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Non-equilibrium relaxation for Driven Lattice Gases

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Introduction

According to the atomic hypothesis, matter consists of atoms or molecules that move according to the laws of classical or quantum mechanics. This dynamical description of the macroscopic world turns out to be impossible, due to the very large number of degrees of freedom of any macroscopic system. Fortunately, at the beginning of the 20th century, a fundamental set of ideas emerged and became what now is called Statistical Mechanics. Statistical Mechanics provides a method for relating the microscopic properties of individual atoms and molecules to the properties of macroscopic systems. In the framework given by Gibbs, Statistical Mechanics states that the probability distribution $P(S)$ of a macroscopic state S *at the equilibrium*, is given, in terms of the Hamiltonian of the system $\mathcal{H}(S)$, by $P(S) \propto e^{-\beta\mathcal{H}(S)}$. For the non-equilibrium systems a general approach, similar to equilibrium Statistical Mechanics is still missing. The concept of equilibrium is intended to be, following Richard P. Feynman, *when all the fast things have happened but the slow things have not*. The main reason for studying non-equilibrium systems is that most systems found in nature aren't in thermodynamic equilibrium. These non-equilibrium systems can be divided into two big classes, those with a slow dynamics which, prevents them to relax in an observable time, and those with an external perturbation, which does not allow the system to equilibrate with its environment. Often, the systems belonging to the last class settle down in a *Non Equilibrium Steady State*, determined by the dynamics of the system and whose probability distribution is a priori different from the Gibbs one. This prominent role played by the dynamics of the system is a consequence of the fact that in non-equilibrium systems time reversal is not a symmetry of the dynamics and this leads to a violation of

the detailed balance condition.

The main difficulty in the description of non-equilibrium phenomena, compared to equilibrium ones, is that while the latter are described by averages over well defined Gibbs ensembles, non-equilibrium systems don't have universal and simple behaviour. Consequently, a reasonable approach consists in investigating systems which are, just as the Ising model for equilibrium critical phenomena, as simple as possible. The Driven Lattice Gas (DLG) model, whose introduction was partially motivated by the physics of solid electrolytes, is a straightforward generalization of the Ising model that displays a non-equilibrium continuous phase transition, and that's why it became one of the main benchmark for non-equilibrium theories and field theoretical approaches. It consists of an ordinary Ising lattice gas, where the transition rates for the jumps to nearest neighbours empty sites are specified not only by the internal energy, but also by an external driving field that biases the jumping rates in a particular direction. A very interesting aspect of this model is that it displays a second-order phase transition at a finite temperature T_c . For $T > T_c$ the system lies in a disordered phase, with short ranged correlations, while for $T < T_c$ the system separates into two co-existing phases of particle-rich and hole-rich regions. Many studies have focused on the DLG, using both continuous field theories and numerical simulations, but after many efforts, the universality class of the DLG, and therefore also its critical exponents, remain a debated issue. Janssen, Schmittmann, Leung and Cardy (JSLC) developed a field theory that describes the critical behaviour of the DLG, but some discrepancies between the theory and the numerical simulations are still present. These ambiguities in determining the critical exponents are mainly caused by some subtleties in implementing a proper anisotropic finite-size scaling, and by the slowdown of dynamics close to the critical point. This led Garrido, de los Santos and Muñoz to propose an alternative field theory for the DLG, in particular when the driving force is infinite.

An interesting aspect of statistical systems with a continuous phase transition is that they exhibit a scale invariance even during their evolution toward equilibrium or non-equilibrium steady states. Very intriguing is the fact that,

surprisingly, this behaviour can be appreciated also in the short-time regime, when the correlation length of the system ξ is still very small compared to its equilibrium counterpart. The main advantage of this technique is that one can measure the critical exponents using quite little computer resources.

In this work we tried to clarify some aspect about the short time behaviour of two variants of driven lattice gases, the infinitely driven lattice gas (IDLG) and the randomly driven lattice gas (RDLG)

This thesis is organised as follow:

- The first Chapter is a general introduction to the lattice gas models.
- The second Chapter is an introduction to dynamical field theory, with a brief description of JSLC theory and RDLG theory, the theories that are commonly believed to describe the critical behaviour of driven lattice gas models.
- The third Chapter is about the finite-size scaling, both for isotropic and anisotropic systems.
- The fourth Chapter focuses on the short-time scaling.
- The fifth Chapter is about the short-time scaling for the DLG models. It contains the simulations we performed to verify the scaling hypothesis for these models.

Chapter 1

The Model

One of the simplest models which describe a system reaching a Non Equilibrium Steady State, is the model introduced at the beginning of the eighties by Katz, Lebowitz and Spohn [9]. This model, sometimes referred as KLS or DLG, is a simple modification of an Ising model on a periodic lattice with Kawasaki dynamics [6]. The particles jump rates to nearest neighbour empty sites are not only specified by the variation of the energy of the system, but there is an uniform driving field which biases the jump in a specific direction.

1.1 Ising Lattice Gas

An Ising lattice gas is a model of classical particles moving on a d -dimensional hypercubic lattice $\Lambda \subset \mathbb{Z}^d$, in which at most one particle may occupy each site. A configuration of the system C is specified by the occupation number $n_{\vec{x}}$ of each site of the lattice, then $n_{\vec{x}} \in \{0, 1\}$, $\vec{x} \in \Lambda$ and $C \in \mathcal{C}(\Lambda) = \{0, 1\}^\Lambda$. Clearly, instead of the occupation number, one can use the spin variables: $\phi_{\vec{x}} \in \{-1, 1\}$, with $\phi_{\vec{x}} = 2n_{\vec{x}} - 1$. We are interested in the dynamics of such a system that keep fixed the total number of particles, or, in the spin language, fixed magnetization. In particular, we are interested in half filled lattices, i.e. with density

$$\rho = \frac{1}{\Lambda} \sum_{\vec{x} \in \Lambda} n_{\vec{x}} = \frac{1}{2}. \quad (1.1)$$

We assume that the internal energy of such a system is the classic Ising Hamiltonian:

$$\mathcal{H} = -4J \sum_{\langle \vec{x}, \vec{y} \rangle} n_{\vec{x}} n_{\vec{y}}, \quad (1.2)$$

where the sum runs over all the lattice nearest-neighbours sites and the coupling constant J may be either positive (ferromagnetic models) or negative (antiferromagnetic ones). In the following, we will be interested in attractive models and we will set, without loss of generality, $J = 1$. The model presented here has the same equilibrium distribution and critical properties of the Ising model and so, for $d \geq 2$, we expect a second order phase transition occurring for a critical temperature T_c . In particular, in the case $d = 2$, $T_c = 2/\log(1 + \sqrt{2})$, known as the Onsager temperature.

Given a configuration C , and two nearest neighbours sites \vec{x} and \vec{y} , we will denote by $C_{\vec{x}\vec{y}}$ the configuration obtained by C exchanging the occupation numbers of the sites \vec{x} and \vec{y} . The dynamics of the system (Kawasaki dynamics [6]) is constituted by a succession of moves of this type, so that the total number of particles, or the particle density (1.1), remains fixed.

The following Master Equation describes this Markov Process in the phase space $\mathcal{C}(\Lambda)$

$$\partial_t P(C, t) = \sum_{C' \in \mathcal{C}(\Lambda)} \{W(C' \rightarrow C)P(C', t) - W(C \rightarrow C')P(C, t)\}, \quad (1.3)$$

where $P(C, t)$ is the probability that the configuration of the system is C at time t , and $W(C \rightarrow C')$ is the transition rate from the configuration C to another configuration C' . The transition $W(C \rightarrow C')$ is non zero only if C and C' are connected by a spin exchange $C_{\vec{x}\vec{y}}$. If the dynamics is irreducible, i.e. any configuration C' can be reached starting from any configuration C , then there is a unique stationary solution $P^s(C, t)$ such that $\partial_t P^s(C, t) = 0$.

If the system is symmetric under time reversal, i.e. the time reversed trajectory in the configuration space has the same probability to be realized, then the following equation holds

$$W(C \rightarrow C')P^s(C) = W(C' \rightarrow C)P^s(C'), \quad (1.4)$$

for every configurations C and C' in the configuration space. This equation is the so called detailed balance condition for the stationary distribution P^s . If this equation is satisfied, then each term in the right hand side of the Master Equation (1.3) vanishes and P^s is in fact stationary, but not all the stationary measures satisfy detailed balance. Hence, it's easy to understand that the stationary solution $P^s(C, t)$ is the usual equilibrium distribution function $e^{-\beta\mathcal{H}(C)}$, if the rates are chosen according to

$$\frac{W(C \rightarrow C')}{W(C' \rightarrow C)} = e^{-\beta(\mathcal{H}(C') - \mathcal{H}(C))}. \quad (1.5)$$

One may thus choose rates of the form

$$W(C \rightarrow C') = w(\beta\Delta\mathcal{H}), \quad (1.6)$$

with an appropriate w such that

$$w(-x) = w(x)e^x. \quad (1.7)$$

Different dynamics correspond to different choices of the function $w(x)$, but the equilibrium distribution is independent from this choice. As we will see later, this is not true for non-equilibrium system, where the dynamics play a more fundamental role because it's not known any a priori probability distribution for the stationary state.

1.2 Driven Lattice Gas

A driven lattice gas is a generalization of the model previously introduced, in which we introduce an uniform electric field pointing in a particular direction of the lattice. It will biases the transition rates favouring the jumps along the field and unfavouring the jumps opposite to the field, while the jumps in the other directions remain unaffected. This is done adding a term to the rate W so that equation (1.6) becomes

$$W(C \rightarrow C') = w(\beta(\Delta\mathcal{H} + lE)), \quad (1.8)$$

where l is $-1(0, 1)$ for jumps against (transversal, along) the field. If we consider *periodic boundary conditions* this field cannot be treated adding a global potential term in the Hamiltonian and the system reaches a stationary state which is not in thermal equilibrium. We can explicitly see that this dynamics violates the detailed balance condition (1.4). A direct method to verify if a given dynamical system displays detailed balance is to consider if it's fulfilled the following condition:

$$\begin{aligned} W(C_1 \rightarrow C_2)W(C_2 \rightarrow C_3)\dots W(C_n \rightarrow C_1) = \\ W(C_2 \rightarrow C_1)W(C_3 \rightarrow C_2)\dots W(C_1 \rightarrow C_n), \end{aligned} \quad (1.9)$$

for all possible cycles $\{C_1, \dots, C_n\}$. This condition is proven to be equivalent to the detailed balance condition (1.4). Suppose now we have a configuration of the system where a single particle is isolated from all the other particles. This particle could make a cycle in the direction of the field with a non-zero probability p , while the probability for the particle to accomplish the backward cycle is in general less than p (zero in the limit of infinite driven field). This example explicitly proves that DLG dynamics violates detailed balance.

For half filled lattices, the system shows a continuous phase transition located at a temperature $T_c(E)$. Surprisingly $T_c(E)$ increases with E and saturates to a finite value $T_c(\infty) \simeq 1,4T_c(0)$, where $T_c(0)$ is the Onsager temperature.

In the following, we will be mainly interested in two special cases of the DLG model previously exposed, the IDLG (Infinitely Driven Lattice Gas) and the RDLG (Randomly Driven Lattice Gas). The IDLG is obtained making the formal limit $E \rightarrow \infty$, so that we assign probability 1(0) for jumps along (opposite to) the field. Instead, in the RDLG the field is always infinite but it is no more deterministically known but, chosen a direction for the field, it points with probability 1/2 to the right and with probability 1/2 to the left.

1.3 Symmetries

In determining the critical properties of a system, a decisive role is played by the symmetry properties. For the DLG (and also for the infinite field version), with the field pointing in the x direction, the transition rates (1.8) are invariant under **any pair** of the following transformation:

$$n_i \rightarrow 1 - n_i, \quad E \rightarrow -E, \quad \text{and} \quad x \rightarrow -x. \quad (1.10)$$

We will refer at these properties, respectively, as charge coniugation (C), field reflection (R) and parity (P).

By contrast, for a RDLG, the randomness of E means that both C and P are separately preserved, as in the Ising model. An obvious consequence of this fact is that for a RDLG any equal-time correlation with an odd number of spin must vanish [12]. We should remark that, below criticality, the situation is more complex because, for both the IDLG and the RDLG, the formation of the strip spontaneously breaks the symmetry of both C and traslation. Hence, the difference beetween the two models is more clear for $T > T_c$ where the simmetries remains unbroken.

1.4 Observables

Considering a lattice gas defined on a two dimensional lattice $\Lambda = L_x \times L_y$, we assign to each site of the lattice an occupation number variable $n_{\vec{x}}$ or, equivalently, a spin variable, $\phi_{\vec{x}} = 2n_{\vec{x}} - 1$. Let's consider now the Fourier Transform:

$$\phi(\vec{k}) = \sum_{\vec{x} \in \Lambda} e^{i\vec{k} \cdot \vec{x}} \phi_{\vec{x}}. \quad (1.11)$$

On a finite lattice, \vec{k} can assume only the values:

$$\vec{k}_{n,m} = \left(\frac{2\pi}{L_x} n, \frac{2\pi}{L_y} m \right), \quad (1.12)$$

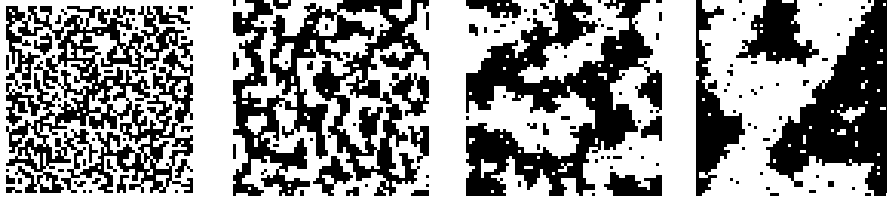


Figure 1.1: Relaxation towards the equilibrium ordered phase for a 64×64 Ising lattice gas, from left to right.

with $n \in (0, L_x)$ and $m \in (0, L_y)$. We are interested in half-filled lattice which corresponds to zero magnetization:

$$\sum_{\vec{x} \in \Lambda} \phi_{\vec{x}} = 0, \quad (1.13)$$

i.e. $\phi(\vec{k}_{0,0}) = 0$.

When we consider the driven lattice gas, we add E along the x direction, so we have $L_x = L_{\parallel}$ and $L_y = L_{\perp}$. In all the observed simulation, the ordered phase consists in a single strip parallel to the direction of the field, so the maximum of $\phi(\vec{k})$ is for $\vec{k}_{0,1}$. A order parameter can then be defined as:

$$m(L_x, L_y) = \frac{1}{\Lambda} \langle | \phi(\vec{k}_{0,1}) | \rangle. \quad (1.14)$$

An alternative choice for the order parameter is the one introduced by Albano and Saracco [28], defined as the excess density in the direction parallel to the applied field:

$$OP(L_x, L_y) = \sum_{x \in (0, L_x)} \left\langle \left| \sum_{y \in (0, L_y)} \phi_{\vec{x}} \right| \right\rangle. \quad (1.15)$$

During the relaxation towards the ordered phase, the system shows a metastable phase of multi-stripped ordering, as shown in Figure 1.2. The main difference between the two previously introduced order parameters is that, while m detect mainly the onset of the single-strip ordering, OP is independent to which phase is the system in.

An immediate result, very common in the realm of non-equilibrium phe-



Figure 1.2: Relaxation towards the stationary ordered phase for a 64×64 driven lattice gas, from left to right. The ordered phase consist in a single strip parallel to the driven field E .

nomena, is that the system display **long range correlations**. If we consider the two-point function:

$$G(\vec{x} - \vec{y}, L_x, L_y) = \langle n_{\vec{x}} n_{\vec{y}} \rangle, \quad (1.16)$$

In the disordered phase, when \vec{x} is taken in the subspace transverse to the electric field, at large distances G shows a power law behaviour

$$G(\vec{x}, L_x, L_y) \sim |x|^{-d}. \quad (1.17)$$

where d is the dimension of the system. We mention that the onset of these long range correlations is a typical feature of many non-equilibrium systems, and an explanation for this behaviour may be found in Section 2.2. In momentum space the static structure factor, i.e. the Fourier Transform of the two-point function, has the expression

$$\tilde{G}(\vec{k}, L_x, L_y) = \frac{1}{\Lambda} \langle |\phi(\vec{k})|^2 \rangle. \quad (1.18)$$

Following the same considerations as for m , \tilde{G} assumes its maximum for $\vec{k}_{0,1}$, then it's natural to define the transverse susceptibility as

$$\chi(L_x, L_y) = \tilde{G}(\vec{k}_{0,1}, L_x, L_y). \quad (1.19)$$

Another quantity of great interest is the correlation length. Usually, correla-

tion length is defined, for infinite volume systems, as the parameter that rules the long distance exponential decay of the two-point function, or, similarly, the small momenta behaviour of the static structure factor. In this context, this is troublesome because of the onset of these long range correlations, that result in a discontinuity of the static structure factor for $\vec{k} = 0$. Moreover, it is not positive definite because of negative correlations in the transverse direction. Furthermore, in finite systems, there is no *a priori* unique way to define a finite volume correlation length with the correct infinite volume limit. In [22], it's shown that

$$\xi(L_x, L_y) = \frac{1}{2} \sqrt{\frac{1}{\sin^2(3\pi/L_y) - \sin^2(\pi/L_y)} \left(\frac{\tilde{G}(\vec{k}_{0,1}, L_x, L_y)}{\tilde{G}(\vec{k}_{0,3}, L_x, L_y)} - 1 \right)} \quad (1.20)$$

is a good definition for a transverse correlation length with a correct finite-size scaling, i.e. with regular finite-size scaling functions and correct anomalous behaviour above the upper critical dimension. The basic observations is that in the DLG the infinite volume *wall-wall* two-point function decays exponentially.

We also define an important parameter, the Binder parameter:

$$g(L_x, L_y) = 2 - \frac{\langle |\phi(\vec{k}_{0,1})|^4 \rangle}{\langle |\phi(\vec{k}_{0,1})|^2 \rangle^2} \quad (1.21)$$

In the high temperature phase, the stationary state is expected to have a Gaussian distribution, so that $\langle |\phi(\vec{k}_{0,1})|^4 \rangle = 2\langle |\phi(\vec{k}_{0,1})|^2 \rangle^2$ and $g(L_x, L_y) \sim 0$. By contrast, at low temperature when the ordered phase is formed, we expect $\langle |\phi(\vec{k}_{0,1})|^n \rangle \sim \langle |\phi(\vec{k}_{0,1})| \rangle^n$ and $g(L_x, L_y) \sim 1$.

Chapter 2

Critical dynamics

For every lattice model displaying a second order phase transition, in the neighborhood of the critical point, we can limit ourselves to consider slowly varying observables. At criticality, the lattice spacing a becomes negligible compared to the typical length scale of the physics involved (the correlation length ξ), hence the limit $T \rightarrow T_c$ is intimately connected to the limit $a \rightarrow 0$ [2]. In this way, it's possible to give a description of the system in term of different variables, hereafter called *mesoscopic* variables, defined on a continuum spacetime. In principle, the dynamics of these variables is obtained from the dynamics of the lattice model (1.3) through a coarse graining process. Because of the difficulty of performing a rigorous coarse graining procedure [4], usually one postulates a dynamical equation, in the form of a **Langevin equation** for the coarse grained degrees of freedom, that keeps into account all the symmetries of the initial lattice model. Within this broad constraints, one could think that an infinite set of different possible coarse grained Langevin equation could give us an infinite set of different predictions. However, for what concern the behaviour near the critical point, *universality* tells us that more or less any equation of motion should lead to the same universal properties. We know that this statement is rigorously proven only for equilibrium critical phenomena, by means of the *Renormalization Group* (RG) approach, while to what extent universality applies in the context of dynamical critical phenomena is still unclear.

2.1 Dynamical field theory

In principle, the laws of classical or quantum mechanics, should describe the dynamics for all the possible observables of a physical system. But these laws are time reversal invariant, and it's not easy to understand how the dissipative processes, characteristic of macroscopic systems at finite temperatures, appear in this context. Our ignorance about the detailed origin of this mechanism is usually hidden in the coarse graining process, that transforms a deterministic equation in a stochastic equation. The Langevin equation is one of the most simple type of stochastic equation. After a coarse graining process, a physical system is described in terms of a reduced set of discrete variables \vec{q} , and the Langevin equation reads

$$\frac{d}{dt}q_i(t) = -\frac{1}{2}F_i(\vec{q}(t)) + \zeta_i(t), \quad (2.1)$$

where F is an external deterministic force, while ζ is a noise which models the microscopic degrees of freedom eliminated after that a proper coarse graining procedure has been performed. This noise can be defined by a probability distribution $[dP(\zeta)]$ and we shall specialize to what is usually called Gaussian white noise:

$$[dP(\zeta)] = [d\zeta] \exp\left(-\frac{1}{2\sigma} \int \zeta_i^2(t) dt\right). \quad (2.2)$$

Alternatively, the Gaussian white noise can be characterized by the first and the second moments

$$\langle \zeta_i(t) \rangle = 0, \quad \langle \zeta_i(t) \zeta_j(t') \rangle = \sigma \delta_{ij} \delta(t - t'). \quad (2.3)$$

Given the initial condition $\vec{q}(t_0) = \vec{q}_0$, the Langevin equation generates a time dependent probability distribution $P(\vec{q}, t)$ for $\vec{q}(t)$, whose dynamics can be described in terms of the Fokker-Plank equation associated to the Langevin equation (2.1)

$$\frac{d}{dt}P(\vec{q}, t) = \frac{1}{2} \frac{\partial}{\partial q_i} \left(\sigma \frac{\partial P}{\partial q_i} + F_i(\vec{q}) P \right). \quad (2.4)$$

The deterministic force F is usually assumed to originate from an Hamiltonian, so that

$$F_i(\vec{q}) = \frac{\delta H[q]}{\delta q_i}. \quad (2.5)$$

In this section, we would like to generalise these facts to the case when, instead of a discrete set of variables $\vec{q}(t)$, we have a scalar field $\phi(x, t)$. Generalization where the order parameter $\phi(x, t)$ is a vector field is straightforward, but the choice of restricting to simpler scalar fields, for spin models, is proven to be sufficient. The simplest model where the dynamics is expressed in terms of a field stochastic Langevin equation, is the so called Model A:

$$\partial_t \phi(x, t) = -\lambda \frac{\delta H[\phi]}{\delta \phi} + \zeta(x, t). \quad (2.6)$$

where the Hamiltonian $H[\phi]$ is the usual Landau-Ginzburg

$$H[\phi] = \int d^d x \left\{ \frac{1}{2} (\nabla \phi)^2 + \frac{\tau}{2} \phi^2 + \frac{g}{4!} \phi^4 \right\} \quad (2.7)$$

and $\zeta(x, t)$ is a white noise

$$\langle \zeta_i(x, t) \rangle = 0, \quad \langle \zeta_i(x, t) \zeta_j(x', t') \rangle = N_{ij} \delta(t - t') \delta(x - x'). \quad (2.8)$$

When the above dynamics is intended to describe critical fluctuations around thermal equilibrium, N_{ij} must be chosen such that the fluctuation-dissipation theorem (FDT) is satisfied, i.e.

$$N_{ij} = 2\lambda k_B T \delta_{ij} \quad (2.9)$$

Model A describes for example the critical dynamics of a kinetic spin model on the lattice.

The expectation value of a generic observable $O[\phi]$ over all possible real-

ization of the noise ζ can be written as

$$\langle O \rangle = \int [d\zeta] O[\phi_\zeta] P[\zeta] = \int [d\phi] O[\phi] \left\{ \int [d\zeta] \delta(\phi - \phi_\zeta) P[\zeta] \right\}, \quad (2.10)$$

where $P[\zeta]$ is the Gaussian probability distribution of the noise and ϕ_ζ is the solution of equation (2.6) for a given realization of the noise ζ . We use

$$\delta(\phi - \phi_\zeta) = \delta\left(\partial_t \phi + \lambda \frac{\delta H[\phi]}{\delta \phi} - \zeta\right) \det \left[\partial_t + \lambda \frac{\delta^2 H}{\delta \phi^2} \right] \quad (2.11)$$

and we express the delta function as an exponential, using the auxiliary Martin-Siggia-Rose field $\tilde{\phi}$ [5] :

$$\delta(\psi) = \int [d\tilde{\phi}] \exp \left\{ \int dt d^d x \tilde{\phi}(x, t) \psi(x, t) \right\}. \quad (2.12)$$

Hence, the average over the noise can be computed as

$$\langle O \rangle = \int [d\phi d\tilde{\phi}] O[\phi] e^{-J[\phi, \tilde{\phi}]} \quad (2.13)$$

in terms of the dynamical functional

$$J[\phi, \tilde{\phi}] = \int_0^\infty dt \int d^d x \left\{ \tilde{\phi} \left[\dot{\phi} + \lambda(\tau - \nabla) \phi + \frac{\lambda g}{6} \phi^3 \right] - \lambda \tilde{\phi}^2 \right\}. \quad (2.14)$$

Up to now, we didn't mention the initial conditions $\phi(x, t_0)$, because usually one is not interested in the influence of them, and so specify them in the infinite past, assume that the dynamics is ergodic, whence the equilibrium values of any observable become independent of $\phi(x, t_0)$. Later, when we will be interested in the short time dynamics of these models, we will see how the initial condition will enter in this context.

In some cases Model A is not suited to describe the dynamics of a physical system. When the order parameter is conserved, for instance when the order parameter is related to the density of particles as in a gas model, or for the critical behaviour of spin models with spin-exchange sampling, one expect the Langevin equation to have the form of a continuity equation, because of

the conservation of the order parameter

$$\frac{d}{dt} \int dx \phi(x, t) = 0. \quad (2.15)$$

The simplest model where this conservation law is implemented is known as Model B, whose Langevin equation is

$$\partial_t \phi(x, t) + \nabla \cdot \vec{J}(x, t) = 0, \quad (2.16)$$

where

$$\vec{J}(x, t) = -\lambda \nabla \frac{\delta H[\phi]}{\delta \phi(x, t)} + \vec{J}_\zeta(x, t). \quad (2.17)$$

A classification of these dynamical universality classes, named with capital letters from A to J, has been done in the early seventies and is reviewed in the classical paper by Hohenberg and Halperin [1].

2.2 JSLC theory

A field theory for the driven lattice gas has been proposed independently by Janssens and Schmittmann, and by Leung and Cardy, in 1986 [10] [11], and it gives exact predictions for critical exponents for $2 < d < d_c = 5$. We will sketch here the main result of the theory, for more details we refer to the review book written by Zia and Schmittmann [8]. This theory is a proper modification of Model B. The presence of the electric field \vec{E} will generate an additional contribution \vec{j}_E to the current \vec{J} , whose form is postulated to be the simplest form consistent with the symmetries of the lattice model (1.10). Moreover, \vec{j}_E must vanish if no particle or no holes are present locally, so we take it to be proportional to $\rho(1 - \rho)$. The simplest form consistent with the above constraint is

$$\vec{j}_E = 4\rho(1 - \rho)\vec{\mathcal{E}} = (1 - \phi^2)\vec{\mathcal{E}} \quad (2.18)$$

where $\vec{\mathcal{E}}$ is a coarse grained odd function of the electric field \vec{E} . The presence of this current doesn't mean simply that we have to add a term like

$$\nabla \cdot \vec{j}_E = \mathcal{E} \partial^2 \phi \quad (2.19)$$

in (2.16-2.17), because we expect this new term to generate anisotropies in (2.16-2.17) and to describe a non-equilibrium critical dynamics. This term is then expected to violate the fluctuation dissipation relations (2.9). We then will have couplings associated with longitudinal gradients different from those describing transverse gradients.

The resulting Langevin equation is, denoting *transverse* gradients by $\vec{\nabla}$ and *longitudinal* ones by ∂

$$\begin{aligned} \frac{\partial}{\partial t} \phi(\vec{x}, t) = & \lambda \left((\tau_{\perp} - \nabla^2) \nabla^2 \phi + (\tau_{\parallel} - \alpha_{\parallel} \partial^2) \partial^2 \phi - 2\alpha_{\times} \partial^2 \nabla^2 \phi + \right. \\ & \left. + \frac{u}{3!} (\nabla^2 \phi^3 + \kappa \partial^2 \phi^3) + \mathcal{E} \partial \phi^2 \right) - (\vec{\nabla} \cdot \vec{\xi} + \partial \zeta), \end{aligned} \quad (2.20)$$

where we have divided the noise current, expected to be anisotropic, previously indicated as \vec{J}_{ζ} , in the longitudinal and transverse components, ζ and $\vec{\xi}$. Therefore (2.8) can be rewritten as

$$\langle \vec{\nabla} \cdot \vec{\xi}(\vec{x}, t) \vec{\nabla}' \cdot \vec{\xi}'(\vec{x}', t') \rangle = n_{\perp} (-\nabla^2) \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (2.21)$$

and

$$\langle \partial \zeta(\vec{x}, t) \partial \zeta(\vec{x}', t') \rangle = n_{\parallel} (-\partial^2) \delta(\vec{x} - \vec{x}') \delta(t - t'). \quad (2.22)$$

For an anisotropic system at the equilibrium, it can be shown that the FDT theorem (2.9) becomes $\tau_{\perp}/\tau_{\parallel} = n_{\perp}/n_{\parallel}$ but, with the electric field E , we expect

$$\frac{\tau_{\perp}}{\tau_{\parallel}} \neq \frac{n_{\perp}}{n_{\parallel}}. \quad (2.23)$$

We will see that one of the first consequences of these equations is the onset of long range correlations, specific of non-equilibrium systems. For temperatures far above T_c , a good approximation for the Langevin equation (2.20)

is given by the linearized version

$$\begin{aligned} \frac{\partial}{\partial t} \phi(\vec{x}, t) = & \lambda \left((\tau_{\perp} - \alpha_{\perp} \nabla^2) \nabla^2 \phi + (\tau_{\parallel} - \alpha_{\parallel} \partial^2) \partial^2 \phi - \right. \\ & \left. 2\alpha_{\times} \partial^2 \nabla^2 \phi \right) - (\vec{\nabla} \cdot \vec{\xi} + \partial \zeta). \end{aligned} \quad (2.24)$$

Defining $\phi(\vec{x}, t) = \int e^{i(\vec{k} \cdot \vec{x} + \omega t)} \phi(\vec{k}, \omega)$, the solution to (2.24) is simply

$$\phi(\vec{k}, \omega) = \left[i\omega + \Lambda(\vec{k}) \right]^{-1} (-i) \left[\vec{k}_{\perp} \cdot \vec{\xi} + k_{\parallel} \zeta \right] \quad (2.25)$$

where

$$\Lambda(\vec{k}) = \lambda \left[\tau_{\perp} k_{\perp}^2 + \tau_{\parallel} k_{\parallel}^2 + (\alpha_{\perp} k_{\perp}^4 + 2\alpha_{\times} k_{\parallel}^2 k_{\perp}^2 + \alpha_{\parallel} k_{\parallel}^4) \right]. \quad (2.26)$$

Since the noise has zero mean, we obtain $\langle \phi \rangle$ as expected. Furthermore, using (2.21) and (2.22) we obtain the full dynamic structure factor

$$\tilde{G}(\vec{k}, \omega) = \langle \phi(\vec{k}, \omega) \phi(-\vec{k}, -\omega) \rangle = \frac{\vec{k} \cdot \mathbb{N} \cdot \vec{k}}{\omega^2 + \Lambda^2(\vec{k})} \quad (2.27)$$

where \mathbb{N} is the diagonal noise matrix, i.e.

$$\vec{k} \cdot \mathbb{N} \cdot \vec{k} = n_{\perp} k_{\perp}^2 + n_{\parallel} k_{\parallel}^2. \quad (2.28)$$

Integrating over ω , we obtain the static structure factor:

$$\tilde{G}(\vec{k}) = \frac{\vec{k} \cdot \mathbb{N} \cdot \vec{k}}{2\Lambda(\vec{k})}. \quad (2.29)$$

Writing $2\Lambda(\vec{k}) = \vec{k} \cdot \mathbb{D} \cdot \vec{k} + O(k^4)$, where \mathbb{D} is the diffusion matrix, i.e.

$$\vec{k} \cdot \mathbb{D} \cdot \vec{k} = 2\lambda(\tau_{\perp} k_{\perp}^2 + \tau_{\parallel} k_{\parallel}^2). \quad (2.30)$$

Therefore, for small \vec{k} , the static structure factor is

$$\tilde{G}(\vec{k}) \rightarrow \frac{\vec{k} \cdot \mathbb{N} \cdot \vec{k}}{\vec{k} \cdot \mathbb{D} \cdot \vec{k}}. \quad (2.31)$$

The FDT theorem $\mathbb{N} \propto \mathbb{D}$ is not expected to hold for non-equilibrium system, hence \tilde{G} has a discontinuity singularity for $\vec{k} = 0$. This singularity translates into a power law decay for the equal-time two-point function in the real space; we have

$$G(\vec{x}) = \langle \phi(\vec{x}, 0) \phi(\vec{0}, 0) \rangle = \int e^{i\vec{k} \cdot \vec{x}} \tilde{G}(\vec{k}). \quad (2.32)$$

The large distance behaviour of G is controlled by the small momenta behaviour of \tilde{G} , so we can use the expression (2.31). Since \mathbb{D} is a positive and diagonal matrix, we can perform a change of variables defining $\vec{k}' = \mathbb{D}^{1/2} \cdot \vec{k}$ and $\vec{x}' = \mathbb{D}^{-1/2} \cdot \vec{x}$ so that the previous equation reads

$$G(\vec{x}') \sim \vec{\nabla}' \cdot \mathbb{M} \cdot \vec{\nabla}' \int \frac{e^{i\vec{k}' \cdot \vec{x}'}}{k'^2} = \vec{\nabla}' \cdot \mathbb{M} \cdot \vec{\nabla}' (x')^{2-d}, \quad (2.33)$$

where the matrix \mathbb{M} is $\mathbb{M} = \mathbb{D}^{-1/2} \mathbb{N} \mathbb{D}^{1/2}$. For equilibrium systems \mathbb{M} is expected to be proportional to the identity matrix. This proportionality is expected to fail for our non-equilibrium system, hence we can extract a non-trivial traceless part of \mathbb{M} , which we call $\bar{\mathbb{M}}$. Thus we find for large x'

$$G(x') \sim \frac{\vec{r}' \cdot \bar{\mathbb{M}} \cdot \vec{r}'}{r'^{d+2}}, \quad (2.34)$$

which is the announced power law decay.

Let's return now to the analysis of the complete Langevin equation (2.20). Using the formalism outlined in the previous section for Model A, we can write down a dynamical generating functional for this theory, that reads

$$J[\phi, \tilde{\phi}] = \int d^d x dt \lambda \left\{ \tilde{\phi} [\lambda^{-1} \partial_t + \nabla^2 (\nabla^2 - \tau) - \tau_{\parallel} \partial^2] \phi + \mathcal{E}(\partial \tilde{\phi}) \phi^2 + \tilde{\phi} \nabla^2 \tilde{\phi} \right\}, \quad (2.35)$$

neglecting irrelevant terms. In particular, the operator associated to the coupling $\partial^2 \phi^3$ in the Langevin equation (2.20), turns out to be dangerously irrelevant.

By means of standar RG techniques, renormalized expressions for response and correlation functions, and critical exponents were exactly computed in generic dimension d , for $2 < d < 5$. An important feature of

this theory is that it reflects the *strong anisotropy* characteristic of the DLG model, i.e. one finds different critical exponents for the longitudinal and for the trasversal directions. For instance, from this field-theoretical model we can extract a scaling relation for the dynamical structure factor

$$\tilde{G}(k_{\parallel}, \vec{k}_{\perp}, t, \tau) = \mu^{-2+\eta} \tilde{G}(k_{\parallel}/\mu^{1+\Delta}, \vec{k}_{\perp}/\mu, t\mu^z, \tau/\mu^{1/\nu}), \quad (2.36)$$

where, apart from the usual critical exponents and the dynamical exponent z , we have to introduce one more exponent, called anisotropy exponent, denoted by Δ . Assuming $\mu = \tau^{\nu}$, the longitudinal and transverse momenta scale with different exponents

$$\vec{k}_{\perp} \sim \tau^{\nu}, \quad k_{\parallel} \sim \tau^{\nu(1+\Delta)}. \quad (2.37)$$

Thus, in the scaling form (2.36), two different correlation lengths emerge from the small momenta behaviour, and so we have two different critical exponents ν

$$\nu_{\perp} = \nu, \quad \nu_{\parallel} = \nu(1 + \Delta) \quad (2.38)$$

Similarly, by setting $\mu = t^{-1/z}$ one gets

$$\vec{k}_{\perp} \sim \tau^{-1/z}, \quad k_{\parallel} \sim \tau^{-(1+\Delta)/z}, \quad (2.39)$$

thus two different dynamical critical exponents emerge for the transverse and longitudinal direction

$$z_{\perp} = z, \quad z_{\parallel} = z/(1 + \Delta) \quad (2.40)$$

The η exponent is derived from the scaling form of the static structure factor ($t = 0$) at criticality ($\tau = 0$)

$$\tilde{G}(k_{\parallel}, \vec{k}_{\perp}) = k_{\perp}^{-2+\eta_{\perp}} \Sigma_{\perp}(k_{\parallel}/k_{\perp}^{1+\Delta}) = k_{\parallel}^{-2+\eta_{\parallel}} \Sigma_{\parallel}(k_{\perp}/k_{\parallel}^{1/(1+\Delta)}) \quad (2.41)$$

one gets

$$\eta_{\perp} = \eta, \quad \eta_{\parallel} = \frac{\eta + 2\Delta}{1 + \Delta} \quad (2.42)$$

Finally, the γ exponent controlling the critical behaviour of susceptibility. Defining transverse and longitudinal susceptibility as

$$\chi_{\perp}(\tau) = \tilde{G}(k_{\parallel} = 0, \vec{k}_{\perp} \rightarrow 0, t = 0, \tau), \quad \chi_{\parallel}(\tau) = \tilde{G}(k_{\parallel} \rightarrow 0, \vec{k}_{\perp} = 0, t = 0, \tau), \quad (2.43)$$

we can also define the γ exponents as

$$\gamma_{\perp} = \gamma_{\parallel} = \nu(2 - \eta) = \gamma. \quad (2.44)$$

We note that η and γ may also be defined from the space dependent definition of the two-point function. At variance with the isotropic case, the η -like and γ -like exponents so obtained differ from the ones previously defined, because anisotropy enters into the intergration measure when performing the Fourier Transform.

2.3 Is the current a relevant parameter?

Despite the fact that JSLC theory provides exact predictions for the critical exponents, and several computer simulations provided good support to this field-theoretical prediction, still some discrepancies remained when the driving field is infinite [13]. This leads Garrido, de los Santos and Muñoz, to propose an alternative field theory [14],[15],[16]. Following their analysis, they suggested that the infinitely driven lattice gas (IDLG) should not behave as predicted by JSLC but should rather belong to the universality class of the randomly driven lattice gas (RDLG). Because this theory is commonly believed to describe the critical behaviour of the RDLG lattice model, in literature it is referred as RDLG theory. We point out that this choice might be misleading, because the same name is used for both the lattice model and the continuum field theory. We point out that the RDLG theory is expected to be appropriate to describe also the critical properties of a Ising lattice gas under two temperatures, in which particle hops in the parallel direction are coupled to a different temperature bath than in the transverse direction.

In their paper, Garrido, de los Santos and Muñoz suggested that the dis-

	JSLC – exact		RDLG – $O(\varepsilon^2)$	
	$\varepsilon = 5 - d$	$d = 2$	$\varepsilon = 3 - d$	$d = 2$
Δ	$1 + \varepsilon/3$	2	$1 - 2\varepsilon^2/243$	0.992
z	4	4	$4 - 4\varepsilon^2/243$	3.984
ν	1/2	1/2	$\frac{1}{2} + \frac{\varepsilon}{12} + \frac{\varepsilon^2}{18} \left[\frac{67}{108} + \ln\left(\frac{2}{\sqrt{3}}\right) \right]$	0.626
η	0	0	$4\varepsilon^2/243$	0.016
β	1/2	1/2	$\frac{\nu}{2} (1 + \Delta - \varepsilon + \eta)$	0.315
ζ	$1 - \varepsilon/6$	1/2	$(2 - \varepsilon + \frac{2\varepsilon^2}{243}) / (4 - \frac{4\varepsilon^2}{243})$	0.253

Figure 2.1: Critical exponents for JSLC and RDLG theory, from [30]

crepancy in the value of the exponent β is due to the fact that when the field is infinite, the anisotropy, and not the current, becomes the basic ingredient controlling the critical behaviour. They suggested that the coefficient of the nonlinear term $\partial\phi^2$ should vanish in the limit of infinite driving field, and so should not appear in the final Langevin equation that should be:

$$\begin{aligned} \frac{\partial}{\partial t}\phi(\vec{x}, t) = & \lambda \left((\tau_{\perp} - \nabla^2)\nabla^2\phi + (\tau_{\parallel} - \alpha_{\parallel}\partial^2)\partial^2\phi - 2\alpha_{\times}\partial^2\nabla^2\phi + \right. \\ & \left. + \frac{u}{3!}(\nabla^2\phi^3 + \kappa\partial^2\phi^3) \right) - (\vec{\nabla} \cdot \vec{\xi} + \partial\zeta). \end{aligned} \quad (2.45)$$

This theory has been extensively studied in [17], its critical dimension is $d_c = 3$ (instead $d_c = 5$ for the JSLC theory) and the critical exponents are known up to two loops perturbatively in $\varepsilon = 3 - d$. The resulting critical exponents, for both JSLC and RDLG theory, are listed in Figure 2.1, for generic d dimensional system, and specifically for $d = 2$.

Chapter 3

Finite-size scaling

Continuous phase transitions are characterized by a non-analytic behaviour of the observables at the critical point. Of course, these non-analyticities are observed only in the infinite volume limit. However, even large but finite systems show an universal behaviour called **Finite-size scaling** (FSS). The FSS, formulated for the first time by Fisher [18], is a powerful method to find the values for the critical exponents and the transition temperature, by observing how measured observables vary for different lattice sizes. To compare numerical results (obtained in a finite system) with theoretical predictions (observed only in the thermodynamic limit), FSS analysis is of fundamental importance.

3.1 Isotropic finite-size scaling

For systems displaying a second order phase transitions, there are observables \mathcal{O} which, in the infinite volume limit, behave as

$$\langle \mathcal{O} \rangle_{\infty} \sim |\tau|^{-x_{\mathcal{O}}} \quad , \quad \text{for } \tau \rightarrow 0, \quad (3.1)$$

where $\tau = (T - T_c)/T_c$ is the reduced temperature. An important observable that display this diverging behaviour is the correlation length

$$\xi \sim |\tau|^{-\nu}. \quad (3.2)$$

In a finite system, for instance a 2-dimensional lattice $L \times L$, the finite volume mean values $\mathcal{O}_L = \langle \mathcal{O} \rangle_L$ are analytic functions of τ . This analytic dependence on control parameters implies that the interchange of the infinite volume limit with the limit $\tau \rightarrow 0$ is in general not permitted:

$$\lim_{\tau \rightarrow 0} \lim_{L \rightarrow \infty} \mathcal{O}_L(\tau) \neq \lim_{L \rightarrow \infty} \lim_{\tau \rightarrow 0} \mathcal{O}_L(\tau), \quad (3.3)$$

However, for large L , the FSS theory predicts that, when the limit is taken such that $\tau \rightarrow 0$, $L \rightarrow \infty$, keeping $\tau L^{1/\nu}$ constant, for these mean values holds a scaling behaviour of the form [19]

$$\mathcal{O}_L \sim L^{x_{\mathcal{O}}/\nu} F_{1,\mathcal{O}}(\tau^{-\nu}/L), \quad (3.4)$$

where the function $F_{1,\mathcal{O}}$ is finite, non-vanishing in zero and tends to zero when its argument approaches ∞ as

$$F_{1,\mathcal{O}}(z) \sim |z|^{-x_{\mathcal{O}}} \quad \text{for } z \rightarrow \infty. \quad (3.5)$$

The equation (3.4) can be rewritten in terms of the correlation length ξ , in order to avoid the knowledge of the critical temperature T_c ,

$$\mathcal{O}_L \sim L^{x_{\mathcal{O}}/\nu} F_{2,\mathcal{O}}\left(\frac{\xi(\tau)}{L}\right) \quad (3.6)$$

where $F_{2,\mathcal{O}}$ is finite for $z \rightarrow \infty$ and

$$F_{2,\mathcal{O}}(z) \sim |z|^{x_{\mathcal{O}}/\nu} \quad \text{for } z \rightarrow 0. \quad (3.7)$$

Equation (3.6) indicates that the only relevant length scale is given by the correlation length ξ , so it's natural to expect that in FSS for a system with length scale L only the ratio ξ/L is relevant. This equation can be used, in a numerical simulation, for the determination of the critical temperature. Considering an observable as the Binder's cumulant $g_L(\tau)$, for which $x_g = 0$, at the critical point it will be independent from L . Collecting data taken from lattices with different sizes, and plotting them as a function of the

temperature T , in the point corresponding to $T = T_c$ all the curves cross each other. Similarly, once T_c has been determined, one may determine ν using ξ_L/L and then all the others critical exponents may be derived with the same technique.

From equation (3.6) we can derive a general relation for the ratio of \mathcal{O}_L at two different sizes L and αL . We obtain

$$\frac{\mathcal{O}_{\alpha L}(\tau)}{\mathcal{O}_L(\tau)} = F_{\mathcal{O}}\left(\alpha, \frac{\xi_L(\tau)}{L}\right) \quad (3.8)$$

where $\xi_L(\tau)$ can be obtained by inverting $\xi_L = L F_{2,\xi}(\xi/L)$.

Defining $z = \xi_L/L$, the function $F_{\mathcal{O}}(\alpha, z)$ is universal and is directly accessible numerically, for instance by Monte Carlo simulations, because all quantities appearing in (3.8) are directly measurable and the precise knowledge of the critical temperature is not necessary. Therefore, using this equations we can have an improved method for the determination of the critical exponents [20]. Varying τ in a fixed geometry, $z = \xi_L/L$ varies between 0 and the critical value z^* defined by

$$z^* = F_{2,\xi}(\infty). \quad (3.9)$$

For ordinary phase transitions z^* turns out to be finite, at the critical point $\tau = 0$ we have

$$\mathcal{O}_L(\tau = 0) \propto L^{x_{\mathcal{O}}/\nu}. \quad (3.10)$$

Then

$$F_{\mathcal{O}}(z^*) = \frac{\mathcal{O}_{\alpha L}(\tau = 0)}{\mathcal{O}_L(\tau = 0)} = \alpha^{x_{\mathcal{O}}/\nu}. \quad (3.11)$$

The previous equation, once it has been found the values of z^* and ν , can be used to determine any critical exponent $x_{\mathcal{O}}$, just by inverting it

$$\frac{x_{\mathcal{O}}}{\nu} = \frac{\log F_{\mathcal{O}}(z^*)}{\log \alpha}. \quad (3.12)$$

For the determination of z^* , we procede as follow: we measure the finite volume correlation length $\xi_L(T)$ for several values of the temperature T and

we determine F_ξ by means of equation (3.8):

$$\frac{\xi_{\alpha L}(T)}{\xi_L(T)} = F_\xi(\xi_L(T)/L), \quad (3.13)$$

fitting the data of the left hand side as a function of $\xi_L(T)/L$ gives us an estimate of the function F_ξ . Then the value of z^* can be derived from

$$\alpha = F_\xi(z^*), \quad (3.14)$$

which follows directly from (3.8) and (3.10). To determine the ν exponent we note that, from equation (3.4), we can expand the functions $z_L = \xi_L/L$ around $\tau = 0$ and we have

$$z_L = z^* + F'_{1,\xi}(0) \cdot (\tau L^{1/\nu}) + O((\tau L^{1/\nu})^2). \quad (3.15)$$

Hence we have

$$\frac{z_{\alpha L}}{z_L} = 1 + \frac{F'_{1,\xi}(0)}{z^*} (\alpha^{1/\nu} - 1) (\tau L^{1/\nu}) + O((\tau L^{1/\nu})^2). \quad (3.16)$$

The two previous equations imply that

$$\left. \frac{d}{dz_L} \frac{z_{\alpha L}}{z_L} \right|_{z_L=z^*} = \frac{\alpha^{1/\nu} - 1}{z^*}, \quad (3.17)$$

and then, using $\alpha = F_\xi(z^*)$,

$$z \left. \frac{d}{dz} \log F_\xi(z) \right|_{z=z^*} = \alpha^{1/\nu} - 1. \quad (3.18)$$

Therefore, the value of ν can be recovered only by means of the FSS function F_ξ .

3.2 Anisotropic finite-size scaling

In strongly anisotropic systems it happens that the correlations length diverges at the critical point with different ν exponents, so that we should clarify how to manage the two correlation length ξ_{\parallel} , ξ_{\perp} . In the specific case of the DLG, both the continuum field theory introduced in the previous Chapter predict the presence of an anisotropy exponent Δ . Following the approach of [21], we will assume that all the observables have a correct FSS limit taking both the longitudinal size L_{\parallel} and the transverse one L_{\perp} to infinity keeping fixed both combinations $\tau L_{\parallel}^{1/\nu_{\parallel}}$ and $\tau L_{\perp}^{1/\nu_{\perp}}$, and therefore also the *anisotropic aspect ratio* S_{Δ}

$$S_{\Delta} = \frac{L_{\parallel}^{1/(1+\Delta)}}{L_{\perp}}. \quad (3.19)$$

Hence, the FSS equations (3.4) and (3.6) can be rewritten

$$\mathcal{O}_{L_{\perp}} \sim L_{\perp}^{x_{\mathcal{O}}/\nu_{\perp}} F_{1,\mathcal{O}}(\tau^{-\nu_{\perp}}/L_{\perp}) \sim L_{\perp}^{x_{\mathcal{O}}/\nu_{\perp}} F_{2,\mathcal{O}}(\xi_{\perp}/L_{\perp}) \quad (3.20)$$

Similarly, the equation (3.8) can be cast in the form

$$\frac{\mathcal{O}_{\alpha L_{\perp}}(\tau, S_{\Delta})}{\mathcal{O}_{L_{\perp}}(\tau, S_{\Delta})} = F_{\mathcal{O}}\left(\alpha, \frac{\xi_{\perp, L_{\perp}}(\tau)}{L_{\perp}}, S_{\Delta}\right). \quad (3.21)$$

We mention here that, because the FSS limit must be taken at fixed S_{Δ} , a deficiency of any numerical simulation is that the exponent Δ has to be fixed since the beginning of the analysis.

3.3 FSS for the DLG model

A detailed study of the FSS for the DLG have been done by S. Caracciolo, A. Gambassi, M. Gubinelli and A. Pelissetto [23] at the beginning of this century, whose results confirmed the Gaussian nature of the transverse fluctuations, in according with the JSLC theory. For the JSLC theory, we

can compute exactly the two-point function (see Section 5.2):

$$\tilde{G}_\perp(k) = \frac{1}{k^2 + \tau'}, \quad (3.22)$$

where τ' is the field theory "bare mass" and not the reduced temperature τ . However, we expect $\tau' = b\tau + O(\tau^2)$ for $\tau \rightarrow 0$ where b is a positive constant. Moreover, equation (3.22) refers to the mesoscopic field $\phi(x, t)$ but for the lattice function we are interested in, we expect in the scaling limit $\tau \rightarrow 0$, $k \rightarrow 0$ with k^2/τ fixed

$$\tilde{G}_{\perp, \text{lat}}(k) = \frac{Z}{k^2 + \tau'}, \quad (3.23)$$

where Z is another positive constant and the momentum k is quantized. Using the previous expression and the definition (1.20) we find

$$\frac{\xi_{\perp, L_\perp}(\tau)}{L_\perp} = [(2\pi)^2 + b\tau L_\perp^2]^{-1/2}. \quad (3.24)$$

This expression is expected to be valid in the FSS limit, hence for $\tau \rightarrow 0$, $L_\parallel, L_\perp \rightarrow \infty$ with τL_\perp^2 , and keeping S_Δ constant. Thus we can express the previous equation also in the form

$$\frac{1}{|\xi_{\perp, L_\perp}(\tau)|^2} = \frac{1}{\xi_\infty(\tau)^2} + \frac{4\pi^2}{L_\perp^2}, \quad (3.25)$$

where we can see the finite volume corrections to the correlation length. Moreover we can compute the scaling function $F_\xi(\alpha, z, S_\Delta)$ defined in equation (3.21). We obtain

$$F_\xi(\alpha, z, S_\Delta) = [1 - (1 - \alpha^{-2})(2\pi)^2 z^2]^{-1/2}, \quad (3.26)$$

from which, using the implicit definition $F_\xi(\alpha, z^*, S_\Delta) = \alpha$ we can see

$$z^* = \frac{1}{2\pi}. \quad (3.27)$$

In references ([23]) a numerical check of these theoretical predictions has been carried out, finding a good agreement with the fact that the critical

behaviour of transverse fluctuations in the DLG is Gaussian, as predicted by JSLC theory. Furthermore, using the method sketched in the previous section, the disputed β exponent is calculated, giving a value $\beta \sim 1/2$, in accordance with JSLC theory.

We give a final comment on the order parameter m defined in (1.14). In JSLC theory the first transverse mode of the Fourier Transform $\phi(\vec{k}_{0,1})$ is expected to have a Gaussian distribution

$$N \exp\left(-\frac{|\phi(\vec{k}_{0,1})|^2}{L_{\parallel}L_{\perp}\chi_{\perp}}\right) d[\phi(\vec{k}_{0,1})]d[\phi^*(\vec{k}_{0,1})], \quad (3.28)$$

where N is some normalization factor. The mean value $m = \langle\phi(\vec{k}_{0,1})\rangle$ can be computed with the result

$$m^2 = \frac{\pi}{4} \frac{\chi_{\perp}}{L_{\parallel}L_{\perp}}. \quad (3.29)$$

Chapter 4

Short-time scaling

Understanding the non-equilibrium properties of physical systems is currently one of the most challenging problems in statistical physics. In recent times, particular interest has been devoted in the study of the universal behaviour for the **dynamics** of systems relaxing at criticality. It's pretty known that, for dynamic systems in the *long time regime*, where the equilibrium is almost reached, one finds a dynamical scaling form. Due to critical slowing down, however, numerical study of critical dynamics is very difficult, such that even for the simple two-dimensional Ising model the dynamic exponent z has not yet rigorously been determined.

More recently, it has been argued that universal scaling behaviour already emerges in the *short time regime* of the relaxation process, even though the correlation length ξ is still very short compared with the system size. This fact has been first showed to be true for Model A by Janssen, Schaub and Schmittmann [24], and then applied to some extent for equilibrium models. The advantages of this technique are easy to understand, mainly very short simulation are needed to obtain critical exponents, while equilibrium (or steady-state) or longer-time methods are usually very demanding in computer time. In some system, like Model A, there is also the presence of a new independent exponent.

When a system is prepared at high temperature, and then allowed to relax at criticality, for (microscopically) short times after the start of the relax-

ation, the behaviour is governed by non-universal microscopic processes and the physics of the system can be described only on the basis of a microscopic theory. Later, the system display an intermediate stage of the relaxation process, termed "critical initial slip", which eventually crosses over to the long-time time behaviour. The main feature of the critical initial slip is that display *universal behaviour*.

If Janssen, Schaub and Schmittmann in their seminal work [24] laid the foundations for the theoretical study of the early stages of the relaxation process, much more theoretical and numerical work has been performed since then [25]. Furthermore, it has been shown that the same set of ideas can be applied for a dynamical study of the long-time regime, in order to understand and describe the critical ageing [26]. In particular, the method has been extended to systems not yet rigorously treated theoretically, such as non-equilibrium phase transitions.

We revise these aspect in this Chapter and then, in the next Chapter we will see how this approach has been applied for the case of driven lattice gases.

4.1 Model A

Consider an Ising model at a temperature $T \gg T_c$. Being far away from criticality, all correlations will be short-ranged. With a non-zero initial magnetization and quenching the system rapidly to a temperature $T \sim T_c$, the process will display a critical initial slip, when the magnetization will increase with a power law behaviour $m \sim t^{\theta'}$. Later, the system enters in a transition regime, that is not well understood. After that, we find the long-time scaling regime, where the magnetization decreases to zero (its equilibrium value), characterised by the power law $m \sim t^{-\beta/\nu z}$. When we are at the critical temperature, such a regime of nonlinear relaxation would extend for every arbitrary big time t , thanks to the critical slowing down. This qualitative picture is summarized in Figure 4.1.

The short-time scaling behaviour for the order parameter, as well as analogous scaling behaviour for the dynamic susceptibility and the order parame-

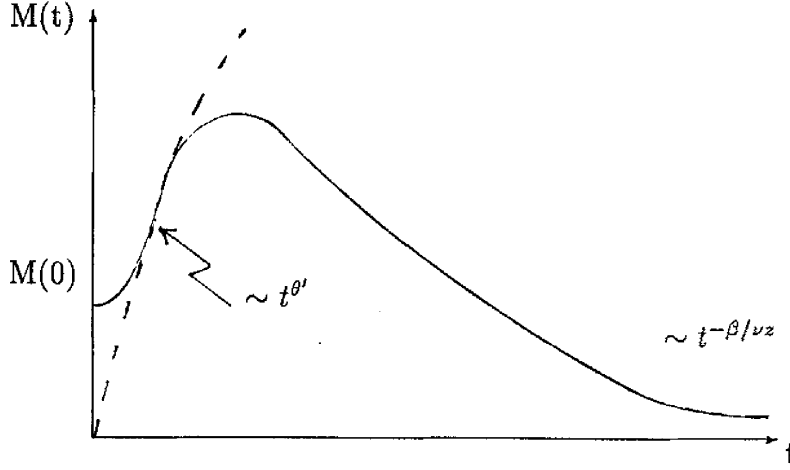


Figure 4.1: Qualitative time dependence of the magnetization $M(t)$ at criticality ($\tau = 0$), from [24].

ter correlation function, can be understood in the framework of dynamic field theory. The dynamics of the fluctuation of the local magnetization $\phi(x, t)$ can be expressed in the form of the Langevin equation (2.6) of Model A:

$$\partial_t \phi(x, t) = -\lambda \frac{\delta H[\phi]}{\delta \phi} + \zeta(x, t), \quad (4.1)$$

As seen in Section 2.1, the average over the noise can be computed as

$$\langle O \rangle = \int [d\phi d\tilde{\phi}] O[\phi] e^{-J[\phi, \tilde{\phi}]} \quad (4.2)$$

in terms of the dynamical functional

$$J[\phi, \tilde{\phi}] = \int_0^\infty dt \int d^d x \left\{ \tilde{\phi} \left[\dot{\phi} + \lambda(\tau - \nabla) \phi + \frac{\lambda g}{6} \phi^3 \right] - \lambda \tilde{\phi}^2 \right\}. \quad (4.3)$$

This weigh functional $\exp\{-J[\phi, \tilde{\phi}]\}$ can be interpreted as a path probability density for the time and space dependent path $\{\phi(x, t), \tilde{\phi}(x, t)\}$. Hence, it is

a functional of the initial configuration at $t = 0$, $\phi(x, t = 0) = \phi_0(x)$. If we want to study the short-time behaviour of the dynamics we can't ignore this initial condition. Hence, in addition to averaging with respect to $\exp\{-J\}$, we also average over $\phi_0(x)$ with a probability distribution $\exp(-H_0[\phi_0])$ with

$$H_0[\phi_0] = \int d^d x \left[\frac{\tau_0}{2} (\phi_0(x) - a(x))^2 \right]. \quad (4.4)$$

This specifies a *macroscopically prepared* initial state $a(x)$ with correlations

$$\langle [\phi_0(x) - a(x)][\phi_0(x') - a(x')] \rangle = \frac{1}{\tau_0} \delta(x - x'), \quad (4.5)$$

where the average is computed with respect to the probability distribution specified by H_0 . Thus, in the following, all the dynamical response and correlation functions can be obtained from the functional

$$J[\phi, \tilde{\phi}, \phi_0] = J[\phi, \tilde{\phi}] + H_0[\phi_0] \quad (4.6)$$

Using standard dimensional analysis, if μ is an external momentum scale, $\tau_0 \sim \mu^2$. Therefore, the fixed point value for τ_0 correspond to $\tau_0 = \pm\infty$ or $\tau_0 = 0$ [24]. Since $\tau_0 = 0$ and $-\infty$ yield non-normalizable probability distribution functions, the fixed point of physical interest will be $\tau_0 = \infty$. Hence, corrections due to finite value of τ_0 will be irrelevant in the RG sense, thus we can set $\tau_0^{-1} = 0$ from the beginning of the calculation.

Using standard RG techniques [24], the response and correlations functions may be obtained by a perturbative expansion of the weight $\exp\{-J[\phi, \tilde{\phi}, \phi_0]\}$ about its Gaussian part, in terms of the coupling g . Even if a complete description of the renormalization procedure is beyond the scope of this section, we mention here that the breaking of time translation invariance, which reflects the presence of a time surface $t = 0$, does not allow the factorization of the connected correlation functions in terms of one-particle irreducible ones (1 PI), making necessary a perturbative computations in terms of connected functions only.

The main results are the following:

The correlation function

$$C_q(t, t') = \langle \phi(q, t) \phi(-q, t') \rangle \quad (4.7)$$

for $t' \rightarrow 0$ is given by

$$C_q(t, t') = q^{-2+\eta} \left(\frac{t}{t'} \right)^{\theta-1} f_C(q\xi, q^z t) \quad (4.8)$$

and the magnetization

$$M(t) = \int dx \langle \phi(x, t) \rangle, \quad (4.9)$$

at criticality, for $t \rightarrow 0$ is

$$M(t) = M(0) t^{\theta'} f_M(t^{\theta'+\beta/\nu z} M(0)) \quad (4.10)$$

where

$$f_M(x) \sim \begin{cases} 1 & \text{if } x \rightarrow 0 \\ 1/x & \text{if } x \rightarrow \infty, \end{cases} \quad (4.11)$$

and

$$\theta' = \theta + (2 - z - \eta)/z \quad (4.12)$$

Finally, the same analysis can be carried out also for Model B dynamics. One finds that all the correlation and response function are convergent in the short-time limit, hence the theory becomes trivial, in the sense that no new exponent can be found in this limit

4.2 Scaling Forms

The results obtained by Janssens, Schaub and Schmittmann for the order parameter (4.10),(4.11) are usually written using a scaling form:

$$M(t, \tau, M_0) = b^{-\beta/\nu} f_M(b^{-z} t, b^{1/\nu} \tau, b^{x_0} M_0), \quad (4.13)$$

where the parameter b represents a spatial rescaling factor. Here β , ν are the well known static critical exponents, z is the dynamic exponent and x_0 is the scaling dimension of the initial magnetization M_0 . If we set $b = t^{1/z}$, the main t -dependence on the right is cancelled and the previous equation becomes, at criticality ($\tau = 0$)

$$M(t, \tau, M_0) = t^{-\beta/\nu z} f_M(1, 0, t^{x_0/z} M_0), \quad (4.14)$$

This scaling form reproduces the results (4.10),(4.11), with $\theta' = (x_0 - \beta/\nu)/z$ assuming that f_M at criticality is a regular function in its third argument

$$f_M(1, 0, y) \sim y \quad (4.15)$$

for *short* times and *small* initial magnetization. For almost all the statistical systems studied up to now, the exponent θ' is positive, i.e. the magnetization undergoes a critical initial increase. Furthermore, f_M is supposed to approach a constant value for large times, losing its dependence from the initial magnetization. Hence, for longer times we get the usual critical decrease

$$M(t) \sim t^{\beta/\nu z}. \quad (4.16)$$

An extension to finite system have been proposed:

$$M(t, \tau, L, M_0) = b^{-\beta/\nu} f_M(b^{-z} t, b^{1/\nu} \tau, b^{-1} L, b^{x_0} M_0). \quad (4.17)$$

In this generalization, when setting $b = t^{1/z}$, the same argument is expected to be valid. And it's expected to be valid even when we are close to criticality ($\tau \sim 0$), but not necessary at criticality. Indeed, in the short time regime, the time dependent correlation length $\xi(t) \sim t^{1/z}$ is expected to be small with respect to both the equilibrium correlation length $\xi_{\text{eq}} \sim \tau^{-\nu}$ and the size of the system L , so that we expect $t^{-1/z} L \gg 1$ and $t^{-1/z} \tau^{-\nu} \gg 1$.

Note that, in order to observe this initial increase in the magnetization $M(t)$, a small but non-zero value for M_0 is needed, because the magnetization is antisymmetric under reflections of M_0 , i.e $M(t, M_0) = -M(t, -M_0)$,

implying that $M(t, 0) = 0$ for all t . When we attempt to generalize a scaling form such as (4.17) for different order parameters and for different systems, the same argument is valid for all the order parameters that have this property.

Other observables that display a power law behaviour at short times are the moments of the magnetization $M^{(k)}$, for which we can write an analogous scaling form [3]

$$M^{(k)}(t, \tau, L, M_0) = b^{-k\beta/\nu} f_M(b^{-z}t, b^{1/\nu}\tau, b^{-1}L, b^{x_0}M_0). \quad (4.18)$$

Using more than a scaling form for different moments of the magnetization can be useful for a cross-check test on the critical exponents.

Up to now a completely disordered initial state has been considered as starting point of the relaxation process, i.e. a state with very high temperature. Another interesting possibility is starting from a completely ordered state, and suddenly heat up the system to the critical temperature. Starting from equation (4.14) and setting $b = t^{1/z}$, if we assume that we can drop the dependence from M_0 we see that for short times the magnetization decays by a power law $M(t) \sim t^{-\beta/\nu z}$.

Another interesting aspect is the extent to which one can use the short time approximation. The determination of the crossover time is troublesome, because the previous approximations can break up for different time scales, which can be determined from the relations $M_0 t^{x_0/z} \sim 1$, $\xi(t) \sim L$ or $\xi(t) \sim \xi_{eq}$. A possible generalization is [27]:

$$M^{(k)}(t, \tau, L, M_0) = b^{-k\beta/\nu} f_M(b^{-z}t, b^{1/\nu}\tau, b^{-1}L, \phi(b, M_0)), \quad (4.19)$$

where the scaling behaviour of the initial magnetization M_0 is specified by the universal function $\phi(b, M_0)$. In the limit $M_0 \rightarrow 0$, $\phi(b, M_0)$ tends to the simple form $b^{x_0}M_0$ used in the previous equations, but pretend also to describe the crossover behaviour between the short time regime and the long time one.

Chapter 5

Short-time scaling for the DLG

We studied the critical relaxation for the DLG in two dimensions by Monte Carlo simulations. Our aim is to check the validity of some proposed scaling forms for the order parameters of this model (1.14) and (1.15) in the present literature [28] [30]. We use the dynamics discussed in Sect. 1.2, with Metropolis rates, i.e.

$$w(x) = \min(1, e^{-x}). \quad (5.1)$$

Simulations were performed at infinite driving field E , i.e. forward (backward) jumps in the direction of the field are always accepted (rejected). We studied the relaxation process both keeping fixed the direction of the field (IDLG) and with a random field (RDLG). In all cases the density ρ is $1/2$ while time t is measured in units of Monte Carlo time steps (mcs) involving $L_{\parallel} \times L_{\perp}$ proposed moves. The initial configuration is generated in a disordered state, but fixing the row magnetization to zero. Then the system is allowed to evolve for a time \bar{t} that is very short compared to the time required for the equilibration. Hence, this protocol is repeated N times in order to obtain an averaged time evolution for the order parameters. In our simulations, we set $\bar{t} = 1000$ mcs and $N = 10000$. The computational time requested for this simulations was always less than 24 hours, even for the biggest lattices considered (682, 40), on the average computers of the local computer lab. This time is very small compared to the time necessary for the study of the steady state properties, which suffers from a strong critical

slowing down since the dynamical exponent of these models is pretty big $z = 4$.

5.1 Phenomenological scaling

In the previous Chapter we've seen how scaling laws for the temporal evolution of the various observables are generally assumed. We expect that a generalization is possible for the driven lattice gas. For a generical observable \mathcal{O} , we expect a scaling form

$$\mathcal{O}(t, \tau, L_{\parallel}, L_{\perp}, m_0) = b^{\gamma_{\mathcal{O}}/\nu_{\perp}} f_{\mathcal{O}}(b^{-z_{\perp}}t, b^{1/\nu_{\perp}}\tau, b^{-\nu_{\parallel}/\nu_{\perp}}L_{\parallel}, b^{-1}L_{\perp}, b^{x_0}M_0) \quad (5.2)$$

where the anisotropy nature of the DLG has been taken into account. Here M_0 is the initial value of some order parameter for the DLG, such as (1.14) or (1.15). We note here that these order parameter, unlike the magnetization for the Ising model, are not antisymmetric with respect to their initial value M_0 , so we can set $M_0 = 0$ from the beginning. This condition can be ensured for both the order parameters, preparing the system fixing the row magnetization at zero, such that all transversal quantities vanish at zero time. The big advantage of this procedure, known as *sharp preparation*, is that we can significantly improve the scale invariance of our results, but still having statistically independent initial configurations. In fact, since the order parameters (1.14) and (1.15) contain absolute values in their definitions, their mean value for completely disordered state is different from zero, and it is size dependent. Therefore, merely generating random initial configuration, is not appropriate for observing a scale invariance in the evolution of these observables.

Specializing to the case $\tau = 0$, we set $b = L_{\perp}$

$$\mathcal{O}(t, 0, L_{\parallel}, L_{\perp}, 0) = L_{\perp}^{\gamma_{\mathcal{O}}/\nu_{\perp}} f_{\mathcal{O}}(L_{\perp}^{-z_{\perp}}t, 0, L_{\perp}^{-\nu_{\parallel}/\nu_{\perp}}L_{\parallel}, 1, 0). \quad (5.3)$$

	JSLC – exact		RDLG – $O(\varepsilon^2)$	
	$\varepsilon = 5 - d$	$d = 2$	$\varepsilon = 3 - d$	$d = 2$
Δ	$1 + \varepsilon/3$	2	$1 - 2\varepsilon^2/243$	0.992
z	4	4	$4 - 4\varepsilon^2/243$	3.984
ν	1/2	1/2	$\frac{1}{2} + \frac{\varepsilon}{12} + \frac{\varepsilon^2}{18} \left[\frac{67}{108} + \ln\left(\frac{2}{\sqrt{3}}\right) \right]$	0.626
η	0	0	$4\varepsilon^2/243$	0.016
β	1/2	1/2	$\frac{\nu}{2} (1 + \Delta - \varepsilon + \eta)$	0.315
ζ	$1 - \varepsilon/6$	1/2	$(2 - \varepsilon + \frac{2\varepsilon^2}{243}) / (4 - \frac{4\varepsilon^2}{243})$	0.253

Figure 5.1: Critical exponents for JSLC and RDLG theory, from [30]. This is the same figure as Figure 2.1, reported here for the readers convenience.

This scaling form depends on L_{\parallel} only through the anisotropic ratio

$$S_{\Delta} = \frac{L_{\parallel}^{\nu_{\perp}/\nu_{\parallel}}}{L_{\perp}} = \frac{L_{\parallel}^{1/(1+\Delta)}}{L_{\perp}}, \quad (5.4)$$

hence we can write

$$\mathcal{O}(t, L_{\parallel}, L_{\perp}) = L_{\perp}^{\gamma_{\mathcal{O}}/\nu_{\perp}} F_{\mathcal{O}}(L_{\perp}^{-z_{\perp}} t, S_{\Delta}). \quad (5.5)$$

For an order parameter $M(t, L_{\parallel}, L_{\perp})$

$$M(t, L_{\parallel}, L_{\perp}) = L_{\perp}^{-\beta/\nu_{\perp}} F_M(L_{\perp}^{-z_{\perp}} t, S_{\Delta}). \quad (5.6)$$

Firstly, this scaling form can be proven with **no assumption** on the form of the function F_M . With fixed S_{Δ} , we can eliminate one variable between L_{\parallel} and L_{\perp} and the previous scaling form reads

$$M(t, L_{\parallel}) = L_{\parallel}^{-\beta/\nu_{\parallel}} F_M(L_{\parallel}^{-z_{\parallel}} t). \quad (5.7)$$

or, equivalently

$$M(t, L_{\perp}) = L_{\perp}^{-\beta/\nu_{\perp}} \tilde{F}_M(L_{\perp}^{-z_{\perp}} t). \quad (5.8)$$

We present here our Monte Carlo simulation results, employing different lattices geometries. We performed simulations on lattices with fixed S_{Δ} with $\Delta = 1$: (414, 36), (512, 40), (620, 44) for which $S_1 \simeq 0.32$ and simulations on lattices with fixed S_{Δ} with $\Delta = 2$: (374, 36), (512, 40) (682, 44), for which $S_2 \simeq 0.008$. We took particular care in considering only lattices with even L_{\parallel} so that we can initialize the system fixing the row magnetization at zero. We attempted to scale the data via equation (5.7) using both JSLC and RDLG exponent values, using for JSLC exponents the exact values $z_{\parallel} = \frac{4}{3}$ and $\beta/\nu_{\parallel} = \frac{1}{3}$, while for RDLG theory the two-loops values $z_{\parallel} \sim 2$ and $\beta/\nu_{\parallel} \sim 0.252$, see Figure 5.1. As already pointed out in [28] the scaling of the Albano-Saracco order parameter OP turns out to be independent from the anisotropic exponent Δ used in the simulation. At variance with the results obtained in [28], we found a good data collapse for both the continuum field theory, see Figure (5.2) and Figure (5.3). By contrast, the Leung order parameter m proves to be Δ -dependent. In [30], Daquila and Tauber showed how a superior data collapse is obtained using the exponents of JSLC theory, but they used only anisotropic lattices with $\Delta = 2$. On the contrary, simulations performed using lattices with $\Delta = 1$ provide exactly the opposite result, giving a good data collapse with the RDLG theory exponents, see Figure 5.4 and Figure 5.5. All these results turn out to be independent from the simulation performed using either IDLG model or RDLG model.

We can push a little further our analysis on the scaling form (5.6) by making some assumption on the scaling function F_M . For short time we expect a regime where, for a generic order parameter M , the scaling relation (5.6) reads:

$$M(t, L_{\parallel}, L_{\perp}) \sim L_{\perp}^{-\beta/\nu_{\perp}} \left(\frac{t}{L_{\perp}^{z_{\perp}}} \right)^x S_{\Delta}^y. \quad (5.9)$$

Particular attention should be paid to the fact that M depends on three parameters, t , L_{\parallel} and L_{\perp} . If we fix the dependence of $M(t, L_{\parallel}, L_{\perp})$ from two of them, the field theory will automatically fix the dependence from the third one. For both the order parameters m and OP , phenomenologically we

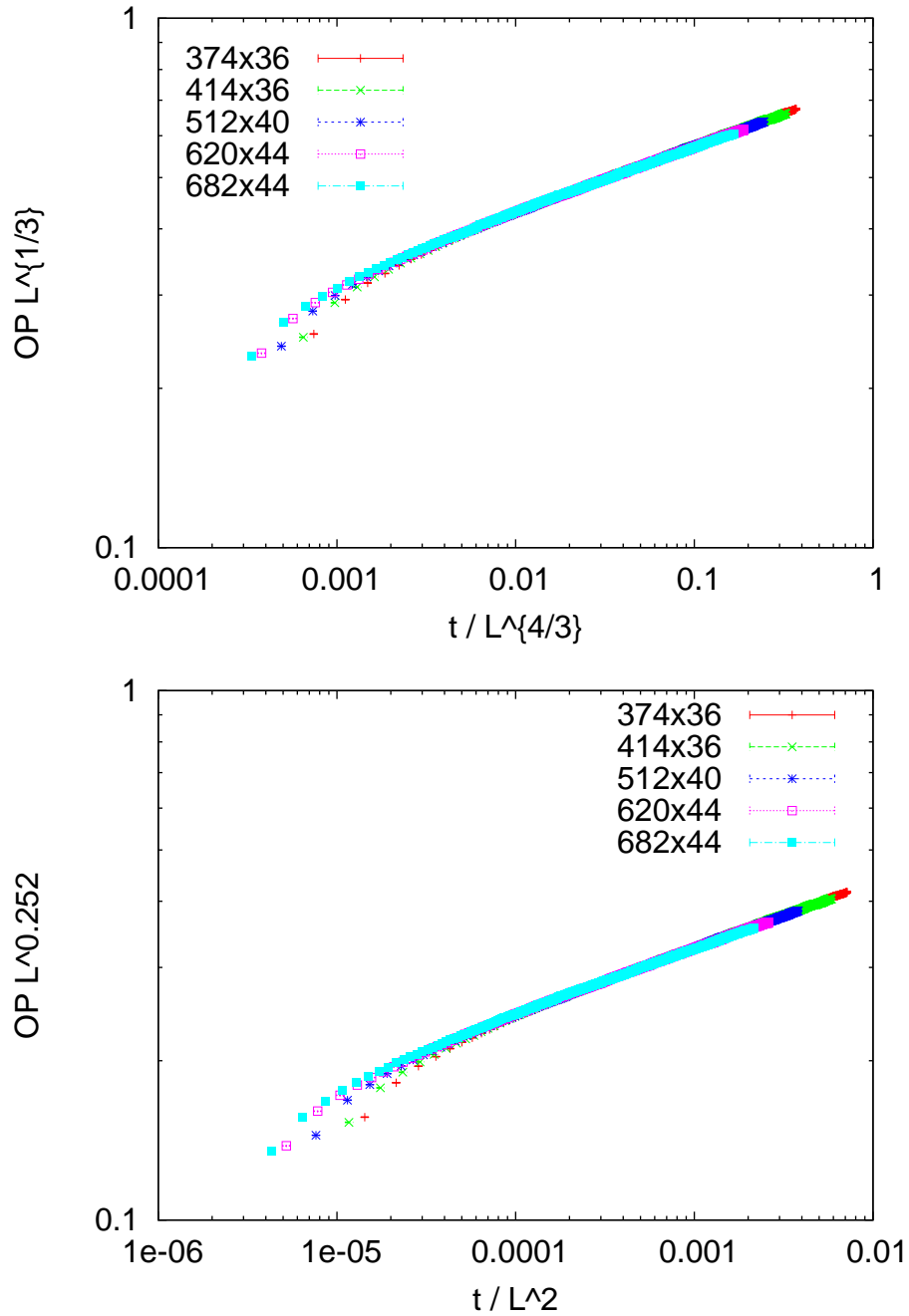


Figure 5.2: Log-log plot of the order parameter OP for the IDLG model, according to equation (5.7). Here L stands for L_{\parallel} . A good data collapse, not depending from the anisotropic aspect ratio S_{Δ} , is obtained for both the JSLC (upper panel) and the RDLG exponents (bottom panel)

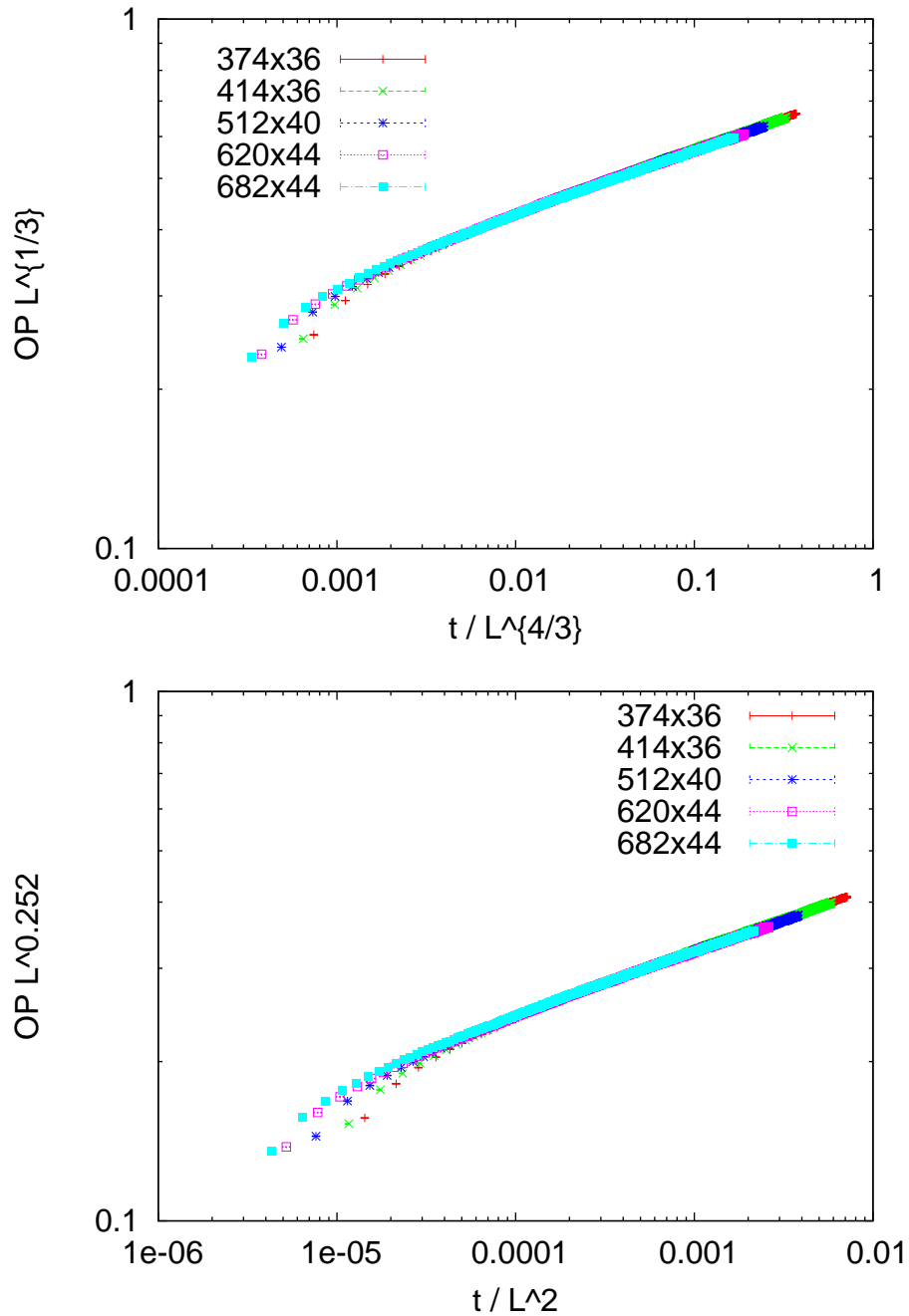


Figure 5.3: Log-log plot of the order parameter OP for the RDLG model, according to equation (5.7). Here L stands for L_{\parallel} . A good data collapse, not depending from the anisotropic aspect ratio S_{Δ} , is obtained for both the JSLC (upper panel) and the RDLG exponents (bottom panel)

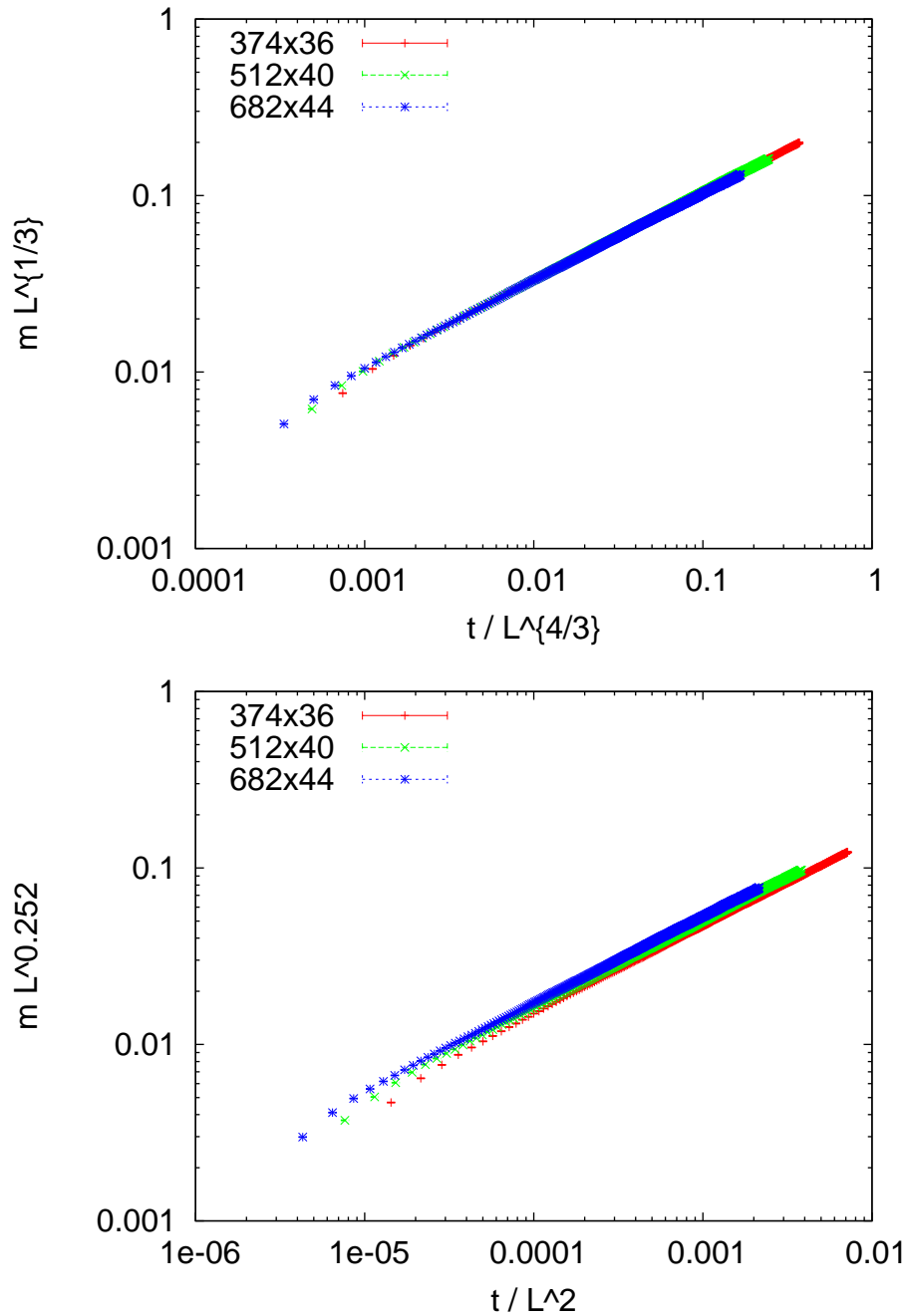


Figure 5.4: Log-log plot of the order parameter m for the IDLG model, according to equation (5.7). Here L stands for L_{\parallel} . For these geometries, with $\Delta = 2$ a superior data collapse is obtained with the JSLC exponents (upper panel) than with RDLG exponents (bottom panel)

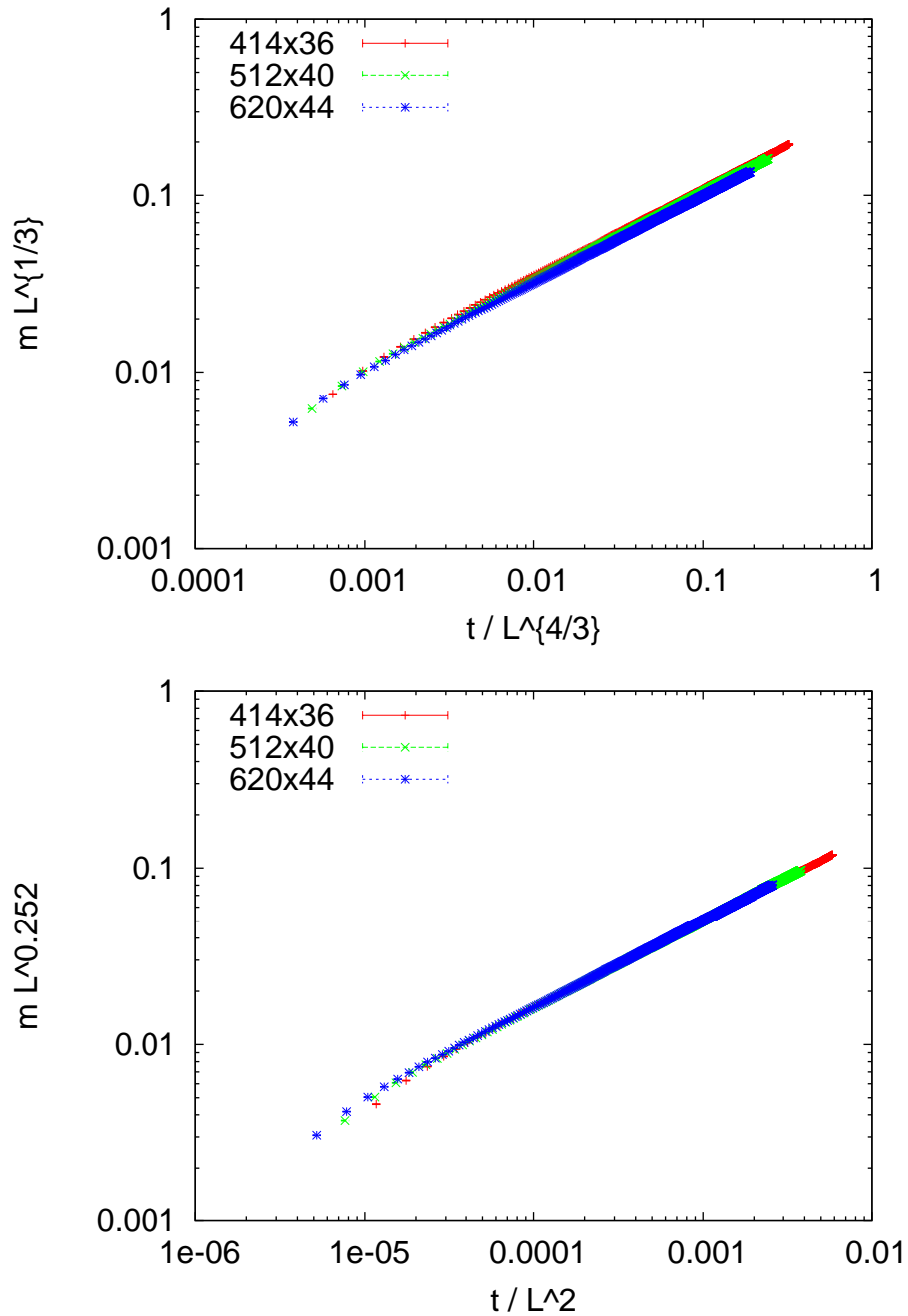


Figure 5.5: Log-log plot of the m order parameter for the IDLG model, according to equation (5.7). Here L stands for L_{\parallel} . For these geometries, with $\Delta = 1$ a superior data collapse is obtained with the RDLG exponents (bottom panel) than with JSLC exponents (upper panel)

can assume a dependence from L_{\parallel} of the form $L_{\parallel}^{-1/2}$. In equation (5.9) the dependence L_{\parallel} is contained only in the anisotropic aspect ratio S_{Δ} , hence it must be

$$y = -\frac{(1 + \Delta)}{2}. \quad (5.10)$$

Therefore we have

$$M(t, L_{\parallel}, L_{\perp}) \sim \frac{L_{\perp}^{\frac{1}{2}(1+\Delta-\frac{2\beta}{\nu_{\perp}})}}{L_{\parallel}^{1/2}} \left(\frac{t}{L_{\perp}^{z_{\perp}}} \right)^x, \quad (5.11)$$

for both order parameters OP and m . Similarly, from the simulations we see that the order parameter OP does not depend from L_{\perp} . Therefore, from equation (5.11) it must be

$$x = \frac{1}{2z_{\perp}} \left[1 + \Delta - \frac{2\beta}{\nu_{\perp}} \right] = \frac{1}{2z_{\parallel}} \left[1 - \frac{2\beta}{\nu_{\parallel}} \right]. \quad (5.12)$$

We can see that, using both the JSLC exponents and the RDLG exponents, it turns out that

$$x = \frac{1}{8} \quad (5.13)$$

and so for OP we can write a scaling law

$$OP(t, L_{\parallel}) \sim L_{\parallel}^{-1/2} t^{1/8}, \quad (5.14)$$

which is valid independently from the geometries of the lattice. With our simulations, we find a good agreement with the scaling law (5.14) as shown in Figure 5.6. In ref. [28] the same equation is derived in a different way, apart from the fact that they do not take properly into account the anisotropic nature of the DLG, assuming $z = z_{\perp} = z_{\parallel}$ as pointed out in [29]. The dependence from $L_{\parallel}^{1/2}$ is explained from the presence of density fluctuation along the rows when the lattice is filled at random. We point out that this argument is misleading, because we obtain the same dependence with a sharp preparation, i.e. with no fluctuation in the initial value of OP . Moreover they find the dependence from $t^{1/2}$ using a regularity assumption on the space dependence of their scaling function. We'll see later in this Section how this

assumption can easily leads to wrong results.

Similarly, we proceeded for the Leung order parameter m (1.14). Phenomenologically we see that the time dependence of this order parameter during its evolution process is $m \sim \sqrt{t}$ so that in equation (5.11) it must be

$$x = 1/2. \quad (5.15)$$

Surprisingly, also in this case both the JSLC field theory and RDLG field theory leads to a single scaling function for m that is expected to be valid for lattices of any geometry

$$m(t, L_{\parallel}, L_{\perp}) \sim \sqrt{\frac{t}{L_{\parallel}L_{\perp}^3}}. \quad (5.16)$$

In Figure 5.7 and Figure 5.8 it's shown an excellent agreement between this scaling law and the simulation data for all the considered lattices. For simulation with fixed $\Delta = 2$, we see that equation (5.16) reads

$$m(t, L_{\parallel}) \sim L_{\parallel}^{-1}t^{1/2} \quad (5.17)$$

In [30], Daquila and Tauber obtain the same equation assuming the regularity of the scaling function F_M in their scaling relation

$$m(t, L_{\parallel}) = t^{-\beta/\nu_{\parallel}z_{\parallel}} F_M(t^{1/z_{\parallel}} L_{\parallel}^{-1}). \quad (5.18)$$

Hence they get

$$m(t, L_{\parallel}) \sim L_{\parallel}^{-1}t^x, \quad \text{with } x = (1 - \beta/\nu_{\parallel})/z_{\parallel}. \quad (5.19)$$

Therefore, using JSLC exponents they have $x = \frac{1}{2}$ while using RDLG exponents they expect $x \sim 0.37$. Hence, finding experimental results in according to ours, they conclude that the DLG model is adequately described by the JSLC theory and not by the RDLG theory. Unfortunately, also the simulation performed with the RDLG lattice model, yields a good data collapse with the scaling (5.16) and (5.17), as is shown in Figure 5.8, hence the derivation

contained in [30] is misleading.

Finally, we attempted to verify these scaling laws also for the ordinary Ising lattice gas. In this case, to define an order parameter is troublesome, because the ordered phase doesn't have the strip form of the DLG. This is the reason why most of the studies on the dynamical aspects of the Ising lattice gas focus on different observables [31]. However, since we are studying an equilibrium model, we can change the boundary conditions. Fixing the boundary condition such that the lattice sites of the first row are in contact with a bath of positive-valued spins, and the lattice sites of the last row are in contact with a bath of negative-valued spins, keeping periodic boundary conditions for the left and right boundaries, the system quenches in a phase with a single horizontal strip. Thus we can use the order parameters m and OP of the DLG also for the equilibrium Ising lattice gas. Since we found that the scaling law (5.14) and (5.16) describes accurately both the IDLG model and the RDLG model, which are supposed to belong to different universality classes, we tried to figure out if they were valid also for the Ising model. In the isotropic case we can set $L_{\parallel} = L_{\perp} = L$, then (5.14) and (5.16) become

$$OP(t, L) = L^{-1/2}t^{1/8} \quad (5.20)$$

and

$$m(t, L) = L^{-2}t^{1/2}. \quad (5.21)$$

The first of these equations, when compared to the numerical results, turns out to be incorrect. Conversely, the second one turns out to be consistent with the simulations, see Figure 5.9. Multiplying both sides of the last equation by $L^{1/8}$ we get

$$m(t, L)L^{1/8} = L^{-15/8}t^{1/2} = (L^{-15/4}t), \quad (5.22)$$

which is consistent with the scaling form 5.7, with the Ising exponents $\beta/\nu = 1/8$ and the dynamical exponent $z = 15/4$.

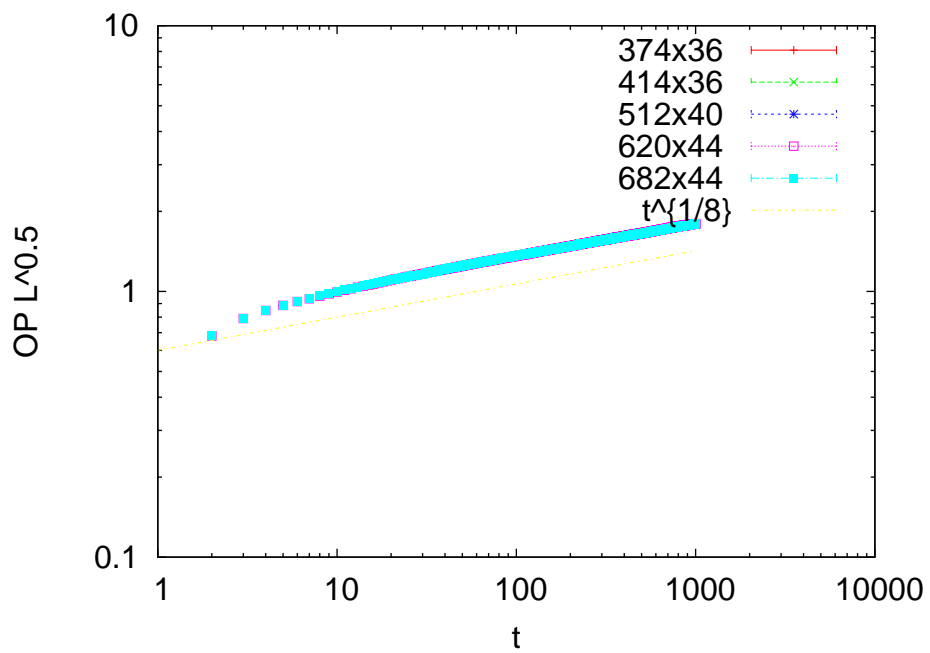


Figure 5.6: Log-log plot of the order parameter OP , for the IDLG model, here L stands for L_{\parallel} . Independently from the anisotropic aspect ratio S_{Δ} we find a good data collapse, in accordance with the scaling assumption (5.14).

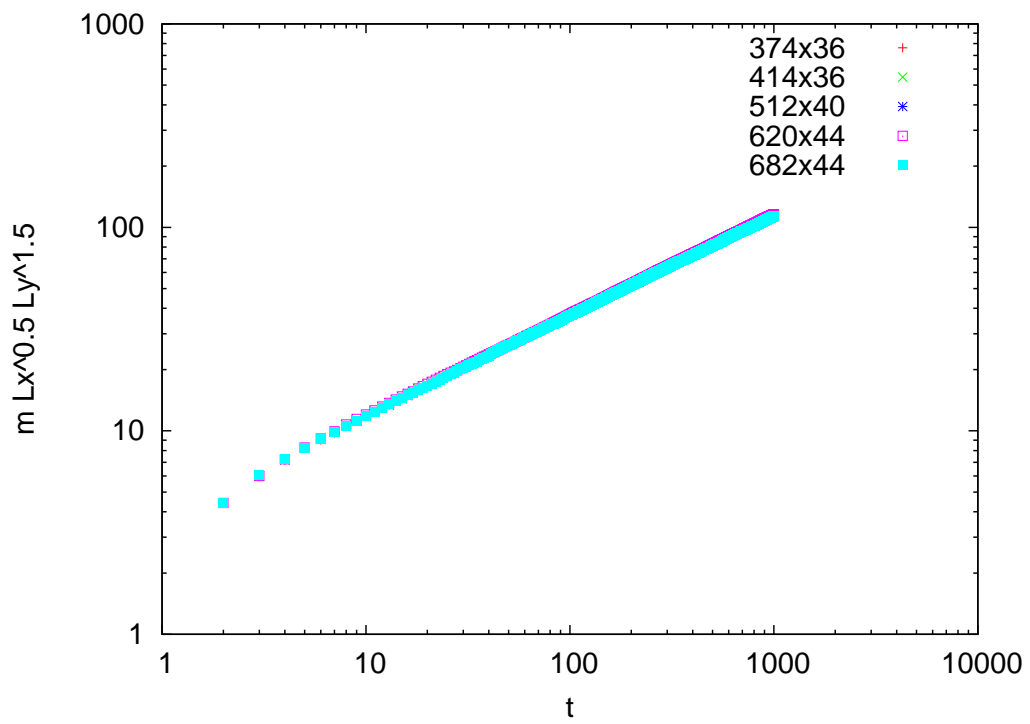


Figure 5.7: Log-log plot of the order parameter m , for the IDLG model, here L_x stands for L_{\parallel} and L_y stands for L_{\perp} . Independently from the anisotropic aspect ratio S_{Δ} we find a good data collapse, in accordance with the scaling assumption (5.16).

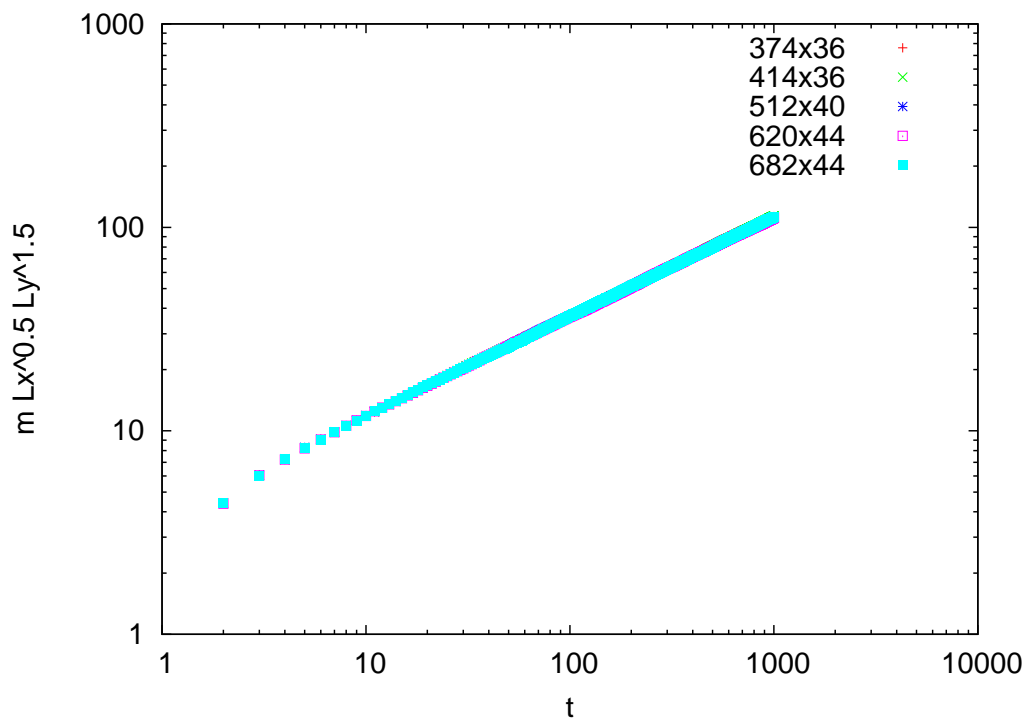


Figure 5.8: Log-log plot of the order parameter m , for the RDLG model, here L_x stands for L_{\parallel} and L_y stands for L_{\perp} . Independently from the anisotropic aspect ratio S_{Δ} we find a good data collapse, in accordance with the scaling assumption (5.16).

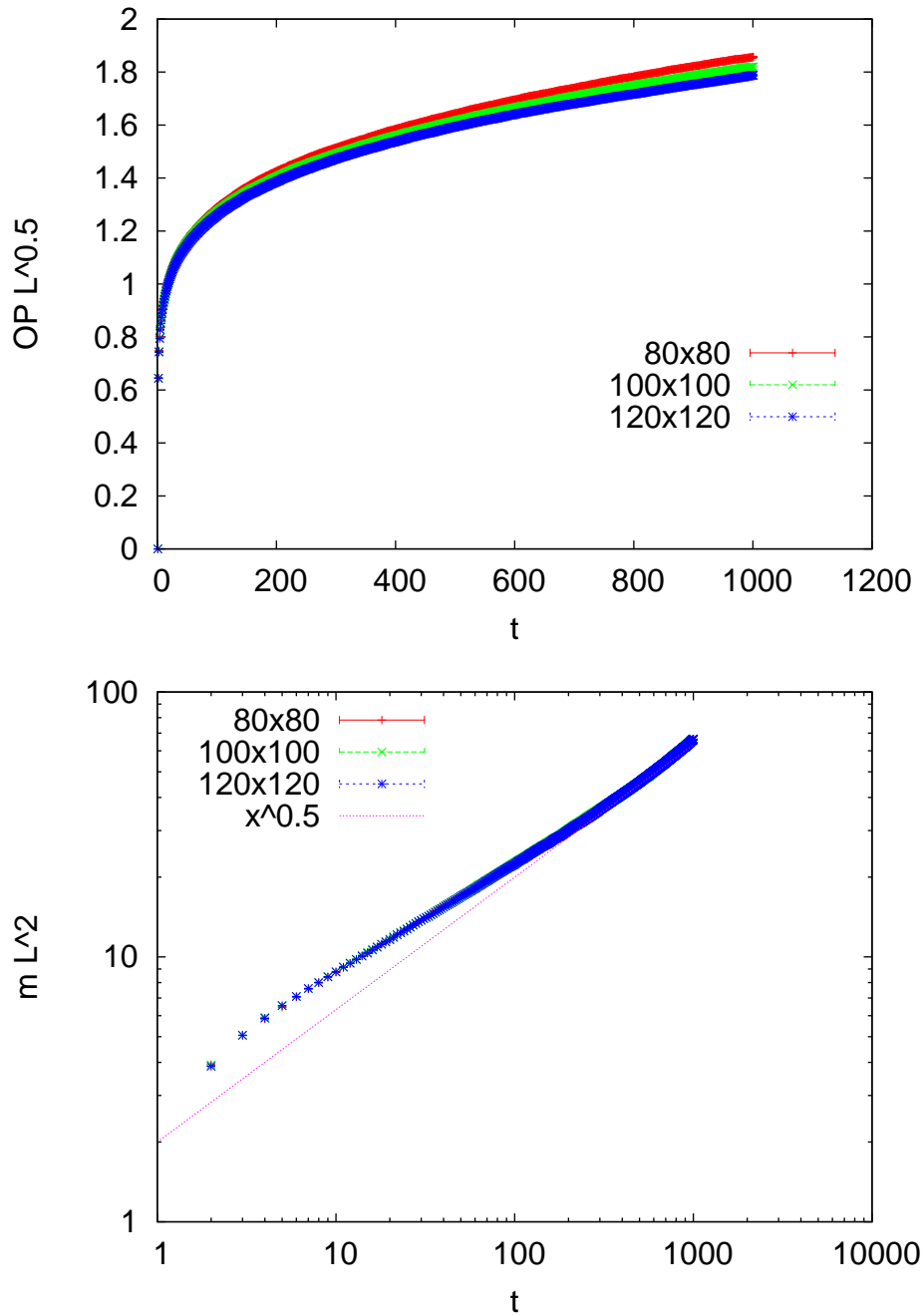


Figure 5.9: Plots for the OP and m order parameter for the Ising model with Kawasaki dynamics. The first of these plots doesn't show a good data collapse, in contrast with equation (5.20). Instead, the second plot provides a good evidence for the validity of equation (5.21).

5.2 Short-time scaling and field theory

In this section we will show how, one of the main result of the previous section, equation (5.16), which describe the short-time evolution of the order parameter m , can be derived in the framework of JSLC theory. The dynamical functional generator for the JSLC theory, neglecting the dangerously irrelevant operator is

$$J[\phi, \tilde{\phi}] = \lambda \int dt d^d x \tilde{\phi} [\lambda^{-1} \partial_t + \nabla^2 (\nabla^2 - \tau) - \tau_{\parallel} \partial^2] \phi + \mathcal{E}(\partial \tilde{\phi}) \phi^2 + \tilde{\phi} \nabla^2 \tilde{\phi}. \quad (5.23)$$

If we are interested only in transverse observables, we can look only at the transverse theory, that can be obtained setting $k_{\parallel} = 0$. Because every non-perpendicular term has a k_{\parallel} leg attached, we obtain the following functional for the transverse theory

$$J[\phi, \tilde{\phi}] = \lambda \int dt d^d x \tilde{\phi} [\lambda^{-1} \partial_t + \nabla^2 (\nabla^2 - \tau)] \phi + \tilde{\phi} \nabla^2 \tilde{\phi}. \quad (5.24)$$

Therefore, we note the transverse Gaussian part of the JSLC theory is the same than the transverse Gaussian part of Model B. An average is needed also on initial conditions, so the dynamical generating functional of connected functions is

$$W[h, \tilde{h}] = \int [\phi, i\tilde{\phi}] \exp \left\{ -J[\phi, \tilde{\phi}] - H_0[\phi_0] + \int_0^{\infty} dt \int d^d x h \phi + \tilde{h} \tilde{\phi} \right\} \quad (5.25)$$

This functional integral $W[h, \tilde{h}]$ can be easily calculated, expanding it around the solutions of the variational equations:

$$\begin{aligned} [-\partial_t + k^2(k^2 + \tau)] \tilde{\phi}_c(k, t) &= h(k, t) \\ [\partial_t + k^2(k^2 + \tau)] \phi_c(k, t) - 2k^2 \lambda \tilde{\phi}_c(k, t) &= \tilde{h}(k, t) \end{aligned} \quad (5.26)$$

with initial conditions

$$\begin{aligned} \tilde{\phi}(k, t = \infty) &= 0 \\ \phi(k, t = 0) - a(k) &= \tau_0^{-1} \tilde{\phi}(k, 0). \end{aligned} \quad (5.27)$$

The general solutions for these equation is

$$\tilde{\phi}_c(k, t) = \int_0^\infty dt' e^{-\lambda k^2(k^2+\tau)(t'-t)} h(k, t') \quad (5.28)$$

$$\begin{aligned} \phi_c(k, t) = & \int_0^\infty dt' \left[e^{-\lambda k^2(k^2+\tau)(t'-t)} \tilde{h}(k, t') + \right. \\ & \left. + \frac{e^{-\lambda k^2(k^2+\tau)|t'-t|} - (1 - \tau_0^{-1}(k^2 + \tau))e^{-\lambda k^2(k^2+\tau)(t'+t)}}{k^2 + \tau} \right] + \\ & + a(k)e^{-\lambda k^2(k^2+\tau)t}. \end{aligned} \quad (5.29)$$

Then we obtain

$$W[h, \tilde{h}] = \frac{1}{2} \int_0^\infty dt \int d^d x h(x, t) \phi_c(x, t) + \tilde{h}(x, t) \tilde{\phi}_c(x, t) + a(x) \delta(t) \tilde{\phi}_c(x, t). \quad (5.30)$$

Hence, the functional generator for connected functions can be expressed in terms of the propagator $G(k, t, t')$ and the correlator $C(k, t, t')$

$$G(k, t, t') = \theta(t - t') e^{\lambda k^2(k^2+\tau)(t-t')} \quad (5.31)$$

and

$$C(k, t, t') = \frac{e^{\lambda k^2(k^2+\tau)|t-t'|} - (1 - \tau_0^{-1}(k^2 + \tau))e^{\lambda k^2(k^2+\tau)(t-t')}}{k^2 + \tau}, \quad (5.32)$$

as

$$W[h, \tilde{h}] = \left(h, G\tilde{h} + \frac{1}{2}Ch \right) + \int_0^\infty dt \int d^d k h(-k, t) G(k, t, 0) a(k). \quad (5.33)$$

Here, we have introduced a short hand notation for integration over k and t . In principle, we expect the JSLC theory to derive from a coarse-graining of the master equation of a lattice model. Therefore we assume that the static structure factor \tilde{G} , intended as a measurable observable of the DLG lattice model, may be

$$\tilde{G}(k, k_{\parallel} = 0, t, t') = \langle \phi(-k, t') \phi(k, t) \rangle_{\text{lattice}} = Z C(k, t, t'), \quad (5.34)$$

where Z is a positive constant. In particular, we will be interested in the equal time dynamical structure factor, so we are interested in the equal time correlation function $C(k, t, t)$. This expression can be simplified assuming for τ_0 its fixed point value, so $\tau_0^{-1} = 0$

$$\tilde{G}(k, k_{\parallel} = 0, t) = Z \frac{1 - e^{-2\lambda k^2(k^2 + \tau)t}}{k^2 + \tau}. \quad (5.35)$$

We note that this equation, for large t , reduces to the well-known transverse structure factor, used in Section 3.3:

$$\tilde{G}(k, k_{\parallel} = 0) = Z \frac{1}{k^2 + \tau}. \quad (5.36)$$

Of course, on a finite lattice with periodic boundary conditions, the momentum k is quantized, so we have $k = \frac{2\pi}{L}n$, with n integer. Hence

$$\tilde{G}\left(\frac{2\pi}{L}n, k_{\parallel} = 0, t\right) = Z L^2 \frac{1 - e^{-2\lambda 4\pi^2 n^2(4\pi^2 n^2 + L^2 \tau) \frac{t}{L^4}}}{4\pi^2 n^2 + \tau L^2}. \quad (5.37)$$

Therefore, in the short-time regime we have

$$\tilde{G}\left(\frac{2\pi}{L}n, k_{\parallel} = 0, t\right) \sim 2\lambda Z \frac{t}{L^2} 4\pi^2 n^2, \quad (5.38)$$

which turns out to be independent from τ .

The susceptibility, being the first transversal mode of the static structure factor is

$$\chi = \tilde{G}\left(\frac{2\pi}{L}, k_{\parallel} = 0, t\right) \sim 2\lambda Z \frac{t}{L^2} 4\pi^2. \quad (5.39)$$

Therefore, using equation (3.29) we have

$$m^2 = \frac{\pi}{4} \frac{\chi}{L_{\parallel} L_{\perp}} = 2\lambda Z \frac{\pi^3}{L_{\parallel} L_{\perp}^3} t \quad (5.40)$$

which is exactly the scaling observed for this order parameter, as shown in the previous Section.

Conclusions

The aim of this work was the study of the critical properties of the Driven Lattice Gas. In particular we focused on the dynamical properties of the critical relaxation processes.

We studied the short-time behaviour of the order parameter m and OP , defined in (1.14) (1.15). The main virtue of this method is that short-time MC simulations do not suffer the problem of critical slowing down, and thus using this method shorter simulations are necessary. [25]. Our results extend previous studies of these quantities [28] [30]; in particular we found two scaling laws for these order parameters valid for lattices of any geometries:

$$m(t, L_{\parallel}, L_{\perp}) \sim \sqrt{\frac{t}{L_{\parallel} L_{\perp}^3}} \quad \text{and} \quad OP(t, L_{\parallel}) \sim L_{\parallel}^{-1/2} t^{1/8}.$$

Furthermore, all the results we found concerning the short-time behaviour of these order parameters turn out to be valid both for the IDLG and the RDLG lattice models.

Even though other authors [28] [30] state that they were able to discern which is the correct field theory that describes the critical behaviour of the IDLG, we point out that unfortunately these results aren't strong enough to distinguish which one is the correct theory. Furthermore, we found that the first of the previous equations, describes the critical behaviour of the order parameter m also for an Ising lattice gas with Kawasaki dynamics, proving to be very general. Hence, in Section 5.2 we showed how this equation can be derived in the framework of JSLC theory, or more generally in the framework of any theory which is Gaussian in its transverse part. This may be a hint of

the fact that, in the short-time limit in $2d$, the critical behaviour is the same for all these universality classes, regardless of higher order couplings.

The present results and methods could be extended and developed in several directions. In particular, for systems in $d > 2$ dimensions, the methods outlined in Section 5.1 are expected to give different predictions, using the JSLC theory rather than the RDLG theory. In [32] Lee and Okabe performed a similar analysis using the Binder parameter instead of the order parameter. This allows the determination of the dynamic critical exponent z alone, hence it may be an additional independent test for these scaling hypothesis.

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Riassunto

La Meccanica Statistica fornisce un insieme di metodi per collegare le proprietà dei singoli atomi e molecole alle proprietà dei sistemi macroscopici che essi compongono. Ad esempio, la Meccanica Statistica afferma che la distribuzione di probabilità $P(S)$ di uno stato macroscopico S all'equilibrio con un termostato a temperatura T , è data, in termini della Hamiltoniana microscopica $\mathcal{H}(S)$ del sistema, da $P(S) \propto e^{-\frac{1}{k_B T} \mathcal{H}(S)}$.

Per i sistemi fuori equilibrio un approccio generale, simile a quello della Meccanica Statistica di equilibrio, non è ancora disponibile. Data la mancanza di un contesto generale in cui studiare tali sistemi, un approccio ragionevole è quello di studiare modelli che siano il più semplici possibili. Di particolare interesse sono quei sistemi soggetti ad una perturbazione esterna, che non permette al sistema di equilibrarsi con l'ambiente circostante. Spesso, questi sistemi evolvono verso uno stato stazionario di non equilibrio, la cui distribuzione di probabilità è diversa da quella di Gibbs e solo in pochissimi casi è calcolabile esattamente. Il Driven Lattice Gas (DLG), essendo una semplice generalizzazione del modello di Ising, è diventato uno dei modelli più studiati in questo ambito, sia attraverso simulazioni numeriche che attraverso teorie di campo. Esso consiste in un ordinario gas su reticolo di Ising con un campo forzante uniforme, il quale modifica il tasso dei salti delle particelle lungo una determinata direzione. Caratteristica molto interessante di tale modello è che esso mostra una transizione di fase ad una certa temperatura finita T_c , sopra la quale il sistema appare disordinato e sotto la quale

invece il sistema si segrega in due fasi particelle/vuoto. Dopo molti anni dalla sua introduzione, quale sia la classe di universalità del DLG è tuttora una questione aperta. Janssen, Schmittmann, Leung e Cardy (JSLC) hanno sviluppato una teoria di campo che cattura le caratteristiche principali della transizione di fase del DLG. Siccome però rimangono delle discrepanze tra gli esponenti critici misurati dalle simulazioni numeriche e quelli predetti dalla teoria JSLC, Garrido, de los Santos e Muñoz hanno previsto che il DLG nel limite di forzante infinita non sia correttamente descritto dalla teoria JSLC, ma da un'altra teoria (RDLG).

Un aspetto molto interessante dei sistemi statistici sul punto critico è che essi mostrano un'invarianza di scala anche durante il processo di rilassamento verso lo stato stazionario. Sorprendentemente, questo comportamento è apprezzabile anche nei primi istanti dell'evoluzione, quando la lunghezza di correlazione del sistema ξ è ancora molto piccola. Questo fenomeno è noto in letteratura sotto il nome di short-time scaling. Il principale vantaggio del short-time scaling è che permette di ottenere informazioni sugli esponenti critici di un modello attraverso simulazioni molto più brevi.

In questa tesi sono state analizzate diverse ipotesi di scaling relative all'andamento di diversi parametri d'ordine nei primi istanti di evoluzione per diverse varianti del DLG: per mezzo di una simulazione Monte Carlo, il sistema viene inizializzato casualmente e successivamente lasciato rilassare verso lo stato stazionario di non equilibrio. Questa procedura, ripetuta molte volte, permette di ricavare l'andamento medio di diverse osservabili del sistema. L'invarianza di scala che mostra quest'andamento permette di ricavare informazioni sugli esponenti critici del modello.