\chi^+_{-2}\chi_{-3}\cdot\chi_{-1}\Omega$			$(2,1[2,1])$	
$(304L_{-5}+258L_{-4}L_{-1}+112L_{-3}L_{-2}L_{-1}+96L_{-3}L^2_{-1}+46L^2_{-2}L_{-1}+20L_{-2}L^3_{-1}+L^5_{-1})\lvert 1\rangle^{**}$					$(1[6])$	
$(4L_{-5}+8L_{-4}L_{-1}+12L_{-3}L_{-2}L_{-1}-4L_{-3}L^2_{-1}-4L^2_{-2}L_{-1}-5L_{-2}L^3_{-1}+L^5_{-1})\lvert 1\rangle$					$(1[4,1])$	
$(10L_{-5}-33L_{-4}L_{-1}-34L_{-3}L_{-2}L_{-1}+L_{-3}L^2_{-1}+L^2_{-2}L_{-1}+L_{-2}L^3_{-1}+L^5_{-1})\lvert 1\rangle$					$(2[3,1])$	
$(10L_{-5}-96L_{-4}L_{-1}+8L_{-3}L_{-2}L_{-1}+L_{-3}L^2_{-1}+L_{-2}L^3_{-1}+L^5_{-1})\lvert 1\rangle^{**}$					$(1[3,2])$	
$(12L_{-3}+7L_{-2}L_{-1}+L^3_{-1})\lvert 3\rangle$					$(1[5,1])$	
$(-6L_{-3}-2L_{-2}L_{-1}+L^3_{-1})\lvert 3\rangle$					$(1[4,2])$	
$\lvert 6\rangle$					$(1[3,2,1])$	

Table 4.1. Fermionic structure of the modules $\mathcal{Q}^{(n)}$ up to $w(\mathcal{D}) = 6$, the $**$ means that the states are degenerate within the same module, and therefore linear combinations fall within the same eigenspace.

Bibliography

[1] VV. Bazanov, S.L. Lukyanov, A.B. Zamolodchikov, arXiv:hep-th/9412229 (1996), arXiv:hep-th/9604044 (1997), arXiv:hep-th/9805008 (1999) .

[2] V.F.R. Jones, *Planar algebras I*, math.QA/9909027.

[3] P.A. Pearce, J. Rasmussen, J.-B. Zuber, J. Stat. Mech. (2006) P11017.

[4] P.A. Pearce, J. Rasmussen, J. Stat. Mech. (2007) P02015.

[5] B. Duplantier, F. David, J. Stat. Phys. **51** (1988) 327–434.

[6] A. Sedrakyan, Nucl. Phys. **B554** (1999) 514-536.

[7] M.R. Gaberdiel, H.G. Kausch, Phys. Lett. **B386** (1996) 131–137.

[8] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, Nucl. Phys. **B757** (2006) 303-343; *Kazhdan-Lusztig dual quantum group for logarithmic extensions of Virasoro models*, hep-th/0606506 (2006).

[9] M.R. Gaberdiel, I. Runkel, J. Phys. **A39** (2006) 14745-14780.

[10] Horst G. Kausch . DAMTP-95-52, Oct 1995. 26pp. e-Print: hep-th/9510149

[11] H.G. Kausch, Nucl. Phys. **B583** (2000) 513–541.

[12] S. Mahieu, P. Ruelle, Phys. Rev. **E64** (2001) 066130; P. Ruelle, Phys. Lett. **B539** (2002) 172–177; G. Piroux, P. Ruelle, J. Stat. Mech. 0410 (2004) P005; J. Phys. **A38** (2005) 1451–1472; Phys. Lett. **B607** (2005) 188–196.

[13] N.S. Izmailian, V.B. Priezzhev, P. Ruelle, C.-K. Hu, Phys. Rev. Lett. **95** (2005) 260602.

[14] J.L. Jacobsen, N. Read, H. Saleur, Phys. Rev. Lett. **93** (2004) 038701.

[15] E.V. Ivashkevich, C.-K. Hu, Phys. Rev. **E71** (2005) 015104 (R).

[16] R. E. Behrend, P. A. Pearce and D. L. O'Brien, "Interaction - round - a - face models with fixed boundary conditions: The ABF fusion hierarchy", J. Stat. Phys.84 (1996) 1-48, arXiv:hep-th/9507118.

[17] A. Klümper and P.A. Pearce, "Conformal weights of RSOS models and their fusion hierarchies", Physica A **183** (1992) 304-350.

[18] D.O'Brien, P.Pearce, S.O.Waarnar, Finitized Conformal Spectrum of the Ising Model on the Cylinder and Torus, Physica A 228 (1996) pp.63 .

[19] A. Nigro, On the Integrable Structure of the Ising Model, J.Stat.Mech.(2008) P01017, (arxiv:07104213 hep-th)

[20] P.Grinza, G.Feverati, Integrals of Motion from TBA and lattice-conformal dictionary, arXiv:hep-th/0405110 (2004) .

[21] Paul.A.Pearce, Leung Chim, Changrim Ahn, Excited TBA Equations I: Massive Tricritical Ising Model, arXiv:hep-th/0012223 (2000).

[22] P. A. Pearce and B. Nienhuis, "Scaling Limit of RSOS Models and TBA Equations", Nucl. Phys. B **519** (1998) 579 [arXiv:hep-th/9711185].

[23] W.M. Koo, H. Saleur, Representations of the Virasoro algebra from lattice models, arxiv:hep-th/9312156, (1993)

[24] P.A. Pearce, J. Rasmussen, P. Ruelle, Integrable Boundary conditions and $\mathcal{W}-$extended Fusion in the Logarithmic Minimal Models $\mathcal{LM}(1,p)$, arxiv:08030785 (2008).

[25] C. Gomez,J. Gunnesson, R. Hernandez, The Ising model and planar $\mathcal{N} = 4$ Yang Mills, arxiv:07113404 (hep-th)

Chapter 5

Acknowledgements