Functional Renormalization Group and symmetries: $\mathbb{Z}_N$-models

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To my parents.

At the top of the mountains.

Contents

Introduction

One of the more conspicuous and salient properties of Nature is the great diversity of size or length scales in the structure of the world. In general, events distinguished by a great disparity in size have little influence on each other; they don't communicate, and so phenomena associated with each scale can be fairly treated independently. Actually the success of almost all practical theories in Physics depends on isolating some limited range of length scale.

A class of phenomena does exist, however, where events at many length scales make contribution of equal importance. An example is the behaviour of a ferromagnet or permanent magnet. Ferromagnetic materials have a critical point called the Curie point or Curie temperature (for iron the Curie temperature is 1,044 K). At higher temperatures iron has no spontaneous magnetization. As iron is cooled, the magnetization remains zero until the Curie temperature is reached, and then the material smoothly becomes magnetized. If the temperature is reduced after, the strength of the magnetization increases smoothly. This means that the phase transition taking place at the critical temperature T_c is the result of an extraordinary collective phenomenon that involves all the spins of the system at once.

Multiple scales of length complicate many of the outstanding problems in theoretical physics. Exact solutions have been found for only a few of these problems, and for some others even the best known approximations are unsatisfactory. A method called renormalization group has been introduced for dealing with problems that have multiple scales of length. It has by no means made the problems easy, but some that have resisted all other approaches may yield to this one.

The renormalization group (RG) is not a descriptive theory of nature but a general method for constructing theories. The RG theory consists of a set of concepts and methods which can be used to understand phenomena in many different fields of Physics, ranging from quantum field theory over classical statistical mechanics to non-equilibrium phenomena, to a fluid at the critical point, but also to a ferromagnetic material at the temperature where spontaneous magnetization first sets in, or to a mixture of liquids at the temperature where they become fully miscible, or to an alloy at the temperature where two kinds of metal atoms take on an orderly distribution. Other problems that have a suitable form include turbulent flow, the onset of superconductivity and of superfluidity, the conformation of polymers and the binding together of elementary particles called quarks.

RG methods are particularly useful to understand phenomena where fluctuations involving many different length or time scales lead to the emergence of new collective behaviour in complex many-body systems. A remarkable hypothesis that seems to be confirmed by work with the renormalization group is that some of these phenomena, which superficially seem quite distinct, are identical at a deeper level. For example, the critical behaviour of fluids, ferromagnets, liquid mixtures and alloys can be described by a single theory.

This fundamental aspect of critical phenomena is called universality. In short, this means that despite the differences that two systems may have at their microscopic level, as long as they share some specific features we will explain after, their critical behaviours are surprisingly identical. It is for these universal aspects that the theory of phase transitions is one of the pillars of statistical mechanics and, simultaneously, of theoretical physics. As a matter of fact, it embraces concepts and ideas that have proved to be the building blocks of the modern understanding of the fundamental interactions in Nature. Universality finds in the framework of the renormalization group, its most elegant and natural demonstration.

The Wilsonian RG idea has been implemented in many different ways, depending on the particular problem at hand, and there seems to be no canonical way of setting up the RG procedure for a given problem. Fortunately, the formulation of the renormalization group in terms of a formally exact functional differential equation for the effective average action of a given theory, developed in early nineties by C.Wetterich (FRG), has unified the field by providing a mathematically elegant and yet simple way of expressing Wilson's idea of successive mode elimination.

By FRG or sometimes called "exact renormalization group (ERG)", we mean the continuous (i.e. not discrete) implementation of the Wilson renormalization group (RG) transformation of the action in which no approximation is made and also no expansion is involved with respect to some small parameter of the action. This technique has successfully been applied in many different areas of Physics: magnetic frustrated systems [14], out of equilibrium phase transitions [7], disordered systems [27] [44], BCS and Bose-Einstein condensation [15] [20], gauge theories [41] [40] [39] [6], gravity [11] [42] [2] and many others.

Interesting, and not yet explored from a FRG viewpoint, are those models known as $\mathbb{Z}_N$-models which exhibit global abelian discrete symmetry. In the past, these models played a major role in several fields of physics ranging from condensed-statistical physics to elementary particle physics.

In this thesis we started an analysis of these models by the non-perturbative functional renormalization group. The central issue has been the understanding of the phase structure of the various $\mathbb{Z}_N$ symmetric theories. In the spirit of the renormalization group ideas future work will be concentrated on the classification of all the universality class to which these theories, in continuos dimension, belong. Particular attention will be dedicated to the special bi-dimensional case where, from conformal field theory, we know a whole exciting world provided by parafermion statistics.

Part I

Foundation of the Renormalization Group

1

Phase transitions and the scaling hypothesis

In the vicinity of continuous phase transitions, thermodynamic quantities and correlation functions typically behave as power laws characterized by universal critical exponents, which are independent of the short distance details of a system. The universal behaviour showed in phase transitions results in the exact coincidence of the critical exponents for systems which are very different at a microscopic level.

In this spirit, the study of phase transitions is deeply related to the classification of all possible universality classes. The development of the Wilsonian RG in the 1970s was driven by the desire to gain a microscopic understanding of this universality.

1.1 Classification of phase transitions

According the modern classification scheme, phase transitions are divided into two broad categories: first-order and second-order phase transitions.

First-order phase transitions are those that involve a latent heat and they are characterized by a finite value of the correlation length. This implies the presence of a mixed-phase regime, in which some parts of the system have completed the transition and others have not. Familiar examples are the melting of ice or the boiling and many other important phase transitions including Bose-Einstein condensation.

Second order phase transitions (continuos phase transitions) instead have no associated latent heat. Simple Examples of second-order phase transitions are the ferromagnetic transitions , superconductors, and the superfluid He transition. The simplest lattice model that exhibits such a phase transition is the famous Ising model.

These transitions are characterized by a collective behaviour on large scales near the transition temperature (the critical temperature T_c). For example, the correlation length, which characterizes the scale of distance over which a collective behaviour is observed, becomes infinite at the transition. Near T_c, these systems thus depend on two very different length-scales, a microscopic scale given by the size of atoms, the lattice spacing or the range of forces, and another scale dynamically generated, given by the correlation length. To the latter scale are associated non-trivial large-distance or macroscopic phenomena.

1.2 Critical exponents and universality

As the temperature approaches the critical temperature T_c of a continuous phase transition, an increasing number of microscopic degrees of freedom start to be coupled to each other and effectively act as a single entity.

The correlation length ξ sets the typical length scale over which the degrees of freedom are strongly coupled. For example, in the vicinity of the paramagnet-ferromagnet transition (with the temperature slightly larger than the critical temperature) microscopic spins within regions of linear size ξ tend to point in the same direction, while spins belonging to different regions whose distance is large compared with ξ remain uncorrelated.

At the critical point associated with a continuous phase transition, the correlation length ξ is infinite and fluctuations of the order parameter extend on all length scales, so that the system is scale invariant. This actually means that the phase transition taking place at T_c is the result of an extraordinary collective phenomenon involving the entire system.

As a consequence, it can be shown thermodynamic observables are homogeneous functions of the relevant thermodynamic variables so that they exhibit power-law behavior. The exponents which characterize the leading behavior of thermodynamic observables for $T \to T_c$ are called critical exponents.

For simplicity, consider the paramagnet-ferromagnet transition. It is convenient to measure the distance from the critical point on the temperature axis in terms of the reduced temperature

$$t = \frac{T - T_c}{T_c}. \tag{1.1}$$

Historically, one introduces the critical exponents α, β, and γ to characterize the asymptotic behavior of the following observables for $t \to 0$,

Table 1.1: Definition of the critical exponents

Observable	Exponent	Definition
Specific heat	α	$C(t) \sim \|t\|^{-\alpha}$
Spontaneous magnetization	β	$m(t) \sim (-t)^{\beta} \quad t \leq 0$
Magnetic susceptibility	γ	$\chi(t) \sim \|t\|^{-\gamma}$
Magnetic field	δ	$B(M) \sim \|M\|^{\delta}$
Correlation length	ν	$\xi(t) \sim \|t\|^{-\nu}$
Correlation function	η	$G \sim r^{-(d-2+\eta)}$

In addition to these thermodynamic exponents, one usually introduces two more exponents ν and η via the behavior of the order-parameter correlation function

$G(\vec{i}-\vec{j})$ for large distances $\vec{i}-\vec{j}$. If the system is homogeneous, the correlator is a function of the absolute value of the distance $r = |\vec{i}-\vec{j}|$. For a magnetic phase transition with Ising symmetry, $G(r)$ is defined as follows.

Let us denote by $m(r)$ the operator representing the local magnetization density at space point r. The thermal average

$$m = \langle m(r) \rangle := \frac{\operatorname{Tr} e^{-\beta H} m(r)}{\operatorname{Tr} e^{-\beta H}} \tag{1.2}$$

is then independent of r, so that $\delta m(r) = m(r) - \langle m(r) \rangle = m(r) - m$. The order parameter correlation function is then defined by the thermal average

$$G(r) = \langle m_{\vec{i}}\, m_{\vec{j}} \rangle - m^2. \tag{1.3}$$

Typically, one finds for the asymptotic behavior of $G(r)$ at distances large compared with the correlation length ξ

$$G(r) \sim \frac{e^{-r/\xi}}{\sqrt{\xi^{d-3} r^{d-1}}}. \tag{1.4}$$

For $T \to T_c$, the correlation length diverges as a power law

$$\xi \sim |t|^{-\nu}, \tag{1.5}$$

where ν is called correlation length exponent. When the system approaches the critical point, the correlation length diverges and the regime of validity of (1.4) is pushed to infinity. Precisely at the critical point ξ is infinite and the order-parameter correlation function decays with a power that is different from the power in the pre-factor in (1.4); in d dimensions we write

$$G(r) \sim \frac{1}{r^{d-2+\eta}}, \qquad T = T_c, \tag{1.6}$$

which defines the correlation function critical exponent η. Because in (1.6) the exponent η can be viewed as a correction to the physical dimension d of the system, η is called anomalous dimension.

It turns out that the the exponents are universal in the sense that entire classes of materials consisting of very different microscopic constituents can have the same exponents. As a consequence of this, a uniaxial ferromagnet and a simple fluid for example are believed to have exactly the same critical exponents.

All materials can thus be divided into universality classes, which are characterized by the same critical exponents. The universality class in turn is determined by some rather general properties of a system, such as its dimensionality, the symmetry of its order parameter, or the range of the interaction. For example, in Table 1.2 we list the critical exponents for the Ising universality class in two and three dimensions and for the XY and Heisenberg universality classes in $d = 3$.

Experimental evidence for the universality of the critical exponents has been consolidated, and the microscopic understanding of this universality was only achieved in the 1970s with the help of the renormalization group, which also provided a systematic method for explicitly calculating critical exponents in cases where mean-field theory fails.

Table 1.2: Critical exponents of the Ising, XY, and Heisenberg universality classes. The corresponding symmetry groups of the order parameter are $\mathbb{Z}_2$ for the Ising universality class, $O(2)$ for the XY universality class, and $O(3)$ for the Heisenberg universality class. The small subscripts in the first line denote the dimensionality. While the exponents of the two-dimensional Ising universality class are exact, the exponents in three dimensions are only known approximately. The numbers for Ising_3 and the error estimates are from the review by Pelissetto and Vicari [36]. For XY_3 we give rounded values for α, γ, ν and η up to two significant figures, as compiled in Pelissetto and Vicari (2002, Table 19). For Heisenberg_3 we quote the results by Holm and Janke (1993).

Exponent	Ising_2	Ising_3	XY_3	Heisenberg_3
α	0	0.110(1)	−0.015	−0.10
β	1/8	0.3265(3)	0.35	0.36
γ	7/4	1.2372(5)	1.32	1.39
δ	15	4.789(2)	4.78	5.11
ν	1	0.6301(4)	0.67	0.70
η	1/4	0.0364(5)	0.038	0.027

1.3 The scaling hypothesis

Under quite general conditions only two of the six exponents α, β, γ, δ, ν, and η are independent, so that we can obtain the thermodynamic exponents α, β, γ and δ from the exponents ν and η related to the scaling of the correlation function using so-called scaling relations.

Let us first discuss the scaling form of the free energy. For simplicity, consider again a magnet with free energy density $f(t, h)$, which is a function of the reduced temperature and the magnetic field. In the vicinity of a continuous phase transition, we expect that $f(t, h)$ can be decomposed into a singular and a regular part,

$$f(t, h) = f_{\text{sing}}(t, h) + f_{\text{norm}}(t, h). \tag{1.7}$$

Sufficiently close to the critical point, the singular part $f_{\text{sing}}(t, h)$ is assumed to be determined by power laws characteristic of a given critical point. According to the scaling hypothesis for the free energy, its singular part satisfies the following generalized homogeneity relation

$$f_{\text{sing}} = b^{-d} f_{\text{sing}}(b^{y_t} t, b^{y_h} h) \tag{1.8}$$

where b is an arbitrary (dimensionless) scale factor and the exponents y_t and y_h are characteristic for a given universality class. It is now easy to show that with this assumption the four thermodynamic exponents α, β, γ and δ can all be expressed in terms of y_t and y_h and the dimensionality d of the system. Using the fact that b is arbitrary, we may set $b^{y_t} = 1/|t|$, or equivalently $b = |t| - 1/y_t$. Then we obtain from (1.8),

$$f_{\text{sing}}(t,h) = t^{d/y_t} f_{\text{sing}}\left(\pm 1, \frac{h}{t^{\frac{y_h}{y_t}}}\right) = t^{d/y_t} \mathcal{F}_\pm\left(\frac{h}{t^{\frac{y_h}{y_t}}}\right) \tag{1.9}$$

where we have defined the scaling function $\mathcal{F}_\pm(x) = f_{\text{sing}}(\pm 1, x)$. We obtain the desired relations between the thermodynamic exponents α, β, γ and y_t, y_h by taking appropriate derivatives of (1.9) with respect to t and h and then setting $h = 0$, assuming that close to the critical point the derivatives of the free energy density are dominated by its singular part

$$C \simeq T_c^{-1} \frac{\partial^2 f_{\text{sing}}}{\partial t^2}\bigg|_{h=0} \sim |t|^{\frac{d}{y_t}-2} = |t|^{-\alpha}, \tag{1.10a}$$

$$m \simeq -\frac{\partial f_{\text{sing}}}{\partial h}\bigg|_{h=0} \sim (-t)^{\frac{d-y_h}{y_t}} = (-t)^{\beta}, \tag{1.10b}$$

$$\chi \simeq \frac{\partial^2 f_{\text{sing}}}{\partial h^2}\bigg|_{h=0} \sim |t|^{\frac{d-2y_h}{y_t}} = |t|^{-\gamma}. \tag{1.10c}$$

Hence

$$\alpha = 2 - \frac{d}{y_t}, \tag{1.11a}$$

$$\beta = \frac{d - y_h}{y_t}, \tag{1.11b}$$

$$\gamma = \frac{2y_h - d}{y_t}. \tag{1.11c}$$

To express the exponent δ associated with the critical isotherm in terms of y_h and d, we consider the derivative of (1.9) for finite $h > 0$,

$$m(t,h) \simeq -\frac{\partial f_{\text{sing}}}{\partial h} = |t|^{\frac{d-y_h}{y_t}} \mathcal{F}'_\pm\left(\frac{h}{|t|^{\frac{y_h}{y_t}}}\right). \tag{1.12}$$

To obtain a finite value of m for $t \to 0$, the scaling function must behave as $x^{\frac{d}{y_h}-1}$ for $x \to \infty$. We thus obtain for the critical isotherm

$$m(h) \sim h^{\frac{d}{y_h}-1} = h^{\frac{1}{\delta}} \tag{1.13}$$

and hence

$$\delta = \frac{y_h}{d - y_h} \tag{1.14}$$

We may now eliminate the two variables y_t and y_h from the previous equations to obtain two scaling relations involving only the experimentally measurable exponents α, β, γ , and δ,

$$2 - \alpha = 2\beta + \gamma \tag{1.15}$$

$$2 - \alpha = \beta(\delta + 1). \tag{1.16}$$

In order to express also the exponents ν and η in terms of y_t and y_h, we need another scaling hypothesis involving the correlation function $G(r)$. We assume that sufficiently close to the critical point, $G(r)$ is dominated by the singular part $G_{\text{sing}}(r);t,h)$ which satisfies

$$G_{\text{sing}}(r;t,h) = b^{-2(d-y_h)} G_{\text{sing}}\left(\frac{r}{b}; b^{y_t}t, b^{y_h}h\right). \tag{1.17}$$

For simplicity, consider the case $h = 0$. Setting again $b = |t| - 1/y_t$, we obtain

$$G_{\text{sing}}(r;t,0) = |t|^{\frac{2(d-y_h)}{y_t}} G_{\text{sing}}\left(\frac{r}{|t|^{-1/y_t}}, \pm 1, 0\right). \tag{1.18}$$

For $|t| \neq 0$ and $r \to \infty$, we expect $G_{\text{sing}}(r) \propto e^{-\frac{r}{\xi}}$, so that we obtain

$$\xi \sim |t|^{-\frac{1}{y_t}} = |t|^{-\nu} \tag{1.19}$$

where we have used the definition of the correlation length exponent ν. We conclude that

$$\nu = \frac{1}{y_t}. \tag{1.20}$$

Using (1.11a) to express y_t in terms of the specific heat exponent α, we obtain from (1.20) the so-called hyperscaling relation

$$2 - \alpha = d\nu. \tag{1.21}$$

Finally we relate the critical exponent η to y_h and d by observing that for $|t| \to 0$ the function $G_{\text{sing}}(r;t,0)$ can only be finite if the rescaled part is proportional to $|\frac{r}{b}|^{-2(d-y_h)}$ for large $|\frac{r}{b}|$. At the critical point we therefore obtain

$$G_{\text{sing}}(r;0,0) \sim r^{-2(d-y_h)} = r^{-(d-2+\eta)} \tag{1.22}$$

where we have used the definition of the correlation function exponent η. We therefore identify

$$\eta = d + 2 - 2y_h, \tag{1.23}$$

and expressing y_h in terms of the susceptibility exponent γ using (1.11c), we obtain another hyperscaling relation

$$\gamma = (2 - \eta)\nu. \tag{1.24}$$

Equations (1.21) and (1.24) are called hyperscaling relations because they connect singularities in thermodynamic observables with singularities related to the correlation function.

It turns out, however, that the underlying scaling hypothesis (1.17) is only valid if the dimension d of the system is smaller than a certain upper critical dimension d_{up}, which depends on the universality class (for the Ising universality class $d_{\text{up}} = 4$). As will be discussed in Chapter 2, for $d > d_{\text{up}}$ the Gaussian approximation is sufficient to calculate the critical behavior of the system. The failure of hyper-scaling for $d > d_{\mathrm{up}}$ is closely related to the existence of a so-called dangerously irrelevant coupling. If hyperscaling is satisfied, we may combine the two thermodynamic scaling relations (1.15), (1.16) with the two hyperscaling relations (1.21) and (1.24) to express the four thermodynamic exponents α, β, γ, and δ in terms of the two correlation function exponents η and ν,

$$\alpha = 2 - d\nu, \tag{1.25a}$$

$$\beta = \frac{\nu}{2}(d - 2 + \eta), \tag{1.25b}$$

$$\gamma = \nu(2 - \eta), \tag{1.25c}$$

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta}. \tag{1.25d}$$

This is really a great simplification in the calculation of critical exponents.

2

Mean field theory and the Gaussian approximation

The Wilsonian renormalization group (RG) was invented in order to study the effect of strong fluctuations and the mutual coupling between different degrees of freedom in the vicinity of continuous phase transitions. Two less sophisticated methods of dealing with this problem, namely the mean-field approximation and the Gaussian approximation were proposed before Wilsonian RG.

Within the mean-field approximation, fluctuations of the order parameter are completely neglected and the interactions between different degrees of freedom are taken into account in some simple average way.

The Gaussian approximation is in some sense the leading fluctuation correction to the mean-field approximation. Although these methods are very general and can also be used to study quantum mechanical many-body systems, for our purpose it is sufficient to introduce these methods using the nearest-neighbour Ising model in d dimensions as an example.

The Ising model is the prototype of a statistical system exhibiting a phase transition and is defined in terms of the following Hamiltonian,

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_{i}^{N} \sigma_i \tag{2.1}$$

Here, $\langle ij \rangle$ denotes the summation over all distinct pairs of nearest neighbours of a d-dimensional hypercubic lattice with N sites, the variables $\sigma_i = \pm 1$ correspond to the two possible states of the z-components of spins localized at the lattice sites, J denotes the interaction strength between spins, and h is the Zeeman energy associated with an external magnetic field in the z-direction. When $h = 0$ the model has a global discrete symmetry $\mathbb{Z}_2$, implemented by the transformation $\sigma_i \to -\sigma_i$.

In order to obtain thermodynamic observables, we have to calculate the partition function,

$$Z(T,h) = \sum_{\sigma_i} e^{-\beta H} \equiv \sum_{\sigma_1=\pm1} \sum_{\sigma_2=\pm1} \cdots \sum_{\sigma_N=\pm1} \exp\left[\beta J \sum_{\langle ij\rangle} \sigma_i\sigma_j + \beta h \sum_i \sigma_i\right] \tag{2.2}$$

While in one dimension it is quite simple to carry out this summation, the corresponding calculation in $d = 2$ is much more difficult and until now the exact $Z(T,h)$ for $h \neq 0$ is not known. For $h = 0$ the partition function of the two-dimensional Ising model was first calculated by Onsager [34], who also presented an exact expression for the spontaneous magnetization at a conference in 1949. In dimensions $d = 3$ there are no exact results available, so one has to rely on approximations. The simplest is the mean-field approximation discussed in the following section.

2.1 Mean-field approximation

2.1.1 Landau free energy

Mean-field theory is based on the assumption that the fluctuations around the average value of the order parameter are so small that they can be neglected. As we will see it is quantitatively correct only above the upper critical dimension but it anyway gives a qualitatively correct picture of the phase diagram. Consider a system with finite magnetization,

$$m = \langle\sigma_i\rangle \equiv \frac{\sum_{\{\sigma_j\}} e^{-\beta H}\sigma_i}{\sum_{\{\sigma_j\}} e^{-\beta H}} \tag{2.3}$$

where we have used the fact that by translational invariance the thermal expectation values $\langle\sigma_i\rangle$ are independent of the site label i . We have:

$$\begin{aligned}
\sigma_i\sigma_j &= [\sigma_j + \langle\sigma_j\rangle - \langle\sigma_j\rangle]\,[\sigma_i + \langle\sigma_i\rangle - \langle\sigma_i\rangle] = \\
&= \langle\sigma_i\rangle\langle\sigma_j\rangle + [\sigma_i - \langle\sigma_i\rangle]\langle\sigma_j\rangle + [\sigma_j - \langle\sigma_j\rangle]\langle\sigma_i\rangle + \\
&+ \underbrace{[\sigma_i - \langle\sigma_i\rangle][\sigma_j - \langle\sigma_j\rangle]}_{\text{small}} = \\
&= m^2 + \sigma_i m_j + m_i\sigma_j,
\end{aligned} \tag{2.4}$$

where we discarded the quadratic fluctuation term since we are assuming that the fluctuations are small. Within this approximation, the Ising Hamiltonian (2.1) is replaced by the mean-field Hamiltonian for nearest-neighbour interactions

$$H_{\mathrm{MF}} = m^2 N \frac{zJ}{2} - \sum_i (h + zJm)\sigma_i \tag{2.5}$$

where $z = 2d$ is the number of nearest neighbours (coordination number) of a given site of a d-dimensional hypercubic lattice. Now the partition function can be easily computed,

$$
\begin{aligned}
Z_{\mathrm{MF}} &= e^{-\frac{\beta N z J m^2}{2}} \sum_{\{\sigma_i\}} e^{\beta(h+zJm)\sum_i \sigma_i} \\
&= e^{-\frac{\beta N z J m^2}{2}} \prod_i \left[e^{\beta(h+zJm)} e^{-\beta(h+zJm)} + \right] \\
&= e^{-\frac{\beta N z J m^2}{2}} \left[2\cosh[\beta(h+zJm)] \right]^N ,
\end{aligned}
\tag{2.6}
$$

where we have introduced the notation $z = 2d$. Writing this as

$$
Z_{\mathrm{MF}}(T,h) = e^{-\beta N f_{\mathrm{MF}}(t,h;m)}, \tag{2.7}
$$

we obtain the following expression for the free-energy per site in mean-field approximation

$$
f_{\mathrm{MF}}(t,h;m) = \frac{zJ}{2} m^2 - \beta^{-1} \log \left[2\cosh[\beta(h+zJm)] \right]. \tag{2.8}
$$

The function $f_{\mathrm{MF}}(T,h;m)$ is an example for a Landau function, which describes the probability distribution of the order parameter: the probability density of observing for the order parameter the value m is proportional to $e^{[-\beta N f_{\mathrm{MF}}(T,h;m)]}$.

2.1.2 Mean-field phase diagram

The equilibrium state of the system is now determined from the condition that the physical value m_0 of the order parameter maximizes its probability distribution, corresponding to the minimum of the Landau function,

$$
\frac{\partial f_{\mathrm{MF}}(T,h;m)}{\partial m}\Big|_{m_0} = 0. \tag{2.9}
$$

From (2.8) we find that the magnetization m_0 in mean-field approximation satisfies the self-consistency condition

$$
m_0 = \tanh[\beta(h + 2dJm_0)]. \tag{2.10}
$$

This mean-field self-consistent equation can easily be solved graphically. As shown in Fig. (2.1) below, for $h \neq 0$ the global minimum of $F_{\mathrm{MF}}(T,h;m)$ occurs always at a finite $m_0 \neq 0$. On the other hand, for $h = 0$ the existence of nontrivial solutions with $m_0 \neq 0$ depends on the temperature. In the low-temperature regime $T < zJ$ there are two nontrivial solutions with $m_0 \neq 0$, while at high temperatures $T > zJ$ our self-consistency equation (2.10) has only the trivial solution $m_0 = 0$, see figure below.

In d dimensions the mean-field estimate for the critical temperature is therefore

$$
T_c = \frac{zJ}{k_b} = 2\frac{dJ}{k_b}. \tag{2.11}
$$

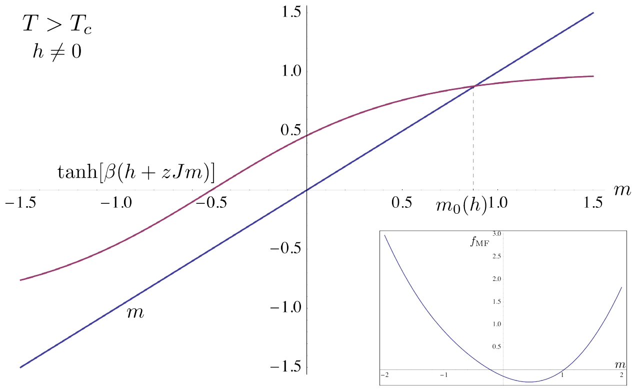


Figure 2.1: Graphical solution of the mean-field self-consistency equation for $h > 0$. The inset shows the behaviour of the corresponding mean-field free energy: for $T > T_c$ it exhibits only one minimum at $m_0 > 0$.

For $d = 1$ this is certainly wrong, because we know from the exact solution that $T_c = 0$ in one dimension. In two dimensions the exact critical temperature of the nearest-neighbour Ising model satisfies $\sinh(2J/T_c) = 1$ [34], which yields $T_c \approx 2.269J$ and is significantly lower than the mean-field prediction of $4J$. As a general rule, in lower dimensions fluctuations are more important and tend to disorder the system or at least reduce the critical temperature.

2.1.3 Critical exponents

For temperatures close to T_c and small $\beta|h|$ the value m_0 of the magnetization at the minimum of $f_{\mathrm{MF}}(T, h; m)$ is small. We may therefore approximate the Landau function (2.8) by expanding the right-hand side of (2.8) up to fourth order in m and linear order in h

$$f_{\mathrm{MF}}(T, h; m) = f + \frac{r}{2} m^2 + \frac{u}{4!} m^4 - hm + \dots \tag{2.12}$$

with

$$f = -T \log 2, \tag{2.13a}$$

$$r = \frac{zJ}{T}(T - zJ) \sim T - T_c, \tag{2.13b}$$

$$u = 2T\left(\frac{zJ}{T}\right)^4 \sim 2T_c, \tag{2.13c}$$

where these approximations are valid close to the critical temperature, where $|T - T_c| \ll T_c$ and $zJ/T \sim 1$. Obviously, the sign of the coefficient r changes at

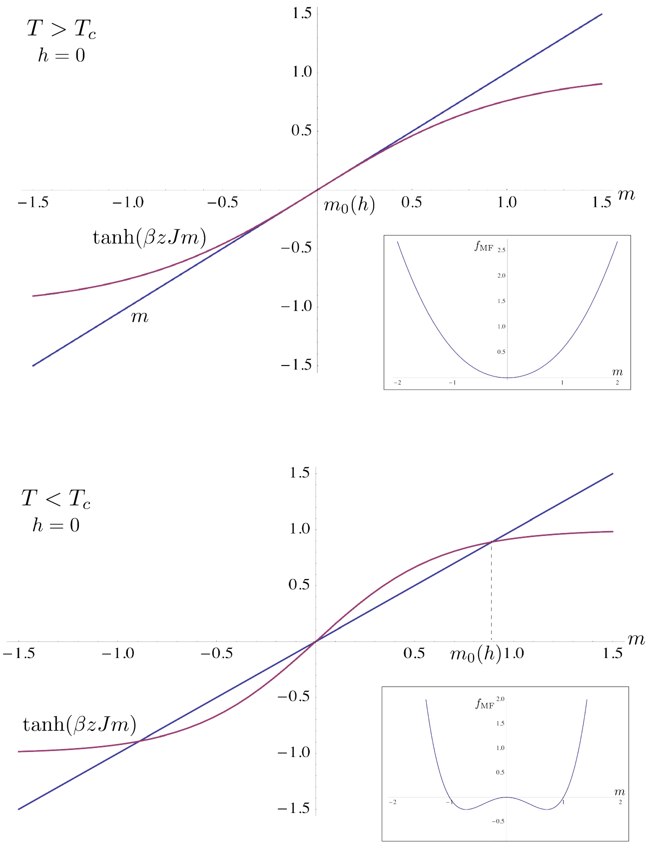


Figure 2.2: Graphical solution of the mean-field self-consistency equation for $h = 0$. The upper figure shows the typical behavior in the disordered phase $T > T_c$, while the lower figure represents the ordered phase $T < T_c$. The behavior of the Landau function is shown in the insets: while for $T > T_c$ it has a global minimum at $m_0 = 0$, it develops for $T < T_c$ two degenerate at $\pm m_0$.

$T = T_c$, so that for $h = 0$ the global minimum of $f_{\rm MF}(T, h; m)$ for $T > T_c$ evolves into a local maximum for $T < T_c$, and two new minima emerge at finite values of m, as shown in Fig. (2.2).

The crucial point is now that for a small reduced temperature $t = (T - T_c)/T_c$ the value of m at the minima of $f_{\rm MF}(T, h; m)$ is small compared with unity, so that our expansion in powers of m is justified a posteriori. Taking the derivative of (2.12) with respect to m, it is easy to see that (2.10) simplifies to

$$\frac{\partial f_{\rm MF}(T, h; m)}{\partial m}\Big|_{m_0} = rm + \frac{u}{6} m_0^3 - h = 0. \qquad (2.14)$$

The behavior of thermodynamic observables close to T_c is now easily obtained:

- Spontaneous magnetization:

Setting $h = 0$ in (2.14) and solving for m_0, we obtain for $T < T_c$ (with $r \leq 0$)

$$m_0 = \sqrt{\frac{-6r}{u}} \sim (-t)^{1/2} \tag{2.15}$$

Comparing this with the definition of the critical exponent β, we conclude that the mean-field approximation predicts for the Ising universality class $\beta^{\mathrm{MF}} = 1/2$, independently of the dimension d.

- Zero-field susceptibility:
  To obtain the mean-field result for the susceptibility exponent γ, we note that for small but finite h and $T \geq T_c$ we may neglect the terms of order m_0^3 in (2.14), so that $m_0(h) \sim h/r$, and hence the zero-field susceptibility behaves for $t \to 0$ as

$$\chi = \frac{\partial m_0(h)}{\partial h}\Big|_{h=0} \sim \frac{1}{r} \sim \frac{1}{T - T_c}. \tag{2.16}$$

  It is a simple to show that $\chi \sim |T - T_c|^{-1}$ also holds for $T > T_c$. The susceptibility exponent is therefore $\gamma^{\mathrm{MF}} = 1$ within mean-field approximation.

- Critical isotherm:
  The equation of state at the critical point can be obtained by setting $r = 0$ in (2.14), implying

$$m_0(h) \sim \left(\frac{h}{u}\right)^{1/3} \tag{2.17}$$

  and hence the mean-field result $\delta^{\mathrm{MF}} = 3$.

- Specific heat:
  The specific heat C per lattice site can be obtained from the thermodynamic relation

$$C = -T \frac{\partial^2 f_{\mathrm{MF}}(T, h)}{\partial T^2}. \tag{2.18}$$

  Setting $h = 0$, we find from for $T > T_c$ that $f_{\mathrm{MF}}(T, h) = f$ because $m_0 = 0$ in this case. On the other hand, for $T < T_c$ we may substitute (2.15) so that

$$f_{\mathrm{MF}}(T, 0) = f - \frac{3r^2}{2u}, \qquad T < T_c. \tag{2.19}$$

  Setting $r \approx T - T_c$ and taking two derivatives with respect to T, we obtain

$$\begin{aligned} C &\approx -T_c \frac{\partial^2 f}{\partial T^2}, \qquad T > T_c, \\ &\approx -T_c \frac{\partial^2 f}{\partial T^2} + 3\frac{T_c}{u}, \qquad T < T_c. \end{aligned} \tag{2.20}$$

  Note that according to (2.13c) $u \approx 2T_c$ so that $3T_c/u \approx 3/2$. We conclude that within the mean-field approximation the specific heat is discontinuous at T_c, so that $C \sim |t|^0$, implying $\alpha^{\mathrm{MF}} = 0$. Note that the mean-field results are consistent with the scaling relations.

2.1.4 Correlation Function

In order to obtain the exponents ν and η, we need to calculate the correlation function $G(r)$ we need to relax the assumption of constant magnetic field. In this case the partition function reads:

$$\begin{aligned} Z^{\mathrm{MF}} &= \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle}[-m_i m J + \sigma_i m_j + \sigma_j m_i] + \beta \sum_i h_i \sigma_i} = \\ &= e^{-\beta J \sum_{\langle i,j\rangle} m_i m_j \sum_{\{\sigma_i\}} e^{\beta J} \sum_{\langle i,j\rangle}[\sigma_i m_j + m_i \sigma_j] + \beta \sum_i h_i \sigma_i}. \end{aligned} \tag{2.21}$$

We need now to evaluate the sum:

$$\begin{aligned} \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle}[\sigma_i m_j + m_i \sigma_j] + \beta h \sum_i \sigma_i} &= \sum_{\{\sigma_i\}} e^{2\beta J \sum_{\langle i,j\rangle} \sigma_i m_j + \beta \sum_i h_i \sigma_i} \\ &= \sum_{\{\sigma_i\}} e^{\beta \sum_i \sigma_i [J \sum_{j(i)} m_j h_i]} \\ &= \prod_i 2\cosh\beta \left[J \sum_{j(i)} m_j + h_i \right], \end{aligned} \tag{2.22}$$

where $j(i)$ are the nearest-neighbour of i. The free energy reads:

$$f^{\mathrm{MF}} = \frac{J}{N} \sum_{\langle i,j\rangle} m_i m_j - \frac{1}{\beta N} \sum_i \log 2\cosh\left[\beta \left(J \sum_{j(i)} m_j + h_i \right) \right] \tag{2.23}$$

and minimizing the free energy we obtain:

$$m_i = \tanh\left[\beta \left(J \sum_{j(i)} m_j + h_i \right) \right]. \tag{2.24}$$

In order to compute the correlation function we need to determine the solutions of (2.24) to linear order in the external magnetic field. Since we are interested in the critical region, we can assume that also the magnetization is small and expand the hyperbolic tangent to first order:

$$\beta J \sum_{j(i)} m_j + m_i = \beta h_i. \tag{2.25}$$

To solve the previous equation we need to perform a Fourier transform and we give here the result for the correlator:

$$G(r) = \frac{1}{T_c} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\boldsymbol{r}\boldsymbol{k}}}{t + \frac{1}{z}\sum_a 2(1-\cos k_a)} \tag{2.26}$$

At large r we can wtite:

$$\begin{aligned} G(r) &\sim \frac{1}{T_c}\int \frac{d^dk)}{(2\pi)^d}\frac{e^{-i\boldsymbol{rk}}}{t+k^2} \\ &= \frac{1}{T_c}\frac{1}{(2\pi)^d}\left(\frac{\sqrt{t}}{r}\right)^{d-2} K_{\frac{d}{2}-1}(\sqrt{t}r) \\ &\sim \frac{1}{T_c}\frac{1}{(2\pi)^d}\left(\frac{\sqrt{t}}{r}\right)^{d-2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{\sqrt{t}r}}e^{-\sqrt{t}r}. \end{aligned} \tag{2.27}$$

We see therefore that the correlation length diverges at the critical point like:

$$\xi = t^{-1/2} \equiv t^{-\nu}. \tag{2.28}$$

The mean field correlation length exponent is thus $\nu^{\mathrm{MF}} = \frac{1}{2}$.

At the critical point we find instead:

$$G(r) \sim \frac{1}{r^{d-2}} \equiv \frac{1}{r^{d-2+\eta}} \tag{2.29}$$

which therefore identifies the mean-field anomalous dimension $\eta^{\mathrm{MF}} = 0$.

2.1.5 Ginzburg criterion

The Ginzburg criterion establishes when the mean-field analysis is valid, i.e. under which conditions fluctuations are small. We start from:

$$\sqrt{\langle M^2\rangle - \langle M\rangle^2} \ll \langle M\rangle, \tag{2.30}$$

which is equivalent to

$$NT\chi \ll \langle M\rangle^2. \tag{2.31}$$

At criticality fluctuations are on all scales and we have (for $t \to 0^-$):

$$\xi \sim |t|^{-\nu} \qquad \langle M\rangle^2 \sim (-t)^{2\beta}L^{2d} \qquad N\chi \sim (-t)^{-\gamma}L^d. \tag{2.32}$$

Thus, setting $\xi = L$, we find:

$$T(-t)^{-\gamma}\xi^d \ll (-t)^{2\beta}\xi^{2d}, \tag{2.33}$$

or

$$T(-t)^{-\gamma-2\beta+\nu d} \ll 1, \tag{2.34}$$

which may happen when $\gamma - 2\beta + \nu d > 0$ or:

$$\frac{\gamma+2\beta}{\nu} < d, \tag{2.35}$$

which is the Ginzburg criterion. If we insert the mean-field exponent we find $d > 4$. Thus the mean field predictions are valid in dimension greater than four, which for this reason is called *upper critical dimension* d_c.

2.2 Gaussian approximation

The Gaussian approximation retaining quadratic fluctuations includes the lowest-order correction to the mean-field approximation in an expansion in fluctuations around the saddle point.

In the field-theory language, the Gaussian approximation corresponds to describing fluctuations in terms of a free field theory, where the fluctuations with different momenta and frequencies are independent. In condensed matter physics, the Gaussian approximation is closely related to the so-called random phase approximation.

The Gaussian approximation, however, satisfies Ginzburg criterion too and therefore it is not sufficient to describe the critical behavior of Ising magnets in experimentally accessible dimensions. In this section we set $h = 0$ and we anticipate a field-theory language we will anyway present in detail later.

2.3.1 Exact effective field theory

The functional integral representation of the partition functions reads:

$$Z = \int \mathcal{D}\varphi \; e^{-S_{\Lambda_0}[\varphi]}. \tag{2.36}$$

Defining the real-space Fourier transforms of the continuum fields $\varphi(\boldsymbol{k})$ via

$$\int_k e^{i\boldsymbol{k}\boldsymbol{r}} \varphi(\boldsymbol{k}), \tag{2.37}$$

we can write our φ^4 toy-model action as:

$$S_{\Lambda_0}[\varphi] = \int d^d r \; \left[f_0 + \frac{r_0}{2}\varphi^2(\boldsymbol{r}) + \frac{c_0}{2}[\nabla\varphi(\boldsymbol{r})]^2 + \frac{u_0}{4!}\varphi^4(\boldsymbol{r}) \right] \tag{2.38}$$

Let us decompose the field $\varphi(\vec{r})$ describing the coarse-grained fluctuating magnetization as follows,

$$\varphi(\vec{r}) = \bar{\varphi}_0 + \delta\varphi(\vec{r}), \tag{2.39}$$

or in wave vector space

$$\varphi(\vec{k}) = (2\pi)^d \delta(\vec{k})\bar{\varphi}_0 + \delta\varphi(\vec{k}). \tag{2.40}$$

Here $\bar{\varphi}_0$ is the mean-field value of the order parameter satisfying the saddle point condition:

$$\left. \frac{\partial S_{\Lambda_0}[\bar{\varphi}]}{\partial \bar{\varphi}} \right|_{\bar{\varphi}_0} = 0 \tag{2.41}$$

and $\delta\varphi(\vec{r})$ describes inhomogeneous fluctuations around the saddle point. Substituting (2.39) into the action (2.38) and retaining all terms up to quadratic order in the fluctuations, we obtain in momentum space

$$
\begin{aligned}
S_{\Lambda_0}[\bar{\varphi}_0 + \delta\varphi] \approx V\left[f_0 + \frac{r_0}{2}\bar{\varphi}_0^2 + \frac{u_0}{4!}\bar{\varphi}_0^4\right] + \left[r_0\bar{\varphi}_0 + \frac{u_0}{6}\bar{\varphi}_0^3\right]\delta\varphi(\vec{k}=0) \\
+ \frac{1}{2}\int_{\vec{k}}\left[r_0 + \frac{u_0}{2}\bar{\varphi}_0^2 + c_0\vec{k}^2\right]\delta\varphi(-\vec{k})\delta\varphi(\vec{k})
\end{aligned}
\tag{2.42}
$$

This is the Gaussian approximation for the Ginzburg-Landau-Wilson action (2.38). To further simplify (2.42), we note that the second term on the right-hand side of (2.42) vanishes because the coefficient of $\delta\varphi(\vec{k}=0)$ satisfies the saddle point condition (2.41). The first line on the right-hand side of (2.42) can be identified with the mean-field free energy. Explicitly substituting for $\bar{\varphi}_0$ in (2.42) the saddle point value,

$$
\bar{\varphi}_0 = \begin{cases} 0 & \text{for} \quad r_0 > 0 \\ \sqrt{-\dfrac{6r_0}{u_0}} & \text{for} \quad r_0 < 0 \end{cases}
$$

we obtain for the *effective action* in Gaussian approximation for $T > T_c$, where $\bar{\varphi}_0$ and $\delta\varphi = \varphi$,

$$
S_{\Lambda_0}[\varphi] = Vf_0 + \frac{1}{2}\int_k [r_0 + c_0\vec{k}^2]\varphi(\vec{k})\varphi(-\vec{k}) \qquad T > T_c. \tag{2.43}
$$

On the other hand, for $T < T_c$, where $r_0 < 0$, we have

$$
\begin{aligned}
\frac{r_0}{2}\bar{\varphi}_0^2 + \frac{u_0}{4!}\bar{\varphi}_0^4 &= -\frac{3r_0^2}{2u_0} \\
r_0 + \frac{u_0}{2}\bar{\varphi}_0^2 &= -2r_0
\end{aligned}
\tag{2.44}
$$

and hence

$$
S_{\Lambda_0}[\varphi] = V\left[f_0 - \frac{3r_0^2}{2u_0}\right] + \frac{1}{2}\int_k [-2r_0 + c_0\vec{k}^2]\delta\varphi(\vec{k})\delta\varphi(-\vec{k}) \qquad T < T_c. \tag{2.45}
$$

With the help of the Gaussian effective action given in (2.43) and (2.45), we may now estimate the effect of order-parameter fluctuations on the mean-field results for the thermodynamic critical exponents.

Because in the thermodynamic limit both the Gaussian approximation and the mean-field approximation predict the same contribution from the homogeneous fluctuations represented by $\bar{\varphi}$ to the free energy, Gaussian fluctuations do not modify the mean-field predictions for the exponents β, γ, and δ, which are related to the homogeneous order-parameter fluctuations. Within the Gaussian approximation, we therefore still obtain $\beta = 1/2$, $\gamma = 1$, $\delta = 3$.

On the other hand, the Gaussian approximation for the specific heat exponent α is different from the mean-field prediction $\alpha = 0$, because the fluctuations with finite wave vectors give a nontrivial contribution Δf to the free energy per lattice site. It can be shown that the general result in Gaussian approximation for the specific heat exponent is given by:

$$\alpha = \begin{cases} 2 - \dfrac{d}{2} & \text{for} \quad d < d_{\mathrm{up}} \\ 0 & \text{for} \quad d \geq d_{\mathrm{up}}. \end{cases}$$

It should be noted that within the Gaussian model considered here the scaling relations $2 - \alpha = 2\beta + \gamma = \beta(\delta + 1)$ are violated for $d < 4$.

The same argument can be applied to the correlation length critical exponent ν and to the anomalous dimension η. We give here the result:

$$\begin{aligned} \eta &= 0 \\ \nu &= \frac{1}{2} \end{aligned} \tag{2.46}$$

Note that for $d < d_{\mathrm{up}} = 4$, where the Gaussian approximation yields $\alpha = 2 - d/2$ for the specific heat exponent, the Gaussian results $\eta = 0$ and $\nu = 1/2$ are also consistent with the hyper-scaling relations connecting ν and η with the thermodynamic critical exponents.

3

Renormalization group: an historical overview

Nowadays the renormalization group (RG) can be considered a meta-theory, which is a theory about theories. RG theory consists of a set of concepts, methods and techniques which historically have been used to understand phenomena in so many different fields of Physics, ranging from quantum field theory over classical statistical mechanics to non-equilibrium phenomena and many others. RG methods revealed to be particularly useful to understand phenomena where fluctuations involving many different length or time scales lead to the emergence of new collective behavior in complex many-body systems.

Since its first formulation, RG theory has been successfully applied to very different fields and so, naturally, a variety of apparently different implementations were given. At the first sight, the field-theoretical formulation of the RG idea looks very different from the RG approach later pioneered by Wilson: the first was employed mainly to cure the divergences in QFT from loop Feynman diagrams while the latter being based on the concept of the effective action which is iteratively calculated by successive integration of the degrees of freedom we're not interested in.

It is worth to give here a brief historical insight on the different implementations of the RG theory trying to capture the salient features and underlying ideas they share in common.

3.1 Field-theoretical renormalization

Originally the concept of Renormalizaton was introduced in the context of quantum field theory. After 1928 [16] quantum electrodynamics (QED) was introduced in order to describe the electromagnetic interactions between protons and electrons more precisely and physicists realized soon that the theory was plagued by divergencies.

The idea is that in field theories, space-time is treated as a continuum, so that in momentum-frequency dual space there is no ultraviolet cutoff. As a consequence, correlation functions, depending on the dimensionality of the space, can be ultraviolet divergent because there is no upper-bound for momenta and frequencies circulating around closed loops one introduces in perturbation theory.

3.1.1 Loops and divergencies

Consider the familiar φ^4 theory. Fluctuations induced by the φ^4 term around the gaussian model are large and they lead in perturbation expansion to integrals, corresponding to loops in Feynman diagrams. Consider the following *bare* action

$$S\,[\varphi_0; m_0, \lambda_0] = \int d^d x\ \mathcal{L}[\varphi_0; m_0, \lambda_0], \tag{3.1}$$

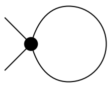
with a Lagrangian density

$$\mathcal{L}[\varphi_0; m_0, \lambda_0] = \frac{1}{2}[\overrightarrow{\nabla}\varphi_0(x)]^2 + \frac{m_0^2}{2}\varphi_0^2(x) + \frac{\lambda_0}{4!}\varphi_0^4(x). \tag{3.2}$$

In momentum space (3.1) reads

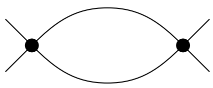
$$\begin{aligned} S\,[\varphi_0; m_0, \lambda_0] = \int \frac{d^d k}{(2\pi)^d}\ [k^2 + m_0^2]\varphi(-k)\varphi(k) + \\ + \frac{\lambda_0}{4!}\prod_{i=1}^{4}(2\pi)^d \delta_{k_1+k_2+k_3+k_4}\varphi_0(k_1)\varphi_0(k_2)\varphi_0(k_3)\varphi_0(k_4). \end{aligned} \tag{3.3}$$

This field theory doesn't have an ultraviolet cutoff so that momentum integrations are unrestricted, which leads to ultraviolet divergencies. In fact, for instance one can consider the first-order correction to the proper self energy



$$\begin{aligned} = \delta\Sigma = \frac{\lambda_0}{2}\int \frac{d^d k}{(2\pi)^d}\ \frac{1}{k^2 + m_0^2} = \frac{\lambda_0}{2}\Omega_d \int_0^\infty dk\ \frac{k^{d-1}}{k^2 + m_0^2} = \\ = \frac{\lambda_0}{2}\Omega_d m_0^{d-2}\int_0^\infty dx \frac{x^{d-1}}{1+x^2} \nearrow \infty \quad \text{for} \quad d \geq 2. \end{aligned} \tag{3.4}$$

Moreover, fluctuation corrections to correlation functions with more than two external legs can also be ultraviolet divergent. For example, the leading interaction correction to the effective interaction is



$$\begin{aligned} = \delta\Gamma = -\frac{3\lambda_0^2}{2}\int \frac{d^d k}{(2\pi)^d}\ \frac{1}{[k^2 + m_0^2]^2} = \\ = -\frac{3\lambda_0^2}{2}\Omega_d m_0^{d-4}\int_0^\infty dx \frac{x^{d-1}}{[1+x^2]^2} \nearrow \infty \quad \text{for} \quad d \geq 4. \end{aligned} \tag{3.5}$$

In order to cure divergent integrals appearing in perturbation series, we have to make them finite introducing some kind of ultraviolet cutoff and this procedure is called *regularization.* Among the regularization procedures (dimensional regularization, lattice regularization, etc.), one can choose a regularization via a sharp momentum cutoff Λ:

$$\int \frac{d^d k}{(2\pi)^d} \longrightarrow \int^{\Lambda} \frac{d^d k}{(2\pi)^d} \tag{3.6}$$

with which, for example, the correction to the proper self energy in $d = 4$ becomes

$$\delta\Sigma(\Lambda) = \frac{\lambda_0(\Lambda)}{32\pi^2} \left(\Lambda^2 - m_0^2(\Lambda) \log \frac{\Lambda^2 + m_0^2(\Lambda)}{m_0^2(\Lambda)} \right), \tag{3.7}$$

so this term has a quadratic and a logarithmic divergence, both coming from the large k integration region. Similarly, we obtain for the regularized correction to the interaction in four dimensions,

$$\delta\Gamma(\Lambda) = -\lambda_0(\Lambda) + \frac{3}{16\pi^2} \lambda_0^2(\Lambda) \log \frac{\Lambda}{m_0(\Lambda)}. \tag{3.8}$$

Of course in the end we should somehow remove these artificial cutoffs introduced by hand to obtain a renormalized theory which is independent of the regularization procedure. Perturbative renormalization is the method that allows to reparametrize the perturbation expansion in such a way that the sensitive dependence on Λ has been eliminated.

3.1.2 Renormalized quantities

During the years between 1947-1949 incredible theoretical developments were achieved mainly by Feynman, Schwinger, Tomonaga, Dyson *et al.* and an empirical general method to eliminate and cure divergencies called *renormalization* was proposed and this led in QED to finite results for all physical observables. The strategy was to reabsorb all infinities encountered in perturbation theory by a redefinition of physical renormalized quantities defined in terms of bare quantities

$$\begin{aligned} \varphi_0 &= Z_\varphi^{\frac{1}{2}} \varphi_R \\ m_0^2 &= Z_m m_R^2 \\ \lambda_0 &= Z_\lambda \mu \lambda_R \end{aligned} \tag{3.9}$$

where the dimensionless multiplicative renormalization constants Z_φ, Z_m and Z_λ will be determined iteratively order by order in perturbation theory, and μ is an arbitrary mass (or energy or momentum) scale which is introduced to make the renormalized coupling λ_R dimensionless.

The idea is that only the renormalized quantities have a physical meaning and can be related to physical observables. We therefore require that the renormalized quantities have finite values, while the bare quantities are infinite due to the singularities contained in the Z-factors.

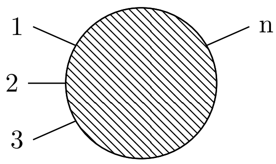
The important point, in this winter season for the field theory, is that renormalized quantities, by definition, does not depend on the cutoff Λ. We adjust the bare coupling, bare mass and field renormalization in such a way that all renormalized correlation functions are finite. The fact that this really works to all orders in perturbation theory is equivalent with the statement of renormalizability of the theory.

Consider a generic n-point function. A general renormalized n-point function Γ_R depends on the external momenta k_i, on the renormalized coupling λ_R (we also assume for simplicity that there is just one coupling, but the generalization is straightforward), and on the scale μ used to define the renormalization procedure

$$\Gamma_R = \Gamma_R(k_i, \lambda_R, \mu). \tag{3.10}$$

Note that bare correlation functions are independent on μ; the dependence on μ enters only when we remove the cutoff dependence by rescaling the fields and eliminating λ_0 in favour of the renormalized coupling λ_R.

We shall refer here to $\Gamma \equiv \Gamma^{(n)}(k_1, \ldots, k_n)$ as a one-particle-irreducible (1PI) vertex function with n external legs represented graphically by



meaning that any diagram contributing to the shaded circle cannot be separated into two disconnected parts by cutting a single propagator line. For example, in the special case $n = 2$ we recover the irreducible (proper) self-energy introduced in (3.4) $\Gamma^{(2)}(k, -k) = \Sigma(k)$.

The relation between bare and renormalized irreducible vertices is

$$\Gamma_0^{(n)}(k_i; m_0, \lambda_0, \Lambda) = Z_\varphi^{n/2} \Gamma_R^{(n)}(k_i; m_R, \lambda_R, \mu), \tag{3.11}$$

and since Γ_R is independent on Λ, we can write

$$\Lambda \frac{d\Gamma_R}{d\Lambda} = 0. \tag{3.12}$$

Using (3.11) we get

$$\boxed{\left[\Lambda \partial_\Lambda + \beta(\lambda_0)\partial_{\lambda_0} - n\eta(\lambda_0)\right] \Gamma_0(k_i; \lambda_0, \Lambda) = 0} \tag{3.13}$$

where

$$\beta(\lambda_0) := \Lambda \frac{d\lambda_0}{d\Lambda} \tag{3.14}$$

$$\eta(\lambda_0) := \frac{1}{2} \Lambda \frac{d}{d\Lambda} \log Z_\varphi \tag{3.15}$$

Equation (3.13) is called a renormalization group equation, and eqs. (3.14) and (3.15) define the β-function and the η-function of the theory. From their definition, we see that they are proportional to the shift in the coupling constant and the shift in field renormalization as the scale Λ is varied since, after renormalization, the coupling constants are not constant at all, but they depend on the energy. The behaviour of the coupling constants as a function of Λ is very important, since it determines the strength of the interaction and the conditions under which perturbation theory is valid.

In general the renormalization of the coupling has the form

$$\lambda_R = \lambda_0 - \beta_0 \lambda_0^2 \log \Lambda + O(\lambda^3), \tag{3.16}$$

so that (3.14) gives

$$\beta(\lambda_0) := \frac{d\lambda_0}{d \log \Lambda} = \beta_0 \lambda_0^2 + O(\lambda_0^3). \tag{3.17}$$

This shows that there is always a zero of the beta function at $\lambda_0 = 0$, and it is possible to remove the cutoff while at the same time sending $\lambda_0(\Lambda) \to 0$. In other words, given a regularized theory, with a cutoff in momentum space or, for example, on a space-time lattice (which is another possible UV regulator) we find the limit $\Lambda \to \infty$ (or the continuum limit in the case of a lattice) tuning the bare couplings toward a zero of the beta function. This is a way to see things that has very important applications to statistical mechanics and critical phenomena and we will come back on this later.

There is another way to extract information from (3.11), which is more useful from the point of view of particle physics. We use the fact that Γ_0 is independent of the renormalization point μ. Instead Γ_R depends on μ explicitly, and also through the renormalized mass and coupling. Let us neglect all mass terms. Then we write

$$[\mu \partial_\mu + \beta(\lambda_R) \partial_{\lambda_R} + n\eta(\lambda_R)]\, \Gamma_R(k_i; \lambda_R, \mu) = 0. \tag{3.18}$$

where now

$$\beta(\lambda_R) = \mu \frac{d\lambda_R}{d\mu} \tag{3.19}$$

and

$$\gamma(\lambda_R) = \frac{1}{2} \mu \frac{d}{d\mu} \log Z_\varphi. \tag{3.20}$$

Equation (3.18) is called *Callan-Symanzik* equation. This equation is formally very similar to the one previously studied, but now we have the renormalized coupling λ_R rather than the bare coupling λ_0. It tells us how Γ_R changes if we change the renormalization point μ.

3.2 Wilson's RG

Ideally, one should take the further step of trying to understand physically why the divergences appear and why their effects are more severe in some theories than in others. This direct approach to divergence problem was pioneered in the 1960s by Kenneth Wilson. The crucial insights needed to solve this problem emerged from a correspondence, discovered by Wilson and others, between quantum field theory and statistical mechanics [37], [32], [26], [48].

3.2.1 The connection between Statistical Mechanics and QFT

The connection emerges from the euclidean formulation of the path integral in which (expectation values) *correlation functions* are computed as statistical averages in which each (quantum field) configuration is weighted with e^{-S}:

$$\langle \hat{\varphi}(x_1) \dots \hat{\varphi}(x_n) \rangle = \frac{\int \mathcal{D}\varphi \; \varphi(x_1) \dots \varphi(x_n) e^{-S}}{\int \mathcal{D}\varphi \; e^{-S}} \tag{3.21}$$

where S is the Euclidean action. In order to compute the path integral in eq. (3.21) it is convenient to put the system in a finite volume and to discretize the four-dimensional Euclidean space, using for instance a four-dimensional lattice with spacing a. Then the system has a finite number of variables $\varphi_i = \varphi(x_i)$, corresponding to the lattice sites, and the path integral is a well-defined statistical sum. The question is how to take the continuum limit $a \to 0$ so that a non-trivial QFT emerges.

At this concern consider a point x which is separated from the origin by n lattice sites, in general the correlation function behaves as:

$$\langle \varphi_x \varphi_0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2 + m^2} \sim e^{-|x|m}. \tag{3.22}$$

In statistical mechanics, the behaviour $\langle \varphi_n \varphi_0 \rangle \sim e^{-n/\xi}$ defines the dimensionless correlation length

$$\xi = \frac{1}{am} \tag{3.23}$$

and we see therefore that, if we want to take the continuum limit $a \to 0$ while keeping the physical mass m fixed, the correlation length must go to infinity.

The correlation length is a function of the couplings of the statistical system and therefore in order to obtain a continuum QFT in which the physical masses are finite, we must tune the parameters of the corresponding statistical system so that the correlation length diverges. Here we are rephrasing, in the language of statistical mechanics, the renormalization procedure introduced in the previous section: the dependence of the bare couplings on the cutoff (here the lattice spacing a) is tuned so to obtain finite values for the renormalized masses and couplings.

In this analogy the procedure of removing the cutoff in a QFT corresponds to tuning a statistical system toward a critical point where the correlation length diverges.

3.2.2 Kadanoff’s RG in 2d-Ising

In order to introduce the underlying concepts of the renormalization group, let us start by a simple but illuminating example of Wilson’s method implemented in x-space instead of momentum space and without having recourse to field theory. This example has been given by Kadanoff in 1966 [24] and introduce the concepts of spin-blocking, coarse-graining and *effective long-distance theories.*

Consider the two dimensional Ising model, defined placing a spin variable $\sigma_i = \pm 1$ on each site i of a two-dimensional lattice, whose Hamiltonian is:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{3.24}$$

and the sum runs over nearest-neighbour pairs (i, j). We consider the interaction term J to be positive (ferromagnetic interaction) so that it tends to align the spins.

The Hamiltonian (3.24) gives a *microscopic* description of the system in terms of interactions between nearest-neighbour spins separated at a distance a. However, we are usually not interested in the microscopic details of the system, but instead we usually look for the long distance behaviour and universal quantities. Going from a microscopic to a *macroscopic* description is a non-trivial problem, because in particular when the correlation length ξ is infinite, cooperative effects between spins can take place: even if the interaction is between nearest-neighbour spins, the influence of each spin on the others is propagated to the entire system resulting in an extraordinary cooperative effect.

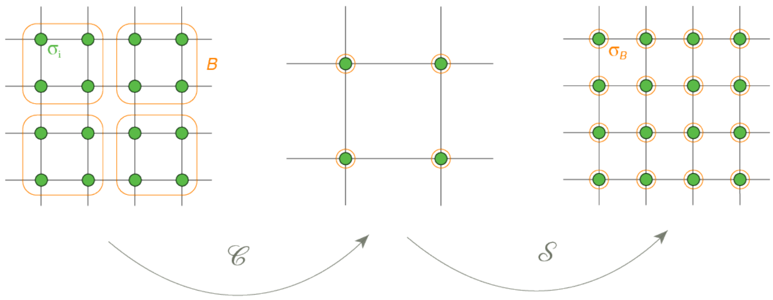


Figure 3.3: RG implementation $\mathcal{R}$ in direct space.

To construct an effective long distance theory we can proceed integrating out the short distance details of the model. Kadanoff's recipe is via a *spin-blocking* transformation: since at criticality the correlation length $\xi \gg 1$, we can regroup the lattice sites into blocks B made up of four neighbouring spins as shown in Fig. (3.3). This part of the RG transformation is known as *coarse-graining* $\mathcal{C}$: we decide how to reduce the short-distance degrees of freedom of the model. But a rule has to be established so that the system, which has now a lattice spacing twice as big as before $a' = 2a$ is scaled back to its original lattice spacing a and we define a new spin variable $\sigma_{B_i} = 1/4 \sum_{i \in B} \sigma_i = \pm 1$. This rule is known as *scale transformation* $\mathcal{S}$ and it is fundamental to preserve the partition function (and thus its critical behaviour) and to describe the same long distance Physics. This way we can consider the blocks B as *effective* spin variables living on the centre of their respective squares.

$$Z = \sum_{\sigma_i} e^{-H(\{\sigma_i\},\vec{K})} = \sum_{\sigma_B} \sum_{\sigma_i \in B} e^{-H(\{\sigma_i\},\vec{K})} = \sum_{\sigma_B} e^{-H(\{\sigma_B\},\vec{K}')} \tag{3.25}$$

The important thing to notice here is that the blocks, after a RG transformation, will interact not only with nearest-neighbour interaction, and the new interaction terms are parametrized by new coupling constants we indicate with $\vec{K}'$.

Therefore during the first step of a RG transformation $\mathcal{R}$, the couplings $\vec{K}$ evolve in new couplings $\vec{K}_2 = \mathcal{R}(\vec{K})$ which describes the system at a scale $2a$. The evolution step-by-step should thus be followed in a multiparameter space of coupling constants we call *theory space* Ω. We can iterate this recipe of regrouping and rescaling n times:

$$\vec{K}_n = \mathcal{R}(\vec{K}_{n-1}) = \underbrace{\mathcal{R} \circ \cdots \circ \mathcal{R}}_{n\text{-times}}(\vec{K}). \tag{3.26}$$

the coupling constant $\vec{K}_n$ being the effective coupling which describes the Physics at the scale $l = 2^n a$.

What we have found again, in this parallelism with standard QFT, is that the theory is described by running coupling constants which, from a statistical point of view depend on the length scale l over which microscopic fluctuations have been integrated out and in QFT depend on the cut off $1/l$ in momentum space. Therefore, as for the β-functions, all we need to know is encoded in the function $\mathcal{R}$ that describes how couplings evolve in theory space. Knowledge of the *exact* $\mathcal{R}$, in some sense, is equivalent to the solution of the theory at long distances.

3.2.3 Flow in theory space

We have seen that Kadanoff's idea of reducing degrees of freedom is translated into the problem of following the coupling constants flow in the theory space.

At criticality we have seen that for a second order phase transition $\xi = \infty$. Thus the point $\vec{K}_\xi$ is mapped into $\vec{K}'_{\xi'}$ under a RG transformation for which ξ' is infinite again and the blocked system we are describing is equally critical. This naturally defines the critical manifold $\mathcal{M}_c$ as the manifold spanned by all coupling constants for which $\xi = \infty$:

$$\mathcal{M}_c := \{\vec{K} \in \Omega \mid \xi = \infty\}, \tag{3.27}$$

every point on this manifold is a theory which describes a system at criticality.

Following the flow in theory space is a powerful method in connection with the notion of *fixed point*. A fixed point $\vec{K}_* \in \mathcal{M}_c$ is defined as a solution of the following equation:

$$\vec{K}_* = \mathcal{R}(\vec{K}_*), \tag{3.28}$$

and the RG flow in theory space is determined mainly by the fixed point structure: the RG trajectory will flow either to infinity or toward a fixed point.

Fixed points can be repulsive or attractive if, starting infinitesimally close to $\vec{K}_*$ in any direction, the RG transformation will bring us respectively away or toward

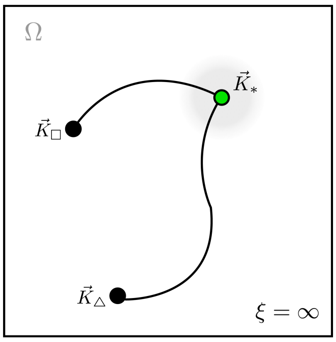


Figure 3.4: Typical topolgy of the RG flow in theory space: we can start for example with two points which represent an Ising model whose underlying lattice is respectively triangular and squared. Despite the microscopic theory for these two models is different, the RG flow drives them in the same basin of attraction since they belong to the same universality class.

$\vec{K}_*$. All fixed points on the critical surface have at least one unstable (repulsive) direction, which is the one orthogonal to the critical manifold itself: if we start close to $\mathcal{M}_c$ but not exactly on it, RG transformations constantly decreases ξ toward the trivial fixed point $\xi = 0$.

Since a RG trajectory connects equivalent theories (theories describing the same macroscopic physics), all the theories which lie in the same *basin of attraction* of the same fixed point can be considered equivalent. We can say that the theory space of renormalizable QFTs is split into equivalence classes called *universality classes*.

We understand that universality explains naturally why theories which are very different at a microscopic level turn out to have the same critical behaviour in terms of critical exponents. Consider two different point on $\mathcal{M}_c$, which therefore describe two very different theories at a microscopic level; if the RG flow drives them toward the same fixed point then it means that despite the diversity in the microscopic details the are describing the same long-distance physics.

3.2.5 Wilson's recipe in momentum space.

The implementation of the RG transformation in momentum space was given first by Wilson and Kogut in a very famous and celebrated report [46]. The idea is the same as before: in order to build an effective theory for the long distance degrees of freedom one should integrate out the short distance details which is not interested in. In other words, the best way to study a subset of degrees of freedom of a system is to build an effective theory for them integrating out the others.

Back to the field theory description in momentum space where the spin variables $\sigma_i(x)$ are substituted by fields $\varphi_i(k)$, the short distance details correspond here to the high energy modes of the field $\varphi(k)$, those for which $k \in [\Lambda - d\Lambda, \Lambda]$, with $\Lambda = 1/a$ a cutoff in momentum space.

The coarse graining operation consists here in integrating out the high energy modes of the field $\varphi(k)$ in the shell $[\Lambda - d\Lambda, \Lambda]$ and in computing the effective hamiltonian for the remaining modes. Iterating this procedure down to a scale $\bar{k}$ one remains with an effective hamiltonian for the low energy modes $k < \bar{k}$. The long distance physics is recovered here in the limit $\bar{k} \to 0$ since no fluctuation remains in this limit and we coped with computing the partition function.

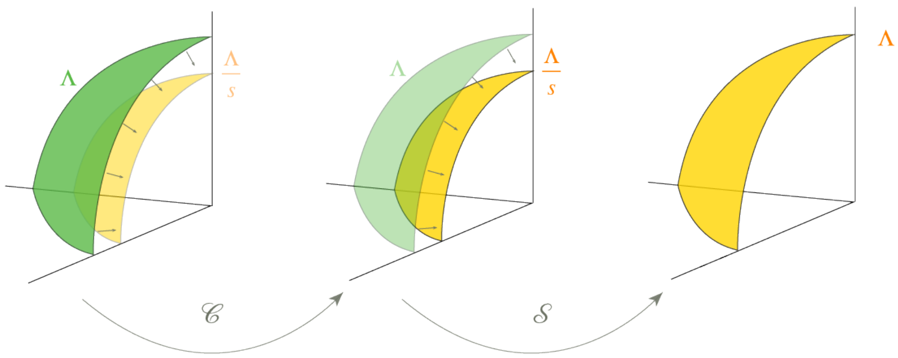


Figure 3.5: RG implementation $\mathcal{R}$ in momentum space.

In order to implement this ideas one can divide the fields $\varphi(k)$ into two pieces:

- $\varphi_>(k)$ that involves the rapid modes $k \in [\Lambda/s, \Lambda]$
- $\varphi_<(k)$ that involves the slow modes $k \in [0, \Lambda/s]$

and integrate out the rapid modes so that the partition function reads

$$\begin{aligned} Z &= \int^{\Lambda} \mathcal{D}\varphi(k)\; e^{-H[\varphi,\vec{K}]} \\ &= \int^{\Lambda/s} \mathcal{D}\varphi_<(k) \left[\int_{\Lambda/s}^{\Lambda} \mathcal{D}\varphi_>(k)\; e^{-H[\varphi_<,\varphi_>,\vec{K}]}\right] \\ &= \int^{\Lambda} \mathcal{D}\varphi'(k')\; e^{-H[\varphi',\vec{K}']}. \end{aligned} \tag{3.29}$$

Again we see that we start with couplings $\vec{K}$ and after the RG transformation $\mathcal{R}$ we obtain new couplings $\vec{K}' = \mathcal{R}(\vec{K})$ each associated with a given scale ($\Lambda \to \vec{K}$, $\Lambda/s \to \vec{K}'$, $\Lambda/s^2 \to \vec{K}''$, ...). In other words, in the coarse graining and rescaling procedure we generate new couplings the flow of which will determine a trajectory in theory space.

Compared to perturbation theory, in this approach *a la Wilson*, the coupling constants $\vec{K}$ are naturally associated with a scale instead of the complicated by-product of field-theoretical regularization and renormalization.

We conclude that renormalizable QFTs can emerge as effective theories describing the large-distance properties of critical phenomena. The RG of QFT then appears as an asymptotic form of the general Wilson-Kadanoff RG.

Part II

Functional Renormalization Group

4

Functional Renormalization Group

In order to cover Physics across different scales, it is important to have a systematic scheme of integrating out quantum fluctuations. Fortunately, in the last decade the development of the so-called functional renormalization group (FRG) method has somewhat unified the field by providing a mathematically elegant and yet simple way of expressing Wilson's idea of successive mode elimination in terms of a formally *exact* functional differential equation for the suitably defined generating functionals of a given theory.

4.1 Effective action

In quantum field theories, all the physical informations are encoded in correlation functions. In standard QFT we obtain correlation functions by definition from the product of n field operators at different spacetime points $\varphi(x_n)$ averaged over all possible field configurations (quantum fluctuations). In euclidean QFT, the field configurations are weighted with the exponential of the euclidean action $S[\varphi]$

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \frac{\int \mathcal{D}\varphi \; \varphi(x_1) \dots \varphi(x_n) e^{-S[\varphi]}}{\int \mathcal{D}\varphi \; e^{-S[\varphi]}}. \tag{4.1}$$

Minkowskian Green's function of quantum field theory in d spatial dimension plus time $\langle 0| T\{\varphi(x_1) \dots \varphi(x_n)\} |0\rangle$, can be obtained by computing the correlation functions of a classical statistical system living in $d+1$ spatial dimension, and performing the analytic continuation back to Minkowskian space.

We also assume that a proper regularized definition of the measure can be given (for instance, using a spacetime lattice discretization), which we formally write as

$$\int \mathcal{D}\varphi \; \longrightarrow \int_\Lambda \mathcal{D}\varphi \tag{4.2}$$

where Λ denotes an ultraviolet (UV) cutoff. The regularized measure should also preserve the symmetry of the theory. Here we work with scalar fields φ despite the following discussion holds identically for other fields like spinors fields, see e.g. [4], [25].

All n-point correlation functions are summarized by the generating functional $Z[J]$

$$Z[J] =: e^{W[J]} = \int \mathcal{D}\varphi \; e^{-S[\varphi]+\int_x J\varphi} \tag{4.3}$$

where sources J are introduced as a mathematical trick to compute all n-point correlation functions by functional differentiation:

$$\langle \varphi(x_1)\dots\varphi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right)\Bigg|_{J=0}. \tag{4.4}$$

In (4.3) we introduced the generating functional of *connected correlators*, $W[J] = \ln Z[J]$, which, loosely speaking, is a more efficient way to store all relevant physical informations.

In analogy with the construction of the Gibbs free energy in statistical mechanics, an even more efficient way to store informations is obtained by a Legendre transform of $W[J]$, namely, the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J]\right). \tag{4.5}$$

For any given ϕ, there is a particular value $J = J_{\text{sup}}$ for which $\int J\phi - W[J]$ is maximum. At $J = J_{\text{sup}}$

$$0 = \frac{\delta}{\delta J(x)}\left(J\phi - W[J]\right) \quad \Rightarrow \quad \phi := \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \langle\varphi\rangle_J. \tag{4.6}$$

This implies that ϕ corresponds to the expectation value of φ in the presence of the sources J. The field ϕ is related to φ in the same way that the magnetization is related to the local spin field in statistical mechanics: it is a weighted average over all possible fluctuations. The meaning of Γ becomes clear by studying its derivative w.r.t. ϕ at $J = J_{\text{sup}}$:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -\int_y \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(x)} + \int_y \frac{\delta J(y)}{\delta\phi(x)}\phi(y) + J(x) = J(x). \tag{4.7}$$

This is the *quantum equation of motion* by which the effective action $\Gamma[\phi]$ governs the dynamics of the field expectation value, taking the effects of all quantum fluctuations into account.

As the Gibbs free energy gives, in statistical mechanics, a picture of the preferred thermodynamic state that includes all effects of thermal fluctuations, when (4.6) is equal to zero we get the values of $\varphi(x)$ in the stable quantum states of the theory.

From the definition of the generating functional, the effective action can be written as an implicit functional integral in the presence of a background field ϕ:

$$e^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\chi \; \exp\left(-S[\varphi+\chi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\chi\right) \tag{4.8}$$

where we introduced fluctuations $\chi := \varphi - \phi$ by a shift of the integration variable. We observe here that the effective action is determined by a nonlinear first-order functional differential equation, the structure of which is itself a result of a functional integration. We notice here that an exact determination of $\Gamma[\phi]$, and thus an exact solution, has been found for rare special cases.

Introducing the effective action formalism, we coped with the problem of switching from $W[J]$, which is a functional of the rather mathematical sources J we introduced to compute correlation functions, to $\Gamma[\phi]$ which is a functional of the more physical objects $\phi = \langle\varphi\rangle$.

Moreover a solution of (4.8) attempted by a vertex expansion of $\Gamma[\phi]$

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) \tag{4.9}$$

where the expansion coefficients $\Gamma^{(n)}$ correspond to the *one-particle irreducible proper vertices* (1PI). Just as an example, the 3-point and 4-point connected correlation functions can be expressed diagrammatically as

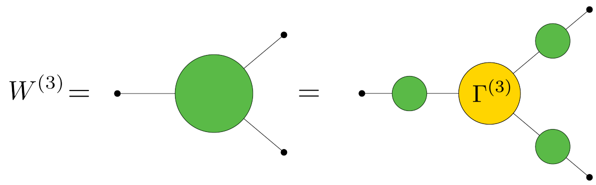


Figure 4.6: On the l.h.s. the connected 3-point function (green) expressed (on the r.h.s.) in terms of the 1PI 3-point function.

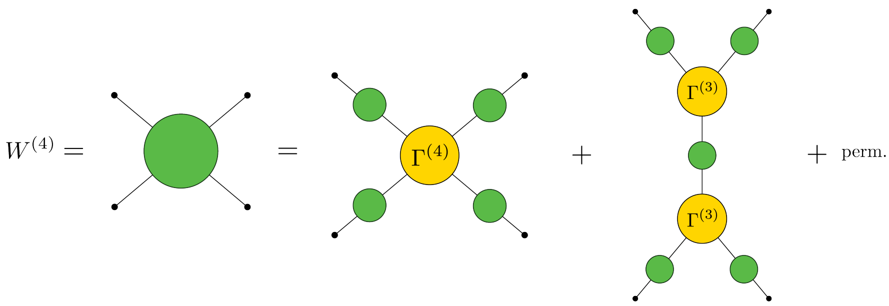


Figure 4.7: On the l.h.s. the connected 4-point function (green) expressed (on the r.h.s.) in terms of the sum of connected diagrams (green) plus the 1PI 3-point function and 1PI 4-point function.

The effective action is therefore the generating functional of the one-particle irreducible proper vertices and, as a functional, is obtained after integrating out the quantum fluctuations. Furthermore, the field equations derived from the effective action are exact as all quantum effects are included. The effective action is the interesting macroscopic object we will concentrate on since it gives a defined physical insight of the quantum theory, because in some sense knowledge of Γ is somehow equivalent to the solution of a theory.

4.2 A scale-dependent effective action

In principle a computation via equation (4.8) or via Scwhinger-Dyson equations correspond to integrating-out all fluctuations at once. We can instead approach this rather difficult problem implementing Wilson's idea of integrating out iteratively momentum shells. Our goal is to introduce an *averaged effective action* Γ_k with a momentum-shell parameter k so that:

- it is a generalization of the effective action which includes *only* fluctuations with $q^2 \gtrsim k^2$;
- it represents a coarse-grained effective action averaged over volumes $\sim k^{-d}$, so that quantum fluctuations on smaller scales are integrated out;
- in the limit $k \to \Lambda$ ($\Rightarrow$ small length scales) it reduces to the bare action $S[\varphi]$ since no fluctuation has been already integrated out;
- in the limit $k \to 0$ ($\Rightarrow$ large lenght scales) it corresponds to the full quantum action Γ since all the fluctuations are integrated out: lowering k results in a successive inclusion of fluctuations with momenta $q^2 \gtrsim k^2$ and therefore permits to explore the theory on larger and larger length scales;
- can be derived from the generating functional.

It is worth to notice here that the effective averaged action Γ_k has the very important property that it interpolates between the classical bare action S and the full effective action Γ as k is lowered from the UV cutoff to zero. The ability to follow the evolution to $k \to 0$ is equivalent to the ability to solve the theory. Most importantly, the dependence of the average action on the scale k is described by an *exact* non-perturbative flow equation presented below which is the key-point of the modern point of view on renormalization.

4.2.1 Derivation of the effective averaged action

We look at the derivation of such an effective action starting from the generating functional for n-point correlation functions

$$Z[J] = \int \mathcal{D}\chi \; e^{-S[\chi]+\int_x J\chi} \tag{4.10}$$

referring to scalar fields in d euclidean dimensions with classical action S. We define a scale-dependent generating functional by inserting a cutoff term ΔS_k

$$e^{W_k[J]} \equiv Z_k[J] = \int \mathcal{D}\chi \; e^{-S[\chi]-\Delta S_k[\chi]+\int_x J\chi} \tag{4.11}$$

where ΔS_k is choosen to be quadratic in the fields

$$\Delta S_k[\chi] = \frac{1}{2}\int_q \chi^*(q)R_k(q)\chi(q) \tag{4.12}$$

because this choice, as we will see, ensures than a *one-loop* flow equation can be exact [29]. The kernel regulator $R_k(q)$ should:

- actually implement an IR regularization

  $$\lim_{q^2\to 0} R_k(q) > 0 \tag{4.13}$$

  e.g. $R_k(q) \to k^2$ for $q^2 \to 0$ so the regulator screens the IR modes in a mass like fashion;

- be dominated by the stationary point of the action in the limit

  $$\lim_{k\to\Lambda} R_k(q) \to \infty \tag{4.14}$$

  so that $\Gamma_{k\to\infty}[\phi] = \Gamma_\Lambda[\phi] = S[\phi]$;

- vanish in the limit $k \to 0$ at fixed q

  $$\lim_{k\to 0} R_k(q) = 0 \tag{4.15}$$

  so we recover the standard generating functional as well as the full effective action in this limit $\Gamma_{k\to 0}[\phi] = \Gamma[\phi]$.

Typical regulators that satisfies these three conditions well known in literature [4], [28], are plotted below.

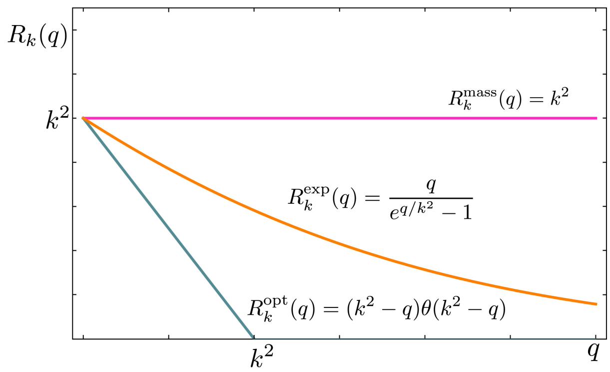


Figure 4.8: Typical cutoff functions

Now that we checked that the interpolating effective action Γ_k exhibits the correct limiting behaviour upon inserting a cutoff function, we can proceed further. The expectation value of χ, i.e. the macroscopic field ϕ, in the presence of $\Delta S_k[\chi]$ and J reads

$$\phi(x) := \langle \chi(x) \rangle = \frac{\delta W_k[J]}{\delta J(x)} \tag{4.16}$$

and again we employ a (modified) Legendre transform to define the scale dependent effective action as

$$\Gamma_k[\phi] = -W_k[J] + \int_x J(x)\phi(x) - \Delta S_k[\phi] \tag{4.17}$$

where we have subtracted the term $\Delta S_k[\phi]$ in the r.h.s. This subtraction of the IR cutoff term as a function of the macroscopic field ϕ is crucial for the definition of a reasonable coarse-grained free energy with the property $\Gamma_\Lambda \equiv S$ at the UV scale. The quantum equation of motion receives a regulator modification

$$\begin{aligned} \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} &= -\int_y \frac{\delta W_k[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \phi(x)} + \int_y \frac{\delta J(y)}{\delta \phi(x)} \phi(y) + J(x) - \frac{\delta \Delta S_k[\phi]}{\delta \phi(x)} = \\ &= J(x) - \frac{\delta}{\delta \phi(x)} \Delta S_k[\phi] = J(x) - (R_k \phi)(x). \end{aligned} \tag{4.18}$$

4.2.2 Derivation of the flow equation

We now derive a *flow equation* which describes the change of the scale-dependent effective action at a scale k with a change of the RG scale, and thus it describes how the effective actions on different scales are connected. In the modern functional formulation of the renormalization group it is the central equation and it is the starting point of all our further investigations.

In order to derive it we have to take the scale derivative of the modified Legendre transform (4.17). Defining the RG time $t := \log(k/\Lambda) \;\Rightarrow\; \partial_t = k\partial_k$, we get:

$$\begin{aligned} \partial_t \Gamma_k[\phi] &= -\partial_t W_k[J] - \int_x \frac{\delta W_k[J]}{\delta J(x)} \partial_t J + \int_x \phi(x) \partial_t J - \partial_t \Delta S_k[\phi] \\ &= -\partial_t W_k[J] - \partial_t \Delta S_k[\phi] \end{aligned} \tag{4.19}$$

where the derivative of the cutoff term (remember that ϕ is the independent variable in $\Gamma_k[\phi]$) reads:

$$\partial_t \Delta S_k[\phi] = \frac{1}{2} \partial_t \int_x \phi^*(q) R_k(q) \phi(q) = \frac{1}{2} \int_x \phi^*(x) (\partial_t R_k(q)) \phi(q). \tag{4.20}$$

We need to express the scale derivative of $W_k[J]$. Let's first express the derivative in terms of $\exp(W_k[J])$:

$$\begin{aligned} \partial_t W_k[J] &= \exp(-W_k[J]) \exp(+W_k[J]) \partial_t W_k[J] = \\ &= \exp(-W_k[J]) \left[\partial_t \exp(W_k[J]) \right]. \end{aligned} \tag{4.21}$$

Now, going back to the path integral representation and noticing that the scale dependence appears only in cutoff term, we get

$$\begin{aligned}
\partial_t W_k[J] &= e^{-W_k[J]}\ \partial_t \int \mathcal{D}\chi\ e^{-S[\chi]+\int_x J\chi - \Delta S_k[\chi]} = \\
&= e^{-W_k[J]}\ \partial_t \int \mathcal{D}\chi\ (-\partial_t \Delta S_k[\chi]) e^{-S[\chi]+\int_x J\chi - \Delta S_k[\chi]} = \\
&= e^{-W_k[J]}\ \partial_t \int \mathcal{D}\chi\ \left(-\frac{1}{2}\int_q \chi^*(q)\partial_t R_k(q)\chi(q)\right) e^{-S[\chi]+\int_x J\chi - \Delta S_k[\chi]} = \\
&= -\frac{1}{2}\int_q \partial_t R_k(q) e^{-W_k[J]} \int \mathcal{D}\chi\ \chi^*(q)\chi(q) e^{-S[\chi]+\int_x J\chi - \Delta S_k[\chi]}.
\end{aligned} \tag{4.22}$$

Now express this result in terms of the connected Green function:

$$\begin{aligned}
e^{-W_k[J]} \frac{\delta^2}{\delta J(q)\delta J^*(q)} e^{+W_k[J]} &= \frac{\delta^2 W_k[J]}{\delta J(q)\delta J^*(q)} + \frac{\delta W_k[J]}{\delta J^*(q)}\frac{\delta W_k[J]}{\delta J(q)} = \\
&= \langle \chi^*(q)\chi(q)\rangle_c + \phi^*(q)\phi(q) = \\
&= G_k(q) + \phi^*(q)\phi(q),
\end{aligned} \tag{4.23}$$

we therefore obtain the flow of $W_k[J]$

$$\begin{aligned}
\partial_t W_k[J] &= -\frac{1}{2}\int_q\ \partial_t R_k(q)\,(G_k(q) + \phi^*(q)\phi(q)) \\
&= -\frac{1}{2}\int_q\ \partial_t R_k(q) G_k(q) - \frac{1}{2}\int_q \phi^*(q)(\partial_t R_k(q))\phi(q) \\
&= -\frac{1}{2}\int_q\ \partial_t R_k(q)(G_k(q) + \partial_t \Delta S_k[\phi]).
\end{aligned} \tag{4.24}$$

Now insert (4.22) in the flow equation for Γ_k (4.24), we get:

$$\begin{aligned}
\partial_t \Gamma_k[\phi] &= -\partial_t W_k[J] - \partial_t S_k[\phi] \\
&= \frac{1}{2}\int_q \partial_t R_k(q) G_k(q)
\end{aligned} \tag{4.25}$$

This should be expressed as a functional differential equation for the scale-dependent effective action and in order to do it, let's express $G(p,q)$ in terms of this effective action

$$G(p,q) = \frac{\delta^2 W_k[J]}{\delta J^*(p) J(q)}, \qquad \phi(q) = \frac{\delta W_k[J]}{\delta J^*(q)}. \tag{4.26}$$

Using the variational condition on the effective action (4.18), the second variation w.r.t. $\phi^*(q')$

$$\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^*(q')\delta\phi(q)} = \frac{\delta J^*(q)}{\delta\phi(q')} - R_k(q)\delta(q-q'). \tag{4.27}$$

Now consider the following identity:

$$\begin{aligned}\delta(q-q') =& \frac{\delta}{\delta\phi^*(q)}\frac{\delta W_k[J]}{\delta J(q)} = \int_{q''}\frac{\delta^2 W_k[J]}{\delta J^*(q'')\delta J(q)}\frac{\delta J^*(q'')}{\delta\phi^*(q')} \\ \overset{(4.27)}{=}& \int_{q''}\frac{\delta^2 W_k[J]}{\delta J^*(q'')\delta J(q)}\left[\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^*(q')\delta\phi(q'')} + R_k(q)\delta(q'-q'')\right]\end{aligned} \tag{4.28}$$

and we finally obtain the scale-dependent inverse propagator:

$$G(q,q') = \left[\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^*(q)\delta\phi(q')} + R_k(q)\delta(q-q')\right]^{-1} \tag{4.29}$$

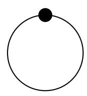
The result for the effective averaged action flow equation is:

$$\boxed{\partial_t\Gamma_k[\phi] = \frac{1}{2}\int_q\left[\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^*(q)\delta\phi(q)} + R_k(q)\right]^{-1}\partial_t R_k(q)} \tag{4.30}$$

4.3 Flow equation

The dependence of the average action Γ_k on the coarse graining scale k is described by the exact non-perturbative flow equation (4.30) and it represents the key-object in the functional formulation of RG [4], [30]:

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left\{\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\partial_t R_k\right\} = \frac{1}{2} \tag{4.31}$$



Hence, let us carefully discuss its most important properties and consequences:

- the flow equation is an exact functional integro-differential equation for Γ_k;
- it is possible to *define* QFTs based on the flow equation. For given suitable initial conditions, for instance, by defining the bare action at a high UV cutoff scale $k = \Lambda$, the flow equation defines a trajectory to the full quantum theory described by the full effective action Γ.
- the flow equation is UV and IR finite: by construction, the presence of R_k in the denominator of (4.31) guarantees the IR regularization. In addition, the properties (4.13) (4.15) also ensure UV regularization, since its predominant support lies on a smeared momentum shell near $p^2 \sim k^2$: the flow is localized in momentum space;

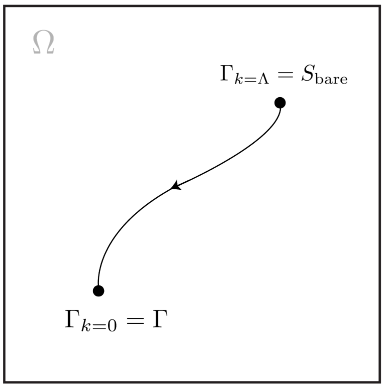


Figure 4.9: Sketch of the RG flow in theory space

- the solution of (4.31), as already mentioned, corresponds to an RG trajectory in the *theory space* Ω, introduced as the space of all action functionals compatible with the symmetries. The important property is that the two endpoints of a trajectory are given by the initial condition $\Gamma_{k\to\Lambda} = S_{\text{bare}}$ and the full effective action $\Gamma_{k\to 0} = \Gamma$;
- the regulator R_k can be choosen arbitrarily; obviously the trajectory depends on the regulator and this reflects the RG scheme dependence. Nevertheless, the final point of the trajectory ($\Rightarrow$ universal quantities) is independent of R_k;
- the equation has a very simple graphical expression as a one-loop structure equation and this has a very important practical consequence: only one integral has to be computed. This is very different from perturbation theory where l-loop diagrams require l d-dimensional integrals. This is a tremendous simplification of the exact renormalization group approach w.r.t perturbation theory. Anyway perturbation theory can be retrieved from (4.31) as we will show in appendix A1.

There is no general method to solve equation (4.31). This is the first hint to the necessity of introducing approximations in order to study it.

5

Approximation procedures

The fundamental FRG flow equation (4.31) for the generating functional of the one particle irreducible vertices is a very complicated mathematical object which in general cannot be solved exactly and which naturally addresses the question on how to solve it. It is therefore important to develop reliable approximation strategies for solving it. Three different types of strategies have been proposed. The first is based on an iterative approach in which one typically selects an initial seed for the effective action and insert it in (4.31) to obtain a new effective action an so on till self-consistency is reached. The second one relies on a suitable truncation of the hierarchy of integro-differential equations for the vertices. This vertex expansion approach was pioneered by Morris (1994) [30] and has been extensively used in the condensed matter community. The third approximation strategy, which is the one on which we will concentrate more, has been preferentially used in field theory and statistical mechanics [4], is based on the expansion of the functional $\Gamma_\Lambda[\phi]$ in powers of gradients of the field ϕ. Apart the first method, the strategy consists in solving the RG equation in a *restricted functional space*, projecting consistently the exact RG equation in the functional space chosen. The quality of the result, of course, depends crucially on the choice of the space in which we search for a solution.

5.1 Vertex expansion

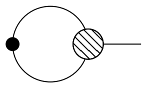
A hint of such an approximation has already been given in (4.9), which, for the effective averaged action reads:

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma_k^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n). \tag{5.1}$$

The idea is to insert such an expansion into the flow equation (4.31) and obtain an *infinite hierarchy of flow equations* for the vertex function $\Gamma_k^{(n)}$ which interpolate between the bare and the fully dressed vertices. Let us give an example for $\Gamma_k^{(1)}[\phi]$:

$$\partial_t \frac{\delta \Gamma_k[\phi]}{\delta \phi} = \frac{1}{2} \operatorname{Tr} \left\{ (\partial_t R_k)(-1)[\Gamma_k^{(2)} + R_k]^{-1} \frac{\delta \Gamma_k^{(2)}}{\delta \phi} [\Gamma_k^{(2)} + R_k]^{-1} \right\}$$
$$= -\frac{1}{2} \operatorname{Tr} \left\{ (\partial_t R_k)[\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(3)} [\Gamma_k^{(2)} + R_k]^{-1} \right\} \tag{5.2}$$

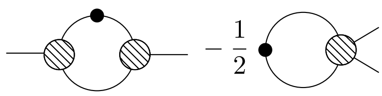
which graphically

$$\partial_t \Gamma_k^{(1)}[\phi] = -\frac{1}{2}$$ 

and if we take a further derivative to get the flow equation for the 2-point function:

$$\partial_t \frac{\delta \Gamma_k[\phi]}{\delta \phi \delta \phi} = - \operatorname{Tr} \left\{ (\partial_t R_k)[\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(3)} [\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(3)} [\Gamma_k^{(2)} + R_k]^{-1} \right\} +$$
$$- \frac{1}{2} \operatorname{Tr} \left\{ \partial_t R_k)[\Gamma_k^{(2)} + R_k]^{-1} \Gamma_k^{(4)} [\Gamma_k^{(2)} + R_k]^{-1} \right\} \tag{5.3}$$

which has a the following graphical representation

$$\partial_t \Gamma_k^{(2)}[\phi] = - \quad (\ldots) \quad - \frac{1}{2} \quad (\ldots)$$


We realize that to find the flow equation for $\Gamma_k^{(2)}$, we need $\Gamma_k^{(3)}$ and $\Gamma_k^{(4)}$. As a general rule, the flow equation of $\Gamma_k^{(n)}$ requires $\Gamma_k^{(n+1)}$ and $\Gamma_k^{(n+2)}$. This establishes the hierarchy of flow equations we mentioned before. If we want to solve this infinite tower of equations, we have to *truncate* it.

A possible truncation consists in keeping $\Gamma_k^{(2)}$ and $\Gamma_k^{(4)}$ and to neglect $\Gamma_k^{(6)}$ in the equation for $\partial_t \Gamma_k^{(4)}$. A better method could be to write down most general ansatz compatible with the symmetries for the effective action, expressed in terms of $\Gamma_k^{(2)}$ and $\Gamma_k^{(4)}$. In both cases the system of equations becomes closed and could, in principle, be solved.

It is important to notice here that the vertex expansion is *not* an expansion in some small parameter (although of course the assumption is that high order operators will be irrelevant and suppressed due to the existence of a large scale). For most practical applications, this is obviously the most problematic part, and it requires a lot physical insight to make the correct physical choices.

5.2 Derivative Expansion

The other main strategy to obtain approximate solutions of the FRG flow equations is the derivative expansion. This method has been successfully applied in statistical physics as well as in field theory where one is usually interested in long distance physics, that is the $|q| \to 0$ region of the correlation function. The idea is thus to keep and retain only the lowest orders of the expansion of Γ_k in powers of momenta (derivatives):

$$\begin{aligned}\Gamma_k[\phi] = \int d^d x \, \Big\{ & V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial\phi)^2 + \frac{1}{2} W_{1,k}(\phi)(\partial^2\phi)^2 + \\ & + \frac{1}{2} W_{2,k}(\phi)(\partial\phi)^2 \phi \partial^2 \phi + \frac{1}{4} W_{3,k}(\phi)(\partial\phi)^4 \Big\} + O(\partial^6)\end{aligned} \tag{5.4}$$

Here we fully understand what a *projection* in the theory space is: taking the scale derivative of (5.4) correspond to project the flow equation into the theory space spanned by all effective actions truncated up to the desired order. With this kind of ansatz, the RG equation on Γ_k becomes a set a ordinary differential equations for the running functions retained in the ansatz:

$$\begin{aligned}\partial_t V_k(\phi) &= \beta_V[\{V_k(\phi), Z_k(\phi), W_{i,k}(\phi)\}] \\ \partial_t Z_k(\phi) &= \beta_Z[\{V_k(\phi), Z_k(\phi), W_{i,k}(\phi)\}] \\ \partial_t W_{i,k}(\phi) &= \beta_{W_i}[\{V_k(\phi), Z_k(\phi), W_{i,k}(\phi)\}].\end{aligned} \tag{5.5}$$

The lowest level only includes the scalar potential and a standard kinetic term with constant field renormalization $Z_k(\phi)$: this approximation is therefore called *local potential approximation* (LPA) and it was introduced by Nicoll, Chang and Stanley [33].

5.3 LPA: Ising model

In order to have an idea on how the local potential approximation works, let us study here the simple case of the Ising universality class. When we work in LPA, equation (5.4) reduces to:

$$\Gamma_k[\phi] = \int d^d x \, \left\{ \frac{1}{2}(\partial\phi)^2 + V_k(\phi) \right\}, \tag{5.6}$$

and in what follows we consider this *ansatz* for $\mathbb{Z}_2$-invariant models.

The problem is to *project* the RG flow equation (4.31) on the theory space spanned by all effective potentials V_k. This is naturally performed by *defining* the effective potential as Γ_k computed for *uniform* field configurations $\phi = \phi_c$:

$$V_k(\phi_c) := \frac{1}{\Omega} \Gamma[\phi_c] \tag{5.7}$$

To compute the RG flow, we act on both sides of this equation with ∂_t and we calculate the right hand side thanks to Wetterich's equation (4.31). We notice here that the from the quantum equation of motion,

$$\frac{\delta\Gamma[\phi_*]}{\delta\phi} = 0 \quad \Longleftrightarrow \quad V'(\phi_*) = 0 \tag{5.8}$$

and therefore we deduce that if we are in the situation that the ground state is independent on the position (field configuration constant), the vacuum of the theory actually corresponds to an extremum of the effective potential.

Coming back to the projection procedure we have:

$$\partial_t \Gamma_k[\phi_c] = \Omega \partial_t V_k[\phi_c] = \frac{1}{2} \int_q \left[\Gamma^{(2)}_{k,|\mathrm{LPA}} + R_k \right]^{-1}_{q,-q} \partial_t R_k, \tag{5.9}$$

so we have to invert $[\Gamma^{(2)}_k + R_k]_{q,-q}$ within LPA. From now on, we omit the superscript LPA on Γ_k and c on ϕ_c. An elementary calculation leads to:

$$\Gamma^{(2)}_{k,(q,q')} = (2\pi)^{-d} \left(\frac{\delta V_k}{\delta q \delta q'} + q^2 \right) \delta(q + q') \tag{5.10}$$

and we immediately find

$$\partial_t V_k(\phi) = \frac{1}{2} \int_q \frac{\partial_t R_k(q^2)}{q^2 + V''_k(\phi) + R_k(q^2)}. \tag{5.11}$$

It is convenient to compute the d-dimensional integral in generalized spherical coordinates:

$$\int_q \longrightarrow \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^\infty dq\; q^{d-1} \tag{5.12}$$

and to introduce a change of variable $q^2 =: z$ so that (5.11) reads:

$$\partial_t V_k(\phi) = \frac{1}{2(4\pi)^{d/2}\Gamma(d/2)} \int dz\; z^{\frac{d}{2}-1} \frac{\partial_t R_k(z)}{z + V''_k(\phi) + R_k(z)}. \tag{5.13}$$

This form is particularly useful now because, in order to proceed further, we have to explicitly choose a cutoff function R_k. If we choose the optimized Litim's regulator cutoff

$$R_k(z) = (k^2 - z)\theta(k^2 - z) \tag{5.14}$$

the flow equation for the effective potential becomes:

$$\partial_t V_k(\phi) = c_d k^d \frac{1}{1 + \frac{V''_k(\phi)}{k^2}}, \tag{5.15}$$

where we have defined $c_d := [(4\pi)^{d/2}\Gamma(d/2)]^{-1}$. Equation (5.15) is the *exact* projected flow equation for the effective potential V_k. Up to know we implemented the coarse-graining part of the full RG transformation, and so we have to switch to dimensionless variables to implement the scale transformation. Let us therefore introduce

$$\begin{aligned} \phi &= k^{d/2-1}\tilde{\phi} \\ V_k(\phi) &= k^d \tilde{V}_k(\tilde{\phi}). \end{aligned} \tag{5.16}$$

Once these dimensionless variables are introduced in (5.15) we obtain the exact flow equation for the dimensionless effective potential $\tilde{V}_k$:

$$\boxed{\partial_t \tilde{V}_k(\tilde{\phi}) = -d\tilde{V}_k(\tilde{\phi}) + \left(\frac{d}{2}-1\right)\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) + c_d \frac{1}{1+\tilde{V}_k''(\tilde{\phi})}.} \tag{5.17}$$

Equation (5.17) is quite general because, apart from the fact that the trace operation in (4.31) is reduced to an integral implying no internal indeces to be summed on, nowhere we have already specified the mentioned $\mathbb{Z}_2$-symmetry.

All we can learn about the model at this approximation is contained in the solution of this equation. We clearly see that the flow of $\tilde{V}_k$ has two parts, one that comes from the dimension of V_k and ϕ and one that comes from the dynamics of the model. This RG equation on $\tilde{V}_k$ is a rather simple partial differential equation that can be easily integrated numerically. We are now in the position to discuss the critical behaviour of the Ising model and to look for fixed points.

The effective potential V_k has another very nice property: it is the *generating functional* $\mathcal{B}$ of all the β-functions. In fact if we decide for example to expand $\tilde{V}_k$ in even powers of the field ($\mathbb{Z}_2$-symmetry):

$$\tilde{V}_k(\tilde{\phi}) = \sum_{n=1}^{\mathrm{N_{tr}}} \frac{\lambda_{2n}}{(2n)!}\tilde{\phi}^{2n} \tag{5.18}$$

we obtain every β-function we want by the following formula:

$$\tilde{\beta}_{2n} = \partial_t \lambda_{2n} = \frac{(2n)!}{n!}\frac{\partial^n}{\partial(\tilde{\phi}^2)^n}\partial_t \tilde{V}_k\bigg|_{\tilde{\phi}=0} =: \mathcal{B}(\tilde{V},\tilde{V}',\tilde{V}'')\big|_{\tilde{\phi}=0}\,. \tag{5.19}$$

For example the first β-functions are:

$$\begin{aligned}
\tilde{\beta}_2 &= -2\lambda_2 - c_d\frac{\lambda_4}{(1+\lambda_2)^2}\\
\tilde{\beta}_4 &= (d-4)\lambda_4 + 6c_d\frac{\lambda_4^2}{(1+\lambda_2)^3} - c_d\frac{\lambda_6}{(1+\lambda_2)^2}\\
\tilde{\beta}_6 &= (2d-6)\lambda_6 - 90c_d\frac{\lambda_4^3}{(1+\lambda_2)^4} + 30c_d\frac{\lambda_4\lambda_6}{(1+\lambda_2)^3} - c_d\frac{\lambda_8}{(1+\lambda_2)^2}\\
\tilde{\beta}_8 &= \ldots
\end{aligned} \tag{5.20}$$

If we want to discuss the symmetry breaking (critical behaviour) in the Ising model at the lowest possible approximation we can choose $\mathrm{N_{tr}} = 2$ so that V_k is expanded up to ϕ^4 term and equation (5.6) reads:

$$\Gamma_k[\phi] = \int d^d x \; \left\{\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_k^2(\phi)^2 + \frac{1}{4}\lambda_k(\phi)^4\right\}, \tag{5.21}$$

with relative β-functions:

$$\tilde{\beta}_m = -2m_k^2 - c_d \frac{\lambda_k}{(1+m_k^2)^2}$$
$$\tilde{\beta}_\lambda = (d-4)\lambda_k + 6c_d \frac{\lambda_k^2}{(1+m_k^2)^3}. \tag{5.22}$$

Fixed points are essential in the RG theory. In field theory, they determine the nature of the continuum limits while in statistical physics they control the large distance physics of a critical system. A fixed point is a *scaling solution* of $\partial_t \tilde{V}_k = 0$. At a fixed point of the RG flow the beta functions vanish:

$$\beta_{2n}(m_*, \lambda_*) = 0, \tag{5.23}$$

the solutions are (we are restricting ourselves in the case $d \geq 3$):

$$\text{Gauss FP} \begin{cases} m_* = 0 \\ \lambda_* = 0 \end{cases}$$
$$\text{Wilson-Fisher FP} \begin{cases} m_* = -\frac{4-d}{16-d} \\ \lambda_* = \frac{288}{c_d}\frac{4-d}{(16-d)^3}. \end{cases} \tag{5.24}$$

We soon realize that we have the following structure:

$$\begin{array}{ll} d > 4 & \text{spurious FPs} \\ d = 4 & \text{only Gaussian FP} \\ d < 4 & \text{non trivial FPs.} \end{array} \tag{5.25}$$

It is easy to see from (5.24) that it is only in $d < 4$ that the mass term m_* is negative opening for a broken phase. For example in $d = 3$ the coordinates of the WF fixed point are $m_{\text{WF}} = -1/13$ and $\lambda_{\text{WF}} \simeq 7,75$. This is actually what we expect, since the upper critical dimension for the Ising universality class is $d_c = 4$ and indeed we find that in $d > 4$ we have only spurious fixed points which don't represent any physical solution.

In $2 < d < 3$ multicritical fixed points appear at the dimensional thresholds $d_{c,i} = 2i/(i-1)$, $i = 2, 3, \ldots, \infty$. The Wilson-Fisher FP still exists and the same analysis which is carried on for $d = 3$ can be done to estimate the critical exponents in this multicritical regime. These kind of calculations have been performed brilliantly in continuous dimensions by Alessandro Codello [9], confirming also the special case of $d = 2$ which yields the infinite set of non-perturbative and multicritical fixed points expected from conformal field theory (CFT) minimal models.

In order to test the stability properties of the fixed point and to extract universal quantities, we linearize the flow around the (WF) fixed point:

$$\mathcal{M}_{nm} = \left.\frac{\partial \beta_{2n}}{\partial \lambda_{2m}}\right|_* \tag{5.26}$$

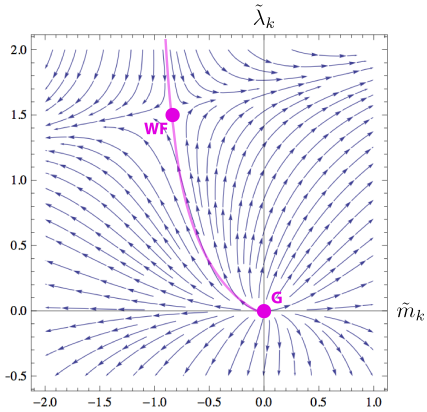


Figure 5.10: Flow in the bidimensional theory space. The flow is governed by the gaussian FP ($m = 0$, $\lambda = 0$) and the Wilson-Fisher fixed point ($m = m^*$ and $\lambda = \lambda^*$). The attractive submanifold is shown (purple).

the largest negative eigenvalue of the stability matrix is related to the correlation length exponent ν:

$$\nu = -\frac{1}{\Lambda_{\max}} \tag{5.27}$$

The critical exponent ν^{LPA} for the $3d$-Ising model is calculated to be $\nu^{\mathrm{LPA}} = 0.65(1)$ while the world best result given in [36], [17] gives $\nu = 0.63(0)$ see Tab. 1.2 . LPA qualitatively and quantitatively captures the critical behaviour very well even in the worst approximation among derivative expansion.

Within LPA, there is no renormalization of the derivative term. The exponent η is therefore vanishing at this order of the derivative expansion ($\eta^{\mathrm{LPA}} = 0$). In principle, from ν and η all the other exponents could be calculated thanks to scaling relations (1.25). We understand that, in order to close that system we should move to the next leading approximation in the derivative expansion.

5.4 LPA: $O(N)$ models

$O(N)$ models are important vector models which express a rich-full physical insight. For instance, the four dimensional $N = 4$ model describes the scalar sector of the electroweak standard model in the limit of vanishing gauge and Yukawa couplings. It is also used as an effective model for the chiral phase transition in QCD in the limit of two quark flavours. In condensed matter physics $N = 3, 2, 1$ corresponds respectively to the Heisenberg model, XY model and Ising model used to describe the ferromagnetic phase transition. There are other applications like the helium superfluid transition $N = 2$ and of course, liquid-vapour transition $N = 1$ or statistical properties of long polymer chains (self-avoiding walks) $N = 0$.

We start the analysis writing a suitable effective action that depends only on the $O(N)$ invariant $\rho = \frac{1}{2}\phi^a\phi_a$.

$$\Gamma_k[\phi] = \int_x \left\{ V_k(\rho) + \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi_a \right\} \tag{5.28}$$

The flow equation is then projected in the submanifold of the theory space spanned by all effective potentials compatible with the symmetry. The derivation proceeds identically w.r.t. the Ising universality class example given above, apart from the fact that now the trace operation in (4.31) implies a summation over internal indexes. The flow equation for the dimensionless effective potential reads:

$$\boxed{\partial_t \tilde{V}_k(\tilde{\rho}) = (d-2)\tilde{\rho}\tilde{V}_k' - d\tilde{V}_k + c_d \left[(N-1)\frac{1}{1+\tilde{V}_k'} + \frac{1}{1+\tilde{V}_k' + 2\tilde{\rho}\tilde{V}_k''} \right]} \tag{5.29}$$

Every solution of (5.29), together with its domain of attraction, represents a different $O(N)$ universality class. For every d and N one finds a discrete set of solutions corresponding to multi-critical potentials of increasing order, i.e. with i minima, which describe multi-critical phase transitions (in which one needs to tune multiple parameters to reach the critical point).

Truncating again V_k up to ϕ^4 as the lowest possible approximation for symmetry breaking, we have

$$\Gamma_k[\phi] = \int_x \left\{ \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi_a + \frac{m_k}{2}\phi_a \phi^a + \frac{\lambda_k}{4!}(\phi_a\phi^a)^2 \right\}, \tag{5.30}$$

and the flow equation is therefore projected into the theory space of all effective potentials spanned by the coupling constants m_k and λ_k. Switching to dimensionless quantities and taking advantage of the definition of $\tilde{V}_k$ as the generating functional of all the β-functions, we immediately write the *non-perturbative* β-functions for m_k and λ_k:

$$\begin{aligned} \tilde{\beta}_m &= -2\tilde{m}_k^2 - \frac{N+2}{3} c_d \frac{\tilde{\lambda}_k}{[1+\tilde{m}_k^2]^2} \\ \tilde{\beta}_\lambda &= (d-4)\tilde{\lambda}_k + \frac{2}{3}(N+8)c_d \frac{\tilde{\lambda}_k^2}{[1+\tilde{m}_k^2]^3}. \end{aligned} \tag{5.31}$$

which of course reduce to the Ising β-functions (5.22) in the case $N = 1$. In the general $O(N)$ case, the fixed point solutions exhibit an explicit N dependence:

$$\tilde{m}_*^2 = -\frac{(4-d)(N+2)}{8(N+5) - d(N+2)} \tag{5.32}$$

$$\tilde{\lambda}_* = \frac{96(4-d)(n+8)^2}{c_d[8(N+5) - d(N+2)]^3}. \tag{5.33}$$

The analysis of these fixed point solution revealed again [10] that for $d > 4$ and for *any* N, in accordance with the Ginzburg criterion, one finds only the gaussian fixed point ($i = 1$). Starting at $d = 4$, the upper critical dimension for $O(N)$-models, the Wilson-Fisher fixed points ($i = 2$) branch away from the gaussian fixed point. When $d = 3$ these fixed points describe the known universality classes of the Ising, XY, Heisenberg and other models.

5.5 Anomalous dimension

The local potential approximation (LPA) is the simplest approximation among the derivative expansion. It anyway captures qualitatively well the critical behaviours of the models analized. If we want to take a step forward and calculate critical exponents we should work in the next leading approximation scheme, namely the *improved local potential approximation* (LPA').

In this approximation we consider the field-renormalization function $Z_k(\rho)$ *not* to be constant. Working again with $O(N)$-models, the standard effective averaged action becomes:

$$\Gamma_k[\phi] = \int_x \left\{ V_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \phi^a \partial^\mu \phi_a \right\}. \tag{5.34}$$

Following [10], we could derive the flow equation for the dimensionless effective potential which in these approximation scheme reads:

$$\boxed{\partial_t \tilde{V}_k(\tilde{\rho}) = (d-2)\tilde{\rho}\tilde{V}_k' - d\tilde{V}_k + c_d(N-1)\frac{1-\frac{\eta}{d+2}}{1+\tilde{V}_k'} + c_d \frac{1-\frac{\eta}{d+2}}{1+\tilde{V}_k' + 2\tilde{\rho}\tilde{V}_k''}.} \tag{5.35}$$

We see that naturally the anomalous dimension η enters the flow equation because it fixes the scaling properties of the field at a particular fixed-point.

The computation of the anomalous dimension η_k requires the computation of the flow of Z_k. As we did for the potential this is possible only after a definition of Z_k in terms of Γ_k has been found. It is clear that Z_k corresponds to the term in Γ_k which is quadratic in ϕ and in q. In fact, this definition is not sufficient to completely characterize Z_k since it is the first term in the expansion of the function $Z_k(\tilde{\rho})$ and it is necessary to specify around which value of $\tilde{\rho}$ the expansion is performed. Here again, we choose the minimum $\tilde{\rho}_0$ of the potential. A precise calculation [4] shows that

$$Z_k = \frac{(2\pi)^d}{\delta(p=0)} \lim_{p^2 \to 0} \frac{d^2}{dp^2} \left(\bar{\Gamma}^{(2)}_{(2,p),(2,-p)} \Big|_{\min} \right) \tag{5.36}$$

where $\bar{\Gamma}^{(2)}_{(2,p),(2,-p)}$ is the second derivative of Γ_k with respect to $\phi_2(p)$ and $\phi_2(-p)$. The flow of Z_k is then obtained by acting on both sides of (5.36) with ∂_t. The result is:

$$\eta_k = \frac{16 v_d}{d} \tilde{\rho}_0 \lambda_k^2 m_{2,2}^d (2\tilde{\rho}_0 \lambda_k) \tag{5.37}$$

where we defined the threshold function $m_{2,2}^d$ as

$$m^d_{n_1,n_2}(w) := -\frac{1}{2}Z_k^{-1}k^{d-6}\int_0^\infty dx\; x^{d/2}\partial_t \frac{(\partial_x(Z_k x + R_k(x)))}{(Z_k x + R_k(x))^{n_1}(Z_k x + R_k(x) + w)^{n_2}} \tag{5.38}$$

To lowest order its value is related to the running dimensionless effective potential by:

$$\eta = c_d \frac{4\tilde{\rho}_0\left[\tilde{V}_*''(\tilde{\rho}_0)\right]^2}{\left[1 + 2\tilde{\rho}_0\tilde{V}_*''(\tilde{\rho}_0)\right]^2} \tag{5.39}$$

In general we solve equations (5.35) and (5.39) iteratively as proposed in [9] and we report here the most interesting results obtained for $\mathbb{Z}_2$ and $O(N)$ universality classes.

For the Ising universality class we plot below in Fig. 5.11 the anomalous dimensions η_i of the first five multicritical scaling solutions as a function of d. It is interesting to see how the anomalous dimension η_i vanishes at the corresponding upper critical dimension d_c, confirming that in $d > d_c$ mean field solution applies where there is no field renormalization.

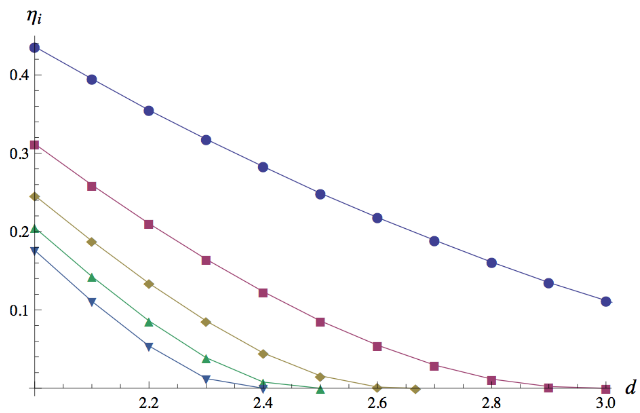


Figure 5.11: Anomalous dimensions η_i of the first five multicritical scaling solutions as a function of d.

Here we give the result for the critical exponents ν and η as given in [8] up to ∂^4 order as compared with 7-loop perturbation theory [21] and MonteCarlo simulations [23].

Method	ν	η
LPA	0.6505	0
∂^2	0.6281	0.0443
∂^4	0.632	0.033
7-loop	0.6304(13)	0.0335(25)
MC	0.6297(5)	0.0362(8)

For $O(N)$ models, approaching $d = 2$ one observes that only the $N = 1$ anomalous dimension continues to grow: for all other values of $N \geq 2$ the anomalous dimension bends downward to become zero exactly when $d = 2$. This non- trivial fact, not evident from the structure of equation (5.35) alone, is the manifestation of the Mermin-Wagner-Hohenberg theorem.

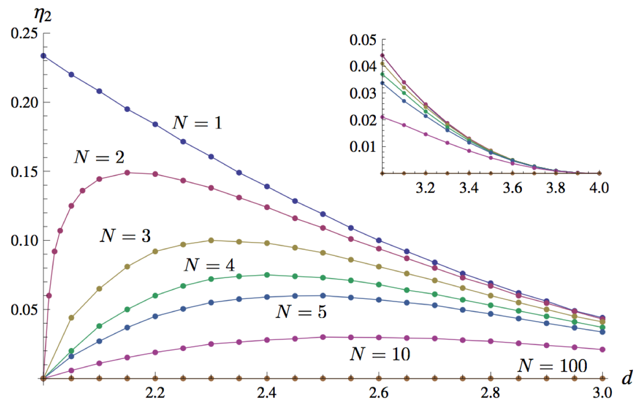


Figure 5.12: η_2 as a function of d for (from above) $N = 1, 2, 3, 4, 5, 10, 100$. In the inset we show the anomalous dimensions in the range $3 \leq d \leq 4$.

Part III

Results

6

$\mathbb{Z}_N$ models

$\mathbb{Z}_N$ models are the simplest statistical mechanics models which exhibit discrete global symmetries. These models have been studied extensively in the past years, mainly because they are interesting, both as lattice models and in the continuum limit as quantum field theories.

In two-dimensional statistical physics, $\mathbb{Z}_N$-models played a major role: these models exhibit intriguing properties such as semi-locality which gives rise to excitations called parafermions and characterized by fractional spin. Parafermions are well understood in critical system in the continuum limit, since the discrete symmetry they are associated with combine with conformal invariance and permit an extensive classification of the so called parafermionic conformal field theories. The first of such theories, constructed by Fateev and Zamolodchikov [19] [47], describes the critical points of $\mathbb{Z}_N$-symmetric lattice models (clock models) as the Ising model ($N = 2$), three state Potts model ($N = 3$) and Ashkin-Teller model ($N = 4$).

In elementary particle physics, the importance of the centre ($\mathbb{Z}_3$) of the SU(3) colour gauge group has been emphasized by 't Hooft [43] and Polyakov [38], who have argued that quark confinement is characterized as triality confinement, so that studies of the phases of the $\mathbb{Z}_3$ gauge theories may lead to important insights. Moreover, the phase structure of the $\mathbb{Z}_3$ gauge theory in four dimensions is related to that of the $\mathbb{Z}_3$ spin model in two dimensions [18].

In condensed matter physics $\mathbb{Z}_N$ symmetric theories appear naturally in the problem of two-dimensional melting [22]. A crystal in two dimensions has discrete rotational invariance in the plane of the crystal. For a square lattice this symmetry is $\mathbb{Z}_4$, while for a triangular lattice the symmetry is $\mathbb{Z}_6$. Part of the problem of melting is associated with the restoration of full rotational symmetry which can be broken to a $\mathbb{Z}_4$ or $\mathbb{Z}_6$ at low enough temperatures.

Besides theoretical motivations, there are some possible experimental realizations of the effective $\mathbb{Z}_N$ symmetry. Substances and experimental systems have been suggested and identified and relevant experiments have been performed.

For example magnetic systems well approximated by simple Ising systems are numerous and well known and we mention here notable examples $CoCs_2Br_5$ in $d = 2$ and Dy_3PO_5 in $d = 3$ [45]. The case $N = 3$ which stimulated curiosity because it coincides with the three-state Potts model, was experimentally realized in cubic ferromagnets like $DyAl_2$ [3], and substances like the stressed $SrTiO_3$ [1]. Higher n started to be investigate, for example, the stacked triangular antiferromagnetic Ising (STI) model with effective $N = 6$ symmetry may correspond to materials such as $CsMnI_3$.

For all these examples, the central issue is to understand the phase structure of the various $\mathbb{Z}_N$ symmetric theories. A major work would be the *classification* of *all* the universality classes to which these theories, in continuos dimension, belong. Today, in the framework of the functional renormalization group, this can be achieved in terms of the so called *spike-plot* technique [31] which will be our main interest in future works.

6.1 $\mathbb{Z}_N$ flow equation: derivation

In this section we give the derivation of the flow equation for the dimensionless effective potential we will use as the starting point to discuss the functional RG approach to $\mathbb{Z}_N$ invariant models. The derivation proceeds analogously to the one given in Chapters 4 and 5.
$\mathbb{Z}_N$ global discrete symmetry is well parametrized by complex scalar fields $(\phi, \bar{\phi})$ and we start specifying the microscopic model, in terms of a suitable bare action:

$$S[\phi] = \int_x \left[\bar{\phi}(-\Delta)\phi + V(\phi, \bar{\phi})\right] \tag{6.1}$$

where integration over x is again d-dimensional $\int_x = \int d^d x$. The microscopic action (6.1) is chosen to be invariant under the global abelian $\mathbb{Z}_N$ symmetry

$$S[\phi] \to S[\omega_n \phi] = S[\phi] \tag{6.2}$$

where ω_n is the suitable group representation

$$\omega_n = e^{\frac{2\pi i}{k} n} \qquad k = 1, \ldots, n-1. \tag{6.3}$$

In what follows, however, we make no reference to a particular symmetry and the result will be completely general.

After Fourier transformation, the kinetic term reads

$$\int_q \bar{\phi}(q)(q^2)\phi(q), \tag{6.4}$$

with $\int_q = (2\pi)^{-d} \int d^d q$. The standard functional definition of the partition function is:

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int \mathcal{D}\chi \; e^{-S[\chi]} \tag{6.5}$$

and as already introduced in previous chapters, we generalize the last equation by introducing a source term J for the fields χ and write

$$Z[J] = e^{W[J]} = \int \mathcal{D}\chi \; e^{-S[\chi] + \int_x \; J\chi}. \tag{6.6}$$

At this point we introduce the effective action $\Gamma[\phi]$ as usual, by a Legendre transform

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right). \tag{6.7}$$

In order proceed we include the infrared cutoff term $\Delta S_k[\chi]$ in (6.6) and define

$$e^{W_k[J]} = \int \mathcal{D}\chi \; e^{-S[\chi]-\Delta S_k[\chi]+\int J\chi}. \tag{6.8}$$

As we have already seen, the effective average action is defined as a modified Legendre transform of W_k

$$\Gamma_k[\phi] = \left(-W_k[J] + \int_x J\phi \right) - \Delta S_k[\phi]. \tag{6.9}$$

Our method to determine $\Gamma_k[\phi]$ (and for $k \to 0$ also $\Gamma[\phi]$) relies on the existence of the exact flow equation (4.31):

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \operatorname{Tr} \left\{ \left[\Gamma_k^{(2)}[\phi] + R_k \right]^{-1} \partial_t R_k \right\}. \tag{6.10}$$

Here the trace operation includes a momentum integration $\int_q$, as well as a sum over internal indices $i = 1, 2$, according to the two real composition $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. What we're actually doing is to consider the fluctuating real fields as the radial mode ϕ_1 and the Goldstone mode ϕ_2 as a convenient parametrization of the complex fields we naturally use to describe $\mathbb{Z}_N$ models. On the r.h.s. of (6.10) $\Gamma_k^{(2)}$ stands for the second functional derivative of $\Gamma_k[\phi]$

$$(\Gamma_k^{(2)}[\phi])_{ij}(q, p) = \frac{\overrightarrow{\delta}}{\delta\phi_i(-q)} \Gamma_k[\phi] \frac{\overleftarrow{\delta}}{\delta\phi_j(p)}. \tag{6.11}$$

It is therefore a matrix in internal and momentum space. Correspondingly, R_k in eq (6.10) stands for $R_k(q)\delta_{ij}\delta(q-p)$.

Approximate solutions of the exact flow equations obtain from a truncation of the general form of the effective action. In what follows we'll work in the LPA as introduced in chapter 5; in the absence of anomalies, the effective action $\Gamma[\phi]$ is invariant under the same symmetries as the microscopic action $S[\phi]$. This holds also for the effective average action $\Gamma_k[\phi]$, provided that cutoff term $\Delta S_k[\phi]$ is invariant too. We shall work here in LPA and this approximation will respect all the symmetries of the classical (microscopic) action

$$\begin{aligned} \Gamma_k[\phi] &= \int_x \left[V_k(\phi, \bar{\phi}) + \frac{1}{2}(\bar{\phi}(-\Delta)\phi + \phi(-\Delta)\bar{\phi}) \right] \\ &= \int_q \left[V(\phi_1, \phi_2) + \frac{1}{2}[\phi_1(q^2)\phi_1 + \phi_2(q^2)\phi_2] \right]. \end{aligned} \tag{6.12}$$

In order to proceed further toward the flow equation, the first thing to compute is the Hessian appearing in the flow equation. Its matrix form is given by

$$\Gamma^{(2)} = \begin{bmatrix} \overrightarrow{\delta}_{\phi_1(-q)}\Gamma_k \overleftarrow{\delta}_{\phi_1(p)} & \overrightarrow{\delta}_{\phi_1(-q)}\Gamma_k \overleftarrow{\delta}_{\phi_2(p)} \\ \overrightarrow{\delta}_{\phi_2(-q)}\Gamma_k \overleftarrow{\delta}_{\phi_1(p)} & \overrightarrow{\delta}_{\phi_2(-q)}\Gamma_k \overleftarrow{\delta}_{\phi_2(p)} \end{bmatrix} = \begin{bmatrix} \frac{q^2}{2} + \frac{\delta V_k}{\delta\phi_1\delta\phi_1} & \frac{\delta V_k}{\delta\phi_1\delta\phi_2} \\ \frac{\delta V_k}{\delta\phi_2\delta\phi_1} & \frac{q^2}{2} + \frac{\delta V_k}{\delta\phi_2\delta\phi_2} \end{bmatrix}$$

Despite the fact that other *equivalent* representations could be considered ((ρ,θ), $(\phi,\bar{\phi})$), it turns out that the most convenient one is the (ϕ_1,ϕ_2). Now we add the diagonal cutoff term $R_k(q)\delta_{ij}\delta(q-p)$ to the matrix $\Gamma^{(2)}$ and then, we first compute the inverse $(\Gamma_k^{(2)}+R_k)^{-1}$ and finally we compute the trace.
The result is as follows:

$$\mathrm{Tr}[(\Gamma_k^{(2)}+R_k)^{-1}] = \frac{q^2+2R_k+A+B}{(q^2/2+A+R_k)(q^2/2+B+R_k)-C^2}, \tag{6.13}$$

where

$$\begin{aligned} A :=& \frac{\delta V_k}{\delta\phi_1\delta\phi_1} \\ B :=& \frac{\delta V_k}{\delta\phi_2\delta\phi_2} \\ C :=& \frac{\delta V_k}{\delta\phi_1\delta\phi_2}. \end{aligned}$$

At this point the flow equation (6.10) is a projected flow equation for the effective potential V_k:

$$\partial_t V_k = \frac{1}{2}\int_q \frac{(q^2+2R_k+A+B)\partial_t R_k}{(q^2/2+A+R_k)(q^2/2+B+R_k)-C^2} \tag{6.14}$$

This form, as we have already seen, is particularly useful upon Litim's regulator cutoff:

$$R_k(z) = (k^2-z)\theta(k^2-z), \tag{6.15}$$

and making the change of variables $q^2 \to z$ so that $dq = \frac{1}{2}z^{-1/2}dz$, equation (6.14) becomes:

$$\begin{aligned} \partial_t V_k =& \frac{1}{2(4\pi)^{d/2}\Gamma(d/2)}\int_0^{2k^2} dz\; z^{d/2-1}\frac{(2k^2+A+B)2k^2}{(k^2+A)(k^2+B)-C^2} \\ =& \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^{k^2} dz\; z^{d/2-1}\frac{(2+A/k^2+B/k^2)}{(1+A/k^2)(1+B/k^2)-C^2/k^4} \\ =& c_d k^d \frac{(2+A/k^2+B/k^2)}{(1+A/k^2)(1+B/k^2)-C^2/k^4}. \end{aligned} \tag{6.16}$$

Equation (6.16) represents again the coarse graining part of the full RG transformation and we finally have to scale back to dimensionless variables to obtain the *exact* flow equation for the dimensionless effective potential in the case of complex scalar fields. Indicating again the dimensionless variables with tilde-notation:

$$\begin{cases} \phi = k^{\frac{d}{2}-1}\tilde{\phi} \\ V_k(\phi) = k^d \tilde{V}_k(\tilde{\phi}) \end{cases} \Longrightarrow \begin{cases} \tilde{\phi} = k^{1-\frac{d}{2}}\phi \\ \tilde{V}_k(\tilde{\phi}) = k^{-d} V_k(\phi) \end{cases} \tag{6.17}$$

the l.h.s. of (6.16) becomes:

$$\begin{aligned} \partial_t V_k =& k\partial_k (k^d \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2)) = \\ =& dk^d \tilde{V}_k(\tilde{\phi}) + \left[k \frac{\partial \tilde{\phi}_1}{\partial_k} \partial_{\tilde{\phi}_1} \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) + \frac{\partial \tilde{\phi}_2}{\partial_k} \partial_{\tilde{\phi}_2} \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) \right] k^d + \\ & + k^d \partial_t \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) = \\ =& k^d \Big\{ d\tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) + (1 - d/2)[\tilde{\phi}_1 \partial_{\tilde{\phi}_1} \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) + \\ & + \tilde{\phi}_2 \partial_{\tilde{\phi}_2} \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2)] + \partial_t \tilde{V}_k(\tilde{\phi}_1, \tilde{\phi}_2) \Big\} . \end{aligned} \tag{6.18}$$

Plugging eq. (6.18) in (6.16) we finally obtain the flow equation for the dimensionless effective potential in the case of complex scalar fields:

$$\boxed{\begin{aligned} & \partial_t \tilde{V}_k + d\tilde{V}_k + \left(1 - \frac{d}{2}\right) (\tilde{\phi}_1 \partial_{\tilde{\phi}_1} \tilde{V}_k + \tilde{\phi}_2 \partial_{\tilde{\phi}_2} \tilde{V}_k) = \\ & \quad = c_d \frac{2 + \partial^{(2)}_{\tilde{\phi}_1} \tilde{V}_k + \partial^{(2)}_{\tilde{\phi}_2} \tilde{V}_k}{(1 + \partial^{(2)}_{\tilde{\phi}_1} \tilde{V}_k)(1 + \partial^{(2)}_{\tilde{\phi}_2} \tilde{V}_k) - (\partial^{(2)}_{\tilde{\phi}_1 \tilde{\phi}_2} \tilde{V}_k)^2} . \end{aligned}} \tag{6.19}$$

Note that nowhere we made reference to the $\mathbb{Z}_N$ symmetry apart claiming that the bare action (and its evolution) will respect it. Our result is therefore completely general and suitable to be used wherever the problem at hand necessitates complex scalar fields to be approached.

6.2 $\mathbb{Z}_N \to O(2)$

The $\mathbb{Z}_N$ symmetry is fundamentally different from $O(2)$ symmetry because of its discrete nature. On the other hand, for large N (in the limit $N \to \infty$), it's natural to expect the $\mathbb{Z}_N$-symmetry to have similar effects to that of the $O(2)$ symmetry. Understanding this apparently contradictory aspects is an interesting problem. In this framework we checked that (6.19), in the desired limit, reduces exactly to the flow equation (5.29) for XY-models well known in FRG literature [4], [13].

In order to compare these results, it's convenient to shift to the (ρ, θ) representation. Then the limit $N \to \infty$ should be recovered as an independence of the effective potential on the angular variable θ. In this limit, indeed, we expect that the $\mathbb{Z}_N$-invariant terms which brakes the $O(2)$ symmetry of the potential should play no role. We start from the $\mathbb{Z}_N$-flow equation (6.19):

$$\partial_t \tilde{V}_k + d\tilde{V} + (1 - \frac{d}{2})(\tilde{\phi}_1 \partial_{\tilde{\phi}_1} \tilde{V}_k + \tilde{\phi}_2 \partial_{\tilde{\phi}_2} \tilde{V}_k) = c_d \frac{2 + \partial^{(2)}_{\tilde{\phi}_1} \tilde{V}_k + \partial^{(2)}_{\tilde{\phi}_2} \tilde{V}_k}{(1 + \partial^{(2)}_{\tilde{\phi}_1} \tilde{V}_k)(1 + \partial^{(2)}_{\tilde{\phi}_2} \tilde{V}_k) - (\partial^{(2)}_{\tilde{\phi}_1 \tilde{\phi}_2} \tilde{V}_k)^2} . \tag{6.20}$$

Changing to (ρ, θ) representation:

$$\begin{aligned} \tilde{\phi} &= \frac{1}{\sqrt{2}}(\tilde{\phi}_1 + i\tilde{\phi}_2) = \sqrt{\rho}e^{i\theta} \\ \bar{\tilde{\phi}} &= \frac{1}{\sqrt{2}}(\tilde{\phi}_1 - i\tilde{\phi}_2) = \sqrt{\rho}e^{-i\theta} \end{aligned} \tag{6.21}$$

The interesting term on the l.h.s. of (6.19) reads

$$\begin{aligned} (\tilde{\phi}_1\partial_{\tilde{\phi}_1}\tilde{V}_k + \tilde{\phi}_2\partial_{\tilde{\phi}_2}\tilde{V}_k) &= 2\rho\left[\cos^2\theta\right]\partial_\rho\tilde{V}_k - \left[\cos\theta\sin\theta\right]\partial_\theta\tilde{V}_k + \\ &+ 2\rho\left[\sin^2\theta\right]\partial_\rho\tilde{V}_k + \left[\sin\theta\cos\theta\right]\partial_\theta\tilde{V}_k = \\ &= 2\rho\partial_\rho\tilde{V}_k. \end{aligned} \tag{6.22}$$

The terms on the r.h.s. are transformed accordingly to

$$\begin{aligned} \partial^{(2)}_{\phi_1}\tilde{V}_k = \Bigg\{ & 2\rho\left[\cos^2\theta\right]\partial^{(2)}_\rho\tilde{V}_k + \frac{1}{2\rho}\left[\sin\theta\cos\theta\right]\partial_\theta\tilde{V}_k + \\ & + \left[\sin^2\theta\right]\partial_\rho\tilde{V}_k + \left[\cos^2\theta\right]\partial_\rho\tilde{V}_k + \frac{1}{\sqrt{2\rho}}\left[\sin\theta\cos\theta\right]\partial_\theta\tilde{V}_k + \\ & + \frac{1}{2\rho}\left[\sin^2\theta\right]\partial^{(2)}_\theta\tilde{V}_k - 2\left[\cos\theta\sin\theta\right]\partial^{(2)}_{\rho\theta}\tilde{V}_k\Bigg\} = \\ & \overset{N\to\infty}{=} 2\rho\left[\cos^2\theta\right]\partial^{(2)}_\rho\tilde{V}_k + \partial_\rho\tilde{V}_k \end{aligned} \tag{6.23}$$

while $\partial^{(2)}_{\phi_2}\tilde{V}_k$ is obtained from (6.23) with the substitution $\cos\theta \rightleftharpoons \sin\theta$, and the mixed partial derivative reads:

$$\begin{aligned} \partial^{(2)}_{\phi_1\phi_2}\tilde{V}_k = \Bigg\{ & \left[\cos^2\theta\right]\partial^{(2)}_{\rho\theta}\tilde{V}_k - \left[\sin^2\theta\right]\partial^{(2)}_{\rho\theta}\tilde{V}_k + \\ & - \frac{1}{2\rho}\left[\sin\theta\cos\theta\right]\partial^{(2)}_\theta\tilde{V}_k + 2\rho\left[\sin\theta\cos\theta\right]\partial^{(2)}_\rho\tilde{V}_k + \\ & - \frac{1}{\sqrt{2\rho}}\left[\cos^2\theta\right]\partial_\theta\tilde{V}_k + \frac{1}{2\rho}\left[\sin^2\theta\right]\partial_\theta\tilde{V}_k\Bigg\} = \\ & \overset{N\to\infty}{=} 2\rho\left[\sin\theta\cos\theta\right]\partial_\rho\tilde{V}_k. \end{aligned} \tag{6.24}$$

Putting everything together in (6.20) we obtain the flow equation in LPA for a $\mathbb{Z}_N$-invariant model in the limit $N \to \infty$ ($\mathbb{Z}_N \to O(2)$):

$$\partial_t\tilde{V}_k + d\tilde{V}_k + (2-d)\tilde{\rho}\partial_{\tilde{\rho}}\tilde{V}_k = \frac{2c_d(1 + \partial_{\tilde{\rho}}\tilde{V}_k + \tilde{\rho}\partial^{(2)}_{\tilde{\rho}}\tilde{V}_k)}{1 + 2\partial_{\tilde{\rho}}\tilde{V}_k + (\partial_{\tilde{\rho}}\tilde{V}_k)^2 + 2\tilde{\rho}\partial^{(2)}_{\tilde{\rho}}\tilde{V}_k + 2\tilde{\rho}(\partial_{\tilde{\rho}}\tilde{V}_k)(\partial^{(2)}_{\tilde{\rho}}\tilde{V}_k)} \tag{6.25}$$

which alternatively can be written in the following form:

$$\partial_t \tilde{V}_k + d\tilde{V}_k + (2-d)\tilde{\rho}\partial_{\tilde{\rho}}\tilde{V}_k = c_d \left[\frac{1}{1+\partial_{\tilde{\rho}}\tilde{V}_k} + \frac{1}{1+\partial_{\tilde{\rho}}\tilde{V}_k + \tilde{\rho}\partial^{(2)}_{\tilde{\rho}}\tilde{V}_k} \right] \tag{6.26}$$

which exactly matches the flow equation (5.29) for $O(N)$ models in the case $N = 2$. So, as expected, we recover the $O(2)$-models flow equation as a special case of the effective dimensionless potential flow equation for complex scalar fields (6.19).

We also tried a numerical simulation for the correlation exponent ν in the framework of the local potential approximation. Despite this approximation is not suitable to discuss numerical results since it is convenient to move to next leading orders in the derivative expansion, we anyway obtained $\nu^{\text{LPA}} = 0.7001(5)$ w.r.t. the XY-model correlation length critical exponent $\nu = 0.67155(27)$ [36].

We therefore conclude that the flow equation (6.19) both analytically and numerically reproduces the expected results for $O(2)$-models in the limiting behaviour $N \to \infty$ where we expect that the $\mathbb{Z}_N$ perturbation to the $O(2)$-models plays no role.

6.3 Perturbing $O(2)$ symmetry

A natural question then would be the effect of the symmetry breaking from the continuous $O(2)$ to the discrete $\mathbb{Z}_N$ symmetry. In doing so we can work with a *reduced* $\mathbb{Z}_N$-symmetric potential which anyway gives us a picture of how this perturbation affects the fixed point structure of the theory.

A generic $\mathbb{Z}_N$-symmetric model may me mapped, in the long distance limit, to the following ϕ^4-type field theory with euclidean action

$$S = \int_x \left[(\partial\phi)^2 + \frac{m}{2}|\phi|^2 + \frac{g}{4!}|\phi|^4 + \frac{\lambda_N}{N!}(\phi^N + \bar{\phi}^N) \right]. \tag{6.27}$$

The corresponding dimensionless effective potential is:

$$\tilde{V}_k(\tilde{\phi}, \bar{\tilde{\phi}}) = \frac{m_k}{2}|\tilde{\phi}|^2 + \frac{g_k}{4!}|\tilde{\phi}|^4 + \frac{\lambda_{k,N}}{N!}(\tilde{\phi}^N + \bar{\tilde{\phi}}^N). \tag{6.28}$$

The $\lambda_{k,N}$ term is the lowest-order term in ϕ which breaks the symmetry from $O(2)$ to $\mathbb{Z}_N$. This ϕ^4 model in three dimension, in the absence of the symmetry breaking term λ_N, reproduces of course the XY_3 universality class.

Its stability under the symmetry breaking to $\mathbb{Z}_N$ is determined by the scaling dimension of λ_N at the XY_3 fixed point. In order to derive it, we again take advantage of the definition of the dimensionless effective potential as the generating functional of all the β-functions as developed in chapter 5:

$$\lambda_{k,N} := \partial^{(N)}_{\tilde{\phi}} \tilde{V}_k \Big|_{\tilde{\phi},\bar{\tilde{\phi}}=0} \qquad \Longrightarrow \qquad \partial_t \lambda_{k,N} = \partial^{(N)}_{\tilde{\phi}} \left(\partial_t \tilde{V}_k \right) \Big|_{\tilde{\phi},\bar{\tilde{\phi}}=0}. \tag{6.29}$$

We have chosen to write the potential (6.28) in terms of $(\tilde{\phi}, \bar{\tilde{\phi}})$ and so the first thing to do is to cast eq. (6.19) into the right representation. The derivation is trivial and we give here the result:

$$\boxed{\begin{aligned}\partial_t \tilde{V}_k = -d\tilde{V}_k + (d/2-1)[(\tilde{\phi}\partial_{\tilde{\phi}}\tilde{V}_k + \bar{\tilde{\phi}}\partial_{\bar{\tilde{\phi}}}\tilde{V}_k)] + \\ + c_d \frac{2(1+\partial^{(2)}_{\tilde{\phi}\bar{\tilde{\phi}}}\tilde{V}_k)}{\left[(1+\partial^{(2)}_{\tilde{\phi}\bar{\tilde{\phi}}}\tilde{V}_k)^2 - \partial^{(2)}_{\tilde{\phi}}\tilde{V}_k\ \partial^{(2)}_{\bar{\tilde{\phi}}}\tilde{V}_k\right]}.\end{aligned}} \tag{6.30}$$

We are now in the position to plug the effective potential (6.28) into the flow equation (6.30) just derived. We obtain:

$$\begin{aligned}\partial_t \tilde{V}_k = & -d\left[\frac{m_k}{2}(\tilde{\phi}\bar{\tilde{\phi}}) + \frac{g_k}{4!}(\tilde{\phi}\bar{\tilde{\phi}})^2 + \frac{\lambda_{k,N}}{N!}(\tilde{\phi}^N + \bar{\tilde{\phi}}^N)\right] + \\ & + \left(\frac{d}{2}-1\right)\left[m_k(\tilde{\phi}\bar{\tilde{\phi}}) + \frac{g_k}{6}(\tilde{\phi}\bar{\tilde{\phi}})^2 + N\frac{\lambda_{k,N}}{N!}(\tilde{\phi}^N + \bar{\tilde{\phi}}^N)\right] + \\ & + 2c_d\left[1 + \frac{m_k}{2} + \frac{g_k}{6}(\tilde{\phi}\bar{\tilde{\phi}})\right]\Bigg[\left(1 + \frac{m_k}{2} + \frac{g_k}{6}(\tilde{\phi}\bar{\tilde{\phi}})\right)^2 + \\ & + \left(\frac{g_k}{12}\bar{\tilde{\phi}}^2 + \lambda_{k,N}\frac{N(N-1)}{N!}\tilde{\phi}^{N-2}\right)\left(\frac{g_k}{12}\tilde{\phi}^2 + \lambda_{k,N}\frac{N(N-1)}{n!}\bar{\tilde{\phi}}^{N-2}\right)\Bigg]^{-1}.\end{aligned} \tag{6.31}$$

Taking the N^{th}-derivative w.r.t. ϕ and keeping terms up to $\tilde{\phi}\bar{\tilde{\phi}}$, we obtain the following scaling relation at the XY fixed point:

$$\boxed{\partial_t \lambda_{k,N} = \lambda_{k,N}\left\{-(4-d)\frac{N(N-1)}{10} + d\left(1-\frac{N}{2}\right) + N\right\}.} \tag{6.32}$$

The scaling dimension of $\lambda_{k,N}$ is thus parametrically expressed in terms of the continuous dimension d:

$$y_N(d) := -(4-d)\frac{N(N-1)}{10} + d\left(1-\frac{N}{2}\right) + N. \tag{6.33}$$

and we can therefore analize how it behaves in the dimension range $2 \leq d \leq 4$ for different N. What we observe is that the λ_N perturbation is

- relevant in $2 \leq d \leq 4$ for both $N = 2$ and $N = 3$;
- marginal in the limiting special case $(N = 4, d = 4)$;
- irrelevant for $n = 4$ and $2 \leq d < 4$;
- irrelevant in $2 \leq d \leq 4$ for $N \geq 5$;

Here it is very interesting to notice that if we choose the dimension d to be $d = 4-\epsilon$, we *exactly* recover the result given by Oshikawa in [35] who approached the same problem in the framework of the ϵ-expansion:

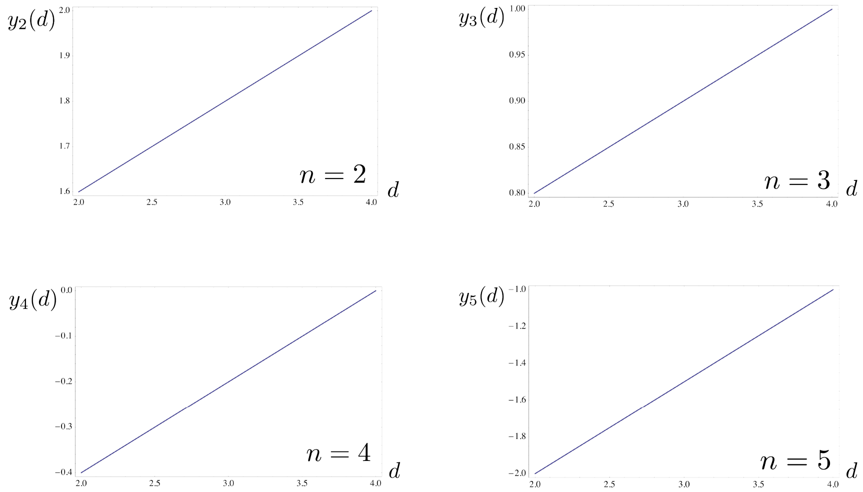


Figure 6.13: Scaling dimension $y_N(d)$ for $N = 2, 3, 4, 5$ in the range $2 \leq d \leq 4$.

$$y_N = 4 - N + \epsilon \left(\frac{N}{2} - 1 - \frac{N(N-1)}{10} \right) + O(\epsilon^2), \tag{6.34}$$

and this is an example on how FRG techniques parametrically extends to continuos dimension d previous results obtained within ϵ-expansion.

In the absence of the symmetry breaking λ_N, the transition belongs to the so-called XY_3 universality class. Extrapolating the result to $d = 3$, we see that the $\mathbb{Z}_N$ perturbation is irrelevant at the XY_3 fixed point for $N \geq N_c$. We know that the cases $N = 2$ (Ising model), $N = 3$ (three-states Potts model) do not belong to the XY_3 universality class and from the linearization procedure above N_c is expected to be $N_c = 4$. On the other hand, from a naive scaling analysis:

$$\begin{aligned} [\phi] &= k^{1-d/2} \\ [\lambda_N] &= k^{N(d/2-1)-d} \quad \overset{d=3}{\Longrightarrow} \quad N_c = 6 \end{aligned} \tag{6.35}$$

we find $N_c = 6$ and so we expect the λ_N perturbation to be for sure irrelevant for $N \geq N_c = 6$.

A numerical analysis of the fixed point structure for different N gives instead the following general picture:

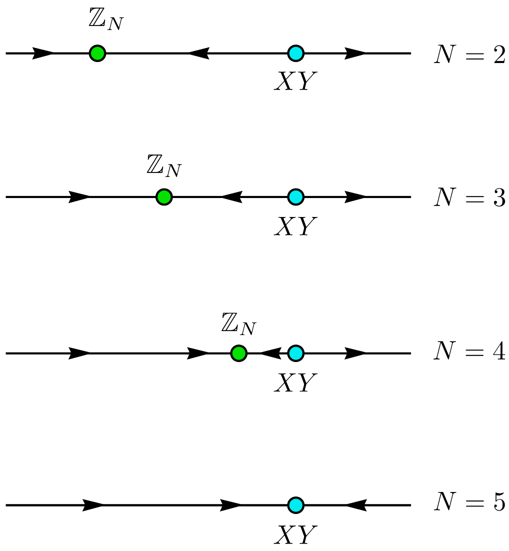


Figure 6.14: Phase diagram of the reduced $\mathbb{Z}_N$ theory in $d = 3$. For $p \geq 5$ the broken $\mathbb{Z}_N$ (low-temperature) phase disappears and we recover the typical BKT-type transition of the XY model.

$$
\begin{aligned}
&\text{N<5}\begin{cases}\text{Gauss FP:} & \{m_* = 0,\ g_* = 0,\ |\lambda_*| = 0\} \\ \text{Wilson Fisher FP:} & \{m_* < 0,\ g_* > 0,\ |\lambda_*| = 0\} \\ \mathbb{Z}_N\ \ \text{FP :} & \{m_* < 0,\ g_* > 0,\ |\lambda_*| \neq 0\}\end{cases} \\
&\text{N}\geq\text{5}\begin{cases}\text{Gauss FP:} & \{m_* = 0,\ g_* = 0,\ |\lambda_*| = 0\} \\ \text{Wilson Fisher FP:} & \{m_* < 0,\ g_* > 0,\ |\lambda_*| = 0\}\,.\end{cases}
\end{aligned}
\tag{6.36}
$$

The phase transition between the ordered phase and the disordered phase is governed by the XY_3 fixed point. This means that the critical exponents are identical to those of the XY_3 model. This is consistent with the numerical results: the correlation length critical exponent is estimated to be $\nu^{\text{LPA}} \simeq 0.70$. However it is not surprising to obtain an inaccurate result in the lowest order of the derivative expansion. It would be interesting to carry out the calculation to higher orders.

While in the disordered phase above T_c $(m > 0)$, there is no essential effect of the $\mathbb{Z}_N$ perturbation, the nature of the ordered phase $(m < 0)$ is more interesting. The Z_N perturbation λ_N is eventually enhanced in the ordered phase below T_c for $N \leq N_c = 5$.

From the flow diagram in Fig. 6.15 we understand that the $\mathbb{Z}_N$ perturbation λ_N

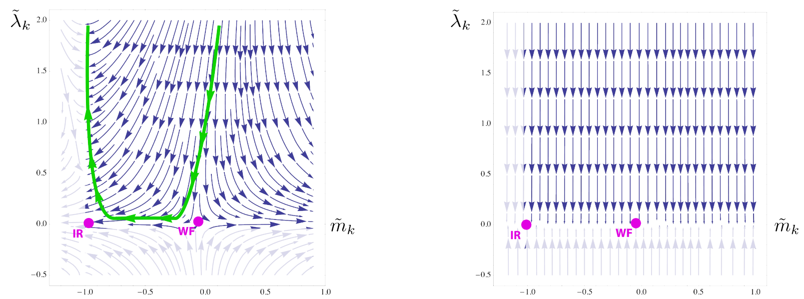


Figure 6.15: RG flow diagram of the $\mathbb{Z}_N$ models into the two-dimensional theory space spanned by $\tilde{m}_k$ and $\tilde{\lambda}_k$. On the left we select the case $N = 2$ in which the $\mathbb{Z}_N$ perturbation λ_N is irrelevant at the XY_3 fixed point, but it becomes relevant at the IR fixed point and it flows toward the $\mathbb{Z}_N$-fixed point (not shown in figure). On the right we plot the limiting case $N = 5$ which clearly shows that the λ_N perturbation is strictly irrelevant.

is irrelevant at the XY_3 fixed point but is relevant at the IR fixed point. For $T < T_c$ and $N < N_c$ the symmetry breaking perturbation λ_N is renormalized to a small value by the RG flow, and remains small until the RG flow reaches the IR fixed point. Then it is renormalized to the $\mathbb{Z}_{\mathbb{N}}$ fixed point.

Summing up, we clarified a FRG picture of the phase structure for $3d$ $\mathbb{Z}_N$ symmetric models. We conclude that there is no finite region of intermediate phase with a spontaneously broken $O(2)$ symmetry, but only a crossover to a phase where the discreteness of $\mathbb{Z}_N$ is relevant. Based on the FRG picture, we have derived a scaling law of the order parameter models which is valid for any continuos dimension d.

6.4 $\mathbb{Z}_N$ symmetric effective potential

In the previous section we analized how a *reduced* $\mathbb{Z}_N$ perturbation affects the $O(2)$ symmetry in the action (6.27). We would like to make a few remarks. First, we note that the RG argument used in the previous section does not contradict the transition to other than XY_3 universality class, because only the local stability of the XY_3 fixed point was discussed. It is possible that a lattice model with Z_N symmetry is renormalized to another *unknown* RG fixed point. Second, the perturbation term λ_N is not enough to describe theories with higher N ($N \geq 5$) since we are neglecting operators which are at least as relevant as λ_N.

In order to approach the problem from a more general viewpoint it is necessary to have a map of *all* the $\mathbb{Z}_N$ fixed points in continuos dimension and for any N: in this way we pursue the classification of *all* the universality classes of $\mathbb{Z}_N$-invariant models in continuos dimension.

Anyway, the standard direct analysis of the β-functions, as it was presented in the previous chapters is complicated as we will show below. The classification of *all* the universality classes of $\mathbb{Z}_N$-invariant models in continuos dimension should rely on a *spike-plot* technique as introduced by Morris in [31] and later developed by others [9] [12]. The spike-plot technique is a powerful tool which permits to select from all the spurious fixed points solutions of the β-functions only those who really corresponds to a physical solution. In this way the problem is reversed: we first spot the physical solutions of the theory among spurious ones and then we fully characterize its stability properties through the analysis of the relative β-functions. Up to now a spike-plot for $\mathbb{Z}_N$-invariant models still lacks and it will be our main interest in future works.

As already seen in the Ising-like example in chapter 5, the standard way to approach the problem is to give an ansatz for the most general potential respecting the particular symmetry at hand and the reality condition. In doing so we can take advantage of the definition of the effective potential as the generating functional of all β-functions and so we can start analizing them.

In the theory space spanned by all dimensionless effective potentials, the *most general* $\mathbb{Z}_N$-symmetric one is:

$$\boxed{\begin{aligned}\tilde{V}_k(\tilde{\phi},\bar{\tilde{\phi}}) = &\sum_i \left[\lambda_{k,i}\frac{(\tilde{\phi})^{Ni}}{(Ni)!} + \bar{\lambda}_{k,i}\frac{(\bar{\tilde{\phi}})^{Ni}}{(Ni)!}\right] + \\ &+ \sum_j \mu_{k,j}\frac{(\bar{\tilde{\phi}}\tilde{\phi})^j}{j!} + \\ &+ \sum_{l,m}\left[\rho_{k,lm}\frac{(\tilde{\phi})^{Nl}(\bar{\tilde{\phi}}\phi)^m}{(Nl+m)!\ m!} + \bar{\rho}_{k,lm}\frac{(\bar{\tilde{\phi}})^{Nl}(\bar{\tilde{\phi}}\tilde{\phi})^m}{(Nl+m)!\ m!}\right].\end{aligned}} \tag{6.37}$$

The potential introduced is parametrized by 5 different complex coupling constants $(\lambda, \bar{\lambda}, \mu, \rho, \bar{\rho})$ which can be obtained by functional derivatives:

$$
\begin{aligned}
\partial_{\tilde{\phi}}^{(Ni)} \tilde{V}_k \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0} &=: \lambda_{k,i} \\
\partial_{\tilde{\bar{\phi}}}^{(Ni)} \tilde{V}_k \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0} &=: \bar{\lambda}_{k,i} \\
\frac{1}{(j!)}\, \partial_{\tilde{\bar{\phi}}}^{(j)} \partial_{\tilde{\phi}}^{(j)} \tilde{V}_k \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0} &=: \mu_{k,l} \\
\partial_{\tilde{\phi}}^{(Nl+m)} \partial_{\tilde{\bar{\phi}}}^{(m)} \tilde{V}_k \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0} &=: \rho_{k,lm} \\
\partial_{\tilde{\bar{\phi}}}^{(Nl+m)} \partial_{\tilde{\phi}}^{(m)} \tilde{V}_k \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0} &=: \bar{\rho}_{k,lm}
\end{aligned}
\tag{6.38}
$$

where, of course, the index k in the coupling constants stands for the cutoff index. The coupling constants flow, and thus the β-functions, can be obtained acting on both sides of these equations with ∂_t and taking advantage of the flow equation in the $(\phi, \bar{\phi})$ representation (6.30). For example the $\lambda_{k,i}$ flow is given by:

$$
\partial_t \lambda_{k,i} = \partial_{\tilde{\phi}}^{(Ni)} \left(\partial_t \tilde{V}_k\right) \Big|_{\tilde{\phi},\tilde{\bar{\phi}}=0}. \tag{6.39}
$$

What we can do is therefore to analize them case by case and order by order. For example in the case $N = 2$ the first $\lambda\phi^4$-*type* truncated potential in which we consider terms with same order of relevance is:

$$
\begin{aligned}
V_k(\phi, \bar{\phi}) = &\frac{1}{2}\mu_1 |\phi|^2 + \frac{1}{4}\mu_2 |\phi|^4 + \\
&+ \frac{1}{6}\left\{\bar{\rho}_{1,1}|\phi|^2 \bar{\phi}^2 + \rho_{1,1}|\phi|^2 \phi^2\right\} + \\
&+ \frac{1}{2}\left\{\lambda_1 \phi^2 + \bar{\lambda}_1 \bar{\phi}^2\right\} + \frac{1}{24}\left\{\lambda_2 \phi^4 + \bar{\lambda}_2 \bar{\phi}^4\right\}.
\end{aligned}
\tag{6.40}
$$

The potential is of course $\mathbb{Z}_2$-symmetric and represents an effective theory in which we are describing a *generalized* Ising universality class. The corresponding β-functions are given in appendix A2.

In the case $N = 3$, a *generalized* Potts universality class truncated potential in which we consider terms up to ϕ^6 is:

$$
\begin{aligned}
V_k(\phi, \bar{\phi}) = &\frac{1}{2}\mu_1 |\phi|^2 + \frac{1}{4}\mu_2 |\phi|^4 + \frac{1}{12}\mu_2 |\phi|^6 \\
&+ \frac{1}{24}\left\{\bar{\rho}_{1,1}\bar{\phi}^3 |\phi|^2 + \rho_{1,1}|\phi|^2 \phi^3\right\} + \\
&+ \frac{1}{6}\left\{\lambda_1 \phi^3 + \bar{\lambda}_1 \bar{\phi}^3\right\} + \frac{1}{720}\left\{\lambda_2 \phi^6 + \bar{\lambda}_2 \bar{\phi}^6\right\}.
\end{aligned}
\tag{6.41}
$$

Since, as shown in appendix A2, the structure of the β-functions is very complicated, we have not found a general method to solve them. The method we addressed to solve for scaling solutions, was to nest β-functions so that one is left only with a system of five equations in five couplings. Indeed, at any order, the β-function in the coupling x_i is *always* a linear function in the next coupling x_{i+1}. Therefore we can solve β_i for x_{i+1} and use the last equation β_m, which is a function of x_m alone to close the system. This has somehow simplified the problem and we were able to compute critical exponents. We have found however that the critical exponents computed do not converge to a finite value because it is necessary to take into consideration next orders in the expansion of the potential. Moving to next orders has proved the problem to be more and more complicated and simulations failed to produce satisfactory results.

As a divertissement, we plot below the scaling solutions obtained in the nesting procedure in which we're left with two equations in two coupling constants for a multicritical Ising-like potential in fractional dimension: each intersection represents a possible fixed point solution, the majority of which are of course spurious fixed points.

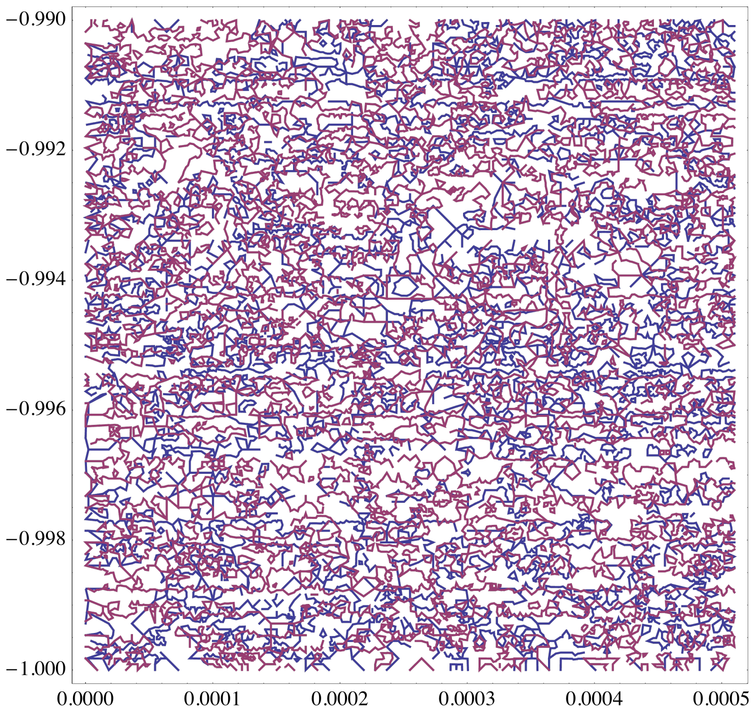


Figure 6.16: Contour plot of nested β-functions: each intersection here representing a possible fixed point scaling solution.

As it is clear, this direct approach to the β-functions for the most general $\mathbb{Z}_N$-invariant effective potential failed to give a defined and satisfactory picture of the universality classes. In future works we will try to derive a spike-plot for $\mathbb{Z}_N$-models from which we expect instead to have a simpler and more transparent picture of $\mathbb{Z}_N$ universality classes.

7

Conclusions

$\mathbb{Z}_N$ models, both as lattice models or as quantum field theories, are the simplest models which exhibit discrete global symmetries and they played a central role in several fields ranging from statistical-condensed matter Physics to elementary particle Physics. In this thesis we have explored a new way to study the flow of $\mathbb{Z}_N$ models in theory space within the framework of the Functional Renormalization Group (FRG) based on the effective average action (EAA).

Within the local potential approximation (LPA), our main result is an *exact* functional RG equation (6.19) for the dimensionless effective potential of complex scalar field theories. In deriving it we do not specify any particular symmetry and so, despite the fact that we focused it in the case of $\mathbb{Z}_N$-invariant theories, the result is completely general: it describes how the effective potential evolves in theory space as the scale at which we observe the system is enlarged from UV to IR for *any* complex scalar field theory and for *any* symmetry content.

In order to check analytically the exactness of the flow equation (6.19), we take advantage of the correspondence between $\mathbb{Z}_N$-models and $O(2)$-models in the limiting behaviour $N \to \infty$ where it is natural to expect they have similar effects. In this limit we *exactly* recover the $O(2)$ flow equation (5.29) for XY-model proving that the the large N limit is correct.

As an inverse problem we studied the effect of the symmetry breaking from the continuous $O(2)$ to the discrete $\mathbb{Z}_N$ symmetry. We worked with the lowest order effective potential (6.28) which brakes the $O(2)$-symmetry to $\mathbb{Z}_N$, but which anyway gives us a picture of how this perturbation affects the fixed point structure of the theory. In order to test the stability of the ϕ^4 model under the symmetry breaking $\mathbb{Z}_N$ we derived an *exact* scaling relation of the perturbation coupling λ_N which is valid in *any* continuos dimension d. The scaling relation reveals that the $\mathbb{Z}_N$ perturbation, in $2 \leq d \leq 4$, is relevant for $N = 2$ and $N = 3$ and irrelevant for $N \geq 5$, while it is marginal in the limiting case $(d = 4, N = 4)$ and irrelevant for $(d < 4, N = 4)$. This analysis extended and clarified the known results from Blankschtein, Oshikawa et al. [35], [5] but we anyway showed that the scaling relation we derived reduces exaclty to the scaling relation given in [35], in the framework of ϵ-expansion. In doing so we give a proof of how the ϵ-expansion can be recovered from FRG.

We specified the previous analysis to the case $d = 3$ giving a FRG picture of the phase structure of $\mathbb{Z}_N$-symmetric models, which has been extensively studied in the past years. We found that for $2 \leq N < 5$ the phase diagram opens for a new $\mathbb{Z}_N$ fixed point which disappears as soon as $N \geq 5$: there is no finite region of intermediate phase with a spontaneously broken $O(2)$-symmetry, but only a crossover to a phase where the discreteness of $\mathbb{Z}_N$ is relevant. The flow diagram we derived showed that the λ_N perturbation is irrelevant at the XY fixed point and becomes relevant at the IR fixed point.

Motivated by the desire of classifying *all* the universality classes in continuos dimension for $\mathbb{Z}_n$-invariant theories in the framework of the spike-plot technique, we introduced the *most* general $\mathbb{Z}_n$-invariant and real effective potential (6.37) which will be our starting point for future works.

We finally numerically computed the correlation length critical exponent ν in the XY_3-limit within LPA approximation: our best result is $\nu_{\text{this work}} = 0,7001(5)$ to be compared to $\nu = 0.67155(27)$ [36], which despite inaccuracy is perfectly acceptable in the approximation retained.

A1

Comparison to perturbation theory

Although the FRG equation (4.31) is a *one-loop* RG flow equation (which should not be confused with a standard perturbative loop as it contains the *full* propagator), it contains effects to arbitrary high loop order: it is possible to reproduce higher loop order perturbation theory. The derivation follows Litim and Pawlowski [29]. In order to show how perturbation theory can be retrieved, we start from the effective action within a loop expansion:

$$\Gamma_k[\phi] = S_{\mathrm{B}}[\phi] + \sum_{n=1}^{\infty} \Delta\Gamma_{n,k}[\phi] \tag{8.1}$$

where the functional S_{B} plays the role of a bare action and $\Delta\Gamma_{n,k}[\phi]$ comprises the n-th loop order. In terms of the flow equation, contributions of different loop orders can be identified:

$$\partial_t \Gamma_k[\phi] = \sum_{n=1}^{\infty} \partial_t \Delta\Gamma_{n,k}[\phi] \tag{8.2}$$

where we use the following notation for the m-point correlation function to n-loop order at a scale k:

$$\Gamma_{n,k}^{(m)} \tag{8.3}$$

We want to show here how we can retrieve the *two-loop* result for the effective action, which is the first non-trivial result. As a road map for the expansion in loop order, here is an outline of the calculation

- start from the effective action at n-loop level and calculate the two-point function

$$\Gamma_{n,k} \xrightarrow{\frac{\delta^2}{\delta\phi\delta\phi}} \Gamma_{n,k}^{(2)} = \Gamma_{n-1,k}^{(2)} + \Delta\Gamma_{n,k}^{(2)} \tag{8.4}$$

- insert the n-loop two-point function into the one-loop flow equation (4.31)

$$\frac{\partial_t R_k}{\Gamma_{n,k}^{(2)} + R_k} = \frac{\partial_t R_k}{\Gamma_{n-1,k}^{(2)} + \Delta\Gamma_{n.k}^{(2)} + R_k} \tag{8.5}$$

- isolate the $n+1$-loop correction to the effective action

$$\partial_t \Delta\Gamma_{n+1,k} = (\partial_t R_k)\frac{1}{\Gamma^{(2)}_{n-1,k}+R_k}\Delta\Gamma^{(2)}_{n,k}\frac{1}{\Gamma^{(2)}_{n-1,k}+R_k} \tag{8.6}$$

- integrate the flow equation to obtain the $n+1$-loop correction to the effective action

$$\Gamma_{n+1,k} = \Gamma_{n,k} + \Delta\Gamma_{n+1,k} \tag{8.7}$$

The effective action at one loop is calculated using the tree-level two-point function

$$\begin{aligned} \Gamma_{1,k=} &= S_\mathrm{B} + \Delta\Gamma_{1,k} \\ \Gamma^{(2)}_{0,k} &= S^{(2)}_\mathrm{B} \end{aligned} \tag{8.8}$$

The flow equation for the one loop correction to the effective action reads:

$$\partial_t \Delta\Gamma_{1,k} = \frac{1}{2}(\partial_t R_k)_{pq}\left[S^{(2)}_\mathrm{B} + R_k\right]^{-1}_{qp}. \tag{8.9}$$

Integrate this to get the one-loop correction:

$$\Delta\Gamma_{1,k} = \int_\Lambda^k dk'\ \frac{1}{k'}\partial_{t'}\frac{1}{2}[\log(S^{(2)}+R_k)]_{pp}, \tag{8.10}$$

and we see that the one-loop result corresponds with ordinary result:

$$\begin{aligned} \Gamma_{1,k} &= S_\mathrm{B} + \frac{1}{2}\left[\log(S^{(2)}_\mathrm{B}+R_k)\right]_{pp} - \frac{1}{2}\left[\log(S^{(2)}_\mathrm{B}+R_\Lambda)\right]_{pp} \\ &= S_\mathrm{B} + \frac{1}{2}\left[\log(S^{(2)}_\mathrm{B}+R_{k'})\right]_{pp}\Big|_\Lambda^k \end{aligned} \tag{8.11}$$

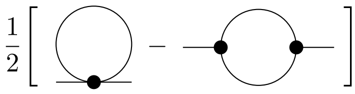
The one-loop contribution to the two-point function is obtained by taking the variation of the one-loop effective action

$$\begin{aligned} \left(\Delta\Gamma^{(2)}_{1,k}\right)_{qq'} &= \frac{\delta}{\delta\phi(q)\delta\phi(q')}\Delta\Gamma_{1,k} \\ &= \frac{1}{2}\frac{\delta^2}{\delta\phi(q)\delta\phi(q')}\left[\log(S^{(2)}_\mathrm{B}+R'_k)\right]_{pp}\Big|_\Lambda^k \\ &= \frac{1}{2}\left[G_{pp'}S^{(4)}_{\mathrm{B},\{pp',qq'\}} - G_{pq'''}S^{(3)}_{\mathrm{B},\{q'''q''q\}}G_{q''p'}S^{(3)}_{\mathrm{B},\{p'pq'\}}\right]\Big|_\Lambda^k \end{aligned} \tag{8.12}$$

where we used:

$$\frac{\delta}{\delta\phi(q)}G_{pp'} = \frac{\delta}{\delta\phi(q)}\left[S^{(2)}_\mathrm{B}+R_k\right]^{(-1)}_{pp'} = (-1)G_{pq'}S_{\mathrm{B},\{q'q''q\}}G_{q''p'} \tag{8.13}$$

with the following graphical representation:



Now we need to find the two loop correction to the flow equation for the effective action. In order to do this we insert the correction to the propagator into the flow equation and then we isolate the two-loop part:

$$\begin{aligned}\partial_t\Gamma_{2,k} &= \frac{1}{2}(\partial_t R_k)_{qp}\left[\Gamma^{(2)}_{1,k}+R_k\right]^{-1}_{pq} = \frac{1}{2}(\partial_t R_k)_{qp}\left[S^{(2)}_{\mathrm{B}}+\Delta\Gamma^{(2)}_{1,k}+R_k\right]^{-1}_{pq} = \\ &= \frac{1}{2}(\partial_t R_k)_{qp}\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{pq} + \\ &+ \frac{1}{2}(\partial_t R_k)_{qp}(-1)\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{pq''}\left(\Delta\Gamma^{(2)}_{1,k}\right)_{q''q'}\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{q'q}+\ldots \\ &= \partial_t\Delta\Gamma_{1,k}+\partial_t\Delta\Gamma_{2,k}+\ldots\end{aligned} \tag{8.14}$$

We therefore find for the correction:

$$\partial_t\Delta\Gamma_{2,k} = -\frac{1}{2}(\partial_t R_k)_{qp}G_{pq'}\left(\Delta\Gamma^{(2)}_{1,k}\right)_{q''q'}G_{q''q} \tag{8.15}$$

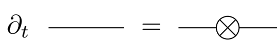
This needs to be integrated over all scales k. Calling, in abbreviation, the tree-level propagator (with cutoff at scale k)

$$G_{pq} = \left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{pq}, \tag{8.16}$$

the derivatives of the propagator w.r.t. the RG scale k writes:

$$\begin{aligned}\partial_t G_{pq} &= \partial_t\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{pq} = (-1)\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{pq'}(\partial_t R_k)_{q'q''}\left[S^{(2)}_{\mathrm{B}}+R_k\right]^{-1}_{q''q} = \\ &= (-1)G_{pq'}(\partial_t R_k)_{q'q''}\end{aligned} \tag{8.17}$$

which graphically



and this allows to rewrite terms in the flow equations as total derivatives with the correct combinatorial factors that come from inserting $(\partial_t R_k)$ in all possible propagators. We can therefore write

$$\begin{aligned}G_{pp'}S^{(4)}_{\mathrm{B},\{pp'q'q''\}}(\partial_t' G)_{q''q'} &= \frac{1}{2}\partial_{t'}(G_{pp'}S^{(4)}_{\mathrm{B},\{pp'q'q''\}}(\partial_t' G)_{q''q'}) \\ G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q'''p'q'\}}(\partial_{t'}G)_{q'q} &= \frac{1}{3}\partial_{t'}(G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q'''p'q'\}})\end{aligned} \tag{8.18}$$

Using now the result for the scale derivative of G_{pq}, we can write for the two-loop flow correction

$$\begin{aligned}\partial_t \Delta\Gamma_{2,k} &= -\frac{1}{2}(\partial_t R_k)_{qp} G_{pq'} \left(\Delta\Gamma^{(2)}_{1,k}\right)_{q''q'} G_{q''q} \\ &= \frac{1}{2}\left(\Delta\Gamma^{(2)}_{1,k}\right)_{q''q'} (\partial_t G)_{q'q''}.\end{aligned} \tag{8.19}$$

We now insert the expression for the one-loop propagator correction and we use the result for $\partial_t G_{pq}$ to write this as a total derivative:

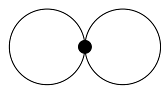
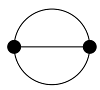
$$\begin{aligned}&\frac{1}{2}\frac{1}{2}\left[G_{pp'}S^{(4)}_{\mathrm{B},\{pp',qq'\}} - G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q''p'q'\}}\right]^k_\Lambda (\partial_{t'}G)_{qq'} = \\ &= \frac{1}{2}\frac{1}{2}\partial_{t'}\left[\frac{1}{2}G_{pp'}S^{(4)}_{\mathrm{b},\{pp',qq'\}}G_{qq'} - \frac{1}{3}G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q''p'q'\}}G_{q'q}\right]\end{aligned} \tag{8.20}$$

and now we perform the scale integration with regard to the renormalization scale k

$$\begin{aligned}\Delta\Gamma_{2,k} &= \int_\Lambda^k dk'\ \frac{1}{k'}\frac{1}{2}\frac{1}{2}\left[G_{pp'}S^{(4)}_{\mathrm{B},\{pp',qq'\}} + \right. \\ &\qquad \left. -G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q''p'q'\}}\right]^k_\Lambda (\partial_{t'}G)_{qq'} \\ &= \int_\Lambda^k dk'\ \frac{1}{k'}\frac{1}{2}\frac{1}{2}\partial_{t'}\left[\frac{1}{2}G_{pp'}S^{(4)}_{\mathrm{B},\{pp',qq'\}}G_{qq'} + \right. \\ &\qquad \left. -\frac{1}{3}G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q''p'q'\}}G_{q'q}\right]\end{aligned} \tag{8.21}$$

so we finally obtain, as a result from integration, the correct perturbative two-loop result:

$$\Delta\Gamma_{2,k} = \left[\frac{1}{8}G_{pp'}S^{(4)}_{\mathrm{b},\{pp',qq'\}}G_{qq'} - \frac{1}{12}G_{pp'}S^{(3)}_{\mathrm{B},\{pp''q\}}G_{p''q''}S^{(3)}_{\mathrm{B},\{q''p'q'\}}G_{q'q}\right]^k_\Lambda \tag{8.22}$$

$= \Big[\frac{1}{8}$  $-\ \frac{1}{12}$  $\Big].$

A2

Full potential: $\mathbb{Z}_2$ β-functions

We give here the β-functions obtained for the following $\mathbb{Z}_2$ $\lambda\phi^4$-*type* truncated potential:

$$\begin{aligned} V_k(\phi,\bar\phi) &= \frac{1}{2}\mu_1|\phi|^2 + \frac{1}{4}\mu_2|\phi|^4 + \\ &+ \frac{1}{6}\left\{\bar\rho_{1,1}|\phi|^2\bar\phi^2 + \rho_{1,1}|\phi|^2\phi^2\right\} + \\ &+ \frac{1}{2}\left\{\lambda_1\phi^2 + \bar\lambda_1\bar\phi^2\right\} + \frac{1}{24}\left\{\lambda_2\phi^4 + \bar\lambda_2\bar\phi^4\right\}. \end{aligned} \tag{9.1}$$

In the full potential (6.37) with $N = 2$ we could consider next orders in the expansion of the potential, but the β-functions structure complicates more and more.

$$\begin{aligned} \beta_{\lambda_1} &= \frac{1}{3}\Bigg\{-6\lambda_1 + \frac{\rho_{1,1}}{\pi^2(-|\lambda_1|^2 + \frac{1}{4}(2+\mu_1)^2)} + \\ &\quad - \frac{8(2+\mu_1)(-\bar\lambda_1\lambda_2 - \lambda_1\mu_2 + (2+\mu_1)\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2}\Bigg\} \\ \beta_{\bar\lambda_1} &= \bar\beta_{\lambda_1} \\ \beta_{\lambda_2} &= -\lambda_2 + \frac{16[2\rho_{1,1}(\bar\lambda_1\lambda_2 + \lambda_1\mu_2 - (2+\mu_1)\rho_{1,1}) + (2+\mu_1)(\lambda_2\mu_2 - \rho_{1,1}^2)]}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2} + \\ &\quad + \frac{64(2+\mu_1)(\bar\lambda_1\lambda_2 + \lambda_1\mu_2 - (2+\mu_1)\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3} \\ \beta_{\bar\lambda_2} &= \bar\beta_{\lambda_2} \\ \beta_{\mu_1} &= \frac{1}{3}\Bigg\{-3\mu_1 + \frac{\mu_2}{\pi^2(-|\lambda_1|^2 + \frac{1}{4}(2+\mu_1)^2)} + \\ &\quad - \frac{8(2+\mu_1)(-\bar\rho_{1,1}\lambda_1 - \bar\lambda_1\rho_{1,1} + (2+\mu_1)\mu_2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2}\Bigg\} \end{aligned}$$

$$\beta_{\mu_2} = \frac{1}{6}\left\{-3\mu_2 - \frac{8(2+\mu_1)(|\lambda_2|^2 + 2|\omega_{1,1}|^2 - 3\mu_2^2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2} + \right.$$
$$+ \frac{16\rho_{1,1}(|\lambda_2|^2\lambda_1 - |\rho_{1,1}|^2(2+\mu_1) + |\lambda_1|^2\mu_2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2} +$$
$$+ \frac{64\mu_2(|\rho_{1,1}|^2\lambda_1 - (2+\mu_1)\mu_2 + |\lambda_1|^2\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2} +$$
$$- \frac{128(2+\mu_1)(|\rho_{1,1}|^2\lambda_1 - (2+\mu_1)\mu_2 + |\lambda_1|\rho_{1,1})^2}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3} +$$
$$+ \frac{16|\rho_{1,1}|^2(|\lambda_1|^2\lambda_2 + \lambda_1\mu_2 - (2+\mu_1)\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^2} - \frac{64(2+\mu_1)|\lambda_2|^2\lambda_1}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3} +$$
$$\left. \frac{-|\rho_{1,1}|^2(2+\mu_1) + |\lambda_1|^2\mu_2)(|\lambda_1|^2\lambda_2 + \lambda_1\mu_2 - (2+\mu_1)\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3}\right\}$$

$$\beta_{\omega_{1,1}} = -\omega_{1,1} + \frac{8(2|\rho_{1,1}|^2\lambda_1(4|\lambda_1|^2 + 3(2+\mu_1)^2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3}$$
$$- \frac{|\omega_{1,1}|(2+\mu_1)(4\lambda_2(|\lambda_1|^2 + (2+\mu_1)^2) + 8\lambda_1^2\mu_2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3} +$$
$$+ \frac{\mu_2^2(2\lambda_1)(4|\lambda|^2 + 3(2+\mu_1)^2) - (2+\mu_1)(28|\lambda_1|^2 + 3(2+\mu_1)^2\rho_{1,1})}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3} +$$
$$+ \frac{6|\lambda_1|^2(2+\mu_1)^2(\lambda_2\mu_2 + \rho_{1,1}^2) + 8|\lambda_1|^2(-\lambda_2(2+\mu_1)\rho_{1,1} + \lambda_1(\lambda_2\mu_2 + \rho_{1,1}^2)}{\pi^2(-4|\lambda_1|^2 + (2+\mu_1)^2)^3}$$
$$\beta_{\bar{\omega}_{1,1}} = \bar{\beta}_{\omega_{1,1}}$$

In the mass-like β-functions we can recognise a structure similar to that derived for the simpler exact $\mathbb{Z}_2$ universality class in (5.20).

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