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Ph.D. Thesis

The statistical mechanics of spanning forests

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Preface

This work collects the results of my research activity of the last three years.

The main part is the study of the spanning forest model in a mean field approximation by the use of a recently introduced fermionic representation. These results have been published in the following papers in collaboration with S. Caracciolo and A. Sportiello.

- A. Bedini, S. Caracciolo, and A. Sportiello, *Hyperforests on the complete hypergraph by Grassmann integral representation*, Journal of Physics A: Mathematical and Theoretical **41**, 205003 (2008).
- A. Bedini, S. Caracciolo, and A. Sportiello, *Phase transition in the spanning-hyperforest model on complete hypergraphs*, Nuclear Physics B **822**, 493 (2009).

A second an more recent subject of research concerns the development of a new general algorithm for the exact computation of statistical mechanics partition function on arbitrary graphs. This work has been done in collaboration with J. L. Jacobsen and its publication is still in progress.

- A. Bedini, J. L. Jacobsen, *Fast solution of NP-hard problems on large random graphs*, in progress.

Contents

Part I

The spanning forest model

Chapter 1

Introduction

This thesis is devoted to the study of the *spanning forest model*. This model can be thought as a specialization of the Potts/random-cluster model in which the cycles have weight zero. Critical features in a model of spanning forests are particularly interesting because only the geometric properties of connection of different parts are involved and this extremely reduced structure is probably at the root of many other critical phenomena, within, and even outside, natural sciences.

The spanning forest model is defined by considering, for a given graph or hypergraph (which is a natural generalization of the concept of graph where the edges can connect more than two vertices at once) $G = (V, E)$, the set of all spanning subgraphs not containing any circuits. Such subgraphs are called *forests* and their connected components *trees*. For reader's convenience basic definitions about graphs and hypergraphs can be found in Appendix A. In mathematical terms we are therefore interested in the following partition function (or generating function in the combinatorics language)

$$F_G(\mathbf{w}) = \sum_{F \in \mathcal{F}(G)} \prod_{A \in F} w_A, \tag{1.1}$$

where the sum is extended over all spanning forests configurations $\mathcal{F}(G)$ in G and $\mathbf{w} = (w_A)_{A \in E}$ are (hyper-)edges weights. Rescaling the edge weights by a proper factor $w_A \to w_A/\lambda^{|A|-1}$, where $|A|$ is the cardinality of the hyperedge (which is two for an ordinary edge), we can give an additional weight λ to each connected components, indeed

$$F_G(\lambda, \mathbf{w}) = \lambda^V F_G(\{w_e/\lambda^{|A|-1}\}) = \lambda^V \sum_{F \in \mathcal{F}(G)} \lambda^{-\sum_{A\in F}(|A|-1)} \prod_{A \in F} w_A \tag{1.2}$$

$$= \sum_{F \in \mathcal{F}(G)} \lambda^{k(F)} \prod_{A \in F} w_A\,, \tag{1.3}$$

where we use the Euler theorem (see A.2.1 in Appendix A) and $k(F)$ denotes the number of connected components in the subgraph F.

A close relative of (1.1) is the generating function of *rooted* spanning forests. Here the term "rooted" indicates that forests configurations comprehend one or more marked vertex (the *roots*) from each of which originates a separate tree (hence the name). It should be clear that, while in (1.3) the weight is independent from the trees sizes, in this case connected components get a weight proportional to their size (since there are more choices in the roots placement). So we have the generating function of rooted (opposed to *unrooted*) spanning forests

$$E_G(\mathbf{t},\mathbf{w}) = \sum_{\substack{F\in\mathcal{F}(G)\\ F=(F_1,\dots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} \left(\sum_{i\in V(F_\alpha)} t_i\right), \tag{1.4}$$

where $\mathbf{t} = \{t_i\}_{i\in V}$ are weight for the roots and the sum extends to forests with connected components $(F_1,\dots,F_\ell)$. A well known particular case of (1.4) is when we put $t_j = 0$ for all vertices $j \in V$ but one. In this case the only configurations surviving in (1.4) are the ones composed of a single tree giving origin to the *spanning tree* generating function

$$T_G(\mathbf{w}) = \sum_{T\in\mathcal{T}(G)} \left(\prod_{A\in F} w_A\right), \tag{1.5}$$

where $\mathcal{T}(G)$ is the set of spanning trees configurations in G. The distinction between *rooted* and *unrooted* trees is trivial since all configuration would take a factor proportional to the size of the only existing tree, which is fixed since the trees are spanning.

The spanning tree generating function can be obtained also form (1.3), as the first term in the λ expansion

$$F_G(\lambda,\mathbf{w}) = \lambda\, T_G(\mathbf{w}) + O(\lambda^2)\,. \tag{1.6}$$

It is worth noting that F_G (respectively E_G) is multilinear in the edge weights $\{w_A\}_{A\in E}$ (respectively edge weights $\{w_A\}_{A\in E}$ and vertex weights $\{t_i\}_{i\in V}$), meaning that they appear with degree at most one.

Computing (1.4), at least in the ordinary graph case, can be done easily using the following slight generalization of the Kirchhoff's matrix-three theorem [1]:

Theorem 1.0.1 *Let $G = (V,E)$ be a finite graph and let w_{ij} be weights associated to edges $e = (i,j) \in E$. Define the* Laplacian matrix $L = (L_{ij})_{i,j\in V}$ *for the graph G by*

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum\limits_{k\neq i} w_{ik} & \text{if } i = j \end{cases} \tag{1.7}$$

Then we have, for each set of k vertices $\{i_1,\dots,i_k\} \subset V$

$$\det L(i_1,\dots,i_k) = \sum_{F\in\mathcal{F}(i_1,\dots,i_k)} \left(\prod_{A\in F} w_A\right) \tag{1.8}$$

where $\det L(i_1,\dots,i_k)$ *is the determinant of the matrix L without the rows and columns corresponding to vertices $i_1,\dots,i_k$ and $\mathcal{F}(i_1,\dots,i_k)$ is the set of rooted spanning forests with roots placed in $i_1,\dots,i_k$.*

Indeed the set $\mathcal{F}(i_1, \ldots, i_k)$ can be enumerated by the function (1.4) by giving weight one to vertices $i_1, \ldots, i_k$ and weight zero otherwise. Due to the multi-linearity, this is equivalent to taking derivatives respect to $t_{i_1}, \ldots, t_{i_k}$ evaluated in $\mathbf{t} = 0$, therefore

$$\det L(i_1, \ldots, i_k) = \frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_k}} E_G(\mathbf{t}, \mathbf{w})\Big|_{\mathbf{t}=0} \tag{1.9}$$

Our approach to the study of (1.1) and (1.4) is based on a novel representation, first presented in [2], in terms of fermionic fields, which means that the partition functions can be written as a multiple Berezin integral over anti-commuting variables belonging to a Grassmann algebra. Indeed, as all determinants, the determinant appearing in the Kirchhoff's thorem can be rewritten in terms of a Grassman–Berezin integration

$$\det L(i_1, \ldots, i_k) = \int \mathcal{D}(\psi, \bar{\psi})\ \bar{\psi}_{i_1}\psi_{i_1} \cdots \bar{\psi}_{i_k}\psi_{i_k}\ \exp\left(\sum_{j,k} \bar{\psi}_j L_{jk} \psi_k\right), \tag{1.10}$$

whose precise meaning will be elucidated in the next chapter. Always in the ordinary graph case, it is simple to obtain a representation for (1.4) by resumming over all possible root choices with weights $\{t_i\}_{i\in V}$:

$$E_G(\mathbf{t}, \mathbf{w}) = \int \mathcal{D}(\psi, \bar{\psi})\ \exp\left(\sum_i t_i\, \bar{\psi}_i \psi_i + \sum_{j,k} \bar{\psi}_j L_{jk} \psi_k\right), \tag{1.11}$$

from which we can clearly see the resemblance to a massive fermionic field theory. How to obtain a similar representation for the *unrooted* spanning forest partition function (1.1) for general graphs or hypergraphs will be the subject of Chapter 3. In the ordinary graph case the action will contain an additional nearest-neighbour four-fermion term with a fixed coupling, while in the general hypergraph case the action will contain term of higher order in the fields.

A model of spanning forests is interesting for a variety of reasons. Perhaps the most appealing reason, at least from a physicist's point of view, is its connection with two very different generalization of the Ising model, the Potts model and the O(N) non-linear σ-model.

We have already mentioned that the spanning forest model emerge from the q-states Potts model in the limit in which $q \to 0$, the analytical continuation being possible thanks to the Fortuin–Kasteleyn expansion.

In addition, hidden in the fermionic representation of the spanning forest model lies a OSP(1|2) symmetry, that is the symmetry group of rotations over the super-sphere in $\mathbb{R}^{(1|2)}$. Indeed, by introducing an auxiliary bosonic field σ_i completing the triplet

$$\mathbf{n}_i = (\sigma_i, \bar{\psi}_i, \psi_i),$$

it is possible to show that the spanning forest model is perturbatively equivalent (with an important inversion of the coupling's sign) to an OSP(1|2) symmetric non-linear σ-model. This is very interesting since a σ model with such symmetry it is itself perturbatively equivalent to a σ-model with an O(N) symmetry at $N = -1$.

The role of supersymmetry in statistical mechanics has a long history. Already in the 70's, it has been observed that the n-vector model in the limit in which $n \to -2$ is equivalent to a free fermion theory [3]. Successively, in 1980 Parisi and Sourlas [4] showed the equivalence of the n-vector model in the limit $n \to 0$, which was already known to describe the critical behavior of polymers [5], with a supersymmetric OSP(2|2) model in which the loops in Feynman graphs (which vanish when $n \to 0$) give zero contribution due to the cancellation between bosonic and fermionic degrees of freedom. This cancellation, which is at the base of our approach, is independent of the graph G as only exploits the symmetry properties of the target space.

This fermionic formulation is also well-suited to the use of standard field-theoretic machinery. For example, in [2] it has been obtained the renormalization-group flow near the spanning-tree (free-field) fixed point for the spanning-forest model on the square lattice, and in [6] this was extended to the triangular lattice.

Plan of the work

In Chapter 2 we shall elucidate the relation with the Potts/random-cluster model by obtaining the spanning forest model as a limiting case of the Potts model, that is when the number of colors are analytically continued to zero.

The Grassmann representation of the spanning forests partition function, along with the related partition function of rooted spanning forests, will be covered extensively in Chapter 3.

Thereafter, in Chapter 4, we shall be concerned mainly with the problem of evaluating the weight of rooted and unrooted hyperforests in the complete hypergraph with n vertices $\overline{\mathcal{K}}_n$ when the weight of a hyperedge depends only on its cardinality. Physically this corresponds to a mean-field approximation allowing the presence of many-body interactions.

All the results can also be obtained by starting from recursion relations in the number of vertices, but we shall show here how the same problem can be directly and more easily solved by means of the Grassmann representation. Once having obtained the general solutions we shall restrict to particular cases to recover more explicit results. In particular we shall consider the case of the k-uniform complete hypergraph, where only edges of cardinality k are present. With the edge weight set to one, we shall reduce to a counting problem obtaining a generalization of many known results in the case $k = 2$ (namely of ordinary forests on the complete graph). In the case of unrooted hyperforests we shall also recover a novel explicit expression for their number with p connected components, that is hypertrees, in terms of the associated Laguerre polynomials, for any k.

In Chapter 5 we will show that the spanning forest model has a hidden OSP(1|2) symmetry and can actually be obtained also as a non-linear σ model with the fields taking values in the unit supersphere in $\mathbb{R}^{1|2}$.

Finally, in Chapter 6, we shall study the phase transition of the spanning forest model on the k-uniform complete hypergraph for any $k \geq 2$. Different k are studied at once by using a microcanonical ensemble in which the number of hyperforests is fixed. The low-temperature phase is characterized by the appearance of a giant hyperforest. The phase transition occurs when the number of hyperforests is a fraction $(k-1)/k$ of the total number of vertices. The behaviour at criticality is also studied by means of the coalescence of two saddle points. As the Grassmann formulation exhibits a global supersymmetry we show that the phase transition is second order and is associated to supersymmetry breaking and we explore the pure thermodynamical phase at low temperature by introducing an explicit breaking field.

Chapter 2

From Potts to spanning forests

Introduction

The aim of this Chapter is to place the spanning forests model in the context of statistical mechanics models. Indeed a possible way to attack this model by the tools of statistical mechanics goes back to the formulation as a Potts model [10, 11, 12] in the limit of vanishing number q of states, which we will review in Section 2.1. In Section 2.2 we will recall the Fortuin-Kasteleyn representation [13, 14, 15] which expresses the partition function of the Potts model as a sum on all subgraphs $H \subseteq G$ of monomials in both q and the edge couplings v_e's. The limit in which $q \to 0$ will be discussed in Section 2.4.

2.1 The Potts model

Potts model is a generalization of the Ising model [16] to more than two components. In studying for his doctoral degree Ernst Ising focused on the special case of a model of ferromagnetism that his then supervisor Wilhelm Lenz had introduced in 1920 [17]. Ising considered a linear chain of magnetic moments that can adopt only two positions, up and down, and that are coupled by interactions between nearest neighbors.

In 1951, while in Oxford, Cyril Domb pointed out to his then research student Renfrey Burnard Potts that the transformation discovered by Kramers and Wannier (1941) for the two-dimensional Ising model could be generalized to a planar vector model having three symmetric orientations at angles of 0, $2\pi/3$, $4\pi/3$ with the axis. Hence the critical point could be located for this model. He suggested that it might be possible to extend the result to a planar vector model with q symmetric orientations.

After a detailed investigation Potts [10] came to the conclusion that the transformation did not generalize to a planar vector model with q orientations, but instead to a q-state model in which there are two different energies of interaction which correspond to nearest neighbors being in the same state or different states; the case $q = 4$ for this model had been considered previously by Ashkin and Teller [18]. For the planar model with $q = 4$ it was possible to locate the critical temperature by an alternative method, but this failed for higher values of q.

In order to differentiate between the two types of model the following terminology, due to Domb [19] is used: the q-state two-energy model is referred to simply as the Potts model (or the *standard* Potts model), and the planar q-orientation model as the *planar* Potts model (or *vector* Potts model). It is worth noting that they both are vector models, indeed the q-state two-energy model correspond to a vector model in which spins can take the q symmetric directions of a simplex in $q-1$ dimensions.

Given a generic finite graph $G = (V, E)$, the q-state Potts model on G is defined as follows: at each site $i \in V$ we first place a spin (or *color*) σ_i which takes value in the set $S = \{0, 1, \dots, q-1\}$, we then make the spins interact via the Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j\rangle \in E} J_{ij}\, \delta(\sigma_i, \sigma_j), \tag{2.1}$$

where J_{ij} measures the interaction energy between site i and site j. A coupling J_{ij} is called *ferromagnetic* if $J_e > 0$ and antiferromagnetic if $J_{ij} < 0$. Hamiltonian is symmetric under any permutation of the state's set S.

Although Potts models with multisite interactions do not appear to have been much studied (but see [20], [21],[22],[23]), such generalization is straightforward. Thus I will adopt from the beginning a notation convenient to deal with multisite interactions.

Consider an hyper-graph $G = (V, E)$ where the hyper-edges $A = \{i_1, i_2, \dots\} \in E$ are the subsets of interacting vertices and suppose that each interaction involves at last two vertices, so that each hyper-edge has at least degree two. Let's assign again to each site $i \in V$ a spin $\sigma_i \in S$ and to each hyper-edge $A \in E$ an interaction energy J_A, the Hamiltonian now reads:

$$\mathcal{H} = -\sum_{A \in E} J_A\, \delta_A(\sigma) \tag{2.2}$$

where, for $A = \{i_1, \dots, i_k\}$ we introduced the symbol $\delta_A(\sigma)$ defined as follows

$$\delta_A(\sigma) = \begin{cases} 1, & \text{if } \sigma_{i_1} = \cdots = \sigma_{i_k} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

If the model has only ferromagnetic couplings the model will have q symmetric ordered phases at low temperature where the spins are all in the same state and a disordered phase at high temperature where on the contrary the spins are uncorrelated at large distance.

If instead the model has antiferromagnetic couplings some spins will prefer being in a different state than their neighbors. It is worth noting that understanding the antiferromagnetic ground-state is non trivial because it depends strongly on the properties of the lattice (as being or nor bipartite).

The partition function reads

$$Z_G(q, \mathbf{J}) = \sum_{\sigma} e^{-\mathcal{H}} = \sum_{\sigma} \prod_{A \in E} e^{J_A\, \delta_A(\sigma)}, \tag{2.4}$$

where the sum extends over all the spin assignment $\sigma : V \to S$.

2.2 Fortuin–Kasteleyn expansion

While the hamiltonian (2.1) is defined only for natural $q > 1$, Fortuin and Kasteleyn showed [13] that its meaning can be extended to all real (or even complex) values of q. The argument goes as follows: since the Kronecker delta can take only two values, 0 and 1, we can rewrite the partition function as

$$Z_G(q, \mathbf{v}) = \sum_{\sigma} \prod_{e \in E} \left[1 + v_A \, \delta_A(\sigma)\right]. \tag{2.5}$$

where we exchanged the parameter J_A for $v_A = e^{J_A} - 1$. Now we expand out the product over the edges and consider the subgraph G' induced by the edges for which the term $v_A \, \delta_A(\sigma)$ is taken. The delta forces spins belonging the a same connected component to be in the same state, therefore the sum becomes trivial and gives the following result:

$$Z_G(q, \mathbf{v}) = \sum_{E' \subseteq E} q^{k(E')} \prod_{A \in E'} v_A, \tag{2.6}$$

where the sum runs over all the spanning subgraphs $G' = (V, E')$ of G.

The Fortuin–Kasteleyn expansion (2.6) shows that the Potts model, which in its original formulation is a *vertex model* (since the degrees of freedom sit on the vertices of the graph), can be reformulated as an *edge model* with an additional factor counting connected components (or *clusters*). It is worth underlining that the presence of this factor makes the model *non-local* in the edge variables at variance with the original definition which is *local* in the site variables.

The parameters $-1 \le v_A < \infty$ control the presence of edges in E', the ferromagnetic and antiferromagnetic being mapped to $v_e > 0$ and $-1 \le v_A \le 0$ respectively. If $v_A < -1$ the state weights are no longer positive and the models becomes unphysical.

If $q > 1$ the presence of many clusters gets favored, while if $q < 1$ is favored the presence of few clusters. If q is exactly one, the weight is independent of the number of clusters in the configuration and the measure factorizes over the edges. In this case the model is equivalent to the percolation model.

2.3 Relation with graph coloring

The problem of finding the number of ways in which the vertices of a given graph can be colored with not more than a given number of colors q so that adjacent vertices have different colors (q-colorings) has a longer tradition than the models mentioned above; in the form of the four-color conjecture it has a history which goes back to the middle of the 19th century. In his research on the coloring problem, Birkhoff introduced in the 1912 the chromatic polynomial $P_G(q)$, which is an extension of the number of q-colorings from integral values to arbitrary integer real values of q.

It is easy to see that the number of q-colorings of a graph is equal to the degeneracy of the ground state of an antiferromagnetic Potts model. To see this fact, consider in (2.4) the limit in which all the couplings J_{ij} go to minus infinity: the weight of any configuration in which two neighbors share the same state tends to zero, and the only surviving configurations in the partition function (2.4) are proper colorings, all with weight one, therefore

$$\lim_{J_{ij} \to -\infty} Z_G(q, \mathbf{J}) = P_G(q). \tag{2.7}$$

This simple connection between the Potts model partition function and the chromatic polynomial is valid for any graph G.

The above expansion also gives a representation for the chromatic polynomial (2.7). Since the limit $J \to -\infty$ corresponds to $v = -1$, we have:

$$P_G(q) = \sum_{E' \subseteq E} q^{k(E')} (-1)^{|E'|}. \tag{2.8}$$

This formula was already known to Birkhoff in 1912.

2.4 The $q \to 0$ limit

There are different ways in which the $q \to 0$ limit can be taken in the q-state Potts model. For brevity in what follows we will suppose G connected, the general case can be re-obtained easily. Let me recall the Fortuin–Kasteleyn expansion

$$Z_G(q, \mathbf{v}) = \sum_{E' \subseteq E} q^{k(E')} \prod_{A \in E'} v_A, \tag{2.6}$$

The simplest limit is to take $q \to 0$ with fixed couplings $\mathbf{v}$. We see that this selects out the subgraphs $E' \subseteq E$ having the smallest possible number of connected components; the minimum achievable value is one, being G connected. We therefore have

$$\lim_{q \to 0} q^{-k(G)} Z_G(q, \mathbf{v}) = C_G(\mathbf{v}), \tag{2.9}$$

where

$$C_G(\mathbf{v}) = \sum_{\substack{E' \subseteq E \\ k(A) = k(G)}} \prod_{A \in E'} v_A \tag{2.10}$$

is the partition function of "connected spanning subgraphs".

A different limit can be obtained by taking $q \to 0$ with fixed values of $w_A = v_A / q^{|A|-1}$, where $|A|$ is the cardinality of the hyperedge A which is two if A is an ordinary edge. From (2.6) we have

$$Z_G(q, \{q^{|A|-1} w_A\}) = \sum_{E' \subseteq E} q^{k(E') + \sum_{A \in E'} (|A|-1)} \prod_{A \in E'} w_A. \tag{2.11}$$

Using proposition A.2.1, we see that the limit $q \to 0$ selects out the spanning forests:

$$\lim_{q \to 0} q^{-|V|} Z_G(q, \{q^{|A|-1} w_A\}) = F_G(\mathbf{w}), \tag{2.12}$$

where

$$F_G(\mathbf{w}) = \sum_{E' \in \mathcal{F}(G)} \prod_{A \in E'} w_A \tag{2.13}$$

$$\begin{tikzcd}
& C_G(\mathbf{v}) \arrow[dr, "\mathbf{v}\text{ infinitesimal}"] & \\
Z_G(q,\mathbf{v}) \arrow[ur, "{q\to 0,\ \mathbf{v}\text{ fixed}}"] \arrow[dr, "{q\to 0,\ \mathbf{w}=\mathbf{v}/q\text{ fixed}}"'] & & T_G(\mathbf{v}\text{ or }\mathbf{w}) \\
& F_G(\mathbf{w}) \arrow[ur, "\mathbf{v}\text{ infinite}"'] &
\end{tikzcd}$$

Figure 2.1: The $q \to 0$ limits of the Potts model

is the partition function (1.1) of *unrooted* spanning forests.

Finally, suppose that in $C_G(\mathbf{v})$ we replace each edge weight v_A by $\lambda^{|A|-1}v_A$ and then take $\lambda \to 0$. This obviously selects out, from among the maximally connected spanning subgraphs, those having the minimum value of $\sum_{e\in E}(|A|-1)$, which are precisely the spanning trees:

$$\lim_{\lambda\to 0} \lambda^{1-|V|} C_G(\{\lambda^{|A|-1}v_A\}) = T_G(\mathbf{v}), \tag{2.14}$$

where

$$T_G(\mathbf{v}) = \sum_{E'\in\mathcal{T}(G)} \prod_{A\in E'} v_A \tag{2.15}$$

is the partition function of spanning trees. Alternatively, suppose that in $F_G(\mathbf{w})$ we replace each edge weight w_A by $\lambda^{|A|-1}w_A$ and then take $\lambda \to \infty$. This selects of, from among the spanning forests, those having the maximum value of $\sum_{A\in E}(|A|-1)$: these are once again the spanning trees:

$$\lim_{\lambda\to\infty} \lambda^{1-|V|} F_G(\{\lambda^{|A|-1}w_A\}) = T_G(\mathbf{w}). \tag{2.16}$$

In summary, we have for the $q \to 0$ limits of the Potts model the scheme in picture 2.1

It is worth underlining that spanning trees can also be obtained directly from $Z_G(q,\mathbf{v})$ by a one-step process in which the limit $q \to 0$ is taken at fixed $x_A = v_A/q^{\alpha(|A|-1)}$, where $0 < \alpha < 1$. Indeed, simple manipulation of (2.6) yields

$$Z_G(q, q^\alpha \mathbf{x}) = q^{\alpha|V|} \sum_{E'\subseteq E} q^{\alpha\, c(E')+(1-\alpha)\, k(E')} \prod_{A\in E'} x_A. \tag{2.17}$$

The quantity $\alpha\, c(A) + (1-\alpha)\, k(A)$ is minimized only if A is a spanning tree, where it takes the value $1-\alpha$. Hence

$$\lim_{q\to 0} q^{-\alpha|V|-1+\alpha} Z_G(q, q^\alpha \mathbf{x}) = T_G(\mathbf{x}). \tag{2.18}$$

There is, however, one important difference between the graph case and the hypergraph case: every connected graph has a spanning tree, but not every connected hypergraph as a spanning hypertree. So the limits (2.14), (2.16), and (2.18) can be zero.

Chapter 3

The fermionic representation

The use of Grassmann–Berezin calculus [26] has provided an interesting short-cut toward the classical matrix-tree theorem [1, 27, 28], which express the generating polynomials of spanning trees and rooted spanning forests in a graph as determinants associated to the graph's Laplacian matrix, as well as generalizations thereof [29, 30, 31]. Indeed, like all determinants, those arising in Kirchhoff's theorem 1.0.1 can be rewritten as Gaussian integrals over Grassmann variables.

A generalization of Kirchhoff's theorem, in which a large class of combinatorial objects are represented by suitable *non-Gaussian* Grassmann integrals, has been recently proved by Caracciolo et al. [2]. In particular, they shown how the partition function of spanning forests in a graph can be represented as a Grassmann integral involving a quadratic term together with a special nearest-neighbor four-fermion interaction. Furthermore as we will discuss later in Chapter 5 this fermionic model possesses an OSP(1|2) supersymmetry.

This chapter, that reproduces the content of [7], provides a full treatment of this fermionic formulation. We will first review basic aspects of the Grassmann–Berzin calculus. Then we will introduce, in Section 3.2 a Grassmann sub-algebra suitable to represent forests on a hypergraph. We will then see, in Section 3.3 how this algebra can be used to build a representation of the partition function for rooted and unrooted spanning forests. In Section 3.4 we understand how correlation functions can be expressed as sums over partially rooted spanning hyperforests constrained to satisfy particular conditions.

Finally, Section 3.5 presents a simple graphical representation of the whole construction, providing a "graphical" proof of the classical matrix-tree theorem as well as the the mentioned generalizations.

3.1 Grassmann–Berezin calculus

A n-dimensional Grassmann algebra is the algebra generated (with coefficients in $\mathbb{R}$ or $\mathbb{C}$) by a set of variables $\{\psi_i\}$, with $i = 1, \dots, n$, satisfying

$$\{\psi_i, \psi_j\} = 0 \qquad \forall i, j \tag{3.1}$$

i.e. they anticommute, which implies in particular that $\psi_i^2 = 0$. The algebra generated by these symbols contains all expressions of the form

$$f(\psi) = f^{(0)} + \sum_i f^i \psi_i + \sum_{i<j} f^{ij} \psi_i \psi_j + \sum_{i<j<k} f^{ijk} \psi_i \psi_j \psi_k + \cdots \tag{3.2}$$

$$= \sum_{0 \le k \le n} \frac{1}{k!} \sum_{\{i\}} f^{i_1,\ldots,i_k} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_k} \tag{3.3}$$

where the coefficients are antisymmetric tensors with k indices, each ranging from 1 to n. Since there are $\binom{n}{k}$ such linearly independent tensors, summing over k from 0 to n produces a 2^n-dimensional algebra. The anticommunting rule allows us to define an associative product

$$f(\psi)g(\psi) = f^{(0)}g^{(0)} + \sum_i \left(f^0 g^i + f^i g^0\right) \psi_i + \frac{1}{2} \sum_{ij} \left(f^{ij} g^0 + f^i g^j - f^j g^i + f^0 g^{ij}\right) \psi_i \psi_j + \cdots \tag{3.4}$$

Please note that in general fg is not equal to $\pm gf$. Nevertheless the subalgebra containing terms with an even number (possibly zero) of ψ variables commutes with any element f.

One can present the above expansion in yet another form which clearly exhibits the relation with Fermi statistics, namely

$$f(\psi) = \sum_{a_i=0,1} f_{a_1,a_2,\ldots,a_n} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n} \tag{3.5}$$

The integers $a_i = 0, 1$ can be thought of as occupation numbers of "states" described by ψ_i. A similar expression in the commuting case has no bounds on the occupation numbers a_i.

Having defined sum and products in the Grassmann algebra we now define a left derivative $\partial/\partial\psi_i$. The latter gives zero on a monomial which does not contain the variable ψ_i. If the monomial does contain ψ_i, the latter is moved to the left (with the appropriate sign due to the exchanges) and then suppressed. The operation is extended by linearity to any element of the algebra. A right derivative can be defined similarly. From this definition the following rules can be obtained:

$$\left\{ \frac{\partial}{\partial\psi_i}, \frac{\partial}{\partial\psi_j} \right\} = 0 \qquad \left\{ \frac{\partial}{\partial\psi_i}, \psi_j \right\} = \delta_{ij}. \tag{3.6}$$

Integrals are defined as linear operations over the functions $f(\psi)$ with the seemingly paradoxical property that they can be identified with the (left) derivatives. Correspondingly

$$\int \mathrm{d}\psi\, f(\psi) = \frac{\partial f}{\partial \psi} \qquad \int \mathrm{d}\psi_2 \mathrm{d}\psi_1\, f(\psi) = \frac{\partial}{\partial\psi_2} \frac{\partial}{\partial\psi_1} f(\psi) \qquad \ldots \tag{3.7}$$

Using the notation of (3.3):

$$\int \mathrm{d}\psi_n \mathrm{d}\psi_{n-1} \ldots \mathrm{d}\psi_1\, f(\psi) = \frac{\partial}{\partial\psi_n} \frac{\partial}{\partial\psi_{n-1}} \cdots \frac{\partial}{\partial\psi_1} f(\psi) = f^{1,2,\ldots,n} \tag{3.8}$$

More generally if τ is a permutation of $[n]$, we have

$$\int \mathrm{d}\psi_{\tau(n)}\mathrm{d}\psi_{\tau(n-1)}\cdots\mathrm{d}\psi_{\tau(1)}\, f(\psi) = \epsilon(\tau)\int \mathrm{d}\psi_n\mathrm{d}\psi_{n-1}\cdots\mathrm{d}\psi_1\, f(\psi), \tag{3.9}$$

where $\epsilon(\tau)$ denotes the signature of τ.

It is evident that this definition fulfills the constraint of translational invariance

$$\int \mathrm{d}\psi\ (a+b\psi) = \int \mathrm{d}\psi\ [a+b\,(\psi+\eta)] \tag{3.10}$$

which requires

$$\int \mathrm{d}\psi\, 1 = 0 \qquad \text{and} \qquad \int \mathrm{d}\psi\, \psi = 1. \tag{3.11}$$

Changes of coordinates are required to preserve the anti-commuting structure of the Grassmann algebra, this allows non-singular linear transformations of the form $\eta_i = \sum_{j=1}^n A_{ij}\,\psi_j$. One then can verify that by setting $f(\psi) = F(\eta)$ one can obtain the following relation:

$$\int \mathrm{d}\psi_n\mathrm{d}\psi_{n-1}\cdots\mathrm{d}\psi_1\, f(\psi) = \det A\int \mathrm{d}\eta_n\mathrm{d}\eta_{n-1}\cdots\mathrm{d}\eta_1\, F(\eta) \tag{3.12}$$

at variance with the commuting case in which the factor on the right hand side would have been $|\det A|^{-1}$.

Very often it is profitable to consider a 2^{2n}-dimensional *complexified* version of the Grassmann algebra comprising two set of generators $\{\psi_i\}$ and $\{\bar\psi_i\}$ with anti-commuting relations:

$$\{\psi_i,\psi_j\}=0 \qquad \{\bar\psi_i,\bar\psi_j\}=0 \qquad \{\psi_i,\bar\psi_j\}=0 \tag{3.13a}$$

$$\begin{gathered}\left\{\frac{\partial}{\partial\psi_i},\bar\psi_j\right\}=0 \qquad \left\{\frac{\partial}{\partial\psi_i},\psi_j\right\}=\delta_{ij}\\ \left\{\frac{\partial}{\partial\bar\psi_i},\bar\psi_j\right\}=\delta_{ij} \qquad \left\{\frac{\partial}{\partial\bar\psi_i},\psi_j\right\}=0\end{gathered} \tag{3.13b}$$

In what follows, we will make often use of the following shorthands:

$$\mathcal{D}_I(\psi,\bar\psi) = \prod_{i\in I}\mathrm{d}\psi_i\mathrm{d}\bar\psi_i, \tag{3.14}$$

where the product runs over the set of indices I. If $I=[n]$, we will just write $\mathcal{D}_n(\psi,\bar\psi)$.

We now want to show a first simple application of this formalism by proving the following formula:

$$\int \mathcal{D}_n(\psi,\bar\psi)\ \exp\left(\sum_{i,j=1}^n \bar\psi_i A_{ij}\psi_j\right) = \det A. \tag{3.15}$$

We first make a change of variables from ψ_i to $\eta_i = \sum_{j=1}^n A_{ij}\psi_j$, obtaining

$$\int \mathcal{D}_n(\psi,\bar\psi)\ \exp\left(\sum_{i,j=1}^n \bar\psi_i A_{ij}\psi_j\right) = \det A\int \mathcal{D}_n(\eta,\bar\psi)\ \exp\left(\sum_{i=1}^n \bar\psi_i\eta_i\right), \tag{3.16}$$

then we observe that due to the nilpotency of the Grassman variables

$$\exp\left(\bar{\psi}_i \eta_i\right) = 1 + \bar{\psi}_i \eta_i, \tag{3.17}$$

so that

$$\int \mathcal{D}_n(\eta, \bar{\psi}) \ \exp\left(\sum_{i=1}^{n} \bar{\psi}_i \eta_i\right) = \int \mathcal{D}_n(\eta, \bar{\psi}) \ \prod_{i=1}^{n} \left(1 + \bar{\psi}_i \eta_i\right). \tag{3.18}$$

Due to properties (3.11), the integral is non zero only if the integrand contains every variable in the integration measure. Therefore, in the product expansion, only the term $\bar{\psi}_1 \eta_1 \cdots \bar{\psi}_n \eta_n$ contributes to the result. Moreover the variables order in the integration fixes the factor of this term to $+1$, proving (3.15).

The above result can be generalized to expectation values of monomials. Denoting with $A(I|J)$ the submatrix obtained from A deleting the rows $I = (i_1, i_2, \ldots, i_k)$ and columns $J = (j_1, j_2, \ldots, j_k)$, we can prove the following more general formula

$$\int \mathcal{D}_n(\psi, \bar{\psi}) \ \bar{\psi}_{i_1} \psi_{j_1} \bar{\psi}_{i_2} \psi_{j_2} \cdots \bar{\psi}_{i_k} \psi_{j_k} \exp\left(\sum_{i,j=1}^{n} \bar{\psi}_i A_{ij} \psi_j\right) = \epsilon(I|J) \det A(I|J). \tag{3.19}$$

Indeed the presence of ψ_k (respectively $\bar{\psi}_k$) annihilates the contribution of terms of the form $\bar{\psi}_i A_{ij} \psi_k$ (respectively $\bar{\psi}_k A_{kj} \psi_j$) and $\epsilon(I|J) = \pm 1$ accounts for the number of interchanges needed to order the variables before the integration.

In the next chapter we shall make use of very simple results for Grassmann integrals. We shall take the chance to present them here. In the following we shall denote by $[z^n]\, f(z)$ the coefficient of z^n in the taylor expansion of $f(z)$ around the origin.

Now consider the linear combination of variables $\bar{\psi}\psi = \sum_{i=1}^{n} \bar{\psi}_i \psi_i$, that can be seen as an internal scalar product. We then have the following

Lemma 3.1.1 *Let $|V| = n$ be the number of vertices, then*

$$\int \mathcal{D}_n(\psi, \bar{\psi})\, (\bar{\psi}\psi)^s = s!\, \delta_{s,n}$$

Proof. It trivially follows from induction in n. □

We soon derive, by expansion in powers, that

Corollary 3.1.2 *Let g be a generic function of the scalar product $\bar{\psi}\psi = \sum_{i=1}^{n} \bar{\psi}_i \psi_i$, that is a polynomial as the scalar product is nilpotent of degree n, then*

$$\int \mathcal{D}_n(\psi, \bar{\psi})\, g(\bar{\psi}\psi) = n!\ [z^n]\ g(z) = \frac{n!}{2\,\pi\, i} \oint \frac{dz}{z^{n+1}}\, g(z)$$

where the contour integral is performed in the complex plain constrained to encircle the origin.

These are the ingredients for the following

Lemma 3.1.3 *Let $|V| = n$ be the number of vertices, g a generic function, then*

$$\begin{aligned}\int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)_{i_1}\cdots(\bar\psi\psi)_{i_r}\,g(\bar\psi\psi) &= \frac{(n-r)!}{n!}\int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,g(\bar\psi\psi)\\ &= (n-r)!\,\left[z^{n-r}\right]\,g(z)\end{aligned}$$

Proof. By integrating over $\bar\psi_{i_1},\psi_{i_1},\cdots,\bar\psi_{i_r},\psi_{i_r}$ on the left hand side we get an integral of the form used in the previous Lemma, where both the integration measure and the scalar product were restricted on the remaining $n-r$ vertices, so that

$$\int \mathcal{D}_{n-r}(\psi,\bar\psi)\,g(\bar\psi\psi) = (n-r)!\,\left[z^{n-r}\right]\,g(z)\,.$$

By expanding instead on the right hand side we get

$$\sum_{s\geq 0}\frac{(n-r)!}{n!}\int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^{r+s}\,[z^s]\,g(z) = (n-r)!\,\left[z^{n-r}\right]\,g(z)$$

and we get our result by using the previous Lemma 3.1.1. □

Let J the matrix with unit entries for each $i,j\in V$, and denote by $(\bar\psi \mathrm{J}\psi)$ the quantity

$$(\bar\psi\mathrm{J}\psi) = \sum_{i,j}\bar\psi_i \mathrm{J}_{ij}\psi_i = \sum_{i,j}\bar\psi_i\psi_i\,. \tag{3.20}$$

Our common tool is the following

Lemma 3.1.4 *Let $|V| = n$ be the number of vertices, g and h generic function, then*

$$\int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,e^{h(\bar\psi\psi)+(\bar\psi\mathrm{J}\psi)g(\bar\psi\psi)} = \int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,e^{h(\bar\psi\psi)}\left[1+\bar\psi\psi\,g(\bar\psi\psi)\right] \tag{3.21}$$

Proof. Let us expand the second part of the exponential

$$\begin{aligned}\int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,e^{h(\bar\psi\psi)}\sum_s\frac{(\bar\psi\mathrm{J}\psi)^s}{s!}\,g(\bar\psi\psi)^s &= \int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,e^{h(\bar\psi\psi)}\left[1+(\bar\psi\mathrm{J}\psi)\,g(\bar\psi\psi)\right]\\ &= \int \mathcal{D}_n(\psi,\bar\psi)\,(\bar\psi\psi)^r\,e^{h(\bar\psi\psi)}\left[1+\bar\psi\psi\,g(\bar\psi\psi)\right]\end{aligned}$$

because all higher powers of $(\bar\psi\mathrm{J}\psi)$ vanish. We get the final line because in the rest of the integral for each i the field $\bar\psi_i$ is always multiplied by the companion ψ_i and thus the only contribution in $(\bar\psi\mathrm{J}\psi)$ comes from the diagonal part, that is $(\bar\psi\psi)$. □

3.2 A peculiar Grassmann subalgebra

Let V be a finite set of cardinality n. For each $i \in V$ we introduce a pair ψ_i, $\bar{\psi}_i$ of generators of a complex Grassmann algebra. For each subset $A \subseteq V$, we associate the monomial $\tau_A = \prod_{i\in A} \bar{\psi}_i \psi_i$, where $\tau_\emptyset = 1$. Please note that all these monomials are even elements of the Grassmann algebra; in particular, they commute with the whole Grassmann algebra. Clearly, the elements $\{\tau_A\}_{A\subseteq V}$ span a vector space of dimension 2^n. In fact, this vector space is a subalgebra, by virtue of the obvious relations

$$\tau_A \, \tau_B \;=\; \begin{cases} \tau_{A\cup B} & \text{if } A \cap B = \emptyset \\ 0 & \text{if } A \cap B \neq \emptyset \end{cases} \tag{3.22}$$

Let us now introduce another family of even elements of the Grassmann algebra, also indexed by subsets of V, which possesses very interesting and unusual properties. For each subset $A \subseteq V$ and each number λ (in $\mathbb{R}$ or $\mathbb{C}$), we define the Grassmann element

$$f_A^{(\lambda)} \;=\; \lambda(1-|A|)\tau_A + \sum_{i\in A} \tau_{A\smallsetminus i} - \sum_{\substack{i,j\,\in\, A \\ i\neq j}} \bar{\psi}_i \psi_j \tau_{A\smallsetminus\{i,j\}} \,. \tag{3.23}$$

or equivalently

$$f_A^{(\lambda)} \;=\; \left[\lambda(1-|A|) + \sum_{i,j\in A} \partial_i \bar{\partial}_j\right] \tau_A \;=\; \left[\lambda(1-|A|) + \partial\bar{\partial}\right] \tau_A, \tag{3.24}$$

where $\partial = \sum\limits_{i\in V} \partial_i$ and $\bar{\partial} = \sum\limits_{i\in V} \bar{\partial}_i$. For instance, we have

$$f_\emptyset^{(\lambda)} \;=\; \lambda \tag{3.25a}$$

$$f_{\{i\}}^{(\lambda)} \;=\; 1 \tag{3.25b}$$

$$\begin{aligned} f_{\{i,j\}}^{(\lambda)} \;&=\; -\lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j + \bar{\psi}_i \psi_i + \bar{\psi}_j \psi_j - \bar{\psi}_i \psi_j - \bar{\psi}_j \psi_i \\ &=\; -\lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \end{aligned} \tag{3.25c}$$

and in general

$$f_{\{i_1,\ldots,i_k\}}^{(\lambda)} \;=\; \lambda(1-k)\tau_{\{i_1,\ldots,i_k\}} + \sum_{\alpha=1}^{k} \tau_{\{i_1,\ldots,\not{i_\alpha},\ldots,i_k\}} - \sum_{\substack{1\,\leq\,\alpha,\beta\,\leq\,k \\ \alpha\neq\beta}} \bar{\psi}_{i_\alpha} \psi_{i_\beta} \tau_{\{i_1,\ldots,\not{i_\alpha},\ldots,\not{i_\beta},\ldots,i_k\}} \,. \tag{3.26}$$

(Whenever we write a set $\{i_1,\ldots,i_k\}$, it is implicitly understood that the elements $i_1,\ldots,i_k$ are all distinct.) Clearly, each $f_A^{(\lambda)}$ is an even element in the Grassmann algebra, and in particular it commutes with all the other elements of the Grassmann algebra.

Let us observe that

$$f_A^{(\lambda)} \, \tau_B \;=\; \begin{cases} \tau_{A\cup B} & \text{if } |A \cap B| = 1 \\ 0 & \text{if } |A \cap B| \geq 2 \end{cases} \tag{3.27}$$

as an immediate consequence of (3.22) [when $A \cap B = \{k\}$, only the second term in (3.23) with $i = k$ survives]. Note, finally, the obvious relations

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} f_A^{(\lambda)} \;=\; (1 - |A|)\tau_A \tag{3.28}$$

and

$$f_A^{(\lambda)} - f_A^{(\lambda')} \;=\; (\lambda - \lambda')(1 - |A|)\tau_A \;. \tag{3.29}$$

We are interested in the subalgebra of the Grassmann algebra that is generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V, for an arbitrary fixed value of λ.[1] The key to understanding this subalgebra is the following amazing identity:

Lemma 3.2.1 *Let $A, B \subseteq V$ with $A \cap B \neq \emptyset$. Then*

$$f_A^{(\lambda)} f_B^{(\lambda)} \;=\; \begin{cases} f_{A\cup B}^{(\lambda)} & \textit{if } |A \cap B| = 1 \\ 0 & \textit{if } |A \cap B| \geq 2 \end{cases} \tag{3.30}$$

More generally,

$$f_A^{(\lambda)} f_B^{(\lambda')} \;=\; \begin{cases} f_{A\cup B}^{(\lambda'')} & \textit{if } |A \cap B| = 1 \\ 0 & \textit{if } |A \cap B| \geq 2 \end{cases} \tag{3.31}$$

where λ'' is the weighted average

$$\lambda'' \;=\; \frac{(|A|-1)\lambda + (|B|-1)\lambda'}{|A| + |B| - 2} \;=\; \frac{(|A|-1)\lambda + (|B|-1)\lambda'}{|A \cup B| - 1} \;. \tag{3.32}$$

PROOF.
Since $\partial^2 = \bar{\partial}^2 = 0$, we have

$$\left(\partial\bar{\partial}\tau_A\right)\left(\partial\bar{\partial}\tau_B\right) \;=\; \partial\bar{\partial}\left(\tau_A \partial\bar{\partial}\tau_B\right) \;=\; \partial\bar{\partial}\left(\tau_B \partial\bar{\partial}\tau_A\right) \;, \tag{3.33}$$

so that

$$\begin{aligned} f_A^{(\lambda)} f_B^{(\lambda')} \;=\;& \lambda(1-|A|)\tau_A\partial\bar{\partial}\tau_B \;+\; \lambda'(1-|B|)\tau_B\partial\bar{\partial}\tau_A \\ & + \lambda\lambda'(1-|A|)(1-|B|)\tau_A\tau_B \;+\; \partial\bar{\partial}\left(\tau_A\partial\bar{\partial}\tau_B\right) \;. \end{aligned} \tag{3.34}$$

If $|A \cap B| \geq 1$, then $\tau_A \tau_B = 0$ and

$$\tau_A\partial\bar{\partial}\tau_B \;=\; \tau_B\partial\bar{\partial}\tau_A \;=\; \begin{cases} \tau_{A\cup B} & \text{if } |A \cap B| = 1 \\ 0 & \text{if } |A \cap B| \geq 2 \end{cases} \tag{3.35}$$

This proves (3.31). □

As a first consequence of Lemma 3.2.1, we have:

[1] One can also consider the smaller subalgebras generated by the elements $f_A^{(\lambda)}$ as A ranges over some collection $\mathcal{S}$ of subsets of V.

Corollary 3.2.2 *Let $A \subseteq V$ with $|A| \geq 2$. Then the Grassmann element $f_A^{(\lambda)}$ is nilpotent of order 2, i.e.*

$$\big(f_A^{(\lambda)}\big)^2 = 0\,.$$

In particular, a product $\prod_{i=1}^m f_{A_i}^{(\lambda)}$ vanishes whenever there are any repetitions among the $A_1, \ldots, A_m$. By iterating Lemma 3.2.1 and using Proposition A.2.2, we easily obtain:

Corollary 3.2.3 *Let $G = (V, E)$ be a* connected *hypergraph. Then*

$$\prod_{A \in E} f_A^{(\lambda)} \;=\; \begin{cases} f_V^{(\lambda)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases} \tag{3.36}$$

More generally,

$$\prod_{A \in E} f_A^{(\lambda_A)} \;=\; \begin{cases} f_V^{(\lambda_\star)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases} \tag{3.37}$$

where $\lambda_\star$ is the weighted average

$$\lambda_\star \;=\; \frac{\sum\limits_{A \in E} (|A|-1)\lambda_A}{\sum\limits_{A \in E} (|A|-1)} \;=\; \frac{\sum\limits_{A \in E} (|A|-1)\lambda_A}{\big|\bigcup\limits_{A \in E} A\big| - 1}\;. \tag{3.38}$$

We are now ready to consider the subalgebra of the Grassmann algebra that is generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V. Recall first that a *partition* of V is a collection $\mathcal{C} = \{C_\gamma\}$ of disjoint nonempty subsets $C_\gamma \subseteq V$ that together cover V. We denote by $\Pi(V)$ the set of partitions of V. If V has cardinality n, then $\Pi(V)$ has cardinality $B(n)$, the n-th *Bell number* [32, pp. 33–34]. We remark that $B(n)$ grows asymptotically roughly like $n!$ [33, Sections 6.1–6.3].

The following corollary specifies the most general product of factors $f_A^{(\lambda)}$. Of course, there is no need to consider sets A of cardinality 1, since $f_{\{i\}}^{(\lambda)} = 1$.

Corollary 3.2.4 *Let E be a collection (possibly empty) of subsets of V, each of cardinality ≥ 2.*

(a) If the hypergraph $G = (V, E)$ is a hyperforest, and $\{C_\gamma\}$ is the partition of V induced by the decomposition of G into connected components, then $\prod\limits_{A \in E} f_A^{(\lambda)} = \prod\limits_{\gamma} f_{C_\gamma}^{(\lambda)}$. More generally, $\prod\limits_{A \in E} f_A^{(\lambda_A)} = \prod\limits_{\gamma} f_{C_\gamma}^{(\lambda_\gamma)}$, where λ_γ is the weighted average (3.38) *taken over the hyperedges contained in C_γ.*

(b) If the hypergraph $G = (V, E)$ is not a hyperforest, then $\prod\limits_{A \in E} f_A^{(\lambda)} = 0$, and more generally $\prod\limits_{A \in E} f_A^{(\lambda_A)} = 0$.

Proof. It suffices to apply Corollary 3.2.3 separately in each set C_γ, where $\{C_\gamma\}$ is the partition of V induced by the decomposition of G into connected components. □

It follows from Corollary 3.2.4 that any polynomial (or power series) in the $\{f_A^{(\lambda)}\}$ can be written as a linear combination of the quantities $f_{\mathcal{C}}^{(\lambda)} = \prod_\gamma f_{C_\gamma}^{(\lambda)}$ for partitions $\mathcal{C} = \{C_\gamma\} \in \Pi(V)$.

$$\mathcal{F}(\{f_A^{(\lambda)}\}_{A\in\mathcal{S}(V)}) = \sum_{P=\{C_\alpha\}\in\Pi(V)} W_{\mathcal{F}}(P) \prod_\alpha f_{C_\alpha}^{(\lambda)} . \tag{3.39}$$

This holds of course also for functions of the form

$$\mathcal{F}(\{f_A^{(\lambda)}\}_{A\in\mathcal{S}(V)}) = \exp(\mathcal{H}(\{f_A^{(\lambda)}\}_{A\in\mathcal{S}(V)}) . \tag{3.40}$$

Incidentally, as a consequence of nilpotency of Grassmann Algebra, every function $\mathcal{F}$ with non-zero coefficient for the monomial of degree zero in the algebra, i.e.

$$W(\{1\},\{2\},\cdots,\{n\}) \neq 0,$$

has a well-defined logarithm, defined by Taylor expansion.

We will now use the foregoing results to simplify the Boltzmann weight associated with a Hamiltonian of the form

$$\mathcal{H} = -\sum_{A\in E} w_A f_A^{(\lambda)}. \tag{3.41}$$

Corollary 3.2.5 *Let $G=(V,E)$ be a hypergraph. Then*

$$\exp\left(\sum_{A\in E} w_A f_A^{(\lambda)}\right) = \sum_{\substack{F \in \mathcal{F}(G) \\ F=(F_1,\dots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} f_{V(F_\alpha)}^{(\lambda)} , \tag{3.42}$$

where the sum runs over spanning hyperforests F in G with components $F_1,\dots,F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α. More generally,

$$\exp\left(\sum_{A\in E} w_A f_A^{(\lambda_A)}\right) = \sum_{\substack{F \in \mathcal{F}(G) \\ F=(F_1,\dots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} f_{V(F_\alpha)}^{(\lambda_\alpha)} , \tag{3.43}$$

where λ_α is the weighted average (3.38) *taken over the hyperedges contained in the hypertree F_α.*

Proof. Since the $f_A^{(\lambda_A)}$ are nilpotent of order 2 and commuting, we have

$$\exp\left(\sum_{A\in E} w_A f_A^{(\lambda_A)}\right) = \prod_{A\in E}\left(1 + w_A f_A^{(\lambda_A)}\right) \tag{3.44a}$$

$$= \sum_{E'\subseteq E}\left(\prod_{A\in E'} w_A\right)\left(\prod_{A\in E'} f_A^{(\lambda_A)}\right) . \tag{3.44b}$$

Using now Corollary 3.2.4, we see that the contribution is nonzero only when (V,E') is a hyperforest, and we obtain (3.42)/(3.43). □

3.3 A model for forests

Our principal goal in this section is to provide a combinatorial interpretation, in terms of spanning hyperforests, for the general partition function

$$\int \mathcal{D}_V(\psi,\bar\psi)\, \exp\biggl[\sum_i t_i\bar\psi_i\psi_i \,+\, \sum_{A\in E} w_A f_A^{(\lambda)}\biggr], \tag{3.45}$$

where $G=(V,E)$ is an arbitrary hypergraph and the $\{w_A\}_{A\in E}$ are arbitrary hyperedge weights. We also handle the slight generalization in which a separate parameter λ_A is used for each hyperedge A.

For any subset $A\subseteq V$ and any vector $\mathbf{t}=(t_i)_{i\in V}$ of vertex weights, let us define the integration measure

$$\mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi) \;=\; \prod_{i\in A} d\psi_i\, d\bar\psi_i\, e^{t_i\bar\psi_i\psi_i}\,, \tag{3.46}$$

so that (3.45) can also be written in a more compact form as

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\, \exp\biggl[\sum_{A\in E} w_A f_A^{(\lambda)}\biggr]. \tag{3.47}$$

Our basic results are valid for an arbitrary vector $\mathbf{t}=(t_i)_{i\in V}$ of "mass terms". However, as we shall see, the formulae simplify notably if we specialize to the case in which $t_i=\lambda$ for all $i\in V$. This is not an accident, as it corresponds to the case in which the action is OSP(1|2)-invariant as we will see in Chapter 5.

We begin with some formulae that allow us to integrate over the pairs of variables $\psi_i,\bar\psi_i$ one at a time:

Lemma 3.3.1 *Let $A\subseteq V$ and $i\in V$. Then:*

(a) $$\int d\psi_i\, d\bar\psi_i\, e^{t_i\bar\psi_i\psi_i}\,\tau_A \;=\; \begin{cases} \tau_{A\smallsetminus i} & \text{if } i\in A\\ t_i\tau_A & \text{if } i\notin A\end{cases}$$

(b) $$\int d\psi_i\, d\bar\psi_i\, e^{t_i\bar\psi_i\psi_i}\, f_A^{(\lambda)} \;=\; \begin{cases} f_{A\smallsetminus i}^{(\lambda)} + (t_i-\lambda)\tau_{A\smallsetminus i} & \text{if } i\in A\\ t_i f_A^{(\lambda)} & \text{if } i\notin A\end{cases}$$

PROOF. (a) is obvious, as is (b) when $i\notin A$. To prove (b) when $i\in A$, we write

$$f_A^{(\lambda)} \;=\; \lambda(1-|A|)\tau_A + \sum_{j\in A}\tau_{A\smallsetminus j} - \sum_{\substack{j,k\,\in\, A\\ j\neq k}} \bar\psi_j\psi_k\tau_{A\smallsetminus\{j,k\}} \tag{3.48}$$

and integrate with respect to $d\psi_i\, d\bar\psi_i\, e^{t_i\bar\psi_i\psi_i}$. We obtain

$$\lambda(1-|A|)\tau_{A\smallsetminus i} \,+\, t_i\tau_{A\smallsetminus i} \,+\, \sum_{j\in A\smallsetminus i}\tau_{A\smallsetminus\{i,j\}} \,-\, \sum_{\substack{j,k\,\in\, A\smallsetminus i\\ j\neq k}} \bar\psi_j\psi_k\tau_{A\smallsetminus\{i,j,k\}} \tag{3.49}$$

(in the last term we must have $j, k \neq i$ by parity), which equals $f^{(\lambda)}_{A\setminus i} + (t_i - \lambda)\tau_{A\setminus i}$ as claimed. $\square$

Applying Lemma 3.3.1 repeatedly for i lying in an arbitrary set $B \subseteq V$, we obtain:

Corollary 3.3.2 *Let $A, B \subseteq V$. Then*

$$\int \mathcal{D}_{B,\mathbf{t}}(\psi,\bar{\psi})\, f^{(\lambda)}_A \;=\; \left(\prod_{i\in B\setminus A} t_i\right) \left[f^{(\lambda)}_{A\setminus B} \,+\, \Big(\sum_{i\in B\cap A} (t_i - \lambda) \Big) \tau_{A\setminus B} \right] . \tag{3.50}$$

In particular, for $B = A$ we have

$$\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar{\psi})\; f^{(\lambda)}_A = \lambda + \sum_{i\in A} (t_i - \lambda). \tag{3.51}$$

and, if all $t_i = \lambda$,

$$\int \mathcal{D}_{A,\lambda}(\psi,\bar{\psi})\; f^{(\lambda)}_A = \lambda\,. \tag{3.52}$$

Proof. The factors t_i for $i \in B\setminus A$ follow trivially from the second line of Lemma 3.3.1(b). For the rest, we proceed by induction on the cardinality of $B\cap A$. If $|B\cap A| = 0$, the result is trivial. So assume that the result holds for a given set B, and consider $B' = B \cup \{j\}$ with $j \in A \setminus B$. Using Lemma 3.3.1(a,b) we have

$$\begin{aligned}
&\int d\psi_j\, d\bar{\psi}_j\, e^{t_j \bar{\psi}_j \psi_j} \left[f^{(\lambda)}_{A\setminus B} \,+\, \left(\sum_{i\in B\cap A} (t_i - \lambda) \right) \tau_{A\setminus B} \right] \\
&\quad = \; f^{(\lambda)}_{(A\setminus B)\setminus\{j\}} \,+\, (t_j - \lambda)\tau_{(A\setminus B)\setminus\{j\}} \,+\, \left(\sum_{i\in B\cap A} (t_i - \lambda) \right) \tau_{(A\setminus B)\setminus\{j\}} && \text{(3.53a)} \\
&\quad = \; f^{(\lambda)}_{A\setminus B'} \,+\, \left(\sum_{i\in B'\cap A} (t_i - \lambda) \right) \tau_{A\setminus B'} \,, && \text{(3.53b)}
\end{aligned}$$

as claimed. $\square$

Applying (3.51) once for each factor C_α, we have:

Corollary 3.3.3 *Let $\{C_\alpha\}$ be a partition of V. Then*

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar{\psi}) \prod_\alpha f^{(\lambda_\alpha)}_{C_\alpha} \;=\; \prod_\alpha \Big(\lambda_\alpha + \sum_{i\in C_\alpha} (t_i - \lambda_\alpha) \Big)\,. \tag{3.54}$$

The partition function (3.45) can now be computed immediately by combining Corollaries 3.2.5 and 3.3.3. We obtain the main result of this section:

Theorem 3.3.4 *Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A\in E}$ be hyperedge weights. Then*

$$\int \mathcal{D}_V(\psi,\bar{\psi}) \exp\left[\sum_{i\in V} t_i \bar{\psi}_i \psi_i + \sum_{A\in E} w_A f_A^{(\lambda_A)}\right] = \sum_{\substack{F\in\mathcal{F}(G)\\ F=(F_1,\ldots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} \left(\sum_{i\in V(F_\alpha)} t_i - \sum_{A\in E(F_\alpha)} (|A|-1)\lambda_A\right) , \qquad (3.55)$$

where the sum runs over spanning hyperforests F in G with components $F_1, \ldots, F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α. In particular, if λ_A takes the same value for all A, we have

$$\int \mathcal{D}_V(\psi,\bar{\psi}) \exp\left[\sum_{i\in V} t_i \bar{\psi}_i \psi_i \; + \sum_{A\in E} w_A f_A^{(\lambda)}\right] = \sum_{\substack{F\in\mathcal{F}(G)\\ F=(F_1,\ldots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} \left(\lambda + \sum_{i\in V(F_\alpha)} (t_i - \lambda)\right) , \qquad (3.56)$$

Proof. We apply (3.54), where (according to Corollary 3.2.5) λ_α is the weighted average (3.38) taken over the hyperedges contained in the hypertree F_α. Then

$$\lambda_\alpha + \sum_{i\in V(F_\alpha)} (t_i - \lambda_\alpha) = \sum_{i\in V(F_\alpha)} t_i \;-\; \lambda_\alpha(|V(F_\alpha)| - 1) \qquad (3.57a)$$

$$= \sum_{i\in V(F_\alpha)} t_i \;-\; \sum_{A\in E(F_\alpha)} (|A|-1)\lambda_A \,. \qquad (3.57b)$$

□

If we specialize (3.56) to $t_i = \lambda$ for all vertices i, we obtain:

Corollary 3.3.5 *Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A\in E}$ be hyperedge weights. Then*

$$\int \mathcal{D}_V(\psi,\bar{\psi}) \exp\left[\lambda \sum_{i\in V} \bar{\psi}_i \psi_i \; + \sum_{A\in E} w_A f_A^{(\lambda)}\right] = \sum_{F\in\mathcal{F}(G)} \lambda^{k(F)} \prod_{A\in F} w_A \,, \qquad (3.58)$$

where the sum runs over spanning hyperforests F in G, and $k(F)$ is the number of connected components of F.

This is the partition function of *unrooted* spanning hyperforests, with a weight w_A for each hyperedge A and a weight λ for each connected component. Note that the second equality in (3.58) uses Proposition A.2.1.

If, on the other hand, we specialize (3.56) to $\lambda = 0$, we obtain:

Corollary 3.3.6 *Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A\in E}$ be hyperedge weights. Then*

$$\int \mathcal{D}_V(\psi,\bar\psi)\, \exp\left[\sum_{i\in V} t_i\bar\psi_i\psi_i + \sum_{A\in E} w_A f_A^{(0)}\right] = \sum_{\substack{F\in\mathcal{F}(G)\\ F=(F_1,\dots,F_\ell)}} \left(\prod_{A\in F} w_A\right) \prod_{\alpha=1}^{\ell} \left(\sum_{i\in V(F_\alpha)} t_i\right) ,$$

where the sum runs over spanning hyperforests F in G with components $F_1, \dots, F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α.

This is the partition function of *rooted* spanning hyperforests, with a weight w_A for each hyperedge A and a weight t_i for each root i.

In the ordinary graph case formula (3.3.6) takes a very simple form. To see this, let $G = (V, E)$ be an ordinary graph, so that each edge $e \in E$ is simply an unordered pair $\{i, j\}$ of distinct vertices $i, j \in V$, to which there is associated an edge weight $w_{ij} = w_{ji}$, (absent edges are considered having weight zero). Then by definition (3.23) we have

$$f^{(\lambda)}_{\{i,j\}}(\psi,\bar\psi) \;=\; -\lambda\,\bar\psi_i\psi_i\bar\psi_j\psi_j + \bar\psi_i\psi_i + \bar\psi_j\psi_j - \bar\psi_i\psi_j - \bar\psi_j\psi_i \,, \tag{3.59}$$

therefore we have

$$\sum_{\{i,j\}\in E} w_{ij} f^{(\lambda)}_{\{i,j\}} = \sum_{i,j\in V} \bar\psi_i L_{ij}\psi_j - \lambda \sum_{\{i,j\}\in E} w_{ij}\bar\psi_i\psi_i\bar\psi_j\psi_j \tag{3.60}$$

$$= \sum_{i,j\in V} \bar\psi_i L_{ij}\psi_j + \frac{\lambda}{2} \sum_{i,j\in V} \bar\psi_i\psi_i L_{ij}\bar\psi_j\psi_j \,, \tag{3.61}$$

where L_{ij} is Laplacian matrix as defined in Theorem 1.0.1. Therefore we have

$$\int \mathcal{D}_V(\psi,\bar\psi)\, \exp\left[\sum_{i,j\in V} \bar\psi_i L_{ij}\psi_j + \sum_{i\in V} t_i\bar\psi_i\psi_i + \frac{\lambda}{2}\sum_{i,j\in V} \bar\psi_i\psi_i\bar\psi_j L_{ij}\psi_j\right] . \tag{3.62}$$

More generally, consider the case in which $G = (V, E)$ is a k-uniform hypergraph (an ordinary graph corresponds to the case $k = 2$). Let $w_{i_1,\dots,i_k}$ (assumed completely symmetric in the indices $i_1, \dots, i_k$) be the weight associated to the hyperedge $\{i_1, \dots, i_k\}$ when $i_1, \dots, i_k$ are all distinct, and let $w_{i_1,\dots,i_k} = 0$ when at least two indices are equal. Define the *(weighted) Laplacian tensor* (a rank-k symmetric tensor) by

$$L_{i_1,\dots,i_k} \;=\; \begin{cases} -w_{i_1,\dots,i_k} & \text{if } i_1,\dots,i_k \text{ are all different} \\ \frac{1}{k-1}\sum_{i'_s} w_{i_1,\dots,i'_s,\dots,i_k} & \text{if } i_r = i_s\ (r\neq s) \text{ and the others are all different} \\ 0 & \text{otherwise} \end{cases} \tag{3.63}$$

Then we have

$$\sum_{A\in E} w_A f_A^{(\lambda)} = \sum_{i_1,\dots,i_k\in V} \frac{L_{i_1,\dots,i_k}}{(k-2)!} \left[\bar\psi_{i_1}\psi_{i_2}\bar\psi_{i_3}\psi_{i_3}\cdots\bar\psi_{i_k}\psi_{i_k} \;+\; \frac{\lambda}{k}\bar\psi_{i_1}\psi_{i_1}\cdots\bar\psi_{i_k}\psi_{i_k}\right] , \tag{3.64}$$

so that the "action" is given by (3.64) plus the "mass term" $\lambda \sum_i \bar{\psi}_i \psi_i$. Combining Corollary 3.3.5 with (3.64), we obtain a formula for the partition function of spanning hyperforests in a k-uniform hypergraph:

$$\sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} w_A \right) \lambda^{k(F)} = \int \mathcal{D}_V(\psi, \bar{\psi}) \; \exp \Bigg\{ \lambda \sum_{i \in V} \bar{\psi}_i \psi_i + \\ + \sum_{i_1, \ldots, i_k \in V} \frac{L_{i_1, \ldots, i_k}}{(k-2)!} \left[\bar{\psi}_{i_1} \psi_{i_2} \bar{\psi}_{i_3} \psi_{i_3} \cdots \bar{\psi}_{i_k} \psi_{i_k} + \frac{\lambda}{k} \bar{\psi}_{i_1} \psi_{i_1} \cdots \bar{\psi}_{i_k} \psi_{i_k} \right] \Bigg\} . \qquad (3.65)$$

Let us remark that while the Laplacian matrix for an ordinary graph has vanishing row and column sums (i.e., $\sum_j L_{ij} = 0$), the Laplacian tensor (3.63) for a hypergraph satisfies $\sum_{i_k} L_{i_1, \ldots, i_k} = 0$ when $i_1, \ldots, i_{k-1}$ are all distinct, but not in general otherwise.

3.4 Extension to correlation functions

In the preceding section we saw how the partition function (3.45) of a particular class of fermionic theories can be given a combinatorial interpretation as an expansion over spanning hyperforests in a hypergraph. In this section we will extend this result to give a combinatorial interpretation for a class of Grassmann integrals that correspond to (unnormalized) correlation functions in this same fermionic theory; we will obtain a sum over partially rooted spanning hyperforests satisfying particular connection conditions.

Given ordered k-tuples of vertices $I = (i_1, i_2, \ldots, i_k) \in V^k$ and $J = (j_1, j_2, \ldots, j_k) \in V^k$, let us define the operator

$$\mathcal{O}_{I,J} \;=\; \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_k} \psi_{j_k} \; , \qquad (3.66)$$

which is an even element of the Grassmann algebra. Of course, the $i_1, i_2, \ldots, i_k$ must be all distinct, as must the $j_1, j_2, \ldots, j_k$, or else we will have $\mathcal{O}_{I,J} = 0$. We shall therefore assume henceforth that $I, J \in V^k_{\neq}$, where $V^k_{\neq}$ is the set of ordered k-tuples of *distinct* vertices in V. Note, however, that there can be overlaps between the sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$. Note finally that $\mathcal{O}_{I,J}$ is antisymmetric under permutations of the sequences I and J, in the sense that

$$\mathcal{O}_{I \circ \sigma, J \circ \tau} \;=\; \operatorname{sgn}(\sigma) \, \operatorname{sgn}(\tau) \, \mathcal{O}_{I,J} \qquad (3.67)$$

for any permutations σ, τ of $\{1, \ldots, k\}$.

Our goal in this section is to provide a combinatorial interpretation, in terms of partially rooted spanning hyperforests satisfying suitable connection conditions, for the general Grassmann integral ("unnormalized correlation function")

$$[\mathcal{O}_{I,J}] \;=\; Z \langle \mathcal{O}_{I,J} \rangle = \int \mathcal{D}_{V,\mathbf{t}}(\psi, \bar{\psi}) \, \mathcal{O}_{I,J} \; \exp \left[\sum_{A \in E} w_A f_A^{(\lambda)} \right] \qquad (3.68a)$$

$$= \int \mathcal{D}_V(\psi, \bar{\psi}) \, \mathcal{O}_{I,J} \; \exp \left[\sum_{i \in V} t_i \bar{\psi}_i \psi_i \;+\; \sum_{A \in E} w_A f_A^{(\lambda)} \right] . \qquad (3.68b)$$

The principal tool is the following generalization of (3.51):

Lemma 3.4.1 *Let $A \subseteq V$, and let $I = (i_1, i_2, \ldots, i_k) \in A^k_{\neq}$ and $J = (j_1, j_2, \ldots, j_k) \in A^k_{\neq}$. Then*

$$\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)\, \mathcal{O}_{I,J}\, f_A^{(\lambda)} \;=\; \begin{cases} \lambda + \sum\limits_{i\in A}(t_i-\lambda) & \text{if } k=0 \\ 1 & \text{if } k=1 \\ 0 & \text{if } k\geq 2 \end{cases} \tag{3.69}$$

PROOF. The case $k=0$ is just (3.51). To handle $k=1$, recall that

$$f_A^{(\lambda)} = \lambda(1-|A|)\tau_A + \sum_{\ell\in A} \tau_{A\smallsetminus l} - \sum_{\substack{\ell,m\in A\\ \ell\neq m}} \bar\psi_\ell \psi_m \tau_{A\smallsetminus\{l,m\}}\,. \tag{3.70}$$

Now multiply $f_A^{(\lambda)}$ by $\bar\psi_i\psi_j$ with $i,j\in A$, and integrate with respect to $\mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)$. If $i=j$, then the only nonzero contribution comes from the term $\ell=i$ in the single sum, and $\bar\psi_i\psi_i\tau_{A\smallsetminus i} = \tau_A$, so the integral is 1. If $i\neq j$, then the only nonzero contribution comes from the term $\ell=j$, $m=i$ in the double sum, and $(\bar\psi_i\psi_j)(-\bar\psi_j\psi_i)\tau_{A\smallsetminus\{i,j\}} = \tau_A$, so the integral is again 1.

Finally, if $|I|=|J|=k\geq 2$, then every monomial in $\mathcal{O}_{I,J} f_A^{(\lambda)}$ has degree $\geq 2|A|-2+2k > 2|A|$, so $\mathcal{O}_{I,J} f_A^{(\lambda)} = 0$. $\square$

Of course, it goes without saying that if $m(\psi,\bar\psi)$ is a monomial of degree k in the variables ψ_i $(i\in A)$ and degree k' in the variables $\bar\psi_i$ $(i\in A)$, and k is *not* equal to k', then $\int \mathcal{D}_{A,\mathbf{t}}(\psi,\bar\psi)\, m(\psi,\bar\psi)\, f_A^{(\lambda)} = 0$.

Now go back to the general case $I=(i_1,i_2,\ldots,i_k)\in V^k_{\neq}$ and $J=(j_1,j_2,\ldots,j_k)\in V^k_{\neq}$, let $\mathcal{C}=\{C_\alpha\}_{\alpha=1}^m$ be a partition of V, and consider the integral

$$\mathcal{I}(I,J;\mathcal{C}) \;=\; \int \mathcal{D}_{V,\mathbf{t}}(\psi,\bar\psi)\, \mathcal{O}_{I,J} \prod_{\alpha=1}^m f_{C_\alpha}^{(\lambda)}\,. \tag{3.71}$$

The integral factorizes over the sets C_α of the partition, and it vanishes unless $|I\cap C_\alpha| = |J\cap C_\alpha|$ for all α; here $I\cap C_\alpha$ denotes the subsequence of I consisting of those elements that lie in C_α, kept in their original order, and $|I\cap C_\alpha|$ denotes the length of that subsequence (and likewise for $J\cap C_\alpha$). So let us decompose the operator $\mathcal{O}_{I,J}$ as

$$\mathcal{O}_{I,J} = \sigma(I,J;\mathcal{C}) \prod_{\alpha=1}^m \mathcal{O}_{I\cap C_\alpha, J\cap C_\alpha}\,, \tag{3.72}$$

where $\sigma(I,J;\mathcal{C})\in\{\pm1\}$ is a sign coming from the reordering of the fields in the product. Applying Lemma 3.4.1 once for each factor C_α, we see that the integral (3.71) is nonvanishing only if $|I\cap C_\alpha| = |J\cap C_\alpha| \leq 1$ for all α: that is, each set C_α must contain either one element from I and one element from J (possibly the same element) or else no element from I or J. Let us call the partition $\mathcal{C}$ *properly matched* for (I,J) when this is the case. (Note that this requires in particular that $m\geq k$.) Note also that for properly matched partitions $\mathcal{C}$ we can express the combinatorial sign $\sigma(I,J;\mathcal{C})$ in a simpler way: it is the sign of the unique permutation π of $\{1,\ldots,k\}$ such that i_r

and $j_{\pi(r)}$ lie in the same set C_α for each r ($1 \le r \le k$). (Note in particular that when $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_k\} \equiv S \neq \emptyset$, the pairing π has to match the repeated elements [i.e., $i_r = j_{\pi(r)}$ whenever $i_r \in S$], since a vertex cannot belong simultaneously to two distinct blocks C_α and C_β.) We then deduce immediately from Lemma 3.4.1 the following generalization of Corollary 3.3.3:

Corollary 3.4.2 *Let $I, J \in V^k_{\neq}$ and let $\mathcal{C} = \{C_\alpha\}$ be a partition of V. Then*

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi, \bar{\psi})\, \mathcal{O}_{I,J} \prod_\alpha f^{(\lambda)}_{C_\alpha} = \begin{cases} \operatorname{sgn}(\pi) \displaystyle\prod_{\alpha\colon |I \cap C_\alpha| = 0} \Big(\lambda + \sum_{i \in C_\alpha} (t_i - \lambda)\Big) & \textit{if } \mathcal{C} \textit{ is properly matched for } (I,J) \\ 0 & \textit{otherwise} \end{cases} \tag{3.73}$$

where π is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same set C_α for each r.

We then have

$$\mathcal{I}(I, J; \mathcal{C}) = \begin{cases} \sigma(I, J; \mathcal{C}) \prod_{\alpha\colon |I \cap C_\alpha| = 0} \Big(\lambda + \sum_{i \in C_\alpha} (t_i - \lambda)\Big) & \text{if } \mathcal{C} \text{ is } \textit{properly matched} \text{ for } (I,J) \\ 0 & \textit{otherwise} \end{cases} \tag{3.74}$$

We can now compute the integral (3.68) by combining Corollaries 3.2.5 and 3.4.2. If $G = (V, E)$ is a hypergraph and G' is a spanning subhypergraph of G, let us say that G' is *properly matched* for (I, J) [we denote this by $G' \sim (I, J)$] in case the partition of V induced by the decomposition of G' into connected components is properly matched for (I, J). We then obtain the main result of this section:

Theorem 3.4.3 *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A \in E}$ be hyperedge weights, and let $I, J \in V^k_{\neq}$. Then*

$$\int \mathcal{D}_V(\psi, \bar{\psi})\, \mathcal{O}_{I,J} \exp\!\left[\sum_{i \in V} t_i \bar{\psi}_i \psi_i + \sum_{A \in E} w_A f^{(\lambda)}_A\right] = \sum_{\substack{F \in \mathcal{F}(G) \\ F \sim (I,J) \\ F = (F_1, \ldots, F_\ell)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A \in F} w_A\right) \prod_{\alpha\colon |I \cap F_\alpha| = 0} \left(\lambda + \sum_{i \in V(F_\alpha)} (t_i - \lambda)\right), \tag{3.75}$$

where the sum runs over spanning hyperforests F in G, with components $F_1, \ldots, F_\ell$, that are properly matched for (I, J), and $V(F_\alpha)$ is the vertex set of the hypertree F_α; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component F_α for each r.

If we specialize (3.75) to $t_i = \lambda$ for all vertices i, we obtain:

Corollary 3.4.4 *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A\in E}$ be hyperedge weights. and let $I, J \in V^k_{\neq}$. Then*

$$\int \mathcal{D}_V(\psi,\bar\psi)\,\mathcal{O}_{I,J}\,\exp\left[\lambda\sum_{i\in V}\bar\psi_i\psi_i \;+\; \sum_{A\in E} w_A f_A^{(\lambda)}\right] = \sum_{\substack{F\in\mathcal{F}(G)\\ F\sim(I,J)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A\in F} w_A\right) \lambda^{k(F)-k}\,, \quad (3.76)$$

where the sum runs over spanning hyperforests F in G that are properly matched for (I, J), and $k(F)$ is the number of connected components of F; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component of F for each r.

This is the partition function of spanning hyperforests that are rooted at the vertices in I, J and are otherwise unrooted, with a weight w_A for each hyperedge A and a weight λ for each unrooted connected component.

If, on the other hand, we specialize (3.75) to $\lambda = 0$, we obtain:

Corollary 3.4.5 *Let $G = (V, E)$ be a hypergraph, let $\{w_A\}_{A\in E}$ be hyperedge weights, and let $I, J \in V^k_{\neq}$. Then*

$$\int \mathcal{D}_V(\psi,\bar\psi)\,\mathcal{O}_{I,J}\,\exp\left[\sum_{i\in V} t_i\bar\psi_i\psi_i \;+\; \sum_{A\in E} w_A f_A^{(0)}\right] = \sum_{\substack{F\in\mathcal{F}(G)\\ F\sim(I,J)\\ F=(F_1,\ldots,F_\ell)}} \operatorname{sgn}(\pi_{I,J;F}) \left(\prod_{A\in F} w_A\right) \prod_{\alpha\colon\, |I\cap F_\alpha|=0} \left(\sum_{i\in F_\alpha} t_i\right)\,, \quad (3.77)$$

where the sum runs over spanning hyperforests F in G, with components $F_1, \ldots, F_\ell$, that are properly matched for (I, J), and $V(F_\alpha)$ is the vertex set of the hypertree F_α; here $\pi_{I,J;F}$ is the permutation of $\{1, \ldots, k\}$ such that i_r and $j_{\pi(r)}$ lie in the same component F_α for each r.

This is the partition function of rooted spanning hyperforests, with a weight w_A for each hyperedge A and a weight t_i for each root i *other than* those in the sets I, J.

Let us conclude by making some remarks about the *normalized* correlation function $\langle\mathcal{O}_{I,J}\rangle$ obtained by dividing (3.68) by (3.45). For simplicity, let us consider only the two-point function $\langle\bar\psi_i\psi_j\rangle$. We have

$$\langle\bar\psi_i\psi_j\rangle \;=\; \left\langle \gamma_{ij}\left(\lambda + \sum_{k\in\Gamma(i)}(t_k-\lambda)\right)^{-1}\right\rangle\,, \quad (3.78)$$

where the expectation value on the right-hand side is taken with respect to the "probability distribution"[2] on spanning hyperforests of G in which the hyperforest $F = (F_1, \ldots, F_\ell)$

[2] We write "probability distribution" in quotation marks because the "probabilities" will in general be complex. They will be true probabilities (i.e., real numbers between 0 and 1) if the hyperedge weights w_A are nonnegative real numbers.

gets weight

$$Z^{-1}\left(\prod_{A\in F} w_A\right)\prod_{\alpha=1}^{\ell}\Big(\lambda+\sum_{k\in F_\alpha}(t_k-\lambda)\Big)\,, \tag{3.79}$$

where γ_{ij} denotes the indicator function

$$\gamma_{ij} \;=\; \begin{cases} 1 & \text{if } i \text{ and } j \text{ belong to the same component of } F \\ 0 & \text{if not,} \end{cases} \tag{3.80}$$

Z is (3.45), and $\Gamma(i)$ denotes the vertex set of the component of F containing i. The factor $\Big(\lambda+\sum\limits_{k\in\Gamma(i)}(t_k-\lambda)\Big)^{-1}$ in (3.78) arises from the fact that in (3.56) each component gets a weight $\lambda+\sum_{k\in\Gamma(i)}(t_k-\lambda)$, while in (3.75) only those components *other than* the one containing i and j get such a weight. So in general the correlation function $\langle\bar\psi_i\psi_j\rangle$ is *not* simply equal to (or proportional to) the connection probability $\langle\gamma_{ij}\rangle$. However, in the special case of Corollaries 3.3.5 and 3.4.4 — namely, all $t_i=\lambda$, so that we get unrooted spanning hyperforests with a "flat" weight λ for each component — then we have the simple identity

$$\langle\bar\psi_i\psi_j\rangle \;=\; \lambda^{-1}\langle\gamma_{ij}\rangle\;. \tag{3.81}$$

Behind combinatorial identities like (3.81), there are Ward identities arising from the OSP(1|2) supersymmetry that will be discussed in Chapter 5.

3.5 Graphical proof of some generalized matrix-tree theorems

We now present a "graphical" proof of the classical matrix-tree theorem as well as a number of extensions thereof, by interpreting in a graphical way the terms of a formal Taylor expansion of an action belonging to the even subalgebra of a Grassmann algebra. (We require the action to belong to the even subalgebra in order to avoid ordering ambiguities when exponentiating a sum of terms.)

We already have seen some of these extensions proven by an "algebraic" method based on Lemma 3.2.1 and its corollaries.

Other more exotic extensions are described here with an eye to future work; they could also be proven by suitable variants of the algebraic technique.

Curiously enough, it turns out that the more general is the fact we want to prove, the *easier* is the proof; indeed, the most general facts ultimately become almost tautologies on the rules of Grassmann algebra and integration. The only extra feature of the most general facts is that the "zoo" of graphical combinatorial objects has to become wider (and wilder).

So, in this exposition we shall start by describing the most general situation, and then show how, when special cases are chosen for the parameters in the action, a corresponding simplification occurs also in the combinatorial interpretation.

Consider a hypergraph $G = (V, E)$, as usual we introduce a pair $\psi_i, \bar\psi_i$ of Grassmann generators for each $i \in V$. We shall consider actions of the form

$$\mathcal{S}(\psi, \bar\psi) \;=\; \sum_{A \in E} \mathcal{S}_A(\psi, \bar\psi) \;, \tag{3.82}$$

where

$$\mathcal{S}_A(\psi, \bar\psi) = w^*_A \, \tau_A + \sum_{i \in A} w_{A;i} \, \tau_{A \setminus i} + \sum_{\substack{i,j \in A \\ i \neq j}} w_{A;ij} \, \psi_i \bar\psi_j \, \tau_{A \setminus \{i,j\}} \tag{3.83}$$

and $\tau_A = \prod_{i \in A} \bar\psi_i \psi_i$. is the shorthand defined already at the beginning of Section 3.2. Please note that the form 3.83 resembles the definition (3.23) of $f_A^{(\lambda)}$ the same monomials appear, but now each one is multiplied by an independent indeterminate. Thus, for each hyperedge A of cardinality k we have k^2+1 parameters: w^*_A, $\{w_{A;i}\}_{i \in A}$ and $\{w_{A;i,j}\}_{(i \neq j) \in A}$. [We have chosen, for future convenience, to write the last term in (3.23) as $+\psi_i \bar\psi_j$ rather than $-\bar\psi_i \psi_j$.]

Please note that, for $|A| > 2$, all pairs of terms in $\mathcal{S}_A(\psi, \bar\psi)$ have a vanishing product, because they contain at least $2(2|A| - 2) = 4|A| - 4$ fermions in a subalgebra (over A) that has only $2|A|$ distinct fermions. As a consequence, we have in this case

$$\exp[\mathcal{S}_A(\psi, \bar\psi)] \;=\; 1 + \mathcal{S}_A(\psi, \bar\psi) \,. \tag{3.84}$$

On the other hand, if $|A| = 2$ (say, $A = \{i, j\}$), we have two nonvanishing cross-terms:

$$(w_{A;i} \, \bar\psi_j \psi_j) \, (w_{A;j} \, \bar\psi_i \psi_i) \;=\; w_{A;i} w_{A;j} \, \bar\psi_i \psi_i \bar\psi_j \psi_j \tag{3.85a}$$

$$(w_{A;ij} \, \psi_i \bar\psi_j) \, (w_{A;ji} \, \psi_j \bar\psi_i) \;=\; -w_{A;ij} w_{A;ji} \, \bar\psi_i \psi_i \bar\psi_j \psi_j \tag{3.85b}$$

where the minus sign comes from commutation of fermionic fields. So we can write in the general case

$$\exp[\mathcal{S}_A(\psi, \bar\psi)] \;=\; 1 + \widehat{\mathcal{S}}_A(\psi, \bar\psi) \;, \tag{3.86}$$

where $\widehat{\mathcal{S}}_A(\psi, \bar\psi)$ is defined like $\mathcal{S}_A(\psi, \bar\psi)$ but with the parameter w^*_A replaced by

$$\widehat{w}^*_A \;=\; \begin{cases} w^*_A + w_{A;i} w_{A;j} - w_{A;ij} w_{A;ji} & \text{if } A = \{i, j\} \\ w^*_A & \text{if } |A| \geq 3 \end{cases} \tag{3.87}$$

Consider now a Grassmann integral of the form

$$\int \mathcal{D}_V(\psi, \bar\psi) \, \mathcal{O}_{I,J} \, \exp\!\left[\sum_i t_i \bar\psi_i \psi_i \;+\; \sum_{A \in E} \mathcal{S}_A(\psi, \bar\psi) \right] , \tag{3.88}$$

where $\mathbf{t} = (t_i)_{i \in V}$ are parameters, $I = (i_1, i_2, \ldots, i_k) \in V^k$ and $J = (j_1, j_2, \ldots, j_k) \in V^k$ are ordered k-tuples of vertices, and

$$\mathcal{O}_{I,J} \;=\; \bar\psi_{i_1} \psi_{j_1} \cdots \bar\psi_{i_k} \psi_{j_k} \tag{3.89}$$

[cf. (3.66)]. Here the $i_1, \ldots, i_k$ must be all distinct, as must the $j_1, \ldots, j_k$, but there can be overlaps between the sets $\mathsf{I} = \{i_1, i_2, \ldots, i_k\}$ and $\mathsf{J} = \{j_1, j_2, \ldots, j_k\}$.[3] We intend to show that (3.88) can be interpreted combinatorially as a partition function for rooted oriented[4] spanning sub(hyper)graphs of G, in which each connected component is either a (hyper-)tree or a (hyper-)unicyclic. In the case of a unicyclic component, the rest of the component is oriented towards the cycle, and no vertex from $\mathsf{I} \cup \mathsf{J}$ lies in the component. In the case of a tree component, either (a) no vertex from $\mathsf{I} \cup \mathsf{J}$ is in the component, and then there is either a special "root" vertex or a "root" hyperedge, all the rest of the tree being oriented towards it, or (b) the component contains a single vertex from $\mathsf{I} \cap \mathsf{J}$, which is the root vertex, and the tree is again oriented towards it, or (c) the component contains exactly one vertex from I and one from J, a special oriented path connecting them, and all the rest is oriented towards the path. The weight of each configuration is essentially the product of t_i for each root $i \notin \mathsf{I} \cup \mathsf{J}$ and an appropriate weight ($\widehat{w}^*_A$, $w_{A;i}$ or $w_{A;ij}$) for each occupied hyperedge, along with a $-$ sign for each unicyclic using the $w_{A;ij}$'s and a single extra $\pm$ sign corresponding to the pairing of vertices of I to vertices of J induced by being in the same component. (This same sign appeared already in Section 3.4.)

Kirchhoff's matrix-tree theorem arises when all the hyperedges A have cardinality 2 (i.e. G is an ordinary graph), $\mathsf{I} = \mathsf{J} = \{i_0\}$ for some vertex i_0, all $t_i = 0$, all $w^*_A = 0$, and $w_{A;i} = w_{A;ij} = w_A$. The principal-minors matrix-tree theorem is obtained by allowing $\mathsf{I} = \mathsf{J}$ of arbitrary cardinality k, while the all-minors matrix-tree theorem is obtained by allowing also $\mathsf{I} \neq \mathsf{J}$. Rooted forests with root weights t_i can be obtained by allowing $t_i \neq 0$. On the other hand, unrooted forests are obtained by taking all $t_i = \lambda$, $\mathsf{I} = \mathsf{J} = \emptyset$, $w^*_A = -\lambda w_A$ and the rest as above. [More generally, unrooted hyperforests are obtained by taking all $t_i = \lambda$, $\mathsf{I} = \mathsf{J} = \emptyset$, $w^*_A = -\lambda(|A| - 1)w_A$ and the rest as above.] The sequences I and J are used mainly in order to obtain expectation values of certain connectivity patterns in the relevant ensemble of spanning subgraphs.

Let us now prove all these statements, and give precise expressions for the weights of the configurations, which until now have been left deliberately vague in order not to overwhelm the reader.

We start by manipulating (3.88), exponentiating the action to obtain

$$\int \mathcal{D}_V(\psi, \bar{\psi})\, \mathcal{O}_{I,J} \left(\prod_{i \in V} (1 + t_i \bar{\psi}_i \psi_i) \right) \left(\prod_{A \in E} (1 + \widehat{\mathcal{S}}_A) \right) \tag{3.90}$$

or, expanding the last products,

$$\sum_{\substack{V' \subseteq V \setminus (\mathsf{I} \cup \mathsf{J}) \\ E' \subseteq E}} \left(\prod_{i \in V'} t_i \right) \int \mathcal{D}_V(\psi, \bar{\psi})\, \mathcal{O}_{I \cup V', J \cup V'} \left(\prod_{A \in E'} \widehat{\mathcal{S}}_A \right), \tag{3.91}$$

where $I \cup V'$ consists of the sequence I followed by the list of elements of V' in any chosen order, and $J \cup V'$ consists of the sequence J followed by the list of elements of V' in the *same* order.

[3] Please note the distinction between the *ordered* k-tuple $I = (i_1, i_2, \ldots, i_k)$, here written in italic font, and the *unordered* set $\mathsf{I} = \{i_1, i_2, \ldots, i_k\}$, here written in sans-serif font.

[4] We shall define later what we mean by "orienting" a hyperedge A: it will correspond to selecting a single vertex $i \in A$ as the "outgoing" vertex.

Factors coming from $\mathcal{O}_{I\cup V', J\cup V'}$		
$\bar{\psi}_i \psi_i$	$i\,\odot$	root vertex
$\bar{\psi}_i$	$i\,\circ$	sink vertex
ψ_i	$i\,\times$	source vertex

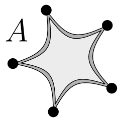
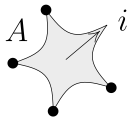
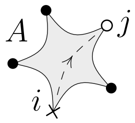
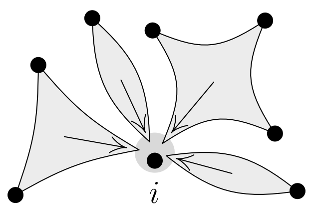
Factors coming from $\prod \widehat{\mathcal{S}}_A$		
τ_A		root hyperedge
$\tau_{A\setminus i}$		pointing hyperedge
$\psi_i \bar{\psi}_j \tau_{A\setminus\{i,j\}}$		dashed hyperedge

Table 3.1: Graphical representation of the various factors in the expansion (3.91).

We now give a graphical representation and a fancy name to each kind of monomial in the expansion (3.91), as shown in Table 3.1. Please note that in this graphical representation a solid circle $\bullet$ corresponds to a factor $\bar{\psi}_i \psi_i$, an open circle $\circ$ corresponds to a factor $\bar{\psi}_i$, and a cross $\times$ corresponds to a factor ψ_i.

According to the rules of Grassmann algebra and Grassmann–Berezin integration, we must have in total exactly one factor $\bar{\psi}_i$ and one factor ψ_i for each vertex i. Graphically this means that at each vertex we must have either a single $\bullet$ or else the superposed pair $\otimes$ (please note that in many drawings we actually draw the $\circ$ and $\times$ slightly split, in order to highlight which variable comes from which factor). At each vertex i we can have an arbitrary number of "pointing hyperedges" pointing towards i, as they do not carry any fermionic field:



Aside from pointing hyperedges, we must be, at each vertex i, in one of the following situations (Figure 3.1):

1. If $i \in V'$ or $i \in \mathsf{I} \cap \mathsf{J}$ [resp. cases (a) and (b) in the figure], the quantity $\mathcal{O}_{I\cup V', J\cup V'}$ provides already a factor $\bar{\psi}_i \psi_i$; therefore, no other factors of $\bar{\psi}_i$ or ψ_i should come from the expansion of $\prod \widehat{\mathcal{S}}_A$.

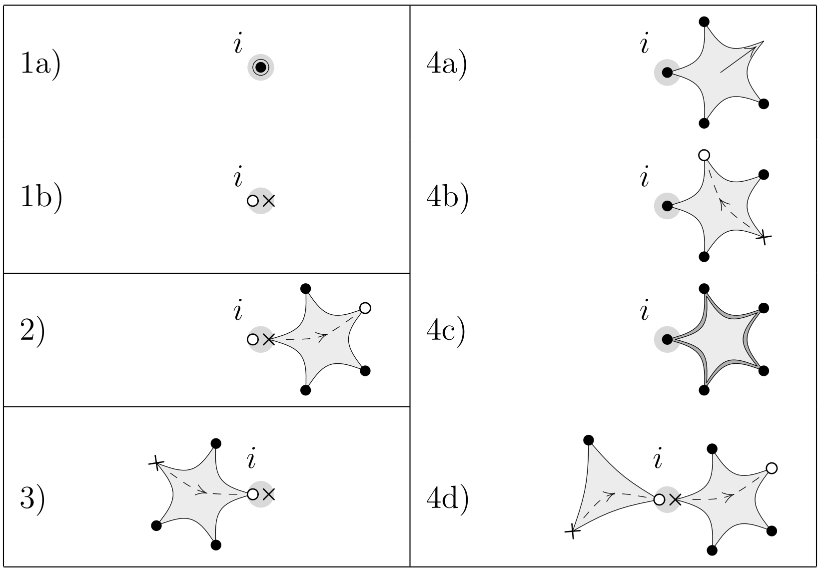


Figure 3.1: Possible ways of saturating the Grassmann fields on vertex i (indicated by the small gray disk).

2. If $i \in \mathsf{I} \setminus \mathsf{J}$, the quantity $\mathcal{O}_{I \cup V', J \cup V'}$ provides already a factor $\bar{\psi}_i$; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide ψ_i, i.e. we must have one dashed hyperedge pointing from i.

3. If $i \in \mathsf{J} \setminus \mathsf{I}$, the quantity $\mathcal{O}_{I \cup V', J \cup V'}$ provides already a factor ψ_i; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide $\bar{\psi}_i$, i.e. we must have one dashed hyperedge pointing towards i.

4. If $i \notin \mathsf{I} \cup \mathsf{J} \cup V'$, then the quantity $\mathcal{O}_{I \cup V', J \cup V'}$ provides neither $\bar{\psi}_i$ nor ψ_i; therefore, the expansion of $\prod \widehat{\mathcal{S}}_A$ must provide both $\bar{\psi}_i$ and ψ_i, so that at i we must have one of the following configurations:

   a) a non-pointed vertex of a pointing hyperedge;

   b) a vertex of a dashed hyperedge that is neither of the two endpoints of the dashed arrow;

   c) a vertex of a root hyperedge;

   d) two dashed hyperedges, one with the arrow incoming, one outgoing.

Having given the local description of the possible configurations at each vertex i, let us now describe the possible global configurations. Note first that at each vertex we can have at most two incident dashed arrows, and if there are two such arrows then they must have opposite orientations. As a consequence, we see that dashed arrows must either form cycles, or else form open paths connecting a source vertex of $\mathsf{I} \setminus \mathsf{J}$ to a sink vertex of $\mathsf{J} \setminus \mathsf{I}$. Let us use the term *root structures* to denote root vertices, root hyperedges, cycles of dashed hyperedges, and open paths of dashed hyperedges.

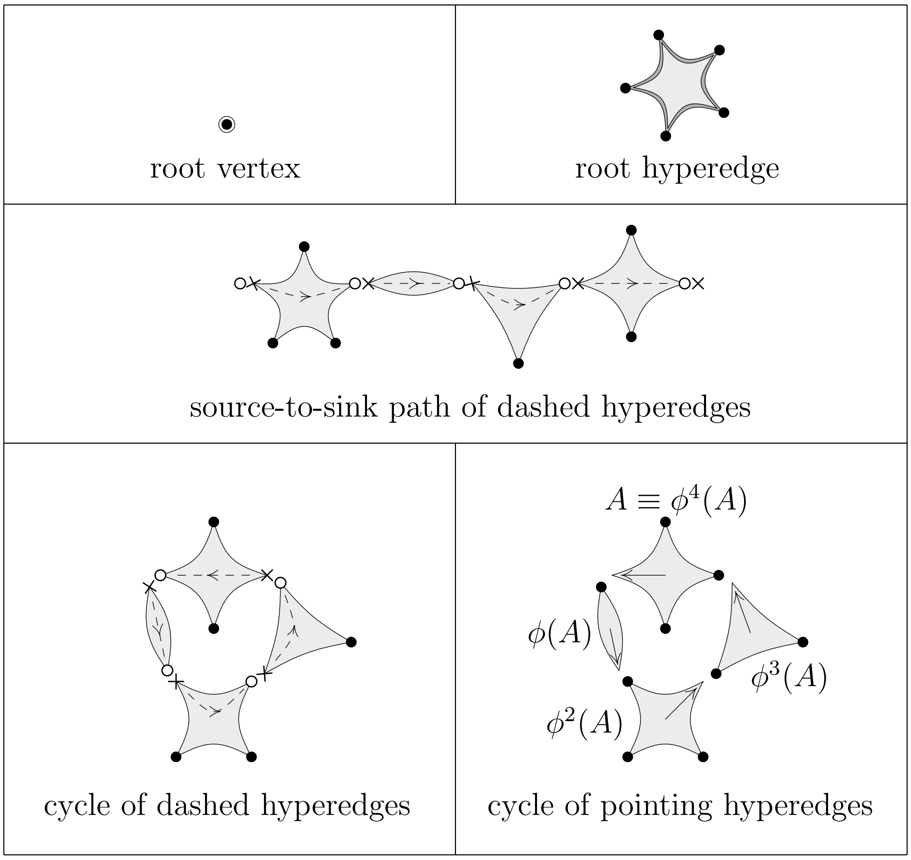
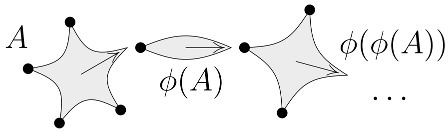


Figure 3.2: The five kinds of root structures.

As for the solid arrows in the pointing hyperedges, the reasoning is as follows: If a pointing hyperedge A points towards i, then either i is part of a root structure as described above, or else it is a non-pointed vertex of another pointing hyperedge $\varphi(A)$. We can follow this map iteratively, i.e. go to $\varphi(\varphi(A))$, and so on:



Because of the finiteness of the graph, either we ultimately reach a root structure, or we enter a cycle. Cycles of the "dynamics" induced by φ correspond to cycles of the pointing hyperedges. We now also include such cycles of pointing hyperedges as a fifth type of root structure (see Figure 3.2 for the complete list of root structures).

All the rest is composed of pointing hyperedges, which form directed arborescences, rooted on the vertices of the root structures. In conclusion, therefore, the most general configuration consists of a bunch of disjoint root structures, and a set of directed arborescences (possibly reduced to a single vertex) rooted at its vertices, such that the whole is a spanning subhypergraph H of G.

As each root structure is either a single vertex, a single hyperedge, a (hyper-)path or a (hyper-)cycle, we see that each connected component of H is either a (hyper-)tree or a (hyper-)unicyclic. Furthermore, all vertices in $\mathsf{I} \cup \mathsf{J}$ are in the tree components, and each tree contains either one vertex from I and one from J (possibly coincident) or else no vertices at all from $\mathsf{I} \cup \mathsf{J}$.

We still need to understand the weights associated to the allowed configurations. Clearly, we have a factor $w_{A;i}$ per pointing hyperedge in the arborescence. Root vertices coming from V' have factors t_i, and root hyperedges have factors $\widehat{w}^*_A$. Cycles $\gamma = (i_0, A_1, i_1, A_2, ..., i_\ell = i_0)$ of the dynamics of φ (*bosonic cycles*) have a weight $w_{A_1;i_1} \cdots w_{A_\ell;i_\ell}$. All the foregoing objects contain Grassmann variables only in the combination $\bar\psi_i \psi_i$, and hence are commutative. Finally, we must consider the dashed hyperedges, which contain "unpaired fermions" ψ_i and $\bar\psi_j$, and hence will give rise to signs coming from anticommutativity. Let us first consider the dashed cycles $\gamma = (i_0, A_1, i_1, A_2, ..., i_\ell = i_0)$, and note what happens when reordering the fermionic fields:

$$\begin{aligned}
&(w_{A_1;i_\ell i_1}\psi_{i_\ell}\bar\psi_{i_1})(w_{A_2;i_1 i_2}\psi_{i_1}\bar\psi_{i_2})\cdots(w_{A_\ell;i_{\ell-1}i_\ell}\psi_{i_{\ell-1}}\bar\psi_{i_\ell}) \\
&\quad = \; -w_{A_1;i_\ell i_1} w_{A_2;i_1 i_2}\cdots w_{A_\ell;i_{\ell-1}i_\ell}\, \bar\psi_{i_1}\psi_{i_1}\cdots\bar\psi_{i_\ell}\psi_{i_\ell}
\end{aligned} \tag{3.92}$$

because ψ_{i_ℓ} had to pass through $2\ell - 1$ fermionic fields to reach its final location. This is pretty much the result one would have expected, *but* we have an overall minus sign, irrespective of the length of the cycle (or its parity), which is in a sense "non-local", due to the fermionic nature of the fields ψ and $\bar\psi$. For this reason we call a dashed cycle a *fermionic cycle*.

A similar mechanism arises for the open paths of dashed hyperedges $\gamma = (i_0, A_1, i_1, A_2, \ldots, i_\ell)$, where i_0 is the source vertex and i_ℓ is the sink vertex. Here the weight $w_{A_1;i_0 i_1} w_{A_2;i_1 i_2} \cdots w_{A_\ell;i_{\ell-1}i_\ell}$ multiplies the monomial $\psi_{i_0}\bar\psi_{i_1}\psi_{i_1}\bar\psi_{i_2}\psi_{i_2}\cdots\bar\psi_{i_{\ell-1}}\psi_{i_{\ell-1}}\bar\psi_{i_\ell}$, in which the only unpaired fermions are ψ_{i_0} and $\bar\psi_{i_\ell}$. in this order. Now the monomials for the open paths must be multiplied by $\mathcal{O}_{I,J}$, and each source (resp. sink) vertex from an open path must correspond to a vertex of I (resp. J). This pairing thus induces a permutation of $\{1, \ldots, k\}$, where $k = |\mathsf{I}| = |\mathsf{J}|$: namely, i_r is connected by an open path to $j_{\pi(r)}$. We then have

$$\left(\prod_{r=1}^{k} \bar\psi_{i_r}\psi_{j_r} \right) \left(\prod_{r=1}^{k} \psi_{i_r}\bar\psi_{j_{\pi(r)}} \right) , \tag{3.93}$$

where the first product is $\mathcal{O}_{I,J}$ and the second product comes from the open paths. This can easily be rewritten as

$$\prod_{r=1}^{k} \bar\psi_{i_r}\psi_{j_r}\psi_{i_r}\bar\psi_{j_{\pi(r)}} \;=\; \prod_{r=1}^{k} \bar\psi_{i_r}\psi_{i_r}\bar\psi_{j_{\pi(r)}}\psi_{j_r} \tag{3.94a}$$

$$= \left(\prod_{r=1}^{k} \bar\psi_{i_r}\psi_{i_r} \right) \left(\prod_{r=1}^{k} \bar\psi_{j_{\pi(r)}}\psi_{j_r} \right) \tag{3.94b}$$

$$= \operatorname{sgn}(\pi) \left(\prod_{r=1}^{k} \bar\psi_{i_r}\psi_{i_r} \right) \left(\prod_{r=1}^{k} \bar\psi_{j_r}\psi_{j_r} \right) . \tag{3.94c}$$

Putting everything together, we see that the Grassmann integral (3.88) can be represented as a sum over rooted oriented spanning subhypergraphs $\vec{H}$ of G, as follows:

- Each connected component of H (the unoriented subhypergraph corresponding to $\vec{H}$) is either a (hyper-)tree or a (hyper-)unicyclic.
- Each (hyper-)tree component contains either one vertex from I (the *source vertex*) and one from J (the *sink vertex*, which is allowed to coincide with the source vertex), or else no vertex from $\mathsf{I} \cup \mathsf{J}$. In the latter case, we choose either one vertex of the component to be the *root vertex*, or else one hyperedge of the component to be the *root hyperedge*.
- Each unicyclic component contains no vertex from $\mathsf{I} \cup \mathsf{J}$. As a unicyclic, it necessarily has the form of a single (hyper-)cycle together with (hyper-)trees (possibly reduced to a single vertex) rooted at the vertices of the (hyper-)cycle.
- Each hyperedge other than a root hyperedge is *oriented* by designating a vertex $i(A) \in A$ as the *outgoing vertex*. These orientations must satisfy following rules:
  (i) each (hyper-)tree component is directed towards the sink vertex, root vertex or root hyperedge,
  (ii) each (hyper-)tree belonging to a unicyclic component is oriented towards the cycle, and
  (iii) the (hyper-)cycle of each unicyclic component is oriented consistently.

  Thus, in each (hyper-)tree component the orientations are fixed uniquely, while in each unicyclic component we sum over the two consistent orientations of the cycle.

The *weight* of a configuration $\vec{H}$ is the product of the weights of its connected components, which are in turn defined as the product of the following factors:

- Each root vertex i gets a factor t_i.
- Each root hyperedge A gets a factor $\widehat{w}^*_A$.
- Each hyperedge A belonging to the (unique) path from a source vertex to a sink vertex gets a factor $w_{A;ij}$, where j is the outgoing vertex of A and i is the outgoing vertex of the preceding hyperedge along the path (or the source vertex if A is the first hyperedge of the path).
- Each hyperedge A that does not belong to a source-sink path or to a cycle gets a factor $w_{A;i(A)}$ [recall that $i(A)$ is the outgoing vertex of A].
- Each oriented cycle $(i_0, A_1, i_1, A_2, ..., i_\ell = i_0)$ gets a weight

$$\prod_{\alpha=1}^{\ell} w_{A_\alpha;i_\alpha} - \prod_{\alpha=1}^{\ell} w_{A_\alpha;i_{\alpha-1}i_\alpha} . \tag{3.95}$$

- There is an overall factor $\operatorname{sgn}(\pi)$.

Summary

- The Grassmann subalgebra generated by (3.23)

$$f_A^{(\lambda)} = \Big[\lambda(1-|A|) + \partial\bar{\partial}\Big] \prod_{i\in A} \bar{\psi}_i \psi_i$$

has the following property (Corollary 3.2.3): given an hypergraph $G = (V, E)$, then

$$\prod_{A\in E} f_A^{(\lambda)} = \begin{cases} f_V^{(\lambda)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases}$$

Furthermore, for any subset $A \subseteq V$ and any vector $\mathbf{t} = (t_i)_{i\in V}$ of weights, we have Corollary 3.3.2, which states that

$$\int \mathcal{D}_{A,\mathbf{t}}(\psi, \bar{\psi})\, f_A^{(\lambda)} = \lambda + \sum_{i\in A} (t_i - \lambda)$$

where the integration measure is defined as

$$\mathcal{D}_{A,\mathbf{t}}(\psi, \bar{\psi}) \;=\; \prod_{i\in A} \mathrm{d}\psi_i \mathrm{d}\bar{\psi}_i \, e^{t_i \bar{\psi}_i \psi_i}$$

- Using the above properties, we proved Theorem 3.3.4:

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi, \bar{\psi}) \exp\Big[\sum_{A\in E} w_A f_A^{(\lambda)}\Big] = \sum_{\substack{F\in\mathcal{F}(G) \\ F=(F_1,\ldots,F_\ell}} \Big(\prod_{A\in F} w_A\Big) \prod_{\alpha=1}^{\ell} \Big(\lambda + \sum_{i\in V(F_\alpha)} (t_i - \lambda)\Big)$$

- This general result has two main specializations. The first (Corollary 3.3.5) is when $t_i = \lambda$ for every $i \in V$:

$$\int \mathcal{D}_{V,\lambda}(\psi, \bar{\psi}) \exp\Big[\sum_{A\in E} w_A f_A^{(\lambda)}\Big] = \sum_{F\in\mathcal{F}(G)} \lambda^{k(F)} \prod_{A\in F} w_A$$

which is the the partition function of unrooted spanning hyperforests with a weight w_A for each edge $A \in E$ and a weight λ for each connected component.

The second (Corollary 3.3.6) is when $\lambda = 0$:

$$\int \mathcal{D}_{V,\mathbf{t}}(\psi, \bar{\psi}) \exp\Big[\sum_{A\in E} w_A f_A^{(0)}\Big] = \sum_{\substack{F\in\mathcal{F}(G) \\ F=(F_1,\ldots,F_\ell)}} \Big(\prod_{A\in F} w_A\Big) \prod_{\alpha=1}^{\ell} \Big(\sum_{i\in V(F_\alpha)} t_i\Big)$$

which is the partition function of rooted spanning hyperforests with a weight w_A for each edge $A \in E$ and a weight t_i for each root at vertex $i \in V$.

Chapter 4

On the complete hypergraph

Introduction

In this chapter we shall be concerned mainly with the problem of evaluating the weight of rooted and unrooted hyperforests for the case of the hypergraph with n vertices $\overline{\mathcal{K}}_n$ which is complete in hyperedges of all possible cardinality, when the weight w_A of a hyperedge depends only on its cardinality $|A|$, i.e. $w_A = w_{|A|}$. These questions are usually analyzed by using the exponential generating function and the Lagrange inversion formula [34, 35], eventhough it seems that they have been posed and solved in the context of statistical mechanics [36].

These results could in principle and in many cases have been already derived by using the standard methods of enumerative combinatorics, that is Lagrange inversion formula in connection with the formalism of the exponential generating functions. We hope to convince the reader that also in these cases the Grassmann formalism provides an alternative, simple and compact way to recover the total weights for rooted and unrooted hyperforests on n labeled vertices, which is to say spanning on the complete hypergraph $\overline{\mathcal{K}}_n$.

Considering the complete hypergraph is one of the standard ways to achieve a mean-field approximation, approximation that in our case will allow also the presence of many-body interactions. As one would expect, we will see that in this approximation the action becomes function of a single (commuting thus bosonic) variable. This fact will lead us to a complex-integral representation suitable for the saddle point analysis of the next chapter.

This chapter is organized as follow. In Section 4.1 we report the derivation of the number of (hyper-)trees in a unified way by the standard exponential generating function formalism and Lagrange inversion formula and we illustrate how, at least in the complete case, the generating function of unrooted hyperforests

$$F_n(\lambda, \mathbf{w}) = \int \mathcal{D}_{n,\lambda}(\psi, \bar{\psi}) \exp\left[\sum_{A\in E} w_A f_A^{(\lambda)}\right] = \sum_{F\in\mathcal{F}(G)} \lambda^{k(F)} \prod_{A\in F} w_A \tag{4.1}$$

can be deduced from that for the rooted hypertrees

$$E_n(t, \mathbf{w}) = \int \mathcal{D}_{n,t}(\psi, \bar{\psi}) \exp\left[\sum_{A \in E} w_A f_A^{(0)}\right] = \sum_{\substack{F \in \mathcal{F}(G) \\ F=(F_1,\dots,F_\ell)}} t^\ell \left(\prod_{A \in F} w_A\right) \prod_{\alpha=1}^{\ell} |F_\alpha| \ , \tag{4.2}$$

where every vertex has the same weight $t_i = t$. In Section 4.2 we show the relation between our Grassmann integrals and the explicit solutions achieved by standard methods.

In Section 4.3 we deal with rooted hyperforests, while Section 4.4 is devoted to unrooted hyperforests. By restricting our general model to the case in which only one weight in nonzero, that is $w_p = \delta_{p,k}$, we obtain the explicit evaluation of the number of rooted and unrooted spanning hyperforests on the k-uniform complete hypergraphs $\mathcal{K}_n^{(k)}$ with n-vertices. These results are presented respectively in Section 4.3.1 and Section 4.4.1. Here we also derive a novel general simple expression for the number of unrooted hyperforests with p hypertrees in terms of associated Laguerre polynomials and its asymptotic expansion for large number of vertices.

We consider also the case in which all the weights are equal, that is $w_p = 1$ for all p, this is done in Section 4.3.2 for the rooted hyperforests and in Section 4.4.2 for unrooted hyperforests.

To test the generality of our approach we shall examine another special case that is the k-partite graph in which the vertices can be written as union of k mutually disjoint sets $V = V_1 \cup V_2 \cdots V_k$ such that for each hyperedge $E = (i_1, \dots, i_k)$ we have $i_\alpha \in V_\alpha$. It is interesting that even in this case the partition function can be rewritten in terms of quadratic combinations of the fields.

4.1 Exponential generating functions for hypertrees and hyperforests

Let us consider the complete hypergraph $\overline{\mathcal{K}}_n$ for every n, with general hyperedge-weights w_A which vary only with the cardinality of the hyperedge A, i.e. $w_A = w_{|A|}$. The k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ corresponds to the case in which the only non-vanishing weight is w_k.

Let t_n be the total weight of rooted hypertrees in the case of n vertices, $\mathbf{w} = \{w_k\}_{k \geq 2}$ and let

$$T(z) = T(z, \mathbf{w}) = \sum_{n \geq 0} t_n(\mathbf{w}) \, \frac{z^n}{n!} \tag{4.3}$$

be the exponential generating function for the sequence $\{t_n\}$. The exponential generating function for rooted hyperforests is therefore $e^{\,t\,T(z)}$, where t counts the number of connected components. We can also consider the exponential generating function for unrooted trees

$$U(z) = U(z, \mathbf{w}) = \sum_{n \geq 0} u_n(\mathbf{w}) \, \frac{z^n}{n!} \tag{4.4}$$

where u_n is the total weight of unrooted trees in the case of n vertices. Of course as the root of a trees on n vertices can be chosen in n ways

$$t_n = n\, u_n, \tag{4.5}$$

giving us the relation

$$T(z) = z\,\frac{d}{dz}U(z) \tag{4.6}$$

and conversely

$$U(z) = \int_0^z \frac{d\omega}{\omega}\, T(\omega)\,. \tag{4.7}$$

Counting the number of unrooted trees u_n on the complete graph $\mathcal{K}_n^{(2)}$ is presented in [34, Chapter 7] as a simple application of the formalism of the exponential generating function.

For $n > 0$ the recurrence

$$u_n = \sum_{m>0} \frac{1}{m!} \sum_{\substack{a_1,a_2,\ldots,a_m \\ a_1+\cdots+a_m=n-1}} \binom{n-1}{a_1,\ldots,a_m} a_1 \cdots a_m\, u_{a_1} \cdots u_{a_m} \tag{4.8}$$

can be obtained as follows: a given vertex is attached to m components of sizes $a_1, \ldots, a_m$. There are $\binom{n-1}{a_1,\ldots,a_m}$ ways to assign $n-1$ vertices to those components and $a_1 \cdots a_m$ ways to connect the given vertex to them. There are $u_{a_1} \cdots u_{a_m}$ ways to connect those individual components with spanning trees; and we divide by $m!$ because the m components are not ordered.

In virtue of (4.5), the recurrence relation can be re-written as

$$\frac{t_n}{n!} = \sum_{m>0} \frac{1}{m!} \sum_{\substack{a_1,a_2,\cdots,a_m \\ a_1+\cdots+a_m=n-1}} \frac{t_{a_1}}{a_1!} \cdots \frac{t_{a_m}}{a_m!}\,. \tag{4.9}$$

By introducing the exponential generating function for the sequence $\{t_n\}$

$$T(z) = \sum_{n\geq 0} t_n\, \frac{z^n}{n!} \tag{4.10}$$

it follows that the inner sum in (4.9) is the coefficient of z^{n-1} in $T(z)^m$

$$\frac{t_n}{n!} = \left[z^{n-1}\right] \sum_{m\geq 0} \frac{1}{m!}\, T(z)^m = \left[z^{n-1}\right]\, e^{T(z)} \tag{4.11}$$

where we denoted by $[z^n]\, f(z)$ the coefficient of z^n in the expansion and we have included also the case $n = 1$ by adding the contribution $m = 0$. In the end we have

$$T(z) = z\, e^{T(z)}. \tag{4.12}$$

This result is usually attributed to Cayley in 1889 [37], but in his paper he refers to a previous result by Borchardt in 1860 [38].

More generally when $\theta(u)$ is a formal power series in u with $\theta(0) = 1$, a relation for the formal power series $T(z)$ of the form

$$T(z) = z\,\theta\,(T(z)) \tag{4.13}$$

has a unique solution, which is given by Lagrange inversion formula [35]

$$[z^n]\,T(z) = \frac{1}{n}\left[T^{n-1}\right]\theta\,(T)^n\ . \tag{4.14}$$

Furthermore

$$[z^n]\,T(z)^r = \frac{r}{n}\left[T^{n-r}\right]\theta\,(T)^n\ . \tag{4.15}$$

In our application to the trees, $\theta(T) = e^T$ and therefore

$$u_n = \frac{t_n}{n} = \frac{n!}{n}\,[z^n]\,T(z) = \frac{(n-1)!}{n}\,\left[T^{n-1}\right]e^{n\,T} = \frac{n^{n-1}}{n}\,. \tag{4.16}$$

While the number of rooted forests with r trees is given by

$$t_{n,r} = \frac{n!}{r!}\,[z^n]\,T(z)^r = \frac{(n-1)!}{(r-1)!}\,\left[T^{n-r}\right]e^{n\,T} = \binom{n-1}{r-1}\,n^{n-r}\,. \tag{4.17}$$

In the case of the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ with weights w_k the recurrence relation for the weight of unrooted hypertrees is

$$u_n = \sum_{\substack{m>0\\ m/(k-1)}} \frac{w_k^{\frac{m}{k-1}}}{\left(\frac{m}{k-1}\right)!\ [(k-1)!]^{\frac{m}{k-1}}} \sum_{\substack{a_1,a_2,\cdots,a_m\\ a_1+\cdots+a_m=n-1}} \binom{n-1}{a_1,\cdots,a_m} a_1\cdots a_m\,u_{a_1}\cdots u_{a_m} \tag{4.18}$$

where at variance with respect to (4.8) the sum on m is restricted to integers that can be divided by $k-1$ and appears a combinatorial factor $\frac{m!}{\left(\frac{m}{k-1}\right)!\,[(k-1)!]^{\frac{m}{k-1}}}$ because this is the number of ways in which the m sub-hypertrees can be hooked to the starting vertex by using hyperedges of cardinality k. As a consequence the equation for the rooted hypertrees becomes

$$\frac{t_n}{n!} = \left[z^{n-1}\right]\sum_{l\geq 1}\frac{w_k^l}{[(k-1)!]^l\ l!}T(z)^{(k-1)\,l}! = \left[z^{n-1}\right]\,\exp\left\{w_k\,\frac{T(z)^{k-1}}{(k-1)!}\right\} \tag{4.19}$$

which is to say

$$T(z) = z\exp\left[w_k\,\frac{T(z)^{k-1}}{(k-1)!}\right] \tag{4.20}$$

We can now apply again the Lagrange inversion formula with $\theta(T) = e^{w_k\,\frac{T(z)^{k-1}}{(k-1)!}}$ and therefore

$$\begin{aligned} u_n &= \frac{t_n}{n} = \frac{n!}{n}\,[z^n]\ T(z) = \frac{(n-1)!}{n}\left[T^{n-1}\right]\ \exp\left\{n\,w_k\,\frac{T(z)^{k-1}}{(k-1)!}\right\} \\ &= \frac{1}{n}\,\frac{(n\,w_k)^{\frac{n-1}{k-1}}}{\left(\frac{n-1}{k-1}\right)!\,[(k-1)!]^{\frac{n-1}{k-1}}}\,. \end{aligned} \tag{4.21}$$

While the weight for the rooted hyperforests with r hypertrees is

$$t_{n,r} = \frac{n!}{r!}\,[z^n]\;T(z)^r = \frac{(n-1)!}{(r-1)!}\,\left[T^{n-r}\right]\;\exp\left\{n\,\frac{T^{k-1}}{(k-1)!}\right\} \tag{4.22}$$

$$= \frac{(n-1)!}{(r-1)!}\,\frac{1}{\left(\frac{n-r}{k-1}\right)!}\,\frac{(n\,w_k)^{\frac{n-r}{k-1}}}{[(k-1)!]^{\frac{n-r}{k-1}}} \tag{4.23}$$

when $(n-r)/(k-1)$ is an integer. It is indeed the total number of hyperedges.

In the general case of the complete hypergraph $\overline{\mathcal{K}}_n$ the recurrence relation for the total weight of unrooted hypertrees is more involved, but the possibilities of attaching hyperedges of different cardinality at the starting vertex are mutually avoiding and this makes the recursion affordable. It follows that in this case the generating function satisfies the equation

$$T(z) = z\exp\left[\sum_{k\ge 2} w_k\,\frac{T(z)^{k-1}}{(k-1)!}\right] \tag{4.24}$$

so that

$$t_{n,r} = \frac{(n-1)!}{(r-1)!}\,\left[T^{n-r}\right]\exp\left\{n\sum_{k\ge 2} w_k\,\frac{T^{k-1}}{(k-1)!}\right\}\,. \tag{4.25}$$

In the simpler case in which all the weights are equal to, say, x, the recurrence relation for the unrooted hypertrees is

$$u_n = \sum_{m\ge 0}\sum_{l\ge 0}\frac{1}{m!}\left\{ {m \atop l} \right\} x^l \sum_{\substack{a_1,a_2,\cdots,a_m\\ a_1+\cdots+a_m=n-1}} \binom{n-1}{a_1,\cdots,a_m} a_1\cdots a_m\,u_{a_1}\cdots u_{a_m} \tag{4.26}$$

where, at variance with respect to (4.8) there appears a factor $\left\{ {m \atop l} \right\}$ [1]because this is the number of ways in which the m sub-hypertrees can be hooked to the starting vertex by using l generic hyperedges. As a consequence the equation for the rooted hypertrees becomes

$$\frac{t_n}{n!} = \left[z^{n-1}\right]\sum_{m\ge 0}\sum_{l\ge 0}\frac{1}{m!}\left\{ {m \atop l} \right\} x^l\,T(z)^m = \left[z^{n-1}\right]\,e^{x\left(e^{T(z)}-1\right)} \tag{4.28}$$

which is to say

$$T(z) = z\,e^{x\left(e^{T(z)}-1\right)} \tag{4.29}$$

that is (4.24) for $w_k = x$ for all k, a relation that in the case $x = 1$ is reported in Warme's Ph. D. Thesis [39] as due to W. D. Smith, but see also [40]. We can now apply the Lagrange inversion formula with $\theta(T) = e^{x\left(e^T-1\right)}$ and therefore

$$u_n = \frac{t_n}{n} = \frac{n!}{n}\,[z^n]\,T(z) = \frac{(n-1)!}{n}\,\left[T^{n-1}\right]e^{n\,x\,\left(e^T-1\right)} = \frac{b_{n-1}(n\,x)}{n}\,. \tag{4.30}$$

[1]The Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$ stands for the number of ways to partition a set of cardinality n into k *nonempty* subsets. Their exponential generating function is

$$\sum_{n\ge 0}\left\{ {n \atop k} \right\}\frac{z^n}{n!} = \sum_{n\ge k}\left\{ {n \atop k} \right\}\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!} \tag{4.27}$$

While the total weight of rooted hyperforests with r hypertrees is

$$t_{n,r} = \frac{n!}{r!} \, [z^n] \, T(z)^r = \frac{(n-1)!}{(r-1)!} \, \left[T^{n-r}\right] e^{n\,x\,(e^T - 1)} = \binom{n-1}{r-1} b_{n-r}(n\,x)\,. \tag{4.31}$$

Here we introduced the Bell polynomials $b_n(x)$, also called exponential polynomials, which are defined by

$$b_n(x) = \sum_{k \geq 0} \left\{ n \atop k \right\} x^k \tag{4.32}$$

whose generating function is indeed

$$\sum_{n \geq 0} b_n(x) \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{k \geq 0} \left\{ n \atop k \right\} x^k \frac{z^n}{n!} = \sum_{k \geq 0} \frac{[x\,(e^z - 1)]^k}{k!} = e^{x(e^z - 1)}\,. \tag{4.33}$$

Finally, we want to show that relation (4.24) can be used to derive an explicit formula allowing us to write $U(z)$ in terms of $T(z)$, indeed we have from (4.24)

$$z = T(z) \, \exp\left[- \sum_{k \geq 2} w_k \, \frac{T(z)^{k-1}}{(k-1)!} \right] \tag{4.34}$$

and by changing variables from ω to $T(\omega)$ in the integral in (4.7) we easily get

$$U(z) = T(z) + \sum_{k \geq 2} w_k \, (1 - k) \, \frac{T(z)^k}{k!} \tag{4.35}$$

that is the exponential generating function for unrooted hypertrees can be expressed in terms of the exponential generating function of rooted hypertrees [36].

4.2 Relation with the classical approach

We now want to understand how our Grassmann approach is related with the classical one. Let us use the just obtained relation between $U(z)$ and $T(z)$ to to re-obtain, at least in the here considered case of the complete hypergraph, the generating function of unrooted hyperforests in the Grassmann representation from the generating function of rooted hyperforests.

Formula (4.2) for the generating function (or partition function) of rooted hyperforests for $\overline{\mathcal{K}}_n$, at $t_i = t$ for every vertex, means that

$$E_n(t, \mathbf{w}) = n! \; [z^n] \; e^{t\,T(z)} = \int \mathcal{D}_n(\psi, \bar\psi) \, \exp\left[t \, \bar\psi\psi + \sum_{A \in E} w_{|A|} f_A^{(0)} \right] \tag{4.36}$$

where we again used the short notation

$$\bar\psi\psi = \sum_{i=1}^{n} \bar\psi_i \psi_i\,. \tag{4.37}$$

It follows that, for every power r, the coefficient of t^r is equal to

$$n!\ [z^n]\ T(z)^r = \int \mathcal{D}_n(\psi, \bar{\psi})\, (\bar{\psi}\psi)^r\ \exp\left[\sum_{A \in E} w_{|A|} f_A^{(0)}\right] \tag{4.38}$$

and therefore for each function L defined by a formal power series

$$n!\ [z^n]\ L\left(T(z)\right) = \int \mathcal{D}_n(\psi, \bar{\psi})\, L(\bar{\psi}\psi)\ \exp\left[\sum_{A \in E} w_{|A|} f_A^{(0)}\right]\ . \tag{4.39}$$

Now, the exponential generating function for unrooted hyperforests is $e^{\lambda U(z)}$, where λ counts the hypertrees in the hyperforests and we know by (4.1) that

$$F_n(\lambda, \mathbf{w}) = n!\ [z^n]\ e^{\lambda U(z)} = \int \mathcal{D}_n(\psi, \bar{\psi})\ \exp\left[\lambda\, \bar{\psi}\psi + \sum_{A \in E} w_{|A|} f_A^{(\lambda)}\right] \tag{4.40}$$

but

$$\sum_{A \in E} w_{|A|} f_A^{(\lambda)} = \sum_{A \in E} w_{|A|} \left[\lambda\ (1 - |A|)\ \tau_A + f_A^{(0)}\right] \tag{4.41}$$

and

$$\sum_{A \in E} w_{|A|}\ (1 - |A|)\ \tau_A = \sum_{k \geq 2} w_k\ (1 - k) \sum_{A : |A| = k} \tau_A = \sum_{k \geq 2} w_k\, (1 - k)\, \frac{(\bar{\psi}\psi)^k}{k!} \tag{4.42}$$

so that

$$F_n(\lambda, \mathbf{w}) = \int \mathcal{D}_n(\psi, \bar{\psi}) \exp\left\{\lambda \left[\bar{\psi}\psi + \sum_{k \geq 2} w_k (1 - k)\, \frac{(\bar{\psi}\psi)^k}{k!}\right] + \sum_{A \in E} w_{|A|} f_A^{(0)}\right\} \tag{4.43}$$

But this is exactly formula (4.39) when

$$L(z) = e^{\lambda K(z)} \tag{4.44}$$

with

$$K(z) = z + \sum_{k \geq 2} w_k\, (1 - k)\, \frac{z^k}{k!} \tag{4.45}$$

which is such that $U(z) = K\left(T(z)\right)$ by (4.35).

The Grassmann integrals for the generating functions of rooted and unrooted hyperforests at fixed number of vertices can be expressed as a unique contour integral of a complex variable. Let us show the change of variables which explicitly maps those integrals into the coefficient of the corresponding exponential generating function in the number of vertices, without using the Lagrange inversion formula.

The sum on all the edges appears in both main formulas (4.36) and (4.43) and in our model it becomes

$$\sum_{A\in E} w_{|A|} f_A^{(0)} = \sum_{k\geq 2} w_k \sum_{A:|A|=k} f_A^{(0)} \tag{4.46a}$$

$$= \sum_{k\geq 2} w_k \left[(n-k+1)\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi,(\mathrm{J}-\mathrm{I})\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right] \tag{4.46b}$$

$$= \sum_{k\geq 2} w_k \left[n\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi \mathrm{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right] \tag{4.46c}$$

and according to Lemma 3.1.4, for any function h of the scalar product $\psi\bar\psi$

$$\int \mathcal{D}_n(\psi,\bar\psi)\ h(\bar\psi\psi)\ \exp\left[\sum_{A\in E} w_{|A|} f_A^{(0)}\right] =$$

$$\int \mathcal{D}_n(\psi,\bar\psi)\ h(\bar\psi\psi)\ \exp\left[n\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-1)!}\right]\ \left[1-\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-2)!}\right] \tag{4.47}$$

so the Grassmann integrals reduces to what has been formally obtained in Corollary 3.1.2 and we have for (4.36)

$$E_n(t,\mathbf{w}) = \int \mathcal{D}_n(\psi,\bar\psi)\ \left[1-\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-2)!}\right] \exp\left[t\,\bar\psi\psi + n\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-1)!}\right] \tag{4.48}$$

$$= \frac{n!}{2\,\pi\, i}\oint \frac{d\xi}{\xi^{n+1}}\ \left[1-\sum_{k\geq 2} w_k\, \frac{\xi^{k-1}}{(k-2)!}\right]\ \exp\left[t\,\xi + n\sum_{k\geq 2} w_k\, \frac{\xi^{k-1}}{(k-1)!}\right] \tag{4.49}$$

which is nothing but

$$E_n(t,\mathbf{w}) = n!\,[z^n]\ e^{\,t\,T(z)} = \frac{n!}{2\,\pi\, i}\oint \frac{dz}{z^{n+1}}\, e^{\,t\,T(z)} \tag{4.50}$$

with the change of variables (4.34) with $T(z)=\xi$, as

$$\frac{dz}{z} = \frac{d\xi}{\xi}\ \left[1-\sum_{k\geq 2} w_k\, \frac{\xi^{k-1}}{(k-2)!}\right]\,. \tag{4.51}$$

Analogously for (4.43)

$$F_n(\lambda,\mathbf{w}) = \int \mathcal{D}_n(\psi,\bar\psi)\ \left[1-\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-2)!}\right]$$

$$\times\ \exp\left[\lambda\ \left(\bar\psi\psi + \sum_{k\geq 2} w_k\,(1-k)\,\frac{(\bar\psi\psi)^{k}}{k!}\right) + n\sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-1)!}\right] \tag{4.52a}$$

$$= \frac{n!}{2\,\pi\, i}\oint \frac{d\xi}{\xi^{n+1}}\ \left[1-\sum_{k\geq 2} w_k\, \frac{\xi^{k-1}}{(k-2)!}\right]$$

$$\times\ \exp\left[\lambda\ \left(\xi\ + \sum_{k\geq 2} w_k\,(1-k)\,\frac{\xi^{k}}{k!}\right) + n\sum_{k\geq 2} w_k\, \frac{\xi^{k-1}}{(k-1)!}\right] \tag{4.52b}$$

which, by using the same change of variables, is nothing but

$$F_n(\lambda, \mathbf{w}) = n!\,[z^n]\; e^{\lambda\, U(z)} = \frac{n!}{2\,\pi\, i} \oint \frac{dz}{z^{n+1}}\, e^{\lambda\, U(z)} \tag{4.53}$$

$$= \frac{n!}{2\,\pi\, i} \oint \frac{dz}{z^{n+1}}\, \exp\left\{ \lambda \left[T(z) \,+\, \sum_{k\geq 2} w_k\, (1-k)\, \frac{T(z)^k}{k!} \right] \right\} \tag{4.54}$$

4.3 Rooted hyperforests

Let us begin considering the weight of rooted hyperforests on r vertices which on the complete hypergraph $\overline{\mathcal{K}}_n$ does not depend on the particular choice of the vertices. The expansion in power series of t of (4.2)

$$E_n(t, \mathbf{w}) = \sum_{r\geq 0} t_{n,r}(\mathbf{w})\, t^r \tag{4.55}$$

provides the total weight of rooted hyperforests with r connected components

$$t_{n,r} = t_{n,r}(\mathbf{w}) = n!\,[z^n]\,[t^r]\, e^{t\,T(z)} = [t^r]\, E_n(t, \mathbf{w}) \tag{4.56}$$

then

$$t_{n,r} = \int \mathcal{D}_n(\psi, \bar\psi)\, \frac{(\bar\psi\psi)^r}{r!} \left[1 - \sum_{k\geq 2} w_k \frac{(\bar\psi\psi)^{k-1}}{(k-2)!} \right] \exp\left[n \sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-1)!} \right] \tag{4.57}$$

while then the total weight of rooted hyperforests

$$E_n(\mathbf{w}) = E_n(1, \mathbf{w}) = \sum_{r\geq 0} t_{n,r}(\mathbf{w}) \tag{4.58}$$

is given by the generating function at $t = 1$.

Let us now introduce the function

$$\theta(x, y; \mathbf{w}) = \exp\left[x \sum_{k\geq 2} w_k\, \frac{y^{k-1}}{(k-1)!} \right] = \sum_{s\geq 0} P_s(x; \mathbf{w})\, \frac{y^s}{s!} \tag{4.59}$$

which is the exponential generating function for the exponentials $P_s(x; \mathbf{w})$ in the variable x, which varies with the choice of the weights $\mathbf{w}$. We recognize that

$$\sum_{k\geq 2} w_k \frac{y^{k-1}}{(k-2)!}\, \theta(x, y; \mathbf{w}) = \frac{y}{x} \frac{\partial}{\partial y} \theta(x, y; \mathbf{w}) = \frac{1}{x} \sum_{s\geq 1} P_s(x; \mathbf{w})\, \frac{y^s}{(s-1)!} \tag{4.60}$$

Therefore $E_n(\mathbf{t}, \mathbf{w})$ can be re-expressed by using

$$e^{t\,y}\, \theta(x, y; \mathbf{w}) = \sum_{s\geq 0} \sum_{r\geq 0} P_s(x; \mathbf{w})\, t^r\, \frac{y^{r+s}}{r!\, s!} \tag{4.61}$$

and

$$\sum_{k\geq 2} w_k \frac{y^{k-1}}{(k-2)!}\, e^{t\,y}\, \theta(x,y;\mathbf{w}) = \sum_{s\geq 0}\sum_{r\geq 0} P_s(x;\mathbf{w})\, t^r\, \frac{y^{r+s}}{r!\,s!}\, \frac{s}{x} \tag{4.62}$$

The same expression could be written also with the help of the derivative with respect to the variable t, let $D = \frac{\partial}{\partial t}$, then

$$\sum_{k\geq 2} w_k \frac{y^{k-1}}{(k-2)!}\, e^{t\,y}\, \theta(x,y;\mathbf{w}) = \tag{4.63}$$

$$= \sum_{k\geq 2} \frac{w_k}{(k-2)!}\, D^{k-1}\, e^{t\,y}\, \theta(x,y;\mathbf{w}) \tag{4.64}$$

$$= \sum_{k\geq 2} \frac{w_k}{(k-2)!}\, D^{k-1} \sum_{s\geq 0}\sum_{r\geq 0} P_s(x;\mathbf{w})\, t^r\, \frac{y^{r+s}}{r!\,s!} \tag{4.65}$$

$$= \sum_{s\geq 0} P_s(x;\mathbf{w}) \sum_{r\geq k-1} \frac{y^{r+s}}{s!} \sum_{k\geq 2} \frac{w_k}{(k-2)!}\, \frac{1}{[r-(k-1)]!}\, t^{r-(k-1)} \tag{4.66}$$

$$= \sum_{k\geq 2}\sum_{s\geq k-1} P_{s-(k-1)}(x;\mathbf{w}) \sum_{r\geq 0} t^r\, \frac{y^{r+s}}{r!\,[s-(k-1)]!}\, \frac{w_k}{(k-2)!} \tag{4.67}$$

so that by comparing term by term in 4.62 and 4.67 we recover a recursion relation for the polynomials $P_s(x,\mathbf{w})$

$$P_s(x;\mathbf{w}) \;=\; x \sum_{k\geq 2} w_k \binom{s-1}{k-2} P_{s-(k-1)}(x;\mathbf{w})\,. \tag{4.68}$$

In terms of the polynomials $P_s(x,\mathbf{w})$ we soon get for the generating function of rooted hyperforests

$$\begin{aligned} E_n(t,\mathbf{w}) \;&=\; \sum_{s\geq 0}\sum_{r\geq 0} P_s(n;\mathbf{w})\, t^r \int \mathcal{D}_n(\psi,\bar\psi)\, \frac{(\bar\psi\psi)^{r+s}}{r!\,s!} \left[1-\frac{s}{n}\right] &(4.69)\\ &=\; \sum_{r\geq 1} \binom{n-1}{r-1} P_{n-r}(n;\mathbf{w})\, t^r &(4.70)\end{aligned}$$

(4.68). Therefore the total weight of rooted hyperforests is

$$E_n(\mathbf{w}) = \sum_{r\geq 1} \binom{n-1}{r-1} P_{n-r}(n;\mathbf{w}) \tag{4.71}$$

and the total weight of rooted hyperforests with r hypertrees is

$$t_{n,r} = \binom{n-1}{r-1} P_{n-r}(n;\mathbf{w}) \tag{4.72}$$

from which in particular we obtain for $r=0$

$$t_{n,0} = 0 \tag{4.73}$$

for all choices of the weights $\mathbf{w}$, a generalization of what occurs for the case of ordinary trees because the determinant of the weighted Laplacian on the graph is always vanishing.

Also, as $P_0(x;\mathbf{w}) = 1$ for all choices of the weights $\mathbf{w}$, of course

$$t_{n,n} = 1 \tag{4.74}$$

as there is only one possible hyperforest with n hypertrees, the trivial one in which each hypertree is a vertex.

The weight of rooted hypertrees t_n is given by the case $r = 1$

$$t_n = t_{n,1} = P_{n-1}(n;\mathbf{w})\,. \tag{4.75}$$

A more explicit expression for the polynomials $P_s(x;\mathbf{w})$ is obtained by expanding the exponential in the definition (4.59)

$$\begin{aligned} P_s(x;\mathbf{w}) &= s!\,[y^s]\,\theta(x,y;\mathbf{w}) \\ &= s! \prod_{j\geq 2} \sum_{l_j} \frac{1}{l_j!} \left(\frac{x w_j}{(j-1)!}\right)^{l_j} y^{l_j(j-1)} \\ &= s! \sum_{\{l_j\}} \delta_{s,\sum_{j\geq 2} l_j(j-1)} \left[\prod_{j\geq 2} \frac{1}{l_j!} \left(\frac{x w_j}{(j-1)!}\right)^{l_j}\right] \end{aligned} \tag{4.76}$$

so that if we define the coefficients $p_{s,l}(\mathbf{w})$ by

$$P_s(x;\mathbf{w}) = \sum_{l\geq 0} p_{s,l}(\mathbf{w})\, x^l \tag{4.77}$$

we get

$$\begin{aligned} p_{s,l} = p_{s,l}(\mathbf{w}) &= s!\,[y^s]\,[x^l]\,\theta(x,y;\mathbf{w}) \\ &= s! \sum_{\{l_j\}} \delta_{l,\sum_{j\geq 2} l_j} \delta_{s,\sum_{j\geq 2} l_j(j-1)} \left[\prod_{j\geq 2} \frac{1}{l_j!} \left(\frac{w_j}{(j-1)!}\right)^{l_j}\right]. \end{aligned}$$

In order to understand the constraint which is imposed in the sum on the coefficients l_j's, remember that from Proposition A.2.1, if l_j is the number of hyperedges of cardinality j, n is the number of vertices and r is the number of connected components, which in our case is the number of hypertrees

$$0 = \sum_{A\in E} (|A| - 1) - |V| + c(G) = \sum_{j\geq 2} l_j(j-1) - n + r \tag{4.78}$$

and this is exactly the constraint which is imposed. The number l is instead n_E the total number of hyperedges.

4.3.1 On the k-uniform complete hypergraph

In the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ the hyperedges are all the subsets $A \subset V$ of k vertices: $|A| = k$. This is therefore the particular case of our model in which if we introduce the vectors $\mathbf{e}_k$ such that their components are

$$(\mathbf{e}_k)_s \;=\; \delta_{ks} \tag{4.79}$$

we have weights

$$\mathbf{w} = w\,\mathbf{e}_k \tag{4.80}$$

and as we wish to count configurations we have to set $w = 1$ so that in the general formulas $w_k = 1$ and all the others weights for the hyperedges have to be set to zero. We have

$$\theta(x, y; \mathbf{e}_k) = \exp\left[x \frac{y^{k-1}}{(k-1)!}\right] \tag{4.81}$$

and therefore

$$P_s(x; \mathbf{e}_k) = \begin{cases} \frac{s!}{\left(\frac{s}{k-1}\right)![(k-1)!]^{\frac{s}{k-1}}}\, x^{\frac{s}{k-1}} & \text{if } s = l(k-1) \text{ for integer } l \\ 0 & \text{otherwise} \end{cases} \tag{4.82}$$

which satisfy the recursion relation (4.68) which for $\mathbf{w} = \mathbf{e}_k$ takes the form

$$P_s(x; \mathbf{e}_k) = x \binom{s-1}{k-2} P_{s-(k-1)}(x; \mathbf{e}_k)\,. \tag{4.83}$$

We easily get that

$$p_{s,l}(\mathbf{e}_k) = \begin{cases} \frac{s!}{l![(k-1)!]^l} & \text{if } s = l(k-1) \text{ for integer } l \\ 0 & \text{otherwise} \end{cases} \tag{4.84}$$

On $\mathcal{K}_n^{(k)}$, the numbers $n_E = l$ of hyperedges and the number of connected components $c(G) = r$ are related by (4.78)

$$l\,(k-1) - n + r = 0 \tag{4.85}$$

that is

$$n_E = l = \frac{n-r}{k-1} \tag{4.86}$$

is the number of hyperedges (of degree k).

For the number of rooted hyperforests with r hypertrees on the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$, we have when $n - r$ can be divided by $k - 1$

$$\begin{aligned} t_{n,r}(\mathbf{e}_k) &= \binom{n-1}{r-1} P_{n-r}(n; \mathbf{e}_k) && (4.87) \\ &= \binom{n-1}{r-1} \frac{(n-r)!}{\left(\frac{n-r}{k-1}\right)![(k-1)!]^{\frac{n-r}{k-1}}}\, n^{\frac{n-r}{k-1}} && (4.88) \\ &= \binom{(k-1)n_E + r - 1}{r-1} \frac{[(k-1)n_E]!}{n_E!\,[(k-1)!]^{n_E}}\, [(k-1)n_E + r]^{n_E} && (4.89) \end{aligned}$$

where the prefactor in (4.88)

$$\frac{(n-r)!}{\left(\frac{n-r}{k-1}\right)![(k-1)!]^{\frac{n-r}{k-1}}} \tag{4.90}$$

is exactly the number of ways in which $n-r$ vertices can be divided into $(n-r)/(k-1)$ groups of $k-1$ elements and in (4.89) we have replaced the dependence from the number of vertices n with that from the number of hyperedges n_E.

In the case of simple graphs ($k=2$) it follows that

$$t_{n,r}(\mathbf{e}_2) = \binom{n-1}{r-1} n^{n-r} \tag{4.91}$$

which at $r = 1$ provides the well-known result by Cayley about the number $u_n^{(2)}$ of spanning unrooted trees on the complete graph with n vertices

$$u_n(\mathbf{e}_2) = \frac{t_n(\mathbf{e}_2)}{n} = n^{n-2}\,. \tag{4.92}$$

Also

$$E_n(t,\mathbf{e}_2) = \sum_{r\geq 1} \binom{n-1}{r-1} n^{n-r}\, t^r = t\,(n+t)^{n-1} \tag{4.93}$$

which could be obtained by direct evaluation as

$$E_n(t,\mathbf{e}_2) = \int \mathcal{D}_n(\psi,\bar\psi)\ \left[1-(\bar\psi\psi)\right] e^{(t+n)\,(\bar\psi\psi)} = (t+n)^n \left[1-\frac{n}{n+t}\right] \tag{4.94}$$

This relation says at $t=1$ that the total number of rooted forests is

$$E_n(\mathbf{e}_2) = (n+1)^{n-1}\,. \tag{4.95}$$

In this simple case also the whole generating function can be expressed in terms of the generalized exponential [34] (the usual exponential is at $\alpha = 0$)

$$\mathcal{E}_\alpha(z) = \sum_{n\geq 1} (\alpha n+1)^{n-1}\, \frac{z^n}{n!} \tag{4.96}$$

which satisfies

$$\mathcal{E}_\alpha(z)^{-\alpha} \ln \mathcal{E}_\alpha(z) = z \qquad \mathcal{E}_\alpha(z) = \mathcal{E}(\alpha z)^{\frac{1}{\alpha}} \tag{4.97}$$

where $\mathcal{E}(z)$ is a shorthand for $\mathcal{E}_1(z)$. Indeed

$$\begin{aligned} e^{t\,T(z)} &= \sum_{n\geq 1} E_n(t,\mathbf{e}_2)\,\frac{z^n}{n!} = \sum_{n\geq 1}\left(\frac{n}{t}+1\right)^{n-1}\frac{(t\,z)^n}{n!} \\ &= \mathcal{E}_{\frac{1}{t}}(t\,z) = \mathcal{E}(z)^t = e^{t\,z\,\mathcal{E}(z)} \end{aligned} \tag{4.98}$$

4.3.2 On the complete hypergraph

We shall consider here the complete hypergraph $\overline{\mathcal{K}}_n$ when all the hyperedge-weights w_d are set to one, that is

$$\mathbf{w} = \mathbf{1} \tag{4.99}$$

where $\mathbf{1}$ is the vector with 1 on all components. We have

$$\theta(x, y; \mathbf{1}) = \exp\left[\, x\ (e^y - 1)\right] = \sum_{s\geq 0} b_s(x) \frac{y^s}{s!} \tag{4.100}$$

where $b_s(x)$ are again the Bell polynomials and therefore

$$P_s(x; \mathbf{1}) = b_s(x) = \sum_{l\geq 0} \left\{ {s \atop l} \right\} x^l \tag{4.101}$$

so that

$$p_{s,l}(\mathbf{1}) = \left\{ {s \atop l} \right\} \tag{4.102}$$

where $\left\{ {s \atop l} \right\}$ is a Stirling number of the second kind.

The recursion relation (4.68) becomes here

$$b_s(x) = x \sum_{k\geq 1} \binom{s-1}{k-1} b_{s-k}(x) \tag{4.103}$$

The number of rooted hyperforests with r hypertrees on the complete hypergraph $\overline{\mathcal{K}}_n$ is therefore

$$t_{n,r}(\mathbf{1}) = \binom{n-1}{r-1} b_{n-r}(n) = \binom{n-1}{r-1} \sum_{n_E \geq 0} n^{n_E} \left\{ {n-r \atop n_E} \right\} . \tag{4.104}$$

and the total number of rooted hyperforests is

$$E_n(\mathbf{1}) = \sum_{r\geq 1} t_{n,r}(\mathbf{1}) = \sum_{k\geq 1} \binom{n-1}{k-1} b_{n-k}(n) = \frac{b_n(n)}{n} \tag{4.105}$$

because of (4.103) for $x = n$.

4.4 Unrooted hyperforests

According to our general formula the generating function for unrooted hyperforests on n vertices is given by the Grassmann integral (4.1) (taking into account the results of Section 4.2, see equations (4.43) and (4.46c))

$$\begin{aligned} F_n(\lambda, \mathbf{w}) = n!\,[z^n]\, e^{\lambda U(z)} = \int \mathcal{D}_n(\psi, \bar\psi)\, \exp\left\{ \lambda \left[\bar\psi\psi + \sum_{k\geq 2} w_k\,(1-k) \frac{(\bar\psi\psi)^k}{k!} \right] \right\} \\ \times\ \exp\left[n \sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi \mathrm{J} \psi) \sum_{k\geq 2} w_k\, \frac{(\bar\psi\psi)^{k-2}}{(k-2)!} \right] \end{aligned} \tag{4.106}$$

which we expand in λ

$$F_n(\lambda, \mathbf{w}) = \sum_{p=0}^{\infty} u_{n,p}(\mathbf{w})\, \lambda^p \tag{4.107}$$

where $u_{n,p}(\mathbf{w})$ is the total weight of unrooted hyperforests with p hypertrees.

We find convenient to introduce the polynomials $\Pi_s(\lambda, \mathbf{w})$ and the coefficients $\pi_{s,r}(\mathbf{w})$ according to

$$\exp\left[\lambda\left(y + \sum_{k\geq 2} w_k\,(1-k)\,\frac{y^k}{k!}\right)\right] = \sum_{s\geq 0} \Pi_s(\lambda, \mathbf{w})\,\frac{y^s}{s!} = \sum_{s\geq 0}\sum_{p\geq 0} \pi_{s,p}(\mathbf{w})\,\lambda^p\,\frac{y^s}{s!} \tag{4.108}$$

It soon follows that

$$\begin{aligned}
F_n(\lambda, \mathbf{w}) &= \sum_{s\geq 1} \Pi_s(\lambda, \mathbf{w}) \int \mathcal{D}_n(\psi, \bar\psi)\,\frac{(\bar\psi\psi)^s}{s!} \\
&\qquad \exp\left[n \sum_{k\geq 2} w_k\,\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi \mathrm{J} \psi) \sum_{k\geq 2} w_k\,\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right] && (4.109)\\
&= \sum_{s\geq 1} \Pi_s(\lambda, \mathbf{w})\, t_{n,s}(\mathbf{w}) && (4.110)\\
&= \sum_{s\geq 1} \binom{n-1}{s-1} \Pi_s(\lambda, \mathbf{w})\, P_{n-s}(n; \mathbf{w}) && (4.111)
\end{aligned}$$

The total weight of unrooted hyperforests on the set on n vertices, irrespective from the number of hypertrees, is obtained from the partition function at $\lambda = 1$

$$F_n(\mathbf{w}) = F_n(1, \mathbf{w}) = \sum_{s\geq 1} \Pi_s(1, \mathbf{w})\, t_{n,s}(\mathbf{w})\,. \tag{4.112}$$

Also we get

$$\begin{aligned}
u_{n,p}(\mathbf{w}) &= \sum_{s\geq 1} \pi_{s,p}(\mathbf{w})\, t_{n,s}(\mathbf{w}) && (4.113)\\
&= \sum_{s\geq 1} \binom{n-1}{s-1} \pi_{s,p}(\mathbf{w})\, P_{n-s}(n; \mathbf{w}) && (4.114)
\end{aligned}$$

Remark that from the definition

$$\pi_{s,p}(\mathbf{w}) = 0 \qquad \text{when } p > s \tag{4.115}$$

so that $\Pi_s(\lambda, \mathbf{w})$ is a polynomial of degree s. It is monic because

$$\pi_{s,s}(\mathbf{w}) = 1\,. \tag{4.116}$$

And remark also that $\pi_{s,0}(\mathbf{w}) = 0$ while

$$\pi_{s,1}(\mathbf{w}) = \begin{cases} 1 & \text{for } s = 1 \\ w_s\,(1-s) & \text{otherwise.} \end{cases} \tag{4.117}$$

Accordingly $u_{n,0}(\mathbf{w}) = 0$ and $u_{n,n}(\mathbf{w}) = 1$, while it follows that the weight of unrooted hypertrees on n vertices is simply the weigth of the rooted hypertrees divided by n, indeed from (4.113)

$$u_{n,1}(\mathbf{w}) = P_{n-1}(n;\mathbf{w}) + \sum_{s\geq 2} w_s\,(1-s)\binom{n-1}{s-1} P_{n-s}(n;\mathbf{w}) \tag{4.118a}$$

$$= P_{n-1}(n;\mathbf{w}) - \;(n-1)\sum_{s\geq 2} w_s \binom{n-2}{s-2} P_{n-s}(n;\mathbf{w}) \tag{4.118b}$$

$$= \frac{P_{n-1}(n;\mathbf{w})}{n} \tag{4.118c}$$

$$= \frac{t_n(\mathbf{w})}{n} \tag{4.118d}$$

where we used the recursion relation (4.68) for the polynomials $P_s(x;\mathbf{w})$ at $x = n$ and $s = n-1$.

More formally we can follow a different strategy. Let $D = \frac{\partial}{\partial t}$ then

$$\exp\left[\lambda\left(y + \sum_{k\geq 2} w_k\,(1-k)\,\frac{y^k}{k!}\right)\right] \;=\; \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!}\right]\,\exp\,(t\,y)\Bigg|_{t=\lambda} \tag{4.119}$$

so that

$$\Pi_s(\lambda,\mathbf{w}) = \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!}\right]\; t^s\,\Bigg|_{t=\lambda} \tag{4.120}$$

and therefore

$$F_n(\lambda,\mathbf{w}) = \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!} + n\sum_{k\geq 2} w_k\,\frac{D^{k-1}}{(k-1)!}\right] \left[1 - \sum_{k\geq 2} w_k\,\frac{D^{k-1}}{(k-2)!}\right]\int \mathcal{D}_n(\psi,\bar\psi)\;\; e^{t\,(\bar\psi\psi)}\Big|_{t=\lambda} \tag{4.121}$$

$$= \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!} + n\sum_{k\geq 2} w_k\,\frac{D^{k-1}}{(k-1)!}\right] \left[t^n - \;n\sum_{k\geq 2} w_k\binom{n-1}{k-2} t^{n-k+1}\right]\Bigg|_{t=\lambda} \tag{4.122}$$

now, we expand first the second exponential, to get once more

$$F_n(\lambda,\mathbf{w}) = \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!}\right]\; E_n(t,\mathbf{w})\Bigg|_{t=\lambda} \tag{4.123}$$

$$= \exp\left[\lambda\sum_{k\geq 2} w_k\,(1-k)\frac{D^k}{k!}\right]\;\left[\sum_{s\geq 0}\binom{n-1}{s-1} P_{n-s}(n,\mathbf{w})\,t^s\right]\Bigg|_{t=\lambda}$$

$$= \sum_{s\geq 0}\binom{n-1}{s-1}\Pi_s(\lambda,\mathbf{w})\;P_{n-s}(n,\mathbf{w})\,.$$

4.4.1 On the k-uniform complete hypergraph

When $\mathbf{w} = \mathbf{e}_k$ the formula (4.108) becomes

$$\exp\left[\lambda\left(y + (1-k)\,\frac{y^k}{k!}\right)\right] = \sum_{s\geq 0} \Pi_s(\lambda;\mathbf{e}_k)\,\frac{y^s}{s!} = \sum_{s\geq 0}\sum_{p\geq 0} \pi_{s,p}(\mathbf{e}_k)\,\lambda^p\,\frac{y^s}{s!}\,. \tag{4.124}$$

We introduce a family of generalized Hermite polynomials $H_s^{(k)}(x)$ as defined by the generating function

$$\exp\left[x\,z + (1-k)\,\frac{z^k}{k!}\right] = \sum_{s\geq 0} H_s^{(k)}(x)\,\frac{z^s}{s!} \tag{4.125}$$

which when $k=2$ are related to the ordinary Hermite polynomials H_s by

$$H_s^{(2)}(x) = \mathrm{He}_s(x) = \frac{1}{2^{\frac{s}{2}}}\,H_s\left(\frac{x}{2^{\frac{1}{2}}}\right)\,. \tag{4.126}$$

where He_s are sometimes used [41]. Similar generalizations of the Hermite polynomials can be found in [42, 43, 44]. We then get

$$\Pi_s(\lambda;\mathbf{e}_k) = \lambda^{\frac{s}{k}}\,H_s^{(k)}\left(\lambda^{\frac{k-1}{k}}\right)\,. \tag{4.127}$$

Thus the generating function of unrooted hyperforests is

$$F_n(\lambda;\mathbf{e}_k) = \sum_{\substack{p\geq 0\\ p:\,(n-p)|(k-1)}} \binom{n-1}{p-1}\,\frac{(n-p)!}{\left(\frac{n-p}{k-1}\right)![(k-1)!]^{\frac{n-p}{k-1}}}\,n^{\frac{n-p}{k-1}}\,\lambda^{\frac{p}{k}}\,H_p^{(k)}\left(\lambda^{\frac{k-1}{k}}\right) \tag{4.128}$$

where the sum is restricted to the values of p such that $n-p$ can be divided by $k-1$. By using (4.121) we get instead

$$\begin{aligned}
F_n(\lambda;\mathbf{e}_k) &= \exp\left[\lambda\,(1-k)\frac{D^k}{k!} + n\,\frac{D^{k-1}}{(k-1)!}\right]\left[t^n - n\binom{n-1}{k-2}\,t^{n-k+1}\right]\Bigg|_{t=\lambda}\\
&= \lambda^{\frac{n}{k}}\,\exp\left[(1-k)\frac{D^k}{k!} + \frac{n}{\lambda^{\frac{k-1}{k}}}\,\frac{D^{k-1}}{(k-1)!}\right]\left[t^n - \frac{n}{\lambda^{\frac{k-1}{k}}}\binom{n-1}{k-2}\,t^{n-k+1}\right]\Bigg|_{t=\lambda^{\frac{k-1}{k}}}\\
&= \lambda^{\frac{n}{k}}\,\exp\left[\frac{n}{\lambda^{\frac{k-1}{k}}}\,\frac{D^{k-1}}{(k-1)!}\right]\left[H_n^k(t) - \frac{n}{\lambda^{\frac{k-1}{k}}}\binom{n-1}{k-2}\,H_{n-k+1}^k(t)\right]\Bigg|_{t=\lambda^{\frac{k-1}{k}}}\,.
\end{aligned}$$

In the particular case $k=2$ we soon get

$$F_n(\lambda;\mathbf{e}_2) = \sqrt{\lambda}^n\left[\mathrm{He}_n\left(\sqrt{\lambda} + \frac{n}{\sqrt{\lambda}}\right) - \frac{n}{\sqrt{\lambda}}\,\mathrm{He}_{n-1}\left(\sqrt{\lambda} + \frac{n}{\sqrt{\lambda}}\right)\right] \tag{4.129}$$

because $\exp\left[\alpha \frac{\partial}{\partial t}\right]$ is the translation operator from t to $t+\alpha$. The same result can be obtained by using (4.123) and (4.94) as

$$\begin{aligned}
F_n(\lambda;\mathbf{e}_2) &= \exp\left[-\lambda\,\frac{D^2}{2}\right]\; E_n(t,\mathbf{e}_2)\Big|_{t=\lambda} \\
&= \exp\left[-\lambda\,\frac{D^2}{2}\right]\; \left[(t+n)^n - n\,(t+n)^{n-1}\right]\Big|_{t=\lambda} \\
&= \exp\left[-\,\frac{D^2}{2}\right]\; \sqrt{\lambda}^{\,n}\; \left[\left(t+\frac{n}{\sqrt{\lambda}}\right)^n - \frac{n}{\sqrt{\lambda}}\,\left(t+\frac{n}{\sqrt{\lambda}}\right)^{n-1}\right]\Bigg|_{t=\sqrt{\lambda}} \\
&= \sqrt{\lambda}^{\,n}\; \left[\mathrm{He}_n\left(\sqrt{\lambda}+\frac{n}{\sqrt{\lambda}}\right) - \frac{n}{\sqrt{\lambda}}\,\mathrm{He}_{n-1}\left(\sqrt{\lambda}+\frac{n}{\sqrt{\lambda}}\right)\right]\,.
\end{aligned}$$

This formula has been reported in [45] for $\lambda = 1$, where it counts the total number of unrooted forests. In this case (4.128) becomes instead

$$F_n(\mathbf{e}_2) = \sum_{p\geq 1} \binom{n-1}{p-1}\, n^{n-p}\, \mathrm{He}_p(1) \tag{4.130}$$

in agreement with what obtained in [45] and reported as the series A001858 in the *The On-Line Encyclopedia of Integer Sequences* by Sloane [46].

By using $D = \frac{\partial}{\partial x}$ we get

$$\exp\left[x\,z + (1-k)\,\frac{z^k}{k!}\right] \;=\; \exp\left[\frac{1-k}{k!}\,D^k\right]\,\exp\left[x\,z\right] \tag{4.131}$$

and therefore

$$\begin{aligned}
H_s^{(k)}(x) &= \exp\left[\frac{1-k}{k!}\,D^k\right]\,x^s & (4.132)\\
&= \sum_{q\geq 0} \frac{1}{q!}\left(\frac{1-k}{k!}\right)^q D^{kq}\,x^s & (4.133)\\
&= \sum_{q\geq 0} \frac{1}{q!}\left(\frac{1-k}{k!}\right)^q \frac{s!}{(s-kq)!}\,x^{s-kq} & (4.134)
\end{aligned}$$

which implies because of (4.127)

$$\Pi_s(\lambda;\mathbf{e}_k) = \sum_{q\geq 0} \frac{1}{q!}\left(\frac{1-k}{k!}\right)^q \frac{s!}{(s-kq)!}\,\lambda^{s-(k-1)q} \tag{4.135}$$

so that

$$\pi_{s,p}(\mathbf{e}_k) = \sum_{q\geq 0} \frac{1}{q!}\left(\frac{1-k}{k!}\right)^q \frac{s!}{(s-kq)!}\,\delta_{p,s-(k-1)q} \tag{4.136}$$

and therefore, by using (4.113)

$$\begin{aligned}
u_{n,p}(\mathbf{e}_k) &= \sum_{q\geq 0} t_{n,p+q(k-1)}(\mathbf{e}_k)\,\frac{[p+q(k-1)]!}{(p-q)!}\,\frac{1}{q!}\left(\frac{1-k}{k!}\right)^q & (4.137)\\
&= \frac{(n-1)!}{p!}\left[\frac{n}{(k-1)!}\right]^{\frac{n-p}{k-1}} \sum_{q=0}^{p} \binom{p}{q} \frac{p+(k-1)q}{\left(\frac{n-p}{k-1}-q\right)!}\left(\frac{1-k}{kn}\right)^q & (4.138)
\end{aligned}$$

when $n-p$ can be divided by $k-1$, otherwise it vanishes, where we used the relation (4.72) and the explicit expression (4.88). Once more in the simpler case $k=2$ this formula reduces to

$$u_{n,p}(\mathbf{e}_2) = \frac{1}{p!}\sum_{q=0}^{p}\left(-\frac{1}{2}\right)^q \binom{p}{q}\binom{n-1}{p+q-1} n^{n-p-q}\,(p+q)! \tag{4.139}$$

a result which can be found in [47].

In order to proceed we need the sums

$$\frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{(-z)^{-q}}{(v-q)!} = \frac{(-z)^{-p}}{v!}\,L_p^{(v-p)}(z) \tag{4.140}$$

$$\frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{q\,(-z)^{-q}}{(v-q)!} = -\,z\,\frac{d}{dz}\,\frac{(-z)^{-p}}{v!}\,L_p^{(v-p)}(z) \tag{4.141}$$

$$= \frac{(-z)^{-p}}{v!}\,\left[p\,L_p^{(v-p)}(z) + z\,L_{p-1}^{(v-p+1)}(z)\right] \tag{4.142}$$

$$= \frac{(-z)^{-p}}{v!}\,v\,L_{p-1}^{(v-p)}(z) \tag{4.143}$$

where $L_m^{(\alpha)}(x)$ are the associated Laguerre polynomials

$$L_m^{(\alpha)}(x) = \sum_{\nu=0}^{\infty}\binom{m+\alpha}{m-\nu}\frac{(-x)^\nu}{\nu!} \tag{4.144}$$

which satisfy the recursion relation

$$L_{p-1}^{(k)}(z) = \frac{1}{z}\,\left[p\,L_p^{(k)}(z) - (p+k)\,L_{p-1}^{(k)}(z)\right]\,. \tag{4.145}$$

We arrive at the representation

$$\begin{aligned} u_{n,p}(\mathbf{e}_k) \quad = \quad & \frac{(n-1)!}{\left(\frac{n-p}{k-1}\right)!}\left[\frac{n}{(k-1)!}\right]^{\frac{n-p}{k-1}}\left(-\frac{k-1}{k\,n}\right)^p \\ & \left[p\,L_p^{\left(\frac{n-p}{k-1}-p\right)}\left(\frac{k\,n}{k-1}\right) + (n-p)\,L_{p-1}^{\left(\frac{n-p}{k-1}-p\right)}\left(\frac{k\,n}{k-1}\right)\right] \end{aligned} \tag{4.146}$$

for the number of unrooted hyperforests with p hypertrees on the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ with n vertices.

In order to study the asymptotic behaviour of the previous expression in the limit of large n at fixed p we need the following expansion for the Laguerre polynomial

$$L_s^{\left(\frac{n-p}{k-1}-p\right)}\left(\frac{k\,n}{k-1}\right) \simeq \frac{(-n)^s}{s!}\left\{1 + \frac{s\,[s+1+2\,k\,(p-s)]}{2\,n\,(k-1)} + O\left(\frac{1}{n^2}\right)\right\} \tag{4.147}$$

that can be easily obtained from the definition (4.144) (see [8], Appendix C) then

$$p\,L_p^{\left(\frac{n-p}{k-1}-p\right)}\left(\frac{k\,n}{k-1}\right) + (n-p)\,L_{p-1}^{\left(\frac{n-p}{k-1}-p\right)}\left(\frac{k\,n}{k-1}\right) \simeq \frac{(-n)^p}{(p-1)!}\,\frac{1}{n}\,\frac{k}{k-1} \tag{4.148}$$

because the leading terms in the two contributions cancel out. We get

$$u_{n,p}(\mathbf{e}_k) \simeq \binom{n-1}{p-1} \frac{(n-p)!}{\left(\frac{n-p}{k-1}\right)!} \frac{n^{\frac{n-p}{k-1}-1}}{[(k-1)!]^{\frac{n-p}{k-1}}} \left(\frac{k-1}{k}\right)^{p-1} \tag{4.149}$$

Remark that when $p = 1$ this formula is exact, indeed

$$u_{n,1}(\mathbf{e}_k) = \frac{(n-1)!}{\left(\frac{n-1}{k-1}\right)!} \frac{n^{\frac{n-1}{k-1}-1}}{[(k-1)!]^{\frac{n-1}{k-1}}} = \frac{t_{n,1}(\mathbf{e}_k)}{n} \tag{4.150}$$

is the number of unrooted hypertrees in n vertices, because of the general result (4.118d) and the explicit expression (4.88). In [48] this number is quoted as obtained in [50].

The formula (4.149) at $k = 2$ provides the result

$$u_{n,p}(\mathbf{e}_2) \simeq \binom{n-1}{p-1} \frac{n^{n-p-1}}{2^{p-1}} . \tag{4.151}$$

It follows that the partition function is, if λ is such that the relevant contribution to the sum comes from regions which don't change with n, a problem which we address in the next chapter, we get

$$\begin{aligned}\sum_{p=0}^{\infty} u_{n,p}(\mathbf{e}_2)\, \lambda^p &\sim n^{n-2}\, \lambda \sum_{p=0}^{n-1} \binom{n-1}{p} \left(\frac{\lambda}{2\,n}\right)^p = n^{n-2}\, \lambda \left(1 + \frac{\lambda}{2\,n}\right)^{n-1} \\ &\simeq n^{n-2}\, \lambda\, e^{\frac{\lambda}{2}}\end{aligned}$$

which at $\lambda = 1$ provides the well-known result by [51, 52].

More generally, by using the Stirling approximation for large factorials

$$u_{n,p}(\mathbf{e}_k) \simeq \frac{n^{n-2}}{e^{n\frac{k-2}{k-1}}} \frac{\sqrt{k-1}}{[(k-2)!]^{\frac{n-p}{k-1}}} \frac{1}{(p-1)!} \left(\frac{k-1}{k}\right)^{p-1} \tag{4.152}$$

while

$$\sum_{p=0}^{\infty} u_{n,p}(\mathbf{e}_k)\, \lambda^p \simeq \frac{n^{n-2}}{e^{n\frac{k-2}{k-1}}} \frac{\sqrt{k-1}}{[(k-2)!]^{\frac{n}{k-1}}}\, \lambda\, e^{\frac{k-1}{k}[(k-2)!]^{\frac{1}{k-1}}\lambda} \tag{4.153}$$

4.4.2 On the complete hypergraph

When $\mathbf{w} = \mathbf{1}$ (4.108) becomes

$$\exp\left[\lambda\,(1-y)\,(e^y - 1)\right] = \sum_{s\geq 0} \Pi_s(\lambda; \mathbf{1}) \frac{y^s}{s!} = \sum_{s\geq 0} \sum_{p\geq 0} \pi_{s,p}(\mathbf{1})\, \lambda^p \frac{y^s}{s!} \tag{4.154}$$

Now

$$\begin{aligned}\pi_{s,p}(\mathbf{1}) &= s!\, [y^s][\lambda^p] \exp\left[\lambda\,(1-y)\,(e^y-1)\right] && (4.155)\\ &= s!\, [y^s]\, (1-y)^p \frac{(e^y-1)^p}{p!} && (4.156)\\ &= s!\, [y^s] \sum_{m\geq 0} (-1)^m \binom{p}{m} y^m \sum_{q\geq 0} \left\{ {q+p \atop p} \right\} \frac{y^{q+p}}{(q+p)!} && (4.157)\\ &= \sum_{q\geq 0} (-1)^{s-p-q} \binom{p}{s-p-q} \left\{ {p+q \atop p} \right\} \frac{s!}{(p+q)!} && (4.158)\end{aligned}$$

so that the number of unrooted hyperforests with p hypertrees obtained by formula (4.113), by using the number of rooted hyperforests given in (4.104), is

$$
\begin{aligned}
&u_{n,p}(\mathbf{1}) = \\
&= \sum_{s\geq 1} \binom{n-1}{s-1} b_{n-s}(n) \sum_{q\geq 0} (-1)^{s-p-q} \binom{p}{s-p-q} \left\{ {p+q \atop p} \right\} \frac{s!}{(p+q)!} \\
&= \sum_{s\geq 1} \binom{n-1}{s-1} \sum_{r\geq 0} \left\{ {n-s \atop r} \right\} n^r \sum_{q\geq 0} (-1)^{s-p-q} \binom{p}{s-p-q} \left\{ {p+q \atop p} \right\} \frac{s!}{(p+q)!} \, .
\end{aligned}
\tag{4.159}
$$

Of course, because of the general result (4.118d),

$$
u_n(\mathbf{1}) = u_{n,1}(\mathbf{1}) = \frac{t_{n,1}(\mathbf{1})}{n} = \frac{b_{n-1}(n)}{n} = \sum_{r\geq 0} \left\{ {n-1 \atop r} \right\} n^{r-1} \tag{4.160}
$$

a sequence which is reported with the number A030019 in the *The On-Line Encyclopedia of Integer Sequences* by Sloane [46].

4.5 Application to the k-partite hypergraph

In this section we shall use the general formula (4.2) to compute the number of rooted hyperforests on the complete k-partite hypergraph. We will call k-partite an k-uniform hypergraph in which the vertices can be written as union of k mutually disjoint sets $V = V_1 \cup V_2 \cdots V_k$ such that for each hyperedge $E = (i_1, \ldots, i_k)$ we have $i_\alpha \in V_\alpha$. For simplicity in this section we will consider all edge weights equal to one (i.e $w_A = 1$ for all $A \in E$).

Consider the complete k-partite hypergraph with $n_\alpha = |V_\alpha|$ vertices in the subset V_α for $\alpha \in [k]$ and let $|V| = N = n_1 + \cdots + n_k$ be the total number of vertices. As usual we put on each vertex a pair of Grassmann fields $\bar\psi_{i,\alpha}$, $\psi_{i,\alpha}$ where the subscript means the i-th vertex of the set V_α.

The first step consists in simplifying the Boltzmann weight, to this purpose we define the following variables

$$
X_\alpha = \prod_{i=1}^{n_\alpha} \bar\psi_{i,\alpha}\psi_{i,\alpha} \qquad \Psi_\alpha = \sum_{i=1}^{n_\alpha} \psi_{i,\alpha} \qquad \bar\Psi_\alpha = \sum_{i=1}^{n_\alpha} \bar\psi_{i,\alpha} \, , \tag{4.161}
$$

so that the action on the complete graph can be written as follows

$$
\sum_A f_A^{(0)} = \sum_{\alpha=1}^k n_\alpha \prod_{\beta\neq\alpha} X_\beta - \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^k \bar\Psi_\alpha \Psi_\beta \prod_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^k X_\gamma \, . \tag{4.162}
$$

In the complete case, examined in the previous section, we made use of Lemma 3.1.4 to reduced the partition function to a function of the quadratic combination $\bar\psi\psi$, we now want to show that, in perfect analogy, in this case the partition function can be written in terms of X_α's only.

First, due to the nilpotency of Ψ_α and $\bar{\Psi}_\alpha$ we have

$$\exp\left(-\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{k} \bar{\Psi}_\alpha \Psi_\beta \prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{k} X_\gamma\right) = \prod_{\alpha=1}^{k}\left(1 - \bar{\Psi}_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{k} \Psi_\beta \prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{k} X_\gamma\right) \tag{4.163a}$$

$$= \sum_{H\subset[1,k]} (-1)^h \prod_{\alpha\in H}\left(\bar{\Psi}_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{k} \Psi_\beta \prod_{\substack{\gamma=1\\ \gamma\neq\beta,\alpha}}^{k} X_\gamma\right) \tag{4.163b}$$

where we have set $h = |H|$ for the cardinality of the generic subset H in $[k] = \{1, 2, \dots, k\}$.

Then, we observed that since in the remaining part of the integrand variables ψ and $\bar{\psi}$ are paired in the combinations X_α, the variables $\bar{\Psi}_\alpha\Psi_\beta$ have to form cycles to contribute to the integral, furthermore each β must belong the the same subset H. Let $\beta(\alpha)$ be the index of the Ψ following $\bar{\Psi}_\alpha$. Thus $\beta(\alpha)$ must be a permutation of H. Also, by the constraint $\beta(\alpha) \neq \alpha$, it cannot have fixed points, so it must be a derangement $\beta \in S^*(h)$. Hence, we get for the previous sum, under the integral sign, the following expression

$$\sum_{H\subset[k]} (-1)^h \sum_{\beta\in S^*(h)} \left(\prod_{\alpha\in H} \bar{\Psi}_\alpha \Psi_{\beta(\alpha)} \prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta(\alpha)}}^{k} X_\gamma\right) \tag{4.164a}$$

$$= \sum_{H\subset[k]} (-1)^h \sum_{\beta\in S^*(h)} (-1)^{\#\text{ even cycles}} \left(\prod_{\alpha\in H} X_\alpha^{h-1}\right)\left(\prod_{\alpha\in H^c} X_\alpha^{h}\right) \tag{4.164b}$$

$$= \sum_{H\subset[k]} (1-h) \left(\prod_{\alpha=1}^{k} X_\alpha^{h-\chi_H(\alpha)}\right) \tag{4.164c}$$

where

$$\chi_H(\alpha) = \begin{cases} 1 & \text{if } \alpha \in H; \\ 0 & \text{otherwise.} \end{cases}$$

The Grassmann integral is now reduced in the form we were looking for: a function only of the variables X_α. Now we only need to remember the previous Lemma 3.1.1 of Grassmann integration, which we report here in the current notations:

$$\int \left(\prod_{i=1}^{n_\alpha} \mathrm{d}\psi_{i,\alpha} \mathrm{d}\bar{\psi}_{i,\alpha}\right) X_\alpha^s \equiv \int \mathcal{D}_{n_\alpha} X_\alpha \; X_\alpha^s = n_\alpha!\; \delta(s, n_\alpha) \tag{4.165}$$

We now have all the ingredients to approach the computation of (4.2) in the complete

k-partite case:

$$K(\mathbf{t}) = \int \mathcal{D}_{n,\mathbf{t}}(\psi, \bar{\psi}) \, \exp\left(\sum_{A \in E} f_A^{(0)}\right)$$
$$= \sum_{H \subset [k]} (1-h) \int \left(\prod_{\alpha=1}^{k} \mathcal{D}_{n_\alpha} X_\alpha\right) \exp\left(\sum_{\alpha=1}^{k} t_\alpha X_\alpha + n_\alpha \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{k} X_\beta\right) \left(\prod_{\alpha=1}^{k} X_\alpha^{h-\chi_H(\alpha)}\right)$$

Expanding both parts of the exponential, we obtain

$$K(\mathbf{t}) = \sum_{H \subset [k]} (1-h) \sum_{\mathbf{r},\mathbf{m}} \int \left(\prod_{\alpha=1}^{k} \mathcal{D}_{n_\alpha} X_\alpha\right) \prod_{\alpha=1}^{k} \frac{t_\alpha^{r_\alpha}}{r_\alpha!} \frac{n_\alpha^{m_\alpha}}{m_\alpha!} X_\alpha^{r_\alpha + h - \chi_H(\alpha)} \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{k} X_\beta^{m_\alpha}$$
$$= \sum_{H \subset [k]} (1-h) \sum_{\mathbf{r},\mathbf{m}} \prod_{\alpha=1}^{k} \frac{t_\alpha^{r_\alpha}}{r_\alpha!} \frac{n_\alpha^{m_\alpha}}{m_\alpha!} \int \mathcal{D}_{n_\alpha} X_\alpha X_\alpha^{\sum_{\beta=1}^{k} m_\beta - m_\alpha + r_\alpha + h - \chi_H(\alpha)}$$
$$= \sum_{H \subset [k]} (1-h) \sum_{\mathbf{r},\mathbf{m}} \prod_{\alpha=1}^{k} \frac{t_\alpha^{r_\alpha}}{r_\alpha!} \frac{n_\alpha^{m_\alpha}}{m_\alpha!} n_\alpha! \, \delta\left(\sum_{\beta=1}^{k} m_\beta - m_\alpha + r_\alpha + h - \chi_H(\alpha), n_\alpha\right).$$

The constaints imposed by the delta terms can be solved

$$\sum_{\beta=1}^{k} m_\beta - m_\alpha + r_\alpha + h - \chi_H(\alpha) - n_\alpha = 0 \tag{4.166}$$

summing over α, and introducing the total number of roots, that is of connected components, $R = \sum_{\alpha=1}^{k} r_\alpha$, we get when $k-1$ divides $N-R$

$$(k-1) \sum_{\beta=1}^{k} m_\beta + (k-1)h + R = N \tag{4.167a}$$
$$\sum_{\beta=1}^{k} m_\beta = \frac{N-R}{k-1} - h \tag{4.167b}$$
$$m_\alpha = \frac{N-R}{k-1} - \chi_H(\alpha) + r_\alpha - n_\alpha \tag{4.167c}$$

so we have, introducing the number of hyperedges $E = \frac{N-R}{k-1}$, that

$$K(\mathbf{t}) = \sum_{H \subset [k]} (1-h) \sum_{\mathbf{r}} \prod_{\alpha=1}^{k} \frac{t_\alpha^{r_\alpha}}{r_\alpha!} \frac{n_\alpha^{E-\chi_H(\alpha)+r_\alpha-n_\alpha} n_\alpha!}{(E - \chi_H(\alpha) + r_\alpha - n_\alpha)!} \tag{4.168}$$
$$= \sum_{\mathbf{r}} \sum_{H \subset [k]} (1-h) \prod_{\alpha=1}^{k} \left[n_\alpha! \frac{n_\alpha^{E+r_\alpha-n_\alpha}}{(E+r_\alpha-n_\alpha)!} \frac{t_\alpha^{r_\alpha}}{r_\alpha!}\right] \left(\frac{E+r_\alpha-n_\alpha}{n_\alpha}\right)^{\chi_H(\alpha)} \tag{4.169}$$

The sum over H can be evaluated considering the following expression

$$\sum_{H\subset[k]}\prod_{\alpha=1}^{k} B_\alpha^{\chi_H(\alpha)} = \prod_{\alpha=1}^{k}\sum_{\chi=0,1} B_\alpha^{\chi} = \prod_{\alpha=1}^{k}(1+B_\alpha) \tag{4.170}$$

and

$$\sum_{H\subset[k]} h\prod_{\alpha=1}^{k} B_\alpha^{\chi_H(\alpha)} = \frac{d}{dx}\sum_{H\subset[k]} h\prod_{\alpha=1}^{k}(xB_\alpha)^{\chi_H(\alpha)}\Bigg|_{x=1} = \frac{d}{dx}\prod_{\alpha=1}^{k}(1+xB_\alpha)\Bigg|_{x=1} \tag{4.171}$$

$$= \left[\prod_{\alpha=1}^{k}(1+B_\alpha)\right]\sum_{\alpha=1}^{k}\frac{B_\alpha}{1+B_\alpha} \tag{4.172}$$

so that

$$\sum_{H\subset[k]}(1-h)\prod_{\alpha=1}^{k}\left(\frac{E+r_\alpha-n_\alpha}{n_\alpha}\right)^{\chi_H(\alpha)} = \left(\prod_{\alpha=1}^{k}\frac{E+r_\alpha}{n_\alpha}\right)\left[1-\sum_{\alpha=1}^{k}\left(1-\frac{n_\alpha}{E+r_\alpha}\right)\right]$$

$$= \left(\prod_{\alpha=1}^{k}\frac{E+r_\alpha}{n_\alpha}\right)\left[1-k+\sum_{\alpha=1}^{k}\frac{n_\alpha}{E+r_\alpha}\right]. \tag{4.173}$$

Therefore our result is

$$K(\mathbf{t}) = \sum_{\mathbf{r}}\prod_{\alpha=1}^{k}\left[\frac{(E+r_\alpha)n_\alpha^{E+r_\alpha-n_\alpha-1}}{(E+r_\alpha-n_\alpha)!}\frac{t_\alpha^{r_\alpha}}{r_\alpha!}\right]\left[\sum_{\beta=1}^{k}\frac{n_\beta}{E+r_\beta}-(k-1)\right] \tag{4.174}$$

First of all, let us remark that the term in the second square brackets can also be written as

$$\sum_{\beta=1}^{k}\frac{n_\beta}{E+r_\beta}-\frac{N-R}{E} = \sum_{\beta=1}^{k}\left(\frac{n_\beta}{E+r_\beta}-\frac{n_\beta-r_\beta}{E}\right) = \sum_{\beta=1}^{k}\frac{r_\beta(E+r_\beta-n_\beta)}{E(E+r_\beta)} \tag{4.175}$$

from which we see that all the terms in the sum are positive because we cannot have more roots than vertices in any subset V_β. There is a contribution in this sum from the subset V_β only when there is at least a root. Therefore when $r_\alpha = 0$ for all $\alpha\in[k]$ this sum vanishes and

$$K(\mathbf{0}) = 0$$

as it must be.

Consider now the case in which $r_\alpha \neq 0$ only for $\alpha = 1$, that is the roots are taken only in the first subset V_1, then $R = r_1 = N - E(k-1)$ and we have

$$\left(\frac{\partial}{\partial t_1}\right)^{r_1} K(\mathbf{t})\Bigg|_{\mathbf{t}=0} = \left[\prod_{\alpha=1}^{k} n_\alpha!\frac{E\,n_\alpha^{E-n_\alpha-1}}{(E-n_\alpha)!}\right] n_1^{r_1}\frac{E+r_1}{E}\frac{(E-n_1)!}{(E+r_1-n_1)!}\frac{r_1(E+r_1-n_1)}{E(E+r_1)}$$

$$= E^{k-2}\left[\prod_{\alpha=1}^{k} n_\alpha!\frac{n_\alpha^{E-n_\alpha-1}}{(E-n_\alpha)!}\right]\frac{(E-n_1)!}{(E+r_1-n_1-1)!}r_1 n_1^{r_1} \tag{4.176}$$

In particular when $r_1 = 1$, so that in the hyperforest there is only one connected component, that is the hypergraph is a hypertree, the number of *unrooted* hypertrees is obtained by dividing for the number of possible roots in V_1 which is n_1, so that it is

$$\frac{1}{n_1} \left.\frac{\partial K(\mathbf{t})}{\partial t_1}\right|_{\mathbf{t}=0} = E^{k-2} \prod_{\alpha=1}^{k} n_\alpha! \frac{n_\alpha^{E-n_\alpha-1}}{(E-n_\alpha)!} \tag{4.177}$$

where the right and side does not see any difference for the different indices α. In the special case of graphs, that is for $k = 2$, when $E = n_1 + n_2 - 1$ the previous expression reduces to

$$\frac{1}{n_1} \left.\frac{\partial K(\mathbf{t})}{\partial t_1}\right|_{\mathbf{t}=0} = \frac{1}{n_2} \left.\frac{\partial K(\mathbf{t})}{\partial t_2}\right|_{\mathbf{t}=0} = \prod_{\alpha=1}^{2} n_\alpha^{E-n_\alpha} = n_1^{n_2-1} n_2^{n_1-1}\,. \tag{4.178}$$

Summary

- We have studied the generating function of both rooted and unrooted hyperforests in the complete hypergraph with n vertices, when the weight of each hyperedge depends only on its cardinality. The results could also be obtained by starting from recursion relations in the number of vertices, and solving the obtained implicit relations by using the Lagrange inversion formula. However we showed here how the same problem can be directly and more easily solved by means of the Grassmann representation developed in the previous chapter.

- For the number of rooted hyperforests with r hypertrees on the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$, we have obtained (4.89), when $n-r$ can be divided by $k-1$ and $n_E = \frac{n-r}{k-1}$:

$$t_{n,r}(\mathbf{e}_k) = \binom{(k-1)n_E + r - 1}{r-1} \frac{[(k-1)n_E]!}{n_E!\,[(k-1)!]^{n_E}} [(k-1)n_E + r]^{n_E}$$

While on the complete hypergraph we obtained:

$$t_{n,r}(\mathbf{1}) = \binom{n-1}{r-1} b_{n-r}(n) = \binom{n-1}{r-1} \sum_{n_E \geq 0} n^{n_E} \left\{ {n-r \atop n_E} \right\}$$

- In the case of unrooted hyperforests we also recovered a novel explicit expression for their number with p connected components. On the k-uniform complete hypergraph we obtained

$$\begin{aligned} u_{n,p}(\mathbf{e}_k) &= \frac{(n-1)!}{p!} \left[\frac{n}{(k-1)!}\right]^{\frac{n-p}{k-1}} \sum_{q=0}^{p} \binom{p}{q} \frac{p+(k-1)q}{\left(\frac{n-p}{k-1} - q\right)!} \left(\frac{1-k}{kn}\right)^q \\ &\simeq \binom{n-1}{p-1} \frac{(n-p)!}{\left(\frac{n-p}{k-1}\right)!} \frac{n^{\frac{n-p}{k-1}-1}}{[(k-1)!]^{\frac{n-p}{k-1}}} \left(\frac{k-1}{k}\right)^{p-1} \end{aligned}$$

While on the complete hypergraph we obtained

$$u_{n,p}(\mathbf{1}) = \sum_{s\geq 1} \binom{n-1}{s-1} \sum_{r \geq 0} \left\{ {n-s \atop r} \right\} n^r \sum_{q\geq 0} (-1)^{s-p-q} \binom{p}{s-p-q} \left\{ {p+q \atop p} \right\} \frac{s!}{(p+q)!}$$

- Finally we showed that the same technique is straightforward to apply to different scenarios. Considering the complete $k-partite$ hypergraph, that is the $k-$uniform hypergraph in which the vertices can be written as union of k mutually disjoint sets $V = V_1 \cup V_2 \cdots V_k$ such that for each hyperedge $A = (i_1, \ldots, i_k)$ we have $i_\alpha \in V_\alpha$, we obtained the number of unrooted hypertrees:

$$\frac{1}{n_1} \frac{\partial K(\mathbf{t})}{\partial t_1}\bigg|_{\mathbf{t}=0} = E^{k-2} \prod_{\alpha=1}^{k} n_\alpha! \frac{n_\alpha^{E-n_\alpha-1}}{(E-n_\alpha)!}$$

where $E = (N-1)/(k-1)$ is the number of hyperedges.

Chapter 5

The role of OSP(1|2) symmetry

In the previous chapters we saw how the partition function of unrooted spanning forests can be represented as a Berezin-Grassmann integral over anti-commuting variables. In this chapter we will show how this formulation emerges naturally when considering a theory with bosons and fermions taking values in the unit supersphere in $\mathrm{R}^{1|2}$ when the action is quadratic and invariant under rotations in OSP(1|2).

We begin by introducing, at each vertex $i \in V$, a superfield $\mathbf{n}_i = (\sigma_i, \psi_i, \bar{\psi}_i)$ consisting of a bosonic (i.e., real) variable σ_i and a pair of Grassmann variables $\psi_i, \bar{\psi}_i$. We equip the "superspace" $\mathrm{R}^{1|2}$ with the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i \sigma_j \;+\; \lambda(\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j) \;, \tag{5.1}$$

where $\lambda \neq 0$ is an arbitrary real parameter.

The infinitesimal rotations in $\mathrm{R}^{1|2}$ that leave invariant the scalar product (5.1) form the Lie superalgebra osp(1|2) [53, 54, 55]. This algebra is generated by two types of transformations: Firstly, we have the elements of the sp(2) subalgebra, which act on the field as $\mathbf{n}'_i = \mathbf{n}_i + \delta \mathbf{n}_i$ with

$$\delta\sigma_i = 0 \tag{5.2a}$$
$$\delta\psi_i = -\,\alpha\,\psi_i + \gamma\,\bar{\psi}_i \tag{5.2b}$$
$$\delta\bar{\psi}_i = +\,\alpha\,\bar{\psi}_i + \beta\,\psi_i \tag{5.2c}$$

where α, β, γ are bosonic (Grassmann-even) global parameters; it is easily checked that these transformations leave (5.1) invariant. Secondly, we have the transformations parametrized by fermionic (Grassmann-odd) global parameters $\epsilon, \bar{\epsilon}$:

$$\delta\sigma_i = -\lambda^{1/2}(\bar{\epsilon}\psi_i + \bar{\psi}_i \epsilon) \tag{5.3a}$$
$$\delta\psi_i = \lambda^{-1/2}\,\epsilon\,\sigma_i \tag{5.3b}$$
$$\delta\bar{\psi}_i = \lambda^{-1/2}\,\bar{\epsilon}\,\sigma_i \tag{5.3c}$$

(Here an overall factor $\lambda^{-1/2}$ has been extracted from the fermionic parameters for future convenience.) To check that these transformations leave (5.1) invariant, we compute

$$\begin{aligned}
\delta(\mathbf{n}_i \cdot \mathbf{n}_j) &= (\delta\sigma_i)\sigma_j + \sigma_i(\delta\sigma_j) + \lambda\big[(\delta\bar{\psi}_i)\psi_j + \bar{\psi}_i(\delta\psi_j) - (\delta\psi_i)\bar{\psi}_j - \psi_i(\delta\bar{\psi}_j)\big] \\
&= -\lambda^{1/2}(\bar{\epsilon}\psi_i + \bar{\psi}_i\epsilon)\sigma_j - \lambda^{1/2}(\bar{\epsilon}\psi_j + \bar{\psi}_j\epsilon)\sigma_i + \lambda^{1/2}\big[\bar{\epsilon}\psi_j\sigma_i + \bar{\psi}_i\epsilon\sigma_j - \epsilon\bar{\psi}_j\sigma_i - \psi_i\bar{\epsilon}\sigma_j\big] \\
&= 0\,.
\end{aligned}$$

In terms of the differential operators $\partial_i = \partial/\partial\psi_i$ and $\bar{\partial}_i = \partial/\partial\bar{\psi}_i$, the transformations (5.2c) can be represented by the generators

$$X_0 = \sum_{i\in V} (\bar{\psi}_i \bar{\partial}_i - \psi_i \partial_i) \tag{5.4a}$$

$$X_+ = \sum_{i\in V} \bar{\psi}_i \partial_i \tag{5.4b}$$

$$X_- = \sum_{i\in V} \psi_i \bar{\partial}_i \tag{5.4c}$$

corresponding to the parameters α, β, γ, respectively, while the transformations (5.3c) can be represented by the generators

$$Q_+ = \lambda^{-1/2} \sum_{i\in V} \sigma_i \partial_i \, + \, \lambda^{1/2} \sum_{i\in V} \bar{\psi}_i \frac{\partial}{\partial\sigma_i} \tag{5.5a}$$

$$Q_- = \lambda^{-1/2} \sum_{i\in V} \sigma_i \bar{\partial}_i \, - \, \lambda^{1/2} \sum_{i\in V} \psi_i \frac{\partial}{\partial\sigma_i} \tag{5.5b}$$

corresponding to the parameters $\epsilon, \bar{\epsilon}$, respectively. (With respect to the notations of [55] we have $X_\pm = L_\mp$, $X_0 = -2L_0$ and $Q_\pm = \mp 2iR_\mp$.) These transformations satisfy the commutation/anticommutation relations

$$[X_0, X_\pm] = \pm 2X_\pm \qquad\qquad [X_+, X_-] = X_0 \tag{5.6a}$$

$$\{Q_\pm, Q_\pm\} = \pm 2X_\pm \qquad\qquad \{Q_+, Q_-\} = X_0 \tag{5.6b}$$

$$[X_0, Q_\pm] = \pm Q_\pm \qquad [X_\pm, Q_\pm] = 0 \qquad [X_\pm, Q_\mp] = -Q_\pm \tag{5.6c}$$

Note in particular that $X_\pm = Q_\pm^2$ and $X_0 = Q_+Q_- + Q_-Q_+$. It follows that any element of the Grassmann algebra that is annihilated by $Q_\pm$ is also annihilated by the entire osp(1|2) algebra.

Now let us consider a σ-model in which the superfields $\mathbf{n}_i$ are constrained to lie on the unit supersphere in $\mathrm{R}^{1|2}$, i.e. to satisfy the constraint

$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \sigma_i^2 + 2\lambda\bar{\psi}_i\psi_i \;=\; 1\,. \tag{5.7}$$

We can *solve* this constraint by writing

$$\sigma_i \;=\; \pm(1 - 2\lambda\bar{\psi}_i\psi_i)^{1/2} \;=\; \pm(1 - \lambda\bar{\psi}_i\psi_i)\,, \tag{5.8}$$

exploiting the fact that $\psi_i^2 = \bar{\psi}_i^2 = 0$. Let us henceforth take only the + sign in (5.8), neglecting the other solution, so that

$$\sigma_i = 1 - \lambda\bar{\psi}_i\psi_i\,. \tag{5.9}$$

We then have a purely fermionic model with variables $\psi, \bar{\psi}$ in which the sp(2) transformations continue to act as in (5.2c) while the fermionic transformations act via the "hidden" supersymmetry

$$\delta\psi_i = \lambda^{-1/2}\,\epsilon\,(1 - \lambda\bar{\psi}_i\psi_i) \tag{5.10a}$$

$$\delta\bar{\psi}_i = \lambda^{-1/2}\,\bar{\epsilon}\,(1 - \lambda\bar{\psi}_i\psi_i) \tag{5.10b}$$

All of these transformations leave invariant the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; 1 \;-\; \lambda(\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \;+\; \lambda^2 \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \;. \tag{5.11}$$

The generators $Q_\pm$ are now defined as

$$Q_+ \;=\; \lambda^{-1/2} \sum_{i \in V} (1 - \lambda \bar{\psi}_i \psi_i) \partial_i \;=\; \lambda^{-1/2} \partial \;-\; \lambda^{1/2} \sum_{i \in V} \bar{\psi}_i \psi_i \partial_i \tag{5.12a}$$

$$Q_- \;=\; \lambda^{-1/2} \sum_{i \in V} (1 - \lambda \bar{\psi}_i \psi_i) \bar{\partial}_i \;=\; \lambda^{-1/2} \bar{\partial} \;-\; \lambda^{1/2} \sum_{i \in V} \bar{\psi}_i \psi_i \bar{\partial}_i \tag{5.12b}$$

where we recall the notations $\partial = \sum_{i \in V} \partial_i$ and $\bar{\partial} = \sum_{i \in V} \bar{\partial}_i$.

Let us now show that the polynomials $f_A^{(\lambda)}$ defined as in (3.24) are OSP(1|2)-invariant, i.e. are annihilated by all elements of the osp(1|2) algebra. As noted previously, it suffices to show that the $f_A^{(\lambda)}$ are annihilated by $Q_\pm$. Applying the definitions (5.12), we have

$$Q_- \tau_A \;=\; \lambda^{-1/2} \bar{\partial} \tau_A \tag{5.13}$$

and hence

$$Q_+ Q_- \tau_A \;=\; \lambda^{-1} \partial \bar{\partial} \tau_A \;-\; |A| \tau_A \;, \tag{5.14}$$

so that

$$f_A^{(\lambda)} \;=\; \lambda (1 + Q_+ Q_-) \tau_A \;. \tag{5.15}$$

The next step is to compute $Q_+ f_A^{(\lambda)}$: since

$$\begin{aligned} Q_+(1 + Q_+ Q_-) \;&=\; Q_+ + Q_+^2 Q_- \;=\; Q_+ + X_+ Q_- \\ &=\; Q_+ + [X_+, Q_-] + Q_- X_+ \;=\; Q_+ - Q_+ + Q_- X_+ \;=\; Q_- X_+ \end{aligned} \tag{5.16}$$

by the relations (5.6), while it is obvious that $X_+ \tau_A = 0$, we conclude that $Q_+ f_A^{(\lambda)} = 0$, i.e. $f_A^{(\lambda)}$ is invariant under the transformation Q_+. A similar calculation of course works for Q_-.

In fact, the OSP(1|2)-invariance of $f_A^{(\lambda)}$ can be proven in a simpler way by writing $f_A^{(\lambda)}$ explicitly in terms of the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j \in A$. Note first that

$$f_{\{i,j\}}^{(\lambda)} \;=\; -\lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \;+\; (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \tag{5.17a}$$

$$=\; \frac{1}{\lambda} (1 - \mathbf{n}_i \cdot \mathbf{n}_j) \tag{5.17b}$$

$$=\; \frac{(\mathbf{n}_i - \mathbf{n}_j)^2}{2\lambda} \;. \tag{5.17c}$$

By Corollary 3.2.3, we obtain

$$f_{\{i_1, i_2, \ldots, i_k\}}^{(\lambda)} \;=\; \frac{1}{\lambda^{k-1}} (1 - \mathbf{n}_{i_1} \cdot \mathbf{n}_{i_2})(1 - \mathbf{n}_{i_2} \cdot \mathbf{n}_{i_3}) \cdots (1 - \mathbf{n}_{i_{k-1}} \cdot \mathbf{n}_{i_k}) \tag{5.18a}$$

$$=\; \frac{1}{(2\lambda)^{k-1}} (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 (\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \cdots (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2 \;. \tag{5.18b}$$

Note the striking fact that the right-hand side of (5.18) is invariant under all permutations of $i_1, \ldots, i_k$, though this fact is not obvious from the formulae given, and is indeed false for vectors in Euclidean space R^N with $N \neq -1$. Moreover, the path $i_1, \ldots, i_k$ that is implicit in the right-hand side of (5.18) could be replaced by *any tree* on the vertex set $\{i_1, \ldots, i_k\}$, and the result would again be the same (by Corollary 3.2.3).

It follows from (5.17)/(5.18) that the subalgebra generated by the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j \in V$ is identical with the subalgebra generated by the $f_A^{(\lambda)}$ for $A \subseteq V$, for any $\lambda \neq 0$. Therefore, the *most general* OSP(1|2)-symmetric Hamiltonian depending on the $\{\mathbf{n}_i\}_{i \in V}$ is precisely the one in which the action contains all possible products $f_{\mathcal{C}}^{(\lambda)} = \prod_{\alpha} f_{C_\alpha}^{(\lambda)}$, where $\{C_\alpha\}$ is a partition of V.

Furthermore, in [7] it has been shown that $f^{(\lambda)}_{\{i_1, i_2, \ldots, i_k\}}$ can be written as:

$$f^{(\lambda)}_{\{i_1, i_2, \ldots, i_k\}} \;=\; \frac{1}{k!\, \lambda^{k-1}} \det M \tag{5.19}$$

where M is the $k \times k$ matrix of scalar products $M_{rs} = \mathbf{n}_{i_r} \cdot \mathbf{n}_{i_s}$. In this formula, unlike (5.18), the symmetry under all permutations of $i_1, \ldots, i_k$ is manifest. We remark that the determinant of a matrix of inner products is commonly called a *Gram determinant* [56, p. 110].

Finally, we need to consider the behavior of the integration measure in (3.45), namely

$$\mathcal{D}_{V,\mathbf{t}}(\psi, \bar\psi) = \prod_{i \in V} \mathrm{d}\psi_i \, \mathrm{d}\bar\psi_i \, e^{t_i \bar\psi_i \psi_i} \,, \tag{5.20}$$

under the supersymmetry (5.10). In general this measure is *not* invariant under (5.10), but in the special case $t_i = \lambda$ for all i, it *is* invariant, in the sense that

$$\int \mathcal{D}_{V,\lambda}(\psi, \bar\psi) \, \delta F(\psi, \bar\psi) \;=\; 0 \tag{5.21}$$

for any function $F(\psi, \bar\psi)$. Indeed, $\mathcal{D}_{V,\lambda}(\psi, \bar\psi)$ is invariant more generally under *local* supersymmetry transformations in which separate generators $\epsilon_i, \bar\epsilon_i$ are used at each vertex i. To see this, let us focus on one site i and write $F(\psi, \bar\psi) = a + b\psi_i + c\bar\psi_i + d\bar\psi_i\psi_i$ where a, b, c, d are polynomials in the $\{\psi_j, \bar\psi_j\}_{j \neq i}$ (which may contain both Grassmann-even and Grassmann-odd terms). Then

$$\delta F \;=\; \lambda^{1/2} \left[b\epsilon_i\sigma_i \,+\, c\bar\epsilon_i\sigma_i \,+\, d\left(\bar\epsilon_i\sigma_i\psi_i + \bar\psi_i\epsilon_i\sigma_i\right)\right] \tag{5.22a}$$

$$\;=\; \sigma_i\, \lambda^{1/2} \left[b\epsilon_i \,+\, c\bar\epsilon_i \,+\, d(\bar\epsilon_i\psi_i + \bar\psi_i\epsilon_i)\right] . \tag{5.22b}$$

Since $\sigma_i = e^{-\lambda\bar\psi_i\psi_i}$, this cancels the factor $e^{t_i\bar\psi_i\psi_i}$ from the measure (since $t_i = \lambda$) and the integral over $d\psi_i\, d\bar\psi_i$ is zero (because there are no $\bar\psi_i\psi_i$ monomials). Thus, the measure $\mathcal{D}_{V,\mathbf{t}}(\psi, \bar\psi)$ is invariant under the local supersymmetry at site i whenever $t_i = \lambda$. If this occurs for all i, then the measure is invariant under the global supersymmetry (5.3c).

The OSP(1|2)-invariance of $\mathcal{D}_{V,\lambda}(\psi, \bar\psi)$ can be seen more easily by writing the manifestly invariant combination

$$\delta(\mathbf{n}_i^2 - 1)\, \mathrm{d}\mathbf{n}_i \;=\; \delta(\sigma_i^2 + 2\lambda\bar\psi_i\psi_i - 1)\, \mathrm{d}\sigma_i\, \mathrm{d}\psi_i\, \mathrm{d}\bar\psi_i \tag{5.23a}$$

$$\;=\; e^{\lambda\bar\psi_i\psi_i}\, \delta\big(\sigma_i - (1 - \lambda\bar\psi_i\psi_i)\big)\, \mathrm{d}\sigma_i\, \mathrm{d}\psi_i\, \mathrm{d}\bar\psi_i \,, \tag{5.23b}$$

where the factor $e^{\lambda\bar{\psi}_i\psi_i}$ comes from the inverse Jacobian. Integrating out σ_i from (5.23b), we obtain $e^{\lambda\bar{\psi}_i\psi_i}\,\mathrm{d}\psi_i\,\mathrm{d}\bar{\psi}_i$.

As a consequence of (5.18b) and (5.23b), the partition function (3.65) for spanning hyperforests in a k-uniform hypergraph can be rewritten as a many-body non-linear σ-model

$$F(\lambda,\mathbf{w}) = \int \left(\prod_{i\in V}\delta(\mathbf{n}_i^2-1)\,\mathrm{d}\mathbf{n}_i\right) \exp\left[\frac{1}{(2\lambda)^{k-1}}\sum_{i_1,\ldots,i_k\in V}\frac{L_{i_1,\ldots,i_k}}{(k-2)!}(\mathbf{n}_{i_1}-\mathbf{n}_{i_2})^2(\mathbf{n}_{i_2}-\mathbf{n}_{i_3})^2\cdots(\mathbf{n}_{i_{k-1}}-\mathbf{n}_{i_k})^2\right]. \qquad (5.24)$$

In the special case $k=2$ (already discussed in [2]), we have simply

$$F(\lambda,\mathbf{w}) = \int \left(\prod_{i\in V}\delta(\mathbf{n}_i^2-1)\,\mathrm{d}\mathbf{n}_i\right)\exp\left[\frac{1}{2\lambda}\sum_{i,j\in V}L_{ij}(\mathbf{n}_i-\mathbf{n}_j)^2\right]. \qquad (5.25)$$

As a last, but nevertheless important, remark it is worth mentioning that the above correspondence, while valid at all orders of perturbation theory, does not hold non-perturbatively. The error arises from neglecting the second square root when solving the constraint (5.8); we did not, in fact, parametrize a (super)sphere but rather a (super)-hemisphere. Indeed, since $\lambda > 0$ corresponds to an antiferromagnetic model, the terms we have neglected are actually dominant.

Summary

- The quantities $f_A^{(\lambda)}$ present a non-linearly realized OSP(1|2) symmetry.

  The group OSP(1|2) is the group of transformations acting on the superspace $\mathbf{n} = (\sigma, \bar{\psi}, \psi) \in \mathbb{R}^{1|2}$ made of one bosonic and two fermionic variables, that leave invariant the scalar product
$$\mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i \sigma_j \; + \; \lambda(\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j) \, ,$$
  which are generated by
$$\begin{aligned} \delta\sigma_i &= -\lambda^{1/2}(\bar{\epsilon}\psi_i + \bar{\psi}_i \epsilon) \\ \delta\psi_i &= \lambda^{-1/2} \, \epsilon \, \sigma_i \\ \delta\bar{\psi}_i &= \lambda^{-1/2} \, \bar{\epsilon} \, \sigma_i \end{aligned}$$

- Thanks to Grassmann variables nilpotency, the constraint $\mathbf{n}_i^2 = \sigma_i^2 + 2\bar{\psi}_i \psi_i = 1$ can be solved exactly (neglecting the "$-$" solution)
$$\sigma_i = 1 - \lambda \bar{\psi}_i \psi_i$$
  and we are left with the following purely fermionic symmetry
$$\begin{aligned} \delta\psi_i &= \lambda^{-1/2} \, \epsilon \, (1 - \lambda\bar{\psi}_i \psi_i) \\ \delta\bar{\psi}_i &= \lambda^{-1/2} \, \bar{\epsilon} \, (1 - \lambda\bar{\psi}_i \psi_i) \end{aligned}$$

- f_A's can indeed written as combination of scalar products on $\mathbb{R}^{1|2}$ making their OSP(1|2) symmetry manifest
$$f^{(\lambda)}_{\{i_1, i_2, \ldots, i_k\}} = \frac{1}{(2\lambda)^{k-1}} \, (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 \, (\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \, \cdots \, (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2 \, .$$

- Furthermore, at $t_i = \lambda$ the integration measure $\mathcal{D}_{V,\lambda}(\psi, \bar{\psi})$ is also invariant and can be itself written as
$$\mathcal{D}_{V,\lambda}(\psi, \bar{\psi}) = \prod_{i \in V} \mathrm{d}\psi_i \, \mathrm{d}\bar{\psi}_i \, e^{\lambda \bar{\psi}_i \psi_i} = \prod_{i \in V} \delta(\mathbf{n}_i^2 - 1) \, \mathrm{d}\mathbf{n}_i \, ,$$

- The spanning forest model is therefore equivalent to an OSP(1|2) non-linear σ-model, even though the correspondence is only perturbative. In fact, if $\lambda > 0$ the resulting σ-model is antiferromagnetic and the neglected "-"solution becomes relevant.

Chapter 6

The phase transition

Introduction

In two dimensions, the critical behaviour of the ferromagnetic Potts/random-cluster model is quite well understood, thanks to a combination of exact solutions [57], Coulomb gas methods [58], and conformal field theory [59]. Information can also be deduced from the study of the model on random planar lattices [60, 61, 62]. Also in the $q \to 0$ limit detailed results are avalaible both for the tree model, in particular in connection with the abelian sandpile model [63], as for spanning forest on a regular lattice [2, 6] and directly in the continuum [64]. Also the model on random planar lattices has been considered [65].

But in more than two dimensions the only quantitative informations we have about the spanning-forest model come from numerical investigations [66]. Monte Carlo simulations performed at increasing dimensionality ($d = 3, 4, 5$) show a second-order phase transition.

Much less results are available for the case of hyperforests. Also in two dimensions or in the limit of hypertrees. Even the problem of determining whether there exists a spanning hypertree in a given k-uniform hypergraph, is hard, technically NP-complete, for $k \geq 4$, whereas for $k = 3$, there exists a polynomial-time algorithm based on Lovasz' theory of polymatroid matching [67]. See [68] for a randomized polynomial-time algorithm in the case $k = 3$ whose main ingredients is a Pfaffian formula for a polynomial that enumerates spanning hypertrees with some signs [69], which is quite similar to our Grassmann representation [70].

In [71] a phase transition is detected in the random k-uniform hypergraph when a number of hyperedges $|E| = n/k(k-1)$ of the total number of vertices $n = |V|$ is chosen uniformly at random. In the case of random graphs, that is for $k = 2$, Erdös and Rényi showed in their classical paper [25] that at the transition an abrupt change occurs in the structure of the graph, for low density of edges it consists of many small components, while, in the high-density regime a *giant* component occupies a finite fraction of the vertices. Remark that their ensemble of subgraphs is the one occurring in the microcanonical formulation, at fixed number of edges, of the Potts model at number of states $q = 1$. The connected-component structure of the random k-uniform hypergraph has been analyzed in [71] where it has been shown that if $|E| < n/k(k-1)$ the largest component occupies order $\log n$ of vertices, for $|E| = n/k(k-1)$ it has order $n^{2/3}$ vertices and for $|E| > n/k(k-1)$ there is a unique component with order n vertices. More detailed information on the behaviour near the phase transition when $|E| \to n/k(k-1)$ have been recovered in [72, 73] for the case of the random graph, but see also [74, 75], and in [48] for the general case of hypergraphs.

By using the Grassmann representation described in Chapter 3, we present here a study of the phase transition for the hyperforest model on the k-uniform complete hypergraph, for general k, where the case $k = 2$ corresponds to spanning forests on the complete graph. The random-cluster model [49] on the complete graph has already been developed but it cannot be extended to the $q \to 0$ case, exactly like the mean-field solution for the Potts model [76, 77]. The fermionic representation, instead, describes the Potts model directly at $q = 0$ as it provides an exact representation of the partition function of the spanning-hyperforest model.

In Chapter 4 we already saw that, as usual with models on the complete graph, the statistical weight reduces to a function of only one extensive observable, which here is quadratic in the Grassmann variables and, under such a condition, the partition function can be expressed as the integration over a single complex variable in a closed contour around the origin.

Indeed, counting the spanning forests over a complete (hyper-)graph is indeed a typical problem of *analytical combinatorics*. And, exactly like in the case of ordinary graph, when the number of connected components in the spanning forests is macroscopic, that is a finite fraction of the number of vertices, there are two different regimes, which can be well understood by means of two different saddle points of a closed contour integration over a single complex variable as presented in [79] (but see also the probabilistic analysis in [78]). And even the behaviour at the critical point can be studied as the coalescence of these two saddle points.

In this chapter we shall first review how it is possible to recover, in the case of the k-uniform complete hypergraphs, a representation of the partition function suitable for the asymptotic analysis for large number of vertices n by working a micro-canonical ensemble (i.e at fixed number of connected components). Thereafter we shall present, in Section 6.2, a full discussion of the saddle points in the micro-canonical ensemble and of the associated different phases. We shall see that the universality class of the transition is independent from k. The relation with the canonical ensemble is discussed in Section 6.3.

In Section 6.4 we will provide an interpretation of the transition as the appearance of a *giant* component by introducing a suitable observable which is sensible to the size of the different hypertrees in the hyperforest.

More interestingly, our Grassmann formulation exhibits a global continuos supersymmetry, non-linearly realized. We shall show that the phase transition is associated to the spontaneous breaking of this supersymmetry. By the introduction of an explicit breaking we shall be able to investigate the expectation values in the broken pure thermodynamical states. We shall therefore be able to see in Section 6.5 that the phase transition is of second order. This seems at variance with the supersymmetric formulation of polymers given by Parisi and Sourlas [4] where it appeared to be of zeroth order.

In this Chapter for simplicity we will consider only the case in which $\mathbf{w} = \mathbf{e}_k$, i.e only edges of cardinality k are present and they have weight one.

6.1 The microcanonical ensemble

In the previous chapter we saw that, on the complete hypergraph, the partition function of unrooted spanning forests admits the representation (4.52), which we recall here In this chapter we shall set the edge weights to $\mathbf{w} = \mathbf{e}_k$ as already done in Section 4.4.1 and we shall adhere to a more "physical" notation that emphasizes the role of λ naming $\mathcal{Z}(\lambda) = F_n(\lambda, \mathbf{e}_k)$

$$\mathcal{Z}(\lambda) = \int \mathcal{D}_n(\psi, \bar{\psi}) \left[1 - \frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] \exp\left[\lambda \left(\bar{\psi}\psi + (1-k)\frac{(\bar{\psi}\psi)^k}{k!}\right) + n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right].$$

With the definition

$$\mathcal{U} = \sum_{i \in V} \bar{\psi}_i \psi_i + (1-k) \sum_{A:|A|=k} \tau_A = \bar{\psi}\psi + (1-k)\frac{(\bar{\psi}\psi)^k}{k!} \tag{6.1}$$

what above can be written more shortly

$$\mathcal{Z}(\lambda) = \int \mathcal{D}_n(\bar{\psi}, \psi) \left[1 - \frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] \exp\left[t\,\mathcal{U} + n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right]. \tag{6.2}$$

In order to perform an estimate for the asymptotic value of the integral for large n we recall that, due to Corollary 3.1.2, for an analytic function f of the internal product $\bar{\psi}\psi$ we have

$$\int \mathcal{D}_n(\psi, \bar{\psi})\ f(\bar{\psi}\psi) = \frac{n!}{2\pi i} \oint \frac{d\xi}{\xi^{n+1}}\ f(\xi)\,, \tag{6.3}$$

where the integration contour in the complex plane is around the origin. We have the following complex integral representation form for the partition function

$$\mathcal{Z}(\lambda) = n! \oint \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \exp\left\{\lambda\left[\xi + (1-k)\frac{\xi^k}{k!}\right] + n\frac{\xi^{k-1}}{(k-1)!}\right\}\left[1 - \frac{\xi^{k-1}}{(k-2)!}\right]. \tag{6.4}$$

Let us first work at fixed number of hypertrees, in a micro–canonical ensemble in the physics terminology.

Expanding (6.4) in powers of λ we obtain the number $\mathcal{Z}_p$ of spanning hyperforests on the complete k-uniform hypergraph which is the number of states in the micro-canonical ensamble

$$\mathcal{Z}(\lambda) = \sum_p \mathcal{Z}_p \lambda^p \tag{6.5}$$

$$\mathcal{Z}_p = \frac{n!}{p!} \oint \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \left[\xi + (1-k)\frac{\xi^k}{k!}\right]^p \exp\left\{n\frac{\xi^{k-1}}{(k-1)!}\right\}\left[1 - \frac{\xi^{k-1}}{(k-2)!}\right]. \tag{6.6}$$

Since we are interested in obtaining $\mathcal{Z}_p$ in the thermodynamical limit $n \to \infty$ also for large values of p, we define $p = \alpha n$ with fixed α as $n, p \to \infty$. Changing the variable of integration to $\eta = (k-1)\frac{\xi^{k-1}}{k!}$, we obtain the following integral expression:

$$\mathcal{Z}_{\alpha n} = \frac{n!}{\Gamma(\alpha n + 1)} \left[\frac{k-1}{k!}\right]^{n\frac{1-\alpha}{k-1}} I(\alpha) \tag{6.7}$$

where

$$I(\alpha) = \oint \frac{\mathrm{d}\eta}{2\pi i}\, A(\eta)\, e^{nB(\eta)} \tag{6.8}$$

with

$$A(\eta) = \frac{1-k\eta}{\eta} \tag{6.9}$$

$$B(\eta) = \frac{k}{k-1}\eta + \alpha\log(1-\eta) + \frac{\alpha-1}{k-1}\log\eta. \tag{6.10}$$

Please note that the factor $k-1$ coming from the change of variable in the integral is exactly compensated by the fact that a full turn around the origin in the η plane is equivalent to $k-1$ turns of the ξ variable.

Precise estimates of integrals of this kind for $n \to \infty$ can be obtained by the saddle point method (see [79] for a very complete discussion of this method).

6.2 The saddle point method

A *saddle point* of a function $B(\eta)$ is a point η_0 where $B'(\eta_0) = 0$, it is said to be a *simple saddle point* if furthermore $B''(\eta_0) \neq 0$. In this case it is easy to see that the equilevel lines divide a neighborhood of η_0 in four regions where $\Re B(\eta)$ is alternately higher and lower than the saddle point value $\Re B(\eta_0)$. We will refer to the two lower regions as the *valleys*.

Analogously, a *multiple saddle point* has multiplicity p if all derivatives up to $B^{(p)}(\eta_0)$ are equal to zero while $B^{(p+1)}(\eta_0) \neq 0$. In this case there are $p+1$ higher and lower regions.

When evaluating Cauchy contour integrals of the form (6.8), saddle points of $B(\eta)$ play a central role in the asymptotic estimate for large n. The method essentially consists of two basic ingredients: an accurate choice of the contour and Laplace's method for the evaluation of integrals depending on a large parameter.

The contour has to be chosen to pass through a point which is a global maximum of the integrand along the contour and that a neighborhood of which (the *central region*) dominates the rest of the contour (the *tails*) as n grows. Since an analytic function cannot have an isolated maximum, this implies that the contour should pass through a saddle point.

The existence of a contour surrounding the origin and that crosses a saddle point along its direction of steepest descent requires that two of its valleys are topologically connected and the region connecting them surrounds the origin.

Once we have a contour, we proceed neglecting the tails and approximating the functions $A(\eta)$ and $B(\eta)$ with their Taylor series about the chosen saddle point η^*. Then, after having absorbed the factor n into a rescaled variable $x = (\eta - \eta^*)/n^{1/(p+1)}$ (where p is the multiplicity of the saddle point), we can easily obtain an asymptotic expansion of the integral in inverse powers of n.

We collect here the first few terms of the asymptotic expansion for the case of a simple saddle

$$I \simeq \frac{e^{nB(\eta^*)}}{\sqrt{2\pi n B''(\eta^*)}} \left[A(\eta^*) + \frac{1}{n}C(\eta^*) + \frac{1}{n^2}D(\eta^*) + O\left(\frac{1}{n^3}\right)\right] \tag{6.11}$$

where the terms in the square brackets with half-integer inverse-power of n vanish, and of a double saddle

$$I \simeq \frac{e^{nB(\eta^*)}}{n^{\frac{1}{3}} B^{(3)}(\eta^*)^{\frac{1}{3}}} \left[\gamma_0 \, A(\eta^*) + \frac{1}{n^{\frac{1}{3}}} \tilde{C}(\eta^*) + \frac{1}{n} \tilde{D}(\eta^*) + O\left(\frac{1}{n^{\frac{4}{3}}}\right) \right] \tag{6.12}$$

where the terms in the square brackets with powers $n^{-(l+\frac{2}{3})}$, with integer l, vanish. In these formulae C, $\tilde{C}$, D, and $\tilde{D}$ are rational functions of $A(\eta^*)$, $B(\eta^*)$ and their derivatives. Let us use the notation

$$X_{(n)} = \left. \frac{\partial^n}{\partial \eta^n} X(\eta) \right|_{\eta=\eta^*} ,$$

then, for the simple saddle point, C and D are given by

$$C = \frac{1}{24 B_{(2)}^3} \left[12 A_{(1)} B_{(2)} B_{(3)} - 12 A_{(2)} B_{(2)}^2 + A \left(3 B_{(2)} B_{(4)} - 5 B_{(3)}^2 \right) \right] \tag{6.13}$$

$$\begin{aligned} D = \frac{1}{1152 B_{(2)}^6} & \left[385 \, A \, B_{(3)}^4 + 144 \, B_{(2)}^4 A_{(4)} - 210 \, B_{(2)} B_{(3)}^2 \left(4 A_{(1)} B_{(3)} + 3 A B_{(4)} \right) \right. \\ & + 21 \, B_{(2)}^2 \left(40 A_{(2)} B_{(3)}^2 + 40 A_{(1)} B_{(3)} B_{(4)} + 5 A B_{(4)}^2 + 8 A B_{(3)} B_{(5)} \right) \\ & \left. - 24 \, B_{(2)}^3 \left(20 A_{(3)} B_{(3)} + 15 A_{(2)} B_{(4)} + 6 A_{(1)} B_{(5)} + A B_{(6)} \right) \right] . \end{aligned} \tag{6.14}$$

For the double saddle points the necessary combinations are instead

$$\tilde{C} = \frac{A \, B_{(4)}}{B_{(3)}^{4/3}} \frac{\gamma_4}{4!} - \frac{A_{(1)}}{B_{(3)}^{1/3}} \gamma_1 \tag{6.15}$$

$$\begin{aligned} \tilde{D} = & - \frac{A_{(3)}}{B_{(3)}} \frac{\gamma_3}{3!} + \frac{A_{(2)} B_{(4)}}{B_{(3)}^2} \frac{\gamma_6}{2 \cdot 4!} + \frac{A_{(1)} B_{(5)}}{B_{(3)}^2} \frac{\gamma_6}{5!} - \frac{A_{(1)} B_{(4)}^2}{B_{(3)}^3} \frac{\gamma_9}{2(4!)^2} \\ & - \frac{A \, B_{(4)} B_{(5)}}{B_{(3)}^3} \frac{\gamma_9}{4!5!} + \frac{A \, B_{(4)}^3}{B_{(3)}^4} \frac{\gamma_{12}}{3!(4!)^3} + \frac{A \, B_{(6)}}{B_{(3)}^2} \frac{\gamma_6}{6!} . \end{aligned} \tag{6.16}$$

Finally, constants γ_k are given by

$$\gamma_k = -\frac{1}{\pi} \sin\left(2\pi \, \frac{1+k}{3}\right) \int_0^\infty du \, u^k \, e^{-\frac{u^3}{3!}} = -\frac{(3!)^{\frac{1+k}{3}}}{3\pi} \sin\left[2\pi \left(\frac{1+k}{3}\right)\right] \Gamma\left(\frac{1+k}{3}\right) .$$

For our integral (6.8) in the large n limit the relevant saddle-point equation $B'(\eta) = 0$ has two solutions, η_a and η_b:

$$\eta_a = \frac{1}{k} \qquad \eta_b = 1 - \alpha . \tag{6.17}$$

If $\alpha \neq \alpha_c \equiv (k-1)/k$ the two solutions are distinct and correspond to simple saddle points. To understand which one is relevant to our discussion we need to study the landscape of the function $B(\eta)$ beyond the neighborhood of the saddles.

In our specific case, as illustrated in figures from 6.1 to 6.3, when $\alpha < \alpha_c$ among the two saddles only η_a is accessible, while, if $\alpha > \alpha_c$, only η_b is so. When $\alpha = \alpha_c$ the two saddle points coalesce into a double saddle point, thus with three valleys, having

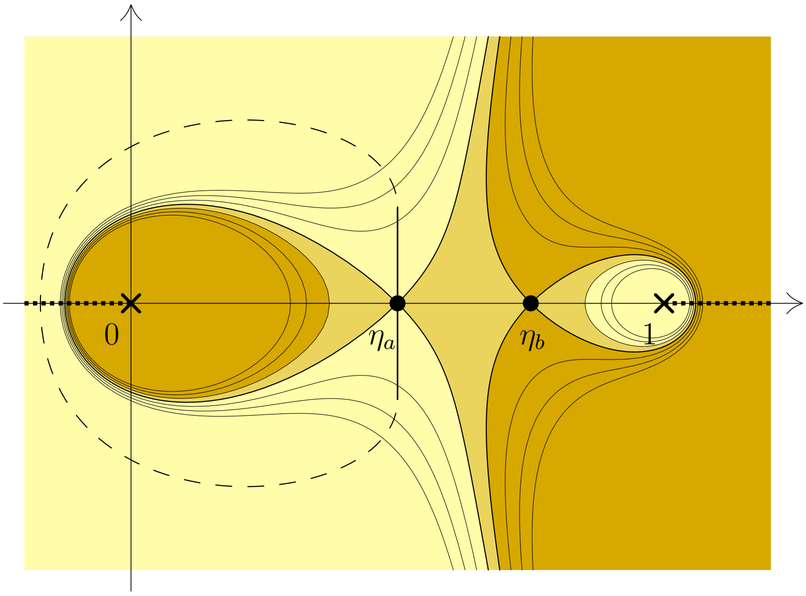


Figure 6.1: Contour levels for $\Re B(\eta)$ when $\alpha < \alpha_c$. More precisely, the figure shows the case $k = 2$ and $\alpha = \frac{1}{2}\alpha_c$. The two bold contour lines describe the level lines of $\Re B(\eta)$ for the values at the two saddle points (located at the bullets). Darker tones denote higher values of $\Re B(\eta)$. The crosses and the dotted lines describe the cut discontinuities due to logarithms in $B(\eta)$. The dashed path surrounding the origin going through one of the saddle points is an example of valid integration contour, and the solid straight portion of the path describes an interval in which the perturbative approach is valid.

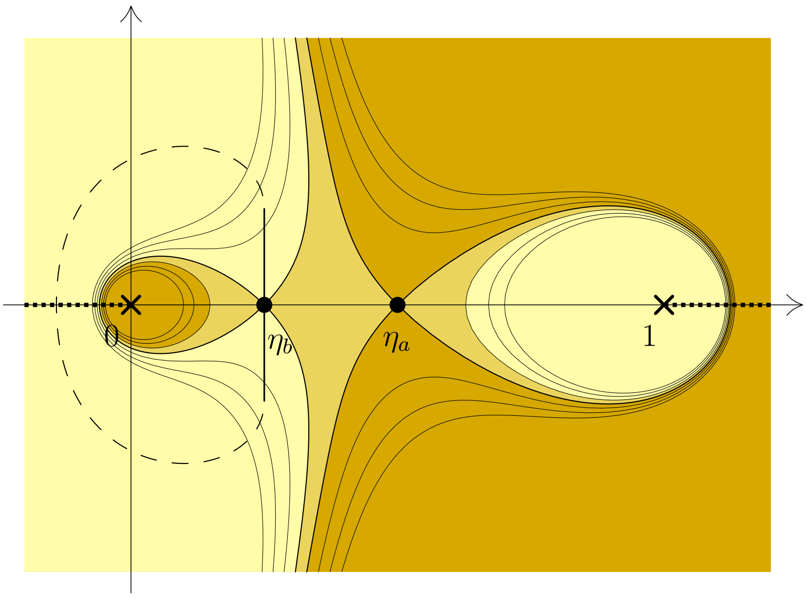


Figure 6.2: Contour levels for $\Re B(\eta)$ when $\alpha > \alpha_c$. More precisely, the figure shows the case $k = 2$ and $\alpha = \frac{3}{2}\alpha_c$. Description of notations is as in figure 6.1.

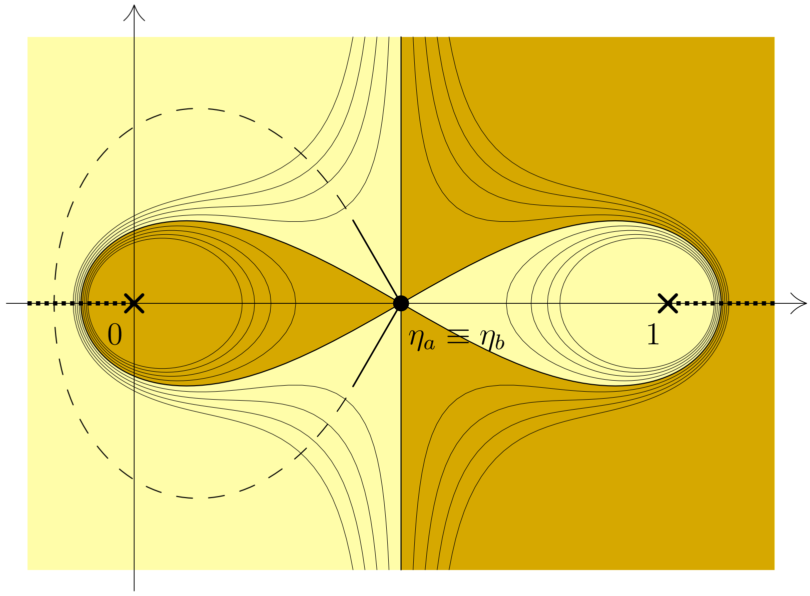


Figure 6.3: Contour levels for $\Re B(\eta)$ when $\alpha = \alpha_c$. More precisely, the figure shows the case $k = 2$. Description of notations is as in figure 6.1.

steepest-descent directions $e^{\frac{2\pi i k}{3}}$, with $k = 0, 1, 2$. Of these valleys, the ones with the appropriate global topology are those with indices $k = 1$ and 2.

As a first result of this discussion, in order to study the asymptotic behaviour of $\mathcal{Z}_{\alpha n}$ will need to distinguish two different phases, and a critical point, upon the value of α being below, above or equal to α_c.

We will name the phases with a smaller and a larger number of hypertrees, respectively, the *low temperature* and *high temperature* phase, the reason being that, as we shall see, in the low temperature phase there is a spontaneous symmetry breaking and the appearance of a non-zero residual *magnetization*.

Low temperature phase

In the case $\alpha < \alpha_c$ the relevant saddle point is $\eta_a = 1/k$. See Fig. 6.1. Since $A(\eta_a) = 0$, we are in the case in which the leading order of (6.11) vanishes and the next order has to be considered. The expansion of $A(\eta)$ and $B(\eta)$ in a neighborhood of the saddle η_a is as follows:

$$A(\eta_a + u) \simeq -k^2 u + k^3 u^2 + O(u^3) \tag{6.18}$$

$$\begin{aligned} B(\eta_a + u) \simeq & \frac{1 + (1-\alpha)\log k}{k\,\alpha_c} + \alpha \log \alpha_c + k\,\frac{\alpha_c - \alpha}{\alpha_c^2}\frac{u^2}{2} \\ & - \left[k^2 \frac{1-\alpha}{\alpha_c} + \frac{\alpha}{\alpha_c^3} \right] \frac{u^3}{3} + O(u^4)\,. \end{aligned} \tag{6.19}$$

Using formula (6.11) we obtain for (6.6) the following asymptotic expression:

$$
\begin{aligned}
\mathcal{Z}_{\alpha n} &\simeq \frac{n!}{\Gamma(\alpha n+1)} \frac{\alpha\sqrt{k-1}}{\sqrt{2\pi n^3}} \frac{e^{\frac{n}{k-1}} \left(\frac{k-1}{k}\right)^{n\alpha-1}}{[(k-2)!]^{n\frac{1-\alpha}{k-1}}} \left(1-\frac{k\alpha}{k-1}\right)^{-5/2} && (6.20)\\
&\simeq \frac{n^{n-2}}{(\alpha n)^{\alpha n-\frac{1}{2}}} \sqrt{\frac{k-1}{2\pi}} \frac{e^{\left(\alpha-\frac{k-2}{k-1}\right)n} \left(\frac{k-1}{k}\right)^{n\alpha-1}}{[(k-2)!]^{n\frac{1-\alpha}{k-1}}} \left(1-\frac{k\alpha}{k-1}\right)^{-5/2} && (6.21)
\end{aligned}
$$

where in the second line we used the Stirling formula to approximate the large factorial $n!$. In Chapter 4 we already saw an asymptotic formula (4.152) for the number of forests with a given number p of connected components. That formula has been obtained keeping p fixed while doing the limit $n \to \infty$, this means taking α infinitesimal. By setting $\alpha n \to p$ in (6.20) and using

$$
\frac{\alpha}{\Gamma(\alpha\, n+1)} = \frac{\alpha}{\alpha n\, \Gamma(\alpha n)} = \frac{n^{-1}}{(p-1)!} \tag{6.22}
$$

and then taking the limit $\alpha \to 0$, we can re-obtain (4.152) by using again the Stirling formula to approximate the large factorial $n!$:

$$
\mathcal{Z}_p \simeq \frac{n^{n-2}}{e^{n\frac{k-2}{k-1}}} \frac{\sqrt{k-1}}{[(k-2)!]^{\frac{n-p}{k-1}}} \frac{1}{(p-1)!} \left(\frac{k-1}{k}\right)^{p-1} . \tag{6.23}
$$

High temperature phase

When $\alpha_c < \alpha < 1$ the relevant saddle point changes into $\eta_b = 1-\alpha$ (see Fig. 6.2) where the functions $A(\eta_b+u)$ and $B(\eta_b+u)$ can be approximated at $O(u^4)$ with

$$
A(\eta_b+u) \simeq k\,\frac{\alpha-\alpha_c}{1-\alpha} - \frac{u}{(\alpha-1)^2} - \frac{u^2}{(\alpha-1)^3} - \frac{u^3}{(\alpha-1)^4} \tag{6.24}
$$

$$
\begin{aligned}
B(\eta_b+u) \simeq{}& \frac{1-\alpha}{\alpha_c}\left[1-\frac{1}{k}\log(1-\alpha)\right] + \alpha\log\alpha \\
&+ \frac{1}{\alpha\,\alpha_c}\frac{\alpha-\alpha_c}{1-\alpha}\frac{u^2}{2} - \left[\frac{1}{\alpha^2} + \frac{1}{k\,\alpha_c\,(1-\alpha)^2}\right]\frac{u^3}{3}
\end{aligned} \tag{6.25}
$$

The situation is quite analogous to the previous one, with the exception that $A(\eta_b) \neq 0$, and using formula (6.11) we obtain for (6.6) the following asymptotic expression:

$$
\begin{aligned}
\mathcal{Z}_{\alpha n} &\simeq \frac{n!\,\alpha^{\alpha n}\,(k-1)}{\Gamma(\alpha n+1)} \left[\frac{e^k}{k\,(1-\alpha)\,(k-2)!}\right]^{n\frac{1-\alpha}{k-1}} \sqrt{\frac{\alpha}{2\pi n(1-\alpha)}} \left(\frac{\alpha k}{k-1}-1\right)^{1/2} && (6.26)\\
&\simeq \frac{n^{(1-\alpha)n}}{\sqrt{2\pi n\frac{1-\alpha}{k-1}}} \left[\frac{e}{k\,(1-\alpha)\,(k-2)!}\right]^{n\frac{1-\alpha}{k-1}} (\alpha k-k+1)^{1/2} . && (6.27)
\end{aligned}
$$

Remark that the saddle point method cannot be applied when $\alpha = 1$, but if we replace

$$
\frac{e^{n\frac{1-\alpha}{k-1}}}{\sqrt{2\pi n\frac{1-\alpha}{k-1}}} \simeq \frac{n^{n\frac{1-\alpha}{k-1}}}{\Gamma\left(n\frac{1-\alpha}{k-1}+1\right)} \tag{6.28}
$$

for $\alpha \simeq 1$ we get

$$\mathcal{Z}_n \simeq 1\,. \tag{6.29}$$

as we should.

The critical phase

When α is exactly $\alpha_c = (k-1)/k$, the saddle points η_b and η_a coalesce into a double saddle point in η_a in which the second derivative vanishes along with the first one. The expansion of $A(\eta)$ and $B(\eta)$ are as in (6.18)-(6.19) with $\alpha = \alpha_c$:

$$A(\eta_c + u) \simeq -k^2 u + k^3 u^2 + O(u^3) \tag{6.30}$$

$$\begin{aligned} B(\eta_c + u) \simeq & \frac{1}{k-1} + \frac{1}{k(k-1)}\log(k-1) + \frac{k-2}{k-1}\log k \\ & - \frac{k^3}{(k-1)^2}\frac{u^3}{3} + \frac{k^4(k-2)}{(k-1)^3}\frac{u^4}{4} + O(u^5) \end{aligned} \tag{6.31}$$

Using (6.12) we obtain the following result

$$\begin{aligned} \mathcal{Z}_{\alpha_c n} =& \frac{n!}{\Gamma\left(\frac{k-1}{k}n + 1\right)} \frac{e^{\frac{n}{k-1}}\left(\frac{k-1}{k}\right)^{\frac{k-1}{k}n}}{[(k-2)!]^{\frac{n}{k(k-1)}}} \frac{3^{1/6}\Gamma(2/3)(k-1)^{4/3}}{2\pi\, n^{2/3}} && (6.32) \\ \simeq & \, n^{\frac{n}{k}} \left[\frac{e}{(k-2)!}\right]^{\frac{n}{k(k-1)}} \frac{3^{1/6}\Gamma(2/3)(k-1)^{4/3}}{2\pi\, n^{2/3}}\,. && (6.33) \end{aligned}$$

This formula, for $k = 2$, can be, in principle, compared with the result presented in [79, Proposition VIII.11], but unfortunately there is a discrepancy in the numerical pre-factor.

6.3 The canonical ensemble

According to the definition (6.7) for the number of forests with $p = \alpha n$ trees $\mathcal{Z}_{\alpha n}$, by evaluating the integral I defined in (6.8) by the saddle-point method, when the saddle point $\eta^*(\alpha)$ is simple and thus away from the critical point α_c, we get the following asymptotic expansion for large number of vertices n:

$$\mathcal{Z}_{\alpha n} \simeq \frac{n!}{\Gamma(\alpha n + 1)} \left[\frac{k-1}{k!}\right]^{n\frac{1-\alpha}{k-1}} \frac{e^{nB(\eta^*)}}{\sqrt{2\pi n B''(\eta^*)}} \left[A(\eta^*) + \frac{1}{n}C(\eta^*)\right]\,. \tag{6.34}$$

We define the entropy density $s(\alpha)$ as

$$s(\alpha) = \frac{1}{n}\log\frac{\mathcal{Z}_{\alpha n}}{n!} \tag{6.35}$$

so that we can recover the partition function $\mathcal{Z}(\lambda)$ by a Legendre transformation

$$\mathcal{Z}(\lambda) = \sum_p \mathcal{Z}_p\,\lambda^p \simeq \int_0^1 d\alpha\, \mathcal{Z}_{\alpha n}\,\lambda^{n\alpha} = n! \int_0^1 d\alpha\, \exp\{n[s(\alpha) + \alpha\log\lambda]\} \tag{6.36}$$

that can be evaluated for large n once more by the saddle-point method. Calling $\bar{\alpha}(\lambda)$ the mean number of trees at given λ, we have:

$$s'(\bar{\alpha}(\lambda)) + \log \lambda = 0\,. \tag{6.37}$$

From (6.34) we see that $s(\alpha)$ still has an α-dependent leading order in n

$$s(\alpha) \simeq -\alpha \log n - \alpha \log \alpha + \alpha + \frac{\alpha - 1}{k-1} \log \left[\frac{k!}{k-1}\right] + B(\eta^*(\alpha)) \tag{6.38}$$

that would shift the solution down to 0. By the rescaling

$$\lambda = n\,\tilde{\lambda} \tag{6.39}$$

which is usual in the complete graph, in order to obtain a correct thermodynamic scaling, we can reabsorb this factor. The saddle-point equation now reads

$$s'(\bar{\alpha}) + \log n + \log \tilde{\lambda} = 0 \tag{6.40}$$

whose solution is

$$\bar{\alpha} = \begin{cases} \frac{k-1}{k}\ [(k-2)!]^{\frac{1}{k-1}}\ \tilde{\lambda} & \text{for} \quad \tilde{\lambda} < \tilde{\lambda}_c \\ 1 - \frac{k-1}{k!} \frac{1}{\tilde{\lambda}^{k-1}} & \text{for} \quad \tilde{\lambda} > \tilde{\lambda}_c \end{cases} \tag{6.41}$$

where $\tilde{\lambda}_c = [(k-2)!]^{-1/(k-1)}$. And by inversion

$$\tilde{\lambda} = \begin{cases} \frac{k}{k-1}\ [(k-2)!]^{-\frac{1}{k-1}}\ \bar{\alpha} & \text{for} \quad \bar{\alpha} < \bar{\alpha}_c \\ \left(\frac{k-1}{k!} \frac{1}{1-\bar{\alpha}}\right)^{\frac{1}{k-1}} & \text{for} \quad \bar{\alpha} > \bar{\alpha}_c\,. \end{cases} \tag{6.42}$$

In the ordinary graph case, this means

$$\bar{\alpha} = \begin{cases} \frac{\tilde{\lambda}}{2} & \tilde{\lambda} < 1 \\ 1 - \frac{1}{2\tilde{\lambda}} & \tilde{\lambda} > 1 \end{cases} \qquad \tilde{\lambda} = \begin{cases} 2\bar{\alpha} & \bar{\alpha} < \frac{1}{2} \\ \frac{1}{2(1-\bar{\alpha})} & \bar{\alpha} > \frac{1}{2}\,. \end{cases} \tag{6.43}$$

6.4 Size of the hypertrees

We have shown in the previous sections that the system admits two different phases. We want now to characterize these two regimes. Our field-theoretical approach provides us a full algebra of observables, as polynomials in the Grassmann fields, which we could study systematically. However, it is interesting to note that some of these observables have a rephrasing in terms of combinatorial properties of the forests (cfr. [7]). Furthermore, we are induced by the results of [66] to investigate the possibility of a transition of percolative nature, with the emergence of a giant component in the *typical* forest for a given ensemble.

A possibility of this sort is captured by the mean square size of the trees in the forest, as the following argument shows at least at a heuristic level. If we have all trees with size of order 1 in the large n limit, (say, with average a and variance σ both of order 1), then the sum of the squares of the sizes of the trees in a forest scales as $(a + \sigma^2/a)n$. If, conversely, in the large n limit one tree occupies a finite fraction p of the whole graph, the same sum as above would scale as $p^2 n^2 + \mathcal{O}(n)$.

Furthermore, it turns out that the combinatorial observable above has a very simple formulation in the field theory, corresponding to the natural susceptibility for the fermionic fields, as we will see in a moment.

Let's start our analysis with the un-normalized expectation

$$\lambda \langle \bar{\psi}_i \psi_i \rangle = \lambda \int \mathcal{D}(\bar{\psi}, \psi)\, \bar{\psi}_i \psi_i \, \exp(-\mathcal{H}) \, = \mathcal{Z}(\lambda) \tag{6.44}$$

because the insertion of the operator $\bar{\psi}_i \psi_i$ simply marks the vertex i as a root of a hypertree, and in a spanning forest every vertex can be chosen as the root of a hypertree. If we now sum over the index i we gain a factor $|T|$ for each hypertree. Therefore we have:

$$\lambda \langle \bar{\psi}\psi \rangle = \lambda \sum_{i \in V} \langle \bar{\psi}_i \psi_i \rangle = \sum_{F \in \mathcal{F}} \lambda^{k(F)} \sum_{T \in F} |T| \prod_{A \in T} w_A = n \, \mathcal{Z}(\lambda) \tag{6.45}$$

as in each spanning hyperforest the total size of the hypertrees is the number of vertices in the graph, that is n. By expanding in the parameter λ and by taking the p-th coefficient we get the relation

$$\frac{1}{\mathcal{Z}_p} \frac{\langle \bar{\psi}\psi \, \mathcal{U}^{p-1} \rangle_{\lambda=0}}{(p-1)!} = n \, . \tag{6.46}$$

For the un-normalized two-point function

$$\langle \bar{\psi}_i \psi_j \rangle = \int \mathcal{D}(\bar{\psi}, \psi)\, \bar{\psi}_i \psi_j \, \exp(-\mathcal{H}) \tag{6.47}$$

we know (see Section 3.4) that

$$\lambda \langle \bar{\psi}_i \psi_j \rangle = \sum_{\substack{F \in \mathcal{F} \\ i,j \text{ connected}}} \lambda^{K(F)} \sum_{T \in F} \prod_{A \in T} w_A \, . \tag{6.48}$$

As i and j are connected if they belong to the same hypertree, if we sum on both indices i and j we gain a factor $|T|^2$ for each hypertree, therefore

$$\lambda \langle (\bar{\psi} \mathrm{J} \psi) \rangle = \lambda \sum_{i,j \in V} \langle \bar{\psi}_i \psi_j \rangle = \sum_{F \in \mathcal{F}} \lambda^{K(F)} \sum_{T \in F} |T|^2 \prod_{A \in T} w_A \, . \tag{6.49}$$

The effect of this observable is to introduce an extra weight for hypertrees in the spanning forests which is the square of its size.

The average of the square-size of hypertrees in the microcanonical ensemble of hyperforests with fixed number p of hypertrees is easily obtained from the previous relation by expanding in the parameter t and by taking the p-th coefficient, so that

$$\langle \, |T|^2 \rangle_p = \frac{1}{\mathcal{Z}_p} \frac{\langle (\bar{\psi} \mathrm{J} \psi) \, \mathcal{U}^{p-1} \rangle_{\lambda=0}}{(p-1)!} \, . \tag{6.50}$$

The very same method of the preceding section can be used to evaluate this quantity.

Still in a mean-field description, we have:

$$\langle(\bar{\psi}\mathrm{J}\psi)\,\mathcal{U}^{p-1}\rangle_{\lambda=0} = \tag{6.51a}$$

$$= \int \mathcal{D}_n(\bar{\psi},\psi)\ (\bar{\psi}\mathrm{J}\psi)\ \mathcal{U}^{p-1} \exp\left[n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right]\left[1-(\bar{\psi}\mathrm{J}\psi)\frac{(\bar{\psi}\psi)^{k-2}}{(k-2)!}\right] \tag{6.51b}$$

$$= \int \mathcal{D}_n(\bar{\psi},\psi)\ \bar{\psi}\psi\ \mathcal{U}^{p-1} \exp\left[n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right] \tag{6.51c}$$

$$= n! \oint \frac{\mathrm{d}\xi}{2\pi i}\,\frac{1}{\xi^{n+1}}\ \xi \left[\xi + (1-k)\frac{\xi^k}{k!}\right]^{p-1} \exp\left[n\frac{\xi^{k-1}}{(k-1)!}\right] \tag{6.51d}$$

$$= n! \left[\frac{k-1}{k!}\right]^{n\frac{1-\alpha}{k-1}} \oint \frac{\mathrm{d}\eta}{2\pi i}\ \tilde{A}(\eta)\ e^{nB(\eta)}, \tag{6.51e}$$

where now

$$\tilde{A}(\eta) = \frac{1}{\eta(1-\eta)}, \tag{6.52}$$

and $B(\eta)$ is the same as before. To evaluate this integral we again use the saddle point method. Please note that since the function $B(\eta)$ is unchanged so are the saddle points.

Using the general expansion for $p = \alpha n$ (6.11) we have

$$\langle\,|T|^2\rangle_{\alpha n} = \frac{1}{\mathcal{Z}_{\alpha n}}\,\frac{\langle(\bar{\psi}\mathrm{J}\psi)\,\mathcal{U}^{\alpha n-1}\rangle_{\lambda=0}}{\Gamma(\alpha n)} = \alpha n\,\frac{\tilde{A}(\eta^*) + \frac{1}{n}\tilde{C}(\eta^*) + O\left(\frac{1}{n^2}\right)}{A(\eta^*) + \frac{1}{n}C(\eta^*) + O\left(\frac{1}{n^2}\right)}. \tag{6.53}$$

Now in the low temperature phase we have $A(\eta_a) = 0$ so in order to get the leading term we need $C(\eta^*)$ and as

$$\tilde{A}(\eta_a) = \frac{k^2}{k-1} \quad \text{and} \quad C(\eta_a) = \frac{\alpha(k-1)}{(\alpha-\alpha_c)^2}, \tag{6.54}$$

(6.53) at leading order gives

$$\langle\,|T|^2\rangle_{\alpha n} \simeq \alpha n^2\,\frac{\tilde{A}(\eta_a)}{C(\eta_a)} = n^2\left(\frac{\alpha_c-\alpha}{\alpha_c}\right)^2 \tag{6.55}$$

so that, as soon as $\alpha < \alpha_c$, a giant hypertree appears in the typical forest, which occupies on average a fraction $1-\alpha/\alpha_c$ of the whole graph. In the high temperature instead we have

$$\tilde{A}(\eta_b) = \frac{1}{\alpha(1-\alpha)} \quad \text{and} \quad A(\eta_b) = k\frac{\alpha-\alpha_c}{1-\alpha}, \tag{6.56}$$

giving (always at leading order)

$$\langle\,|T|^2\rangle_{\alpha n} \simeq \alpha n\,\frac{\tilde{A}(\eta_b)}{A(\eta_b)} = \frac{n}{k}\,\frac{1}{\alpha-\alpha_c}\,. \tag{6.57}$$

So that

$$\lim_{n\to\infty}\frac{1}{n^2}\,\langle\,|T|^2\rangle_{\alpha n} = \begin{cases} \left(\frac{\alpha_c-\alpha}{\alpha_c}\right)^2 & \text{for } \alpha \le \alpha_c \\ 0 & \text{for } \alpha \ge \alpha_c \end{cases} \tag{6.58}$$

is an order parameter, but it is represented as the expectation value of a non-local operator. We shall see in the next Section how to construct a local order parameter.

6.5 The symmetry breaking

In this section we will describe the phase transition in terms of the breaking of the global osp(1|2) supersymmetry. According to the general strategy (see for example [80]) let's add an exponential weight with an external source h coupled to the variation of the fields (5.10):

$$h \sum_{i \in V} (1 - \lambda\, \bar{\psi}_i \psi_i) \;=\; h\,(n - \lambda\, \bar{\psi}\psi), \tag{6.59}$$

The partition function becomes now:

$$\mathcal{Z}(\lambda, h) = \int \mathcal{D}_n(\bar{\psi}, \psi)\; e^{-\mathcal{H} - h(n - \lambda \bar{\psi}\psi)}. \tag{6.60}$$

We have chosen to add the exponential weight with a minus sign because in this way when λ is sent to zero with the product $h\,\lambda$ kept fixed, we get, aside from a vanishing trivial factor, the generating function of rooted hyperforests.

More generally, for finite λ and h, we have that $\mathcal{Z}(\lambda, h)$ can be expressed as a sum over spanning hyperforests with a modified weight

$$\mathcal{Z}(\lambda, h) = \sum_{F \in \mathcal{F}} \prod_{T \in F} \lambda\, e^{-h|T|}\,(1 + h\,|T|) \tag{6.61}$$

which is always positive, at any n, only for $h \geq 0$.

On the k-uniform complete hypergraph the partition function (6.60) is expressed

$$\mathcal{Z}(\lambda, h) = \int \mathcal{D}_n(\bar{\psi}, \psi) \exp\left[\lambda \mathcal{U} + h\lambda\, \bar{\psi}\psi - nh + n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right] \left[1 - \frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] \tag{6.62}$$

To work in the micro-canonical ensemble we again expand in powers of λ

$$\mathcal{Z}(\lambda, h) = \sum_{p=0}^{n} \mathcal{Z}_p(h)\, \lambda^p, \tag{6.63}$$

where each term of the above expansion gives the partition function at fixed number of components:

$$\mathcal{Z}_p(h) = \frac{1}{p!} \int \mathcal{D}_n(\bar{\psi}, \psi)\;\; \big(\mathcal{U} + h\bar{\psi}\psi\big)^p \exp\left[-nh + n\frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!}\right] \left[1 - \frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] \tag{6.64}$$

$$= \frac{e^{-nh}}{p!}\, \langle \big(\mathcal{U} + h\bar{\psi}\psi\big)^p \rangle_{\lambda=0}\,. \tag{6.65}$$

Following the same steps of the previous section, we can write this expression in terms of a complex integral:

$$\mathcal{Z}_p(h) = \frac{n!}{\Gamma(\alpha n + 1)} \left[\frac{k-1}{k!}\right]^{n\frac{1-\alpha}{k-1}} I(\alpha, h) \tag{6.66}$$

with

$$I(\alpha, h) = \oint \frac{\mathrm{d}\eta}{2\pi i}\, A(\eta)\, e^{nB(\eta, h)} \tag{6.67}$$

where $A(\eta)$ is the same as in (6.9) while

$$B(\eta, h) = -h + \frac{k}{k-1}\eta + \alpha \log(1 + h - \eta) + \frac{\alpha - 1}{k-1} \log \eta. \tag{6.68}$$

$I(\alpha, h)$ can be again evaluated with the same technique as above. Let's call $\eta^*(h)$ the position of the relevant saddle point, which is the accessible solution of the saddle point equation

$$\frac{\partial}{\partial \eta} B(\eta, h)\Big|_{\eta=\eta^*(h)} = 0\,. \tag{6.69}$$

If $h > 0$ the two solutions are real valued and distinct for every value of α and the accessible saddle is simple and turns out to be always the one closer to the origin.

In the following we are going to consider all the functions A, B, C and D as evaluated on $\eta^*(h)$ and therefore as functions of the single parameter h.

$$A(h) \equiv A(\eta^*(h)) \qquad\qquad B(h) \equiv B(\eta^*(h), h) \tag{6.70}$$
$$C(h) \equiv C(\eta^*(h), h) \qquad\qquad D(h) \equiv D(\eta^*(h), h) \tag{6.71}$$

The asymptotic behaviour of (6.66) is given by the general formula (6.11):

$$\mathcal{Z}_{\alpha n}(h) \propto \frac{e^{nB(h)}}{\sqrt{2\pi n B''(h)}} \left[A(h) + \frac{C(h)}{n} + O\left(\frac{1}{n^2}\right)\right]\,. \tag{6.72}$$

The density of entropy is obtained by taking the logarithm of the partition function $\mathcal{Z}_{\alpha n}(h)$

$$s(\alpha, h) = \frac{1}{n} \log \frac{\mathcal{Z}_{\alpha n}(h)}{n!}\,. \tag{6.73}$$

The magnetization is then the first derivative of the entropy

$$m(\alpha, h) = -\frac{\partial s}{\partial h} = 1 - \alpha\, \frac{\langle \bar\psi\psi\, \left(\mathcal{U} + h\bar\psi\psi\right)^{\alpha n - 1}\rangle_{\lambda=0}}{\langle \left(\mathcal{U} + h\bar\psi\psi\right)^{\alpha n}\rangle_{\lambda=0}} \tag{6.74}$$

$$= 1 - \alpha\, n\, \frac{\langle \bar\psi_i\psi_i\, \left(\mathcal{U} + h\bar\psi\psi\right)^{\alpha n - 1}\rangle_{\lambda=0}}{\langle \left(\mathcal{U} + h\bar\psi\psi\right)^{\alpha n}\rangle_{\lambda=0}} \tag{6.75}$$

which is written as the expectation of a local operator, and if we set $h = 0$ in this formula we get

$$m(0) = 1 - \frac{1}{\mathcal{Z}_{\alpha n}(0)} \frac{\langle \bar\psi_i\psi_i \mathcal{U}^{\alpha n - 1}\rangle_{\lambda=0}}{\Gamma(\alpha n)} = 0 \tag{6.76}$$

because of (6.46). In order to evaluate first the limit of large number of vertices we use the asymptotic expression for $\mathcal{Z}_{\alpha n}(h)$ to get

$$m(\alpha, h) = -\frac{\partial B(h)}{\partial h} + \frac{1}{2n}\frac{1}{B''(h)}\frac{\partial B''(h)}{\partial h} - \frac{1}{n}\frac{\frac{\partial A(h)}{\partial h} + \frac{1}{n}\frac{\partial C(h)}{\partial h} + O(\frac{1}{n^2})}{A(h) + \frac{1}{n}C(h) + O\left(\frac{1}{n^2}\right)}. \tag{6.77}$$

The vanishing of $A(0)$ in the low temperature phase ($\alpha < \alpha_c$) has the consequence that the two limits $n \to \infty$ and $h \to 0$ do not commute, indeed:

$$\lim_{n\to\infty}\lim_{h\to 0} m(\alpha,h) = -\left.\frac{\partial B(h)}{\partial h}\right|_{h=0} - \frac{1}{C(0)}\left.\frac{\partial A(h)}{\partial h}\right|_{h=0} = 0 \tag{6.78}$$

$$\lim_{h\to 0}\lim_{n\to\infty} m(\alpha,h) = -\left.\frac{\partial B(h)}{\partial h}\right|_{h=0} = \frac{\alpha_c-\alpha}{\alpha_c} \geq 0. \tag{6.79}$$

Remark that the magnetization m vanishes at the critical point linearly and not with critical exponent $1/2$ as it is common in mean-field theory, the reason being that here the order parameter is not linear but quadratic in the fundamental fields.

In the high temperature phase $A(0) \neq 0$ and the two limits above commute.

$$\lim_{n\to\infty}\lim_{h\to 0} m(\alpha,h) = \lim_{h\to 0}\lim_{n\to\infty} m(\alpha,h) = -\left.\frac{\partial B(h)}{\partial h}\right|_{h=0} = 0. \tag{6.80}$$

In the study of phase transitions the thermodynamical limit $n \to \infty$ has to be taken first. Indeed, the ergodicity is broken in the thermodynamical limit first and then a residual spontaneous magnetization appears even when the external field vanishes. Remark that both the free energy and the magnetization vary continuously passing from one phase to the other.

The longitudinal susceptibility χ_L

$$\chi_L(\alpha,h) = \frac{\partial^2 s(\alpha,h)}{\partial h^2} = \alpha\,(\alpha n-1)\,\frac{\langle(\bar\psi\psi)^2\,\left(\mathcal{U}+h\bar\psi\psi\right)^{\alpha n-2}\rangle_{\lambda=0}}{\langle\left(\mathcal{U}+h\bar\psi\psi\right)^{\alpha n}\rangle_{\lambda=0}} - n\,\left[1-m(\alpha,h)\right]^2 \tag{6.81}$$

can be obtained from the magnetization:

$$\begin{aligned}\chi_L(\alpha,h) = -\frac{\partial m(\alpha,h)}{\partial h} &= \frac{\partial^2 B(h)}{\partial h^2} + \frac{1}{2n}\frac{1}{B''(h)^2}\left(\frac{\partial B''(h)}{\partial h}\right)^2 \\ &\quad - \frac{1}{2n}\frac{1}{B''(h)}\frac{\partial^2 B''(h)}{\partial h^2} + \frac{1}{n}\frac{\frac{\partial^2 A(h)}{\partial h^2} + \frac{1}{n}\frac{\partial^2 C(h)}{\partial h^2} + O(\frac{1}{n^2})}{A(h)+\frac{1}{n}C(h)+O(\frac{1}{n^2})} \\ &\quad - \frac{1}{n}\left[\frac{\frac{\partial A(h)}{\partial h} + \frac{1}{n}\frac{\partial C(h)}{\partial h} + O(\frac{1}{n^2})}{A(h)+\frac{1}{n}C(h)+O(\frac{1}{n^2})}\right]^2 + O\left(\frac{1}{n^2}\right),\end{aligned} \tag{6.82}$$

and taking the two limits in the appropriate order we get

$$\lim_{h\to 0}\lim_{n\to\infty}\chi_L(\alpha,h) = \left.\frac{\partial^2 B(h)}{\partial h^2}\right|_{h=0} = \begin{cases} -\frac{\alpha(1-\alpha)}{\alpha_c^2}\left(\frac{\alpha_c-\alpha}{\alpha_c}\right)^{-1} & \alpha < \alpha_c \\ -\frac{1-\alpha_c}{\alpha_c}\left(\frac{\alpha-\alpha_c}{\alpha_c}\right)^{-1} & \alpha > \alpha_c \end{cases} \tag{6.83}$$

which shows that the susceptibility is discontinuous at the transition, with a singularity $\chi(\alpha) \sim |\alpha-\alpha_c|^{-1}$, so that the transition is second order. Remark that the longitudinal susceptibility appears to be negative. This means that in our model of spanning hyperforest there are events negatively correlated. It is well known that in the model of spanning trees on a finite connected graph the indicator functions for the events in which an edge belongs to the tree are negatively correlated. This is proven by Feder and Mihail [81] in the wider context of balanced matroids (and uniform weights). See also [82] for a purely combinatorial proof of the stronger Raileigh condition, in the weighted case. The random cluster model for $q > 1$ is known to be positive associated. When $q < 1$ negative association is conjectured to hold. For an excellent description of the situation about negative association see [83].

Still following the analogy with magnetic systems, let us introduce the transverse susceptibility

$$\chi_T(\alpha, h) = \frac{2}{\mathcal{Z}_{\alpha n}} \frac{\langle (\bar{\psi} \mathrm{J} \psi) \, \mathcal{U}^{\alpha n - 1} \rangle_{\lambda=0}}{n \, \Gamma(\alpha n)} \tag{6.84}$$

which, by comparison with (6.53), provides, at $h = 0$

$$\chi_T(\alpha, 0) = \frac{2}{n} \langle \, |T|^2 \rangle_{\alpha n} \, . \tag{6.85}$$

As we shall see later in Section 6.7 we have the identity

$$m(\alpha, h) = \frac{h}{2} \, \chi_T(\alpha, h) \, . \tag{6.86}$$

This relation is the bridge between the average square-size of hypertrees and the local order parameter.

At finite n, when the symmetry-breaking field h is set to zero, we get

$$m(\alpha, 0) = 0 \tag{6.87}$$

in agreement with formula (6.46) and

$$\chi_T(\alpha, 0) = \lim_{h \to 0} 2 \, \frac{m(\alpha, h)}{h} = 2 \, \frac{\partial m(\alpha, h)}{\partial h} \bigg|_{h=0} = -2 \, \chi_L(\alpha, 0) \tag{6.88}$$

which should be compared with the analogous formula for the $O(N)$-model where it is

$$\chi_T(\alpha, 0) = (N - 1) \, \chi_L(\alpha, 0) \tag{6.89}$$

and, in our case, as the symmetry is osp(1|2), N should be set to -1 as we have one bosonic direction and two fermionic ones which give a negative contribution.

The leading n contribution is

$$\frac{1}{2} \, \chi_T(\alpha, 0) = - \, \chi_L(\alpha, 0) = \begin{cases} n \, \left(\frac{\alpha_c - \alpha}{\alpha_c} \right)^2 & \text{for } \alpha \leq \alpha_c \\ \frac{1}{k} \, \frac{1}{\alpha - \alpha_c} & \text{for } \alpha \geq \alpha_c \end{cases} \tag{6.90}$$

But, for $\alpha \leq \alpha_c$, if we first compute the large n limit and afterwards send $h \to 0$, we know that we get a non-zero magnetization and therefore the transverse susceptibility diverges as

$$\chi_T(\alpha, 0) \sim 2 \, \frac{m(\alpha, 0)}{h} = \frac{2}{h} \, \frac{\alpha_c - \alpha}{\alpha_c} \tag{6.91}$$

which corresponds to the idea that there are massless excitations, Goldstone modes associated to the symmetry breaking. Remark that, at finite h, the transverse susceptibility does not increase with n, which shows that the average square-size of hypertrees stays finite.

The longitudinal susceptibility instead

$$\chi_L(\alpha,0) \sim -\frac{1}{m(\alpha,0)}\,\frac{\alpha\,(1-\alpha)}{\alpha_c^2}\,. \tag{6.92}$$

diverges only at $\alpha = \alpha_c$, when the magnetization vanishes.

6.6 A symmetric average

At the breaking of an ordinary symmetry the equilibrium states can be written as a convex superposition of pure, clustering, states, which cab be obtained, one from the other, by applying the broken symmetry transformations. The pure state we have defined in this Section uses a breaking field in the only direction we have at disposal where the Grassmann components are null. A more general breaking field would involve a direction in the superspace to which we are unable to give a combinatorial meaning. However, if we take the average in the invariant Berezin integral of these fields we give rise to a different, non-pure but symmetric, low-temperature state.

In this Section we shall set $\lambda = 1$.

The most general breaking field, with total strength h, but arbitrary direction in the super-space, would give a weight

$$h\sum_{i=1}^{n}\left[\mu\,(1-\bar\psi_i\psi_i) + \bar\epsilon\psi_i + \bar\psi_i\epsilon\,\right] \tag{6.93}$$

where $(\mu;\bar\epsilon,\epsilon)$ is a unit vector in the $1|2$ supersphere, i.e. ϵ and $\bar\epsilon$ are Grassmann coordinates and μ is a formal variable satisfying the constraint

$$\mu^2 + 2\,\bar\epsilon\,\epsilon = 1.$$

Let us introduce the normalized generalized measure

$$\mathrm{d}\Omega = \mathrm{d}\mu\,\mathrm{d}\epsilon\,\mathrm{d}\bar\epsilon\,\delta\left(\mu^2 + 2\,\bar\epsilon\,\epsilon - 1\right)\,. \tag{6.94}$$

A symmetric equilibrium measure can be constructed by considering the factor

$$F[h;\bar\psi,\psi] = \tag{6.95}$$

$$= \int \mathrm{d}\Omega\,\exp\left\{-\,h\sum_{i=1}^{n}\left[\mu\,(1-\bar\psi_i\psi_i) + \bar\epsilon\psi_i + \bar\psi_i\epsilon\,\right]\right\} \tag{6.96}$$

$$= \int \mathrm{d}\epsilon\,\mathrm{d}\bar\epsilon\,\exp\left\{\bar\epsilon\,\epsilon - h\sum_{i=1}^{n}\left[(1-\bar\epsilon\,\epsilon)(1-\bar\psi_i\psi_i) + \bar\epsilon\psi_i + \bar\psi_i\epsilon\right]\right\} \tag{6.97}$$

$$= \left[1 - h^2\,(\bar\psi \mathrm{J}\psi) + h\,(n-\bar\psi\psi)\right]\,\exp\left[-\,h\,(n-\bar\psi\psi)\right]. \tag{6.98}$$

where only the last expression is specific to our model, but the previous are the appropriate expressions for the model of unrooted spanning hyperforests on an arbitrary weighted hypergraph. This function is symmetric, for every strength h, as it can be easily checked that

$$Q_{\pm}\, F = 0\,. \tag{6.99}$$

If we send $h \to 0$ this factor is simply 1, but if we first take the $n \to \infty$ limit and then $h \to 0$ the expectation value of non-symmetric observables can be different.

The partition function is not changed because of the identity (6.110). Indeed

$$\langle F \rangle = \langle \exp\left[-\,h\,(n - \bar{\psi}\psi)\right] \rangle = \langle 1 \rangle_h \tag{6.100}$$

for un-normalized expectation values, because of the relation between the transverse susceptibility and the magnetization, equation (6.110), which is

$$\begin{aligned} 0 &= \langle \left[-h^2\,(\bar{\psi}\mathrm{J}\psi) + h\,(n - \bar{\psi}\psi)\right]\, \exp\left[-\,h\,(n - \bar{\psi}\psi)\right] \rangle && \text{(6.101a)} \\ &= h\,\langle \left[-h\,(\bar{\psi}\mathrm{J}\psi) + \,(n - \bar{\psi}\psi)\right] \rangle_h && \text{(6.101b)} \end{aligned}$$

for every h, and therefore also for the derivatives with respect to h. But consider for example the magnetization. The insertion of the given factor F in the un-normalized expectation provides the relation

$$\begin{aligned} \langle (n - \bar{\psi}\psi) \rangle_h^{\mathrm{sym}} &= \langle (n - \bar{\psi}\psi)\, F \rangle = \langle (n - \bar{\psi}\psi) \rangle_h + && \text{(6.102a)} \\ &+ \langle \left[-h^2\,(\bar{\psi}\mathrm{J}\psi) + h\,(n - \bar{\psi}\psi)\right] \left(-\frac{\partial}{\partial h}\right) \exp\left[-\,h\,(n - \bar{\psi}\psi)\right] \rangle && \text{(6.102b)} \\ &= \, 2\,\langle (n - \bar{\psi}\psi) \rangle_h - 2\,h\,\langle (\bar{\psi}\mathrm{J}\psi) \rangle_h = 0 && \text{(6.102c)} \end{aligned}$$

Similarly

$$\begin{aligned} \langle (\bar{\psi}\mathrm{J}\psi) \rangle_h^{\mathrm{sym}} &= \langle (\bar{\psi}\mathrm{J}\psi)\, F \rangle && \text{(6.103a)} \\ &= \langle (\bar{\psi}\mathrm{J}\psi) \rangle_h - h \frac{\partial}{\partial h} \langle (\bar{\psi}\mathrm{J}\psi) \rangle_h && \text{(6.103b)} \\ &= \langle (\bar{\psi}\mathrm{J}\psi) \rangle_h - h \frac{\partial}{\partial h} \frac{\langle (n - \bar{\psi}\psi) \rangle_h}{h} && \text{(6.103c)} \\ &= \langle (\bar{\psi}\mathrm{J}\psi) \rangle_h + \left(\frac{1}{h} - \frac{\partial}{\partial h}\right) \langle (n - \bar{\psi}\psi) \rangle_h && \text{(6.103d)} \\ &= 2\,\langle (\bar{\psi}\mathrm{J}\psi) \rangle_h + \langle (n - \bar{\psi}\psi)^2 \rangle_h && \text{(6.103e)} \\ &= \sum_{i,j} \langle \bar{\psi}_i\psi_j + \bar{\psi}_j\psi_i + (1 - \bar{\psi}_i\psi_i)(1 - \bar{\psi}_j\psi_j) \rangle_h && \text{(6.103f)} \\ &= \sum_{i,j} \langle 1 - f_{\{i,j\}} \rangle_h && \text{(6.103g)} \end{aligned}$$

is the *total*, not-connected, susceptibility, that is the sum of the longitudinal and transverse not-connected ones. And also

$$\langle (n - \bar{\psi}\psi)^2 \rangle_h^{\mathrm{sym}} = -\,2\,\langle (\bar{\psi}\mathrm{J}\psi) \rangle_h - \langle (n - \bar{\psi}\psi)^2 \rangle_h \tag{6.104}$$

so that

$$2\,\langle (\bar{\psi}\mathrm{J}\psi) \rangle_h^{\mathrm{sym}} + \langle (n - \bar{\psi}\psi)^2 \rangle_h^{\mathrm{sym}} = 2\,\langle (\bar{\psi}\mathrm{J}\psi) \rangle_h + \langle (n - \bar{\psi}\psi)^2 \rangle_h \tag{6.105}$$

as it must occur for a symmetric observable.

6.7 Ward identities

As a result of the underlying symmetry, there are relations among the correlation functions, called *Ward identities* [80]. In this Section we give a more direct derivation of one of them which simply uses integration by parts.

By definition

$$\mathcal{U}(\xi) \;=\; \xi + (1-k)\frac{\xi^k}{k!} \tag{6.106}$$

so that

$$\frac{\partial \mathcal{U}}{\partial \xi} = 1 - \frac{\xi^{k-1}}{(k-2)!} \tag{6.107}$$

and therefore the un-normalized expectation value of $\bar\psi\psi$ in presence of the symmetry breaking is

$$\lambda \,\langle \bar\psi\psi\rangle_h \;=\; n! \oint \frac{\mathrm{d}\xi}{2\pi i}\,\frac{1}{\xi^{n+1}}\,\exp\left[\lambda\mathcal{U} + n\frac{\xi^{k-1}}{(k-1)!} - h\,\lambda\,(n-\xi)\right]\lambda\,\xi\,\frac{\partial\mathcal{U}}{\partial\xi} \tag{6.108a}$$

$$=\; n! \oint \frac{\mathrm{d}\xi}{2\pi i}\,\frac{1}{\xi^{n+1}}\,\exp\left[n\frac{\xi^{k-1}}{(k-1)!} - h\,\lambda\,(n-\xi)\right]\xi\,\frac{\partial}{\partial\xi}\,e^{\lambda\mathcal{U}}\,. \tag{6.108b}$$

Perform now an integration by parts

$$\lambda \,\langle \bar\psi\psi\rangle_h \;=\; n! \oint \frac{\mathrm{d}\xi}{2\pi i}\,\frac{1}{\xi^{n+1}}\,\left\{n\left[1 - \frac{\xi^{k-1}}{(k-2)!}\right] - h\,\lambda\,\xi\right\}$$

$$\times \exp\left[\lambda\mathcal{U} + n\frac{\xi^{k-1}}{(k-1)!} - h\,\lambda\,(n-\xi)\right] \tag{6.109a}$$

$$=\; n\,\mathcal{Z}(\lambda,h) \;-\; h\,\lambda\,\langle(\bar\psi \mathrm{J}\psi)\rangle_h\,. \tag{6.109b}$$

So that

$$n\,\mathcal{Z}(\lambda,h) \;-\; \lambda\,\langle\bar\psi\psi\rangle \;=\; h\,\lambda\,\langle(\bar\psi\mathrm{J}\psi)\rangle \tag{6.110}$$

which expanded in series of t implies that

$$\frac{n}{p}\,\langle(\mathcal{U}+h\bar\psi\psi)^p\rangle_{\lambda=0} \;-\; \langle\bar\psi\psi\,(\mathcal{U}+h\bar\psi\psi)^{p-1}\rangle_{\lambda=0} \;=\; h\,\langle(\bar\psi\mathrm{J}\psi)(\mathcal{U}+h\bar\psi\psi)^{p-1}\rangle_{\lambda=0} \tag{6.111}$$

or for $p = \alpha n$

$$1 \;-\; \alpha\,\frac{\langle\bar\psi\psi\,(\mathcal{U}+h\bar\psi\psi)^{\alpha n-1}\rangle_{\lambda=0}}{\langle(\mathcal{U}+h\bar\psi\psi)^{\alpha n}\rangle_{\lambda=0}} \;=\; \alpha\,h\,\frac{\langle(\bar\psi\mathrm{J}\psi)(\mathcal{U}+h\bar\psi\psi)^{\alpha n-1}\rangle_{\lambda=0}}{\langle(\mathcal{U}+h\bar\psi\psi)^{\alpha n}\rangle_{\lambda=0}} \tag{6.112}$$

which means that, in the microcanonical ensemble, for every h, we have

$$m(\alpha,h) \;=\; \frac{h}{2}\,\chi_T(\alpha,h)\,. \tag{6.113}$$

Summary

- We have found that in the k-uniform complete hypergraph with n vertices, in the limit of large n, the structure of the hyperforests with p hypertrees has an abrupt change when $p = \alpha_c n$ with $\alpha_c = (k-1)/k$.

- This change of behaviour is related to the appearance of a *giant* hypertree which covers a finite fraction of all the vertices. This change occurs when the number of hyperedges becomes $1/k(k-1)$, which is exactly the critical number of hyperdges in the phase transition of random hypergraphs at fixed number of hyperedges [71].

- If $\mathcal{Z}(\lambda)$ is the generating partition function of hyperforests, where the coefficient of λ^p is the total number of those with p hypertrees, in the limit of large n there is a corresponding singularity at $\lambda_c = n\,[(k-2)!]^{-1/(k-1)}$.

- In the Grassmann formulation this singularity can be described as a second-order phase transition associated to the breaking of a global OSP(1|2) supersymmetry which is non-linearly realized. The equilibrium state occurring in the broken phase can be studied by the introduction of an explicit breaking of the supersymmetry.

Chapter 7

Conclusions

We studied a model of spanning forests, defined by the following partition function

$$F_G(\lambda, \mathbf{w}) = \int \mathcal{D}_V(\psi, \bar{\psi}) \exp\left[\lambda \sum_{i\in V} \bar{\psi}_i \psi_i + \sum_{A\in E} w_A f_A^{(\lambda)}\right] ,$$

where f_As are elements of a particular sub-algebra of the Grassmann algebra generated by the fields. We showed that this simple model has an hidden non-linearly realized (global) supersymmetry which is ultimately responsible for the cancellation of the loops in the Feynman diagrams. The above supersymmetry is of type OSP$(1|2)$ and introducing bosonic field $\sigma = 1 - \lambda\bar{\psi}\psi$ one can see that the spanning forest model is perturbatively equivalent to a supersymmetric non-linear σ model with negative coupling $t = -\lambda$.

The model has been solved in a mean-field approximation by considering it as defined on a complete (hyper-)graph, where the action can be rewritten in terms of a single variable which is a quadratic combination of the fields.

We found two phases separated by a second order transition point at

$$\tilde{\lambda}_c = \frac{1}{[(k-2)!]^{1/(k-1)}} ,$$

where $\tilde{\lambda}$ is the rescaled coupling $\tilde{\lambda} = n\lambda$, associated to the breaking of the OSP$(1|2)$ supersymmetry.

The low-temperature (high-density) phase is characterized by the appearance of a *giant component*, which means that, in that regime, the typical forest configuration has a tree that occupies a finite fraction of the vertexes.

As one can expect on the ground of the correspondence with the σ model, the longitudinal susceptibility is negative, meaning that in the model there are event which are negatively correlated. This fact is also in agreement with one would expect from some conjectures about negative association in random-cluster models with $q < 1$.

In [66] has been conjectured that the upper critical dimension is 6, having now a good understanding of the mean-field theory we plan to investigate analytically the corrections to the mean-field approximation and thus to check this conjecture.

Part II

A fast algorithm for NP-hard problems

Chapter 8

A fast algorithm for NP-hard problems

Perhaps the most important outstanding question in theoretical computer science is whether the class P of decision problems that can be *solved* in polynomial time coincides with the class NP of problems for which a proposed solution can be *verified* in polynomial time. NP-complete problems are those to which any problem in NP can be reduced in polynomial time. At present, no polynomial-time algorithm has been found for any of the thousands of known NP-complete problems and it is hence widely believed that P $\neq$ NP. Likewise one can define a counting analogue of NP, denoted by #P, as the class of enumeration problems in which the structures being counted are recognizable in polynomial time. Clearly #P problems are as hard as problems in NP.

In parallel, in theoretical physics, there is a steady interest for problems related to graph theory and network design. In particular, the approach of statistical physics is to enclose the properties of a physical system in a partition function, which is a weighted sum over the states. Many interesting problems can be therefore restated as counting problems on graphs. Not surprisingly, these problems turn out to be themselves interesting problems in mathematics and theoretical computer science (see, for example, [88]).

To mention but two important examples: the partition function of a Potts antiferromagnet counts, at zero-temperature, the number of *vertex coloring* — an assignation of any of q different colors to each vertex so that neighboring vertices are colored differently — and that of the $O(N)$ model counts, in the $N \to 0$ limit and zero-temperature, the number of *Hamiltonian circuit* — a closed path containing each vertex exactly once. The interest of these counting problems comes from the fact that they, as many others [88], belong to #P class.

Finding efficient algorithms for solving those problems has therefore a two-fold interest: to allow physicists to study their models and to explore the effective computational hardness of NP problems.

Although it is very unlikely to find an efficient (with polynomial time requirements) algorithm, lowering the coefficient of the exponent can still make huge differences.

Algorithmic progress has been made by several, usually widely separated, communities.

On one hand, statistical physicists have shown that the relevant partition functions can be constructed in analogy with the path integral formulation of quantum mechanics. To this end, the configuration of a partially elaborated graph are encoded as suitable quantum states, and the constant-time surface is swept over the graph by means of a time evolution operator known as the transfer matrix. Although rarely stated, this approach is valid not only for regular lattices but also for arbitrary graphs.

On the other hand, graph theorists have used that graphs can be divided into "weakly interacting" subgraphs through a so-called *tree decomposition* (see [89]), and solutions obtained for the subgraphs can be recursively combined into a complete solution.

As one can expect, time and memory requirements for either problem grow exponentially in the number of vertices N. Using either of the above methods, the enumeration of vertex colorings of a randomly chosen planar graph with $N \sim 50$ is a quite lengthy calculation, if at all possible.

We shall show how the tree decomposition and transfer matrix methods can be combined in a very natural way, pushing the limit of feasibility of computations up to around $N \simeq 200$ vertices. The main idea is that the tree decomposition is compatible with a recursive generalization of the time evolution concept. Borrowing ideas from string theory, the combination of partial solutions is obtained by the fusion of suitable quantum spaces. The resulting algorithm works on any graph, and can readily be adapted to many other problems of statistical physics, by suitable modifications of the state spaces and the fusion procedure.

In this chapter we shall consider the problem of computing the Potts model partition function using a transfer matrix based algorithm that work for general, non layered, graphs.

We shall first review the transfer matrix algorithm for the Potts model partition function in the case where the underlying graph has a layered structure, we shall follow the presentation given in [90]. We will later show that the same approach is valid also for arbitrary graphs. Finally, to show how the same algorithm can be adapted to very different problems, we shall carry out the computation of the Hamiltonian chains partition function using the same formalism.

8.1 Classical transfer matrix algorithm

For the reader's convenience let us recall here the Fortuin-Kasteleyn expansion

$$\mathcal{Z}(q, \mathbf{v}) = \sum_{E' \subseteq E} q^{k(E')} \prod_{e \in E'} v_e \,. \tag{8.1}$$

The key point of the transfer matrix approach is in the way it handles the non-local factor $q^{k(E')}$. The technique for handling this nonlocality was foreshadowed in the work of Lieb and Beyer [91] on percolation and was made explicit (for the case of the chromatic polynomial) in the work of Biggs and collaborators [92, 93, 94]. In the physics literature, this approach seems to have been used for the first time by Derrida and Vannimenus [95] in their study of percolation, and was applied to the q-state Potts model (and explained very clearly) by Blöte and Nightingale [96]; it was subsequently employed by several groups [97, 98, 99, 100].

Consider a graph $G_n = (V_n, E_n)$ consisting of n identical "layers", with connections between adjacent layers repeated in a regular fashion.

To make this precise, let V^0 be the set of vertices in a single layer, let E^0 be the set of edges within a single layer (we call these *horizontal* edges), and let E^* be the set of edges connecting each layer to the next one (we call these *vertical* edges). The vertex set of the graph G_n is then

$$V_n \;=\; V^0 \times \{1,\ldots,n\} \;=\; \{(x,i)\colon\; x \in V^0 \text{ and } 1 \le i \le n\}\;, \tag{8.2}$$

while the edge set is, considering free longitudinal boundary conditions,

$$E_n = \bigcup_{i=1}^{n} E_i^{\text{horiz}} \bigcup \bigcup_{i=1}^{n-1} E_i^{\text{vert}} \tag{8.3}$$

where

$$E_i^{\text{horiz}} \;=\; \{\langle (x,i),(x',i)\rangle\colon\; \langle x,x'\rangle \in E^0\} \tag{8.4}$$

and

$$E_i^{\text{vert}} \;=\; \{\langle (x,i),(x',i+1)\rangle\colon\; \langle x,x'\rangle \in E^*\} \tag{8.5}$$

and of course layer $n+1$ is identified with layer 1. We also assume that the couplings are identical from layer to layer: that is, we are given weights $\{v_e\}_{e\in E^0\cup E^*}$, and we define the edge weights for G_n by

$$v_{(x,i)(x',i)} \;=\; v^0_{xx'} \qquad \text{for } \langle x,x'\rangle \in E^0 \tag{8.6a}$$

$$v_{(x,i)(x',i+1)} \;=\; v^*_{xx'} \qquad \text{for } \langle x,x'\rangle \in E^* \tag{8.6b}$$

The basic idea is to build up the subgraph $E' \subseteq E$ layer by layer. At the end we will need to know the number of connected components in this subgraph; in order to be able to compute this, we shall keep track, as we go along, of which sites in the current "top" layer are connected to which other sites in that layer by a path of occupied bonds (i.e. bonds of E') in lower layers. Thus, we shall work in the basis of connectivities of the top layer, whose basis elements $\mathbf{v}_{\mathcal{P}}$ are indexed by partitions $\mathcal{P}$ of the single-layer vertex set V^0. The elementary operators we shall need are the *join operators*

$$\mathrm{J}_{xx'}\mathbf{v}_{\mathcal{P}} = \mathbf{v}_{\mathcal{P}\bullet xx'} \tag{8.7}$$

(note that all these operators commute) and the *detach operators*

$$\mathrm{D}_x\mathbf{v}_{\mathcal{P}} = \begin{cases} \mathbf{v}_{\mathcal{P}\setminus x} & \text{if } \{x\} \notin \mathcal{P} \\ q\,\mathbf{v}_{\mathcal{P}} & \text{if } \{x\} \in \mathcal{P} \end{cases} \tag{8.8}$$

where $\mathcal{P} \bullet xx'$ is the partition obtained from $\mathcal{P}$ by amalgamating the blocks containing x and x' (if they were not already in the same block) and $\mathcal{P} \setminus x$ is the partition obtained from $\mathcal{P}$ by detaching x from its block (and thus making it a singleton).

The action of the operators D_x and $\mathrm{J}_{xx'}$ on the basis vectors $\mathbf{v}_{\mathcal{P}}$ is therefore quite simple: as one might expect, $\mathrm{J}_{xx'}$ joins sites x and x' and D_x detaches site x from the block it currently belongs to, multiplying by a factor q if x is currently a singleton.

The horizontal transfer matrix is then

$$\mathrm{H} = \prod_{\langle x,x'\rangle\in E^0} \left[1 + v^0_{xx'}\mathrm{J}_{xx'}\right]\;, \tag{8.9}$$

where the product extends to the edges E^0 contained in the same layer. The vertical transfer matrix is slightly more complicated:

$$\mathrm{V}\mathbf{v}_{\mathcal{P}} = \sum_{\widetilde{E} \subseteq E^*} q^{A(\mathcal{P},\widetilde{E})} \left(\prod_{\langle x,x' \rangle \in \widetilde{E}} v^*_{xx'} \right) \mathbf{v}_{\mathcal{P}|\widetilde{E}} \tag{8.10}$$

where $A(\mathcal{P}, \widetilde{E})$ is the number of "abandoned clusters", i.e. the number of blocks $P \in \mathcal{P}$ such that no vertex in P is an endpoint of an edge in $\widetilde{E}$; and $\mathcal{P}|\widetilde{E}$ is the partition of V^0 in which vertices x', y' lie in the same block if and only if there exist vertices x, y in the same block of $\mathcal{P}$ such that both (x, x') and (y, y') lie in $\widetilde{E}$.

If the graph G is planar, V can be written as a product of sparse matrices that correspond to the replacement of *one* site x on layer i by the corresponding site x' on layer $i+1$; and these sparse matrices have a simple expression in terms of join and detach operators. Numbering the sites of V^0 as $1, \ldots, m$, we have

$$\mathrm{V} = \prod_{x=1}^{m} \left[v^*_{xx'} 1 + \mathrm{D}_x \right] . \tag{8.11}$$

Finally, the partition function for *free* longitudinal boundary conditions can be obtained by sandwiching the transfer matrix between suitable vectors on right and left:

$$\mathscr{Z}_G(q, \mathbf{v}) = \mathbf{u}^{\mathrm{T}} \mathrm{H} (\mathrm{VH})^{n-1} \mathbf{v}_{\mathrm{id}} , \tag{8.12}$$

where "id" denotes the partition in which each site $x \in V^0$ is a singleton, and $\mathbf{u}^{\mathrm{T}}$ is defined by

$$\mathbf{u}^{\mathrm{T}} \mathbf{v}_{\mathcal{P}} = q^{|\mathcal{P}|} . \tag{8.13}$$

The dimension of the space in which operators $\mathrm{J}_{xx'}$ and D_x act is the number of non-crossing partitions on $n = |V^0|$ vertices. This quantity is known as the Catalan number C_n and its generating function is

$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z} .$$

We have that $C_n = 4^n n^{-3/2} \pi^{-1/2} [1 + O(1/n)]$. If the graph is not planar we need to consider also crossing partitions, in this case the dimensionality is given by the Bell numbers B_n, whose generating function is

$$B(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \exp(\mathrm{e}^z - 1) ,$$

and which grow super-exponentially.

8.2 Transfer matrices for general graphs

Even if rarely stated, the above procedure can be used also for graphs with no particular structure. The idea is that the role of the detach operator D_x in (8.11) is to "forget" the particular configuration of a vertex x by summing over its states and thus supplying the appropriate factor.

Let us show the new procedure by an example, and consider the graph in Figure 8.1, where without any loss of generality we consider every edge having the same coupling v.

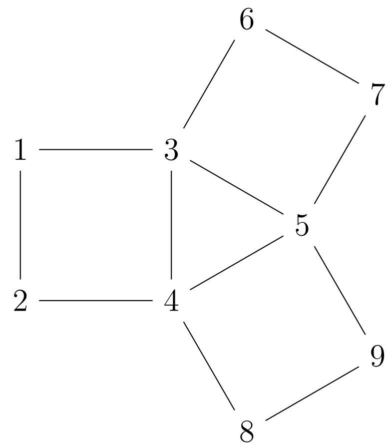


Figure 8.1

In analogy to the previous slicing of the graph in layers, the first step is to choose an order $\{v_i\}$ in which vertices will be processed. This order is the basis for the construction of a "time slicing" of the graph. Indeed, to each step we associate a *bag* of active vertices, a vertex v_i becomes active as soon as one of its neighbors gets processed and it stays active until it gets processed itself. Taking the vertices in lexicographic order we obtain the following time decomposition:

$$\begin{array}{c} (\mathbf{1}\ 2\ 3) \to (\mathbf{2}\ 3\ 4) \to (\mathbf{3}\ 4\ 5\ 6) \to (\mathbf{4}\ 5\ 6\ 8) \\ \downarrow \\ (\mathbf{9}) \leftarrow (\mathbf{8}\ 9) \leftarrow (\mathbf{7}\ 8\ 9) \leftarrow (\mathbf{6}\ 7\ 8\ 9) \leftarrow (\mathbf{5}\ 6\ 7\ 8\ 9) \end{array} \tag{8.14}$$

where we wrote in bold face the vertex processed at each time step.

Each bag has its own basis state consisting of the partitions of the currently active vertices. As in the layered case the partitions represent how the active vertices are interconnected through the restriction E' in (8.1) to the set of edges having been already processed. A *state* is a linear combination of basis states, with coefficients equal to the partially built partition function (8.1).

Processing a vertex consists in processing edges connecting it to its unprocessed neighbors and then deleting it. These operations are implemented in term of operators whose composition will form the terms of the sum over all subgraph in eq. (8.1). Since each edge may or may not be in E' we process an edge $\langle i, j\rangle$ by acting on the state with an operator of the form

$$1 + v\mathrm{J}_{ij}$$

where J_{ij} is the previously defined join operator. Vertex deletion is defined in terms of the deletion operator D_i.

Turning back to our example (8.14), the state just after having processed edges incident to vertex 1 is

$$\big(1{+}v\mathrm{J}_{12}\big)\big(1{+}v\mathrm{J}_{13}\big)\big|\ \overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle = \big|\ \overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle + v\,\big|\ \overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle + v\,\big|\ \overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle + v^2\,\big|\ \overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle\,,$$

and finally, after vertex 1 is processed (deleted) through the action of D_1, this becomes

$$\big|s_1\big\rangle = (q+2v)\big|\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle + v^2\,\big|\ \overset{2}{\circ}\ \overset{3}{\circ}\ \big\rangle\,, \tag{8.15}$$

concluding the first time step.

In this procedure we understood a mapping between a bag and its subsequent in which new vertices are inserted in each partition composing the current state as singletons. After processing the last bag, the complete partition function (8.1) is obtained as the coefficient of the empty partition resulting from the deletion of the last active vertex.

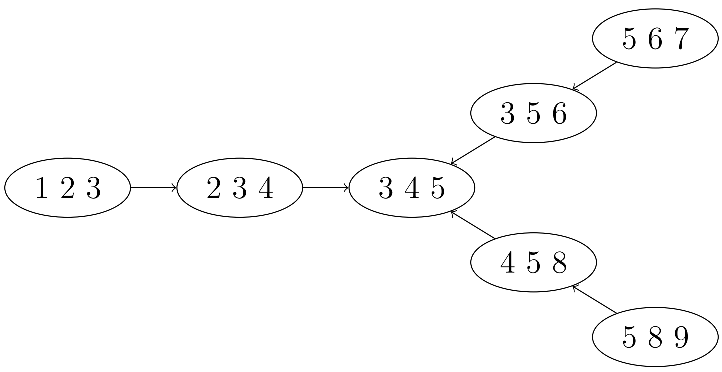
The number of partitions we need to keep track of when processing a bag of size n is given by C_n (B_n if the graph is not planar).

8.3 Tree decompositions

The decomposition (8.14) of G is a special case of a more general construction. By definition, a *tree decomposition* of a graph $G = (V, E)$ is a collection of bags (vertex subsets) organized as a tree, and satisfying the following requirements:

1. For each $i \in V$, there exists a bag containing i
2. For each $\langle i, j\rangle \in E$, there exists a bag containing both i and j
3. For any $i \in V$, the set of bags containing i is connected in the tree

The previous decomposition happens to be a special case of a tree decomposition (a *path decomposition*). As an example of the general construction, consider the following



(8.16)

The transfer matrix approach can be adapted naturally to this new general setting. Properties 1)–2) guarantee that each edge and vertex can be processed within a definite bag. Property 3) implies that each vertex has a definite life time within the recursion, its insertion and deletion being separated by the processing of all edges incident on it.

In this new version the algorithm starts from the root of the tree, which can be chosen arbitrarily, and runs through the tree recursively, going up from one daughter bag D to its parent P implies deleting vertices $D \setminus P$, inserting vertices $P \setminus D$ and finally processing edges in P. A tree decomposition does not specify when edges have to be processed, a simple recipe is to process edges as soon as one comes across a bag containing both its endpoints. However we observe that this freedom of choice can be exploited to optimize the algorithm as we will see later on.

The advantage of working with tree instead of path decompositions relies on the fact that in the former case a decomposition with smaller bags can be obtained (the latter being just a special case). Therefore the number of states one has to keep track of is exponentially smaller and the gain is significant.

8.3.1 The fusion procedure

When a parent bag P has several daughters D_ℓ with $\ell = 1, 2, \ldots, d$, these deletions and insertions are separated by a special *fusion* procedure. Suppose first $d = 2$. Let $\mathcal{P}_1$ be a partition of $D_1 \cap P$ with weight w_1, and $\mathcal{P}_2$ a partition of $D_2 \cap P$ with weight w_2. We can define a set E_1 by writing

$$\mathcal{P}_1 = \prod_{e \in E_1} \mathrm{J}_e\, \mathcal{S}_1\,,$$

where $\mathcal{S}_1$ is the all-singleton partition of $D_1 \cap P$. Similarly define E_2 from $\mathcal{P}_2$. The fused state is therefore

$$\mathcal{P}_1 \otimes \mathcal{P}_2 = \prod_{e \in E_1 \cup E_2} \mathrm{J}_e\, \mathcal{S}_{12}$$

and it occurs with weight $w_1 w_2$, where $\mathcal{S}_{12}$ is the all-singleton partition of $(D_1 \cup D_2) \cap P$. For $d > 2$ daughters, the complete fusion can be accomplished by fusing D_1 with D_2, then fusing the result with D_3, and so on.

Let us illustrate the fusion procedure for G with decomposition (8.16). After processing the bags on the left branch and deleting vertex 2, the state is

$$\mathrm{J}_{34}\mathrm{D}_2(1 + v\mathrm{J}_{24})\big|s_1\big\rangle = \omega_1 \Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ} \,\Big\rangle + \omega_2 \Big|\, \overset{3}{\circ}\!\smile\!\overset{4}{\circ} \,\Big\rangle = (\omega_1 + \omega_2 \mathrm{J}_{34}) \Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ} \,\Big\rangle\,,$$

where for convenience we set $\omega_1 = q^2 + 3qv + 3v^2$ and $\omega_2 = q^2 v + 3qv^2 + 4v^3 + v^4$. By symmetry, the same result is obtained for the top right and bottom right bags replacing respectively 4 by 5 and 3 by 5. The fused state arriving in the central (root) bag is thus

$$\begin{aligned}\big|s_{root}\big\rangle =&(\omega_1 + \omega_2\mathrm{J}_{34})(\omega_1 + \omega_2\mathrm{J}_{35})(\omega_1 + \omega_2\mathrm{J}_{45}) \Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ} \;\; \overset{5}{\circ} \,\Big\rangle \\ =& \,\omega_1^3 \Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ} \;\; \overset{5}{\circ} \,\Big\rangle + \omega_1^2\omega_2 \Big[\Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ}\!\smile\!\overset{5}{\circ} \,\Big\rangle + \Big|\, \overset{3}{\circ}\!\smile\!\overset{4}{\circ} \;\; \overset{5}{\circ} \,\Big\rangle + \Big|\, \overset{3}{\circ} \;\; \overset{4}{\circ} \;\; \overset{5}{\circ} \,\Big\rangle \Big] \\ & + (3\omega_1\omega_2^2 + \omega_2^3) \Big|\, \overset{3}{\circ}\!\smile\!\overset{4}{\circ}\!\smile\!\overset{5}{\circ} \,\Big\rangle\,.\end{aligned}$$

At this last stage we already proceed all edges and we need to delete the remaining vertices, therefore

$$\mathrm{D}_3\mathrm{D}_4\mathrm{D}_5\big|s_{root}\big\rangle = \big[q^3\omega_1^3 + 3q^2\omega_1^2\omega_2 + 3q\omega_1\omega_2^2 + q\omega_2^3\big]\,\big|\big\rangle \tag{8.17}$$

which gives us the final result:

$$\begin{aligned}\mathcal{Z}(q,v) = {} & q^9 + 12\,q^8\,v + 66\,q^7\,v^2 + 219\,q^6\,v^3 + \,q^7\,v^3 + 483\,q^5\,v^4 + 12\,q^6\,v^4 + 729\,q^4\,v^5 \\ & + 63\,q^5\,v^5 + 741\,q^3\,v^6 + 180\,q^4\,v^6 + 3\,q^5\,v^6 + 468\,q^2\,v^7 + 306\,q^3\,v^7 + 18\,q^4\,v^7 \\ & + 144\,q\,v^8 + 297\,q^2\,v^8 + 54\,q^3\,v^8 + 136\,q\,v^9 + 81\,q^2\,v^9 + 3\,q^3\,v^9 + 57\,q\,v^{10} \\ & + 9\,q^2\,v^{10} + 12\,q\,v^{11} + \,q\,v^{12}\end{aligned}$$

8.4 Optimizations and performance

Pruning

In the coloring case, where $v = -1$ the operator associated to an edge happens to be a projector, indeed, since $\mathrm{J}_{ij}^2 = \mathrm{J}_{ij}$, we have $(1 - \mathrm{J}_{ij})^2 = 1 - \mathrm{J}_{ij}$. The edge operator thus annihilates the subspace in which vertices i and j are already connected. This fact can be exploited to reduce the number of partition that one needs to keep track of during the algorithm. Indeed one can discard all the states in which two vertices are connected as soon as one discovers that an edge between them is going to be processed in the following. Especially before fusions this simple trick reduces substantially the number of states and thus leads to a big speed up.

Reordering

For a planar graph, the state of a bag of size n is spanned by C_n basis (For a non-planar graph, replace C_n by B_n.) The memory needed by the algorithm is therefore proportional to $C_{n_{\max}}$, where $n_{\max}$ is the size of the largest bag. The time needed to process one edge in a bag of size n is proportional to C_n.

In practice, however, most of the time is spent fusing states. For a parent P with d daughters D_ℓ, the number of basis state pairs to be fused is

$$\sum_{\ell=1}^{d} C_{|\mathcal{D}_{\ell-1}\cap P|} C_{|D_\ell \cap P|}\,, \tag{8.18}$$

where we have set $\mathcal{D}_k = \cup_{\ell=1}^{k} D_\ell$. Each of these elementary fusions can be done in time linear in the number of participating vertices. Note that we can choose the order of successive fusions so as to minimize the quantity (8.18).

Finding a good tree decomposition

It is essential for the algorithm that one knows how to obtain a good tree decomposition. The minimum of $n_{\max} - 1$ over all tree decompositions is known as the tree width k, but obtaining this is another NP-hard problem. However, the simple algorithm `GreedyFillIn` [102] gives an upper bound k_0 on k and a tree decomposition of width k_0 in time *linear* in the number of vertices N. For uniformly generated planar graphs we find that for $N = 40$—a value enabling comparison with algorithms that determine k exactly—that $k_0 = k$ with probability 0.999 and $k_0 = k + 1$ with probability 0.001.

Benchmarks

We choose to test our algorithm against the one presented by Haggard et al. in [101]. We first generated a uniform sample of 100 planar graphs for each sizes between 20 and 100 using Fusy's algorithm [103], then we ran three different algorithms over this sample: the algorithm of Haggard et al., our first path-based transfer matrix algorithm, the new tree-based version algorithm and tree-based version using the pruning optimization. The average times are presented in Figure 8.2.

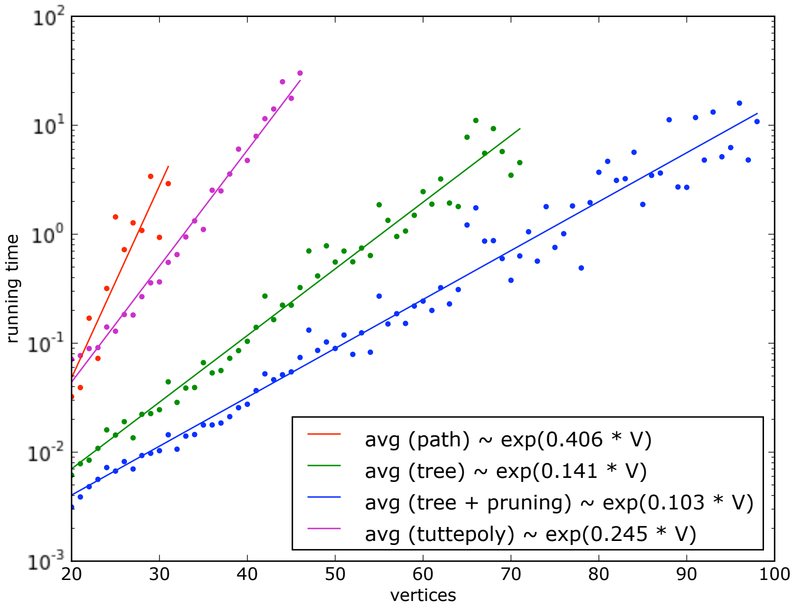


Figure 8.2: Algorithm performance: average running time for the path-algorithm of Section 8.2, the tree-algorithm of Section 8.3, the tree-algorithm with pruning optimization of Section 8.4, and the tuttepoly algorithm [101]. Each point is averaged over 100 planar graphs sampled with the Fusy's algorithm [103].

8.5 Hamiltonian circuits and walks.

To demonstrate the versatility of the tree-decomposed transfer-matrix algorithm, we now show how it can be adopted to the completely different problem of *Hamiltonian chains* [104].

The subject of Hamiltonian circuits and walks plays an important role in mathematics and physics alike. Given a connected undirected graph G, a *Hamiltonian circuit* (or cycle) is a cycle (i.e., a closed loop) through G that visits each of the V vertices of G exactly once [105]. In particular, a Hamiltonian circuit has length V. Similarly, a *Hamiltonian walk* (or path) is an open non-empty path (i.e., with two distinct extremities) of length $V-1$ that visits each vertex exactly once. Note that a Hamiltonian circuit can be turned into a Hamiltonian walk by removing any one of its edges, whereas a Hamiltonian walk can be extended into a Hamiltonian circuit only if its end points are adjacent in G.

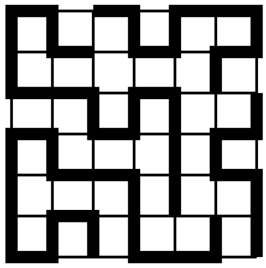


Figure 8.3: Hamiltonian chain of order 4 on a square lattice of size 7×7.

We add now to this list of well-known definitions the set $\mathcal{C}_k$ of *Hamiltonian chains* of order k. Each member in $\mathcal{C}_k$ is a set of k disjoint paths whose union visits each vertex of G exactly once (see Fig. 8.3). The set of Hamiltonian walks is then $\mathcal{C}_1$, and by convention we shall let $\mathcal{C}_0$ denote the set of Hamiltonian circuits. Note that if V is even, $\mathcal{C}_{V/2}$ is the set of dimer coverings of G.

Determining whether G contains a Hamiltonian circuit is a difficult (NP-complete) problem. An even more difficult problem is to determine *how many* distinct Hamiltonian circuits are contained in G. We shall approach this problem by considering the following partition function

$$\mathcal{Z}_G(x) = \sum_{k \geq 0} H_k \, x^k \,, \tag{8.19}$$

where H_0 (respectively H_1) are the numbers of Hamiltonian circuits (respectively walks) on G, and H_k for $k \geq 2$ is the number of ways to cover G with k disjoint paths (with distinct end points) whose union visits each lattice vertex exactly once.

The only necessary modifications to the algorithm presented in the previous section concern the state space description and the operators definitions. In a state, active vertices can have degree 0, 1 or 2. We shall say that it is *empty* if it has degree 0, that is an *endpoint* if it has degree 1, and that is *complete* if it has degree 2. Two endpoints might or might not be connected in pairs, while empty and complete vertices are always singletons. We shall use the graphical representation in Figure 8.4 for these three states:

Again the sprit is to describe a partially built configuration: endpoints are connected when they are extremities of the same chain and complete vertices correspond to vertices inside a chain.

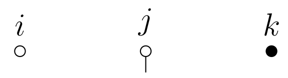


Figure 8.4: An empty vertex i, an endpoint j, and a complete vertex k.

The number of configurations in this state space can be determined considering that a configuration consists either of a smaller configuration followed by either one empty vertex, one endpoint or one complete vertex, either of two linked vertices enclosing another configuration. Therefore we can write the following generating function

$$S(z) = 1 + (1 + a + c)\, z\, S(z) + l\, z^2\, S(z)^2 \tag{8.20}$$

where we gave weight a, c and l respectively to an endpoint, a complete vertex and a pair of joined vertices. Setting $a = c = l = 1$, and solving for $S(z)$ we obtain

$$S(z) = \frac{1 - 3z - \sqrt{(1-z)(1-5z)}}{2z^2}\,, \tag{8.21}$$

whose Taylor expansion around $z = 0$ is

$$S(z) = \sum_{n\geq 0} S_n z^n \simeq 1 + 3z + 10z^2 + 36z^3 + 137z^4 + 543z^5 + O(z)^6 \tag{8.22}$$

From (8.21) we clearly see that S_n grow as 5^n for large n.

The operators acting on these states are now more involved in respect to the previous case. The action of J_{ij} is defined as follows:

$$\mathrm{J}_{ij}\,\big|\ \overset{i}{\bullet}\ \ \overset{j}{*}\ \big\rangle = 0 \qquad\qquad \mathrm{J}_{ij}\,\big|\ \overset{i}{\circ}\ \ \overset{j}{\circ}\ \big\rangle = \big|\ \underbrace{\overset{i}{\circ}\ \ \overset{j}{\circ}}\ \big\rangle \tag{8.23a}$$

$$\mathrm{J}_{ij}\,\big|\ \overset{i}{\circ}\ \ \overset{j}{\underset{|}{\circ}}\ \big\rangle = \big|\ \overset{i}{\underset{|}{\circ}}\ \ \overset{j}{\bullet}\ \big\rangle \qquad\qquad \mathrm{J}_{ij}\,\big|\ \overset{i}{\underset{|}{\circ}}\ \ \overset{j}{\underset{|}{\circ}}\ \big\rangle = x\,\big|\ \overset{i}{\bullet}\ \ \overset{j}{\bullet}\ \big\rangle \tag{8.23b}$$

$$\mathrm{J}_{ij}\,\big|\ \overset{i}{\circ}\ \ \underbrace{\overset{j}{\circ}\ \ \overset{k}{\circ}}\ \big\rangle = \big|\ \underbrace{\overset{i}{\circ}\ \ \overset{j}{\bullet}\ \ \overset{k}{\circ}}\ \big\rangle \qquad\qquad \mathrm{J}_{ij}\,\big|\ \overset{i}{\underset{|}{\circ}}\ \ \underbrace{\overset{j}{\circ}\ \ \overset{k}{\circ}}\ \big\rangle = \big|\ \overset{i}{\bullet}\ \ \overset{j}{\bullet}\ \ \overset{k}{\underset{|}{\circ}}\ \big\rangle \tag{8.23c}$$

$$\mathrm{J}_{ij}\,\big|\ \underbrace{\overset{i}{\circ}\ \ \overset{j}{\circ}}\ \big\rangle = 0 \tag{8.23d}$$

where $*$ means "any state" and the action of J_{ij} is symmetric in i and j.

For the delete operator we have:

$$\mathrm{D}_i\,\big|\ \overset{i}{\circ}\ \big\rangle = 0 \qquad\qquad \mathrm{D}_i\,\big|\ \overset{i}{\bullet}\ \big\rangle = \big|\ \big\rangle \tag{8.24}$$

$$\mathrm{D}_i\,\big|\ \overset{i}{\underset{|}{\circ}}\ \big\rangle = x\,\big|\ \big\rangle \qquad\qquad \mathrm{D}_i\,\big|\ \underbrace{\overset{i}{\circ}\ \ \overset{j}{\circ}}\ \big\rangle = \big|\ \overset{j}{\underset{|}{\circ}}\ \big\rangle \tag{8.25}$$

where the last two equations distinguish whether the endpoint i is a singleton or not. In all cases the vertex i is removed from the configuration.

Returning to the example graph in Figure 8.1, the state after having processed edges incident on vertex 1 is:

$$\big(1 + \mathrm{J}_{12}\big)\big(1 + \mathrm{J}_{13}\big)\big|\ \overset{1}{\circ}\ \ \overset{2}{\circ}\ \ \overset{3}{\circ}\ \big\rangle = \big|\ \overset{1}{\circ}\ \ \overset{2}{\circ}\ \ \overset{3}{\circ}\ \big\rangle + \big|\ \underbrace{\overset{1}{\circ}\ \ \overset{2}{\circ}}\ \ \overset{3}{\circ}\ \big\rangle + \big|\ \underbrace{\overset{1}{\circ}\ \ \overset{2}{\circ}\ \ \overset{3}{\circ}}\ \big\rangle + \big|\ \overset{1}{\bullet}\ \ \underbrace{\overset{2}{\circ}\ \ \overset{3}{\circ}}\ \big\rangle\,,$$

which after vertex 1 is processed becomes

$$\big|s_1\big\rangle = \big|\ \overset{2}{\text{⚲}}\ \ \overset{3}{\circ}\ \big\rangle + \big|\ \overset{2}{\circ}\ \ \overset{3}{\text{⚲}}\ \big\rangle + \big|\ \overset{2}{\text{⚲}}\!\smile\!\overset{3}{\text{⚲}}\ \big\rangle\,.$$

The partition function in 8.19 does not enumerated hamiltonian chains by the number of their edges therefore we need no factor in front of J.

As a last remark we need to observe that by definition the operator J_{ij} cannot close loops. This is always the correct behavior but at the moment of procesing the last edge. In that case, a closed loop is a valid configuration if all the others vertices in the bag are already marked as complete. In that case the operator J_{ij} has to be replaced by $\tilde{\mathrm{J}}_{ij}$ whose definition is identical to J_{ij}'s but allows the possibility of closing a loop. This operator is a bit special since it operates not only on the state but also on its weight (it is non linear)

$$\tilde{\mathrm{J}}_{ij}\,\omega\big|\ \overset{i}{\text{⚲}}\!\smile\!\overset{j}{\text{⚲}}\ \big\rangle = \begin{cases} \omega'\big|\ \overset{i}{\bullet}\ \ \overset{j}{\bullet}\ \big\rangle & \text{if all other vertices in the state are complete} \\ 0 & \text{otherwise} \end{cases}$$

where ω' is the constant (x independent) part of ω. This discard the weight of configurations containing both loops and walks. The definition of a fusion procedure requires a way of "decomposing" each the state $\mathcal{P}$ of the basis into a string of operators O_α.

$$\mathcal{P} = \prod_\alpha \mathcal{O}_\alpha\,\mathcal{S}\,,$$

where $\mathcal{S}$ is the state consisting of empty vertices. This decomposition is best implemented in terms of two more operators (in addition to J) A and C:

$$\mathrm{A}_i\big|\ \overset{i}{\circ}\ \big\rangle = \big|\ \overset{i}{\text{⚲}}\ \big\rangle \qquad\qquad \mathrm{A}_i\big|\ \overset{i}{\text{⚲}}\ \big\rangle = x\big|\ \overset{i}{\bullet}\ \big\rangle \tag{8.26a}$$

$$\mathrm{A}_i\big|\ \overset{i}{\text{⚲}}\!\smile\!\overset{j}{\text{⚲}}\ \big\rangle = \big|\ \overset{i}{\bullet}\ \ \overset{j}{\text{⚲}}\ \big\rangle \qquad\qquad \mathrm{A}_i\big|\ \overset{i}{\bullet}\ \big\rangle = 0 \tag{8.26b}$$

$$\mathrm{C}_i\big|\ \overset{i}{\circ}\ \big\rangle = \big|\ \overset{i}{\bullet}\ \big\rangle \qquad \mathrm{C}_i\big|\ \overset{i}{\text{⚲}}\ \big\rangle = 0 \qquad \mathrm{C}_i\big|\ \overset{i}{\bullet}\ \big\rangle = 0 \tag{8.26c}$$

Given a state $\mathcal{P}$, to obtain its decomposition we associate to each complete vertex i the operator C_i, to each endpoint j the operator A_j and to each pair of connected vertices k and l the operator J_{kl}. Since all those operators act on distinct vertices they commute and thus the order in which the decomposition is done is meaningless.

In our example, after processing the bags one the left branch and deleting vertex 2, the state is

$$\begin{aligned} \mathrm{J}_{34}\mathrm{D}_2(1+\mathrm{J}_{24})\big|s_1\big\rangle &= x\big|\ \overset{3}{\circ}\ \ \overset{4}{\circ}\ \big\rangle + \big|\ \overset{3}{\text{⚲}}\ \ \overset{4}{\circ}\ \big\rangle + \big|\ \overset{3}{\circ}\ \ \overset{4}{\text{⚲}}\ \big\rangle + \big|\ \overset{3}{\text{⚲}}\ \ \overset{4}{\text{⚲}}\ \big\rangle \\ &\quad + \big|\ \overset{3}{\text{⚲}}\ \ \overset{4}{\bullet}\ \big\rangle + \big|\ \overset{3}{\bullet}\ \ \overset{4}{\text{⚲}}\ \big\rangle + x\big|\ \overset{3}{\bullet}\ \ \overset{4}{\bullet}\ \big\rangle + (1+x)\big|\ \overset{3}{\text{⚲}}\!\smile\!\overset{4}{\text{⚲}}\ \big\rangle \qquad (8.27\text{a}) \\ &= \Big[x + \mathrm{A}_3 + \mathrm{A}_4 + \mathrm{A}_3\mathrm{A}_4 + \mathrm{A}_3\mathrm{C}_4 + \mathrm{C}_3\mathrm{A}_4 \\ &\quad + x\mathrm{C}_3\mathrm{C}_4 + (1+x)\mathrm{J}_{34}\Big]\big|\ \overset{3}{\circ}\ \ \overset{4}{\circ}\ \big\rangle \qquad (8.27\text{b}) \end{aligned}$$

Since this is the last stage, when processing edges we are now allowed to close a loop to keep track of the hamiltonian circuit configurations, using the short notation

$$\mathcal{O}_{ij} = x + \mathrm{A}_i + \mathrm{A}_j + \mathrm{A}_i\mathrm{A}_j + \mathrm{A}_i\mathrm{C}_j + \mathrm{C}_i\mathrm{A}_j + x\mathrm{C}_i\mathrm{C}_j + (1+x)\tilde{\mathrm{J}}_{ij}\,,$$

the fused state arriving in the central bag is the product

$$\big|s_{root}\big\rangle = \mathcal{O}_{34}\mathcal{O}_{35}\mathcal{O}_{45}\Big|\overset{3}{\circ}\;\;\overset{4}{\circ}\Big\rangle$$

which is a linear combination of 36 partitions. The result is again obtained deleting the remaining vertices:

$$\mathrm{D}_3\mathrm{D}_4\mathrm{D}_5\big|s_{root}\big\rangle = \big(1 + 15x + 81x^2 + 99x^3 + 24x^4\big)\,\big|\big\rangle \tag{8.28}$$

Giving the result $\mathcal{Z}_G(x) = 1 + 15x + 81x^2 + 99x^3 + 24x^4$.

8.6 Conclusions

We showed that a transfer matrix approach, which previously has been applied only to regular graphs can be generalized to work on general graphs as well, providing a good algorithm to solve enumeration problems located at the border between statistical mechanics and enumerative combinatorics.

Moreover this algorithm can exploit a tree decomposition to increase its efficiency even further. Finding an optimal tree decomposition is again an NP-problem but, at least in our tests, heuristics algorithms for obtaining a tree decomposition have proved to give good results.

We believe that the “modularity” of this framework adds much to its value. Indeed, improvements can come from very different directions, for example one might use more sophisticated algorithms to find a better tree decomposition, or might find a more efficient description of the states (reducing thus memory and time usage).

Appendix **A**

Basic notions about graphs and hypergraphs

A.1 Graphs

A (simple undirected finite) *graph* is a pair $G = (V, E)$, where V is a finite set and E is a collection (possibly empty) of 2-element subsets of V.[1] The elements of V are the *vertices* of the graph G, and the elements of E are the *edges*. Usually, in a picture of a graph, vertices are drawn as dots and edges as lines (or arcs). Please note that, in the present definition, loops ( ) and multiple edges ( ) are not allowed.[2] We write $|V|$ (resp. $|E|$) for the cardinality of the vertex (resp. edge) set; more generally, we write $|S|$ for the cardinality of any finite set S.

A graph $G' = (V', E')$ is said to be a *subgraph* of G (written $G' \subseteq G$) in case $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$, the subgraph is said to be *spanning*. We can, by a slight abuse of language, identify a spanning subgraph (V, E') with its edge set E'.

A *walk* (of length $k \geq 0$) connecting v_0 with v_k in G is a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ such that all $v_i \in V$, all $e_i \in E$, and $v_{i-1}, v_i \in e_i$ for $1 \leq i \leq k$. A *path* in G is a walk in which $v_0, \ldots, v_k$ are distinct vertices of G and $e_1, \ldots, e_k$ are distinct edges of G. A *cycle* in G is a walk in which

(a) $v_0, \ldots, v_{k-1}$ are distinct vertices of G, and $v_k = v_0$

(b) $e_1, \ldots, e_k$ are distinct edges of G; and

(c) $k \geq 2$.[3]

The graph G is said to be *connected* if every pair of vertices in G can be connected by a walk. The *connected components* of G are the maximal connected subgraphs of G. It is

[1] To avoid notational ambiguities it should also be assumed that $E \cap V = \emptyset$. This stipulation is needed as protection against the mad set theorist who, when asked to produce a graph with vertex set $V = \{0, 1, 2\}$, interprets this à la von Neumann as $V = \{\, \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\,\}$, so that the vertex 2 is indistinguishable from the edge $\{0, 1\}$.

[2] This restriction is made mainly for notational simplicity. It would be easy conceptually to allow multiple edges, by defining E as a *multiset* (rather than a *set*) of 2-element subsets of V (cf. also footnote 6 below).

[3] Actually, in a graph as we have defined it, all cycles have length ≥ 3 (because $e_1 \neq e_2$ and multiple edges are not allowed). We have presented the definition in this way with an eye to the corresponding definition for hypergraphs (see below), in which cycles of length 2 are possible.

not hard to see that the property of being connected by a walk is an equivalence relation on V, and that the equivalence classes for this relation are nothing other than the vertex sets of the connected components of G. Furthermore, the connected components of G are the induced subgraphs of G on these vertex sets.[4] We denote by $k(G)$ the number of connected components of G. Thus, $k(G) = 1$ if and only if G is connected.

A *forest* is a graph that contains no cycles. A *tree* is a connected forest. (Thus, the connected components of a forest are trees.) It is easy to prove, by induction on the number of edges, that

$$|E| - |V| + k(G) \geq 0 \tag{A.1}$$

for all graphs, with equality if and only if G is a forest.

In a graph G, a *spanning forest* (resp. *spanning tree*) is simply a spanning subgraph that is a forest (resp. a tree). We denote by $\mathcal{F}(G)$ [resp. $\mathcal{T}(G)$] the set of spanning forests (resp. spanning trees) in G. As mentioned earlier, we will frequently identify a spanning forest or tree with its edge set.

Finally, we call a graph *unicyclic* if it contains precisely one cycle (modulo cyclic permutations and inversions of the sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$). It is easily seen that a connected unicyclic graph consists of a single cycle together with trees (possibly reduced to a single vertex) rooted at the vertices of the cycle.

A.2 Hypergraphs

Hypergraphs are the generalization of graphs in which edges are allowed to contain more than two vertices. Unfortunately, the terminology for hypergraphs varies substantially from author to author, so it is important to be precise about our own usage. For us, a *hypergraph* is a pair $G = (V, E)$, where V is a finite set and E is a collection (possibly empty) of subsets of V, each of cardinality ≥ 2.[5] The elements of V are the *vertices* of the hypergraph G, and the elements of E are the *hyperedges* (the prefix "hyper" can be omitted for brevity). Note that we forbid hyperedges of 0 or 1 vertices (some other authors allow these).[6] We shall say that $A \in E$ is a k-*hyperedge* if A is a k-element subset of V. A hypergraph is called k-*uniform* if all its hyperedges are k-hyperedges. Thus, a graph is nothing other than a 2-uniform hypergraph.

The definitions of subgraphs, walks, cycles, connected components, trees, forests and unicyclics given above for graphs were explicitly chosen in order to immediately generalize to hypergraphs: it suffices to copy the definitions verbatim, inserting the prefix "hyper" as necessary. See Figure A.1 for examples of a forest and a hyperforest.

The analogue of the inequality A.1 is the following:

[4] If $V' \subseteq V$, the *induced subgraph* of G on V', denoted $G[V']$, is defined to be the graph (V', E') where E' is the set of all the edges $e \in E$ that satisfy $e \subseteq V'$ (i.e., whose endpoints are both in V').

[5] To avoid notational ambiguities it is assumed once again that $E \cap V = \emptyset$.

[6] Our definition of hypergraph is the same as that of McCammond and Meier [85]. It is also the same as that of Grimmett [84] and Gessel and Kalikow [40], except that these authors allow multiple edges and we do not: for them, E is a *multiset* of subsets of V (allowing repetitions), while for us E is a *set* of subsets of V (forbidding repetitions).

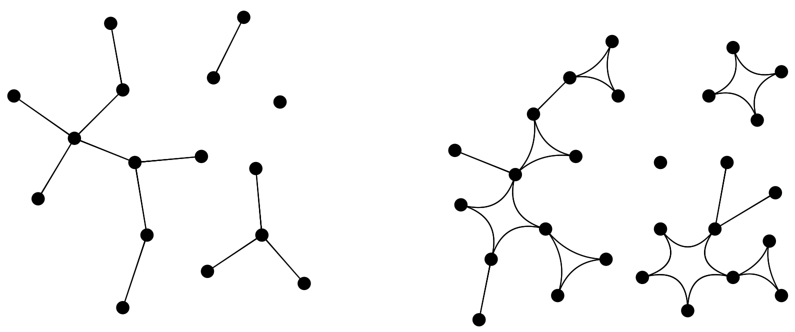


Figure A.1: A forest (left) and a hyperforest (right), each with four components. Hyperedges with more than two vertices are represented pictorially as star-like polygons.

Proposition A.2.1 *Let $G = (V, E)$ be a hypergraph. Then*

$$\sum_{A \in E} (|A| - 1) \;-\; |V| \;+\; k(G) \;\geq\; 0, \tag{A.2}$$

with equality if and only if G is a hyperforest.

Proofs can be found, for instance, in [86, p. 392, Proposition 4] or [40, pp. 278–279, Lemma].

Please note one important difference between graphs and hypergraphs: every connected graph has a spanning tree, but not every connected hypergraph has a spanning hypertree. Indeed, it follows from Proposition A.2.1 that if G is a k-uniform connected hypergraph with n vertices, then G can have a spanning hypertree only if $k-1$ divides $n-1$. Of course, this is merely a necessary condition, not a sufficient one! In fact, the problem of determining whether there exists a spanning hypertree in a given connected hypergraph is NP-complete (hence computationally difficult), even when restricted to the following two classes of hypergraphs:

(a) hypergraphs that are linear (each pair of edges intersect in at most one vertex) and regular of degree 3 (each vertex belongs to exactly three hyperedges); or

(b) 4-uniform hypergraphs containing a vertex which belongs to all hyperedges, and in which all other vertices have degree at most 3 (i.e., belong to at most three hyperedges)

(see [87, Theorems 3 and 4]). It seems to be an open question whether the problem remains NP-complete for 3-uniform hypergraphs.

Finally, let us discuss how a connected hypergraph can be built up one edge at a time. Observe first that if $G = (V, E)$ is a hypergraph without isolated vertices, then every vertex belongs to at least one edge (that is what "without isolated vertices" means!), so that $V = \bigcup\limits_{A \in E} A$. In particular this holds if G is a connected hypergraph with at least two vertices. So let $G = (V, E)$ be a connected hypergraph with $|V| \geq 2$; let us then say that an ordering $(A_1, \ldots, A_m)$ of the hyperedge set E is a *construction sequence* in case all of the hypergraphs $G_\ell = \big(\bigcup_{i=1}^{\ell} A_i,\ \{A_1, \ldots, A_\ell\}\big)$ are connected ($1 \le \ell \le m$). An equivalent condition is that $\left(\bigcup_{i=1}^{\ell-1} A_i\right) \cap A_\ell \neq \emptyset$ for $2 \le \ell \le m$. We then have the following easy result:

Proposition A.2.2 *Let $G = (V, E)$ be a connected hypergraph with at least two vertices. Then:*

(a) There exists at least one construction sequence.

(b) If G is a hypertree, then for any construction sequence $(A_1, \ldots, A_m)$ we have $\left|\left(\bigcup_{i=1}^{\ell-1} A_i\right) \cap A_\ell\right| = 1$ for all ℓ ($2 \le \ell \le m$).

(c) If G is not a hypertree, then for any construction sequence $(A_1, \ldots, A_m)$ we have $\left|\left(\bigcup_{i=1}^{\ell-1} A_i\right) \cap A_\ell\right| \ge 2$ for at least one ℓ.

Proof. (a) The "greedy algorithm" works: Let A_1 be *any* hyperedge; and at each stage $\ell \ge 2$, let A_ℓ be *any* hyperedge satisfying $\left(\bigcup_{i=1}^{\ell-1} A_i\right) \cap A_\ell \neq \emptyset$ (such a hyperedge has to exist, or else G fails to be connected). (b) and (c) are then easy consequences of Proposition A.2.1. $\square$

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