

UNIVERSITÀ DEGLI STUDI DI MILANO

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FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI  
Corso di Laurea Magistrale in Fisica

COMPOSITE BOSON DOMINANCE  
IN RELATIVISTIC FIELD THEORIES AT FINITE  
TEMPERATURE AND CHEMICAL POTENTIAL

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# Riassunto

All'interno del lavoro di tesi si discute l'estensione di un metodo di bosonizzazione a teorie di campo fermioniche relativistiche in  $3 + 1$  dimensioni, a temperatura ( $T$ ) e potenziale chimico ( $\mu$ ) finiti. Questo metodo è stato sviluppato inizialmente per valori nulli dei suddetti parametri [1]. Si analizza anche la possibilità di sostituire un'azione che genera stati legati con una, fisicamente equivalente, in cui i campi associati alle variabili elementari e quelli associati agli stati legati stessi possano essere trattati in identico modo. Un'indagine in questa direzione è di particolare interesse, in quanto, come è noto, esistono sistemi, in cui alcuni gradi di libertà elementari sono fermionici, che, almeno per certe scale di energia, sono più efficacemente descritti tramite gradi di libertà bosonici. Ne sono esempi tutti in sistemi che presentano una rottura di simmetria continua, infatti il parametro d'ordine è sempre una variabile bosonica. Si osservi, inoltre, che questi bosoni risultano essere composti da fermioni. Esempi fisicamente rilevanti sono la superconduttività nei metalli, la dominanza dei mesoni chirali in cromodinamica quantistica (QCD) e il fenomeno della superconduttività di colore in QCD ad alta densità adronica.

La tesi è strutturata come segue. Nel primo capitolo si introduce il contesto in cui si inserisce il lavoro di tesi e si fornisce una breve descrizione di alcuni degli strumenti necessari alla trattazione che segue. Il primo concetto utile che viene introdotto è quello di bosonizzazione. Tale procedura consiste di un cambio di variabili nell'integrale funzionale che esprime la funzione di partizione, in cui le nuove variabili sono di carattere bosonico. Di maggiore interesse è la bosonizzazione di sistemi fermionici. Per i motivi espressi sopra, la bosonizzazione è uno strumento particolarmente efficace in quanto consente di applicare a sistemi fermionici alcuni dei metodi sviluppati per sistemi con gradi di libertà bosonici. In particolare, la possibilità di sostituire, nelle simulazioni numeriche, variabili bosoniche a variabili fermioniche è di grande interesse. In seguito, viene presentata una breve introduzione alle teorie di gauge su reticolo, soffermandosi sulle difficoltà che sorgono introducendo dei fermioni sul reticolo stesso. Tali problemi sono parzialmente risolti con l'utilizzo dei fermioni di Kogut-Susskind. Si introduce, quindi, il formalismo operatoriale della matrice di trasferimento su reticolo e il formalismo degli stati coerenti. Nel formalismo operatoriale, la parte fermionica della funzione di partizione viene scritta come la traccia di una potenza  $L_0$  della matrice di

trasferimento. La temperatura compare nella funzione di partizione tramite la relazione  $T^{-1} = L_0$ . In analogia con la meccanica statistica, si introduce il potenziale chimico nel formalismo come il moltiplicatore di Lagrange associato al numero di particelle. Calcolando la traccia sulla base degli stati coerenti, si ottiene la rappresentazione funzionale della funzione di partizione gran-canonica. In questo modo, il potenziale chimico si accoppia esponenzialmente con i campi fermionici e non risultano essere necessari ulteriori controtermini. Infine, vengono descritti in breve il diagramma di fase della QCD e le problematiche legate al suo studio.

La bosonizzazione è un argomento di grande interesse di ricerca. Le procedure di bosonizzazione per teorie relativistiche in  $(1 + 1)$  dimensioni sono note e hanno portato alla scoperta di risultati esatti [4]. Per quanto riguarda, invece, le tecniche di bosonizzazione in dimensioni maggiori, non si conosce, ad oggi, una tecnica che si possa dire totalmente soddisfacente. In particolare, nei metodi noti, il campo bosonico ha sempre numero fermionico 0. I punti fondamentali che una tecnica di bosonizzazione deve affrontare e risolvere sono due. In primo luogo, come poter introdurre i campi bosonici nell'azione e, successivamente, come ottenerne l'azione effettiva. Nel secondo capitolo si presenta un particolare metodo di bosonizzazione per teorie Abelianie in dimensione  $(d + 1)$ , con  $d > 1$  [21]. In questo metodo, il campo bosonico è un potenziale di Kalb-Ramond antisimmetrico di rango  $(d - 1)$  ed è introdotto tramite un opportuno cambio di variabili nell'integrale funzionale. L'integrale sulle variabili deve essere calcolato in modo esplicito, tuttavia non esiste un metodo generale per valutare il determinante fermionico, come, ad esempio, si verifica per le teorie di gauge. Esiste, inoltre, un secondo metodo di bosonizzazione, che fa uso della trasformata di Hubbard-Stratonovich.

Recentemente, è stato sviluppato un nuovo metodo di bosonizzazione, il metodo ampliato e sviluppato nel lavoro di tesi, in cui la funzione di partizione è scritta nel formalismo operatoriale e in cui si utilizza una regolarizzazione su reticolo. In questo modo, si possono individuare i gradi di libertà dominanti, di tipo bosonico, come sovrapposizione di coppie di quark e antiquark, il cui operatore di creazione è della forma  $\hat{\Phi}^\dagger := \hat{u}^\dagger \Phi^\dagger \hat{v}^\dagger$ , dove  $\hat{u}$  e  $\hat{v}$  sono gli operatori di quark ed antiquark<sup>1</sup>. La funzione  $\Phi$  è la funzione di struttura dei bosoni e questi operatori non costituiscono un'algebra canonica. L'assunzione fisica che i gradi di libertà dominanti siano di carattere bosonico viene implementata valutando la traccia solo sugli stati di bosone composto. L'azione efficace, per i campi di bosone composto, è ottenuta facendo ricorso ad un'unica approssimazione. Infatti, è necessario, al fine di determinare l'azione, esprimere il proiettore sugli stati dominanti tramite stati (semi-)coerenti dei bosoni composti, della forma

$$|\phi\rangle := \exp\left(\hat{u}^\dagger(\phi \cdot \Phi^\dagger)\hat{v}^\dagger\right)|0\rangle. \quad (1)$$

Questa approssimazione è tanto migliore, tanto maggiore è l'indice di nilpotenza di  $\hat{\Phi}^\dagger$ .

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<sup>1</sup>Gli indici di posizione e il numero quantico sono sotto intesi, sia per i fermioni, che per gli operatori composti. Anche la sommatoria sui numeri fermionici è implicita.

Questo approccio consente di utilizzare metodi variazionali. Valutando la funzione di partizione tramite il metodo di punto sella, si ottiene, a livello semi-classico, lo stato fondamentale come in (1), dove  $(\phi \cdot \Phi^\dagger)$  assume il valore  $(\overline{\phi \cdot \Phi^\dagger})$ , soluzione dell'equazione di punto sella<sup>2</sup>. Si determina anche la funzione di struttura del mesone chirale. Nel caso in cui la matrice di trasferimento sia indipendente dal tempo, si conoscono le soluzioni delle equazioni di punto sella e queste sono riportate nella tesi. Questo metodo è stato testato su un modello con interazione a quattro campi e si è trovato un pieno accordo con i risultati noti per il settore bosonico.

Al fine di estendere tale formalismo a temperatura e potenziale chimico finiti, è necessario prendere in considerazione stati con numero fermionico non nullo ed eccitazioni dello stato fondamentale. Questo risultato può essere raggiunto introducendo stati di composto barionico; tuttavia ci siamo limitati a considerare stati fermionici. Allo scopo, si è eseguita una trasformazione di Bogoliubov-Valatin generalizzata sul doppietto degli operatori dei quarks

$$\hat{\psi} \mapsto \hat{\lambda} := U \hat{\psi} \quad \text{dove} \quad \psi := \begin{pmatrix} \hat{u} \\ \hat{v}^\dagger \end{pmatrix}. \quad (2)$$

La condizione di unitarietà per  $U$  è essenziale. Dal doppietto  $\hat{\lambda}$ , è stato possibile estrarre gli operatori di quark  $\hat{a}$  e antiquark  $\hat{\beta}$ . Si è determinata la forma del generatore  $\hat{\mathcal{W}}$  della trasformazione, che risulta essere una composizione di una rotazione nello spazio dei singoli operatori di particella con una rotazione dei doppietti. Si osservi che la procedura ha alcune similitudini con la teoria della superconduttività [26]. Si è determinato lo stato di vuoto della teoria nella nuova base degli operatori, che risulta dipendere dai parametri della rotazione. La forma di tale stato, a meno di costanti moltiplicative, è come quella indicata in Eq. (1), sostituendo a  $\phi \cdot \Phi^\dagger$  una funzione dei parametri della trasformata, che abbiamo determinato. In questa nuova base, si è trovata l'azione effettiva dei campi associati alle quasi particelle, valutando la traccia sugli stati coerenti delle quasi particelle stesse. La derivazione, esatta, dell'azione è stata svolta senza alcuna assunzione e senza fissare il gauge. Tale condizione è necessaria per effettuare efficaci simulazioni numeriche. Abbiamo ottenuto due diverse scritture per l'azione, entrambe gauge invarianti, le cui espressioni sono riportate all'interno della tesi. In entrambe le azioni compaiono i termini cinetici per i quark e per i antiquark, diagonali nello spazio dei doppietti; sono presenti inoltre termini di interazione tra quark e antiquark. La presenza di questi ultimi termini è dovuta al fatto che la matrice di trasferimento, ovviamente, accoppia i quark con i antiquark. I coefficienti di interazione sono dipendenti dai parametri della trasformata. Abbiamo verificato che, nel caso in cui la matrice di trasferimento sia indipendente dal tempo, l'azione non dipende dal parametro della trasformata e il suo valore coincide con il valore esatto. Abbiamo

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<sup>2</sup>In generale la matrice di trasferimento dipende dal tempo, quindi ne dipendono anche l'equazione di punto sella e la sua soluzione.

quindi osservato che, per determinati valori dei parametri della trasformata di Bogoliubov, l'azione si diagonalizza nelle componenti di quasicquarks e di quasiantiquarks. Si è osservato che, per gli stessi valori della trasformata, lo stato fondamentale, nella nuova base, risulta essere lo stato fondamentale ottenuto con il metodo variazionale. Quindi, le eccitazioni di quasicquarks e di quasiantiquarks dello stato fondamentale sono non interagenti. La determinazione dei parametri della trasformata che diagonalizza l'azione richiede la soluzione dell'equazione di punto sella. Nel caso in cui la matrice di trasferimento dipenda dal tempo, ne dipendono anche i parametri. In questo caso, la diagonalizzazione richiede quindi di effettuare un cambiamento di base ad ogni passo temporale. Quindi, si può interpretare il parametro della trasformata come il campo associato al mesone. Il metodo sviluppato è stato testato su una teoria con interazione a quattro campi fermionici e si sono trovati discreti accordi con i risultati presenti in letteratura [31, 32].

Nell'ultima parte dello studio, si sono introdotti nella funzione di partizione, insieme ai mesoni e trascurando le particelle fermioniche, anche stati di bosone composto con numero fermionico 2, i diquark. Tale scelta è stata considerata al fine di estendere il formalismo sviluppato anche alla trattazione di barioni, che possono essere interpretati, almeno alcuni, come un'adatta sovrapposizione di un diquark ed un antiquark. Questi stati devono essere introdotti in maniera tale da proibire il doppio conteggio degli stati stessi. Quindi, i diquark devono essere ortogonali agli stati di mesone e viceversa. La condizione di mutua ortogonalità trova soluzione definendo gli operatori di diquark come segue

$$\hat{B}^\dagger := \hat{\alpha}^\dagger B \hat{\alpha}^\dagger; \quad (3)$$

dove  $\alpha$  indica gli operatori, trasformati secondo (2), a partire dagli operatori  $\hat{u}$ . Come nel caso precedentemente discusso, la determinazione dell'azione efficace richiede l'approssimazione del proiettore sul sottospazio dei mesoni e dei diquark, tramite l'utilizzo di stati coerenti. In questo caso, tuttavia, sorgono alcune complicazioni dovute alla presenza di due tipi di particelle composte e alle regole di commutazione cui soddisfano. Si è derivata l'azione efficace per i campi associati ai due tipi di particelle. Come nel caso dei quasicquark, abbiamo trovato due espressioni dell'azione, che sono riportate nella tesi. Nell'azione efficace compaiono solo variabili olomorfe e, per grandi valori dell'indice di nilpotenza, l'azione è proporzionale all'indice stesso. Quindi, il metodo di punto sella è il primo metodo utile di analisi. Nella tesi sono riportate le equazioni di punto sella ed una discussione delle loro possibili soluzioni.

Nell'ultimo capitolo sono indicati i possibili sviluppi del lavoro di tesi. Innanzitutto, per quanto riguarda lo studio di teorie prive della simmetria di gauge, è possibile eseguire uno sviluppo in potenze di  $1/N_f$  dell'azione dei mesoni e dei diquark. Riteniamo che tale sviluppo possa essere svolto sulla linea di [9], analizzando anche gli aspetti più sottili dello sviluppo come in [33]. Inoltre, l'azione dei quasicquark può essere

ulteriormente analizzata in assenza di campi di gauge, approfondendone le analogie con la teoria della superconduttività. Al fine di analizzare il diagramma di fase della QCD, è interessante studiare le soluzioni delle equazioni di punto sella per l'azione dei mesoni e dei diquark. In seguito, si potrebbero introdurre anche le eccitazioni fermioniche, in aggiunta ai mesoni e ai diquark, e studiarne l'azione efficace. In effetti, tale azione è già stata ottenuta, ma rimane ancora da capire come risolvere la condizione di mutua ortogonalità in presenza di mesoni, diquark e fermioni.

Nello studio delle teorie di gauge la nostra formulazione è particolarmente interessante, in quanto, se si trascurano le eccitazioni fermioniche, permette di applicare metodi variazionali nella determinazione dello stato fondamentale. Chiaramente, la soluzione di queste equazioni può essere trovata solo numericamente. Come indicato in precedenza, una volta ricavata la soluzione per la funzione di struttura del mesone nel suo stato fondamentale, abbiamo diagonalizzato l'azione nei campi fermionici. Ci aspettiamo che questo risultato sia rilevante nello studio a temperatura finita. Infine, abbiamo ragione di credere che questa formulazione costituisca un buon punto di partenza per l'analisi tramite simulazioni numeriche [2, 24].







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# Chapter 1

## Introduction

In this work, we discuss an extension of a new method of bosonization [1, 2] to relativistic field theories at finite chemical potential and temperature. Moreover, we test the formulation we obtain in a Gross-Neveu model. In the sequel, we introduce composite of di-quarks in order to extend our formalism to composite baryonic like and to the study of the phase diagram for large values of the chemical potential.

There exist many systems, with fermionic degrees of freedom, that can be described in a more suitable and easier way, at least for certain energy scales, by means of bosonic degrees of freedom. Namely, in a region of the space of the parameters, the excitations of the system can be described using bosonic degrees of freedom. This is indeed the case in presence of spontaneous breaking of a global symmetry, where the low energy excitations of the symmetry are the Goldstone bosons. Typical examples in non-relativistic systems are the Cooper pairs in metals and the pairing in nuclei with partially filled shells. In the context of relativistic field theories, relevant examples are the vector dominance in strong-electromagnetic interactions, the dominance of chiral mesons in QCD and, in high density QCD, the so called color superconductivity [3]. Notice that in all these systems the elementary particles *couple* in pairs and the result is a *composite* boson, which appears as an effective degree of freedom. These bosons can have fermion number 0 or 2. Therefore, we expect to be able to rewrite every theory that presents the phenomena described above, in terms of another less fundamental one, with dominant bosonic degrees of freedom. Obviously, the latter theory is useful when its analytical studies are less complex than the ones of the original system and when it allows to implement operative numerical simulations.

Usually, in modern physics, a quantum system is described by its partition function

$$Z = \int D[\phi] e^{-S[\phi]}, \quad (1.1)$$

where  $\phi$  denotes all the fields associated to the elementary degrees of freedom of the system,  $S[\phi]$  is the Euclidean action and  $D[\phi]$  is the integral measure. If one considers

a system at finite temperature  $T$ , the integral over the imaginary time in the action runs from 0 to  $1/T$ . To obtain an effective theory, one has to find an approximation  $Z'$  of  $Z$  in (1.1) in terms of the fields  $\phi_{eff}$ ; these fields are associated to the degrees of freedom whose dynamics is governed by an effective action  $S_{eff}[\phi]$ . Usually, this procedure can be performed only in a particular region of the parameters of the theory. An effective theory when the  $\phi_{eff}$  fields are boson-like is particularly successful. The effective theory in terms of boson fields is remarkably operative even in a computational context; in fact, the numerical simulations with anti-commuting variables are usually hard to implement.

### Bosonization

Bosonization consists of a change of variables in the functional integral (1.1), where the new integration variables are boson-like. Usually, the "old" variables are fermion-like. In the cases where bosonization can be applied and when one is able to perform it, the fermion system is replaced by another one, of bosons, which has a completely equivalent physical content, including identical spectra. This procedure is extremely powerful because it permits to re-express the theory in terms of physical and relevant degrees of freedom and to find connections between different theories. For some systems, bosonization can be performed in an exact way and allows to integrate the partition function of the system. For instance, in  $1+1$  dimensions, bosonization has been widely studied and analyzed and gives exact results (see, for example, the equivalence between the Sine-Gordon model and a two fermions model with current-current interaction). However, this is a particular feature of theories in  $1+1$  dimensions and it is well known that it is, in general, not shared by theories in a larger number of dimensions. In fact, bosonization in higher dimensions requires, in general, some kind of approximation.

Since the thesis work under discussion is intended to the study of a four dimensional system, we do not discuss in this introduction, the special features of  $1+1$  dimensional systems. A detailed treatment of this argument is given in [4, 5, 6].

Consider, for the moment, bosonization in non-relativistic theories. One of the first organic approach to bosonization is due to Bogoliubov [7]. In this work, starting from the Fröhlich Hamiltonian of a number of electrons interacting with phonons, the properties of the ground states are derived. Then, the Cooper pairs are mapped into effective bosons whose dynamics is described by the Hamiltonian of a superfluid system of elementary bosons. Notice that, in this theory, the presence of external bosons, the excitations of the lattice, is essential because they produce an attractive force between the electrons.

To the scopes of the thesis work, a key role is played by a recently developed method of bosonization [8, 9]. This procedure can be used when the composite bosons have fermionic number 0 or 2. The starting point is the partition function in its operator

form, i.e. the trace of the transfer matrix in the Fock space of the fermions. The physical assumption of boson dominance is then implemented by restricting the trace to fermion composites. This requires an approximation of the projection operator in the subspace of the composites; the approximation being the better, the higher the number of fermion states in the composites. The approximate projection operator is built in terms of coherent states of composites and the evaluation of the trace, which is done exactly, generates a bosonic action in terms of the holomorphic variables appearing in the coherent states. The approach shares two features with a variational method: the reduction of the starting full space, here the fermions Fock space, to a subspace, that of the composites, and a variational procedure to determine the structure functions. This particular bosonization method has been completely developed for non-relativistic many-body systems and checked on the BCS model of superconductivity and on the pairing model of finite systems, like atomic nuclei and ultra-small superconducting grains. The properties of these systems are indeed exactly reproduced.

Since the thesis studies deal with relativistic theories, we do not go further in deepening the discussion about non-relativistic systems. Furthermore, non-relativistic methods can hardly be extended to treat relativistic field theories; it is, in general, necessary to introduce new techniques. Other approaches to bosonization in non-relativistic systems can be found in [10] and references therein. In Chapter 2, we give a brief introduction to bosonization of relativistic theories.

### **Quantum field theories on a lattice**

Quantum Chromodynamics is a remarkable theory. It is a convincing practical example of the triumph of the quantum field theory. Asymptotic freedom allows QCD to be consistent down to arbitrary short distance scale, enabling us to define the theory completely in terms of the fundamental microscopic degrees of freedom (quarks and gluons). This fundamental definition is very simple, yet the theory describes a wide range of phenomena, from the mass spectrum of hadrons to deep-inelastic scattering. As such, QCD should also possess well defined thermodynamic properties. Analytically, it is very hard to obtain any result and the perturbative approach can be performed only in certain kinematic regions, for example in the study of heavy ions collisions, or in the study of the thermodynamics for large values of the chemical potential and temperature. The above remarks make, at least in principle, lattice approaches, that do not rely on small parameter expansions, an invaluable and the most powerful tool in studying QCD and, in general, non Abelian gauge theory. In fact, lattice gauge theory permits to perform analytical studies for strong coupling [11] and should allow us to extract expectation values from numerical simulations. A detailed discussion about lat-

tice theory can be found in [12, 13]<sup>1</sup>. Over the Euclidean lattice, the partition function for vanishing chemical potential is given by

$$Z := \int D[U]D[\bar{\psi}]D[\psi]e^{-\beta S_G[U]-S_F[U,\bar{\psi},\psi]} = \int D[U]e^{-\beta S_G[U]}\det M[U], \quad (1.2)$$

where  $-S_F[U,\bar{\psi},\psi] = (\bar{\psi}M[U])$ ,  $\beta \propto g^{-2}$ ,  $S_G[U]$  is the pure gauge Wilson action with  $U_{(x,\mu)}$  being the variable associated to the gauge field on the link  $(x, x + \mu)$ ,  $\psi$  is the fermion field and the time direction of the system runs between 0 and  $1/T$ .

However, a price is to be paid in exchange of such advantages; the introduction of fermions over the lattice gives rise to the well known difficulties of the fermions doubling and the breaking of the chiral invariance. Since the Dirac equation contains only derivatives of the first order, a naive regularization on a lattice gives for the propagator of a Dirac field with mass  $m$

$$\Delta_k^{-1} = m + i \sum_{\mu} \gamma_{\mu} \sin p_{\mu}. \quad (1.3)$$

In the limit of zero lattice spacing, the previous equation has 16 poles, corresponding to 16 propagating particles, one for any corner off the Brillouin zone. This phenomenon is called of the fermion doubling: the number of fermions doubles once for any space-time dimension. In a free theory, the doubling problem can be dealt with, but, in a theory with interactions, the extra fermions<sup>2</sup> can be introduced in pairings and change the physical content in the continuum limit. In order to have physically meaningful interacting theories with fermions on lattice, it is thus necessary to avoid the fermion doubling. This scope can be achieved in different ways; the most important approaches are called of the Wilson's and Kogut-Susskind's fermions. In the Wilson's method, the naive action is modified by adding a term proportional to a second order derivative

$$S_{FN} := a^4 \sum_x \bar{\psi}_x m \psi_x + \frac{1}{2a} \sum_{\mu=\pm 1}^{\pm 4} \bar{\psi}_{x+\mu} \gamma_{\mu} \psi_x \mapsto \\ a^4 \sum_x \left[ \bar{\psi}_x \left( m + \frac{4r}{a} \right) \psi_x + \frac{1}{2a} \sum_{\mu=\pm 1}^{\pm 4} \bar{\psi}_{x+\mu} (\gamma_{\mu} + r) \psi_x \right] =: S_{FW}. \quad (1.4)$$

With this action, the propagator is different with respect to the one appearing in (1.3); one observes that the masses of the 15 extra fermions have an additional term proportional to  $ra^{-1}$ . They become infinitely heavy in the continuous limit and finally disappear. The continuum limit of (1.4), carried out with the appropriate tuning of the parameters, corresponds to the Dirac action for  $0 < r \leq 1$  with only one fermion.

<sup>1</sup>In this references, it is possible to find an introduction to all the arguments described in this section. References about the original papers can be found in the textbook.

<sup>2</sup>The Brillouin zone is the area in the momentum space whit  $0 \leq p_{\mu} < 2\pi$ ; the extra fermions correspond to a pole distinct from the origin.



Another possibility, to face the fermion doubling, is to interpret the extra fermions as fermions with different *flavours*; this idea was first introduced by Kogut and Susskind. In order to obtain the action of the Kogut-Susskind fermions, it is first necessary to perform a local change over the fermion fields, in a way such as to obtain the action in a spin diagonal base. In this base one has 4 staggered fermions and, since the action is diagonal in the Dirac index, it is possible to introduce an arbitrary number  $N_f$  of staggered fermions. Once this expression is computed, it is a very nice exercise to assemble all the degrees of freedom, corresponding to each one of the staggered fermions, and to obtain a fermion  $\psi_{x,I}$ , with a new quantum number, namely the taste, that lives on the even sites of the lattice, see appendix B.3. For  $N_f$  staggered fermions, one obtains  $N_f$  Dirac-taste fermions, called the Kogut-Susskind fermions. In the continuum limit they describe a theory with  $4N_f$  *flavours*. The Kogut-Susskind pure fermion gauge invariant action, in the flavour base, is

$$S_{FKS} = \sum'_{x,y} a^4 \bar{\psi}_x \left[ m \delta_{x,y} \mathbf{I} \otimes \mathbf{I} + \sum_{\mu} \gamma_{\mu} \otimes \mathbf{I} \left( \nabla_{\mu}^{(-)} \mathbf{P}_{\mu}^{(+)} - \nabla_{\mu}^{(+)} \mathbf{P}_{\mu}^{(-)} \right) \right] \psi_y \quad (1.5a)$$

$$= \sum'_{x,y} a^4 \bar{\psi}_x \left[ m \delta_{x,y} \mathbf{I} \otimes \mathbf{I} + 2 \sum_{\mu} (\gamma_{\mu} \otimes \mathbf{I} \Delta_{\mu} - 2a \gamma_5 \otimes t_5 t_{\mu} \Delta_{\mu}^2) \right] \psi. \quad (1.5b)$$

Above, all the indices are omitted, included the color index, and we have defined

$$\nabla_{\mu x,y}^{(\pm)} = \frac{\pm 1}{2a} (U_{\pm \mu x} \mathbf{T}_{\mu x,y}^{(\pm)} - \delta_{x,y}), \quad \mathbf{P}_{\mu}^{(\pm)} = \frac{1}{2} (\mathbf{I} \otimes \mathbf{I} \pm \gamma_{\mu} \gamma_5 \otimes t_{\mu} t_5), \quad (1.6a)$$

$$\Delta_{\mu} := \frac{1}{4a} (U_{\mu x}(t) \mathbf{T}_{\mu}^{(+)} - U_{-\mu x}(t)^{\dagger} \mathbf{T}_{\mu}^{(-)}), \quad \mathbf{T}_{\mu x,y}^{(\pm)} := \delta_{x \pm 2\mu, y}. \quad (1.6b)$$

Here  $U_{\pm \mu x}$  is the gauge field associated to the link  $x, x \pm 2\mu$ . Usually, for numerical simulations, one chooses a different normalization of the field. In the following, we always use the action expression (1.5a). From the other form (1.5b), one can observe that the term proportional to  $\Delta_{\mu}^2$  is a lattice artifact that disappears in the continuum limit. Since the fermion fields live on a lattice of double size, with respect to the naive formulation, the Brillouin zone is halved and there is no doubling.

The Wilson's fermions do not have chiral symmetry for any value of the mass. The Kogut-Susskind action, instead, for  $m = 0$ , is symmetrical under the following transformations of the field

$$\psi \mapsto e^{i\theta_5 \gamma_5 \otimes t_5 + i\theta} \psi, \quad \psi \mapsto -\gamma_5 \otimes \mathbf{I} \psi, \quad (1.7)$$

for any real  $\theta_5$  and  $\theta$ . Eq. (1.5) exhibits an other symmetry which we discuss for  $N_f = 1$ , but that can be generalized to any value of  $N_f$ . In fact

$$\psi \mapsto (U_o(1) P_o + U_e(1) P_e) \psi, \quad (1.8)$$

with  $P_{e/o} = \frac{1}{2} (\mathbf{I} \otimes \mathbf{I} \pm \gamma_5 \otimes t_5)$ , and  $U_{o/e}(n)$  are elements of  $U(n)$ , (1.8) leaves (1.5) invariant for  $m = 0$ . This is a lattice version of the chiral symmetry. In the continuum limit, the symmetry (1.8)  $U_o(1) \otimes U_e(1)$  becomes  $U_o(4) \otimes U_e(4)$ .

### Transfer matrix formalism

Usually, the study of the thermodynamical and dynamical properties of a quantum field theory is performed by means of the functional formalism where the partition function is written as in Eq. (1.1). There exists, however, another formalism, namely the operatorial or Hamiltonian formalism; in this case, the partition function, for a gauge theory on a lattice at temperature  $T$ , is

$$Z = \text{Tr} \prod_t^{1/T} \hat{\mathcal{T}}_t. \quad (1.9)$$

Here  $\text{Tr}$  is the trace over the Fock space and  $\hat{\mathcal{T}}_t$  is a selfadjoint operator, called the transfer matrix, that generates the evolution of the state from an imaginary time  $t$  to the time  $t + 1$ . Notice that  $\hat{\mathcal{T}}_t$  is the corresponding of  $e^{-a\hat{H}}$ . A selfadjoint and strictly positive  $\hat{\mathcal{T}}$  assures a selfadjoint and limited from below Hamiltonian defined as  $\hat{H} = \ln \hat{\mathcal{T}}$ . In general, given the transfer matrix of the theory, it is possible to rewrite the partition function in the functional formalism. This can be done by use of a standard procedure [14, 15] that, in short, consists of introducing the identity of the Fock space, written in terms of coherent states for each time slice, and of evaluating the trace in (1.9) over these states. This procedure allows to obtain the corresponding functional expression of  $Z$ . The fields appearing in the functional form come from the variables<sup>3</sup> in the coherent states. However, the inverse is not guaranteed. For example, we can deduce easily the action (1.4) for  $r = 0, 1$  from an Hamiltonian, but this is not trivial or, in general, possible for any value of  $r$ . The existence of the transfer matrix is necessary to prove the physical positivity that is required to obtain an acceptable quantum field theory, i.e. with a selfadjoint Hamiltonian<sup>4</sup>. Normally, for bosonic theories, the existence of the transfer matrix is quite obvious, but for fermionic ones the situation is different. The procedure to construct the transfer matrix of a gauge theory can be summarized in the following way. Firstly, one has to identify the components of the fermion fields  $\psi$  which can be related to fermions  $\psi^{(-)}$  and antifermions  $\psi^{(+)}$ . They propagate, respectively, forwards and backwards in time and are interchanged by charge conjugation. Then, one is to introduce the projection operators over fermions and antifermions  $P_0^{(\mp)}$ ; such that  $P_0^{(-)}\psi = \psi^{(-)}$  and  $P_0^{(+)}\psi = \psi^{(+)}$ . In the sequel, one defines the Hilbert space of the system as the tensor product of the Hilbert space of the fermions ( $\mathcal{H}_F$ ) with the one of the gauge bosons ( $\mathcal{H}_G$ ). The operator associated to the gauge field  $\hat{U}(\mathbf{x}, \mu)$  acts as a multiplication operator of the function of the gauge configuration. Afterwards, one introduces two independent algebras  $\hat{u}_i, \hat{u}_i^\dagger$  and  $\hat{v}_i, \hat{v}_i^\dagger$ , that satisfy anti-canonical

<sup>3</sup>These variables are Holomorphic/Grassmann ones, depending if they are associated to bosons or fermions.

<sup>4</sup>Even if it is clear that the transfer matrix is very similar to  $e^{-a\hat{H}}$ , in general it is difficult to obtain the Hamiltonian.

commutation relations and are associated, respectively, to fermions and antifermions. The index  $i = (\mathbf{x}, I)$  represents all the quantum numbers of the particles, i.e. position  $\mathbf{x}$  and internal quantum numbers  $I$ . The next step is defining an operator  $\hat{\mathcal{T}}$  that is selfadjoint, strictly positive and that, following the standard procedure described above [14, 15], must reproduce the partition function of the theory (1.2). If all the previous steps can be performed, the action describes a consistent quantum field theory and the reflection positivity is automatically verified. This procedure has already been carried out for Wilson's fermions in the gauge  $U_0 \sim 1$  [16] and for Kogut-Susskind's fermions in the gauge  $U_0(n_0)U_0(n_0 + 1) \sim 1$  [17]. In both cases, the transfer matrix assumes the following form

$$\hat{\mathcal{T}}_t = T_F^\dagger(U)T_G^\dagger(U)S(U, U')T_G(U')T_F(U'). \quad (1.10)$$

$T_F(U)$  is the only term in  $\hat{\mathcal{T}}$  that depends on the fermion operator and contains the interacting term for the gauge and fermion fields.  $S(U', U)$  and  $T_G(U)$  come from the plaquette term in the Wilson action. To our proposes, it suffices to write only the fermionic part of the partition function in the operatorial formalism. Since  $\hat{U}(\mathbf{x}, \mu)$  acts as a multiplication operator, it is easy to convert the purely bosons part of the transfer matrix into its functional form, by introducing a s.o.n.c. in  $\mathcal{H}_G$  for each time slice in (1.9). The result is

$$Z = \int D[U] e^{-\beta S'_G[U]} \text{Tr}^F \prod_{t=0}^{1/T} \hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1}, \quad (1.11)$$

where  $\text{Tr}^F$  is the trace in the Fock space of the fermions and the term in the trace, denoted  $Z_F$ , is the fermionic part of the  $Z$ ,  $S'_G[U] = S'_G[U] - 4\sum_t \text{tr}_- M_t$ . Clearly,  $\hat{T}$  is a function of the configuration of the boson fields  $\hat{T} = \hat{T}[U_t]$  and depends of time. The presence of  $\hat{V}_t$  is due to the fact that the axial gauge has not been fixed; in fact  $\hat{V}_t = \mathbb{1}$  in the axial gauge. The expressions of  $\hat{T}_t$  and  $\hat{V}_t$  are

$$\hat{T}_t = \exp\left(-\hat{u}^\dagger M_t \hat{u} - \hat{v}^\dagger M_t^T \hat{v}\right) \exp(\hat{v} N_t \hat{u}), \quad (1.12)$$

$$\hat{V}_t = \exp\left(\hat{u}^\dagger \ln U_{0,t} \hat{u} + \hat{v}^\dagger \ln U_{0,t}^* \hat{v}\right). \quad (1.13)$$

The explicit form of the matrices  $M$ ,  $N$  are reported in appendix B.3 and depend on the choice of the regularization assumed for the fermion field. It is, then, possible to define the pure fermionic transfer matrix operator  $\hat{\mathcal{T}}_t := \hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1}$  and to perform a simplification in the expression of the transfer matrix. One observes that

$$\hat{V}_t \hat{T}_{t+1} = \exp\left(-\hat{u}^\dagger (M_{u,t+1}) \hat{u} - \hat{v}^\dagger (M_{v,t+1}^T) \hat{v}\right) \exp(\hat{v} N_{t+1} \hat{u}), \quad (1.14)$$

with the conventions

$$M_{u,t+1} := \ln(e^{M_{t+1}} U_{0,t}^\dagger), \quad M_{v,t+1}^T := \ln(e^{M_{t+1}^T} U_{0,t}^T). \quad (1.15)$$

This shows that  $\hat{T}_{t+1}$  and  $\hat{V}_t \hat{T}_{t+1}$  have the same structure; the difference is given by the replacements

$$M_{t+1} \mapsto M_{u,t+1} \quad \text{and} \quad M_{t+1}^T \mapsto M_{v,t+1}^T. \quad (1.16)$$

During the procedure we have determined the two components,  $\psi^{(-)}$  and  $\psi^{(+)}$ , of the fermion field  $\psi$ , while in the operatorial formalism the operators associated to the fermions and antifermions are already known. It is useful to introduce the operator doublet

$$\psi := \begin{pmatrix} \hat{u} \\ \hat{v}^\dagger \end{pmatrix}. \quad (1.17)$$

Finally, one can check that

$$P_0^{(-)} \hat{\psi} = \hat{u}, \quad P_0^{(+)} \hat{\psi} = \hat{v}^\dagger. \quad (1.18)$$

### Chemical potential on a lattice

The study of quantum field theories (QFT) at finite chemical potential is very interesting because these theories describe many physical systems, for example neutron stars and nuclear matter. However, some complications arise in the theoretical approach to QFT at finite chemical potential.

In the continuum, the chemical potential is introduced by adding to the Lagrangian density of the theory the term

$$-\mu \bar{\psi} \gamma^0 \psi. \quad (1.19)$$

On a lattice, the same result can be achieved modifying the variables of the time links in (1.2) by a factor

$$U_0 \mapsto e^{a\mu} U_0, \quad U_0^\dagger \mapsto e^{-a\mu} U_0^\dagger. \quad (1.20)$$

One obtains, in this way, a grand canonical partition function  $Z[\mu]$ . The theoretical study does not have particular complications at finite density, on the contrary of Monte Carlo simulations. In fact, the latter are valid only for a real positive integration measure for the gauge fields; condition that does not hold anymore in presence of a non vanishing chemical potential. For finite values of the chemical potential, the fermionic determinant ( $\det M[U]$ ) is a function that highly depends on the configurations of the gauge fields, non local and not necessarily real and positive. These peculiarities make the Monte Carlo simulations hard and inefficient. To avoid these difficulties, a way is to perform simulations with an imaginary chemical potential  $\mu'$ , value for which the fermionic determinant is real. The quantities obtained through numerical simulations for imaginary chemical potential are expanded around the origin in powers of  $\mu'$ . The

information for real values of the chemical potential, therefore the physically relevant information, are obtained from the series in  $\mu'$  by analytical continuation. Obviously, this method is restricted to values of the chemical potential smaller than the convergence radius of the series. In the case of QCD, it is not yet known the analytic structure of the thermodynamical singularities. However, it is possible to study models that exhibit characteristics in common with QCD, as the Gross-Neveu model, in order to extract qualitative information on QCD and to understand useful methods to analyze the latter.

Another difficulty arises while introducing the chemical potential in  $QFT$ , as explained in Ref. [18] and references therein. In a Euclidean lattice path integral formulation, if the chemical potential is coupled as in Eq. (1.19), it is necessary to introduce specific counterterms. This causes some disadvantages in the numerical simulations. By exploiting the analogy of the chemical potential with the time link gauge fields, it is possible to find a solution of this problem:  $e^\mu$  must be coupled linearly to the fermion fields. In this situation no particular counterterms are required. The problem is that the chemical potential loses its physical meaning of Lagrange multiplier of a conserved charge.

In the transfer matrix formalism, this problem can have a solution. In quantum statistical mechanics, the partition function of a non relativistic system is defined as  $Z = \text{Tr} e^{-\beta \hat{H}}$ , where  $\beta = (k_b T)^{-1}$  and  $\hat{H}$  is the Hamiltonian of the system. In this formulation, the chemical potential is introduced as a Lagrange multiplier associated to the number of particles. Then the gran-canonical partition function is defined as  $Z_{gc} = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{n})}$ .

Inspired by the previous procedure, we use the following prescription to introduce the chemical potential in a quantum field theory. The fermionic part of the partition function written in terms of the transfer matrix, when the transfer matrix is known, is

$$Z_F = \text{Tr}^F \prod_{t=0}^{1/T} [\hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1}]. \quad (1.21)$$

As explained, the chemical potential is introduced as the Lagrange multiplier of the fermion number  $\hat{n}_B$ , that is called also baryon number

$$Z_F = \text{Tr}^F e^{\frac{\mu \hat{n}_B}{T}} \prod_{t=0}^{1/T} [\hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1}] = \text{Tr}^F \prod_{t=0}^{1/T} [\hat{T}_t^\dagger e^{\mu \hat{n}_B} \hat{V}_t \hat{T}_{t+1}]. \quad (1.22)$$

Following the procedure described in the last section one can derive the functional form of  $Z_F$ . In this expression, the exponential of the chemical potential is coupled linearly to the fermion fields. In the thesis we take the right side of (1.22) to be the definition of the gran canonical partition function.

### The phase diagram of QCD

The study of relativistic field theories at finite temperature ( $T$ ) and chemical potential ( $\mu$ ) is of great interest, both from an experimental and theoretical point of view. However, the determination of the dynamical and thermodynamic properties of those theories requires the use of complex techniques developed in the context of theoretical physics and of information obtained through nuclear physics experiments. The relevance of these highly non trivial studies is nowadays remarkable in the field of quantum chromodynamics (QCD). The phase diagram of this theory, in the parameters  $T$  and  $\mu$ , is expected to show a number of phase transitions and crossovers, whose nature still has to be completely clarified. Indeed, even at an experimental level, the study of the behavior of hadronic matter is a topical argument.

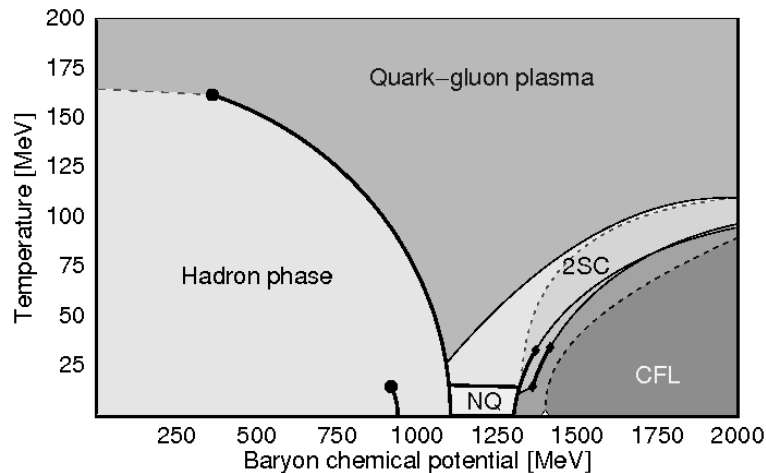


Figure 1.1: The contemporary view of the QCD phase diagram, a semi-quantitative sketch. from [19].

It is believed that the QCD phase diagram should present at least three phases as shown in fig. 1. For small values of  $\mu$  and  $T$  ( $T \lesssim 100\text{MeV}$ ,  $\mu \lesssim 600\text{MeV}$ ), the fundamental state exhibits chiral symmetry breaking. The associated Goldstone bosons are the pions and the region under discussion is said to be of an hadrons gas. Part of this region can be explored through the experimental analysis in heavy ions collisions. At greater temperatures, it is possible to perform a perturbative expansion around a plasma state of quarks and gluons. This investigation predicts that the quarks are massless, free and with the chiral symmetry restored. While the study of the latter region can be carried out by means of perturbative methods, the chiral symmetry breaking in the first zone is clearly a non perturbative phenomenon. An argument by Pisarski-Wilczek suggests that the phase transition between the hadrons gas and the quarks and gluons plasma is

of the first order, at least for a *flavour* number equal to three. The order parameter is given by  $\langle \bar{\psi}\psi \rangle$ . The first order transition line ends in a (tri-)critical point, whose values of  $\mu_c$  and  $T_c$  are still unknown. For non vanishing values of the quarks bare masses and for values of the chemical potential smaller than the critical potential, the phase transition is a crossover. On the contrary, in the case of massless quarks, the transition is of the second order. Therefore, the quarks masses in QCD play a role analogous to the one played by the magnetic field in the Ising's model. The knowledge of the theory for big values of the chemical potential is, instead, poorer. In this region, we expect a phase transition from a quarks and gluons gas to a color superconductivity phase. This transition is characterized by a non vanishing value of the order parameter  $\langle \psi^T C \psi \rangle^5$ . However, the analogies between the QCD phase diagram and the well known water phase diagram suggest that QCD should exhibit other phases beyond the ones described. Other information about the phases of QCD can be found in [19, 20].

Today, the investigation of the QCD phase diagrams is far from conclusion. In fact, the experimental analysis in the accessible regions of the diagram is only at its beginning, as the study through numerical simulations still has to be developed and completed.

The knowledge of QCD thermodynamics is essential for the understanding of such natural phenomena as compact stars and laboratory experiments involving relativistic heavy ions collisions.

An effective theory in terms of dominant degrees of freedom can be very powerful to describe QCD or hadron matter at finite temperature ( $T$ ) and chemical potential ( $\mu$ ). In some of these phases, the excitations should be of composite type, for example mesons and baryons. These new states of matter are accessible in numerical simulation or partially accessible in the experimental investigation of heavy collisions.

The thesis work is organized as follows. Chapter II is devoted to a review of bosonization in relativistic systems. In Chapter III we discuss a new method of bosonization and, in the next Chapter, we extend it to finite  $\mu$  and  $T$ . In Chapter V we test the theory just developed and described with a model with four fields interaction. In Chapter VI, we study the possibility of having di-quarks condensate. Finally, the last Chapter is reserved to a discussion of the conclusions and of the perspectives of the studies carried through in the thesis.

---

<sup>5</sup> $C$  is the charge conjugation matrix. There is a breaking of the color symmetry  $SU_c(3)$  into  $SU_c(2)$  and of the symmetry associated to the baryonic number  $U_B(1)$  into  $Z_2$ .





## Chapter 2

# A method of bosonization

In this chapter, we present an introductory review of the techniques of bosonization. Since we are interested into relativistic field theories, we do not discuss in detail non relativistic systems.

The method of bosonization consists of the replacement of a known system of fermions with a theory of bosons which has a completely equivalent physical content, including identical spectra and interactions. The procedure provides a recipe to express the correlation functions of the fermionic theory in terms of the bosonic ones of the corresponding system. When applicable, it is an extremely useful tool for analyzing fermionic systems, since it permits to apply to them the powerful techniques that have been developed for bosonic systems.

To develop a bosonization method for relativistic theories in higher than  $1 + 1$  dimensions, two points are essential: the introduction of the field associated to the dominant composite boson and the derivation of its action. We now discuss two of the known bosonization methods for relativistic theory. In the procedure we deal with, the bosons are described by Kalb-Ramond fields introduced as Lagrange multipliers for the gauge fields [21, 22]. When it is possible, the action for the boson fields is then derived by explicit computations. There exists another method of bosonization for relativistic field theories, but we do not go into the details of the latter, since it is quite standard. In this method, higher dimensional terms (quartic in the fermion fields) are introduced, if absent, by hands in the action and the boson fields are generated via the Hubbard-Stratonovich transformation [23]. Sometimes, when the added terms are irrelevant in the renormalization-group terminology, they are introduced to stabilize the evaluation, for instance in numerical simulations, of the fermion determinant [24]. However, often, the investigations consist of analytical studies and are substantially restricted to a non-renormalizable framework [23]. In one noticeable exception [25], the Hubbard-Stratonovich transformation is scale-dependent ("re-bosonization") leading to a functional renormalization-group analysis. For these methods, the boson fields are real, and they are related bosons of fermion number zero.

## 2.1 Bosonization in arbitrary dimension by means of gauge forms

It has been proven that, in  $1 + 1$  dimensions, bosonization is a particular duality transformation [5, 6]. The observation that the latter is not restricted by dimensionality permits to generalize the bosonization method to higher dimensional systems. The dualization approach guarantees the existence of a bosonized version of the theory, but do not guarantees that the bosonic theory has the usual proprieties of a field theory, such as, for example, locality. The technique of bosonization as dualization can be used to treat the case where the symmetry of the initial fermionic system is both abelian or non abelian. In what follows, we focus on systems whit abelian symmetry as in [21] and [22]. Usually, bosonization of fermionic systems whit abelian (non-abelian) symmetry is referred to as abelian-bosonization (non-abelian-bosonization).

Consider a system of fermion fields  $\psi_l$  (where  $l = 1 \dots N_f$  is the flavour index) in  $D$  dimensions described by an Euclidean action  $S_F(\psi_l^*, \psi_l)$ . The action under consideration is diagonal in the flavour index and invariant under global  $U(1)$  transformations. In this case, it is possible to omit the flavour index. To study the reaction of the system to the presence of external currents  $J_i^{ext}$ , the action  $S_F(\psi_l^*, \psi_l)$  contains an interaction term of the form  $\psi^* M_i \psi J_i^{ext}$ , where  $M_i$  is an appropriate matrix. The fermionic partition function reads

$$Z_F = \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S_F(\psi_l^*, \psi_l)} . \quad (2.1)$$

To bosonize, we first enlarge the fermionic theory by gauging the global  $U(1)$  symmetry. Replacing the ordinary derivative with the covariant one containing the gauge field  $A$ , the gauge invariant action becomes  $S_F(\psi_l^*, \psi_l, A)$ . The effective action for the gauge field  $A$  is defined by

$$e^{-S_B(A)} := \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S_F(\psi^*, \psi, A)} . \quad (2.2)$$

By Fourier transform, one obtains the action for the variable  $J$ , conjugate to  $A$

$$e^{-\hat{S}_B(J)} := \int \mathcal{D}A e^{-S_B(A) + i \int A_\mu J^\mu d^D x} . \quad (2.3)$$

Since  $S_F(\psi^*, \psi, A)$  is invariant under gauge transformations, it follows that the field  $J$  satisfies the constraint  $\partial_\mu J^\mu = 0$ . By Poincaré lemma, one obtains

$$J = \frac{1}{2\pi} * db , \quad (2.4)$$

where  $*$  denotes the Hodge dual and  $d$  the exterior differential. Notice, that the solution of (2.4) is not unique. In fact, two  $(D-2)$ -forms differing by  $d\lambda^{(D-3)}$ , for a  $(D-3)$ -form  $\lambda^{(D-3)}$ , yield the same  $J$ . Therefore,  $\hat{S}(J) =: \hat{S}(db)$  is invariant under gauge transformations  $b \mapsto b + d\lambda^{(D-3)}$ . In terms of  $b$ , (2.3) can be written as

$$e^{-\hat{S}_B(db)} := \int \mathcal{D}[A] e^{-S_B(A)} e^{\frac{i}{2\pi} \int A \wedge db} , \quad (2.5)$$

where  $\mathcal{D}[A]e^{-S(A)}$  is the measure induced by  $\mathcal{D}Ae^{-S(A)}$  on the space of the gauge orbits, i.e. the gauge equivalent class

$$[A] := \{A' : A - A' = d\Lambda\}$$

and  $\mathcal{D}[b]e^{-\hat{S}_B(db)}$  denotes the measure on the gauge equivalent class

$$[b] := \{b' : b - b' = d\lambda^{(D-3)}\}.$$

It can be proven that

$$Z_F = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-S_F(\psi, \psi^*)} = \int \mathcal{D}[b] e^{-\hat{S}_B(db)} =: Z_B, \quad (2.6a)$$

$$\langle \mathcal{J}(x_1)^{\mu_1} \cdots \mathcal{J}(x_n)^{\mu_n} \rangle_{S_F(\psi, \psi^*)} = (2\pi)^{(-n)} \langle * db(x_1)^{\mu_1} \cdots * db(x_n)^{\mu_n} \rangle_{\hat{S}_B(db)}, \quad (2.6b)$$

where  $\mathcal{J}(x)^\mu := -i \frac{\delta S_F(\psi, \psi^*, A)}{\delta A(x)_\mu}$ . The proof of the previous identities is based over the following observation. Replacing the definition of  $\hat{S}(db)$  in (2.6a), one notices that  $b$  acts as a Lagrange multiplier. The integral over the field  $b$  imposes on  $A$  the constraint  $dA = 0$ . Performing the integration over  $A$ , one finally obtains  $Z_F$ . Instead, if one is able to perform the integral in a different order, first over  $\psi^*, \psi$  and then over  $A$ , one obtains the partition function for the bosonic field  $b$ . The latter, introduced by Fourier transform, is a completely antisymmetric Kalb-Ramond gauge potential and plays the role of a bosonized variable.

The identities (2.6) show the possibility of expressing every quantity in a fermionic theory by an appropriate quantity in a bosonic theory. Clearly, these identities are somewhat abstract and they are useful only if  $\hat{S}_B(db)$  is a tractable quantity. Actually, the obtention of  $S_B(A)$  and then of  $\hat{S}_B(db)$  itself is generally a laborious and not necessarily successful step. Notice that, if  $S_B(A)$  is quadratic in  $A$ ,  $\hat{S}(db)$  is quadratic in  $db$ . This method is valid in relativistic systems [21] as well as in non-relativistic ones [22].

In the following, we study one application of the method under discussion to a sample case where an approximation permits to compute the effective action of the field  $b$ .

### 2.1.1 Bosonization of Dirac fields

Consider the Euclidian free massive fermion Lagrangian density and its corresponding  $U(1)$  gauge invariant version

$$\mathcal{L}_F(\bar{\psi}, \psi) = -\bar{\psi}(\not{\partial} + m + J_i^{ext} M_i)\psi, \quad \mathcal{S}_F(\bar{\psi}, \psi, A) = \int d^D x \mathcal{L}_F(\bar{\psi}, \psi) + \bar{\psi} \not{A} \psi, \quad (2.7)$$

where the interaction term, between the Dirac field  $\psi$  and the external currents  $J_i^{ext}$ , is given by  $\bar{\psi} M_i \psi J_i^{ext} = i\bar{\psi} \gamma^\mu \psi J_\mu^{ext}$ <sup>1</sup>. Notice that, even if omitted for shortness, the

<sup>1</sup>In this case  $\mathcal{J}(x)^\mu := -i \frac{\delta S_F(\psi, \psi^*, A)}{\delta A(x)_\mu} = -i \frac{\delta S_F(\psi, \psi^*, A)}{\delta J^{ext}(x)_\mu}$ .

action depends on  $J^{ext}$ . Following the same procedure described above and carrying on the computations, it is possible to build an action of the kind of  $S_B(A)$  (see Eq. (2.2)) and to obtain the bosonized action for the (D-2)-gauge form  $b$ . The result reads

$$e^{-\hat{S}_B(db)} = \int \mathcal{D}A_\mu \exp \left[ -S_B(A) + \int d^D x \left( i\epsilon^{\mu_1\mu_2\mu_3\cdots\mu_D} \partial_{\mu_1} A_{\mu_2} b_{\mu_3\cdots\mu_D} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) \right]. \quad (2.8)$$

The only difference in the method is that, for convenience, we choose to perform the integral over the whole set of configurations of the gauge field instead that over the gauge equivalence class  $[A]$ . In fact, the last term appearing in (2.8) is the gauge fixing term, necessary for a well posed path integral over  $A$ . In order to investigate the properties of the boson theory, it is necessary to perform the functional integral first over the fermion field and, subsequently, over the field  $A$ . However, the way to evaluate the integral is, in general, not known and appropriate approximations are necessary. One possible choice is to expand the action  $S_B(A)$  in powers of  $1/m$ , for large  $m$ . The leading term in the effective Lagrangian is the one that has the lowest scaling dimension, that, in this case, coincides with the lowest naive dimension. This term has a very simple expression

$$e^{-S_B(A)} = \exp \left[ \frac{1}{2} \int d^D x (J^{ext} + A)_\mu \Pi_D^{\mu\nu} (J^{ext} + A)_\nu + O\left(\frac{1}{m}\right) \right], \quad (2.9)$$

where we have omitted irrelevant  $A$  independent terms. Observe that the integral has been regularized by dimensional regularization and rinormalized by the minimal subtraction scheme  $\overline{\mathcal{MS}}$ . This procedure has the advantage of making it simple to track the  $m$  dependence of the results. The vacuum polarization,  $\Pi_D^{\mu\nu}(x)$ , is well known and takes different expressions for  $D = 3$  or  $D \geq 4$ <sup>2</sup>. Indeed, this difference is at the origin of the different properties of the bosonized theory. As the lowest dimension term is the one quadratic in the sum  $(J^{ext} + A)$ , the evaluation of the integral in (2.8) simplifies remarkably to

$$e^{-\hat{S}_B(db)} = \exp \left\{ \frac{-1}{2} \int d^D x \left[ \Omega_\mu (\hat{\Pi}_D^{\mu\nu})^{-1} \Omega_\nu + 2\Omega_\mu (\hat{\Pi}_D^{\mu\eta})^{-1} \Pi_D^{\eta\nu} J_\nu^{ext} + J_\mu^{ext} \Pi_D^{\mu\eta} (\hat{\Pi}_D^{\eta\lambda})^{-1} \Pi_D^{\lambda\nu} J_\nu^{ext} + J_\mu^{ext} \Pi_D^{\mu\nu} J_\nu^{ext} \right] \right\}. \quad (2.10)$$

Above  $\Omega_\mu := \epsilon^{\mu\mu_2\mu_3\cdots\mu_D} \partial_{\mu_2} b_{\mu_3\cdots\mu_D}$  and  $\hat{\Pi}_D^{\mu\nu} := \Pi_D^{\mu\nu} + \xi^{-1} \partial^\nu \partial^\mu$ . The second term in  $\hat{\Pi}_D^{\mu\nu}$  comes from the gauge fixing. Notice that Eq. (2.10) is valid for all systems for which the effective action of the gauge field is quadratic in  $A$ . In the case in exam, Eq. (2.10) simplifies and using the appropriate expressions of the vacuum polarization 2.1.1, we obtain the bosonized actions

$$S_B(db) = \int d^D x \left[ \frac{1}{2k_D} \Omega_\mu \square^{-1} \Omega_\nu + \Omega^\mu J_\nu^{ext} \right], \quad D \geq 4,$$

$$S_B(db) = \int d^3 x \left[ \epsilon^{\mu\nu\lambda} (b_\mu \partial_\nu b_\lambda + J_\mu^{ext} \partial_\nu b_\lambda) \right], \quad D = 3.$$

<sup>2</sup> $\Pi_3^{\mu\nu} = k_3 \epsilon^{\mu\nu\rho} \partial_\rho$ , where  $k_3$  is a known finite coefficient; while  $\Pi_D^{\mu\nu} = k_D (\square g^{\mu\nu} + \partial_\mu \partial_\nu)$  for  $D \geq 4$ , where  $k_D$  is D-dependent divergent constant.

Both results are quadratic in the  $b$  field and it is possible to check the identity (2.6). For  $D \geq 4$  the action is not local, instead, for  $D = 3$ , the action is local and it is called of Chern-Simons.

### Dualization to scalar variable

It has been proven that, for all spacetime dimension, a theory of antisymmetric Kalb-Ramond fields is dual to a derivatively coupled scalar fields one. Since we have more intuition about the second, the described dualization process is very useful. The dualization method is based mainly on two observations. Firstly, the fact that the functional integral over  $b$  can be rewritten as an integral over  $\Omega$  with the constraint  $\partial \cdot \Omega = 0$ . In fact, in the bosonized theory the Kalb-Ramond fields  $b$  appear in the combination that is equivalent to  $\Omega$ . Secondly, that it is possible to introduce a Lagrange multiplier  $\varphi$ , whose integration gives the constrain  $\partial \cdot \Omega = 0$ .

In general, it is difficult to perform the unconstrained integral over  $\Omega$ , but, in the case of interest here, the functional integral is quadratic and therefore easy to evaluate. The result is the same for all  $D$

$$S'_B(\varphi, J^{ext}) = \frac{1}{2} \int d^D x J_\mu^{ext} \Pi_D^{\mu\nu} J_\nu^{ext} + \frac{1}{\xi} \varphi(\square)^2 \varphi \quad (2.11a)$$

and the partition function can be expressed by a dual scalar theory

$$Z_F = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_F(\bar{\psi}, \psi)} = \int \mathcal{D}[\varphi] e^{-S'_B(\varphi, J^{ext})} = Z'_B. \quad (2.11b)$$

The action for  $\varphi$  is local and free, but the kinetic term for the scalar field is both unusual and irrelevant. Since  $\varphi$  decouples from the external current, the integral in (2.11b) may be ignored; the remaining term is just the large  $m$  limit of the original fermionic theory.



## Chapter 3

# Composite boson dominance

In the present section, we introduce a recently developed method of bosonization for relativistic field theories in arbitrary dimension. This formulation is inspired by a method of bosonization for many-body systems [8, 9], which has been checked to reproduce exactly the properties of the BCS model of superconductivity and of the pairing model of finite systems. However, the treatment of relativistic systems has to be considered carefully and is obviously different in many aspects.

The starting point of the method is the fermionic part of the partition function  $Z$  in its operatorial formalism (1.11), i.e. the trace of a power of the transfer matrix in the Fock space of the fermions  $\hat{\mathcal{T}}$ . We are interested, for the moment, only to the fermionic part  $Z_F$  (1.21) of the partition function. In the operatorial formalism, the dominant degrees of freedom are introduced like bosons composed of fermions in the form of a superposition of pairs of fermions and anti-fermions. The assumption of boson dominance can be implemented by restricting the trace in the partition function only over states of composite bosons.

In general, the functional form of  $Z$  can be found by performing the trace of its operator expression over coherent states [15, 14]. This is nearly the procedure we shall adopt in the following to derive the functional form of the restricted partition function. We introduce particular states, namely similar-coherent states of composite bosons, over which we compute the trace. Then, one is able to deduce the effective action of the composite bosons fields, performing some Berezin integrals; fields are introduced, as usually, as holomorphic variables in the similar-coherent states. Notice that this procedure respects all the symmetries of the system; however, since the transfer matrix formalism does not treat in the same way space and time, Euclidean invariance has to be checked a posteriori.

We adopt lattice regularization because this choice has some advantages, but this approach is, in general, compatible for any regularization. The first advantage is that on a lattice there is an unambiguous definition of composite. Secondly, since we are interested into gauge theories, a lattice formulation allows to extract information from

numerical simulations.

### 3.1 Composite boson operator

Since we are interested in considering only relevant excitations, we must introduce creation and annihilation operators of these states, respectively

$$\hat{\Phi}_I^\dagger := \hat{u}^\dagger \Phi_I^\dagger \hat{v}^\dagger = \sum_{i,j} \hat{u}_i^\dagger (\Phi_I^\dagger)_{i,j} \hat{v}_j^\dagger, \quad \hat{\Phi}_I := \hat{v} \Phi_I^\dagger \hat{u} = \sum_{i,j} \hat{v}_i (\Phi_I)_{i,j} \hat{u}_j, \quad (3.1)$$

where  $I := (\mathbf{x}, K)$ . Here  $\mathbf{x}$  is the spatial coordinate of the composite bosons and  $K$  represents their quantum number. In this notation,  $\Phi_I^\dagger$  is the composite structure function. From the definition of  $\hat{\Phi}_I^\dagger$ , it is possible to see that the composite bosons are an overlap of a fermion and an anti-fermion, with quantum numbers  $i$  and  $j$ , respectively, and coefficients  $(\Phi_I^\dagger)_{i,j}$ . The composite bosons have zero fermion number. The operators  $\hat{\Phi}$  and  $\hat{\Phi}^\dagger$  are boson like but they do not obey canonical commutation relations. In fact, the composite commutation relations are

$$\left[ \hat{\Phi}_I, \hat{\Phi}_J^\dagger \right] = \text{tr}_+ \left( \Phi_I \Phi_J^\dagger \right) - \hat{u}^\dagger \Phi_J^\dagger \Phi_I \hat{u} - \hat{v}^\dagger \Phi_J^* \Phi_I^T \hat{v}, \quad \left[ \hat{\Phi}_I, \hat{\Phi}_J \right] = 0. \quad (3.2)$$

Since the fermion creation operators are nilpotent, the composite boson operators can be classified by their index of nilpotency  $\Omega_I$ , defined by the relation

$$\Omega_I := \text{Max}_{\omega \in \mathbb{N}} \left\{ \omega | (\hat{\Phi}_I^\dagger)^\omega \neq 0 \right\}.$$

Qualitatively, one can observe that the greater the number of fermions and anti-fermions in the composite, i.e. the number of coefficients  $\hat{\Phi}_I^\dagger$  such that  $\hat{\Phi}_I^\dagger \neq 0$ , the greater is  $\Omega_I$ . In the limiting case in which all the matrix elements of  $\hat{\Phi}_I$  are not null,  $\Omega_I$  is equal to the number of fermion states. The composite bosons must have large  $\Omega_I$  to look like canonical bosons, whose nilpotency index is infinity. Therefore, one considers only operators with large nilpotency index. Notice that  $\Omega_I$  can be at most the number of fermion states.

It is possible to generate a sub-space of the fermion Hilbert space by repeatedly acting with  $\hat{\Phi}_I^\dagger$  over  $|0\rangle$ . This Hilbert space is called composite boson Hilbert space. Consider the following state

$$|\phi\rangle := \exp \left( \sum_I \phi_I \hat{\Phi}_I^\dagger \right) |0\rangle, \quad (3.3)$$

where  $\phi_I$  are holomorphic variables. We call this kind of states similar-coherent states of composite or simply coherent states, because they share with the coherent states of elementary bosons the property of a fixed phase relation among the components with



different number of composites. These states look like boson coherent states but they do not satisfy their basic property. In fact, from (3.2), it is easy to deduce that

$$\hat{\Phi}_I|\phi\rangle \neq \phi_I|\phi\rangle. \quad (3.4)$$

By use of the formulae collected in appendix A, one can deduce that the inner product of coherent states is

$$\langle\phi_1|\phi_2\rangle = \det_+(1 + \mathcal{F}_1\mathcal{F}_2^\dagger), \quad (3.5)$$

where we have used the convention

$$\mathcal{F}_{i,j}^\dagger := \sum_I \phi_I(\Phi_I^\dagger)_{i,j} =: \phi_I \cdot (\Phi)_I^\dagger{}_{i,j}.$$

The result (3.5) is different from the inner product of elementary coherent states.

In order to compute the partition function only over composite boson states, it is necessary to know the projector over their subspace

$$\hat{\mathcal{P}}_C := \sum_{\{n_I\}} \frac{\left(\prod_I \hat{\Phi}_I^{\dagger n_I}\right) |0\rangle \langle 0| \left(\prod_I \hat{\Phi}_I^{n_I}\right)}{\langle 0| \left(\prod_I \hat{\Phi}_I^{n_I}\right) \left(\prod_I \hat{\Phi}_I^{\dagger n_I}\right) |0\rangle}. \quad (3.6)$$

For later convenience, we define

$$\begin{aligned} R_{(a,b)} &:= \left(1 + \mathcal{F}_a^\dagger \mathcal{F}_b\right)^{-\frac{1}{2}}, & \overset{\circ}{R}_{(a,b)} &:= \left(1 + \mathcal{F}_b \mathcal{F}_a^\dagger\right)^{-\frac{1}{2}}, \\ R_t &:= R_{(t,t)}, & \overset{\circ}{R}_t &:= \overset{\circ}{R}_{(t,t)}. \end{aligned}$$

Notice that  $R$  and  $\overset{\circ}{R}$  have the same determinant and the only difference between them is that they live in distinct subspaces of the projectors  $P_0^{(-)}$  and  $P_0^{(+)}$ . The definition of the involution  $\circ$  can be found in appendix B.3.

## 3.2 The operator $\hat{\mathcal{P}}$

We would like to rewrite the operator  $\hat{\mathcal{P}}_C$  in terms of coherent states of composite bosons, as

$$\hat{\mathcal{P}} := \int \left[ \frac{d\phi^* d\phi}{2\pi i} \right] \frac{|\phi\rangle \langle \phi|}{\langle \phi | \phi \rangle}, \quad (3.7)$$

where we have defined the measure

$$\left[ \frac{d\phi^* d\phi}{2\pi i} \right] := \prod_I \frac{d\phi_I^* d\phi_I}{2\pi i}. \quad (3.8)$$

Notice that, the identity in a Fock space can be written as in Eq. (3.7) if one replaces the similar-coherent states with fermions coherent states. Since the states  $|\phi\rangle$  are not

strictly coherent, the equivalence between  $\hat{\mathcal{P}}$  and  $\hat{\mathcal{P}}_C$  is not guaranteed. However,  $\hat{\mathcal{P}}$  is a good approximation of  $\hat{\mathcal{P}}_C$  if it satisfies

$$\langle 0 | \left( \prod_I \hat{\Phi}_I^{n_I} \right) \hat{\mathcal{P}} \left( \prod_I \hat{\Phi}_I^{\dagger m_I} \right) | 0 \rangle \simeq \delta_{n_I, m_I} \langle 0 | \left( \prod_I \hat{\Phi}_I^{n_I} \right) \left( \prod_I \hat{\Phi}_I^{\dagger n_I} \right) | 0 \rangle. \quad (3.9)$$

The last equation is generated by the following one

$$\langle \phi_1 | \hat{\mathcal{P}} | \phi_2 \rangle \simeq \langle \phi_1 | \phi_2 \rangle, \quad (3.10)$$

by deriving with respect to  $\phi_1$ ,  $\phi_2$  and then setting both variables to zero. By use of (3.5), it is possible to write (3.10) in the following way

$$\langle \phi_1 | \hat{\mathcal{P}} | \phi_2 \rangle = \int \left[ \frac{d\phi^* d\phi}{2\pi i} \right] e^{-\mathcal{E}(\phi^*, \phi, \phi_1^*, \phi_2)}, \quad (3.11a)$$

where

$$\mathcal{E}(\phi^*, \phi, \phi_1^*, \phi_2) := \text{tr}_+ \left[ \ln \left( 1 + \mathcal{F}\mathcal{F}^\dagger \right) - \ln \left( 1 + \mathcal{F}_1\mathcal{F}_1^\dagger \right) - \ln \left( 1 + \mathcal{F}_2\mathcal{F}_2^\dagger \right) \right]. \quad (3.11b)$$

As it is shown in the following,  $\mathcal{E}$ , for large  $\Omega$ , is proportional to  $\Omega$  itself. The integral (3.14) can be evaluated by saddle point method; whose equations are

$$\frac{\partial \mathcal{E}}{\partial \phi_I} = (\phi_J^* - \phi_{1J}^*) \text{tr}_+ \frac{\Phi_I \Phi_J^\dagger}{(1 + \mathcal{F}\mathcal{F}^\dagger)(1 + \mathcal{F}_2\mathcal{F}_2^\dagger)} = 0, \quad (3.12a)$$

$$\frac{\partial \mathcal{E}}{\partial \phi_I^*} = (\phi_J - \phi_{2J}) \text{tr}_+ \frac{\Phi_J \Phi_I^\dagger}{(1 + \mathcal{F}\mathcal{F}^\dagger)(1 + \mathcal{F}_1\mathcal{F}_1^\dagger)} = 0. \quad (3.12b)$$

The solutions of the saddle point equations are  $\phi = \phi_2$  and  $\phi^* = \phi_1^*$ . Moreover, at the saddle point, the following equations hold

$$\mathcal{E}(\phi_1^*, \phi_2, \phi_1^*, \phi_2) = -\text{tr}_+ \left[ \ln \left( 1 + \mathcal{F}_1\mathcal{F}_2^\dagger \right) \right], \quad (3.13a)$$

$$\left. \frac{\partial^2 \mathcal{E}}{\partial \phi_I \partial \phi_J} \right|_{\substack{\phi = \phi_2 \\ \phi^* = \phi_1^*}} = \left. \frac{\partial^2 \mathcal{E}}{\partial \phi_I^* \partial \phi_J^*} \right|_{\substack{\phi = \phi_2 \\ \phi^* = \phi_1^*}} = 0, \quad \left. \frac{\partial^2 \mathcal{E}}{\partial \phi_I \partial \phi_J^*} \right|_{\substack{\phi = \phi_2 \\ \phi^* = \phi_1^*}} = \text{tr}_+ \frac{\Phi_J \Phi_I^\dagger}{(1 + \mathcal{F}_1\mathcal{F}_2^\dagger)^2}. \quad (3.13b)$$

Considering the gaussian corrections at the saddle point too, the result is

$$\langle \phi_1 | \hat{\mathcal{P}} | \phi_2 \rangle \simeq \langle \phi_1 | \phi_2 \rangle \det^{-1} \left. \frac{\partial^2 \mathcal{E}}{\partial \phi_I \partial \phi_J^*} \right|_{\substack{\phi = \phi_2 \\ \phi^* = \phi_1^*}}, \quad (3.14)$$

where  $\det$  is the determinant over the boson composite quantum number.

In the following, to study the action of  $\hat{\mathcal{P}}$  on the subspace of composite bosons and to fully understand the implication of the formula (3.14), we consider a simple case.

### 3.2.1 The operator $\hat{\mathcal{P}}$ with a unique composite boson

For simplicity, take into consideration the case of a unique composite boson, i.e the case where the index  $I$  in (3.1) assumes only one value. Further assume the following normalization for the structure function

$$\mathrm{tr}_+ \Phi \Phi^\dagger = 1. \quad (3.15)$$

Notice that, if the structure function is smooth and the composite boson is a large numbers of fermion with different quantum number, each matrix element  $\Phi \Phi^\dagger$  is  $\mathcal{O}(\frac{1}{\Omega})$ .

For the moment, we restrict to consider a trivial case, where the structure function has a large nilpotency index and satisfies:

$$\Phi \Phi^\dagger = \mathbb{I} d\Omega^{-1}. \quad (3.16)$$

This hypothesis simplifies the computation; in fact the relation (3.2) becomes

$$[\hat{\Phi}, \hat{\Phi}^\dagger] = 1 - \Omega^{-1} \hat{u}^\dagger \hat{u} - \Omega^{-1} \hat{v}^\dagger \hat{v}. \quad (3.17)$$

The non canonical term appearing above, i.e. the second and the third addends in the last equation, is of  $\mathcal{O}(\frac{1}{\Omega})$ . The inner product is

$$\langle \phi_1 | \phi_2 \rangle = \left( 1 + \frac{1}{\Omega} \phi_1^* \phi_2 \right) \xrightarrow{\Omega \rightarrow \infty} e^{\phi_1^* \phi_2}, \quad (3.18)$$

exactly as the inner product of canonical coherent states. By (3.18), it is clear that  $\mathcal{E}$  is of order  $\Omega$ . In the same hypotheses, one can check that

$$\hat{\Phi} \hat{\Phi}^{\dagger n} |0\rangle = n(1 - 2(n-1)\Omega^{-1}) \hat{\Phi}^{\dagger n-1} |0\rangle. \quad (3.19)$$

From the constraint (3.16), one observes that the composite boson states, with  $n$  composites bosons, satisfy the same identities of canonical boson states up to terms of order  $\mathcal{O}(\frac{n}{\Omega})^1$ . In the subspace of composite bosons with  $n \ll \Omega$ , the states  $|\phi\rangle$  behave as coherent states.

Since the structure function is a dynamical variable, the hypothesis (3.16) is very strong. It is therefore reasonable to return to consider the projector with only one composite but without the constraint given by (3.16). The unwanted term in (3.14) is

$$\left[ \mathrm{tr}_+ \frac{\Phi \Phi^\dagger}{(1 + \mathcal{F}_1 \mathcal{F}_2^\dagger)^2} \right]^{-1}. \quad (3.20)$$

---

<sup>1</sup>It can be proven that

$$\begin{aligned} \langle 0 | \hat{\Phi}^m \hat{\Phi}^{\dagger n} | 0 \rangle &= \delta_{m,n} n! (1 - 2\Omega^{-1})(1 - 4\Omega^{-1}) \cdots (1 - 2(n-1)\Omega^{-1}), \\ \langle \phi | \hat{\Phi}^{\dagger n} | 0 \rangle &= \phi^{*n} (1 - 2\Omega^{-1})(1 - 4\Omega^{-1}) \cdots (1 - 2(n-1)\Omega^{-1}). \end{aligned}$$

To check if  $\hat{\mathcal{P}}$  satisfies (3.9), one needs to perform the derivative, with respect to  $\phi_1^*$  and  $\phi_2$ . In both members of (3.14), only derivatives of the same order with respect to  $\phi_1^*$  and  $\phi_2$  are non null for  $\phi_1^* = \phi_2 = 0$ . Notice that this is true for each factor of the right hand side of (3.14). Performing  $n$  derivatives with respect to  $\phi_1^*$  and  $\phi_2$ , the factor (3.20) gives terms proportional to  $(m+1)! m! \text{tr} (\Phi \Phi^\dagger)^{m+1}$ , for any  $m \leq n$ . These terms do not appear in the right hand side of Eq. (3.14). When the relation

$$\text{tr} (\Phi \Phi^\dagger)^{m+1} \simeq \Omega^{-m+1} \quad (3.21)$$

holds, (3.10) is satisfied for  $n \ll \Omega$  with an error of  $\mathcal{O}(\frac{n!}{\Omega})$ . In this situation, it is verified the idempotency of  $\hat{\mathcal{P}}$  too

$$\hat{\mathcal{P}}^2 \sim \hat{\mathcal{P}}. \quad (3.22)$$

### 3.2.2 The operator $\hat{\mathcal{P}}$ with an arbitrary number of composite bosons

The argument given in the previous section can be generalized to treat an arbitrary number of composites when the index  $\Omega_I$  is very large for all  $I$ . If we assume that

$$\text{tr} (\Phi_I \Phi_J^\dagger)^{m+1} \simeq \delta_{I,J} \Omega^{-m+1}, \quad (3.23)$$

the relations (3.10) and (3.22) are valid in the subspace of a total number  $n \ll \Omega$  of composite bosons.

## 3.3 Derivation of the effective action

We now turn to the derivation of the effective action for the fields  $\phi$  and  $\phi^*$ . For the reader's convenience, we quote the fermion part of the partition function in the transfer matrix formalism

$$Z_F = \mathbf{Tr}^F \prod_{t=0}^{L_0-1} \left[ \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \right]. \quad (3.24)$$

Where  $L_0 = T^{-1}$ . Under the assumption that at low energy the partition function is dominated by boson states, it is possible to restrict the evaluation of the trace over these type of states. The bosonized <sup>2</sup> partition function is

$$Z_C = \mathbf{Tr}^F \prod_{t=0}^{L_0-1} \left[ \hat{\mathcal{P}} \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \right]. \quad (3.25)$$

---

<sup>2</sup>This expression is said to be bosonized, because we write it in terms of boson fields. As explained in Chapter 1, this procedure requires some approximation, in this case, the physical assumption of boson dominance and the approximation of the projector.

Considering periodic boundary conditions for the elementary bosons fields, on which the matrices  $N_t$ ,  $M_t$  and  $U_t$  depend, the partition function becomes

$$Z_C := \mathbf{Tr}^F \left[ \hat{\mathcal{P}} \hat{T}_0^\dagger \hat{V}_0 e^{\mu \hat{n}_B} \hat{T}_1 \hat{\mathcal{P}} T_1^\dagger \hat{V}_1 e^{\mu \hat{n}_B} \hat{T}_2 \cdots \hat{\mathcal{P}} T_{L_0-1}^\dagger \hat{V}_{L_0-1} e^{\mu \hat{n}_B} \hat{T}_0 \right] \quad (3.26)$$

$$= \int \prod_{t=0}^{L_0-1} \left[ \frac{d\phi_t^* d\phi_t}{2\pi i} \right] \frac{\langle \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} | \phi_{t+1} \rangle}{\langle \phi_t | \phi_t \rangle}. \quad (3.27)$$

In order to derive the effective action of composite boson fields we need the matrix element

$$\langle \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} | \phi_{t+1} \rangle. \quad (3.28)$$

Since the subspace of  $\hat{\mathcal{P}}$  includes only neutral states and since  $[e^{\mu \hat{n}_B}, \hat{T}_{t+1}] = 0$ , then the chemical potential disappears in (3.25). In fact,  $\hat{n}_b$  is not active over the subspace of the projector  $\hat{\mathcal{P}}$ . By use of the formulae collected in appendix A, we have

$$a \langle \phi_t | \hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1} | \phi_{t+1} \rangle = \det_+ \left[ e^{-M_t^\dagger} \overset{\circ}{E}_{t+1,t} e^{-M_{v,t+1}} \right] = \det_- \left[ e^{-M_{u,t+1}} E_{t+1,t} e^{-M_t^\dagger} \right], \quad (3.29)$$

where

$$\overset{\circ}{E}_{t+1,t} := \overset{\circ}{\mathcal{F}}_{N,t} e^{M_t^\dagger} e^{M_{v,t+1}} \left( \overset{\circ}{\mathcal{F}}_{N,t+1} \right)^\dagger + \mathcal{F}_t e^{-M_t^\dagger} e^{-M_{u,t+1}} \mathcal{F}_{t+1}^\dagger, \quad (3.30a)$$

$$E_{t+1,t} := (\mathcal{F}_{N,t+1})^\dagger e^{M_{u,t+1}} e^{M_t^\dagger} \mathcal{F}_{N,t} + \mathcal{F}_{t+1}^\dagger e^{-M_{v,t+1}} e^{-M_t^\dagger} \mathcal{F}_t, \quad (3.30b)$$

$$\overset{\circ}{\mathcal{F}}_{N,t} := 1 + B_t N_t^\dagger, \quad (3.30c)$$

$$\mathcal{F}_{N,t} := 1 + N_t^\dagger B_t. \quad (3.30d)$$

The last equality in (3.29) is valid since  $\overset{\circ}{E}_{t,t+1}$  and  $E_{t,t+1}$  (as well  $\overset{\circ}{\mathcal{F}}_{N,t}$  and  $\mathcal{F}_{N,t}$ ) have the same determinant, see appendix B.3. Therefore, by substitution of (3.29) and (3.5) in equation (3.27), we find the functional expression of the partition function in terms of the composite boson fields  $\phi^*$  and  $\phi$

$$Z_C = \int \left[ \frac{d\phi^* d\phi}{2\pi i} \right] e^{-S_C(\phi^*, \phi)}, \quad (3.31)$$

where

$$S_C(\phi^*, \phi) := \sum_t^{L_0-1} \text{tr}_+ \left[ \ln \overset{\circ}{R}_t^{-2} - \ln \left( e^{-M_t^\dagger} \overset{\circ}{E}_{t,t+1} e^{-M_{v,t+1}} \right) \right]. \quad (3.32)$$

The action above describes the composite boson that interacts with the external boson fields on which  $N$  and  $M$  depend. The interaction is no customary. The action in (3.32) is gauge invariant, but it is not invariant under the global  $U(1)$  transformation of the composite boson fields

$$\phi_t \mapsto e^{i\vartheta} \phi_t, \quad \phi_t^* \mapsto e^{-i\vartheta} \phi_t^*. \quad (3.33)$$

This implies that the boson number is not a good quantum number. One side of the dynamical problem is the determination of the structure matrices  $\Phi$ ; the path integral

formulation is a good starting point for the resolution of the problem. For the moment, all present fermions are bosonized, but in the next Chapter we generalize the treatment to the case that include fermion states. Notice that in the action  $S_C(\phi^*, \phi)$ , time and space are not treated in a symmetric way.

It is very important to observe that the derivation of the effective action for composite fields is exact under the physical assumption of boson dominance. The only approximation that has been made is over the form of the projector over the bosons subspace. This derivation is the most general one; in fact, no hypothesis has been taken over the type of fermions and no gauge has been fixed.

### 3.3.1 Evaluation of the partition function by the saddle point equation

At present, we do not know a procedure to exactly perform the integral in (3.31). However, it can be evaluated by saddle point method, as, for large nilpotency index, the integral is dominated by the minimum of the action. The saddle point equations, for (3.31), in  $\mathcal{F}_t$  and  $\mathcal{F}_{t+1}^\dagger$  are, respectively

$$\mathcal{F}_t^\dagger \overset{\circ}{R}_t^2 = \left[ N_t^\dagger e^{M_t^\dagger} e^{M_{v,t+1}} \overset{\circ}{\mathcal{F}}_{N,t+1}^\dagger + e^{-M_t^\dagger} e^{-M_{u,t+1}} \mathcal{F}_{t+1}^\dagger \right] \overset{\circ}{E}_{t+1,t}^{-1}, \quad (3.34a)$$

$$\overset{\circ}{R}_{t+1}^2 \mathcal{F}_{t+1} = \overset{\circ}{E}_{t+1,t}^{-1} \left[ \overset{\circ}{\mathcal{F}}_{N,t} e^{M_t^\dagger} e^{M_{v,t+1}} N_{t+1} + \mathcal{F}_t e^{-M_t^\dagger} e^{-M_{u,t+1}} \right]. \quad (3.34b)$$

In order to look for a solution of the saddle point equations (3.34a) and (3.34b), we multiply the first equation by  $\mathcal{F}_t$  on the left and the second one by  $\mathcal{F}_{t+1}^\dagger$  on the right

$$\overset{\circ}{R}_t^2 = e^{M_t^\dagger} e^{M_{v,t+1}} \overset{\circ}{\mathcal{F}}_{N,t+1}^\dagger \overset{\circ}{E}_{t+1,t}^{-1}, \quad (3.34c)$$

$$\overset{\circ}{R}_{t+1}^2 = \overset{\circ}{E}_{t+1,t}^{-1} \overset{\circ}{\mathcal{F}}_{N,t} e^{M_t^\dagger} e^{M_{v,t+1}}. \quad (3.34d)$$

This equation does not admit  $\phi = 0$  as a solution; this means that the ground state is a condensation of mesons, while there exists the trivial solution  $\phi = \infty$ . The equations (3.34c) and (3.34d) give

$$\overset{\circ}{E}_{t+1,t} = \overset{\circ}{R}_t^{-2} e^{M_t^\dagger} e^{M_{v,t+1}} \overset{\circ}{\mathcal{F}}_{N,t+1}^\dagger = \overset{\circ}{\mathcal{F}}_{N,t} e^{M_t^\dagger} e^{M_{v,t+1}} \overset{\circ}{R}_{t+1}^{-2} \quad (3.34e)$$

and

$$\left( \overset{\circ}{\mathcal{F}}_{N,t+1}^\dagger \right)^{-1} = \overset{\circ}{E}_{t+1,t}^{-1} \overset{\circ}{R}_t^{-2} e^{M_t^\dagger} e^{M_{v,t+1}}, \quad (3.34f)$$

$$\left( \overset{\circ}{\mathcal{F}}_{N,t} \right)^{-1} = e^{M_t^\dagger} e^{M_{v,t+1}} \overset{\circ}{R}_{t+1}^{-2} \overset{\circ}{E}_{t+1,t}^{-1}. \quad (3.34g)$$

Substituting (3.34e) in (3.34c) and in (3.34d), we arrive to the following equations

$$\mathcal{F}_t^\dagger = \mathcal{F}_t^{-1}(e^{-M_{v,t+1}}e^{-M_t^\dagger} - 1) + [N_t^\dagger + e^{-M_t^\dagger}e^{-M_{u,t+1}}\mathcal{F}_{t+1}^\dagger(\overset{\circ}{\mathcal{F}}_{N,t+1}^\dagger)^{-1}]e^{-M_{v,t+1}}e^{-M_t^\dagger}, \quad (3.34h)$$

$$\mathcal{F}_{t+1} = (e^{-M_{v,t+1}}e^{-M_t^\dagger} - 1)\mathcal{F}_{t+1}^{\dagger-1} + e^{-M_{v,t+1}}e^{-M_t^\dagger}[N_{t+1} + \overset{\circ}{\mathcal{F}}_{N,t}^{-1}\mathcal{F}_te^{-M_{v,t+1}}e^{-M_t^\dagger}], \quad (3.34i)$$

The equations (3.34h) and (3.34i) are two non-linear coupled iterative equations and we do not know a solution to them. These equations simplify considerably if we limit to consider the axial gauge and  $M = 0$ . This is the case, for example, of Kogut-Susskind fermions, for which the equations become

$$\mathcal{F}_{t+1} = N_{t+1} + (1 + \mathcal{F}_t N_t^\dagger)^{-1} \mathcal{F}_t, \quad (3.35a)$$

$$\mathcal{F}_t^\dagger = N_t^\dagger + \mathcal{F}_{t+1}^\dagger (1 + N_{t+1} \mathcal{F}_{t+1}^\dagger)^{-1}. \quad (3.35b)$$

The equations (3.35) can be solved by iteration and, imposing boundary condition, they can be transformed into eigenvalue equations. Notice that the equations (3.34a) and (3.34b) are not independent in the axial gauge; indeed the Hermitian Conjugate of (3.34b) is (3.34a) with the substitution of  $t$  with  $t + 1$ .

### Saddle point equation stationary solution

Consider  $N_t$  to be time independent and  $M = 0$ . Therefore, it is necessary that the boson fields, on which  $N$  depends, are time independent. This is not the case of a gauge theory. For a time independent  $N$ , the equations for  $\mathcal{F}_{t+1}$  and  $\mathcal{F}_t^\dagger$  are time independent too; thus, it is possible to find a constant solution for  $\mathcal{F}_t =: \bar{\mathcal{F}}$ . Eq. (3.35a) gives

$$\bar{\mathcal{F}}N^\dagger\bar{\mathcal{F}} - \bar{\mathcal{F}}N^\dagger N - N = 0. \quad (3.36)$$

Supposing that the solution of (3.36) is of the form

$$\bar{\mathcal{F}} = NA, \quad (3.37)$$

then (3.36) becomes

$$N \left( AN^\dagger NA - AN^\dagger N - 1 \right) = 0. \quad (3.38)$$

In the case we are considering

$$H^2 := \frac{1}{4}N^\dagger N$$

is the square of the hamiltonian of the fermions. A time independent  $A$  must commute with it and therefore it is possible to diagonalize  $N^\dagger N$  and  $A$  at the same time. In the eigenvectors base, the equation (3.38) becomes an eigenvalue equation for the matrix  $A$

$$\tilde{A}_q^2 4E_q^2 - \tilde{A}_q E_q^2 - 1 = 0, \quad (3.39)$$

where  $\tilde{A}_q$  and  $E_q^2$  are the eigenvalues of  $A$  and  $H^2$  related to the  $q$ -th eigenvector. The solutions for each eigenvalue are two

$$\tilde{A}_{q\pm} = \left( E_q \pm \sqrt{1 + E_q^2} \right) (2E_q)^{-1}, \quad (3.40)$$

and they satisfy

$$\tilde{A}_{q+} + \tilde{A}_{q-} = 1, \quad \tilde{A}_{q\pm} = - \left( \tilde{A}_{q\mp} 4H_q^2 \right)^{-1}, \quad \tilde{A}_{q\pm}^2 = \frac{1}{4H^2} + \tilde{A}_{q\pm}. \quad (3.41)$$

To a general set of eigenvalues  $\{\tilde{A}_q\}$  corresponds, in a general base, a solution of the saddle point equation (3.38), of the form

$$A_\Gamma = \frac{H + \Gamma\sqrt{1 + H^2}}{2H}. \quad (3.42)$$

Here  $\Gamma$  is a square matrix whose eigenvalues  $\tilde{\Gamma}_q$  are  $\pm 1$ ; its explicit expression depends on the base selected and on the set  $\{\tilde{A}_q\}$ . The family of solutions of the saddle equation is very large. The physical solution is the one that minimizes the action. By use of (3.34e) and (3.41), the action, evaluated over a general solution of (3.36), becomes

$$\bar{S}_{c\Gamma} := S_C(\phi^*, \phi)|_{\mathcal{F}_t = \bar{\mathcal{F}}} = - \sum_t \text{tr}_+ \ln \left( 1 + NA_\Gamma N^\dagger \right) \quad (3.43a)$$

$$= - \sum_t \text{tr}_- \ln \left( E_q + \sqrt{1 + E_q^2} \right)^{\tilde{\Gamma}_q^2}. \quad (3.43b)$$

The  $\text{tr}_\pm$  must be interpreted as the sum over the eigenvalues and the sign in (3.60) depends on the sign in (3.40). The set of eigenvalues that minimizes the action is the one corresponding to the choice of the positive sign in (3.40) for all the eigenvalues. This set corresponds to a  $\Gamma$  with only positive eigenvalues and diagonal in the position base. Instead, the solution with all negative signs in (3.42) is a maximum of the action. The ground state is a condensate of composites with structure function

$$\bar{\mathcal{F}} = \frac{1}{2} NH^{-1} \left( H + \sqrt{1 + H^2} \right). \quad (3.44)$$

This is a time independent minimum of the effective action. In such composites, the occupation number of high momentum states is larger than that of low momentum states, the structure function is different from that of bound pairs. Anyway, if there exist time independent extremes,  $\bar{\mathcal{F}}$  can eventually be only a local minimum.

### 3.4 Second form of the effective boson action

In this section we derive an alternative expression of the effective action of the composite bosons. This can be achieved performing the following transformation of the fermion operators

$$\hat{u} \mapsto \hat{u}' := \hat{v}^\dagger, \quad \hat{v}^\dagger \mapsto \hat{u}'^\dagger := \hat{u}. \quad (3.45)$$



This transformation change particles in hols. The new vacuum is the state  $|0'\rangle$  in the space  $\mathcal{H}_F$ , the Hilbert space of fermions, and is annihilated by all the  $\hat{u}' \hat{v}'$ . Clearly, it is the fully occupied lattice

$$|0'\rangle = \prod_i \hat{v}_i^\dagger \hat{u}_i^\dagger |0\rangle \quad (3.46)$$

The new vacuum is connected to the trivial solution  $\phi = \infty$ . Substituting (3.45) in the definition of  $\hat{\psi}$ , one observes that the transformation change the role of  $P_0^{(\pm)}$ , then it is related to time reversal. To obtain the effective action we must calculate the matrix element in Eq. (3.28). For the canonical boson, whose creation and destruction operator are  $\hat{b}$  and  $\hat{b}^\dagger$ , it is possible to prove that  $\exp(\hat{b}\hat{b}^\dagger)(\hat{b}^\dagger)^n|0\rangle = \exp(\hat{b}^*\hat{b}^\dagger)|0\rangle$  are equivalent in the sub space with  $0, 1, \dots, n$  bosons. It follows that

$$e^{\hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger} |0\rangle = e^{\hat{v}' \mathcal{F}' \hat{u}'} \prod_i \hat{v}_i' \hat{u}_i' |0'\rangle \simeq e^{\hat{u}' \mathcal{F}' \hat{v}' \dagger} |0'\rangle. \quad (3.47)$$

This transformation is only an approximation, but in the case of large  $\Omega$  and for a number of composite bosons  $n \ll \Omega$ , the equivalence should hold. The operator  $\hat{T}$  changes under the transformation; in fact

$$\exp\left(\hat{\psi}^\dagger P_0^{(+)} N P_0^{(-)} \hat{\psi}^\dagger\right) \mapsto \exp\left(\hat{\psi}' \dagger P_0^{(-)} N P_0^{(+)} \hat{\psi}'\right). \quad (3.48)$$

Instead, if we apply the time reversal to the fermion action on a lattice and then perform the Lüscher construction [16] of the transfer matrix, the operator  $\hat{T}$  changes as follows

$$\hat{T}_t = \exp\left(-\hat{u}^\dagger M_t \hat{u} - \hat{v}^\dagger M_t^T \hat{v}\right) \exp\left(\hat{v} N_t \hat{u}\right) \mapsto \exp\left(\hat{v} N_t \hat{u}\right) \exp\left(-\hat{u}^\dagger M_t^T \hat{u} - \hat{v}^\dagger M_t \hat{v}\right). \quad (3.49)$$

From the previous observation, one deduces that the matrix element (3.28) changes, under the transformation (3.45), in this way

$$\langle \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} | \phi_{t+1} \rangle \mapsto \langle \phi_t | \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_t \hat{T}_{t+1}^\dagger | \phi_{t+1} \rangle, \quad (3.50)$$

with the substitutions

$$\mathcal{F}_t \mapsto \mathcal{F}_t^\dagger, \quad N_t \mapsto N_t^\dagger, \quad M_t \mapsto -M_t^*. \quad (3.51)$$

In the right side of (3.50) and of the next expressions, we have omitted the prime. It is now possible to evaluate the matrix element  $\langle \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1}^\dagger | \phi_{t+1} \rangle$ . Firstly, one observes that

$$\hat{T}_{t+1}^\dagger | \phi_{t+1} \rangle = \exp\left(N_{t+1} + e^{M_{t+1}^\dagger} \mathcal{F}_{t+1} e^{M_{t+1}^\dagger}\right) |0\rangle, \quad (3.52)$$

$$\langle \phi_t | \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_t = \langle 0 | \exp\left(N_t + e^{M_{v,t}} \mathcal{F}_t e^{M_{u,t}}\right). \quad (3.53)$$

The chemical potential disappears, as expected. The matrix element (3.50) can be computed introducing the identity in the  $\mathcal{H}_F$ , written in terms of coherent states between the two parts in (3.52). The result is

$$\langle \phi_t | \hat{T}_t \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1}^\dagger | \phi_{t+1} \rangle = \det - \overset{\circ}{E}_{t,t+1}, \quad (3.54)$$

where

$$E'_{t+1,t} := 1 + \left( N_{t+1}^\dagger + e^{-M_{u,t+1}^\dagger} \mathcal{F}_{t+1}^\dagger e^{-M_{v,t+1}^\dagger} \right) (N_t + e^{-M_t} \mathcal{F}_t e^{-M_t}), \quad (3.55a)$$

$$\overset{\circ}{E}'_{t+1,t} := 1 + (N_t + e^{-M_t} \mathcal{F}_t e^{-M_t}) \left( N_{t+1}^\dagger + e^{-M_{u,t+1}^\dagger} \mathcal{F}_{t+1}^\dagger e^{-M_{v,t+1}^\dagger} \right). \quad (3.55b)$$

We have found a second form of the effective action; it is like in (3.31), but with the action

$$S_C(\phi^*, \phi)' = \sum_{t=0}^{L_0} \text{tr}_+ \left( \ln \overset{\circ}{R}_t^{-2} - \ln \overset{\circ}{E}_{t+1,t} \right). \quad (3.56)$$

For  $M_t = 0$  and  $U_{0,t}$ , the action in Eq. (3.32) coincides with the action given in Eq. (3.56), with the substitution

$$\mathcal{F}_t \mapsto (\mathcal{F}_t)^{-1}. \quad (3.57)$$

However, remember that during the derivation of the second form we have performed the substitution in Eq. (3.51). The first form of the partition function is equivalent to the second one, with the substitution  $\mathcal{F}_t \mapsto (\mathcal{F}_t^\dagger)^{-1}$ . The time independent saddle equations (3.36) become, by use of (3.41)

$$N \left( \bar{\mathcal{F}}' N^\dagger N \bar{\mathcal{F}}' + \bar{\mathcal{F}}' N^\dagger N - 1 \right) = 0. \quad (3.58)$$

This equation can be solved in the same way as (3.36). The solution reads

$$\bar{\mathcal{F}}'_\Gamma = -N \frac{H + \Gamma \sqrt{1 + H^2}}{2H}. \quad (3.59)$$

The action  $S_C(\phi^*, \phi)'$ , evaluated over this solution, is

$$S_C(\bar{\phi}^*, \bar{\phi})'_\Gamma := - \sum_t \text{tr}_- \ln \left( E_q + \sqrt{1 + E_q^2} \right)^{-\bar{\Gamma}_q^2}. \quad (3.60)$$

Since a minus appears in the exponential, the physical solution corresponds to the one for which all the eigenvalues of the corresponding  $\Gamma$  are negatives. This means that, under the transformation (3.45), the structure function changes. Since  $S_C(\phi^*, \phi)[\bar{\mathcal{F}} = N A_{\Gamma=+1}] = S_C(\phi^*, \phi)[\bar{\mathcal{F}} = -N A_{\Gamma=-1}]$ , the partition function, are equivalent at the leading order.

The second form of the partition function is easier to handle and allows to evaluate the Gaussian corrections to the saddle point [1].

## Chapter 4

# Finite Temperature and finite chemical potential formalism

In this Chapter we extend the formalism outlined in Chapter 3 to field theories at finite temperature and density. The aim of the studies under discussion is to reformulate the theory in terms of the fields that are related to the dominant degrees of freedom, whose dynamics is controlled by an effective action. The hope that motivates this investigation is that the effective theory may be simpler than the fundamental one.

### 4.1 Quasi-quarks and generalized Bogoliubov transformation

In order to extend the formalism explained in the previous chapters to finite temperature, we must introduce excited states. The aim is to consider baryons composites, but we limit ourselves to introduce, in addition to mesons, excited states with fermion quantum number. In this way, we introduce states with no null fermion number and therefore generalize the formalism to finite density. In Chapter 3 we have determinate, under some assumptions, the ground state, which depends on the dynamical propriety of the theory considered. The new states must be excitations of this ground state. Since we consider quarks and anti-quarks, there are two types of excited states, that we call "quasiquarks" and "quasiantiquarks". The operator associated to these states can be introduced using a generalization of the Bogoliubov-Valatin transformation [7, 26]. This is a linear transformation for operator doublet  $\hat{\psi}$

$$\hat{\psi} \mapsto \hat{\lambda} := U\hat{\psi}. \quad (4.1)$$

This application, defined by the matrix  $U$ , must be unitary with respect to the metric of the fermion space<sup>1</sup>; then  $U$  has to satisfies to

$$\gamma_0 U^+ \gamma_0 U = \mathbb{I}d. \quad (4.2)$$

Choosing the following parametrization for the linear transformation

$$U := \begin{pmatrix} A & -AB^\dagger \\ CD & C \end{pmatrix}, \quad (4.3)$$

and replacing it in (4.2), we obtain the following system with solutions

$$\begin{cases} A^\dagger A + D^\dagger C^\dagger CD & = 1 \\ C^\dagger C + BA^\dagger AB^\dagger & = 1 \\ D^\dagger C^\dagger C - A^\dagger AB^\dagger & = 0 \end{cases}, \quad \begin{cases} B & = D \\ A^\dagger A & = (1 + B^\dagger B)^{-1} \\ CC^\dagger & = (1 + BB^\dagger)^{-1} \end{cases}. \quad (4.4)$$

These equations give the expressions only of the product of  $A^\dagger A$  and  $C^\dagger C$ , that is a constrain over the expressions of  $A$  and  $C$ , namely

$$A = \mathcal{S} \left(1 + B^\dagger B\right)^{-1/2}, \quad C = \mathcal{S}' \left(1 + BB^\dagger\right)^{-1/2}. \quad (4.5)$$

Here  $\mathcal{S}$  ( $\mathcal{S}'$ ) is the root of the identity and satisfies to:  $\mathcal{S} = P_0^{(-)} \mathcal{S} P_0^{(-)}$  ( $\mathcal{S}' = P_0^{(+)} \mathcal{S}' P_0^{(+)}$ ),  $\mathcal{S}^\dagger \mathcal{S} = \mathcal{S} \mathcal{S}^\dagger = P_0^{(-)}$  ( $\mathcal{S}'^\dagger \mathcal{S}' = \mathcal{S}' \mathcal{S}'^\dagger = P_0^{(+)}$ ) and commutes with  $B^\dagger B$  ( $BB^\dagger$ ). The solutions given in Eqs. (4.4) and (4.5) describe the most general unitary transformation in the doublet space

$$U =: \begin{pmatrix} \mathcal{S} (1 + B^\dagger B)^{-\frac{1}{2}} & -\mathcal{S} (1 + B^\dagger B)^{-\frac{1}{2}} B^\dagger \\ \mathcal{S}' (1 + BB^\dagger)^{-\frac{1}{2}} B & \mathcal{S}' (1 + BB^\dagger)^{-\frac{1}{2}} \end{pmatrix}. \quad (4.6)$$

The arbitrariness in the choice of  $\mathcal{S}$  corresponds to the arbitrariness in phases for the coefficients of the Bogoliubov transformation used in the theory of superconductivity [23, 26, 27]. There are other analogies between these two different theories. Notice that the only free parameter of the transformation is the fermion<sup>+</sup> matrix  $B$ , which determines the transformation, while the matrices  $A$  and  $C$  are boson-like and depend on  $B$ ; see appendix (B.3). Observe that for  $B = \mathcal{F}$  these operators annihilate the state  $e^{\hat{u}^\dagger \mathcal{F} \hat{v}^\dagger} |0\rangle$ , i.e. they are excitations of the vacuum.

Given  $\hat{\lambda}$ , it is possible to introduce the quasi-particles  $\hat{\alpha}$  and  $\hat{\beta}$  operators. These operator have to be defined in a way such that  $\hat{\lambda}$  is the doublet operator associated to them, i.e. they have to satisfy the same relation that exist between the field  $\hat{\psi}$  and  $\hat{u}$ ,  $\hat{v}$  (1.17), The definitions are

$$\hat{\alpha} := P_0^{(-)} \hat{\lambda}, \quad \hat{\beta}^\dagger := P_0^{(+)} \hat{\lambda}, \quad \hat{\alpha}^\dagger := \hat{\lambda}^\dagger P_0^{(-)}, \quad \hat{\beta} := \hat{\lambda}^\dagger P_0^{(+)}. \quad (4.7)$$

---

<sup>1</sup>The inner product of the fermion doublet is  $\bar{\hat{\psi}} \hat{\psi} = \hat{u}^\dagger \hat{u} - \hat{v} \hat{v}^\dagger$ ; the metric is  $\text{Diag}(1, -1)$ , the gamma matrices are given in B.

With these definitions of  $\hat{\alpha}$  and  $\hat{\beta}$ ,  $\hat{\lambda}$  is the doublet operator associated to them

$$\hat{\lambda} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta}^\dagger \end{pmatrix}. \quad (4.8)$$

It is necessary to introduce the unitary operator  $\hat{\mathcal{U}}$ , associated to the transformation described by U in (4.1). We suppose that the generator is a composition of a rotation in the space of the operator particles, with generator

$$\hat{\Theta} = \hat{u}^\dagger \Theta_{\hat{u}} \hat{u} + \hat{v}^\dagger \Theta_{\hat{v}}^T \hat{v} \quad (4.9)$$

and of a rotation in the space of the operator doublet

$$\hat{S} := \hat{u}^\dagger X^\dagger \hat{v}^\dagger - \hat{v} X \hat{u}. \quad (4.10)$$

Then

$$\hat{\mathcal{U}} := e^{i\hat{\Theta}} e^{\hat{S}}. \quad (4.11)$$

Obviously, since the transformation described by U is unitary, the operators defined in (4.7) respect anti-canonical commutation relations and they annihilate the new ground state of the theory  $\hat{\mathcal{U}}|0\rangle$ .

#### 4.1.1 Evaluation of the parameters of the Bogoliubov transformation

In Chapter 3 we have determined that, for large  $\Omega$ , the ground state is a condensate of mesons (3.3), whose structure function is a solution of the saddle point equations (3.34). In order to introduce the excited states over the physical ground state, we need to perform a transformation in the space of the doublets, such that the new vacuum is the state  $|\phi\rangle$ . The parameters in the operator, appearing in Eq. (4.11), have to be fixed as to obtain

$$\hat{\mathcal{U}}|0\rangle = \frac{|\phi\rangle}{(\langle\phi|\phi\rangle)^{\frac{1}{2}}}. \quad (4.12)$$

As we explain in appendix C, the generator  $\hat{\mathcal{U}}$  creates the ground state if the parameters of the transformation solve the following equations

$$\cos \sqrt{X^\dagger X} = \frac{1}{\sqrt{1 + \mathcal{F}^\dagger \mathcal{F}}}, \quad e^{i\Theta_{\hat{u}}} X^\dagger e^{i\Theta_{\hat{v}}} = \mathcal{F}^\dagger. \quad (4.13)$$

For these values of the parameters,  $\hat{\psi}$  transforms in this way

$$\hat{\mathcal{U}} \hat{\psi} \hat{\mathcal{U}}^{-1} = \begin{pmatrix} \frac{\exp(-i\Theta_{\hat{u}})}{\sqrt{1 + \mathcal{F}^\dagger \mathcal{F}}} [\hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger] \\ \frac{\exp(+i\Theta_{\hat{u}})}{\sqrt{1 + \mathcal{F} \mathcal{F}^\dagger}} [\hat{v}^\dagger + \mathcal{F} \hat{u}] \end{pmatrix}. \quad (4.14)$$

This transformation rule is equivalent to the one given in Eq. (4.6). Indeed, the quasi-particles,  $\hat{\alpha}$  and  $\hat{\beta}$  defined by (4.7) from  $\hat{U}\hat{\psi}$ , with the parameters of the transformation

$B = \mathcal{F}$ , and  $\hat{\alpha}$  and  $\hat{\beta}$  defined from  $\hat{\mathcal{U}}\hat{\psi}\hat{\mathcal{U}}^{-1}$ , with the parameters that satisfy to (4.13), are mutually orthogonal to the ground state

$$\hat{\alpha}_i|\phi\rangle = \hat{\beta}_i|\phi\rangle = 0. \quad (4.15)$$

The quasi-particles, defined in these two ways, are different for a rotation of the operators.

Notice that the structure function of the mesons  $\mathcal{F}$  is gauge covariant, in order to obtain a gauge invariant state. From the point of view of symmetries we remark that  $\hat{\alpha}$  and  $\hat{u}$  ( $\beta$  and  $\hat{v}$ ) transform in the same way by construction, because of the presence of the dynamical bosonic fields appearing in  $\overset{\circ}{R}$ ,  $R$  and  $\mathcal{F}$ . This is different with respect to the original Bogoliubov's transformation where the matrices  $\overset{\circ}{R}$ ,  $R$  and  $\mathcal{F}$  are kept fixed. Since the  $\hat{\alpha}|0\rangle$  ( $\hat{\beta}^\dagger|0\rangle$ ) has fermion number  $+1$  ( $-1$ ) we call this operator of quaquarks (quasi antiquarks).

#### 4.1.2 The Bogolibov transform fields

In order to obtain the effective action for the fields related to the quasiparticles, we must introduce the coherent states of the latter. These states can be obtained acting with  $\hat{\mathcal{U}}$  over the coherent states of the particles

$$\hat{\mathcal{U}}|\alpha, \beta\rangle = |\alpha\beta; \phi\rangle := \exp\left(-\alpha\hat{\alpha}^\dagger - \beta\hat{\beta}^\dagger\right) \exp\left(\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger\right) |0\rangle \quad (4.16a)$$

$$= \exp\left(\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger - a\hat{\alpha}^\dagger - b\hat{\beta}^\dagger - \beta\mathcal{F}\alpha\right) |0\rangle, \quad (4.16b)$$

where

$$a_i := (R^{-1}\alpha)_i \quad b_i := \left(\beta \overset{\circ}{R}^{-1}\right)_i.$$

The last expression of the coherent states, Eq. (4.16b), is obtained using the formula of *Campbell – Baker – Hausdorff*. Notice, that in the new coherent states we individualize the parameters  $\phi$  of the transformation that has been performed to derive them. The states defined in (4.16), like the ones seen in section 3.1, have the property of a fixed phase relation among components of different number of particles. However, they are not an eigenvectors of  $\Phi$  (3.4). Instead, since the quasiparticelle operators obey a canonical algebra, the states (4.16) are eigenvalues of the operators  $\hat{\alpha}$  and  $\hat{\beta}$ . The fields  $\alpha, \beta$  (and  $a, b$ ) are 2 spinor with quark quantum number. The scalar product of states of the type of (4.16) is

$$\begin{aligned} \langle \alpha_1\beta_1; \phi_1 | \alpha_2\beta_2; \phi_2 \rangle &= \det_+ \overset{\circ}{R}_{2,1}^{-2} \times \\ &\times \exp\left(a_1^* \overset{\circ}{R}_{2,1}^2 a_2 + b_1^* \overset{\circ}{R}_{2,1}^{\circ 2T} b_2 + \right. \\ &\left. + a_1^* \overset{\circ}{\mathcal{F}}_2^\dagger \overset{\circ}{R}_{2,1}^{\circ 2} b_1^* + b_2 \overset{\circ}{R}_{2,1}^{\circ 2} \mathcal{F}_1 a_2 - a_1^* \overset{\circ}{\mathcal{F}}_1^\dagger \overset{\circ}{R}_{1,1}^{\circ 2} b_1^* - b_2 \overset{\circ}{R}_{2,2}^{\circ 2} \mathcal{F}_2 a_2\right). \end{aligned} \quad (4.17)$$

In the first line of (4.17), appears the ground state contribution (or composite boson contribution); while in second the quasiparticelle one. Since  $\hat{\alpha}$  and  $\hat{\beta}$  depend on the parameters of the transformation, the quasi-particelle contribution to (4.17) depends on the configuration of the ground state. The terms in the last line of (4.17) are null for an equal transformation  $\phi_1 = \phi_2$ . This is due to the fact that  $\hat{\alpha}$  and  $\hat{\beta}$ , that satisfy to (4.15) for a value of the  $\phi$  field, do not satisfy the same equation for different values of the field.

In the standard procedure, as we explain in section 1, to derive the functional form of the partition function from the operatorial expression, it is necessary to evaluate the trace over coherent states and therefore to know the identity written in terms of these state. The identity ( $\hat{\mathcal{P}}_{mq\bar{q}}$ ) expressed in terms of the coherent states of the Bogoliubov transform, with parameter  $\phi$ , of particles operator is

$$\hat{\mathcal{P}}_{mq\bar{q}} := \int D[\alpha^*, \alpha, \beta^*, \beta] \frac{|\alpha\beta; \phi\rangle \langle \alpha\beta; \phi|}{\langle \alpha\beta; \phi | \alpha\beta; \phi \rangle}. \quad (4.18)$$

Since we introduce  $\hat{\mathcal{P}}_{mq\bar{q}}$  at each time slice, we can do a change of base at each time. Then, the parameter of the Bogolibov's transform becomes a function of the time.

The Bogolibov's transform can be linked, univocally, if we do not consider a rotation in the space of the parameters, with the structure function of the mesons; we can interpret the Bugolibov's parameter  $B$  as the structure suction  $\mathcal{F}$  of the condensate according to (4.16). They can also be time dependent.

## 4.2 First form of the effective action

In the sequel, we discuss how to obtain the effective action of the quasisquarks and quasiantiquarks in presence of a condensate of mesons. We consider the most general instance: no assumption is made over the form of the transfer matrix and we do not fix any gauge. The fermionic sector of the partition function in terms of the transfer matrix is given by (3.24). We evaluate the trace in the base of the Bogoliubov fields. Notice that we have two fields, one ( $\alpha$ ) for the quasisquarks and the other ( $\beta$ ) for the quasiantiquarks. In order to derive the effective action we need to introduce, for each time slice, the identity written in terms of these states (4.18). By the same procedure described in section 3.3, we obtain the functional expression of the partition function

$$Z_\phi := \mathbf{Tr}^F \prod_{t=0}^{L_0-1} \left[ \hat{\mathcal{P}}_{mq\bar{q}} \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \right] \quad (4.19a)$$

$$= \int D[\alpha^*, \alpha, \beta^*, \beta] \frac{\langle \alpha_t \beta_t; \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} | \alpha_{t+1} \beta_{t+1}; \phi_{t+1} \rangle}{\langle \alpha_t \beta_t; \phi_t | \alpha_t \beta_t; \phi_t \rangle}. \quad (4.19b)$$

The formulae collected in appendix A allow to compute the matrix element appearing in the numerator in (4.19b) and to write the partition function in the following way

$$Z_\phi = \int D[\alpha, \alpha^*, \beta, \beta^*] e^{-S_\phi(\alpha, \beta)}, \quad (4.19c)$$

where

$$S_\phi(\alpha, \beta) := \sum_{t=0}^{L_0} \text{tr}_+ \overset{\circ}{R}_t^{-2} - \text{tr}_+ \left( e^{-M_t^\dagger} \overset{\circ}{E}_{t,t+1} e^{-M_{v,t+1}} \right) + \left[ a_t^* L_t a_t + b_t \overset{\circ}{L}_t b_t^* + b_t I_t^{(2,1)} a_t + a_t^* \overset{\circ}{I}_t^{(1,2)} b_t^* \right], \quad (4.20)$$

and

$$\begin{aligned} L_t &:= -R_t^2 + E_{t+1,t}^{-1} e^\mu T^{(+)}, \\ \overset{\circ}{L}_t &:= \overset{\circ}{R}_t^2 - e^{-\mu} T^{(-)} \overset{\circ}{E}_{t+1,t}^{-1}, \\ I_t^{(2,1)} &:= \left[ \overset{\circ}{R}_t^2 - \overset{\circ}{E}_{t,t-1}^{-1} \overset{\circ}{\mathcal{F}}_{N,t-1} e^{M_{t-1}^\dagger} e^{M_{v,t}} \right] \mathcal{F}_t^{\dagger-1}, \\ \overset{\circ}{I}_t^{(1,2)} &:= \mathcal{F}_t^{-1} \left[ \overset{\circ}{R}_t^2 - e^{M_t^\dagger} e^{M_{v,t+1}} \left( \overset{\circ}{\mathcal{F}}_{N,t+1} \right)^\dagger \overset{\circ}{E}_{t+1,t}^{-1} \right]. \end{aligned}$$

$S_\phi(\alpha, \beta)$  can be interpreted as the effective action of the quasi-particles. The index  $\phi$  denotes the base that has to be used to evaluate the trace. Obviously, the action in (4.20) is gauge invariant and, for null values of the Grassmann variables, reduces to the known one (3.32). In  $S_\phi(\alpha, \beta)$  appears the pure ground state contribution and the last four terms are related to quasi particles. The term proportional to  $L_t$  and  $\overset{\circ}{L}_t$  is the kinetic term of the quasi-particles, that move and interact with the mesons. The terms proportional to  $I_t^{(2,1)}$  and  $\overset{\circ}{I}_t^{(1,2)}$  describe the interaction, mediated by the mesons, between quasi-quarks and quasiaquarks. Integrating over the Grassmann fields, we obtain the effective action for the meson fields

$$S_C(\phi^*, \phi) := \sum_{t=0}^{L_0-1} \left[ \text{tr}_+ \overset{\circ}{R}_t^{-2} - \text{tr}_+ e^{-M_t^\dagger} \overset{\circ}{E}_{t,t+1} e^{-M_{v,t+1}} - 2 \text{tr}_+ \overset{\circ}{R}_t \right] - \text{Tr}_- \ln(-L_t) - \text{Tr}_+ \ln \left[ \overset{\circ}{L}_t + I_t^{(2,1)} \left( -\overset{\circ}{L}_t \right)^{-1} \overset{\circ}{I}_t^{(1,2)} \right]. \quad (4.21)$$

The first two terms in (4.21) describe a pure boson contribution, while the third term comes from a change of variable in the Grassmann integral. The second line, instead, denotes the contribution of the thermal excitations. Given the exact form of the partition function, we must evaluate the sum over the time index too, to know its value. At the moment, it is possible to carry out the trace only in a few cases.



### 4.2.1 Stationary case

Return to consider the case treated in section 4.2.1 and to study the consequences of having introduced quasi-quarks and quasiantiquarks. Recall that we supposed  $M_t = 0$ ,  $U_t = 1$  and  $N$  to be time independent. To evaluate the partition function, we need to perform the sum over the time index. Since the matrix  $N$  is time independent, we assume that the relevant configurations for the composite boson fields  $\phi^*$  and  $\phi$  are time independent too, namely the ground state is a stationary state. Suppose that  $B = NA$ , for any  $B$  that is not necessarily a solution of a saddle point equation, and that  $A$  commutes with  $N^\dagger N$ . From the former hypothesis follows that

$$BN^\dagger = NB^\dagger, \quad N^\dagger B = B^\dagger N. \quad (4.22)$$

Performing some algebra in the expression (4.21) and using the last identity, one can deduce that

$$S_C(\phi^*, \phi) := -\text{Tr}_+ \ln \left[ \left( 2 + NN^\dagger - \left( e^{-\mu T^{(-)}} + e^{\mu T^{(+)}} \right) \right) \right]. \quad (4.23)$$

This expression gives the exact action and it is  $\mathcal{F}$  independent. This means that the transformation given in (4.1) is a symmetry of the system. This result confirms that the operator  $\hat{P}_{mq\bar{q}}$  is not a projector, but it is the change matrix base for different choices of the  $B$  parameter in (4.1). The main difference between our results and the ones obtained in Ref. [2] is that, since we consider the quasiantiquarks too, we do not omit any state in the trace.

## 4.3 Second form of the effective action

Following the same procedure described in section (3.4), we can deduce an alternative expression for the partition function. The first step to perform is the unitary transformation reported in the (3.45). By this transformation the transfer matrix changes as

$$\hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \mapsto \hat{T}_t \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1}^\dagger, \quad (4.24)$$

with the replacements

$$N \mapsto N^\dagger, \quad M \mapsto -M. \quad (4.25)$$

The new Bogoliubov's transformed doublet is not the doublet given in (4.8) with the substitutions (3.45); it is necessary to repeat the procedure of the Bogoliubov's transformation in the new base of the operators  $\hat{u}'$  and  $\hat{v}'$ . Following the same procedure expressed in section 4.2, it is possible to deduce a different expression for the partition

function. The functional form is like the one in Eq. (4.19c), but with the action

$$S_\phi(\alpha\beta)' = \sum_{t=0}^{L_0} \left\{ \text{tr}_+ \left( \ln \overset{\circ}{R}_t^{-2} - \ln \overset{\circ'}{E}_{t+1,t} \right) + \right. \\ \left. - \left[ a_t^* L_t' a_t + b_t \overset{\circ'}{L}_t b_t^* + b_t I_t'^{(2,1)} a_t + a_t^* \overset{\circ'}{L}_t^{(1,2)} b_t^* \right] \right\}, \quad (4.26)$$

where

$$L_t' := - \left[ R_t^2 - e^{-M_t} (E')_{t+1,t}^{-1} e^{-M_{u,t+1}^\dagger} e^\mu T^{(+)} \right] \\ \overset{\circ'}{L}_t := \overset{\circ}{R}_t^2 - e^{-\mu} T^{(-)} e^{-M_{v,t+1}^\dagger} (\overset{\circ'}{E}_{t+1,t})^{-1} e^{-M_t} \\ I_t'^{(2,1)} := - \overset{\circ}{R}_t^2 \mathcal{F}_t + e^{-M_{v,t}^\dagger} (\overset{\circ'}{E}_{t,t-1})^{-1} [N_{t-1} + e^{-M_{t-1}} \mathcal{F}_{t-1} e^{-M_{t-1}}] e^{-M_{u,t}^\dagger} \\ \overset{\circ'}{L}_t^{(1,2)} := - \mathcal{F}_t^\dagger \overset{\circ}{R}_t^2 + e^{-M_t} [N_{t+1}^\dagger + e^{-M_{u,t+1}^\dagger} \mathcal{F}_{t+1}^\dagger e^{-M_{u,t+1}}] (\overset{\circ'}{E}_{t+1,t})^{-1} e^{-M_{t+1}}.$$

The boson matrices  $E'_{t+1,t}$  and  $\overset{\circ'}{E}_{t+1,t}$  are defined in (3.55a). For  $M_t = 0$  and  $U_{0,t}$ , the action (4.26) coincides with the action given in (4.20), under the substitutions

$$B_t \mapsto (B^\dagger)^{-1}, \quad a_t = (B^\dagger)^{-1} a_t, \quad b_t \mapsto b_t (B^\dagger)^{-1}. \quad (4.27)$$

The second form of purely bosonic effective action correspondent to (4.21) is obtained performing the integration over the Grassmann variables. The result reads

$$S_C(\phi^*, \phi) := \sum_{t=0}^{L_0} \left[ \text{tr}_+ \overset{\circ}{R}_t^{-2} - \text{tr}_+ \overset{\circ'}{E}_{t,t+1} - 2 \text{tr}_+ \overset{\circ}{R}_t^{-2} \right] \\ - \text{Tr}_- \ln(-L_t) - \text{Tr}_+ \ln \left[ \overset{\circ}{L}_t + I_t'^{(2,1)} (-L_t)^{-1} \overset{\circ'}{I}_t^{(1,2)} \right]. \quad (4.28)$$

Under the same conditions expressed in section 4.2.1, one can verify that the action (4.28) gives the action (4.23). This means that the transformation (3.45) is a symmetry of the system, at least for time independent configuration of the external bosons. So we have two equivalent expressions of the effective action and we can choose the simplest one. For future developments in the context of gauge theories, we believe that the second form of the action allows for simpler analytical studies.

#### 4.4 Diagonalization of the effective action

We have introduced the quasi-particles as Bogoliubov's transformations, in order to obtain excited states over the physical vacuum. The effective action that we find is given by (4.20). We observe that, for a particular choice of the parameters of the Bogoliubov's transformation, the quasiquarks do not interact with the quasantiquarks.

In fact, this peculiar value of the transformation is such that the interaction terms vanish:  $I_t^{(2,1)} = \overset{\circ}{I}_t^{(1,2)} = 0$ . The whole action is

$$S_\phi(\alpha\beta) := \sum_{t=0}^{L_0} \text{tr}_+ \overset{\circ}{R}_t^{-2} - \text{tr}_+ (e^{-M_t^\dagger} \overset{\circ}{E}_{t,t+1} e^{-M_{v,t+1}}) - [a_t^* L_t a_t + b_t \overset{\circ}{L}_t b_t^*]. \quad (4.29)$$

This value corresponds to a solution of the saddle point equations (3.34a) and (3.34b). The excitations of the ground state do not interact with each other.

The topic can be read also in another way. Suppose we want to evaluate the partition functions. However, the transfer matrix couples particles and antiparticle; in fact in  $\hat{\mathcal{T}}$  there is a term  $e^{\hat{v}N\hat{u}}$ . It is, anyway, preferable to compute the trace of a diagonal operator that does not couple particles and anti-particles. To the purpose, we can operate a change of base in the particle operator, for example a Bogoliubov's transformation, generated by an operator  $\hat{\mathcal{Z}}$  that leads to a diagonal expression like (4.29). It is obvious that the ground state corresponds to the state  $\hat{\mathcal{Z}}|0\rangle$ , where  $|0\rangle$  is the old vacuum. Since the transformation is unitary, we have obtained two representations of the algebra, equivalent in finite system. Anyhow, it is well known that, in the thermodynamic limit, some complications appear [23, 27].



## Chapter 5

# Test of the formalism on a model with 4-fermions interaction

In this chapter we test our method on a fermion system with four fields with Kogut-Susskind fermion 3 + 1 dimension. The action we take into consideration is

$$S_{FGN} = \sum_{x,y} \bar{\psi}_x [m \delta_{x,y} \mathbf{I} \otimes \mathbf{I} + Q] \psi_y - \frac{1}{2} \frac{g^2}{4N_f} \sum_x \sigma_x (\bar{\psi}_x \psi_x)^2. \quad (5.1)$$

It is clear that this system possesses the symmetries (1.7) and (1.8);  $Q$  is defined in (B.4). How explained in Chapter 1, the study of simple systems often allows to understand some phenomena that appear in more complex theories, like QCD. Indeed, the four-fermions model is believed to be an effective theory of quarks and gluons at intermediate energy.

This theory has been deeply studied for non zero values of both temperature [31] and chemical potential [32]. The method that we have discussed in Chapter 3 has already been tested at zero values with the parameters  $T$  and  $\mu$  [1]. It was proven to reproduce all the known results in the boson sector and it also allowed to determine the structure functions of the condensed composite [1]. In this section, instead, we test the method for non null values of the parameters. In order to have an action bi-linear in the fermion fields, as usually, we introduce a scalar field  $\sigma$  whose Gaussian integral gives the starting partition function

$$Z = \int D[\bar{\psi}\psi] e^{-S_{FGN}} = \int D[\bar{\psi}\psi] D[\sigma] e^{-\left(S_{FKS} + \frac{1}{2} \frac{4N_f}{g^2} \sum_x \sigma^2\right)}. \quad (5.2)$$

Here  $S_{FKS}$  is the action of the Kogut-Susskind fermions with a Yukawa interaction with a field  $\sigma$ . In the presence of such an interaction, the matrices  $N$  and  $M$ , for this choice of the regularization, depend on  $\sigma$  and are given in appendix B.1.

## 5.1 The gap equation

In the last section we have derived the effective action for the quasi-particles fields (4.20). In order to evaluate the perturbations around the ground state, we must fix the parameters of Bogoliubov's transform to maximize to ground state contribution to the action. As explained in the former Chapter, these values of the parameters of the Bogoliubov's transform correspond to the structure function of the composite  $\mathcal{F}_t$ . The values  $\phi = \bar{\phi}$  correspond to a value of the structure function that saturates the ground state (solution of the saddle point equations (3.34)); the action of the quasi-quark fields is diagonal in the quasi-particle fields and it is given in (4.29). It is bi-linear in fermion fields. Notice that, in general, the diagonalization of the action requires to solve the gap equations (3.34h) and (3.35b). The effective action (4.29), coming from the fermionic part of (5.2), can be written as

$$S_{\bar{\phi}}(\alpha, \beta, \sigma) = \sum_{t=0}^{\frac{L_0}{2}} \left\{ \text{tr}_- \ln R_t^{-2} E_{t+1,t}^{-1} - \alpha_t^* s (\nabla - \mathcal{H}_t) \alpha_t + \right. \\ \left. - \beta_t \left( \overset{\circ}{\nabla} - \overset{\circ}{\mathcal{H}}_t \right) \beta_t^* \right\}, \quad (5.3)$$

where we have defined

$$\nabla_t := s^{-1} \left( e^\mu U_{0,t} T^{(+)} - 1 \right), \quad \mathcal{H}_t := e^\mu s^{-1} \left( U_{0,t} - R_t^{-1} E_{t+1,t}^{-1} R_{t+1}^{-1} \right) T^{(+)}, \quad (5.4a)$$

$$\overset{\circ}{\nabla}_t := s^{-1} \left( 1 - e^{-\mu} T^{(-)} U_{0,t} \right), \quad \overset{\circ}{\mathcal{H}}_t := - \frac{e^{-\mu}}{s} T^{(-)} \left( U_{0,t} - \overset{\circ}{R}_t^{-1} \overset{\circ}{E}_{t+1,t}^{-1} \overset{\circ}{R}_{t+1} \right). \quad (5.4b)$$

These objects can be interpreted as the covariant and effective Hamiltonians for the bi-spinors field of quasiquarks and quasiaquarks. In the previous formula,  $s = 2$  denotes the step of lattice. Notice that this expression of the action avoid the fermion doubling, since the derivative is not symmetrical.

In the limit  $N_f \rightarrow \infty$ , the partition function is dominated by the saddle point of  $\sigma$ ; in such limit the most important configuration for the  $\sigma$  is space-time independent [31]. For the sake of simplicity, in the following we omit the time index. In this case, we know the expression of  $\mathcal{F}$ , given in (3.44), which is diagonal in the *flavours* index. As we have found in Chapter 3.3.1, the elements that appear in the definition of the Hamiltonian simplify to

$$R^{-1} E^{-1} R^{-1} = R^{-2} E^{-1} = \left( H + (1 + H^2)^{1/2} \right)^{-2}, \quad (5.5a)$$

$$\overset{\circ}{R}^{-1} \overset{\circ}{E}^{-1} \overset{\circ}{R}^{-1} = \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} = \left( \overset{\circ}{H} + (1 + \overset{\circ}{H}^2)^{1/2} \right)^{-2}. \quad (5.5b)$$

The two Hamiltonian can be written in a different way

$$\mathcal{H} = e^\mu H \left[ H + (1 + H^2)^{1/2} \right], \quad \overset{\circ}{\mathcal{H}} = -e^{-\mu} \left[ \overset{\circ}{H} + \left( 1 + \overset{\circ}{H}^2 \right)^{1/2} \right], \quad (5.6a)$$

$$R^{-2} E^{-1} = 1 - s e^{-\mu} \mathcal{H}, \quad \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} = 1 - s e^\mu \overset{\circ}{\mathcal{H}}. \quad (5.6b)$$

The Hamiltonian for the particles,  $\mathcal{H}$ , and the one for the anti-particles,  $\overset{\circ}{\mathcal{H}}$ , as  $H$  and  $\overset{\circ}{H}$  have the same eigenvalues; the only difference is that one is a boson  $^-$ , while the other is a boson  $^+$ . For time independent configurations of the external fields, it is possible to perform the Grassmann integrals and the sum over the Matsubara frequencies<sup>1</sup>. The effective action for the  $\sigma$  fields is

$$\begin{aligned} \bar{S}_{eff}[\sigma] &= \frac{1}{2} \frac{4N_f}{g^2} \sum_x \sigma^2 + N_f \frac{L_0}{2} \text{tr}_- \ln (R^{-2} E^{-1}) \\ &\quad - \text{tr}_- N_f \ln \left( 1 + (R^{-2} E^{-1} e^\mu)^{L_0/2} \right) - N_f \text{tr}_+ \ln \left( 1 + \left( \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} e^{-\mu} \right)^{L_0/2} \right). \end{aligned} \quad (5.7)$$

where the first term is the kinetic term of the auxiliary field, the second is the  $T = 0$  contribution to the quantum fluctuation, while, in the second line, appear the thermal fluctuations. As the action is proportional to  $N_f$ , the partition function in  $\sigma$  is controlled by the minimum of the action. The gap equation for the  $\sigma$  field is

$$\frac{\partial \bar{S}_{eff}[\bar{\sigma}]}{\partial \bar{\sigma}} = \frac{2L_0 N_f \bar{\sigma}}{g^2} - N_f \frac{L_0}{2} \bar{\sigma} f_q(\bar{\sigma}) - N_f \bar{\sigma} f_T(\bar{\sigma}, L_0, \mu) = 0, \quad (5.8)$$

where, as usually, we have split the quantum contribution

$$f_q(\bar{\sigma}) := \text{tr}_- (1 + H^2)^{-\frac{1}{2}} \quad (5.9)$$

from the contribution coming from the thermal fluctuations

$$f_T(\bar{\sigma}, \frac{L_0}{2}, \mu) := \text{tr}_- (1 + H^2)^{-\frac{1}{2}} \left[ \left( 1 + (e^\mu - s\mathcal{H})^{-L_0/2} \right)^{-1} - \left( 1 + (e^{-\mu} - s\overset{\circ}{\mathcal{H}})^{-L_0/2} \right)^{-1} \right]. \quad (5.10)$$

At the saddle point, the fermion number is given by  $n_F := -\frac{2}{L_0} \frac{\partial S_{eff}}{\partial \mu}$  i.e.

$$n_F = \text{tr}_- \frac{1}{1 + (e^\mu - s\mathcal{H})^{-L_0/2}} - \text{tr}_+ \frac{1}{1 + (e^{-\mu} + s\overset{\circ}{\mathcal{H}})^{-L_0/2}} =: n_F^+ - n_F^-. \quad (5.11)$$

Observe that our method is in complete agreement with the solution of the Gross-Neveu model for  $N_f \rightarrow \infty$  [31]. Since we are interested in the chiral transition, we consider the case  $m = 0$ . For this value one observes, from the naive continuum limit of  $\mathcal{H}$  and  $\overset{\circ}{\mathcal{H}}$ ,

<sup>1</sup>  $\frac{L_0}{2}$  is the time dimension of the lattice, i.e  $T^{-1}$ .

that the parameter  $\bar{\sigma}$  is the mass of the quasi-particles. For  $\bar{\sigma} = 0$ , the action possesses the following symmetry

$$\alpha \rightarrow -\gamma_5 \otimes t_5 \alpha \qquad \beta \rightarrow \beta \gamma_5 \otimes t_5. \quad (5.12)$$

The study of the gap equations and of the fermion number, for different values of  $T$  and  $\mu$ , requires the evaluation of the trace in the definitions (5.9) and (5.10). However, this can be performed only by means of numerical methods. In the following, we limit to the discussion of merely qualitative results. Notice that  $H$  is diagonal in the momentum space and that its eigenvalues are given in (B.6); the trace in the momentum space is defined in [1].

### 5.1.1 The case $T = \mu = 0$

Observe that, for  $T = \mu = 0$ , the thermal contribution is null as the fermion number. For this values of the parameters, the gap equations becomes

$$\frac{4\bar{\sigma}}{g^2} = \bar{\sigma} f_q(\bar{\sigma}). \quad (5.13)$$

This equation [1] admits solution  $\bar{\sigma} \neq 0$ , only for values of the coupling constant  $g > g_c$ , where  $g_c^2 = f_q(0)$ . For  $g < g_c$ , we have  $\bar{\sigma} = 0$  and the transformation (5.12) is a symmetry of the system. Instead, for  $g > g_c$ , the physical solution is non zero and then the symmetry of the system is broken; the  $\sigma$  field acquires a mass  $m_\sigma = 2\bar{\sigma}$ .

### 5.1.2 The case $T = 0$

In the limit of zero temperature

$$\begin{aligned} \bar{S}_{eff}[\sigma] = & \frac{1}{2} \frac{4N_f}{g^2} \sum_x \sigma^2 + \frac{L_0}{2} \left\{ \text{tr}_- \ln (R^{-2} E^{-1}) - \text{tr}_- \left[ \theta (R^{-2} E^{-1} e^\mu - 1) \times \right. \right. \\ & \left. \left. \times \ln (R^{-2} E^{-1} e^\mu) \right] + -\text{tr}_+ \left[ \theta \left( \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} e^{-\mu} - 1 \right) \ln \left( \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} e^{-\mu} \right) \right] \right\}. \end{aligned} \quad (5.14)$$

The corresponding gap equations and the number of fermions are

$$n_F = \text{tr}_- \theta (R^{-2} E^{-1} e^\mu - 1) - \text{tr}_+ \theta \left( \overset{\circ}{R}^{-2} \overset{\circ}{E}^{-1} e^{-\mu} - 1 \right), \quad (5.15a)$$

$$\frac{2}{L_0} \frac{\partial \bar{S}_{eff}[\bar{\sigma}]}{\partial \bar{\sigma}} = \frac{4N_f \bar{\sigma}}{g^2} - N_f \bar{\sigma} f_q(\bar{\sigma}) - N_f \bar{\sigma} f_T(\bar{\sigma}, 0, \mu) = 0. \quad (5.15b)$$

These equations give the following qualitative information. For any value of the chemical potential, the number of quasi-anti-quarks is zero (5.15b). Instead, the number of fermions increases as the chemical potential increases and the fermions tend to occupy high momentum states. Notice that on the lattice all the momentum states can be empty and not necessarily in the limit  $\mu \rightarrow 0$ . The contribution of the mass  $\bar{\sigma}$  to the



energy gap of the excitations is exactly reproduced. Indeed, for any value  $\mu$  such that  $e^\mu - 1 < 2\bar{\sigma}$ , the equation for the fermion number gives  $n_F^\pm = 0$ . The value of the chemical potential influences the gap equations too (5.15a); in fact the contribution coming from  $f_T(\bar{\sigma}, L_0, \mu)$  increases if the chemical potential increases; for this the solution,  $\bar{\sigma}$  must be decreasing. At  $\mu \geq \mu_c$ ,  $\bar{\sigma} = 0$ ; for this value the chiral symmetry is restored. The value  $\mu_c$  can be determined solving the equation

$$\frac{4L_0 N_f}{g^2} = \text{tr}_- \frac{L_0}{H\sqrt{1+H^2}} \left[ 1 - \theta(R^{-2}E^{-1}e^\mu - 1) \right] \Big|_{\sigma=0}. \quad (5.16)$$

Finally, for values  $g < g_c$ ,  $\bar{\sigma} = 0$ .

### 5.1.3 The case $\mu = 0$

For null values of the chemical potential, the quasiquarks and quasia antiquarks contributions in the gap equations (5.13) and in the fermion number (5.11) are equal, so  $n_F$  is zero. If  $g > g_c$  and if the temperature increases, the contribution coming from the thermal fluctuations increases. Therefore, the solutions of (5.13) decrease; for high values of the temperature the chiral symmetry is restored. The critical value of the temperature  $L_0$  can be determined from

$$\frac{4L_0 N_f}{g^2} = \text{tr}_- \frac{L_0}{H\sqrt{1+H^2}} \left[ 1 - \frac{2}{1 + (R^{-2}E^{-1})^{-L_0/2}} \right] \Big|_{\sigma=0}. \quad (5.17)$$

For any value of  $g < g_c$ , we have the symmetry phase at any temperature.

In conclusion, we have proven that the main characteristics of the solution, in the limit of large  $N_F$ , are reproduced. Since the evaluation of the trace can be performed only by means of numerical simulations, we can not evaluate it at the critical exponent. We intend to investigate further in this direction. In particular, the fact that the action is diagonal can simplify remarkably both the analytical and numerical investigations.



## Chapter 6

# Condensate of diquarks

The aim of this work is to look for the possibility to write the partition function of the non-abelian gauge theory, that generates bound states for different values of the thermodynamical parameters, in a physical equivalent theory in which the bound states and elementary constituents appear on the same footing. It is obviously that the next step in this direction is the introduction in the trace of baryon states. It is clear that the transfer matrix formalism is a good starting point since it permits, for example, to estimate masses and shapes of bound states as we have been seen in Chapter 3. Since our formalism exactly reproduces the thermodynamical properties in the cases shown in the last chapter, we think it can be adopted to the aim.

Since we are interested in the study of hadron matter and in exotic states of matter, we must add to the mesons different types of states as diquark and baryons. In order to consider different states in the partition function and in the trace, we must avoid the double counting of states. This can be done imposing the condition that states of different particles must be mutually orthogonal. This constraint corresponds to the condition on the wave function renormalization of composite particles in the Lehmann spectral representation of composite operators [29].

In this section, we extend our formalism in presence of diquarks as we suppose that it could be useful in the study of the high chemical potential zone of the phase diagram. At the same time, this can be a first step in order to introduce the baryons. We think that decomposing a baryon as a diquark plus elementary fermions can simplify the computations. The other possibility [30] of introducing the baryons as elementary particles requires that their operator must be cubic in the elementary particle fields. This solution, however, introduces a few complications in the computations.

## 6.1 Composite diquark operator

We introduce diquark composite operators. We choose to define creation and destruction operators in the following way

$$\hat{B}_I^\dagger := \hat{u}_i^\dagger (B_I)_{i,j}^\dagger \hat{u}_j^\dagger, \quad \hat{B}_I := \hat{u}_i (B_I)_{i,j} \hat{u}_j. \quad (6.1)$$

The matrix  $B$  has the same role of the matrix  $\Phi$  for the meson like composite, the only difference is that  $B$  must be antisymmetric. The index  $I$  of the di-quarks structure functions is defined in the same way as the meson one in section 3.1. It is possible to repeat the same observation made about the role of the structure function. The particles that are created by  $\hat{B}^\dagger$  have fermion number two. Notice that

$$\hat{B}_I |\phi\rangle = \hat{v}^\dagger \mathcal{F}^* B_I \mathcal{F}^\dagger \hat{v}^\dagger |\phi\rangle. \quad (6.2)$$

Since we must avoid double counting, diquark states and meson states must be mutually orthogonal; therefore we must change the definition (6.1) of the diquark operator. The diquarks have got to satisfy to

$$\hat{B}_I |\phi\rangle = 0, \quad (6.3)$$

for every  $I$ . One solution<sup>1</sup> of the last equation is

$$\hat{B}_I := \hat{\alpha}_i (B_I)_{i,j} \hat{\alpha}_j, \quad (6.4)$$

where  $\hat{\alpha}$  is defined in (4.7) and this definition guarantees that diquark states are annihilated by  $\hat{\Phi}$ . In the following, we assume (6.6) and its hermitian conjugate

$$\hat{B}_I := \hat{\alpha}_i^\dagger (B_I)_{i,j}^\dagger \hat{\alpha}_j^\dagger \quad (6.5)$$

as the definition of the diquark bosons composite operator. As for the mesons operator, the diquark operator has a nilpotency index. For ours purposes, it is necessary to introduce semi-coherent states

$$|b, \phi\rangle := e^{\frac{1}{2} \hat{\alpha}^\dagger \mathcal{B} \hat{\alpha}^\dagger} e^{\hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger} |0\rangle, \quad (6.6)$$

$$= e^{\frac{1}{2} \hat{u}^\dagger R^{-1} B R^{-1 T} \hat{u}^\dagger + \hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger} |0\rangle, \quad (6.7)$$

where we use the convention

$$\mathcal{B}_{i,j}^\dagger := \sum_I b_I (B_I)_{i,j}^\dagger =: b \cdot B_{i,j}^\dagger.$$

<sup>1</sup>It is not the most general solution of (6.3). We know a more general one, but this one renders the computations more complicated.

From the definition of diquark, it is clear that the state  $|b, \phi\rangle$  is not properly a coherent state. Indeed, the algebras of the mesons and of the diquark are not independent and canonical

$$[\hat{B}_I^\dagger, \frac{1}{2} \hat{B}_j] = \hat{u}^\dagger B_I^\dagger B_j \hat{u} - \frac{1}{2} \text{tr} B_I^\dagger B_j, \quad [\hat{u} \mathcal{F} \hat{v}, \frac{1}{2} \hat{B}_j] = \hat{\alpha}^\dagger B_j R \mathcal{F}^\dagger \hat{v}. \quad (6.8)$$

We can make observations similar to the ones made in section 3.2; the states (6.6) that look like coherent in the subspace with number mesons and diquark not less than nilpotency index  $\Omega$ .  $\Omega_{dq}$ ,  $\Omega_{dq}$  is the nilpotency index of  $B$ . By use of the formulae collected in appendix A, one can evaluate the inner product

$$\langle b_1, \phi_1 | b_2, \phi_2 \rangle = \det -R_{2,1} (\det -\mathcal{U}_{(2,1)})^{1/2}, \quad (6.9)$$

where

$$\mathcal{U}_{(i,j)} := 1 + R_{(i,i)}^{-1} \mathcal{B}_i^\dagger R_{(i,i)}^{-1T} R_{(i,j)}^{2T} R_{(j,j)}^{-1T} \mathcal{B}_j R_{(j,j)}^{-1} R_{(i,j)}^2. \quad (6.10)$$

The computation is quite similar to the ones of the previous Chapter; the only difference is that the integration over the Grassmann variable of the particles in (A.3) has to be performed on the right side formula of (A.1), instead than on the left side of the same formula as in the previous cases. The integration over the anti-particles variable is performed as usually.

## 6.2 The operator $\hat{\mathcal{P}}_{mb}$

As in the case of boson dominance, we want to introduce a projection operator over the sub-space of the composite bosons meson type and diquark type. We prefer to express the projection operator by means of coherent states

$$\hat{\mathcal{P}}_{mb} := \int \left[ \frac{d\phi^* d\phi db^* db}{(2\pi i)^2} \right] \frac{|b, \phi\rangle \langle b, \phi|}{\langle b, \phi | b, \phi \rangle}. \quad (6.11)$$

It is necessary to show that  $\hat{\mathcal{P}}_{mb}$  is an approximation of the projection over that sub-space.  $\hat{\mathcal{P}}_{mb}$  is a projector over that sub space if it satisfies to

$$\langle 0 | \Phi^{m_1} \hat{B}^{n_1} | \hat{\mathcal{P}}_{mb} | \hat{B}^{\dagger n_2} \Phi^{\dagger m_2} | 0 \rangle \simeq \langle 0 | \hat{B}^{n_1} \Phi^{m_1} \hat{B}^{\dagger n_1} \Phi^{\dagger m_2} | 0 \rangle \delta_{n_1, n_2} \delta_{m_1, m_2} \quad (6.12)$$

and to an idempotency condition (3.22). The formula (6.12) is generated by the following one

$$\langle b_1 \phi_1 | \hat{\mathcal{P}}_{mb} | b_2 \phi_2 \rangle \simeq \langle b_1 \phi_1 | b_2 \phi_2 \rangle, \quad (6.13)$$

by derivation with respect to holomorphic variables and then setting both variables to zero. Replacing (6.12) in Eq. (6.13), we obtain

$$\langle b_1 \phi_1 | \hat{\mathcal{P}}_{mb} | b_2 \phi_2 \rangle = \int \left[ \frac{d\phi^* d\phi db^* db}{(2\pi i)^2} \right] e^{-\mathcal{E}(\phi, \phi^*, b, b^*)}, \quad (6.14)$$

where

$$\begin{aligned} \mathcal{E}(\phi_0, \phi_0^*, b_0, b_0^*) &:= & (6.15) \\ &= \text{tr}_- \left[ \ln_- R_{0,0}^{-2} - \ln_- R_{(0,1)}^{-2} - \ln_- R_{(2,0)}^{-2} + \frac{1}{2} (\ln_- \mathcal{U}_{(0,0)} - \ln \mathcal{U}_{(0,1)} - \ln \mathcal{U}_{(2,0)}) \right]. \end{aligned}$$

Since  $\mathcal{E}$ , for large nilpotent index  $\Omega$  and  $\Omega_{dq}$ , is proportional, to the index itself <sup>2</sup>, we can evaluate the integral by the saddle point method. The saddle point equations are

$$\begin{aligned} \frac{\partial \mathcal{E}(\phi, \phi^*, b, b^*)}{\partial b} &= \frac{1}{2} \text{tr}_- \left[ \mathcal{U}_{(0,0)}^{-1} \mathcal{U}_{(0,1)}^{-1} \left( R_{(0,0)}^{-1} B^\dagger R_{(0,0)}^{-1T} R_{(0,0)}^{2T} R_{(0,0)}^{-1T} \mathcal{B}_0 R_{(0,0)}^{-1} R_{(0,0)}^2 + \right. \right. \\ &\quad \left. \left. - R_{(0,0)}^{-1} B^\dagger R_{(0,0)}^{-1T} R_{(1,0)}^{2T} R_{(1,1)}^{-1T} \mathcal{B}_2 R_{(1,1)}^{-1} R_{(0,1)}^2 \right) \right] = 0, \end{aligned} \quad (6.16a)$$

$$\begin{aligned} \frac{\partial \mathcal{E}(\phi, \phi^*, b, b^*)}{\partial b^*} &= \frac{1}{2} \text{tr}_- \left[ \mathcal{U}_{(0,0)}^{-1} \mathcal{U}_{(2,0)}^{-1} \left( R_{(0,0)}^{-1} \mathcal{B}_0^\dagger R_{(0,0)}^{-1T} R_{(0,0)}^{2T} R_{(0,0)}^{-1T} B R_{(0,0)}^{-1} R_{(0,0)}^2 + \right. \right. \\ &\quad \left. \left. - R_{(2,2)}^{-1} \mathcal{B}_2^\dagger R_{(2,2)}^{-1T} R_{(2,0)}^{2T} R_{(0,0)}^{-1T} B R_{(0,0)}^{-1} R_{(2,0)}^2 \right) \right] = 0, \end{aligned} \quad (6.16b)$$

$$\frac{\partial \mathcal{E}(\phi, \phi^*, b, b^*)}{\partial \phi} = \text{tr}_- \left[ \frac{1}{2} \Phi^\dagger \mathcal{F}_0 R_{(0,0)} \left( \mathcal{U}_{(0,1)}^{-1} - \mathcal{U}_{(2,0)}^{-1} \right) - \Phi^\dagger \mathcal{F}_1 R_{(0,1)} \mathcal{U}_{(0,1)}^{-1} \right] = 0, \quad (6.16c)$$

$$\frac{\partial \mathcal{E}(\phi, \phi^*, b, b^*)}{\partial \phi^*} = \text{tr}_- \left[ \frac{1}{2} \left( \mathcal{U}_{(0,1)}^{-1} - \mathcal{U}_{(2,0)}^{-1} \right) R_{(0,0)}^2 \mathcal{F}_0^\dagger \Phi - \mathcal{U}_{(2,0)}^{-1} R_{(2,0)}^2 \mathcal{F}_2^\dagger \Phi \right] = 0. \quad (6.16d)$$

The unique solution of (6.16) is

$$\phi = \phi_2, \quad \phi^* = \phi_1^*, \quad b = b_2, \quad b^* = b_1^*. \quad (6.17)$$

The leading corrections at the saddle point are the gaussian fluctuations around it. This contributions are complex because the algebra of the two different particles are non canonical (6.8). In order to evaluate the gaussian corrections, we have to evaluate the second order derivatives. We do not report the explicit expressions for a matter of space. We know that

$$\left. \frac{\partial^2 \mathcal{E}}{\partial b \partial b} \right|_{\substack{\phi=\phi_2 \\ b=b_2}}^{\substack{\phi^*=\phi_1^* \\ b^*=b_1^*}} = \left. \frac{\partial^2 \mathcal{E}}{\partial b^* \partial b^*} \right|_{\substack{\phi=\phi_2 \\ b=b_2}}^{\substack{\phi^*=\phi_1^* \\ b^*=b_1^*}} = 0. \quad (6.18)$$

The other second order derivatives are all non vanishing and depend, in a complex way, on  $b_1^*$ ,  $b_1$ ,  $\phi_1^*$  and  $\phi_2$ . The quadratic fluctuations around the saddle point give

$$\langle b_1 \phi_1 | \hat{\mathcal{P}}_{mb} | b_2 \phi_2 \rangle \simeq \langle b_1 \phi_1 | b_2 \phi_2 \rangle g(b_1, b_2, \phi_1, \phi_2), \quad (6.19)$$

where  $g(b_1, b_2, \phi_1, \phi_2)$  is the determinant of the Hessian of  $\mathcal{E}$ . Its complicate expression makes difficult the evaluation of its contribution in equation (6.12). We are evaluating the contribute coming from the Hessian, we expect that exist the possibility to erase the possible additional term. At the moment we do have not information about the idempotency of  $\hat{\mathcal{P}}_{mb}$

<sup>2</sup>This means that in the hypothesis of section 3.2.1, we verify that  $\text{tr}_- \ln_- R^{-2} \Omega$  and  $\text{tr}_- \ln_- \mathcal{U}_{(0,0)} \Omega_{dq}$ .

### 6.3 First form of the effective action

In this section, we deduce the expression of the effective action for the composite fields  $\phi$  and  $b$ . For the moment, we consider the most general case: we do not introduce any assumption over the fermion regularization and do not fix the gauge. Under the assumption that the dominant degrees of freedom are meson and di-quarks type, we can approximate the partition function (3.24) evaluating the trace over only such states

$$Z_{Cmc} := \mathbf{Tr}^F \prod_{t=0}^{L_0-1} \left[ \hat{\mathcal{P}}_{mb} \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \right] \quad (6.20a)$$

$$= \int \left[ \frac{d\phi^* d\phi db^* db}{(2\pi i)^2} \right] \frac{\langle b_t \phi_t | \hat{T}_t^\dagger \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} | b_{t+1} \phi_{t+1} \rangle}{\langle b_t \phi_t | b_t \phi_t \rangle}. \quad (6.20b)$$

To evaluate the action one has to compute the new matrix element in (6.20b). It can be verified that

$$\begin{aligned} \langle \gamma, \delta | e^{\hat{v} N_{t+1} \hat{u}} | b_{t+1}, \phi_{t+1} \rangle &= \det -\mathcal{F}_{N,t+1}^\dagger \times \\ &\times \exp \left[ \frac{1}{2} \gamma^* \mathcal{F}_{N,t+1}^{\dagger-1} R_{t+1}^{-1} \mathcal{B}_{t+1}^\dagger R_{t+1}^{T-1} \mathcal{F}_{N,t+1}^{*-1} \gamma^* + \gamma^* \mathcal{F}_{N,t+1}^{\dagger-1} \mathcal{F}_{t+1}^\dagger \delta^* \right], \\ \langle \alpha, \beta | e^{\hat{u}^\dagger X \hat{u} + \hat{v}^\dagger Y \hat{v}} e^{\hat{v} N \hat{v}} | b_{t+1}, \phi_{t+1} \rangle &= \det -\mathcal{F}_{N,t+1}^\dagger \times \\ &\times \exp \left[ \frac{1}{2} \alpha^* e^X \mathcal{F}_{N,t+1}^{\dagger-1} R_{t+1}^{-1} \mathcal{B}_{t+1}^\dagger R_{t+1}^{T-1} \mathcal{F}_{N,t+1}^{*-1} e^{X^T} \alpha^* + \alpha^* e^X \mathcal{F}_{N,t+1}^{\dagger-1} \mathcal{F}_{t+1}^\dagger e^{Y^T} \beta^* \right]. \end{aligned}$$

We can use this and Eq. (6.9) to obtain the effective action. The expression (6.20b) becomes

$$Z_{Cmc} = \int \left[ \frac{d\phi^* d\phi db^* db}{2\pi i} \right] e^{-S_{Cmb}(\phi, b)}, \quad (6.22)$$

where

$$S_{Cmb}(\phi, b) = \sum_{t=0}^{L_0-1} \text{tr}_- \left[ \ln R_t^{-2} - \ln \left( e^{-M_{u,t+1}} E_{t+1,t} e^{-M_t^\dagger} \right) + \frac{1}{2} \ln \mathcal{U}_{(t,t)} - \frac{1}{2} \ln \mathcal{U}_{N(t+1,t)} \right] \quad (6.23a)$$

$$= \sum_{t=0}^{L_0-1} \text{tr}_- \left[ \mu - \ln \left( e^{-M_{u,t+1}} e^{-M_t^\dagger} \right) + \frac{1}{2} \ln \mathcal{W}_{(t,t)} - \frac{1}{2} \ln \mathcal{W}_{N(t+1,t)} \right]. \quad (6.23b)$$

Above, we have defined

$$\mathcal{U}_{N i,j} := 1 + e^{2\mu} R_{(i,i)}^{-1} \mathcal{B}_i^\dagger R_{(i,i)}^{T-1} E_{i,j}^{T-1} R_{(j,j)}^{T-1} \mathcal{B}_j R_{(j,j)}^{-1} E_{i,j}^{-1}, \quad (6.24a)$$

$$\mathcal{W}_{N i,j} := 1 + e^{-2\mu} R_{(i,i)}^T \mathcal{B}_i^{\dagger-1} R_{(j,j)} E_{i,j} R_{(j,j)} \mathcal{B}_j^{-1} R_{(j,j)}^T E_{i,j}^T, \quad (6.24b)$$

$$\mathcal{W}_{i,j} := 1 + R_{(i,i)}^T \mathcal{B}_i^{\dagger-1} R_{(i,i)} R_{(i,j)}^{-2} R_{(j,j)} \mathcal{B}_j^{-1} R_{(j,j)}^T R_{(i,j)}^{-2}. \quad (6.24c)$$

### 6.3.1 Saddle point equations

For the moment, we consider the case where the index  $\Omega$  is large. As explained in section 3.3.1, we can evaluate the partition function (6.22) by saddle point method. Deriving (6.23a) in  $\mathcal{B}_t$  and  $\mathcal{B}_{t+1}^\dagger$  one obtains

$$0 = \frac{1}{2} \left[ \mathcal{U}_{(t,t)}^{-1} \mathcal{B}_t^\dagger - e^{2\mu} R_t^{-1} E_{t+1,t}^{-1} \mathcal{U}_{N(t+1,t)}^{-1} R_{t+1}^{-1} \mathcal{B}_{t+1}^\dagger R_{t+1}^{T-1} E_{t+1,t}^{T-1} R_t^{T-1} \right], \quad (6.25a)$$

$$0 = \frac{1}{2} \left[ \mathcal{B}_{t+1} \mathcal{U}_{(t+1,t+1)}^{-1} - e^{2\mu} R_{t+1}^{T-1} E_{t+1,t}^{T-1} R_t^{T-1} \mathcal{B}_t R_t^{-1} E_{t+1,t}^{-1} \mathcal{U}_{N(t+1,t)}^{-1} R_{t+1}^{-1} \right]. \quad (6.25b)$$

The other gap equations can be obtained in a more suitable form performing the derivate of (6.23b) with respect to  $\mathcal{F}_t$  and  $\mathcal{F}_{t+1}^\dagger$ . The results are, respectively

$$\begin{aligned} 0 = \frac{e^{-2\mu}}{2} \left[ \frac{1}{2} \mathcal{F}_t^* \mathcal{B}_t^{\dagger-1} R_t E_{t,t-1} R_{t-1} \mathcal{B}_{t-1}^{-1} R_{t-1}^T E_{t,t-1}^T \mathcal{W}_{N(t,t-1)}^{-1} R_t^{T3} + \right. \\ \left. + \frac{1}{2} R_t^3 E_{t,t-1} R_{t-1} \mathcal{B}_{t-1}^{-1} R_{t-1}^T E_{t,t-1}^T \mathcal{W}_{N(t,t-1)}^{-1} R_t^T \mathcal{F}_t^\dagger + \right. \\ \left. - R_t \mathcal{B}_{t-1}^{-1} R_t^T E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} \partial_{\mathcal{F}_t} E_{t+1,t} + \right. \\ \left. + \frac{1}{2} R_t^3 \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} \mathcal{F}_t^\dagger + \right. \\ \left. + \frac{1}{2} \mathcal{F}_t^* E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^3 + \right. \\ \left. - \partial_{\mathcal{F}_t^T} E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t \right], \quad (6.25c) \end{aligned}$$

$$\begin{aligned} 0 = \frac{e^{-2\mu}}{2} \left[ \frac{1}{2} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} \mathcal{F}_{t+1}^T + \right. \\ \left. + \frac{1}{2} \mathcal{F}_{t+1} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1}^3 + \right. \\ \left. - \partial_{\mathcal{F}_{t+1}^\dagger} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} + \right. \\ \left. + \frac{1}{2} \mathcal{F}_{t+1} \mathcal{B}_{t+1}^{-1} R_{t+1}^T E_{t+2,t+1} \mathcal{W}_{N(t+2,t+1)}^{-1} R_{t+2}^T \mathcal{B}_{t+2}^{\dagger-1} R_{t+2} E_{t+2,t+1} R_{t+1}^3 + \right. \\ \left. + \frac{1}{2} R_t^{T3} E_{t+2,t+1} \mathcal{W}_{N(t+2,t+1)}^{-1} R_{t+2} \mathcal{B}_{t+2}^{\dagger-1} R_{t+2} E_{t+2,t+1} R_{t+1}^{-1} \mathcal{F}_{t+1}^T + \right. \\ \left. - \mathcal{W}_{N(t+1,t)}^{-1} R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t \partial_{\mathcal{F}_t^T} E_{t+1,t}^T \right]. \quad (6.25d) \end{aligned}$$

In the axial gauge, the hermitian conjugate of the equations (6.25a) is equivalent to (6.25b), if one performs the exchange of  $t$  with  $t+1$ . Multiplying (6.25a) by  $\mathcal{B}_t$  on the left and (6.25b) by  $\mathcal{B}_{t+1}^\dagger$  gives

$$\mathcal{B}_t^\dagger \mathcal{B}_t = e^{2\mu} R_t^{-1} E_{t+1,t}^{-1} R_{t+1}^{-1} \mathcal{B}_{t+1}^\dagger R_{t+1}^{T-1} E_{t+1,t}^{T-1} R_t^{T-1} \mathcal{B}_t, \quad (6.26a)$$

$$\mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} = e^{2\mu} \mathcal{B}_{t+1}^\dagger R_{t+1}^{T-1} E_{t+1,t}^{T-1} R_t^{T-1} \mathcal{B}_t R_t^{-1} E_{t+1,t}^{-1} R_{t+1}^{-1}. \quad (6.26b)$$

Equations (6.25c) and (6.25d) can be recast into a more useful form if one notice that, for example, the first term in trace in Eq. (6.25c) is the transpose of the second, the



sixth is the transpose of the third and so on. These terms can be summed together and, after some algebra, the result is

$$\begin{aligned}
0 = & \frac{1}{2} \left[ R_t^2 \mathcal{F}_t^\dagger + \left( R_t^2 \mathcal{F}_t^\dagger \right)^T - E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} - \left( E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} \right)^T \right. \\
& - R_t^2 \left( 1 + e^{-2\mu} R_t E_{t,t-1} R_{t-1} \mathcal{B}_{t-1}^{-1} R_{t-1}^T E_{t,t-1}^T R_t^T \mathcal{B}_t^{\dagger-1} \right)^{-1} \mathcal{F}_t^\dagger + \\
& - R_t^2 \left( 1 + e^{-2\mu} \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t \right)^{-1} \mathcal{F}_t^\dagger + \\
& \left. + 2 \left( 1 + e^{-2\mu} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} \right)^{-1} E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} \right], \tag{6.27a}
\end{aligned}$$

$$\begin{aligned}
0 = & \frac{1}{2} \left[ \mathcal{F}_{t+1} R_{t+1}^2 + \left( \mathcal{F}_{t+1} R_{t+1}^2 \right)^T - \left( \partial_{\mathcal{F}_{t+1}} E_{t+1,t} \right) E_{t+1,t}^{-1} - \left( \left( \partial_{\mathcal{F}_{t+1}} E_{t+1,t} \right) E_{t+1,t}^{-1} \right)^T \right. \\
& - \mathcal{F}_{t+1} \left( 1 + e^{-2\mu} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} \right)^{-1} R_{t+1}^2 + \\
& - \mathcal{F}_{t+1} \left( 1 + e^{-2\mu} \mathcal{B}_{t+1}^{-1} R_{t+1}^T E_{t+2,t+1}^T R_{t+2}^T \mathcal{B}_{t+2}^{\dagger-1} R_{t+2} E_{t+2,t+1} R_{t+1} \right)^{-1} R_{t+1}^2 + \\
& \left. + 2 \partial_{\mathcal{F}_{t+1}} E_{t+1,t}^{-1} \left( 1 + e^{-2\mu} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} E_{t+1} R_{t+1} \right)^{-1} \right]. \tag{6.27b}
\end{aligned}$$

By use of the formula (6.26), it is possible to recast the matrix appearing in the denominator of the last three terms of (6.27a) and (6.27b) in the following way

$$\begin{aligned}
1 + e^{-2\mu} R_t E_{t,t-1} R_{t-1} \mathcal{B}_{t-1}^{-1} R_{t-1}^T E_{t,t-1}^T R_t^T \mathcal{B}_t^{\dagger-1} &= 1 + \left( \mathcal{B}_t^\dagger \mathcal{B}_t \right)^{-1}, \\
1 + e^{-2\mu} \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} R_t &= 1 + \left( \mathcal{B}_t^\dagger \mathcal{B}_t \right)^{-1}, \\
1 + e^{-2\mu} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} E_{t+1,t} &= R_t \left[ 1 + \left( \mathcal{B}_t^\dagger \mathcal{B}_t \right)^{-1} \right] R_t^{-1}, \\
1 + e^{-2\mu} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} R_{t+1} &= R_{t+1}^{-1} \left[ 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1} \right] R_{t+1}, \\
1 + e^{-2\mu} \mathcal{B}_{t+1}^{-1} R_{t+1}^T E_{t+2,t+1}^T R_{t+2}^T \mathcal{B}_{t+2}^{\dagger-1} R_{t+2} E_{t+2,t+1} R_{t+1} &= 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1}, \\
1 + e^{-2\mu} E_{t+1,t} R_t \mathcal{B}_t^{-1} R_t^T E_{t+1,t}^T R_{t+1}^T \mathcal{B}_{t+1}^{\dagger-1} E_{t+1} R_{t+1} &= R_{t+1}^{-1} \left[ 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1} \right] R_{t+1},
\end{aligned}$$

and to simplify the equations (6.27)

$$\begin{aligned}
0 = & R_t^2 \mathcal{F}_t^\dagger - E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} - \\
& - R_t^2 \left[ 1 + \left( \mathcal{B}_t^\dagger \mathcal{B}_t \right)^{-1} \right]^{-1} \mathcal{F}_t^\dagger + R_t \left[ 1 + \left( \mathcal{B}_t^\dagger \mathcal{B}_t \right)^{-1} \right] R_t^{-1} E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} \tag{6.29a}
\end{aligned}$$

$$\begin{aligned}
0 = & \mathcal{F}_{t+1} R_{t+1}^2 - \left( \partial_{\mathcal{F}_{t+1}} E_{t+1,t} \right) E_{t+1,t}^{-1} \\
& - \mathcal{F}_{t+1} R_{t+1}^{-1} \left[ 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1} \right]^{-1} R_{t+1} R_{t+1}^2 - \mathcal{F}_{t+1} \left[ 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1} \right]^{-1} R_{t+1}^2 + \\
& + 2 \partial_{\mathcal{F}_{t+1}} E_{t+1,t}^{-1} R_{t+1}^{-1} \left[ 1 + \left( \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right)^{-1} \right]^{-1} R_{t+1}. \tag{6.29b}
\end{aligned}$$

To obtain the saddle point equations it is necessary to solve the equation (6.29). If we consider the following condition

$$\left[ \mathcal{F}_t^\dagger \mathcal{F}_t, \mathcal{B}_t^\dagger \mathcal{B}_t \right] = 0, \quad (6.30)$$

the equation (6.29) gives

$$0 = \frac{1}{1 + \mathcal{B}_t^\dagger \mathcal{B}_t} \left[ R_t^2 \mathcal{F}_t^\dagger - E_{t+1,t}^{-1} \partial_{\mathcal{F}_t} E_{t+1,t} \right], \quad (6.31a)$$

$$0 = \left[ \mathcal{F}_{t+1} R_{t+1}^2 - (\partial_{\mathcal{F}_{t+1}} E_{t+1,t}) E_{t+1,t}^{-1} \right] \frac{1}{1 + \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1}}. \quad (6.31b)$$

Notice that these expressions are independent of the structure function of the diquark, The equations (6.31) and (6.26) are iterative equations for the structure function of the composites, they can be solved by numerical method. The equations (6.31) are equivalent to Eqs. (3.34). In the stationary cases, it is known the analytical solution  $\mathcal{F}_\Gamma$ , see section 3.3.1, which expression is given in (3.42). Substituting a solution of the saddle point equations (3.42) in (6.26), the gap equations for  $\mathcal{B}$ , in the time independent case, become

$$\mathcal{B}^\dagger = e^{2\mu} \left( H + \sqrt{1 + H^2} \right)^{2\tilde{\Gamma}} \mathcal{B}^\dagger \left( H + \sqrt{1 + H^2} \right)^{2\tilde{\Gamma}}, \quad (6.32a)$$

$$\mathcal{B} = e^{2\mu} \left( H + \sqrt{1 + H^2} \right)^{2\tilde{\Gamma}} \mathcal{B} \left( H + \sqrt{1 + H^2} \right)^{2\tilde{\Gamma}}. \quad (6.32b)$$

The derivation of the equations is performed with the natural condition that  $\Gamma$ , described in section 3.3.1, commutes with  $H^2$ . It is a very important to observe that in the equation for  $\mathcal{F}$  does not appear  $\mathcal{B}$ , while the equations for  $\mathcal{B}$  depend on  $\mathcal{F}$ . The saddle point equations admit always the solution  $\mathcal{B} = 0$ ; it is the only solution for  $\mu = 0$ . Nowadays, we do not know other solutions different from the null one, but we are proceeding the investigation in this direction. Our belief is that the equations (6.32) admit some non trivial solution  $\bar{\mathcal{B}}_\Gamma$ , for some choice of  $\Gamma$ ; pointedly the solutions will depend on  $\Gamma$ . The physical solution, corresponding to a minimum of the action  $S_{Cmb}(\phi, b)$ , is a function of  $\mu$ . We expect, for values of the chemical potential smaller than some critical value  $\mu_c$ , that the physical solutions will be  $\bar{\mathcal{B}} = 0$  and  $\bar{\mathcal{F}}$  given in (3.44)<sup>3</sup>. Instead, for values of the chemical potential greater than  $\mu_c$ , we expect that the physical solution not to be a null value of  $\bar{\mathcal{B}}$ ; probably the solution for  $\bar{\mathcal{F}}$  is different from (3.44). Indeed, the gap equations (6.31) are  $\mathcal{B}$  independent, but the condition of minimization of the action depends on the value of  $\bar{\mathcal{B}}$ .

## 6.4 Second form of the effective action

In section 3.4, we derived an alternative form of the partition function performing a unitary transformation (3.45) in the operator space. In this section we deduce, by a

<sup>3</sup>For  $\mathcal{B} = 0$ , we obtain, as expected,  $S_C(\phi^*, \phi) = S_{Cmc}(\phi, 0)$ .

different method, a similar result. Considering the cyclicity of the trace, the operatorial form of the partition function (3.24) is equivalent to

$$Z'_C := \text{Tr}^F \left[ \hat{V}_0 e^{\mu \hat{n}_B} \hat{T}_1 T_1^\dagger \hat{V}_1 e^{\mu \hat{n}_B} \hat{T}_2 \hat{T}_2^\dagger \cdots T_{L_0-1}^\dagger \hat{V}_{L_0-1} e^{\mu \hat{n}_B} \hat{T}_0 \hat{T}_0^\dagger \right]. \quad (6.33)$$

Instead, if  $\hat{\mathcal{P}}_{mb}$  is a projector, (6.20b) must be equivalent to

$$\begin{aligned} Z'_{Cmc} := \text{Tr}^F & \left[ \hat{\mathcal{P}}_{mb} \hat{V}_0 e^{\mu \hat{n}_B} \hat{T}_1 T_1^\dagger \hat{\mathcal{P}}_{mb} \hat{V}_1 e^{\mu \hat{n}_B} \hat{T}_2 \hat{T}_2^\dagger \hat{\mathcal{P}}_{mb} \cdots \right. \\ & \left. \cdots \hat{\mathcal{P}}_{mb} \hat{V}_{L_0-1} e^{\mu \hat{n}_B} \hat{T}_{L_0-1} T_{L_0-1}^\dagger \hat{\mathcal{P}}_{mb} \hat{V}_{L_0-1} e^{\mu \hat{n}_B} \hat{T}_0 \hat{T}_0^\dagger \right]. \end{aligned} \quad (6.34)$$

Notice that  $Z_{Cmc}$  and  $Z'_{Cmc}$  do not necessary coincide, because  $\hat{\mathcal{P}}_{mb}$  is only an approximation of a projector. Indeed, at the moment we do not know whether the contribution coming from  $g(b_1, b_2, \phi_1, \phi_2)$  in  $\hat{\mathcal{P}}_{mb}$  can create some additional terms that can make the two expressions to differ. In order to evaluate  $S_{Cmb}(\phi, b)'$  obtained from (6.34), it is necessary to know the matrix element

$$\langle b_t, \phi_t | \hat{V}_t e^{\mu \hat{n}_B} \hat{T}_{t+1} \hat{T}_{t+1}^\dagger | b_{t+1}, \phi_{t+1} \rangle = \det E'_{t+1,t} \det - \left( \mathcal{U}'_{N, (t+1,t)} \right)^{\frac{1}{2}} \quad (6.35)$$

$$= \det^{1/2} \left( e^{2\mu} \Delta_{t+1,t+1}^\dagger \right)^{1/2} \times \quad (6.36)$$

$$\times \det - \left[ \Delta_{u,t+1,t} + E'_{t+1,t} e^{-2\mu} \Delta_{t+1,t+1}^{\dagger-1} E'^T_{t+1,t} \right]^{1/2},$$

where we have defined

$$\begin{aligned} E'_{t+1,t} &:= 1 + \left[ N_{t+1}^\dagger + e^{-M_{t+1}^\dagger} \mathcal{F}_{t+1} e^{-M_{t+1}^\dagger} \right] \left[ N_{t+1}^\dagger + e^{-M_{u,t+1}^\dagger} \mathcal{F}_t e^{-M_{u,t+1}^\dagger} \right] \\ \Delta_{u,t+1,t} &:= e^{-M_{u,t+1}} R_t^{-1} \mathcal{B}_t R_t^T e^{-M_{u,t+1}^T} \\ \Delta_{t+1,t+1}^\dagger &:= e^{-M_{t+1}^*} R_{t+1}^{-1} \mathcal{B}_{t+1}^\dagger R_{t+1}^{\dagger-1} e^{-M_{t+1}^\dagger} \\ \mathcal{U}'_{N, (t+1,t)} &:= 1 + e^{2\mu} \Delta_{t+1,t+1}^\dagger E'^T_{t+1,t} \Delta_{u,t+1,t} E'^{-1}_{t+1,t}. \end{aligned}$$

One of the most important difference between this method of derivation of the second form of the effective action and the previous one is that, in the expression (6.35), the matrices  $N$ ,  $M$  and their adjoints are all evaluated at the same time slice, while, by use of the other method, they are evaluated at different times. In presence of diquark, we must use this technique because we have not an analogue expression of (3.47).

The bosonized functional form of the partition function (6.34) is as in (6.22), but with the second form of the action

$$S_{Cmb}(\phi, b)' := \sum_{t=0}^{L_0-1} \text{tr}_- \left[ \ln R_t^{-2} + \frac{1}{2} \ln \mathcal{U}_{t,t} - \text{tr}_- \ln E'_{t+1,t} - \frac{1}{2} \ln \mathcal{W}'_{t+1,t} \right]. \quad (6.37)$$

#### 6.4.1 Pure diquarks action

In Chapter 3, we have considered the case where the ground state of a system is a composite bosons with the total charge zero (3.1). Now we consider the case where

the ground state is a composite boson with fermion number two. The action of pure diquarks can be obtained from the action of mesons and diquarks, with the additional condition for the structure function of the neutral bosons

$$\Phi = 0.$$

We consider  $M = 0$  and  $U_0 = Id$ . Substituting (6.38a) in the action, one obtains

$$S_{Cmb}(\phi, b) \Big|_{\Phi=0} = \sum_{t=0}^{L_0-1} \text{tr}_- \left[ \frac{1}{2} \ln \left( 1 + \mathcal{B}_t^\dagger \mathcal{B}_t \right) - \frac{1}{2} \ln \left( 1 + e^{2\mu} \mathcal{B}_{t+1}^\dagger \mathcal{B}_t \right) \right] \quad (6.38a)$$

$$S_{Cmb}(\phi, b)' \Big|_{\Phi=0} = \sum_{t=0}^{L_0-1} \text{tr}_- \left[ \frac{1}{2} \ln \left( 1 + \mathcal{B}_{t+1}^\dagger \mathcal{B}_{t+1} \right) - \ln \left( 1 + N_{t+1}^\dagger N_{t+1} \right) + \right. \\ \left. - \frac{1}{2} \ln \left( 1 + \mathcal{B}_{t+1}^\dagger \frac{1}{(1 + N_{t+1}^\dagger N_{t+1})^T} \mathcal{B}_t \frac{e^{2\mu}}{1 + N_{t+1}^\dagger N_{t+1}} \right) \right]. \quad (6.38b)$$

The two forms of the actions have a deep difference. In fact, in the first form of the action there is no dynamical contribution coming from  $N$ . This is no surprise, in fact, we know that<sup>4</sup>

$$e^{\hat{v}N\hat{u}}|b\rangle = |b\rangle.$$

As a matter of fact, the Dirac equation couples the particles part to the anti-particles one of the fields. Then, the evolution of the field requires both particles and anti-particles. In the first form we consider only a subspace of the particles space, this precludes the evolution of particles.

We are still working on the subject, in order to introduce, in addition to the mesons and diquark, also fermionic states. We have indeed already derived the effective actions, but we have not solved yet the condition of mutual orthogonality between the states of composite bosons and fermions. Once we are able to solve this problem, we will have a virtually complete extension of this method at finite temperature.

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<sup>4</sup> $|b\rangle := |b, \phi = 0\rangle.$

## Chapter 7

# Conclusions and outlooks

In this thesis work, we extended a general method of bosonization, for relativistic field theories of fermions in  $3 + 1$  dimension, to non zero values of the chemical potential and temperature. The formalism obtained can be applied to theories whose low energy excitations are dominated by bosonic models; this is always the case for the spontaneous breaking of continuous symmetries. We adopted the formalism of the transfer matrix because it is closed to the Hamiltonian one of non relativistic theories, and therefore very natural while dealing with real and virtual bound states. We chose a lattice regularization, because the main aim is the study of gauge theories and because, on a lattice, composite operators can be defined without any ambiguity. Moreover, the approach developed allows for the application of a variational method.

In order to extend the method, we introduced quasifermions and quasiantifermions states by a generalized Bogoliubov-Valatin's transformations on the doublet operator. This transformation does not break the gauge invariance. By use of coherent states, associated to the quasiparticles, we transformed, without any approximation, the (grand canonical) partition function, written as the trace of transfer matrix, in a functional expression with an effective action. In the latter appears a term coming from the transformation, a term bi-linear in the quasiparticles fields and a term of interaction between the quasifermions and the quasiantifermions. We have determined the generator of the transfer matrix, whose parameters appear in the effective actions. We have been able to prove that, if the external boson fields are time independent, the action coincides with the exact one and it is independent of the choice of the parameter in the transformation. We have found a result analogous to the BCS model [26]. Indeed, the choice of the parameter for which the interacting term from the quasiparticles and antiquasiparticles is null ("compensation of dangerous graph") coincides with a choice of the parameters for which the new ground state is the state that minimize the actions. This result has a particular relevance because the quasiquarks do not interact with the quasiantiquarks. We find an analogy between the Bogoliubov-Valatin's transformations and the Foldy-Wouthuysen's. In the case of interaction with time dependent external bosons field, the

diagonalization of the action, for each time slices, requires to perform a transformation at each time slice. The parameter of the transformation become functions of time, then we can interpret it as a field associate to the mesons. In the sequel, performing a different unitary transform (related to time reversal) we derived an alternative expression of the action. The second form of the action is more suitable to perform calculations. We tested this method with a four fermion fields interaction in the limit  $N_f \rightarrow \infty$ . All the main characteristics of the model are reproduced in both the boson and fermion sector. Moreover, we determined the structure function of the mesons.

In the sequel we introduced, in addition to mesons, composite bosons with fermion number 2, the diquark. This permits to study QFT at high chemical potential. We imposed mutually orthogonality between mesons and diquark. We introduced the projector over the sub-space of meson and diquark. However, we have not been able to perform a completely satisfying study of this object. Then, we derived the effective actions. By variational method we determined the saddle point equations, but we have not found yet any relevant solution. Anyway, we suppose that the saddle point equations should have an interesting physical outcome. In a future work we intend to complete the study of the action for mesons and diquark and to introduce, in addition to mesons and diquark, fermion states. These fermions can either be elementary particles or cubic in the fermion operator. However, the last opportunity has to be discarded in order to have a simple theory from a computational point of view. This permits to introduce in the formalism the baryons, that can be viewed as a diquark plus a fermion. The fermion and boson states must be mutually orthogonal. This leads to an alternative expression of the partition function where the composite and elementary degrees of freedom are treated on the same footing.

The effective actions, the one for the mesons and quasiparticles and the one for mesons and diquark, can be further on analyzed, in the absence of gauge fields, and an expansion of the theory in  $1/N_f$  can be performed. The expansion in the number of flavours should be carried through on the line of [9], analyzing also the more subtle aspects of the expansion itself [33].

The effective action we found, can be useful in the study of gauge theories, or, more in general, of systems of fermions that interact with bosons field. In particular, it could play a meaningful role in the study of the limit of strong coupling and in the analysis of the exotic states of hadronic matter. In this case, the effective bosons carry the quantum number of the chiral bosons and the chiral symmetry is broken by a condensation of sigma mesons. The variational parameters of the expectation value of the  $\sigma$ -mesons should help in numerical simulation [24]. The determination of the structure function of the composite depends on the configuration of the elementary bosons, since  $M$  and  $N$  depend on them. The determination of such structure function can be done by numerical methods.

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We reckon, moreover, that the method developed in the thesis project could be applied, with interesting results, to the study of the phase diagram of QCD. In particular, the determination of the fundamental state can be made through variational methods. The solution of the saddles point equation, that has already been solved in the static case, can be found through iterative procedures in the case of theories with interactions with external boson fields. We think it is going to be possible to determine the structure function of the condensate, at least through numerical ways and under opportune approximations, that at the same time is a minimum of the action and that diagonalized the action in the quasiparticles fields. This is going to be function of the gauge fields. Introducing vacuum states excitations, we obtained a different expression of the QCD partition function. At the moment, we do not know any approximation that would allow to carry on an analytic study of the partition function; however, we have reasons to believe that is going to be possible to proceed in this sense.

The formalism we have developed and studied is certainly a good starting point for numerical simulations. In fact, under the physical hypothesis that all the degrees of freedom are of boson type, i.e. that all the fermions are bosonized, we already obtained an expression of the partition function in terms of only holomorphic variables. The obtained effective theory presents, at least in principle and considering also non bosonized fermions, computational simplifications. It would be of great interest, therefore, to perform numerical simulations for high values of the chemical potential.





# Appendix A

## Notations and conventions

In this section, we report some useful identities and conventions about Berezin Integral and coherent states. These are standard arguments; an introduction about the Grassmann algebra can be found in [14] and a discussion about coherent states formalism can be found in [15]. For the thesis purposes are fundamental the relations that follow.

If  $\theta$  and  $\bar{\theta}$  are two Grassmann algebra it is well known that

$$\int \mathrm{D}\bar{\theta}\mathrm{D}\theta e^{-\bar{\theta}A\theta + \bar{J}\theta + \bar{\theta}J} = \det(A)e^{\bar{J}A^{-1}J}, \quad \int \mathrm{D}\theta e^{-\frac{1}{2}\theta C\theta + \bar{J}\theta} = \mathbf{Pf}(A)e^{\frac{1}{2}\bar{J}A^{-1}J} \quad (\text{A.1})$$

and  $(\mathbf{Pf}(A))^2 = \det(A)$ . Under a general change of base in the integral  $\theta = \theta(\theta')$ , for which  $\det \frac{\partial \theta_i}{\partial \theta'_j} \neq 0$ , the measure of integration changes as  $\mathrm{D}\theta \mapsto \det^{-1} \frac{\partial \theta_i}{\partial \theta'_j} \mathrm{D}\theta'$ .

The coherent states in presence of particles and antiparticles are

$$|\gamma, \delta\rangle := e^{-\alpha\hat{u}^\dagger - \delta\hat{v}^\dagger}|0\rangle. \quad (\text{A.2})$$

The inner product of this states is  $\langle\gamma, \delta|\rho, \sigma\rangle = e^{\gamma^*\rho + \delta^*\sigma}$ . The identity in the Fock space of fermions can be written as

$$\mathbb{Id} = \int \mathrm{D}[\omega^*]\mathrm{D}[\omega]\mathrm{D}[\varphi^*]\mathrm{D}[\varphi] \frac{|\omega, \varphi\rangle\langle\omega, \varphi|}{\langle\omega, \varphi|\omega, \varphi\rangle}. \quad (\text{A.3})$$

In the base of coherent states, the matrix element of the transfer matrix can be easily calculated, for example

$$\langle\omega, \varphi|e^{\hat{v}N\hat{u}}|\rho, \sigma\rangle = e^{\rho N\sigma + \omega^*\rho + \varphi^*\delta} \quad (\text{A.4})$$

$$\langle\omega\varphi|\alpha\beta; \xi\rangle = e^{\omega^*\mathcal{F}\varphi^* - a\omega^* - b\varphi^* - \beta\mathcal{F}\alpha}|0\rangle. \quad (\text{A.5})$$

Since in the transfer matrix  $\hat{\mathcal{T}}$  is a product of the exponential of the operator proportional to  $\hat{v}\hat{u}$ , or to  $\hat{u}^\dagger\hat{u}$ , or to  $\hat{v}^\dagger\hat{v}$ , or to their adjoints, evaluating its matrix elements in coherent states base is easy. This operation can be done introducing, between each exponent in  $\hat{\mathcal{T}}$ , the identity written as in A.3

$$\langle\alpha_t\beta_t; \phi_t|\hat{T}_t^\dagger\hat{V}_t e^{\mu\hat{B}}\hat{T}_{t+1}|\alpha_{t+1}\beta_{t+1}; \phi_{t+1}\rangle = \langle\alpha_t\beta_t; \phi_t|\hat{T}_t^\dagger\mathbb{Id}\hat{V}_t e^{\mu\hat{B}}\hat{T}_{t+1}|\alpha_{t+1}\beta_{t+1}; \phi_{t+1}\rangle.$$

In the next step we need to evaluate two matrix elements; one is

$$\langle \omega, \varphi | \hat{V}_t e^{\mu \hat{B}} \hat{T}_{t+1} | \alpha_{t+1} \beta_{t+1}; \phi_{t+1} \rangle = \langle \omega, \varphi | \hat{V}_t e^{\mu \hat{B}} e^{-\hat{u}^\dagger M_{t+1} \hat{u} - \hat{v}^\dagger M_{t+1}^T \hat{v}} \mathbb{I} d e^{\hat{v} N_{t+1} \hat{u}} | \alpha_{t+1} \beta_{t+1}; \phi_{t+1} \rangle .$$

Then we can repeat the procedure since all the operators leave the way to the Grassmann variables. In the end, it is necessary to evaluate all the integrals over the Grassmann variables. These integrals are all Gaussian. This procedure permits to obtain the functional form of the partition function. Notice that this procedure can be used in presence of the exponential of an operator proportional to  $\hat{u} \hat{u}$ . To evaluate the different matrix elements we report the following identity [16]

$$\langle \omega, \varphi | e^{-\hat{u}^\dagger A \hat{u} - \hat{v}^\dagger B \hat{v}} | \rho, \sigma \rangle = \exp(\omega^* e^{-A} \rho + \varphi^* e^{-B} \sigma) . \quad (\text{A.6})$$

Next we quote some identities, obtained following the former procedure

$$\langle \omega \varphi | e^{w N_t u} | \alpha \beta; \phi \rangle = \det_+ \overset{\circ}{\mathcal{F}}_{N,t} e^{\left( -\beta \mathcal{F} \alpha + b \overset{\circ}{\mathcal{F}}_N N a + \omega^* \mathcal{F}_N a - b \overset{\circ}{\mathcal{F}}_N \varphi^* + \omega^* \mathcal{F}_N \mathcal{F}^\dagger \varphi^* \right)} \quad (\text{A.7})$$

and

$$\begin{aligned} \langle \omega \varphi | \hat{V}_t e^{\mu \hat{B}} \hat{T}_{t+1} | \alpha \beta; \phi \rangle = & \det_+ \mathcal{F}_N \times \\ & \times \exp \left[ -\beta \mathcal{F} \alpha + b \overset{\circ}{\mathcal{F}}_N N a + \omega^* e^{-M_u} \mathcal{F}_N \mathcal{F}^\dagger e^{-M_w} \varphi^* + \right. \\ & \left. + \omega^* e^\mu e^{-M_u} \mathcal{F}_N a - b e^{-\mu} B_N^{(+)} e^{-M_{w,t+1}} \varphi^* \right] . \end{aligned}$$

## Appendix B

# The matrices $M$ , $N$ of the transfer matrix

In this appendix, we quote the explicit forms of the  $M$  and  $N$  matrices, for Wilson and Kogut-Susskind regularization, appearing into the definition of the  $\hat{T}$  in (1.12) and also into the definition of the operator in  $\hat{\mathcal{T}}$ . The general form of  $\hat{T}$  is given in (1.12), the matrices  $M$  and  $N$  depend only on the configuration of the gauge field of the spatial link for both the regularizations. Therefore  $M$  and  $N$  are time dependent:  $M_t = M[U_i(t)]$  and  $N_t = N[U_i(t)]$ ; where  $i = 1 \dots 3$ . The time link gauge variables  $U_0$  are present only in  $\hat{V}_t^1$ .

Notice that the entries of the matrix  $M$ ,  $N$ ,  $\mathcal{F}$  and  $\mathcal{B}$  and of their functions, where the last two are the structure functions of mesons and diquark, are the quantum numbers of fermion. The time index  $t$  must be interpreted as an independent variable and the matrices are functions of it.

### B.1 Kogut-Susskind regularization

The Kogut-Susskind fermion lives on a hypercube lattice whose sides are twice the basic lattice spacing. The sum over basic lattice or block lattice must be distinct. We use the conventions

$$\sum_x' := 2^d \sum_{x \text{ even}}, \quad t_\mu := \gamma_\mu^T. \quad (\text{B.2})$$

Here  $d$  is the space-time dimension and  $x \text{ even}$  means that it is necessary to take into consideration only even sites. The field  $\psi = \psi_{x,I}$  has the index  $x$  of space-time and an

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<sup>1</sup>The gamma matrix in Euclidean space must satisfies  $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu,\nu}$  so they are selfadjoint

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}. \quad (\text{B.1})$$

intrinsic quantum number index  $I$ . For this choice of regularization,  $I = (\alpha, t, j)$ , where the indices are, respectively, Dirac index, taste index and *flavour* index;  $\alpha, t = 1 \cdot 4$  and  $j = 1 \cdot N_f$ . As usually

$$\bar{\psi} \gamma_\mu \otimes t_\nu \psi := \sum_{\alpha, \beta, t, s, j, l} \psi_{x \alpha, t, j} \gamma_\mu^{\alpha, \beta} t_\nu^{t, s} \delta_{j, l} \psi_{x \beta, s, l}.$$

In this formalism [17] the operators  $\hat{u}$  and  $\hat{v}$  and their conjugates have index  $i := (\mathbf{x}, I)$ , where  $\mathbf{x}$  is the space index. It is possible to check that the projectors  $P_0^{(\pm)}$  that select the fermionic or antifermionic part of the field  $\psi$  are those defined in Eq. (1.6a). The  $M$  and  $N$  matrices, in presence of a Yukawa interaction with a field  $\sigma$ , are  $M = 0$  and

$$N = -2 \left\{ (m + \sigma) \gamma_0 \otimes \mathbf{I} + \sum_{n=1}^3 \gamma_0 \gamma_n \otimes \mathbf{I} \left[ \nabla_n^{(-)} P_n^{(+)} - \nabla_n^{(+)} P_n^{(-)} \right] \right\}. \quad (\text{B.3})$$

For a definition of the object in Eq. (B.3), see Eqs. (1.6). It is clear that  $\nabla_{\mu x, y}^{(\pm)}$  are, respectively, lattice covariant forward and back forward derivate. For convenience we define

$$Q = \sum_{\mu} \gamma_\mu \otimes \mathbf{I} \left( \nabla_{\mu}^{(-)} P_{\mu}^{(+)} - \nabla_{\mu}^{(+)} P_{\mu}^{(-)} \right). \quad (\text{B.4})$$

A very important object is

$$H^2 := \frac{1}{4} N^\dagger N = (m + \sigma)^2 - \Delta, \quad (\text{B.5})$$

with  $\Delta = \sum_{n=1}^3 \frac{1}{4} \left[ \nabla_n^{(+)} + \nabla_n^{(-)} - 2 \right]$ . In absence of the gauge field,  $H^2$  can be diagonalized in the momentum space and its eigenvalue is

$$E_q^2 := (m + \sigma)^2 - \tilde{q}^2. \quad (\text{B.6})$$

Above  $\tilde{q}^2 = \sum_{n=1}^3 \tilde{q}_n^2$  and  $\tilde{q}_n^2 = \frac{1}{2} (1 - \cos 2q_n)$ . As explained, the Kogut-Susskind field lives on the lattice of even sites so the Brillouin zone is  $0 \leq p_\mu < \pi$ ; this guarantees that there is no doubling; indeed the action  $S_{KS}$  possesses only first order derivatives. The factor 2 in the definition of  $\tilde{q}_n^2$  comes from the double size of the lattice.

## B.2 Wilson regularization

A derivation of the transfer matrix for this regularization can be found in [16], here we report only the results. The fermion fields  $\psi_{x, I}$  have only the Dirac index as inner quantum number and  $x$  is a position index that can label any site of the lattice. The projector operators are

$$P_0^{(\pm)} := \frac{1}{2} (1 \pm \gamma_0), \quad (\text{B.7})$$

and the matrices  $M$  and  $N$  are

$$M = -\frac{1}{2} \ln \left( \frac{B}{2K} \right) \quad N = 2KB^{-\frac{1}{2}} cB^{-\frac{1}{2}}. \quad (\text{B.8})$$

We have defined

$$B = 1 - k \sum_{j=1}^3 \left( U_j T_j^{(+)} + T_j^{(-)} U_j^\dagger \right) \gamma_j, \quad c = \frac{1}{2} \sum_{j=1}^3 i \left( U_j T_j^{(+)} - T_j^{(-)} U_j^\dagger \right) \sigma_j. \quad (\text{B.9})$$

### B.3 Boson and fermion matrices

From the definitions of  $\hat{\psi}$  and  $P_0^{(\pm)}$ , Eqs. (1.17) and (1.18), it is possible to rewrite (1.12) and (3.1) as

$$\hat{T} = \exp \left( -\hat{\psi}^\dagger \mathbf{M} \hat{\psi} \right) \exp \left( \hat{\psi}^\dagger P_0^{(+)} N P_0^{(-)} \hat{\psi} \right), \quad \hat{\Phi}^\dagger = \hat{\psi}^\dagger P_0^{(-)} \Phi P_0^{(+)} \hat{\psi}, \quad (\text{B.10})$$

where we omit all the indices and define  $\mathbf{M} = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}$ . It is quite natural to extend the definitions of the  $(n \times n)^2$  matrices  $M$ ,  $N$  and  $\Phi$  as matrices in the space of the doublet  $\hat{\psi}$ . In this space, they become square  $(2n \times 2n)$  matrices of the form

$$M := \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \quad N := \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \quad \Phi := \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix}. \quad (\text{B.11})$$

As we have only the  $M$ ,  $N$  and  $\Phi$  matrices, we study the possibilities of their sums and products. In this space there exist two type of matrices. One type is called boson matrix and they are the  $X$  such that  $X^2 \neq 0$ , of the form  $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} =: \mathbf{b}^{(-)}$ , like  $M$  or  $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} =: \mathbf{b}^{(+)}$ , called respectively boson  $\mp$ . The other type is said fermion type and they are such that  $X^2 = 0$  of the form  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{f}^{(-)}$ , as  $N$  and  $\Phi$ , or  $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \mathbf{f}^{(+)}$ , as  $N^\dagger$  and  $\Phi^\dagger$ ; called respectively fermion  $\mp$ . We make this distinction because, if we multiply on the right a fermion  $\pm$  or boson  $\pm$  matrix by  $P_0^{(\mp)}$ , we obtain the null matrix ( $\mathbf{f}^{(\pm)} P_0^{(\mp)} = \mathbf{b}^{(\pm)} P_0^{(\mp)} = 0$ ). Notice that the Hermitian adjoint of a fermion  $\mp$  matrix is a fermion  $\pm$  and that the product of fermion  $\mp$  with fermion  $\pm$  matrix on the right is a boson  $\pm$  matrix. The boson matrices are different: the Hermitian adjoint of a boson  $\mp$  matrix is a boson  $\mp$  and the products are  $\mathbf{b}^{(\mp)} \mathbf{b}_1^{(\pm)} = 0$  and  $\mathbf{b}^{(\mp)} \mathbf{b}_1^{(\mp)} = \mathbf{b}_2^{(\mp)}$ ; the sum of any two elements of boson  $\mp$  is a boson  $\mp$  matrix. Then boson  $\mp$  define two rings. Since the multiplication in fermion  $\mp$  is not defined, this set does not have an interesting algebraic structure. Only functions of boson matrices can be defined. The determinant of boson and fermion matrices is always null. To reproduce the right behavior for the boson  $\pm$  matrices, we must compute the determinant only over the entries of the subspace related  $P_0^{(\pm)}$

$$\det_{\pm} A = \det (P_0^{(\pm)} A). \quad (\text{B.12})$$

The trace is defined as

$$\text{tr}_{\pm} A = \ln \det (P_0^{(\pm)} e^O), \quad (\text{B.13})$$

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<sup>2</sup> $n$  is the number of the different states of fermions.

for any boson $^\pm$  matrix  $O$ . The determinant of fermion matrices is always zero.

Leaving out the matrix  $M$ , the products  $\mathcal{F}^\dagger \mathcal{F}$ ,  $\mathcal{F}^\dagger N$ ,  $N^\dagger N$  and their Hermitian adjoints are the same generators of the boson $^-$ . From these matrices, can be derived some of the generators of the algebra boson $^+$  performing the  $\circ$  transformation, defined as follows

$$\circ : \text{boson}^\pm \mapsto \text{boson}^\mp, \quad (\mathbf{f}^{(\pm)})^\circ \mathbf{f}^{(\mp)} := \mathbf{f}^{(\mp)} \mathbf{f}^{(\pm)}. \quad (\text{B.14})$$

Clearly,  $\circ^2 = \text{Id}$ . This transformation can be extended over all the rings; the transformation of a product is defined as

$$\left( \sum_{j=0 \dots J} \mathbf{f}_j^{(\pm)} \mathbf{f}_j^{(\mp)} \right)^\circ := \sum_{j=0 \dots J} (\mathbf{f}_j^{(\pm)})^\circ \mathbf{f}_j^{(\mp)}, \quad \left( \prod_{j=0 \dots J} \mathbf{f}_j^{(\pm)} \mathbf{f}_j^{(\mp)} \right)^\circ := \left( \prod_{j=0 \dots J} \mathbf{f}_j^{(\pm)} \right)^\circ \mathbf{f}_j^{(\mp)}. \quad (\text{B.15})$$

For the moment we have considered the action of  $\circ$  only over elements of boson $^\pm$  matrices that can be decomposed as sums and/or products of elements of the form  $\mathbf{f}_j^{(\mp)} \mathbf{f}_j^{(\pm)}$ . In the order to extend the definition of the  $\circ$  transformation to the whole ring, it is necessary to introduce a role that gives the transformation of the boson $^\pm$  that can be not decomposed as explain before. In our case, it can be read from the expression of  $\hat{T}$  in Eq. (1.12) and of  $\hat{V}\hat{T}$  in Eq. (1.14). Indeed, for each matrix in boson $^\pm$ , there exists the corresponding boson $^\mp$ ; more explicitly

$$\mu \xrightarrow{\circ} -\mu \quad M_t \xrightarrow{\circ} M_t, \quad M_{u,t+1} \xrightarrow{\circ} M_{v,t+1}. \quad (\text{B.16})$$

Notice that, for Eq. (A), the boson $^\pm$  matrix appears at the exponent. Now we have extended the transformation  $\circ$  over the entire ring  $\text{boson}^\pm$ . It is easy to check that

$$(e^{M_u} + \mathbf{f}^{(\pm)} \mathbf{f}^{(\mp)})^\circ = (e^{\hat{M}_u} + \mathbf{f}^{(\pm)} \mathbf{f}^{(\mp)}), \quad (e^{M_u} \mathbf{f}^{(\pm)} \mathbf{f}^{(\mp)})^\circ = (\mathbf{f}^{(\pm)} \mathbf{f}^{(\mp)})^\circ (e^{\hat{M}_u}). \quad (\text{B.17})$$

The determinant and the trace of the matrix, as  $(\prod_{j=0}^J \mathbf{f}_j^{(\pm)} \mathbf{f}_j^{(\mp)})$ ,  $1 + (\prod_{j=0}^J \mathbf{f}_j^{(\pm)} \mathbf{f}_j^{(\mp)})$ ,  $E$  are  $\circ$  invariant. It is convenient to introduce the convention

$$T^{(\pm)} := T^{(\mp)}. \quad (\text{B.18})$$

Notice that  $I_t^{\circ(2,1)}$ ,  $(I_t^{\circ(2,1)})^\circ$  and  $I_t^{\circ(1,2)}$ ,  $(I_t^{\circ(1,2)})^\circ$  are different.

We have found a solution of the saddle point equations of the form  $\mathcal{F} = NA$  where  $A = A[N^\dagger N]$ ; at the same time, we can try to find a solution of the form  $\mathcal{F} = NA$  with  $A = A[NN^\dagger]$ .

## Appendix C

# Bogoliubov's transform parameters

We have got to determinate the parameters in (4.11) to obtain (4.12). The procedure of the determination of the parameters of  $X$  is quite similar to the procedure performed in BCS theory [26]. The main difference is that in this case we need to perform a rotation in the space of operator particles, while this is not necessary the case in BCS theory.

For convenience, in this section we do not use Einstein notation.  $\hat{S}$  can be written in the form

$$\hat{S} = \sum_i \hat{u}_i^\dagger \hat{\Lambda}_i^\dagger - \hat{\Lambda}_i \hat{u}_i = \sum_i \sigma_i, \quad \hat{\Lambda}_i := \hat{v} \cdot X, \quad \sigma_i := (\hat{u}_i^\dagger \hat{\Lambda}_i^\dagger - \hat{\Lambda}_i \hat{u}_i). \quad (\text{C.1})$$

Evidently,  $\{\hat{u}_j, \tilde{\Lambda}_i\}$ . One observes that

$$[\sigma_i, \sigma_j] = -\hat{u}_i^\dagger \left( X^\dagger \cdot X \right)_{i,j} \hat{u}_j + \hat{u}_j^\dagger \left( X^\dagger \cdot X \right)_{i,j} \hat{u}_i. \quad (\text{C.2})$$

It is possible to perform a change of base in the space of particles:  $\hat{u} \mapsto (O\hat{u}) =: \tilde{u}_k$  to render  $X^\dagger X$  a diagonal matrix  $OX^\dagger XO^{-1} = \delta_{k,k'} \tilde{X}_k^\dagger \tilde{X}_k$ . For convention, each object expressed in this base has a tilde on the top. The commutator in (C.3) vanishes in this base, therefore

$$e^{\hat{S}} = \prod_i e^{\tilde{\sigma}_k}. \quad (\text{C.3})$$

Notice that

$$\tilde{\sigma}_k^2 = -\tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k \tilde{u}_k^\dagger \tilde{u}_k - \tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger \tilde{u}_k \tilde{u}_k^\dagger, \quad \tilde{\sigma}_k^{2n} = (-1)^n \left( \tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k \right)^n \tilde{u}_k^\dagger \tilde{u}_k + \left( \tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger \right)^n \tilde{u}_k \tilde{u}_k^\dagger. \quad (\text{C.4})$$

The previous relations reveal an analogy between  $\sigma^2$  and a projector, they only differ

for a multiplicative factor. From (C.5), it follows that

$$e^{-i\tilde{\sigma}k} = \sum_n \frac{(-)^n}{(2n)!} \left[ 1 + \frac{\tilde{\sigma}_k}{2n+1} \right] \left[ \left( \tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k \right)^n \tilde{u}_k^\dagger \tilde{u}_k + \left( \tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger \right)^n \tilde{u}_k \tilde{u}_k^\dagger \right], \quad (\text{C.5a})$$

$$e^{-i\tilde{\sigma}k}|0\rangle = \sum_n \frac{(-)^n}{(2n)!} \left[ 1 + \frac{\tilde{\sigma}_k}{2n+1} \right] \left( \tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger \right)^n \tilde{u}_k \tilde{u}_k^\dagger |0\rangle. \quad (\text{C.5b})$$

In this base, the computation of (C.5b) is easy, since  $\tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger |0\rangle = \tilde{X}_k^\dagger \tilde{X}_k |0\rangle$ , and one obtains

$$e^{-i\tilde{\sigma}k}|0\rangle = \sum_n \frac{(-)^n}{(2n)!} \left( \tilde{X}_k^\dagger \tilde{X}_k \right)^{\frac{2n}{2}} + \frac{(-)^n}{(2n+1)!} \left( \tilde{X}_k^\dagger \tilde{X}_k \right)^{\frac{2n+1}{2}} \tilde{X}_k^{-\frac{1}{2}} \tilde{X}_k^{\dagger-\frac{1}{2}} |0\rangle \quad (\text{C.6a})$$

$$= \cos \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} \left( 1 + \text{tg} \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} \tilde{X}_k^{-\frac{1}{2}} \tilde{X}_k^{\dagger-\frac{1}{2}} \tilde{u}_k^\dagger \tilde{\Lambda}_k \right) |0\rangle \quad (\text{C.6b})$$

$$= \cos \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} e^{\text{tg} \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} \tilde{X}_k^{-\frac{1}{2}} \tilde{X}_k^{\dagger-\frac{1}{2}} \tilde{u}_k^\dagger \tilde{\Lambda}_k} |0\rangle. \quad (\text{C.6c})$$

Substituting into (C.3) gives

$$\prod_k e^{\tilde{\sigma}k}|0\rangle = \prod_k \cos \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} e^{\sum_k \text{tg} \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} \tilde{X}_k^{-\frac{1}{2}} \tilde{X}_k^{\dagger-\frac{1}{2}} \tilde{u}_k^\dagger \tilde{\Lambda}_k} |0\rangle, \quad (\text{C.7})$$

and returning to the old base

$$e^{\hat{S}}|0\rangle = \det_- \cos \sqrt{X^\dagger X}, e^{\text{tr}-(\text{tg} \sqrt{X^\dagger X} X^{-\frac{1}{2}} X^{\dagger-\frac{1}{2}}) \hat{u}^\dagger \Lambda^\dagger} |0\rangle. \quad (\text{C.8})$$

The action of the operator  $e^{i\hat{\Theta}}$  is well known; the result is

$$\hat{\mathcal{U}}|0\rangle = \left( \det_- \cos \sqrt{X^\dagger X} \right) \exp \left[ \text{tr} - \left( \text{tg} \sqrt{X^\dagger X} X^{-\frac{1}{2}} X^{\dagger-\frac{1}{2}} \right) \hat{u}^\dagger e^{i\Theta_{\hat{u}}} X^\dagger e^{i\Theta_{\hat{v}}} \hat{v}^\dagger \right] |0\rangle. \quad (\text{C.9})$$

This equation is equivalent to (4.12) if the parameters  $\Theta_{\hat{v}}$ ,  $\Theta_{\hat{u}}$  and  $X$  in  $\hat{\mathcal{U}}$  satisfy to

$$\det_- \cos \sqrt{X^\dagger X} = \det_- \frac{1}{\sqrt{1 + \mathcal{F}^\dagger \mathcal{F}}}, \quad (\text{C.10a})$$

$$e^{i\Theta_{\hat{u}}} X^\dagger e^{i\Theta_{\hat{v}}} = \mathcal{F}^\dagger. \quad (\text{C.10b})$$

If Eq. (C.10a) holds, a stronger form of Eq. (C.10b) is satisfied

$$\cos \sqrt{X^\dagger X} = \frac{1}{\sqrt{1 + \mathcal{F}^\dagger \mathcal{F}}}. \quad (\text{C.11})$$

Now we evaluate the action of  $\hat{\mathcal{U}}$  over the doublet  $\hat{\psi}$ . In order to perform this computation, we make use of the following identity

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, \dots [A, B] \dots]]. \quad (\text{C.12})$$



First, we need to evaluate the action of  $e^{\hat{S}}$ . In this case, the first and second terms in (C.12) are

$$\left[ \hat{S}, \hat{u} \right] = -X^\dagger \hat{v}^\dagger, \quad \left[ \hat{S}, \hat{v}^\dagger \right] = X \hat{u}, \quad (\text{C.13})$$

$$\left[ \hat{S}, \left[ \hat{S}, \hat{u} \right] \right] = -X^\dagger X \hat{u} \quad \left[ \hat{S}, \left[ \hat{S}, \hat{v}^\dagger \right] \right] = -X X^\dagger \hat{v}^\dagger \quad (\text{C.14})$$

The particular expressions of this commutators permit to apply a recursive process; the result is

$$\begin{aligned} e^{\hat{S}} \hat{u} e^{-\hat{S}} &= \sum_{n=0}^{\infty} \frac{(-X^\dagger X)^n}{(2n)!} \left[ \hat{u} - \frac{1}{2n+1} X^\dagger \hat{v} \right] \\ &= \cos\left(\sqrt{X^\dagger X}\right) \left[ \hat{u} - \text{tg}\left(\sqrt{X^\dagger X}\right) X^{-1/2} X^{\dagger 1/2} \hat{v}^\dagger \right], \end{aligned} \quad (\text{C.15})$$

$$\begin{aligned} e^{\hat{S}} \hat{v}^\dagger e^{-\hat{S}} &= \sum_{n=0}^{\infty} \frac{(-X X^\dagger)^n}{(2n)!} \left[ \hat{v}^\dagger + \frac{1}{2n+1} X \hat{u} \right] \\ &= \cos\left(\sqrt{X X^\dagger}\right) \left[ \hat{v}^\dagger + \text{tg}\left(\sqrt{X X^\dagger}\right) X^{\dagger -1/2} X^{1/2} \hat{u} \right]. \end{aligned} \quad (\text{C.16})$$

The rotation in the particles operator space gives

$$e^{i\hat{\Theta}} e^{\hat{S}} \hat{u} e^{-\hat{S}} e^{i\hat{\Theta}} = \cos\left(\sqrt{X^\dagger X}\right) e^{-i\Theta_{\hat{u}}} \left[ \hat{u} - e^{+i\Theta_{\hat{u}}} \text{tg}\left(\sqrt{X^\dagger X}\right) X^{-1/2} X^{\dagger 1/2} e^{+i\Theta_{\hat{v}} \hat{v}^\dagger} \right] \quad (\text{C.17})$$

$$e^{i\hat{\Theta}} e^{\hat{S}} \hat{v}^\dagger e^{-\hat{S}} e^{-i\hat{\Theta}} = \cos\left(\sqrt{X X^\dagger}\right) e^{+i\Theta_{\hat{v}}} \left[ \hat{v}^\dagger + e^{-i\Theta_{\hat{v}}} \text{tg}\left(\sqrt{X X^\dagger}\right) X^{\dagger -1/2} X^{1/2} e^{-i\Theta_{\hat{u}} \hat{u}} \right]. \quad (\text{C.18})$$

In the cases where we can solve the equations (C.10) and (C.11), then the doublet operator  $\hat{\psi}$  transforms under  $\hat{\mathcal{U}}$  in the following way

$$\hat{\mathcal{U}} \hat{\psi} \hat{\mathcal{U}}^{-1} = \begin{pmatrix} \frac{\exp(-i\Theta_{\hat{u}})}{\sqrt{1+\mathcal{F}^\dagger \mathcal{F}}} [\hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger] \\ \frac{\exp(+i\Theta_{\hat{u}})}{\sqrt{1+\mathcal{F} \mathcal{F}^\dagger}} [\hat{v}^\dagger + \mathcal{F} \hat{u}] \end{pmatrix}. \quad (\text{C.19})$$



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