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LATTICE $\mathbf{QCD_2}$ EFFECTIVE ACTION WITH BOGOLIUBOV TRANSFORMATIONS

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alla mia Mamma, con affetto
per tutto il suo Amore

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Introduction

A toy model for strong interactions

The theoretical physicist typically spends a lot of efforts to study models whose elementary excitation, or particle, content is not directly observable by his experimental colleague, since the fundamental objects they describe do not propagate individually, but form more complex structures, such as composite particles or bound states. Examples can be found in every branch of Phisics: from the superfluidity and superconductivity theories, where, respectively, atoms and electrons form strongly correlated states responsible for the peculiar characteristics of these phases, to the models that describe the formation of atomic nuclei starting from neutrons and protons.

That's obviously also the case for Quantum Chromodynamics: at low energy, quarks and gluons do not exist as free particles, but are *confined* in composite states, namely mesons and baryons. As a founding part of the Standard Model, QCD shares with Electroweak theory the great success the theory of elementary particles gained in describing the natural phenomena. However, in spite of the efforts of entire generations of physicists, QCD has been proven to be an exceptionally difficult theory, in the first place because of the non-abelian structure of its symmetry gauge group, the colour group $\mathrm{SU}(N_c)$. In order to find clues for the mechanism governing the strong interactions, in the years have been formulated a lot of "toy models", which preserve some of the peculiar features of QCD but are somewhat easier to solve.

In the present work we will study Quantum Chromodynamics in two space-time dimensions, QCD_2. In this case, exploiting its low dimensionality, it is possible to eliminate the gauge fields via a gauge fixing procedure, obtaining in return a confining potential between quarks that can be interpreted as an interaction mediated by gluons. A pioneer in this field has been Gerardus 't Hooft, who published in 1974 the article [15] in which he studied the properties of the model in the limit for large number of colours N_c.

In order to formulate properly a quantum field theory, a regularization procedure is mandatory, to deal with the infinities that arise at various level of the calculations. We will use a lattice regularization scheme: the theory is formulated on a lattice with finite spacing a, which provides a natural momentum cut-off. The limit of zero lattice spacing can be performed at any time to eventually obtain the continuum description. This approach makes possible to formulate the theory in a non-perturbative way, allowing in principle to access informations about the structure of the vacuum and the bound states. However, its main disadvantage is the *fermion doubling problem*: when discretized in a direct (naive) way, a fermion theory becomes redundant in its particle content, acquiring doublers that do not disappear in the continuum theory. To evade this problem, we will use the Wilson method [35], which consists in adding terms in the action which give to the doublers' mass a contribution divergent when $a \to 0$, so that the spurious excitations can never be produced in a physical process.

The Wilson method, however, explicitly breaks a global invariance of the classical action, the *chiral symmetry*, that potentially might be not recovered in the continuum limit, because of marginal terms that modify the conservation law associated. When this happens, we say that the classical symmetry is *anomalous*. The connection between fermion doublers and chiral anomaly is a deep one and deserves to be investigated, but since it is slightly out of the scope of the present work, we present it in appendix E: there, the discussion is self contained and can be used for reference in future studies. However, the reader confident with the Wilson lattice formulation can skip it and start directly from chapter 1.

Effective action and Bogoliubov transformations

As the elementary degrees of freedom for models exhibiting confinement are hidden, the physical phenomena are better described by *effective theories* that are formulated in terms only of the composite states. There is great interest in deriving the characteristics of these "low energy" theories from those of the fundamental ones: in this way, an understanding of Nature in terms of first principles could be achieved. Recently, in [8], a method has been proposed that use to this aim of the formalism of *Bogoliubov transformations*.

These are unitary transformations in the canonical operators algebra that mix creation and annihilation operators. They have been applied before in many branches of physics, from the BCS theory of superconductivity [3], their first historical benchmark, to General Relativity (in the Mukhanov theory of inflation [23], in describing the Unruh effect [11], in a formulation of black holes thermodynamics [34]). Indeed, because of the mixing between operators, they induced also a transformation in the space of physical states, which can now be built acting with the new creation operators on a new vacuum state (that is, the state annihilated by the new destruction operators). Imposing a *variational principle*, that is demanding this new vacuum to be the one with minimal energy, the parameters of the transformation can be fixed and the new vacuum interpreted as the physical one. In BCS, it corresponds to the state where the electrons are coupled in strongly correlated pairs, the Cooper pairs, while in General Relativity the vacuum itself is a matter of point of view (it depends on the observer).

The Bogoliubov transformations have been successfully applied also in QCD_2, starting from [2] to more recent studies (see [17]). Here, the new operators are interpreted as generating *quasiparticles*, which are shown to form bound states, the mesons. However, in spite of their intuitiveness, their application has been somewhat limited in more realistic quantum field theories: the need to start from the canonical (and so Hamiltonian) approach to formulate the theory makes things terribly complicated when a renormalization procedure has to be introduced, because then one cannot use the explicit Lorentz-covariant form of the action to guess the shape of the counterterms.

However, the method mentioned earlier can in principle overcome these problems: noting that the *partition function* of a quantum field theory admits an operatorial representation

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta \hat{H}} \tag{1}$$

and a functional representation

$$\mathcal{Z} = \int [\mathcal{D}\phi] e^{-S[\phi_1, \phi_2, \cdots]} \tag{2}$$

whose identification is a matter of textbooks, we can switch from one formulation to the other. Performing first the transformation in the operatorial approach and then pass to the functional one, an effective action with the same original symmetries can be obtained. In this action we can isolate the vacuum contribution and the quasiparticle one, and so get some relevant information about the confining phase of the theory without the drawbacks of the canonical scheme.

The application of this method to QCD_2 in the limit of large number of colours, whose characteristics are quite well understood and explained in literature, it's a test for its validity: if we could deduce in this formulation the expected informations about vacuum structure, quasiparticles confinement and meson dominance, then we have a remarkable alternative approach to the construction of effective quantum field theories.

Thesis structure

At first, in chapter 1, we introduce our model for lattice QCD_2 and state all its features we will use in the sequel. Then, in chapter 2 we describe the general method to obtain an effective theory using Bogoliubov transformations. The main part of our work start from chapter 3: there, we apply this method to our model, getting an expression for the vacuum energy and explaining why the quasiparticles are confined. Finally, in 4 we bosonize the model, formulating it in terms of effective colourless mesons, with an interaction contribution and a proper kinetic term. The results are summarized in chapter 5. The appendices from A to D simply state conventions and specify some aspects introduced in the main text. As already said, appendix E is a self contained report about the chiral anomaly in Wilson's lattice formulation, included to outline his approach and to deepen this intriguing aspect, barely mentioned in the previous chapters: we hope it can be a valuable addition for the interested reader.

Chapter 1

QCD_2 on the lattice

In this chapter we introduce the lattice formulation of QCD_2 and enunciate its main features. In section 1.1 we write the discrete action of the theory in Wilson's scheme. Then, in section 1.2 we build its space of states and obtain an expression for the transfer matrix, the operator that generates the lattice equivalent of time translations. Finally, in section 1.3 we sketch briefly some results from the weak coupling expansion of the theory, the limit for small gauge coupling constant g.

1.1 Wilson-Dirac action

The action for our model consists of two distinct terms: a fermion and a pure gauge ones, such that

$$S = S_F + S_G \tag{1.1}$$

The fermion part of the action is

$$\begin{aligned} S_F = a^2 \sum_{x \in (a\mathbb{Z})^2} \Bigg\{ &\left(m + \frac{2r}{a} \right) \bar{\psi}(x)\psi(x) \\ &- \left[\bar{\psi}(x) \frac{r - \gamma^0}{2a} U_0(x)\psi(x + a\hat{0}) + \bar{\psi}(x + a\hat{0}) \frac{r + \gamma^0}{2a} U_0^\dagger(x)\psi(x) \right] \\ &- \left[\bar{\psi}(x) \frac{r - \gamma^1}{2a} U_1(x)\psi(x + a\hat{1}) + \bar{\psi}(x + a\hat{1}) \frac{r + \gamma^1}{2a} U_1^\dagger(x)\psi(x) \right] \Bigg\} \end{aligned} \tag{1.2}$$

The terms proportional to the Wilson parameter r are introduced to solve the fermion doubling problem (for more details on this point, see appendix E). In two space-time dimensions the 2×2 Hermitian γ matrices satisfying the (Euclidean) Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \tag{1.3}$$

can be represented as[1]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma^2 \tag{1.4}$$

A third Hermitian matrix, that we call γ^5 in analogy with four space-time dimensions, has the properties

$$\gamma^5 = -i\gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1, \qquad \{\gamma^5, \gamma^\mu\} = 0 \tag{1.5}$$

The fields ψ, $\bar{\psi}$ are two-component spinors in Dirac indices. The operators

$$P_\pm = \frac{1 \pm \gamma^0}{2} \tag{1.6}$$

project, respectively, on the components of the fermion fields that propagate forward and backward in time:

$$\begin{aligned} \psi_+ &= P_+\psi \qquad & \psi_+^\dagger &= \bar{\psi}P_+ \\ \psi_-^\dagger &= P_-\psi \qquad & -\psi_- &= \bar{\psi}P_- \end{aligned} \tag{1.7}$$

The link variable $U_\mu(x)$ belongs to the gauge group SU(N_c) and plays the role of a connection, such that the fermion action becomes invariant under local (point-dependent) SU(N_c) transformations. Therefore, under the gauge transformations

$$\psi(x) \to \Omega(x)\psi(x); \qquad \bar{\psi}(x) \to \bar{\psi}(x)\Omega^\dagger(x) \tag{1.8}$$

it changes according to

$$U_\mu(x) \to \Omega(x)U_\mu(x)\Omega^\dagger(x + a\hat{\mu}) \tag{1.9}$$

It can be represented in terms of elements of the algebra via the exponential map

$$U_\mu(x) = e^{igaA_\mu(x)} \tag{1.10}$$

[1] Our convention agrees with that of reference [36], after taking the basis in which γ^0 is diagonal.

with the algebra-valued Hermitian field[2]

$$A_\mu(x) = \sum_{l=1}^{N_c^2-1} A_\mu^l(x)\Theta^l \tag{1.11}$$

and Θ^l a set of Hermitian generators of the algebra, normalized such that

$$\mathrm{tr}\left(\Theta^l\Theta^m\right) = \frac{1}{2}\delta_{lm} \tag{1.12}$$

The quadratic Casimir operator is

$$\Theta\cdot\Theta \equiv \sum_{l=1}^{N_c^2-1} \Theta^l\Theta^l = \frac{N_c^2-1}{2N_c}\,\mathbb{I}_{N_c} \tag{1.13}$$

We will use also the following Fierz-type identity (see [24]):

$$\sum_{l=1}^{N_c^2-1} \Theta^l_{ab}\Theta^l_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N_c}\delta_{ac}\delta_{bd}\right) \tag{1.14}$$

Giving up axial interchange symmetry, as in [12], we can introduce a different lattice spacing a_μ and Wilson parameter r_μ for each axis: the action becomes

$$\begin{aligned} S_F = a_0a_1 \sum_{x\in(a_0\mathbb{Z})\times(a_1\mathbb{Z})} \Bigg\{ & \left(m + \frac{r_0}{a_0} + \frac{r_1}{a_1}\right)\bar\psi(x)\psi(x) \\ & - \left[\bar\psi(x)\frac{r_1-\gamma^1}{2a_1}U_1(x)\psi(x+a_1\hat{1}) + \bar\psi(x+a_1\hat{1})\frac{r_1+\gamma^1}{2a_1}U_1^\dagger(x)\psi(x)\right] \\ & - \left[\bar\psi(x)\frac{r_0-\gamma^0}{2a_0}U_0(x)\psi(x+a_0\hat{0}) + \bar\psi(x+a_0\hat{0})\frac{r_0+\gamma^0}{2a_0}U_0^\dagger(x)\psi(x)\right]\Bigg\} \end{aligned} \tag{1.15}$$

where

$$U_0(x) = e^{iga_0A_0(x)}, \qquad U_1(x) = e^{iga_1A_1(x)} \tag{1.16}$$

We call this representation of the action the "mass form", because the bare mass parameter m is explicit in it. Introducing the *hopping parameters*

$$\kappa_0 \equiv \frac{1}{2}\left(ma_0 + r_0 + r_1\frac{a_0}{a_1}\right)^{-1}, \qquad \kappa_1 \equiv \frac{1}{2}\left(ma_1 + r_0\frac{a_1}{a_0} + r_1\right)^{-1} \tag{1.17}$$

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[2] We will use the "late" latin letters l, m, $\cdots$ to indicate the N_c^2-1 indices of the adjoint representation, while we will assign the "early" letters a, b, $\cdots$ to the N_c indices of the fundamental one. However, as long as possible, we will omit the latter to lighten notation: when not explicitly written, they are contracted following the natural order (for example, $\bar\psi\Theta^l\psi = \bar\psi_a\Theta^l_{ab}\psi_b$). "tr" is a trace over the fundamental indices.

so that

$$\zeta \equiv \frac{a_0}{a_1} = \frac{\kappa_1}{\kappa_0} \tag{1.18}$$

and rescaling the fermion fields into the dimensionless variables

$$\varphi = \left(\frac{a_1}{2\kappa_0}\right)^{1/2} \psi, \qquad \bar{\varphi} = \left(\frac{a_1}{2\kappa_0}\right)^{1/2} \bar{\psi} \tag{1.19}$$

the action can be written in the "hopping-parameter form"

$$\begin{aligned} S_F = \sum_{n\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \Bigg\{ & \bar{\varphi}_{n,t}\varphi_{n,t} \\ & - \kappa_1 \left[\bar{\varphi}_{n,t}(r_1 - \gamma^1)(U_1)_{n,t}\varphi_{n+1,t} + \bar{\varphi}_{n+1,t}(r_1 + \gamma^1)(U_1^\dagger)_{n,t}\varphi_{n,t} \right] \\ & - \kappa_0 \left[\bar{\varphi}_{n,t}(r_0 - \gamma^0)(U_0)_{n,t}\varphi_{n,t+1} + \bar{\varphi}_{n,t+1}(r_0 + \gamma^0)(U_0^\dagger)_{n,t}\varphi_{n,t} \right] \Bigg\} \end{aligned} \tag{1.20}$$

with dimensionless lattice units

$$t = x_0/a_0, \qquad n = x_1/a_1 \tag{1.21}$$

To avoid inessential complications due to time doublers in the construction of the fermionic transfer matrix, which we will discuss in the next section, we will always take $r_0 = 1$, so that the operators $r_0 \pm \gamma^0$ become the projectors (1.6).[3]

A standard choice of the pure gauge action, that reduces to the Yang-Mills action in the continuum limit, is

$$S_G = \frac{1}{a_0 a_1} \sum_P \frac{1}{g^2} \left[2N_c - \mathrm{Tr}\left(U_P + U_P^\dagger\right)\right] \tag{1.22}$$

where the plaquette sum and variables are defined by

$$\begin{aligned} \sum_P &= \frac{1}{2} \sum_x \sum_{\mu,\nu=0}^{1} \\ U_P &= U_\mu(x) U_\nu(x + a_\mu\hat{\mu}) U_\mu^\dagger(x + a_\nu\hat{\nu}) U_\nu^\dagger(x) \end{aligned} \tag{1.23}$$

As it is clear from (1.22), the bare coupling constant g in QCD$_2$ has dimension of mass (the theory is super-renormalizable) while the field A_μ is

[3]For a construction of the fermionic transfer matrix with a generic r_0, using a double time slice Hilbert space formalism, see [32].

dimensionless. An explicit form for the plaquette variable can be obtained from (1.10) via the Baker-Campbell-Hausdorff formula: setting

$$U_P = e^{iga_\mu a_\nu G_{\mu\nu}(x)} \tag{1.24}$$

then the lattice (algebra-valued) field strength is

$$G_{\mu\nu}(x) = F_{\mu\nu}(x) + o(a) \tag{1.25}$$
$$F_{\mu\nu}(x) = \partial_\mu^{(+)} A_\nu(x) - \partial_\nu^{(+)} A_\mu(x) + ig[A_\mu(x), A_\nu(x)] \tag{1.26}$$

where the right derivative is defined as in (A.2).

1.2 Transfer matrix

The quantum theory descends, in the functional formalism, from the partition function

$$\mathscr{Z} = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,e^{-S[U,\psi,\bar{\psi}]} \tag{1.27}$$

where the total action is

$$S = S_F + S_G \tag{1.28}$$

The fermionc measure is defined by

$$\mathcal{D}\psi\,\mathcal{D}\bar{\psi} = \prod_x \mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x) \tag{1.29}$$

where $\mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x)$ denotes a Berezin integral over the Grassmann variables ψ, $\bar{\psi}$ at each site, while $\mathcal{D}U$ is the Haar measure on SU(N_c). Introducing external sources we could define a generating functional and establish a perturbative series as in the continuum. In principle all the theory can be obtained in this manner, because on the lattice the functional integrals are well defined (provided eventually the use of a finite box and of a procedure of thermodynamic limit): that's exactly the usual reason to adopt lattice regularization. On the other hand, there is a more direct way to find the quantum theory on a lattice, which consists in building up the Hilbert space of physical states, where the fields of the theory act as particles creation and annihilation operators. That's the formalism of second quantization where QFT originated from, today still relevant in many branches of theoretical physics, such as the study of quantum many-body systems. In QCD, the states of the system are in a Fock space $\mathscr{F}$ that can be written as a tensor product of a pure fermion space of states $\mathscr{F}_F$ and a pure gauge field Fock space $\mathscr{F}_G$.

The bridge between the functional and the operatorial formalism is provided by the partition function itself, which, from a quantum statistical point of view, is defined as the volume of the accessible states of the system, each weighted by its Boltzmann factor:

$$\mathcal{Z} = \mathrm{Tr}^{\mathscr{F}}\, e^{-\beta \hat{H}} \tag{1.30}$$

where $\mathrm{Tr}^{\mathscr{F}}$ denotes a sum over states and $\hat{H}$ is the quantum Hamiltonian. The connection between this equation and a path integral representation such as that in (1.27) is a standard textbook matter and we will not discuss it (for details see, for example, [36]). We just point out that, as the Hamiltonian is the generator of continuum time evolution, on the lattice we don't have one readily available, because in our model also time is discrete. Thus, the role of the Boltzmann factor is taken by the *transfer matrix* $\hat{\mathcal{T}}_{t,t+1}$, the operator that maps the Hilbert space of states defined at time t into that defined at time $t+1$. If this operator is self-adjoint and strictly positive, then the lattice Hamiltonian can be defined as

$$\hat{H} = -\frac{1}{a_0} \ln \hat{\mathcal{T}}_{t,t+1} \tag{1.31}$$

The proof of the existence of such an operator for Wilson fermions in the temporal gauge $U_0 = 1$ can be found in [20], [10], [32] and [31]. In order to work in a more general gauge, we will use a slightly different method, as in [8]. To get the explicit form of the transfer matrix in our present model, we will sketch it briefly for completeness.

Let's focus on the fermionic part of the Hilbert space of states. The partition function (1.27) can be written as

$$\mathcal{Z} = \int \mathcal{D}U\, e^{-S_G[U]}\, \mathrm{Tr}^F \prod_t \mathcal{J}_t \hat{\mathcal{T}}_{t,t+1} \tag{1.32}$$

where Tr^F denotes the trace over the fermion Fock space. The operator $\hat{\mathcal{T}}_{t,t+1}$ is the *fermion transfer matrix*

$$\hat{\mathcal{T}}_{t,t+1} = \hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1} \tag{1.33}$$

where

$$\hat{T}_t = \exp\left[-\hat{u}^\dagger \mathcal{M}_t \hat{u} + \hat{v} \mathcal{M}_t \hat{v}^\dagger\right] \exp\left[\hat{v} \mathcal{N}_t \hat{u}\right] \tag{1.34}$$

$$\hat{V}_t = \exp\left[\hat{u}^\dagger \ln \mathcal{U}_{0,t} \hat{u} - \hat{v} \ln \mathcal{U}_{0,t}^\dagger \hat{v}^\dagger\right] \tag{1.35}$$

$$\mathcal{J}_t = \exp\left[\mathrm{tr}\left(\mathcal{M}_t + \mathcal{M}_t^\dagger\right)\right] \tag{1.36}$$

and $\mathcal{M}_t$, $\mathcal{N}_t$, $U_{0,t}$ are matrices in internal (colour) and space (but not time, which is just a label) indices, with the last one defined as

$$[\mathcal{U}_{\mu,t}]_{n_1 n_2} \equiv \delta_{n_1 n_2}(U_\mu)_{n_1,t} \tag{1.37}$$

Therefore, "tr" is a sum over these indices. The symbol $\hat{u}^\dagger$ ($\hat{v}^\dagger$) denotes the creation operator of fermions (antifermions) and carries internal and space indices understood (in two space-time dimensions, they carry no Dirac index because they are single-valued). In a fermion space, the kernel of an operator can be expressed in terms of matrix elements between canonical coherent states; using formulas collected in section A.3 it's easy to get, from the transfer matrix (1.33),

$$\begin{aligned}\langle \rho_t, \sigma_t | \hat{\mathcal{T}}_{t,t+1} | \rho_{t+1}, \sigma_{t+1} \rangle = &\exp\left[\rho_t^\dagger \mathcal{N}_t \sigma_t^\dagger\right] \exp\left[\rho_t^\dagger e^{-\mathcal{M}_t} \mathcal{U}_{0,t} e^{-\mathcal{M}_{t+1}} \rho_{t+1}\right] \\ &\cdot \exp\left[-\sigma_{t+1} e^{-\mathcal{M}_{t+1}} \mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t} \sigma_t^\dagger\right] \exp\left[\sigma_{t+1} \mathcal{N}_{t+1} \rho_{t+1}\right]\end{aligned} \tag{1.38}$$

where $|\rho_t, \sigma_t\rangle$, $|\rho_{t+1}, \sigma_{t+1}\rangle$ are canonical coherent states in the Hilbert spaces defined, respectively, at time t and $t+1$; ρ_t, σ_t, ρ_{t+1}, σ_{t+1} are the corresponding Grasmannian eigenvalues (they carry a space and a colour index understood); we also used the additional hypothesis that the matrices $\mathcal{M}$, $\mathcal{N}$ are Hermitian (for details of the above calculation, see [27]). Inserting appropriately the resolution of unity (A.51) and using equation (A.48) we get, for the trace in (1.32),

$$\mathrm{Tr}^F \prod_t \mathcal{J}_t \hat{\mathcal{T}}_{t,t+1} = \int \prod_t \left[\mathcal{J}_t \, \mathrm{d}\rho_t^\dagger \mathrm{d}\rho_t \mathrm{d}\sigma_t^\dagger \mathrm{d}\sigma_t\right] e^{-\tilde{S}_F} \tag{1.39}$$

with

$$\begin{aligned}\tilde{S}_F = \sum_t \Big[&\rho_t^\dagger \rho_t - \sigma_t \sigma_t^\dagger - \rho_t^\dagger \mathcal{N}_t \sigma_t^\dagger - \sigma_{t+1} \mathcal{N}_{t+1} \rho_{t+1} \\ &- \rho_t^\dagger e^{-\mathcal{M}_t} \mathcal{U}_{0,t} e^{-\mathcal{M}_{t+1}} \rho_{t+1} + \sigma_{t+1} e^{-\mathcal{M}_{t+1}} \mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t} \sigma_t^\dagger\Big]\end{aligned} \tag{1.40}$$

Introducing the two-component spinors χ, $\chi^\dagger$ (in spite of the $\dagger$, these spinors are independent from an algebra construction point of view) via the associations

$$\chi_t = \begin{pmatrix}\rho_t \\ \sigma_t^\dagger\end{pmatrix}, \qquad \chi_t^\dagger = (\rho_t^\dagger, \sigma_t), \qquad \bar{\chi}_t \equiv \chi_t^\dagger \gamma^0 = (\rho_t^\dagger, -\sigma_t) \tag{1.41}$$

we can write

$$\begin{aligned}\tilde{S}_F = \sum_t \Big[&\bar{\chi}_t \chi_t + \bar{\chi}_t \left(i\gamma^1 \mathcal{N}_t\right) \chi_t \\ &- \bar{\chi}_t e^{-\mathcal{M}_t} P_+ \mathcal{U}_{0,t} e^{-\mathcal{M}_{t+1}} \chi_{t+1} - \bar{\chi}_{t+1} e^{-\mathcal{M}_{t+1}} P_- \mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t} \chi_t\Big]\end{aligned} \tag{1.42}$$

Operating the change of variables

$$\varphi_t = \frac{e^{-\mathcal{M}_t}}{\sqrt{2\kappa_0}}\chi_t, \qquad \bar{\varphi}_t = \bar{\chi}_t\frac{e^{-\mathcal{M}_t}}{\sqrt{2\kappa_0}} \tag{1.43}$$

which simplifies the Jacobian $\mathcal{J}$ in the integration measure, we obtain

$$\tilde{S}_F = 2\kappa_0 \sum_t \left[\bar{\varphi}_t \left(e^{2\mathcal{M}_t} + i\gamma^1 e^{\mathcal{M}_t}\mathcal{N}_t e^{\mathcal{M}_t}\right)\varphi_t - \bar{\varphi}_t \mathcal{U}_{0,t} P_+ \varphi_{t+1} - \bar{\varphi}_{t+1}\mathcal{U}^\dagger_{0,t} P_- \varphi_t\right] \tag{1.44}$$

Comparing this result with equation (1.20) we find

$$\mathcal{B}_t \equiv 2\kappa_0 e^{2\mathcal{M}_t} = \mathbb{I} - \kappa_1 r_1 \left(\mathcal{U}_{1,t}T_1 + T_1^\dagger \mathcal{U}^\dagger_{1,t}\right) \tag{1.45}$$

$$\mathcal{N}_t = -i\kappa_1 \mathcal{B}_t^{-1/2}\left(\mathcal{U}_{1,t}T_1 - T_1^\dagger \mathcal{U}^\dagger_{1,t}\right)\mathcal{B}_t^{-1/2} \tag{1.46}$$

where we used the definition (A.1) for the lattice shift operators.

1.3 Gauge fixing and weak coupling expansion

The gauge action (1.22) can be written as a quadratic plus an interaction part in the field $A_\mu(x)$ only in the continuum limit, when it becomes the usual Yang-Mills action for gluons. In order to define a gluon propagator for any finite lattice spacing a we need to formulate a perturbation theory in g on the lattice, the so called *weak coupling expansion*, mirroring the diagrammatic series in the continuum. The lattice procedure has a number of complications compared to the usual formulation, some of which we briefly mention:

- Firstly, in order to keep gauge invariance explicit for any finite lattice spacing a, a lot of new vertices are introduced. These vertices correctly do not survive in the continuum limit (they are *irrelevant*), but nevertheless they complicate considerably the perturbation series when a is kept fixed.

- To express the Haar invariant measure $\mathcal{D}U$ in terms of the adjoint measure $\mathcal{D}A$, a *metric* of the gauge group SU(N_c) must be inserted in the functional integral. This metric can be considered as a new addend of the action, which generates other vertices (usually, these vertices diverge in the continuum limit, because they play the role of mass renormalization counterterms).

- Also, a gauge fixing is mandatory to define correctly the functional integral (as in the continuum). In the Faddeev-Popov picture, ghost fields must be included in the theory to preserve unitarity.

We won't get in the details of the points above: the interested reader can find complete discussions in textbooks such as [22] and [29]. For our future convenience, we only need an expression for the gluon propagator at the lowest order in g (that is, at tree level) in the Coulomb gauge. This gauge in two space-time dimensions is fixed by the condition

$$U_1(x) = 1 \qquad \Longleftrightarrow \qquad A_1(x) = 0 \tag{1.47}$$

so that the only gluon field remaining in the theory is A_0. The gauge action becomes

$$S_G = \frac{1}{a_0 a_1} \sum_x \frac{1}{g^2} \left\{ 2N_c - \mathrm{Tr}\left[U_0(x) U_0^\dagger(x + a_1 \hat{1}) + U_0(x + a_1 \hat{1}) U_0^\dagger(x) \right] \right\} \tag{1.48}$$

Therefore, the quadratic term in A_0 (i.e. the lowest term in the coupling constant g) is

$$\begin{aligned} S_G^{(0)} &= a_0 a_1 \sum_x \sum_{l,m=1}^{N_c^2-1} \partial_1^{(+)} A_0^l(x) \partial_1^{(+)} A_0^m(x) \,\mathrm{Tr}\left(T^l T^m\right) \\ &= \frac{a_0 a_1}{2} \sum_x \sum_{l=1}^{N_c^2-1} \partial_1^{(+)} A_0^l(x) \partial_1^{(+)} A_0^l(x) \end{aligned} \tag{1.49}$$

Using the lattice translational invariance, it is easy to show that the above expression can be written as

$$S_G^{(0)} = -\frac{a_0 a_1}{2} \sum_x \sum_{l=1}^{N_c^2-1} A_0^l(x) \partial_1^{(-)} \partial_1^{(+)} A_0^l(x) \tag{1.50}$$

The free theory is simply ($N_c^2 - 1$ copies of) the one of a free massless scalar boson. The generating functional is

$$\mathcal{Z}_G^{(0)}[J_0] = \frac{1}{\mathcal{Z}_G^{(0)}} \int \mathcal{D}A_0 \, \exp\left[\frac{1}{2} \left(A_0, (\partial_1)^2 A_0 \right) + (J_0, A_0) \right] \tag{1.51}$$

where $\mathcal{Z}_G^{(0)} = \mathcal{Z}_G^{(0)}[0]$, $(\partial_1)^2 = \partial_1^{(-)} \partial_1^{(+)}$ and we used the notation

$$\begin{aligned} (f, g) &= a_0 a_1 \sum_x \sum_{l=1}^{N_c^2-1} f^l(x) g^l(x) \\ (Mf)^l(x) &= \sum_y \sum_{m=1}^{N_c^2-1} M^{lm}(x, y) f^m(y) \end{aligned} \tag{1.52}$$

Solving the gaussian integral we get

$$\mathcal{Z}_G^{(0)}[J_0] = \exp\left[-\frac{1}{2}\left(J_0, (\partial_1)^{-2} J_0\right)\right] \tag{1.53}$$

so that the free gluon propagator is

$$\begin{aligned} \mathcal{G}_{00}^{lm}(x,y) &\equiv \langle A_0^l(x) A_0^m(y)\rangle \\ &= \left(\frac{1}{a_0 a_1}\frac{\delta}{\delta J_0^l(x)}\right)\left(\frac{1}{a_0 a_1}\frac{\delta}{\delta J_0^l(x)}\right)\mathcal{Z}_G^{(0)}[J_0]\Big|_{J_0=0} \\ &= -\frac{1}{a_0 a_1}\delta_{lm}\left[(\partial_1)^{-2}\right]_{xy} \end{aligned} \tag{1.54}$$

Using the Fourier transform (A.10) we get the momentum representation

$$\begin{aligned} \mathcal{G}_{00}^{lm}(p,q) &= (2\pi)^2 \delta^{(2)}(p+q)\mathcal{G}_{00}^{lm}(p) \\ \mathcal{G}_{00}^{lm}(p) &\equiv \delta_{lm}\frac{1}{(\hat{p}^1)^2} \end{aligned} \tag{1.55}$$

with

$$\hat{p}^1 = \frac{2}{a_1}\sin\frac{a_1 p^1}{2} \tag{1.56}$$

so that

$$\mathcal{G}_{00}^{lm}(x,y) = \delta_{lm}\frac{\delta_{x^0,y^0}}{a_0}\int_{-\pi/a_1}^{\pi/a_1}\frac{\mathrm{d}p}{2\pi}\,\frac{e^{ip\left(x^1-y^1\right)}}{\hat{p}^2} \tag{1.57}$$

In the continuum limit $a_0, a_1 \to 0$ this equation gives the well known result

$$\begin{aligned} \mathcal{G}_{00}^{lm}(x,y) &= \delta_{lm}\delta(x^0-y^0)\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\,\frac{e^{ip\left(x^1-y^1\right)}}{p^2} \\ &= -\frac{1}{2}\delta_{lm}\delta(x^0-y^0)\left|x^1-y^1\right| \end{aligned} \tag{1.58}$$

where the last equality can be obtained with a prescription to regularize the integral (see, for reference, [17]). In this form, it is more evident that, in two space-time dimensions, the interaction between quarks mediated by the gluons yields to linear confinement.

Chapter 2

Effective action: general formalism

It's now time to describe the method [8] to obtain an effective action using Bogoliubov transformations. After explaining this scheme in section 2.1, in the following ones we introduce the variational principle that leads to the physical vacuum and its connection with the diagonalization of lattice Hamiltonian.

2.1 Unitary transformations

Let $\hat{\chi}$, $\hat{\bar{\chi}}$ be the two-component spinor fields that act as canonical operators in the fermion space of states $\mathscr{F}_F$, defined at a certain time slice (we neglect the label t for brevity). Our theory allows a unitary involution to distinguish the component, $\hat{u}^\dagger$, that generates a particle, from the one, $\hat{v}^\dagger$, that generates an antiparticle (that is, the component that propagates, respectively, forward and backward in time). As we have already mentioned earlier, this involution is provided by the matrix γ^0:

$$\begin{aligned} \gamma_0 \hat{u} &= \hat{u} \qquad & \gamma_0 \hat{u}^\dagger &= \hat{u}^\dagger \\ \gamma_0 \hat{v} &= -\hat{v} \qquad & \gamma_0 \hat{v}^\dagger &= -\hat{v}^\dagger \end{aligned} \tag{2.1}$$

so that, using the projectors defined in equation (1.6), we found

$$\begin{aligned} \hat{u} &= P_+ \hat{\chi} \qquad & \hat{u}^\dagger &= \hat{\bar{\chi}} P_+ \\ \hat{v}^\dagger &= P_- \hat{\chi} \qquad & -\hat{v} &= \hat{\bar{\chi}} P_- \end{aligned} \tag{2.2}$$

We will always work in the basis (1.4) where γ^0 is diagonal, so we can confuse the two-component spinors $\hat{u}$, $\hat{v}$ with their non-null components:

$$\hat{u} = \begin{pmatrix} \hat{u} \\ 0 \end{pmatrix}, \qquad \hat{v}^\dagger = \begin{pmatrix} 0 \\ \hat{v}^\dagger \end{pmatrix}, \qquad \hat{u}^\dagger = (\hat{u}^\dagger, 0), \qquad \hat{v} = (0, \hat{v}) \tag{2.3}$$

Therefore,

$$\hat{\chi} = \begin{pmatrix} \hat{u} \\ \hat{v}^\dagger \end{pmatrix}, \qquad \hat{\bar{\chi}} = (\hat{u}^\dagger, -\hat{v}) \tag{2.4}$$

A *Bogoliubov transformation* is a unitary transformation in the space of the operators acting on the Fock space, that mixes creation and annihilation operators. A generalized version of this transformation can be used in the transfer matrix formalism of a lattice field theory to obtain results comparable to that of a *Foldy-Wouthuysen transformation* in the Dirac one-particle theory of a free fermion: namely, the separation of fermions from antifermions and the diagonalization of the Dirac Hamiltonian (see [33]). The parameters of the transformation can be used to introduce bosonic fields in the theory, in order to bosonize the model. The general formalism of this method can be found in references [5] and [8] (see also [4]); we will sketch it briefly to apply it to our present model. A generic unitary transformation on the operators (2.4) can be written as

$$\mathscr{U} = W R^{1/2}(1 + F) \tag{2.5}$$

where W is a block diagonal unitary transformation which does not mix creation and annihilation operators; F is an anti-Hermitian operator that anti-commutes with the involution γ^0, so that, in our basis,

$$F = \begin{pmatrix} 0 & -\mathcal{F}^\dagger \\ \mathcal{F} & 0 \end{pmatrix} \tag{2.6}$$

and R is a positive definite Hermitian operator that commutes with γ^0 and can be written as

$$R = \left(1 + F F^\dagger\right)^{-1} = \begin{pmatrix} \left(1 + \mathcal{F}^\dagger \mathcal{F}\right)^{-1} & 0 \\ 0 & \left(1 + \mathcal{F}\mathcal{F}^\dagger\right)^{-1} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathring{\mathcal{R}} \end{pmatrix} \tag{2.7}$$

Applying this transformation to the operators (2.4) we get the new components, which we call *quasiparticles*,

$$\begin{aligned} \hat{a} &= \mathcal{R}^{1/2}\left(\hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger\right) & \qquad \hat{b} &= \left(\hat{v} + \hat{u}^\dagger \mathcal{F}^\dagger\right) \mathring{\mathcal{R}}^{1/2} \\ \hat{a}^\dagger &= \left(\hat{u}^\dagger - \hat{v}\mathcal{F}\right) \mathcal{R}^{1/2} & \qquad \hat{b}^\dagger &= \mathring{\mathcal{R}}^{1/2}\left(\hat{v}^\dagger + \mathcal{F}\hat{u}\right) \end{aligned} \tag{2.8}$$

These operators satisfy canonical anti-commutation relations for any choice of $\mathcal{F}$, as a consequence of the unitarity of $\mathscr{U}$. The new (unnormalized) vacuum, i.e. the state annihilated by $\hat{a}$ and $\hat{b}$, is

$$|\mathcal{F}\rangle = \exp\left(\hat{\mathcal{F}}^\dagger\right)|0\rangle \tag{2.9}$$

where

$$\hat{\mathcal{F}}^\dagger = \hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger \tag{2.10}$$

is the creation operator of a "meson", a boson (in the sense of a commuting degree of freedom, considering that it is a fermion bilinear) composed of a particle and an antiparticle. The proof that (2.9) is actually a vacuum for the quasiparticles can be attained building the unitary operator $\hat{\mathscr{U}}(\mathcal{F})$ that realizes the transformation (2.5) in the space of states and applying it to the original vacuum $|0\rangle$ (see [8]). One can find, to be precise,

$$\hat{\mathscr{U}}(\mathcal{F})|0\rangle = \det\left(\mathcal{R}^{1/2}\right)\exp\left(\hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger\right)|0\rangle \tag{2.11}$$

Starting from $|\mathcal{F}\rangle$ and using the quasiparticle operators, we can build a new set of canonical coherent states, as in (A.50):

$$\begin{aligned}|\alpha,\beta;\mathcal{F}\rangle &= \exp\left(-\alpha\hat{a}^\dagger - \beta\hat{b}^\dagger\right)|\mathcal{F}\rangle \\ &= \exp\left(\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger - \mathcal{R}^{-1/2}\alpha\hat{a}^\dagger - \beta\mathring{\mathcal{R}}^{-1/2}\hat{b}^\dagger - \beta\mathcal{F}\alpha\right)|0\rangle\end{aligned} \tag{2.12}$$

(the last identity can be obtained via the Baker–Campbell–Hausdorff formula). The trace of the transfer matrix can be now expressed using matrix elements between these new coherent states:

$$\begin{aligned}&\langle\alpha_t,\beta_t;\mathcal{F}_t|\hat{T}_t^\dagger\hat{V}_t\hat{T}_{t+1}|\alpha_{t+1},\beta_{t+1};\mathcal{F}_{t+1}\rangle \\ &\qquad\qquad = \langle\alpha_t,\beta_t;\mathcal{F}_t|\hat{T}_t^\dagger\hat{\mathbb{I}}\hat{V}_t\hat{\mathbb{I}}\hat{T}_{t+1}|\alpha_{t+1},\beta_{t+1};\mathcal{F}_{t+1}\rangle\end{aligned} \tag{2.13}$$

where $\hat{\mathbb{I}}$ is the usual form (A.51) of the identity, which can be inserted appropriately also between the two exponential factors which the operators $\hat{T}_t^\dagger$ $\hat{T}_{t+1}$ consist of. Using the properties (A.34) and (A.47) of the coherent states and solving the Gaussian integrals over the intermediate variables with the aid of equation (A.23), one can get, for the partition function,

$$\mathcal{Z} = \int \mathcal{D}U\, e^{-S_G[U]}e^{-S_0[\mathcal{F}]}\int\prod_t\left[\mathrm{d}\alpha_t^\dagger\mathrm{d}\alpha_t\mathrm{d}\beta_t^\dagger\mathrm{d}\beta_t\right]e^{-S_Q[\alpha,\beta;\mathcal{F}]} \tag{2.14}$$

where the *zero-point action*, which depends only on the time-dependent, holomorphic parameters $\mathcal{F}_t$ of the Bogoliubov transformation (and, of course, on the gauge fields), is given by

$$S_0[\mathcal{F}] = -\sum_t \mathrm{tr}\left[\ln(\mathcal{R}_t\mathcal{U}_{0,t}\mathcal{E}_{t+1,t})\right] = -\sum_t \mathrm{tr}\left[\ln(\mathcal{R}_t\mathcal{E}_{t+1,t})\right] \tag{2.15}$$

while the *quasifermion action*, which describes the interaction of the quasiparticles generated from (2.8) with themselves and the boson (gauge and $\mathcal{F}$-type) fields, has the form

$$S_Q[\alpha,\beta;\mathcal{F}] = -\sum_t \Big[\beta_t \mathcal{I}_t^{(2,1)}\alpha_t + \alpha_t^\dagger \mathcal{I}_t^{(1,2)}\beta_t^\dagger \\ + \alpha_t^\dagger(\nabla_t - \mathcal{H}_t)\alpha_{t+1} - \beta_{t+1}(\mathring{\nabla}_t - \mathring{\mathcal{H}}_t)\beta_t^\dagger\Big] \quad (2.16)$$

In the zero-point action we followed [8] adopting the definitions:

$$\mathcal{E}_{t+1,t} = \mathcal{F}_{\mathcal{N},t+1}^\dagger e^{\mathcal{M}_{t+1}} \mathcal{U}_{0,t}^\dagger e^{\mathcal{M}_t^\dagger} \mathcal{F}_{\mathcal{N},t} + \mathcal{F}_{t+1}^\dagger e^{-\mathcal{M}_{t+1}} \mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t^\dagger} \mathcal{F}_t \quad (2.17a)$$

$$\mathring{\mathcal{E}}_{t+1,t} = \mathring{\mathcal{F}}_{\mathcal{N},t} e^{\mathcal{M}_t^\dagger} \mathcal{U}_{0,t} e^{\mathcal{M}_{t+1}} \mathring{\mathcal{F}}_{\mathcal{N},t+1}^\dagger + \mathcal{F}_t e^{-\mathcal{M}_t^\dagger} \mathcal{U}_{0,t} e^{-\mathcal{M}_{t+1}} \mathcal{F}_{t+1}^\dagger \quad (2.17b)$$

with

$$\mathcal{F}_{\mathcal{N},t} = 1 + \mathcal{N}_t^\dagger \mathcal{F}_t \quad (2.18a)$$

$$\mathring{\mathcal{F}}_{\mathcal{N},t} = 1 + \mathcal{F}_t \mathcal{N}_t^\dagger \quad (2.18b)$$

In the same way, in the quasiparticle action the terms mixing fermions and antifermions are

$$\mathcal{I}_t^{(2,1)} = \mathring{\mathcal{R}}_t^{-1/2}\left[\mathring{\mathcal{R}}_t - \mathring{\mathcal{E}}_{t,t-1}^{-1} \mathring{\mathcal{F}}_{\mathcal{N},t-1} e^{\mathcal{M}_{t-1}^\dagger} \mathcal{U}_{0,t-1} e^{\mathcal{M}_t}\right] \left(\mathcal{F}_t^\dagger\right)^{-1} \mathcal{R}_t^{-1/2} \quad (2.19a)$$

$$\mathcal{I}_t^{(1,2)} = \mathcal{R}_t^{-1/2} \mathcal{F}_t^{-1} \left[\mathring{\mathcal{R}}_t - e^{\mathcal{M}_t^\dagger} \mathcal{U}_{0,t} e^{\mathcal{M}_{t+1}} \mathring{\mathcal{F}}_{\mathcal{N},t+1}^\dagger \mathring{\mathcal{E}}_{t+1,t}^{-1}\right] \mathring{\mathcal{R}}_t^{-1/2} \quad (2.19b)$$

while the "particle Hamiltonians" for fermions and antifermions are given by

$$\mathcal{H}_t = \mathcal{U}_{0,t} - \mathcal{R}_t^{-1/2} \mathcal{E}_{t+1,t}^{-1} \mathcal{R}_{t+1}^{-1/2} \quad (2.20a)$$

$$\mathring{\mathcal{H}}_t = \mathcal{U}_{0,t}^\dagger - \mathring{\mathcal{R}}_{t+1}^{-1/2} \mathring{\mathcal{E}}_{t+1,t}^{-1} \mathring{\mathcal{R}}_t^{-1/2} \quad (2.20b)$$

We introduced also the covariant derivatives

$$\nabla_t = \mathcal{U}_{0,t} - T_0^\dagger \quad (2.21a)$$

$$\mathring{\nabla}_t = \mathcal{U}_{0,t}^\dagger - T_0 \quad (2.21b)$$

where T_0, $T_0^\dagger$ are the shift operators in the time direction, defined in (A.1). Using

$$\begin{aligned} \alpha_t^\dagger \nabla_t \alpha_{t+1} &= \alpha_t^\dagger \mathcal{U}_{0,t} \alpha_{t+1} - \alpha_t^\dagger \alpha_t = \alpha_t^\dagger \left(\mathcal{U}_{0,t} T_0 - 1\right) \alpha_t \\ \beta_{t+1} \mathring{\nabla}_t \beta_t^\dagger &= \beta_{t+1} \mathcal{U}_{0,t}^\dagger \beta_t^\dagger - \beta_{t+1} \beta_{t+1}^\dagger = \beta_{t+1} \left(\mathcal{U}_{0,t}^\dagger T_0^\dagger - 1\right) \beta_{t+1}^\dagger \end{aligned} \quad (2.22)$$

we see that these operators are indeed proportional to the covariant derivatives $D_0^{(+)}$, $D_0^{(-)}$ defined in (A.7).

2.2 Extremality

A major task of the method sketched above is to find a persuasive physical interpretation of the parameters of the Bogoliubov transformations at each time slice. So far in the present discussion, the variables $\mathcal{F}_t$, $\mathcal{F}_t^\dagger$ have been kept completely arbitrary and the "effective theory" obtained from (2.14) is unitary equivalent to the original one by construction. Since however the transformation $\hat{\mathscr{U}}(\mathcal{F}_t)$ generates a new vacuum, that is a state annihilated by the quasiparticle destruction operators, the parameters $\mathcal{F}_t$, $\mathcal{F}_t^\dagger$ can be fixed demanding $|\mathcal{F}_t\rangle$ to be of minimal energy. Supposing that the quasiparticles do not contribute to $\langle\mathcal{F}_t|\hat{H}|\mathcal{F}_t\rangle$,[1] this extremality condition can be achieved minimizing the zero-point action S_0 with respect to the parameters $\mathcal{F}_t$, $\mathcal{F}_t^\dagger$. A variation of (2.15) yields the *saddle point equations*

$$\begin{aligned} \mathcal{F}_{t+1} &= \mathcal{N}_{t+1} + e^{-\mathcal{M}_{t+1}}\mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t^\dagger}\mathcal{F}_t\left(\mathcal{F}_{N,t}\right)^{-1}e^{-\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{-\mathcal{M}_{t+1}} \\ \mathcal{F}_t^\dagger &= \mathcal{N}_t^\dagger + e^{-\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{-\mathcal{M}_{t+1}}\left(\mathcal{F}_{N,t+1}^\dagger\right)^{-1}\mathcal{F}_{t+1}^\dagger e^{-\mathcal{M}_{t+1}}\mathcal{U}_{0,t}^\dagger e^{-\mathcal{M}_t^\dagger} \end{aligned} \tag{2.23}$$

(see ref. [8]). Choosing the parameters $\mathcal{F}_t$, $\mathcal{F}_t^\dagger$ to comply with these equations leads the vacuum of the theory to coincide with a state in the ensemble (2.11) that is also an extremum of the action.

In general, these equations are too difficult to be solved, because of their dependence upon time: some sort of approximation must be introduced. A recent proposal (see [7]) is to expand the parameters around some stationary *background field*:

$$\mathcal{F}_t = \bar{\mathcal{F}} + \delta\mathcal{F}_t, \qquad \mathcal{F}_t^\dagger = \bar{\mathcal{F}}^\dagger + \delta\mathcal{F}_t^\dagger \tag{2.24}$$

In this way, the saddle point equations can be solved, in first approximation, for the background field in stationary conditions (that is, with all the coefficients in (2.23) kept fixed in time), while the time-dependent fluctuations are interpreted as dynamical mesons with specific symmetries.[2] The identification works only if the number of fermionic states forming the composite is large, to avoid, as long as possible, the nilpotency of the meson creation operator (2.10) and to use it as a bosonic canonical one. Following these lines, the original fermion theory can be described using these effective bosonic degrees of freedom and, under some physical assumptions such as *composite boson dominance*, whereby the contribution from quasiparticles can be neglected altogether, can be completely *bosonized*. This program has been developed in references [5], [26] and [25].

[1] This is certainly true for QCD_2 in large N_c limit, as we will see in the next chapter.

[2] We call them "$\mathcal{F}$-type mesons", to distinguish them from the fields composed of quasiparticles we will study in chapter 4.

For our present purpose, however, we can overcome these complications simply noting that, for QCD_2 in Coulomb gauge, the matrices $\mathcal{N}$ and $\mathcal{M}$ does not depend on time, and the gluon fields A_0, when integrated out, provides an instantaneous two-dimensional (lattice) Coulomb potential between fermions, as is it clear from the form (1.57) of the propagator. As a matter of fact, using the Gauss' law, the contribution of the gauge fields could be replaced, directly in the continuum Hamiltonian, with a current-current interaction term consisting of four quarks coupled with a linear potential (as should be the Coulomb potential in one spatial dimension: in equation (1.58) we have already noted this property); for reference, see [17]. In section 3.1 we will take a different path and perform the integration over the gluon fields during the calculation of the effective action. Anyway, as the saddle point equations have time independent coefficients, we can search a static solution for $\mathcal{F}$, $\mathcal{F}^\dagger$. However, we point out here a possible source of misconception: as long as the gauge fields have not been integrated out, the stationary hypothesis on the solution of the system (2.23) is not justified, because the coefficients of $\mathcal{F}$, $\mathcal{F}^\dagger$ are function of U_μ and a stationary solution corresponds to a free theory with $U_\mu = 1$. Only when the effective Coulomb potential between quark has been generated via the averaging process, the saddle point equations lead to non-trivial solutions in stationary condition. We will develop this point in the next chapter, where we will explicitly solve the system for our present model.

The extremality conditions we have just stated have also a very interesting side effect: using relations such as

$$\mathcal{F}\mathring{\mathcal{F}}_N = \mathcal{F}_N\mathcal{F} \tag{2.25}$$

we can see easily from (2.19) that (2.23) are equivalent to

$$\mathcal{I}^{(1,2)} = 0 = \mathcal{I}^{(2,1)} \qquad \text{on the saddle point} \tag{2.26}$$

This means that the particular Bogoliubov transformation that minimize the vacuum energy produces quasiparticles that does not mix in the action. In this context, there is full analogy with the celebrated BCS theory of superconductivity, which has the same feature. Also, as stated in [8], this must have something to do with the *Foldy–Wouthuysen transformation*, a unitary transformation that diagonalize the one-particle Dirac Hamiltonian in such a way that the resulting fermion and antifermion propagate separately. To clarify this point, in the next section we will take more directly a lattice Hamiltonian point of view, showing the connection between the two formulations.

2.3 Diagonalization of the lattice Hamiltonian

We would like to derive an expression for the lattice Hamiltonian starting from the transfer matrix, as in (1.31). Using the two-dimensional spinor formalism introduced in section 2.1, the transfer matrix can be written as

$$\begin{aligned}\hat{\mathcal{T}}_{t+1,t} = \exp\left[\hat{\chi}^\dagger N_t^\dagger \hat{\chi}\right] \exp\left[\hat{\chi}^\dagger M_t^\dagger \hat{\chi}\right] \exp\left[\hat{\chi}^\dagger \ln U_{0,t} \hat{\chi}\right] \\ \cdot \exp\left[\hat{\chi}^\dagger M_{t+1} \hat{\chi}\right] \exp\left[\hat{\chi}^\dagger N_{t+1} \hat{\chi}\right]\end{aligned} \tag{2.27}$$

where we used the two-dimensional block matrices

$$M_t \equiv \begin{pmatrix} -\mathcal{M}_t & 0 \\ 0 & \mathcal{M}_t \end{pmatrix} \qquad N_t \equiv \begin{pmatrix} 0 & 0 \\ \mathcal{N}_t & 0 \end{pmatrix} \tag{2.28}$$

$$\ln U_{0,t} \equiv \begin{pmatrix} \ln \mathcal{U}_{0,t} & 0 \\ 0 & -\ln \mathcal{U}_{0,t}^\dagger \end{pmatrix} = \begin{pmatrix} \ln \mathcal{U}_{0,t} & 0 \\ 0 & \ln \mathcal{U}_{0,t} \end{pmatrix} \tag{2.29}$$

In order to take the logarithm of the above expression, we would like to combine the various exponentials in a single one. As the exponents do not commute, this is a not trivial job: in general,

$$\hat{\mathcal{T}}_{t+1,t} = \exp\left[\hat{\chi}^\dagger f_{t,t+1}(M,N,U) \hat{\chi}\right] \tag{2.30}$$

where $f_{t,t+1}(M,N,U)$ is a complicate sum of its arguments and their commutators (see equation (B.10)). If we could find the form of f, then

$$\hat{H} = -\frac{1}{a_0} \hat{\chi}^\dagger f_{t,t+1}(M,N,U) \hat{\chi} \tag{2.31}$$

At this point we observe that, using the results of Appendix B, we can reduce this problem to a pure matrix multiplication task: in (2.27) the transfer matrix is written in the form (B.11), but f can be found in a generic representation of the algebra of the General Linear group, also the fundamental one. In this way,

$$e^{f_{t,t+1}(M,N,U)} = e^{N_t^\dagger} e^{M_t^\dagger} U_{0,t} e^{M_{t+1}} e^{N_{t+1}} \tag{2.32}$$

The matrices N, $N^\dagger$ are nilpotent:

$$N_t^2 = 0 = (N_t^\dagger)^2 \qquad \Longrightarrow \qquad \begin{cases} e^{N_t} = 1 + N_t \\ e^{N_t^\dagger} = 1 + N_t^\dagger \end{cases} \tag{2.33}$$

so we find

$$e^{f_{t,t+1}(M,N,U)} = (1+N_t^\dagger)e^{M_t^\dagger}U_{0,t}e^{M_{t+1}}(1+N_{t+1})$$
$$= \begin{pmatrix} e^{-\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{-\mathcal{M}_{t+1}} + \mathcal{N}_t^\dagger e^{\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{\mathcal{M}_{t+1}}\mathcal{N}_{t+1} & \mathcal{N}_t^\dagger e^{\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{\mathcal{M}_{t+1}} \\ e^{\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{\mathcal{M}_{t+1}}\mathcal{N}_{t+1} & e^{\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{\mathcal{M}_{t+1}} \end{pmatrix} \quad (2.34)$$

We would like to impose stationary conditions to proceed further, as explained in the previous section. However, we could use this hypothesis only after the integration over the gauge fields: as the link variables U_μ appears in our expressions, both explicitly and in the definitions of $\mathcal{M}$ and $\mathcal{N}$, if we look naively for a static solution before the averaging process, we would end in the free theory, where the gauge fields are null (or constant, which is the same). A possible way out is to formulate the stationary condition in a gauge covariant form (see [7]), requiring that the spatial links fields evolve in such a way that at time t they are related to those at a certain initial time $t_0 = 0$ by a pure gauge transformation (1.9):

$$U_j(t,\mathbf{x}) = \Omega_{t,\mathbf{x}}^\dagger U_j(0,\mathbf{x})\Omega_{t,\mathbf{x}+a\hat{\mathbf{j}}} \quad (2.35)$$

Thus, at any time they are gauge equivalent to a constant field, and the non-trivial dynamics remains encoded in the temporal link $U_0(t,\mathbf{x})$. As the spatial link variables appear in the matrices $\mathcal{M}_t$, $\mathcal{N}_t$ and must also be part of the solution $\mathcal{F}_t$, they evolve according to

$$\mathcal{M}_t = \Omega_t^\dagger\mathcal{M}\Omega_t, \qquad \mathcal{N}_t = \Omega_t^\dagger\mathcal{N}\Omega_t, \qquad \mathcal{F}_t = \Omega_t^\dagger\mathcal{F}\Omega_t \quad (2.36)$$

where $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{F}$ are the corresponding values at time 0, whose gauge dependence comes only from the time 0 space link variables $U_j(0,\mathbf{x})$. Requiring the transformation Ω_t to be in a very particular form, we can finally drop the temporal dependence in our expressions. Indeed, if

$$\Omega_{t,\mathbf{x}} = U_0(0,\mathbf{x})U_0(1,\mathbf{x})\cdots U_0(t-1,\mathbf{x}) \quad (2.37)$$

then a typical matrix product in (2.34), for example $e^{-\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{-\mathcal{M}_{t+1}}$, can be written as

$$e^{-\mathcal{M}_t^\dagger}\mathcal{U}_{0,t}e^{-\mathcal{M}_{t+1}} = \Omega_t^\dagger e^{-\mathcal{M}^\dagger}\Omega_t\mathcal{U}_{0,t}\Omega_{t+1}^\dagger e^{-\mathcal{M}^\dagger}\Omega_{t+1} = \Omega_t^\dagger e^{-\mathcal{M}^\dagger}e^{-\mathcal{M}_t^\dagger}\Omega_{t+1} \quad (2.38)$$

because $\mathcal{U}_{0,t}$ provides the extra $U_0(t,\mathbf{x})$ factor that makes possible the simplification between Ω_t and $\Omega_{t+1}^\dagger$. In this way

$$e^{f_{t,t+1}(M,N,U)} = \Omega_t^\dagger e^{f(M,N,U)}\Omega_{t+1} \quad (2.39)$$

with

$$e^{f(M,N,U)} = \begin{pmatrix} e^{-\mathcal{M}^\dagger} e^{-\mathcal{M}} + \mathcal{N}^\dagger e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \mathcal{N} & \mathcal{N}^\dagger e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \\ e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \mathcal{N} & e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \end{pmatrix} \tag{2.40}$$

and all the Ω factors drop out in the product over time in (1.32), *except* for those at the temporal boundaries, which provide the information about the gauge fields dynamics. That's exactly what would happen if we chose from the start to work in the temporal gauge

$$U_0(x) = 1 \quad \Longleftrightarrow \quad A_0(x) = 0 \tag{2.41}$$

In this case we could simply forget the $\mathcal{U}_{0,t}$ in our equations but, as (2.41) does not completely fix the gauge, we should still introduce a projector in the product (1.32) in order to work only in the physical sector of the Fock space, where the states respect the Gauss' law: the projector enforcing this additional constraint is a Polyakov line that has precisely the form (2.37) at the boundaries.

We are now interested in block-diagonalizing (2.40), to decouple fermions from antifermions in the Hamiltonian, which is just $-f(M,N,U)/a$. Using a unitary transformation of the algebra, parametrized as in (2.5), we find

$$e^{\mathscr{U} f(M,N,U) \mathscr{U}^\dagger} = \mathscr{U} e^{f(M,N,U)} \mathscr{U}^\dagger = \begin{pmatrix} \mathcal{A} & \mathcal{C}^{(1,2)} \\ \mathcal{C}^{(2,1)} & \mathcal{D} \end{pmatrix} \tag{2.42}$$

with

$$\begin{aligned} \mathcal{A} &= \mathcal{R}^{1/2} \Big[e^{-\mathcal{M}^\dagger} e^{-\mathcal{M}} + \left(\mathcal{F}^\dagger - \mathcal{N}^\dagger\right) e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \left(\mathcal{F} - \mathcal{N}\right) \Big] \mathcal{R}^{1/2} \\ \mathcal{C}^{(1,2)} &= \mathcal{R}^{1/2} \Big[e^{-\mathcal{M}^\dagger} e^{-\mathcal{M}} \mathcal{F}^\dagger + \left(\mathcal{N}^\dagger - \mathcal{F}^\dagger\right) e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \mathring{\mathcal{F}}_N^\dagger \Big] \mathring{\mathcal{R}}^{1/2} \\ \mathcal{C}^{(2,1)} &= \mathring{\mathcal{R}}^{1/2} \Big[\mathcal{F} e^{-\mathcal{M}^\dagger} e^{-\mathcal{M}} + \mathring{\mathcal{F}}_N e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \left(\mathcal{N} - \mathcal{F}\right) \Big] \mathcal{R}^{1/2} \\ \mathcal{D} &= \mathring{\mathcal{R}}^{1/2} \Big[\mathcal{F} e^{-\mathcal{M}^\dagger} e^{-\mathcal{M}} \mathcal{F}^\dagger + \mathring{\mathcal{F}}_N e^{\mathcal{M}^\dagger} e^{\mathcal{M}} \mathring{\mathcal{F}}_N^\dagger \Big] \mathring{\mathcal{R}}^{1/2} \end{aligned} \tag{2.43}$$

As we can see, the conditions for the transformed matrix to be in block-diagonal form are exactly the saddle point equations (2.23) when the stationary hypothesis is in force! Indeed, as $\mathcal{C}^{(1,2)} = \left[\mathcal{C}^{(2,1)}\right]^\dagger$, the system becomes the single equation

$$\mathcal{C}^{(2,1)} = 0 \tag{2.44}$$

To proceed further and explicitly block-diagonalize the transfer matrix we should find its solution. This in general a too difficult task, because (2.44) is a quadratic matrix equation in $\mathcal{F}$ with non-commuting (matrix) coefficients.

However, the solution can be obtained in closed form, with a simple generalization of the quadratic formula for an ordinary algebraic equation, if we impose that the matrices $\mathcal{N}$ and $\mathcal{M}$ do commute, and thus searching for a commuting $\mathcal{F}$. This is certainly not true for ordinary QCD in four space-time dimension, because these matrices contains the dependence from the time 0 link variables $U_j(0, \mathbf{x})$, which do not commute with each other. However, as in QCD_2 there is only one spatial dimension, this condition is luckily satisfied in our model. Furthermore, for our present scope we can also take $\mathcal{N}$, $\mathcal{M}$ to be Hermitian, as it is clear from (1.45) and (1.46). In this case

$$\mathcal{F}e^{-2\mathcal{M}} + e^{2\mathcal{M}}\mathcal{N} + \mathcal{F}\mathcal{N}e^{2\mathcal{M}}\mathcal{N} - e^{2\mathcal{M}}\mathcal{F} - \mathcal{F}\mathcal{N}e^{2\mathcal{M}}\mathcal{F} = 0 \tag{2.45}$$

Multiplying by $-\mathcal{N}e^{-2\mathcal{M}}$ we find

$$(\mathcal{N}\mathcal{F})^2 + \left(1 - \mathcal{N}^2 - e^{-4\mathcal{M}}\right)\mathcal{N}\mathcal{F} - \mathcal{N}^2 = 0 \tag{2.46}$$

with solutions

$$\mathcal{N}\bar{\mathcal{F}} = \frac{1}{2}\left[-\mathcal{Y} \pm \sqrt{\mathcal{Y}^2 + 4\mathcal{N}^2}\right] \tag{2.47}$$

where we defined

$$\mathcal{Y} = 1 - \mathcal{N}^2 - e^{-4\mathcal{M}} \tag{2.48}$$

Using the convenient parametrization

$$\mathcal{N}^2 \equiv e^{-2\mathcal{M}}\left[(2\sinh\varepsilon)^2 - (2\sinh\mathcal{M})^2\right] = 2e^{-2\mathcal{M}}(\cosh 2\varepsilon - \cosh 2\mathcal{M}) \tag{2.49}$$

we get

$$\mathcal{N}\bar{\mathcal{F}} = e^{-2\mathcal{M}}e^{2\varepsilon} - 1 \tag{2.50}$$

and so

$$\bar{\mathcal{F}} = \mathcal{N}\frac{e^{2\varepsilon} - e^{2\mathcal{M}}}{e^{2\varepsilon} - e^{2\mathcal{M}} + e^{-2\varepsilon} - e^{-2\mathcal{M}}} \tag{2.51}$$

We also find

$$\begin{aligned} \mathcal{A} &= e^{-2\varepsilon} \\ \mathcal{D} &= e^{2\varepsilon} \end{aligned} \tag{2.52}$$

and so we get, finally,

$$f(M, N, U) = \mathscr{U}^\dagger \begin{pmatrix} -2\varepsilon & 0 \\ 0 & 2\varepsilon \end{pmatrix} \mathscr{U} \tag{2.53}$$

Chapter 3

Vacuum and quasiparticles

We will now apply the formalism explained so far to the model described in chapter 1, namely QCD_2 in Wilson's lattice formulation. Using the variational principle we enunciated in section 2.2, we will find an expression for the vacuum energy. Then, we will study the features of the quasiparticle contributions.

3.1 Vacuum energy

In this section, we are interested in solving the saddle point equations (2.23), to derive an expression for the vacuum energy to be confronted with that known in literature, and thus to find confirmations for our approach. We will evaluate explicitly the zero-point action (2.15) for our model which, in stationary conditions, can be written, for our future convenience, as

$$S_0[\mathcal{F}] = -\sum_t \operatorname{tr}\left[\ln(\mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2})\right] \tag{3.1}$$

where we used the cyclicity of the trace. We will work in Coulomb gauge (1.47):

$$U_1(x) = 1 \qquad \Longleftrightarrow \qquad A_1(x) = 0 \tag{3.2}$$

because then the matrices (1.45) and (1.46) have the easy form

$$\mathcal{B} = \mathbb{I} - \kappa_1 r_1 (T_1 + T_1^\dagger) \tag{3.3}$$

$$\mathcal{N} = -i\kappa_1 \mathcal{B}^{-1/2}\left(T_1 - T_1^\dagger\right)\mathcal{B}^{-1/2} \tag{3.4}$$

Using the Fourier representation (A.10) we get[1]

$$\mathcal{B}(p,q) = 2\pi a_1 \delta(p+q)\mathcal{B}(q) \tag{3.5}$$
$$e^{2\mathcal{M}}(p,q) = 2\pi a_1 \delta(p+q) e^{2\mathcal{M}}(q) \tag{3.6}$$

with

$$\mathcal{B}(q) \equiv 1 - 2\kappa_1 r_1 \cos a_1 q \tag{3.7}$$
$$e^{2\mathcal{M}}(q) = \frac{\mathcal{B}(q)}{2\kappa_0} = 1 + m a_0 - r_1 \zeta(\cos a_1 q - 1) \tag{3.8}$$

Using repeatedly (A.15) it's easy to see[2]

$$\mathcal{B}^{-1}(p,q) = 2\pi a_1 \delta(p+q) \left[\mathcal{B}(q)\right]^{-1} \tag{3.9}$$
$$\mathcal{B}^{-1/2}(p,q) = 2\pi a_1 \delta(p+q) \left[\mathcal{B}(q)\right]^{-1/2} \tag{3.10}$$

and so

$$\mathcal{N}(p,q) = 2\pi a_1 \delta(p+q)\mathcal{N}(q) \tag{3.11}$$
$$\mathcal{N}(q) \equiv 2\kappa_1 \mathcal{B}^{-1}(q) \sin a_1 q \tag{3.12}$$

In this way, we can see that in the present model $\mathcal{M}$ and $\mathcal{N}$ are *commuting operators*. In analogy, we define the Fourier representation of $\mathcal{F}$ as

$$\mathcal{F}(p,q) = 2\pi a_1 \delta(p+q)\mathcal{F}(q) \tag{3.13}$$
$$\mathcal{F}(q) = \tan\frac{\theta_q}{2} = \mathcal{F}^\dagger(q) \tag{3.14}$$

where θ_q is the *Bogoliubov-Valatin angle* usually introduced in literature to parametrize the unitary transformation. In this way

$$\mathcal{R}(p,q) = 2\pi a_1 \delta(p+q)\mathcal{R}(q) \tag{3.15}$$
$$\mathcal{R}(q) = \left(1 + \tan^2\frac{\theta_q}{2}\right)^{-1} = \cos^2\frac{\theta_q}{2} \tag{3.16}$$

[1] The a_1 we factorized in front of these and the next equations is a consequence of our choice of normalization (A.10) for the Fourier transform, and will be ultimately dropped in evaluating the trace, as in (A.14). We could easily absorb it by defining summations and traces in a way readily available for the continuum limit:

$$\sum_{x,y} A_{xy} B_{yx} = a \sum_x a \sum_y \frac{A_{xy}}{a}\frac{B_{yx}}{a} = a\sum_x a \sum_y A'_{xy} B'_{yx} \to \int \mathrm{d}x \int \mathrm{d}y\, A'(x,y) B'(y,x)$$

[2] In general, this property works for any function of $\mathcal{B}$ defined as a power series, because of the convolution theorem of the Fourier transform and of the diagonal form of $\mathcal{B}$.

and

$$\begin{aligned}\cos\theta_q &= \frac{1-\mathcal{F}^2(q)}{1+\mathcal{F}^2(q)} = 2\mathcal{R}(q) - 1 \\ \sin\theta_q &= \frac{2\mathcal{F}(q)}{1+\mathcal{F}^2(q)}\end{aligned} \tag{3.17}$$

As the usual results are obtained directly in the continuum theory, we will work at the lowest orders in lattice spacings and ultimately take the limit $a_0, a_1 \to 0$. We get[3]

$$e^{\mathcal{M}}(q) = 1 + \frac{ma_0}{2} + O(a_0 a_1) \tag{3.18a}$$

$$\mathcal{N}(q) = \zeta a_1 q + O(a_0 a_1) = a_0 q + O(a_0 a_1) \tag{3.18b}$$

We will also need to expand the link variables (1.10) to second order in g, using the results sketched in section 1.3. In this way

$$\begin{aligned}U_0 &= 1 + iga_0 A_0 - \frac{1}{2} g^2 a_0^2 A_0^2 + \cdots \\ U_0^\dagger &= 1 - iga_0 A_0 - \frac{1}{2} g^2 a_0^2 A_0^2 + \cdots\end{aligned} \tag{3.19}$$

We remark that, unlike the previous one, this is not actually an expansion in the lattice spacing a_0: in particular, once we have mediated over gauge fields, the quantity $a_0^2 \langle A_0^2 \rangle$ will be not $O(a_0^2)$, but only $O(a_0)$. Indeed, this object is proportional to the gluon propagator we reported in (1.57): one of the a_0 factor simplifies with the one in the denominator and leaves an overall term $O(a_0)$. This will happen to all (even) orders in g, because the powers of a_0 will always be equal to those of A_0. What is really in force here is a weak coupling expansion that, as we will see, is nothing else than the *'t Hooft's limit*

$$N_c \to \infty \qquad \text{with } g^2 N_c \text{ fixed} \tag{3.20}$$

Expanding (2.17) to second order in g we get

$$\mathcal{E} = \mathcal{E}^{(0)} + g\mathcal{E}^{(1)} + g^2 \mathcal{E}^{(2)} \tag{3.21}$$

[3] There is a subtlety in our limit procedure: as a matter of fact, we are first expanding for $a_1 q \simeq 0$ and then taking the limit $a_1 \to 0$. In this way, we can see only the contribution from the physical fermion pole at $q = 0$ and lost the informations about the lattice doublers. To understand what happens of these spurious terms in the continuum limit, we report the solution of the saddle point equations around them in appendix C (where we study a free theory in d space-time dimensions).

with

$$\begin{aligned}
\mathcal{E}^{(0)} &= (1+\mathcal{F}\mathcal{N})\,e^{2\mathcal{M}}\,(1+\mathcal{F}\mathcal{N}) + \mathcal{F}e^{-2\mathcal{M}}\mathcal{F} \\
\mathcal{E}^{(1)} &= -i\,(1+\mathcal{F}\mathcal{N})\,e^{\mathcal{M}}a_0A_0e^{\mathcal{M}}\,(1+\mathcal{N}\mathcal{F}) - i\mathcal{F}e^{-\mathcal{M}}a_0A_0e^{-\mathcal{M}}\mathcal{F} \\
\mathcal{E}^{(2)} &= -\frac{1}{2}\,(1+\mathcal{F}\mathcal{N})\,e^{\mathcal{M}}a_0^2A_0^2e^{\mathcal{M}}\,(1+\mathcal{N}\mathcal{F}) - \frac{1}{2}\mathcal{F}e^{-\mathcal{M}}a_0^2A_0^2e^{-\mathcal{M}}\mathcal{F}
\end{aligned} \tag{3.22}$$

so that, in momentum space,

$$\begin{aligned}
\mathcal{E}^{(0)}(p,q) &= 2\pi a_1\delta(p+q)\left[(1+\mathcal{F}\mathcal{N})\,e^{2\mathcal{M}}\,(1+\mathcal{F}\mathcal{N}) + \mathcal{F}e^{-2\mathcal{M}}\mathcal{F}\right](q) \\
\mathcal{E}^{(1)}(p,q) &= -ia_1a_0\Big\{\left[(1+\mathcal{F}\mathcal{N})\,e^{\mathcal{M}}\right](-p)\,A_0(p+q)\,\left[e^{\mathcal{M}}\,(1+\mathcal{N}\mathcal{F})\right](q) \\
&\qquad + \left[\mathcal{F}e^{-\mathcal{M}}\right](-p)\,A_0(p+q)\,\left[e^{-\mathcal{M}}\mathcal{F}\right](q)\Big\} \\
\mathcal{E}^{(2)}(p,q) &= -\frac{a_1a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}\Big\{ \\
&\qquad \left[(1+\mathcal{F}\mathcal{N})\,e^{\mathcal{M}}\right](-p)\,A_0(p+k)A_0(-k+q)\,\left[e^{\mathcal{M}}\,(1+\mathcal{N}\mathcal{F})\right](q) \\
&\qquad + \left[\mathcal{F}e^{-\mathcal{M}}\right](-p)\,A_0(p+k)A_0(-k+q)\,\left[e^{-\mathcal{M}}\mathcal{F}\right](q)\Big\}
\end{aligned} \tag{3.23}$$

Using (3.18) and keeping only first-order terms in a_0, we get

$$\begin{aligned}
\mathcal{E}^{(0)}(p,q) &= 2\pi a_1\delta(p+q)\left\{1+\mathcal{F}^2(q) + ma_0\left[1-\mathcal{F}^2(q)\right] + 2a_0q\mathcal{F}(q)\right\} \\
\mathcal{E}^{(1)}(p,q) &= -ia_1a_0\left[A_0(p+q) + \mathcal{F}(-p)A_0(p+q)\mathcal{F}(q)\right] \\
\mathcal{E}^{(2)}(p,q) &= -\frac{a_1a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}\Big[A_0(p+k)A_0(-k+q) \\
&\qquad + \mathcal{F}(-p)A_0(p+k)A_0(-k+q)\mathcal{F}(q)\Big]
\end{aligned} \tag{3.24}$$

Then, we multiply on left and right by $\mathcal{R}^{1/2}$ and use, when convenient, (3.17):

$$\begin{aligned}
\mathcal{R}^{1/2}\mathcal{E}^{(0)}\mathcal{R}^{1/2}(p,q) &= 2\pi a_1\delta(p+q)\,(1+ma_0\cos\theta_q + a_0q\sin\theta_q) \\
\mathcal{R}^{1/2}\mathcal{E}^{(1)}\mathcal{R}^{1/2}(p,q) &= -ia_1a_0\Big[\mathcal{R}^{1/2}(-p)A_0(p+q)\mathcal{R}^{1/2}(q) \\
&\qquad + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)A_0(p+q)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\Big] \\
\mathcal{R}^{1/2}\mathcal{E}^{(2)}\mathcal{R}^{1/2}(p,q) &= -\frac{a_1a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}\Big[\mathcal{R}^{1/2}(-p)A_0(p+k)A_0(-k+q)\mathcal{R}^{1/2}(q) \\
&\qquad + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)A_0(p+k)A_0(-k+q)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\Big]
\end{aligned} \tag{3.25}$$

In evaluating the logarithm of the sum of these quantities to first order in a_0 we use the usual formula

$$\ln\left(1+(\mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}-1)\right) \simeq \mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2} - 1 - \frac{1}{2}(\mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}-1)^2 \tag{3.26}$$

and note that the only term that can produce an $O(a_0)$ addend when squared is $\mathcal{R}^{1/2}\mathcal{E}^{(1)}\mathcal{R}^{1/2}$, because of the property of $\langle A_0^2 \rangle$ we already mentioned. As a matter of fact,

$$\begin{aligned} -\frac{1}{2}\left[\mathcal{R}^{1/2}\mathcal{E}^{(1)}\mathcal{R}^{1/2}\right]^2(p,q) = \frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi} \\ \cdot\left[\mathcal{R}^{1/2}(-p)A_0(p+k)\mathcal{R}(k)A_0(-k+q)\mathcal{R}^{1/2}(q)\right. \\ +\mathcal{R}^{1/2}(-p)\mathcal{F}(-p)A_0(p+k)\mathcal{F}(k)\mathcal{R}(k)\mathcal{F}(k)A_0(-k+q)\mathcal{F}(q)\mathcal{R}^{1/2}(q) \\ \left. +2\mathcal{R}^{1/2}(-p)A_0(p+k)\mathcal{R}(k)\mathcal{F}(k)A_0(-k+q)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\right] \end{aligned} \tag{3.27}$$

Summing the $O(g^2)$ terms we get

$$\begin{aligned} &\mathcal{R}^{1/2}\mathcal{E}^{(2)}\mathcal{R}^{1/2}(p,q) - \frac{1}{2}\left[\mathcal{R}^{1/2}\mathcal{E}^{(1)}\mathcal{R}^{1/2}\right]^2(p,q) \\ &= \frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q) \\ &\cdot\left[-\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q) - \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\right. \\ &\quad +\mathcal{R}^{1/2}(-p)\mathcal{R}(k)\mathcal{R}^{1/2}(q) + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(k)\mathcal{R}(k)\mathcal{F}(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q) \\ &\quad \left. +2\mathcal{R}^{1/2}(-p)\mathcal{R}(k)\mathcal{F}(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\right] \end{aligned} \tag{3.28}$$

As we will ultimately take the trace of this expression, we can confound $-p$ with q in the square bracket, to obtain

$$\begin{aligned} &\mathcal{R}^{1/2}\mathcal{E}^{(2)}\mathcal{R}^{1/2}(p,q) - \frac{1}{2}\left[\mathcal{R}^{1/2}\mathcal{E}^{(1)}\mathcal{R}^{1/2}\right]^2(p,q) \\ &= \frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q)\left[-\mathcal{R}(q) - \mathcal{R}(q)\mathcal{F}^2(q) + \mathcal{R}(q)\mathcal{R}(k)\right. \\ &\quad \left. +\mathcal{R}(q)\mathcal{F}^2(q)\mathcal{R}(k)\mathcal{F}^2(k) + 2\mathcal{R}(q)\mathcal{F}(q)\mathcal{R}(k)\mathcal{F}(k)\right] \end{aligned} \tag{3.29}$$

The dependence from θ is

$$\begin{aligned} -1+\mathcal{R}(q)\mathcal{R}(k)\left[1+\mathcal{F}(q)\mathcal{F}(k)\right]^2 &= -1+\left(\cos\frac{\theta_q}{2}\cos\frac{\theta_k}{2}+\sin\frac{\theta_q}{2}\sin\frac{\theta_k}{2}\right)^2 \\ &= -1+\cos^2\frac{\theta_q-\theta_k}{2} = -\sin^2\frac{\theta_q-\theta_k}{2} \end{aligned} \tag{3.30}$$

We can simplify further the expression for the linear term in g making the assumption, which we will verify in a short time, that

$$\mathcal{F}(-p) = -\mathcal{F}(p) \qquad \mathcal{R}(-p) = \mathcal{R}(p) \tag{3.31}$$

With a little bit of trigonometry, our logarithm becomes

$$\begin{aligned}&\left[\ln \mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}\right](p,q) \simeq \\ &2\pi a_0 a_1 \delta(p+q)\left(m\cos\theta_q + q\sin\theta_q\right) - i a_0 a_1 g A_0(p+q)\cos\frac{\theta_p+\theta_q}{2} \\ &\qquad - \frac{a_1 a_0^2 g^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q)\sin^2\frac{\theta_q-\theta_k}{2}\end{aligned} \tag{3.32}$$

It's now time to integrate over the gauge fields, that is perform the gauge integral in (2.14). This means we can substitute the gauge fields with their expectation values. Using (1.55), ultimately we get

$$\begin{aligned}&\left[\ln \mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}\right](p,q) \simeq \\ &\quad 2\pi a_0 a_1 \delta(p+q)\left(m\cos\theta_q + q\sin\theta_q - \frac{g^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}\frac{\Theta\cdot\Theta}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2}\right)\end{aligned} \tag{3.33}$$

We used directly the continuum gluon propagator instead of its lattice version to lighten the notation: that's what we would get in the continuum limit, which is implicit in our choice to maintain only the lowest terms in lattice spacing; we note that at the end the first Brillouin Zone $[-\pi/a_1, \pi/a_1]$ will extend from $-\infty$ to ∞. We also imposed that $\langle A_0\rangle = 0$: that's true for all the odd product of the gauge fields, because the generating functional (1.53) is quadratic in the sources and an odd number of functional derivatives leaves a factor of J that, when evaluated in $J = 0$, cancels the result. Evaluating the trace over space (see (A.14)) and colour indices we find

$$\begin{aligned}\mathrm{tr}\left[\ln \mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}\right] = a_0 2\pi\delta(0)\Bigg[&\frac{N_c}{2\pi}\int_{BZ}\mathrm{d}q\left(m\cos\theta_q + q\sin\theta_q\right) \\ &- \alpha_s\frac{(N_c^2-1)}{2(2\pi)}\int_{BZ}\mathrm{d}q\int_{BZ}\mathrm{d}k\,\frac{1}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2}\Bigg]\end{aligned} \tag{3.34}$$

with

$$\alpha_s = \frac{g^2}{4\pi} \tag{3.35}$$

The factor $2\pi\delta(0)$ is simply the length of the lattice in the spatial direction, as we can see from (A.12):

$$2\pi\delta(0) = a_1\sum_{x^1} 1 \tag{3.36}$$

Summing over t we can finally derive the expression for the vacuum contribution to the action:

$$S_0[\theta] = -\frac{VN_c}{2\pi}\left[\int_{BZ} \mathrm{d}q\,(m\cos\theta_q + q\sin\theta_q) - \frac{\alpha_s\,(N_c^2-1)}{2N_c}\int_{BZ}\mathrm{d}q\int_{BZ}\mathrm{d}k\,\frac{1}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2}\right] \tag{3.37}$$

The space-time volume

$$V = a_0 a_1 \sum_{x^0}\sum_{x^1} 1 \tag{3.38}$$

accounts for the fact that energy is an extensive property; it can be regularized imposing boundaries on the lattice and then perform a thermodynamic limit procedure. We remark here that, in the weak coupling regime $g \to 0$, this result is not trivial (that is, different from what we would obtain in the free theory) only if $N_c \to \infty$, accordingly to the 't Hooft's limit (3.20). In this way, the dispersion law for the vacuum energy density is

$$\omega_q^0 = m\cos\theta_q + q\sin\theta_q - \frac{\gamma}{2}\int_{BZ}\mathrm{d}k\,\frac{1}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2} \tag{3.39}$$

with $\gamma \equiv \alpha_s\,(N_c^2-1)\,/N_c \to \alpha_s N_c$. Imposing parity on this expression for any value of γ, we get

$$\omega_q^0 = \omega_{-q}^0 \quad\Longleftrightarrow\quad \theta_q = -\theta_{-q} \tag{3.40}$$

and so (3.31) is demonstrated. We also note that (3.37) is a $O(N_c)$ contribution in the 't Hooft's limit.

A variation with respect to the Bogoliubov angle θ_q leads to the saddle point equation

$$-\,m\sin\theta_q + q\cos\theta_q - \frac{\gamma}{2}\int_{BZ}\mathrm{d}k\,\frac{\sin(\theta_q-\theta_k)}{(q-k)^2} = 0 \tag{3.41}$$

The solution can be obtained in closed form only in the free ($\gamma = 0$) theory, where it is simply

$$\theta_q = \arctan\frac{q}{m} \tag{3.42}$$

and has a trend plotted in Figure 3.1 for different values of the quark mass m. However, using the numerical methods described in Appendix D, we can find its form also in the interacting theory, as we can see in Figure 3.2. Once that the values of θ_q that solve the saddle point equation are available, the form for the energy density ω_q, which we interpret as the vacuum contribute

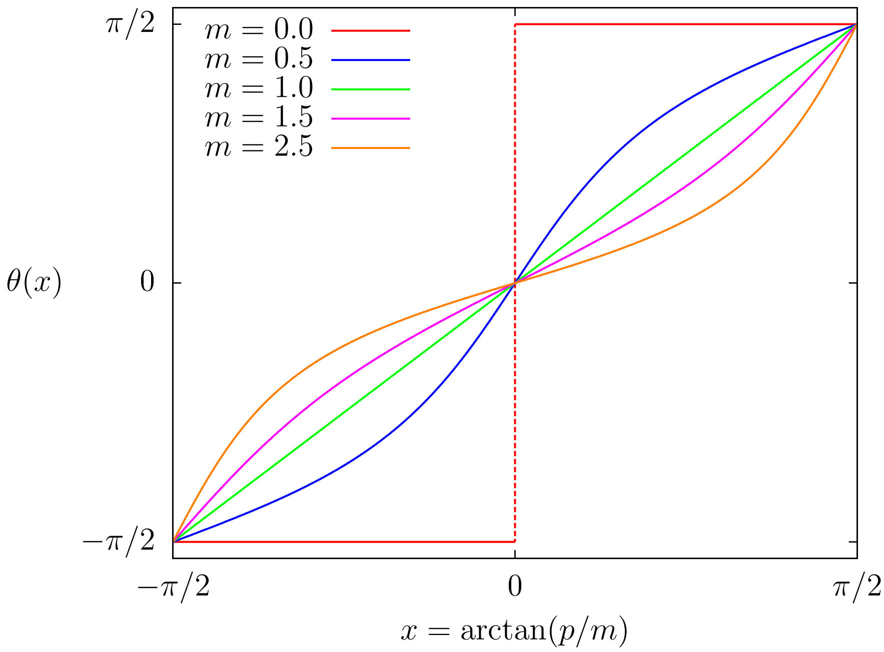

Figure 3.1: Plot for the θ_p that solves the saddle point equation in the free theory ($\gamma = 0$), with different values of the quark mass.

to the energy, follows easily (Figure 3.3). These results are well known in literature: see, for example, reference [17] and graphs within. The fact that we can derive it in our present scheme is a proof for the validity of the method proposed.

3.2 Quasiparticle dynamics

We will now switch on the quasiparticle contribution to the action, in order to find informations about the excited states of the theory. From the perturbative point of view we adopted when we set up the weak coupling expansion (3.19), the fermion action can be written as a series in g that, to second order, reads as

$$S_Q \simeq S_Q^{(0)} + gS_Q^{(1)} + g^2 S_Q^{(2)} \tag{3.43}$$

This series can be calculated, as we will do in the next paragraphs, using the same arguments we formulated to obtain the zero-point action. However, the integration over gauge fields in (2.14) acts on the *exponential* of the action: we should first expand it to the desired order in g and then averaging over

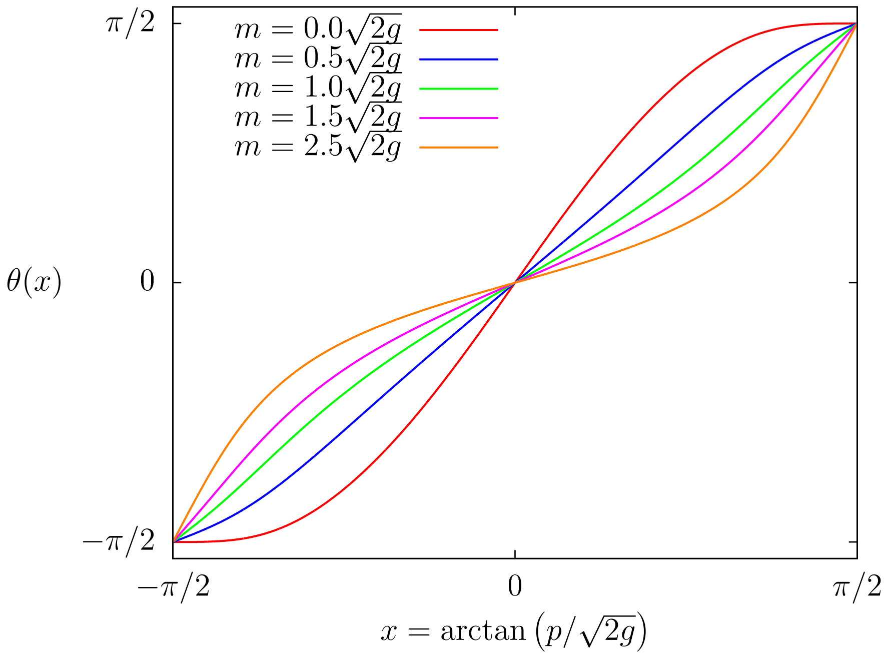

Figure 3.2: Plot for the θ_p that solves the saddle point equation in the interacting theory ($\gamma = 1$), , with different values of the quark mass.

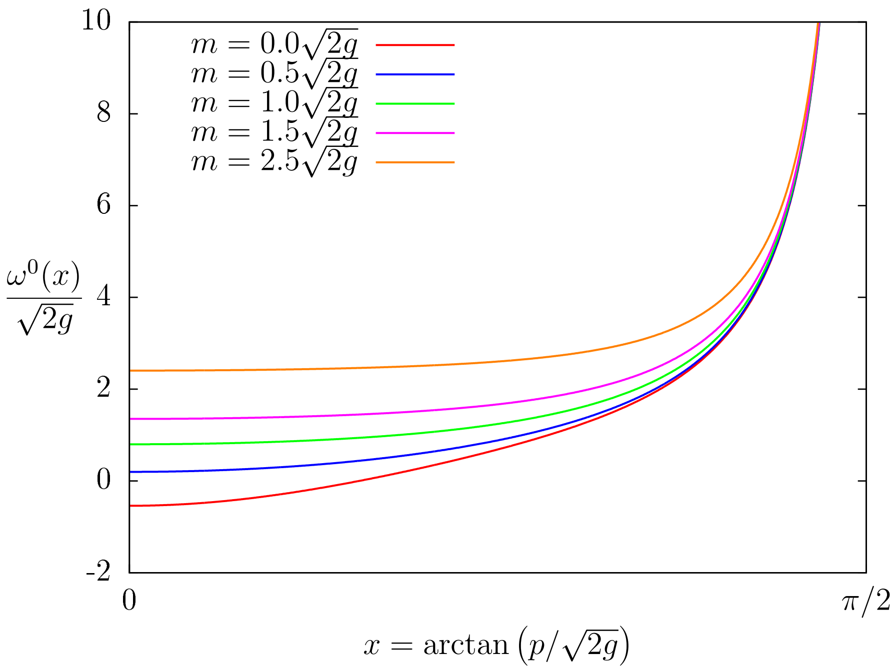

Figure 3.3: Plot for ω_p, evaluated on solutions θ_p of the saddle point equation.

gluons, eventually re-exponentiating the result. We get

$$\exp\left(-S_Q\right) \simeq \exp\left(-S_Q^{(0)} - g^2 S_Q^{(2)}\right)\left[1 - g S_Q^{(1)} + \frac{1}{2} g^2 \left(S_Q^{(1)}\right)^2\right] \tag{3.44}$$

The term proportional to $\left(S_Q^{(1)}\right)^2$, which we would miss if we had not expand the exponential, is *quartic* in the fields α, β: it can be interpreted as an interaction term between effective mesons composed of quasifermions bound by gluons. We will discuss it in section 4.1, while now we will focus on the bilinear terms in α, β, which can be obtained mediating directly the action (3.43).

Mixing terms

To begin the discussion about the quasiparticle bilinear part, we will now calculate explicitly the terms in (2.16) that mix the α and β fields. By Hermiticity and commutativity we have

$$\mathcal{I}_t^{(1,2)} = \mathcal{I}_t = \mathcal{I}_{t+1}^{(2,1)} \tag{3.45}$$

We can write

$$\mathcal{I} = \left[\mathcal{C}\left(\mathcal{R}^{-1/2} \mathring{\mathcal{E}}^{-1} \mathcal{R}^{-1/2}\right)\right] \tag{3.46}$$

with

$$\mathcal{C} = \mathcal{R}^{1/2}\Big[e^{-\mathcal{M}} \mathcal{U}_0 e^{-\mathcal{M}} \mathcal{F} + \left(\mathcal{N} - \mathcal{F}\right) e^{\mathcal{M}} \mathcal{U}_0 e^{\mathcal{M}} \left(1 + \mathcal{N}\mathcal{F}\right)\Big]\mathcal{R}^{1/2} \tag{3.47}$$

Expanding to second order in g as in the previous section we get

$$\mathcal{C} = \mathcal{C}^{(0)} + g\mathcal{C}^{(1)} + g^2 \mathcal{C}^{(2)} \tag{3.48}$$

with

$$\begin{aligned}
\mathcal{C}^{(0)} &= \mathcal{R}^{1/2}\Big[e^{-2\mathcal{M}} \mathcal{F} + \left(\mathcal{N} - \mathcal{F}\right) e^{2\mathcal{M}} \left(1 + \mathcal{N}\mathcal{F}\right)\Big]\mathcal{R}^{1/2} \\
\mathcal{C}^{(1)} &= i a_0 \mathcal{R}^{1/2}\Big[e^{-\mathcal{M}} A_0 e^{-\mathcal{M}} \mathcal{F} + \left(\mathcal{N} - \mathcal{F}\right) e^{\mathcal{M}} A_0 e^{\mathcal{M}} \left(1 + \mathcal{N}\mathcal{F}\right)\Big]\mathcal{R}^{1/2} \\
\mathcal{C}^{(2)} &= -\frac{a_0^2}{2}\mathcal{R}^{1/2}\Big[e^{-\mathcal{M}} A_0^2 e^{-\mathcal{M}} \mathcal{F} + \left(\mathcal{N} - \mathcal{F}\right) e^{\mathcal{M}} A_0^2 e^{\mathcal{M}} \left(1 + \mathcal{N}\mathcal{F}\right)\Big]\mathcal{R}^{1/2}
\end{aligned} \tag{3.49}$$

In Fourier space, keeping only first order terms in a_0,

$$\begin{aligned}
\mathcal{C}^{(0)}(p,q) &= 2\pi a_0 a_1 \delta(p+q)\left(-m\sin\theta_q + p\cos\theta_q\right) \\
\mathcal{C}^{(1)}(p,q) &= i a_0 a_1 A_0(p+q)\Big[\mathcal{R}^{1/2}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q) - \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{R}^{1/2}(q)\Big] \\
\mathcal{C}^{(2)}(p,q) &= -\frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi} A_0(p+k) A_0(-k+q)\Big[\mathcal{R}^{1/2}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q) \\
&\qquad - \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{R}^{1/2}(q)\Big]
\end{aligned} \tag{3.50}$$

We see that the $O(g^2)$ term, once we have mediated over gauge fields and extract a $\delta(p+q)$ factor, will be exactly 0.

In order to evaluate the other factor in (3.46), we note that, in our present model, $\mathring{\mathcal{E}}$ and $\mathcal{E}$ differ only for the sign of the linear term in g, because of the presence of $\mathcal{U}_0$ instead of $\mathcal{U}_0^\dagger$ in (2.17). Therefore, we can find the expression for $\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}$ from that of $\mathcal{R}^{1/2}\mathcal{E}\mathcal{R}^{1/2}$ we got in the previous section, simply sending $A_0 \to -A_0$. In this way

$$\begin{aligned}
\mathcal{R}^{1/2}\mathring{\mathcal{E}}^{(0)}\mathcal{R}^{1/2}(p,q) &= 2\pi a_1 \delta(p+q)\left(1 + m a_0 \cos\theta_q + a_0 q \sin\theta_q\right) \\
\mathcal{R}^{1/2}\mathring{\mathcal{E}}^{(1)}\mathcal{R}^{1/2}(p,q) &= i a_0 a_1 A_0(p+q)\big[\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q) \\
&\qquad + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\big] \\
\mathcal{R}^{1/2}\mathring{\mathcal{E}}^{(2)}\mathcal{R}^{1/2}(p,q) &= -\frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi} A_0(p+k)A_0(-k+q)\big[\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q) \\
&\qquad + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\big]
\end{aligned} \tag{3.51}$$

Inverting this expression with the aid of the Taylor series

$$\left[1 + \left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2} - 1\right)\right]^{-1} \simeq 1 - \left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2} - 1\right) + \left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2} - 1\right)^2 \tag{3.52}$$

we obtain

$$\begin{aligned}
\left[\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(0)}(p,q) &= 2\pi a_1 \delta(p+q)\left(1 - m a_0 \cos\theta_q - a_0 q \sin\theta_q\right) \\
\left[\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(1)}(p,q) &= -i a_0 a_1 A_0(p+q)\big[\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q) \\
&\qquad + \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\big] \\
\left[\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(2)}(p,q) &= \frac{a_1 a_0^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi} A_0(p+k)A_0(-k+q) \\
\cdot \big[\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q) &+ \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(q)\mathcal{R}^{1/2}(q) - 2\mathcal{R}^{1/2}(-p)\mathcal{R}(k)\mathcal{R}^{1/2}(q) \\
- 2\mathcal{R}^{1/2}(-p)\mathcal{R}(k)\mathcal{F}(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q) &- 2\mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{F}(k)\mathcal{R}(k)\mathcal{R}^{1/2}(q) \\
&- 2\mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{R}(k)\mathcal{F}^2(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\big]
\end{aligned} \tag{3.53}$$

where the last 2 lines come from the $O(g)$ term squared, which gives, as usual, an $O(a_0)$ contribution. When multiplied by $\mathcal{C}$, which already starts from first order in a_0, the only terms relevant to this order are the identity in first line and the linear term in g. Indeed,

$$\begin{aligned}\mathcal{C}^{(1)}\left[\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(1)}(p,q) = a_1 a_0^2 \int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q)\\ \cdot\left[\mathcal{R}^{1/2}(-p)\mathcal{F}(k)\mathcal{R}(k)\mathcal{R}^{1/2}(q) - \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{R}(k)\mathcal{R}^{1/2}(q)\right.\\ + \mathcal{R}^{1/2}(-p)\mathcal{F}(k)\mathcal{R}(k)\mathcal{F}(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\\ \left.- \mathcal{R}^{1/2}(-p)\mathcal{F}(-p)\mathcal{R}(k)\mathcal{F}(k)\mathcal{F}(q)\mathcal{R}^{1/2}(q)\right]\end{aligned} \tag{3.54}$$

Ultimately, keeping only first order term in a_0,

$$\begin{aligned}\left[\mathcal{C}\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(0)}(p,q) &= 2\pi a_0 a_1 \delta(p+q)\left(-m\sin\theta_q + q\cos\theta_q\right)\\ \left[\mathcal{C}\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(1)}(p,q) &= i a_0 a_1 A_0(p+q)\sin\frac{\theta_p+\theta_q}{2}\\ \left[\mathcal{C}\left(\mathcal{R}^{1/2}\mathring{\mathcal{E}}\mathcal{R}^{1/2}\right)^{-1}\right]^{(2)}(p,q) &= a_1 a_0^2 \int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q)\\ &\quad\cdot\left[-\frac{1}{2}\sin\frac{\theta_p+\theta_q}{2} + \cos\frac{\theta_q-\theta_k}{2}\sin\frac{\theta_p+\theta_k}{2}\right]\end{aligned} \tag{3.55}$$

Integrating over the gauge fields, we get

$$\begin{aligned}\mathcal{I}(p,q) = 2\pi a_0 a_1 \delta(p+q)\Big[-m\sin\theta_q + q\cos\theta_q\\ - g^2\int_{BZ}\frac{\mathrm{d}k}{2\pi}\frac{\Theta\cdot\Theta}{(q-k)^2}\cos\frac{\theta_q-\theta_k}{2}\sin\frac{\theta_q-\theta_k}{2}\Big]\end{aligned} \tag{3.56}$$

Inserting the expression for the Casimir operator (1.13) and contracting the colour indices we get, for the mixing terms,

$$\beta_t\mathcal{I}\alpha_t + \alpha_t^\dagger\mathcal{I}\beta_t^\dagger = \zeta\int_{BZ}\frac{\mathrm{d}q}{2\pi}\mathcal{I}_q\left[\beta_t(q)\alpha_t(-q) + \alpha_t^\dagger(q)\beta_t^\dagger(-q)\right] \tag{3.57}$$

with

$$\mathcal{I}_q = -m\sin\theta_q + q\cos\theta_q - \gamma\int_{BZ}\mathrm{d}k\,\frac{1}{(q-k)^2}\cos\frac{\theta_q-\theta_k}{2}\sin\frac{\theta_q-\theta_k}{2}\Big] \tag{3.58}$$

Confronting this expression with equation (3.39) we can verify that

$$\mathcal{I}_q = \frac{\mathrm{d}}{\mathrm{d}\theta_q}\omega_q \tag{3.59}$$

as expected from the discussion in section 2.2: if θ is a solution of the saddle point equation, that is respects the extremality condition for the zero-point action, the mixing term in the quasiparticle action is null once the gauge fields have been integrated out.

Quasiparticle energy

At this point we have to perform the calculation of the quasiparticle Hamiltonians (2.20), a relevant aspect in order to understand how the mechanism of confinement is realized in our picture. We sketch it for the α field, as the β field discussion is analogue. The corresponding term in (2.16) is

$$\begin{aligned}\alpha_t^\dagger(\nabla_t - \mathcal{H}_t)\alpha_{t+1} &= \alpha_t^\dagger\left[\left(\mathcal{U}_{0,t} - T_0^\dagger\right) - \left(\mathcal{U}_{0,t} - \mathcal{R}_t^{-1/2}\mathcal{E}_{t+1,t}^{-1}\mathcal{R}_{t+1}^{-1/2}\right)\right]\alpha_{t+1} \\ &= \alpha_t^\dagger\left(\mathcal{R}_t^{-1/2}\mathcal{E}_{t+1,t}^{-1}\mathcal{R}_{t+1}^{-1/2} - T_0^\dagger\right)\alpha_{t+1} \\ &= \alpha_t^\dagger\left(\alpha_{t+1} - \alpha_t\right) - \alpha_t^\dagger\left(1 - \mathcal{R}_t^{-1/2}\mathcal{E}_{t+1,t}^{-1}\mathcal{R}_{t+1}^{-1/2}\right)\alpha_{t+1}\end{aligned} \tag{3.60}$$

The first term in the last line is simply proportional to the lattice time (right) derivative (A.2), so we only need to find the expression of the second term. In stationary condition, we already know the form of $\mathcal{R}^{-1/2}\mathring{\mathcal{E}}^{-1}\mathcal{R}^{-1/2}$ from (3.53); sending $A_0 \to -A_0$ to get $\mathcal{E}$ from $\mathring{\mathcal{E}}$ we obtain:

$$\begin{aligned}\left[1 - \mathcal{R}^{-1/2}\mathcal{E}^{-1}\mathcal{R}^{-1/2}\right](p,q) &= 2\pi a_0 a_1 \delta(p+q)\left(m\cos\theta_q + q\sin\theta_q\right) \\ &+ i a_0 a_1 g A_0(p+q)\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q)\left[1 + \mathcal{F}(-p)\mathcal{F}(q)\right] \\ - \frac{a_1 a_0^2 g^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}&A_0(p+k)A_0(-k+q)\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q)\Big\{1 + \mathcal{F}(-p)\mathcal{F}(q) \\ &- 2\mathcal{R}(k)\left[1 + \mathcal{F}(-p)\mathcal{F}(q)\mathcal{F}^2(k) + \mathcal{F}(-p)\mathcal{F}(k) + \mathcal{F}(k)\mathcal{F}(q)\right]\Big\}\end{aligned} \tag{3.61}$$

and so

$$\begin{aligned}\left[1 - \mathcal{R}^{-1/2}\mathcal{E}^{-1}\mathcal{R}^{-1/2}\right](p,q) &= 2\pi a_0 a_1 \delta(p+q)\left(m\cos\theta_q + q\sin\theta_q\right) \\ &+ i a_0 a_1 g A_0(p+q)\cos\frac{\theta_p + \theta_q}{2} \\ &- \frac{a_1 a_0^2 g^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi}A_0(p+k)A_0(-k+q)\left[\cos\frac{\theta_p+\theta_q}{2}\right. \\ &\left. - 2\cos\frac{\theta_p - \theta_k}{2}\cos\frac{\theta_q - \theta_k}{2}\right]\end{aligned} \tag{3.62}$$

After the integration over gluon fields:

$$\begin{aligned}\left[1 - \mathcal{R}^{-1/2}\mathcal{E}^{-1}\mathcal{R}^{-1/2}\right](p,q) &= 2\pi a_0 a_1 \delta(p+q)\left[m\cos\theta_q + q\sin\theta_q\right. \\ &\left. + \frac{g^2}{2}\int_{BZ}\mathrm{d}k\,\frac{\Theta\cdot\Theta}{(q-k)^2}\cos(\theta_q - \theta_k)\right]\end{aligned} \tag{3.63}$$

Inserting the expression for the Casimir operator (1.13) and contracting the colour indices we get, for the α-energy in the action,

$$\zeta \int_{BZ} \frac{\mathrm{d}q}{2\pi}\, \omega_q^\alpha \alpha_t^\dagger(q)\alpha_{t+1}(-q) \tag{3.64}$$

with

$$\omega_q^\alpha = m\cos\theta_q + q\sin\theta_q + \frac{\gamma}{2}\int_{BZ} \frac{\mathrm{d}k}{2\pi}\,\frac{\cos(\theta_q-\theta_k)}{(q-k)^2} \tag{3.65}$$

Using

$$\begin{aligned}
\left[1-\mathcal{R}^{-1/2}\mathcal{E}^{-1}\mathcal{R}^{-1/2}\right](p,q) &= 2\pi a_0 a_1 \delta(p+q)\left(m\cos\theta_q + q\sin\theta_q\right) \\
&\quad - i a_0 a_1 g A_0(p+q)\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q)\left[1+\mathcal{F}(-p)\mathcal{F}(q)\right] \\
-\frac{a_1 a_0^2 g^2}{2}\int_{BZ}\frac{\mathrm{d}k}{2\pi} & A_0(p+k)A_0(-k+q)\mathcal{R}^{1/2}(-p)\mathcal{R}^{1/2}(q)\Big\{1+\mathcal{F}(-p)\mathcal{F}(q) \\
&\quad - 2\mathcal{R}(k)\left[1+\mathcal{F}(-p)\mathcal{F}(q)\mathcal{F}^2(k)+\mathcal{F}(-p)\mathcal{F}(k)+\mathcal{F}(k)\mathcal{F}(q)\right]\Big\}
\end{aligned} \tag{3.66}$$

the β term follows easily:

$$\zeta \int_{BZ} \frac{\mathrm{d}q}{2\pi}\, \omega_q^\beta \beta_{t+1}(q)\beta_t^\dagger(-q) \tag{3.67}$$

with

$$\omega_q^\beta = \omega_q^\alpha \tag{3.68}$$

It is clear that the integral in $\omega_q^{\alpha,\beta}$ diverges in a neighbourhood of q. This property can also be seen in Figure 3.4, where we plot its trend at the saddle point: as the exact point $k=q$ must be skipped to perform the numerical calculation, the divergence is more evident if we increase the number of points we use to sample the interval. We can interpret this result as a sign of confinement: in the low momentum phase the quasiparticles does not propagate individually because they would require an infinite energy to do so.

As we can see, the quasiparticle contributions are $O(1)$ in N_c, while the term we have evaluated in the previous section is $O(N_c)$: that's the reason we neglected them, in the large N_c expansion, when we were searching for the vacuum energy (see note 1 in section 2.2).

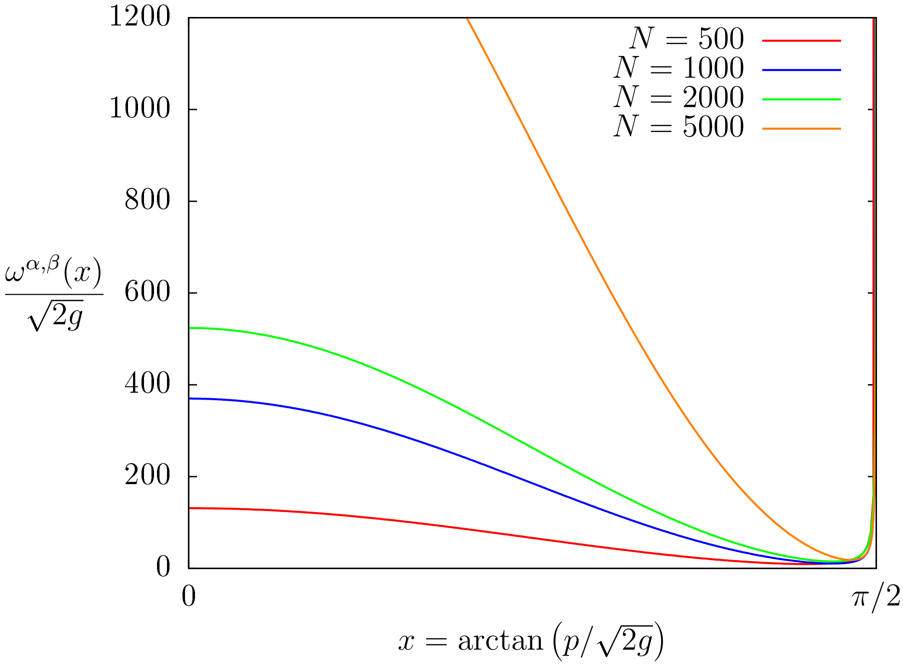

Figure 3.4: The quasiparticle energy at the saddle point as a function of x, for $m = 1$, plotted for different numbers N of samplings in the interval $[-\pi/2, \pi/2]$. The lack of stability for every x less than a critical value is a hint for the divergence in the low momentum phase.

Chapter 4

Effective mesons

4.1 Interaction potential

Let's move on now to the evaluation of the quartic terms in the effective action, as explained at the beginning of section 3.2. Practically, we have to square the sum of the $O(g)$ terms we found in the previous section (in equations (3.55) and (3.62)). We will get objects with a colour structure of the type

$$\left(\psi^1 \Theta^l \psi^2\right)\left(\psi^3 \Theta^m \psi^4\right) \tag{4.1}$$

with ψ^i one of the α, β, $\alpha^\dagger$, $\beta^\dagger$ fields. After the integration over gauge fields, the adjoint indices of the Θ matrices will be summed over, because the gluon propagator is proportional to δ_{lm}, and we can thus rearrange the contractions between fermions using the Fierz identity (1.14). Keeping only the leading part in N_c, we will end with the products of bilinears summarized in Table 4.1. To express them we can introduce effective "mesonic" and "number" fields made of fermion pairs:

$$\begin{aligned}
\Gamma_t(p,q) &= \frac{1}{\sqrt{N_c}}\beta_t(p)\alpha_t(q) & \Gamma_t^\dagger(p,q) &= \frac{1}{\sqrt{N_c}}\alpha_t^\dagger(p)\beta_t^\dagger(q) \\
\Lambda_t^\alpha(p,q) &= \frac{1}{\sqrt{N_c}}\alpha_t^\dagger(p)\alpha_{t+1}(q) & \Lambda_t^\beta(p,q) &= \frac{1}{\sqrt{N_c}}\beta_t^\dagger(p)\beta_{t+1}(q)
\end{aligned} \tag{4.2}$$

We can see that these objects are *non-local* objects in momentum space: this is due to the fact that they do not represent elementary particles, but composite ones. Furthermore, they are *colourless*: they carry no colour indices, because of the contractions in the bilinears; we say that they represent *singlet states* in colour.

Now we have to find the momentum structure. A generic term in the first

action term	bilinears after rearrangement
$(\beta_t \mathcal{I}_t \alpha_t)^2$	$(\beta_t \alpha_t)(\alpha_t \beta_t)$
$(\alpha_t^\dagger \mathcal{I}_t \beta_t^\dagger)^2$	$(\alpha_t^\dagger \beta_t^\dagger)(\beta_t^\dagger \alpha_t^\dagger)$
$(\alpha_t^\dagger \mathcal{H}_t \alpha_{t+1})^2$	$(\alpha_t^\dagger \alpha_{t+1})(\alpha_{t+1} \alpha_t^\dagger)$
$(\beta_{t+1} \mathcal{H}_t \beta_t^\dagger)^2$	$(\beta_{t+1} \beta_t^\dagger)(\beta_t^\dagger \beta_{t+1})$
$(\beta_{t+1} \mathcal{I}_{t+1} \alpha_{t+1})(\alpha_t^\dagger \mathcal{I}_t \beta_t^\dagger)$	$(\beta_{t+1} \beta_t^\dagger)(\alpha_{t+1} \alpha_t^\dagger)$
$(\alpha_t^\dagger \mathcal{H}_t \alpha_{t+1})(\beta_{t+1} \mathcal{H}_t \beta_t^\dagger)$	$(\alpha_t^\dagger \beta_t^\dagger)(\alpha_{t+1} \beta_{t+1})$
$(\beta_{t+1} \mathcal{I}_{t+1} \alpha_{t+1})(\alpha_t^\dagger \mathcal{H}_t \alpha_{t+1})$	$(\beta_{t+1} \alpha_{t+1})(\alpha_{t+1} \alpha_t^\dagger)$
$(\beta_{t+1} \mathcal{I}_{t+1} \alpha_{t+1})(\beta_{t+1} \mathcal{H}_t \beta_t^\dagger)$	$(\beta_{t+1} \beta_t^\dagger)(\alpha_{t+1} \beta_{t+1})$
$(\alpha_t^\dagger \mathcal{I}_t \beta_t^\dagger)(\alpha_t^\dagger \mathcal{H}_t \alpha_{t+1})$	$(\alpha_t^\dagger \alpha_{t+1})(\beta_t^\dagger \alpha_t^\dagger)$
$(\alpha_t^\dagger \mathcal{I}_t \beta_t^\dagger)(\beta_{t+1} \mathcal{H}_t \beta_t^\dagger)$	$(\alpha_t^\dagger \beta_t^\dagger)(\beta_t^\dagger \beta_{t+1})$

Table 4.1: Fierz rearrangement at leading order in N_c.

column of Table 4.1 is of the type

$$\frac{g^2}{2a_1^4}\int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}p'}{2\pi}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}q'}{2\pi}\psi_t^1(-p)\mathcal{L}_t^{\mathrm{I}}(p,-p')\psi_t^2(p')\psi_{t'}^3(-q)\mathcal{L}_{t'}^{\mathrm{II}}(q,-q')\psi_{t'}^4(q') \quad (4.3)$$

with $\mathcal{L}_t^{\mathrm{I,II}}$ one of the operators $\mathcal{I}_t^{(1)}$, $\mathcal{I}_{t-1}^{(1)}$, $\mathcal{H}_t^{(1)}$. Then

$$\begin{aligned}\mathcal{L}_t^{\mathrm{I,II}}(p,-p') \in \Big\{ & ia_0a_1A_{0,t}(p-p')\sin\frac{\theta_p-\theta_{p'}}{2}, \\ & ia_0a_1A_{0,t-1}(p-p')\sin\frac{\theta_p-\theta_{p'}}{2}, \\ & ia_0a_1gA_{0,t}(p-p')\cos\frac{\theta_p-\theta_{p'}}{2}\Big\}\end{aligned} \quad (4.4)$$

Calculating all the combinations, integrating over gauge fields, anticommuting the bilinears to obtain the effective fields (4.2), redefining the momenta integration variables $p \to -p$ when needed, we finally obtain

$$\frac{1}{2}g^2\left(S_Q^{(1)}\right)^2 = -a_0\sum_t \frac{1}{a_1^2}\mathcal{V}_t\left[\Gamma,\Gamma^\dagger,\Lambda^\alpha,\Lambda^\beta\right] \quad (4.5)$$

with

$$
\begin{aligned}
\mathcal{V}_t[\Gamma,\Gamma^\dagger,\Lambda^\alpha,\Lambda^\beta] = & -\frac{\gamma}{2}\frac{1}{(2\pi)^2}\int_{BZ}\mathrm{d}p\,\mathrm{d}p'\,\mathrm{d}q\,\mathrm{d}q'\,\frac{\delta(p+p'+q+q')}{(p+p')^2} \\
& \cdot\Bigg\{\sin\frac{\theta_p+\theta_{p'}}{2}\sin\frac{\theta_q+\theta_{q'}}{2}\Big[\Gamma_t(p,q')\Gamma_t(q,p')+\Gamma_t^\dagger(p,q')\Gamma_t^\dagger(q,p') \\
& \qquad -\Lambda_t^\alpha(p,q')\Lambda_t^\beta(q,p')-\Lambda_t^\beta(p,q')\Lambda_t^\alpha(q,p')\Big] \\
& +\cos\frac{\theta_p+\theta_{p'}}{2}\cos\frac{\theta_q+\theta_{q'}}{2}\Big[\Gamma_{t+1}^\dagger(p,q')\Gamma_t(q,p')+\Gamma_t(p,q')\Gamma_{t+1}^\dagger(q,p') \\
& \qquad +\Lambda_t^\alpha(p,q')\Lambda_t^\alpha(q,p')+\Lambda_t^\beta(p,q')\Lambda_t^\beta(q,p')\Big] \\
& +2\cos\frac{\theta_p+\theta_{p'}}{2}\sin\frac{\theta_q+\theta_{q'}}{2}\Big[-\Lambda_t^\alpha(p,q')\Gamma_t(q,p')+\Gamma_t(p,q')\Lambda_t^\beta(q,p') \\
& \qquad -\Gamma_{t+1}^\dagger(p,q')\Lambda_t^\alpha(q,p')+\Lambda_t^\beta(p,q')\Gamma_{t+1}^\dagger(q,p')\Big]\Bigg\}
\end{aligned}
\tag{4.6}
$$

This contribution to the action is quadratic in our composite fields and proportional to γ: it represents an *effective interaction term*.

4.2 Kinetic term

The identification of the quartic term in the quasiparticle action with an interaction potential between composite fields that we derived in the previous section has required no additional hypothesis and works at the level of the functional path integral, as it is clear from the discussion at the the beginning of section 3.2. However, in order to *bosonize* the model, that is to describe it only in terms of these effective fields, the part interpretable as a kinetic energy for the mesons, quadratic in Γ and so quartic in α and β, is still missing. Nevertheless, the only way we have to produce quartic contributions in the action is to expand in g, and so we can never generate a free part! The quasiparticle energies (3.64) and (3.67) have the right requirements, as they do not depend on g, but they are *quadratic* in α and β. The solution for this problem is provided by a very subtle argument, enunciated in [2], that goes as follows. First, we go back to the operatorial formalism and write the partition function as the trace of the transfer matrix, written in terms of the quasiparticle operators (2.8) generated after the Bogoliubov transformation. The quasiparticle energy is then the *fermionic Hamiltonian*

$$
\hat{H}_F = \int \mathrm{d}p\,E_p\hat{a}^\dagger(-p)\hat{a}(p) + \int \mathrm{d}p\,E_p\hat{b}^\dagger(-p)\hat{b}(p) \tag{4.7}
$$

with $E_p = \omega_p^{\alpha,\beta}$. Then, instead of evaluating the trace expanding on generic canonical coherent states, as in (2.13), we *impose* that the relevant states for the model, in the large N_c limit, are mesonic composites. This means that we can introduce a projector in equation (1.32) to restrict the trace and evaluate the matrix elements only between these states (see [5]):

$$\mathcal{Z}_F = \mathrm{Tr}^F \prod_t \mathcal{J}_t \hat{\mathcal{P}} \hat{\mathcal{T}}_{t,t+1} \tag{4.8}$$

Finally, we have to demonstrate that the matrix elements of the operator $\hat{H}_F$ between these states are equal to those of the operator

$$\hat{H}_M = \int \mathrm{d}p \int \mathrm{d}q \, (E_p + E_q) \hat{\Gamma}^\dagger(p,q) \hat{\Gamma}(-q,-p) \tag{4.9}$$

where $\hat{\Gamma}^\dagger$ and $\hat{\Gamma}$ are creation and annihilation operators for the colourless composites:

$$\hat{\Gamma}(p,q) = \frac{1}{\sqrt{N_c}} \hat{b}(p)\hat{a}(q) \qquad \hat{\Gamma}^\dagger(p,q) = \frac{1}{\sqrt{N_c}} \hat{a}^\dagger(p)\hat{b}^\dagger(q) \tag{4.10}$$

In this way, under this assumption of *boson dominance*, we can substitute the quasiparticle energy with an effective meson kinetic energy in the action.

To start, in analogy with (2.9) we define a “coherent” composite state of quasiparticles as

$$|\Phi\rangle = \exp\!\left(\hat{a}^\dagger \Phi^\dagger \hat{b}^\dagger\right) |0\rangle_\theta \tag{4.11}$$

where $|0\rangle_\theta = |\mathcal{F}\rangle$ is the vacuum state for the operators $\hat{a}$, $\hat{b}$ (in this section, as we will always work with canonical coherent states $|\alpha\beta;\mathcal{F}\rangle$ built up acting on this vacuum with the creation operators $\hat{a}^\dagger$, $\hat{b}^\dagger$, we will omit the $\mathcal{F}$ and write $|0\rangle$ instead of $|0\rangle_\theta$). In the exponent, the contraction between internal (colour) and space (momentum) indices is understood:

$$\hat{\Phi}^\dagger = \hat{a}^\dagger \Phi^\dagger \hat{b}^\dagger = \sum_{a,b=1}^{N_c} \int_{BZ} \mathrm{d}p \, \mathrm{d}q \, \hat{a}_a^\dagger(p) \Phi_{ab}^\dagger(-p,q) \hat{a}_b(-q) \tag{4.12}$$

Let $\mathcal{S}$ be the tensor space from where these indices take values (in our case, $\mathcal{S} = \mathbb{R}_{BZ} \otimes \mathrm{SU}(N_c)$), spanned by the collective indices A, B, etc. At this level, Φ_{AB} is still a generic matrix. The projector in (4.8) is then

$$\hat{\mathcal{P}} = \int \frac{\mathrm{d}\xi \, \mathrm{d}\xi^*}{2\pi i} \frac{1}{\langle \Phi | \Phi \rangle} |\Phi\rangle\langle\Phi| \tag{4.13}$$

where ξ, ξ^* are complex variables parametrizing the matrix Φ. The norm of the composite state is

$$\begin{aligned}
\langle\Phi'|\Phi\rangle &= \int \frac{\mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, \mathrm{d}\beta^\dagger \mathrm{d}\beta}{\langle\alpha\beta|\alpha\beta\rangle}\,\langle\Phi'|\alpha\beta\rangle\langle\alpha\beta|\Phi\rangle \\
&= \int \mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, \mathrm{d}\beta^\dagger \mathrm{d}\beta\, \exp\big(-\alpha^\dagger\alpha - \beta^\dagger\beta + \beta\Phi'\alpha + \alpha^\dagger\Phi^\dagger\beta^\dagger\big) \\
&= \int \mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, \exp\big[-\alpha^\dagger(1+\Phi^\dagger\Phi')\alpha\big] \\
&= \det(1+\Phi^\dagger\Phi')
\end{aligned} \tag{4.14}$$

where det is the determinant in $\mathcal{S}$. We are interested in evaluating matrix elements of the operator products appearing in (4.7) between these states:

$$\frac{\langle\Phi'|\hat{a}^\dagger_A\hat{a}_B|\Phi\rangle}{\langle\Phi'|\Phi\rangle} = \delta_{AB} - \frac{\langle\Phi'|\hat{a}_B\hat{a}^\dagger_A|\Phi\rangle}{\langle\Phi'|\Phi\rangle} \tag{4.15}$$

with (deriving (A.23) with respect to the sources)

$$\begin{aligned}
\langle\Phi'|\hat{a}_B\hat{a}^\dagger_A|\Phi\rangle &= \int \frac{\mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, \mathrm{d}\beta^\dagger \mathrm{d}\beta}{\langle\alpha\beta|\alpha\beta\rangle}\,\langle\Phi'|\hat{a}_B|\alpha\beta\rangle\langle\alpha\beta|\hat{a}^\dagger_A|\Phi\rangle \\
&= \int \mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, \mathrm{d}\beta^\dagger \mathrm{d}\beta\, (\alpha_B\alpha^\dagger_A)\exp\big(-\alpha^\dagger\alpha - \beta^\dagger\beta + \beta\Phi'\alpha + \alpha^\dagger\Phi^\dagger\beta^\dagger\big) \\
&= \int \mathrm{d}\alpha^\dagger \mathrm{d}\alpha\, (\alpha_B\alpha^\dagger_A)\exp\big[-\alpha^\dagger(1+\Phi^\dagger\Phi')\alpha\big] \\
&= \det(1+\Phi^\dagger\Phi')(1+\Phi^\dagger\Phi')^{-1}_{BA}
\end{aligned} \tag{4.16}$$

To evaluate the same expectation value for the $\hat{b}$ bilinears we can note that

$$\hat{b}_A\,|\Phi\rangle = \hat{b}_A \exp\big[(\hat{a}^\dagger\Phi^\dagger)_B\hat{b}^\dagger_B\big)\,|0\rangle = \hat{b}_A\big[1+(\hat{a}^\dagger\Phi^\dagger)_B\hat{b}^\dagger_B+\cdots\big]\,|0\rangle = -(\hat{a}^\dagger\Phi^\dagger)_A\,|\Phi\rangle \tag{4.17}$$

so we can write

$$\begin{aligned}
\langle\Phi'|\hat{b}^\dagger_A\hat{b}_B|\Phi\rangle &= \langle\Phi'|(\Phi'\hat{a})_A(\hat{a}^\dagger\Phi^\dagger)_B|\Phi\rangle = \Phi'_{AC}\Phi^\dagger_{DB}\,\langle\Phi'|\hat{a}_C\hat{a}^\dagger_D|\Phi\rangle \\
&= \langle\Phi'|\Phi\rangle\,\Phi'_{AC}(1+\Phi^\dagger\Phi')^{-1}_{CD}\Phi^\dagger_{DB} = \langle\Phi'|\Phi\rangle\,\big[\Phi'(1+\Phi^\dagger\Phi')^{-1}\Phi^\dagger\big]_{AB}
\end{aligned} \tag{4.18}$$

Using

$$\Phi'(1+\Phi^\dagger\Phi')^{-1}\Phi^\dagger = (1+\Phi'\Phi^\dagger)^{-1}\Phi'\Phi^\dagger = 1-(1+\Phi'\Phi^\dagger)^{-1} \tag{4.19}$$

we arrive at

$$\frac{\langle\Phi'|\hat{a}^\dagger_A\hat{a}_B|\Phi\rangle}{\langle\Phi'|\Phi\rangle} = \big[1-(1+\Phi^\dagger\Phi')^{-1}\big]_{BA} \tag{4.20}$$

$$\frac{\langle\Phi'|\hat{b}^\dagger_A\hat{b}_B|\Phi\rangle}{\langle\Phi'|\Phi\rangle} = \big[1-(1+\Phi'\Phi^\dagger)^{-1}\big]_{AB} \tag{4.21}$$

To evaluate now the matrix elements of $\hat{H}_M$, we need a quartic term of the type

$$\begin{aligned}\langle\Phi'|\hat{a}^\dagger_A\hat{b}^\dagger_B\hat{b}_C\hat{a}_D|\Phi\rangle &= \langle\Phi'|\hat{a}^\dagger_A\hat{a}_D\hat{b}^\dagger_B\hat{b}_C|\Phi\rangle \\ &= \delta_{AD}\,\langle\Phi'|\hat{b}^\dagger_B\hat{b}_C|\Phi\rangle - \delta_{BC}\,\langle\Phi'|\hat{a}_D\hat{a}^\dagger_A|\Phi\rangle + \langle\Phi'|\hat{b}_C\hat{a}_D\hat{a}^\dagger_A\hat{b}^\dagger_B|\Phi\rangle \end{aligned} \tag{4.22}$$

We have already found the expression for the first two addends. The last one, using many times the translational invariance of the Berezin integral, is

$$\begin{aligned}\langle\Phi'|\hat{b}_C\hat{a}_D\hat{a}^\dagger_A\hat{b}^\dagger_B|\Phi\rangle &= \int\frac{\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta}{\langle\alpha\beta|\alpha\beta\rangle}\,\langle\Phi'|\hat{b}_C\hat{a}_D|\alpha\beta\rangle\langle\alpha\beta|\hat{a}^\dagger_A\hat{b}^\dagger_B|\Phi\rangle \\ &= \int\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta\,\beta_C\alpha_D\alpha^\dagger_A\beta^\dagger_B e^{-\alpha^\dagger\alpha-\beta^\dagger\beta+\beta\Phi'\alpha+\alpha^\dagger\Phi^\dagger\beta^\dagger} \\ &= \int\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta\,\beta_C\alpha_D\alpha^\dagger_A\beta^\dagger_B e^{-\alpha^\dagger\alpha+\beta(\beta^\dagger+\Phi'\alpha)+\alpha^\dagger\Phi^\dagger\beta^\dagger} \\ &= \int\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta\,\beta_C\alpha_D\alpha^\dagger_A[\beta^\dagger_B-(\Phi'\alpha)_B] e^{-\alpha^\dagger\alpha+\beta\beta^\dagger+\alpha^\dagger\Phi^\dagger(\beta^\dagger-\Phi'\alpha)} \\ &= \int\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta\,[\beta_C-(\alpha^\dagger\Phi^\dagger)_C]\alpha_D\alpha^\dagger_A[\beta^\dagger_B-(\Phi'\alpha)_B] e^{-\alpha^\dagger(1+\Phi^\dagger\Phi')\alpha+\beta\beta^\dagger} \\ &= \int\mathrm{d}\alpha^\dagger\mathrm{d}\alpha\,\mathrm{d}\beta^\dagger\mathrm{d}\beta\,\left(\beta_C\alpha_D\alpha^\dagger_A\beta^\dagger_B+\Phi^\dagger_{EC}\Phi'_{BF}\alpha^\dagger_E\alpha_D\alpha^\dagger_A\alpha_F\right) e^{-\alpha^\dagger(1+\Phi^\dagger\Phi')\alpha+\beta\beta^\dagger}\end{aligned} \tag{4.23}$$

where, in the last line, we kept only even terms in the Grassmann fields because the odd ones are null (as a consequence of the Wick theorem for Berezin integrals: it can be seen deriving (A.23) with respect to the sources and putting them to 0). In this way

$$\begin{aligned}\langle\Phi'|\hat{b}_C\hat{a}_D\hat{a}^\dagger_A\hat{b}^\dagger_B|\Phi\rangle &= \langle\Phi'|\Phi\rangle\,\big\{\delta_{BC}(1+\Phi^\dagger\Phi')^{-1}_{DA} \\ &\qquad + \Phi^\dagger_{EC}\Phi'_{BF}\big[(1+\Phi^\dagger\Phi')^{-1}_{DE}(1+\Phi^\dagger\Phi')^{-1}_{FA} \\ &\qquad\qquad - (1+\Phi^\dagger\Phi')^{-1}_{DA}(1+\Phi^\dagger\Phi')^{-1}_{FE}\big]\big\} \\ &= \langle\Phi'|\Phi\rangle\,\big\{(1+\Phi^\dagger\Phi')^{-1}_{DA}(1+\Phi'\Phi^\dagger)^{-1}_{BC} \\ &\qquad + \left[(1+\Phi^\dagger\Phi')^{-1}\Phi^\dagger\right]_{DC}\left[\Phi'(1+\Phi^\dagger\Phi')^{-1}\right]_{BA}\big\}\end{aligned} \tag{4.24}$$

Summing and rearranging terms we finally get

$$\begin{aligned}\frac{\langle\Phi'|\hat{a}^\dagger_A\hat{b}^\dagger_B\hat{b}_C\hat{a}_D|\Phi\rangle}{\langle\Phi'|\Phi\rangle} &= \left[\Phi^\dagger\Phi'(1+\Phi^\dagger\Phi')^{-1}\right]_{DA}\left[\Phi'\Phi^\dagger(1+\Phi'\Phi^\dagger)^{-1}\right]_{BC} \\ &\qquad + \left[(1+\Phi^\dagger\Phi')^{-1}\Phi^\dagger\right]_{DC}\left[\Phi'(1+\Phi^\dagger\Phi')^{-1}\right]_{BA}\end{aligned} \tag{4.25}$$

We have now all the ingredient to specialize the structure matrices Φ, Φ', until now completely general. Suppose for the moment that they are singlet in colour and non-local in momentum composite fields, so

$$\Phi^{ab}(p,q) = \delta_{ab}\frac{g(p,q)}{\sqrt{N_c}} \tag{4.26}$$

In this way, keeping only leading terms in large N_c expansion, we have

$$\langle\Phi'|\Phi\rangle = \det(1+\Phi^\dagger\Phi') = \exp\operatorname{tr}\ln(1+\Phi^\dagger\Phi') \simeq \exp\operatorname{tr} g^* g' \tag{4.27}$$

where

$$\operatorname{tr} g^* g' = \int \mathrm{d}q\,\mathrm{d}q'\, g^*(q,q')g'(-q',-q) \tag{4.28}$$

and

$$\frac{\langle\Phi'|\hat{a}_a^\dagger(-p)\hat{a}_a(p)|\Phi\rangle}{\langle\Phi'|\Phi\rangle} \simeq \frac{\delta_{aa}}{N_c}\int \mathrm{d}q\, g^*(p,q)g'(-q,-p) \tag{4.29}$$

$$\frac{\langle\Phi'|\hat{b}_a^\dagger(-p)\hat{b}_a(p)|\Phi\rangle}{\langle\Phi'|\Phi\rangle} \simeq \frac{\delta_{aa}}{N_c}\int \mathrm{d}q\, g'(-p,q)g^*(-q,p) \tag{4.30}$$

$$\frac{\langle\Phi'|\hat{a}_a^\dagger(p)\hat{b}_a^\dagger(q)\hat{b}_b(-p)\hat{a}_b(-q)|\Phi\rangle}{\langle\Phi'|\Phi\rangle} \simeq \frac{\delta_{bb}\delta_{aa}}{N_c} g^*(-q,-p)g'(q,p) \tag{4.31}$$

Summing over colour indices and integrating over momenta, it is now easy to see that

$$\frac{\langle\Phi'|\hat{H}_F|\Phi\rangle}{\langle\Phi'|\Phi\rangle} = \frac{\langle\Phi'|\hat{H}_M|\Phi\rangle}{\langle\Phi'|\Phi\rangle} \tag{4.32}$$

for any choice of g, g'.

We remark here that this identification provides an $O(1)$ kinetic energy contribution in N_c for the colourless composite fields. Since this argument works for the Hamiltonian, that's true also for the transfer matrix, which can be obtained exponentiating it. In this way we are able to introduce in the effective action the the free energy contribution for the singlet mesons we were looking for.

Chapter 5

Conclusions and outlook

In our study of QCD_2 in the large N_c (and weak coupling) expansion, after the integration over gauge fields we have obtained an effective action with a number of features, which we summarize here:

- The vacuum energy contribution, obtained choosing the Bogoliubov transformation to comply with a variational principle (the saddle point equation), is $O(N_c)$ and has the rate depicted in Figure 3.3.
- At the saddle point, the terms in the action that mix quasiquarks and quasiantiquarks are exactly 0. Thus, the transformation associated with the variational principle is also the one that diagonalizes the quasiparticle sector.
- The quasiparticle energy is a contribution of order $O(1)$ in N_c, which diverges for low momentum (see Figure 3.4). That's the reason why they do not propagate as free particles: confinement occurs.
- The dynamics can be described in terms of meson fields that are singlet states in colour. They have an effective interaction term and, under the established hypothesis of boson dominance, also a kinetic $O(1)$ (in N_c) term obtained from the quasiparticle energy contributions with a projection over these composite states. In this way the theory has been bosonized.

As explained, these results are in complete accordance with the ones we found in literature, obtained with the canonical formalism. However, we remark here that the method we applied in the present work is not only an alternative version of the usual continuum approach, whose predictions we re-obtained under the same assumption of weak coupling/large N_c expansion.

These hypothesis are not an essential feature and, in principle, the formulation described in chapter 2 is more general. The form for the effective action

$$S_{\text{eff}} = S_0[\mathcal{F}] + S_Q[\alpha, \beta; \mathcal{F}] \tag{5.1}$$

is unitary equivalent to the original one, so we needed no additional assumptions to write it. Indeed, using the properties of the lattice, we could access to some of the informations about the non-perturbative (strong-coupling) regime. That's an aspect to investigate in the future: the usual approach to effective theories in QFT leads to effective actions that are defined as generating functionals of n-particle irreducible correlation functions (see [9] and, for reference, [14]), so they do not even exist outside perturbation theory. The connection between the two formulations has to be established.

There is also another chance to explore some of the non-perturbative features of the theory, provided by numerical methods to evaluate expectation values of observables on the lattice. As explained in [6], the present method is free from a problem that makes these simulations really difficult to implement. Thus, it should be tested to see if it can be an improvement to the current numerical approaches.

Furthermore, the possibility to pass from the canonical to the functional formalism, and vice versa, that we exploited in the last section, makes the method really powerful and particularly suitable for relativistic theories. In this context, driven by the promising results we got here, we hope to extend it to more realistic models for strong interaction. However, its field of application is not limited to this branch of physics: since it has been formulated in a very general way, it could be applied to other relevant models, for example in solid state physics. In this respect, it preserve the great versatility the Bogoliubov transformations historically proved to have.

Appendix A

Useful formulas

A.1 Lattice operators

We define the free shift operators as

$$\big[T_\mu\big]_{xy} = \delta_{x+a\hat\mu,y}, \qquad \big[T_\mu^\dagger\big]_{xy} = \delta_{x-a\hat\mu,y} \tag{A.1}$$

Lattice right and left derivatives are

$$\partial_\mu^{(+)} = \frac{1}{a}(T_\mu - 1), \qquad \partial_\mu^{(-)} = \frac{1}{a}(1 - T_\mu^\dagger) \tag{A.2}$$

Their action on a generic function is

$$\partial_\mu^{(+)} f(x) = \frac{f(x+a\hat\mu) - f(x)}{a}, \qquad \partial_\mu^{(-)} f(x) = \frac{f(x) - f(x-a\hat\mu)}{a} \tag{A.3}$$

A symmetric choice for the derivative is

$$\partial_\mu^{(\mathrm{s})} = \frac{1}{2}(\partial_\mu^{(+)} + \partial_\mu^{(-)}) = \frac{1}{2a}(T_\mu - T_\mu^\dagger) \tag{A.4}$$

and so

$$\partial_\mu^{(\mathrm{s})} f(x) = \frac{f(x+a\hat\mu) - f(x-a\hat\mu)}{2a} \tag{A.5}$$

The free Laplacian operator on the lattice is

$$\partial^2 = \sum_\mu \partial_\mu^{(+)}\partial_\mu^{(-)} = \sum_\mu \frac{1}{a}(\partial_\mu^{(+)} - \partial_\mu^{(-)}) = \sum_\mu \frac{1}{a^2}(T_\mu + T_\mu^\dagger - 2) \tag{A.6}$$

Introducing a connection, we get the covariant derivatives

$$D_\mu^{(+)}(x,y) = \frac{1}{a}\Big\{U_\mu(x)\big[T_\mu\big]_{xy} - 1\Big\} \tag{A.7a}$$

$$D_\mu^{(-)}(x,y) = \frac{1}{a}\Big\{1 - \big[T_\mu^\dagger\big]_{xy} U_\mu^\dagger(y)\Big\} \tag{A.7b}$$

$$D_\mu(x,y) = \frac{1}{2a}\Big\{U_\mu(x)\big[T_\mu\big]_{xy} - \big[T_\mu^\dagger\big]_{xy} U_\mu^\dagger(y)\Big\} \tag{A.7c}$$

where $U_\mu(x)$ is the parallel transporter in direction μ. As usual,

$$\not{D}(x,y) = \sum_\mu \gamma_\mu D_\mu(x,y) \tag{A.8}$$

The covariant Laplacian is naturally

$$D^2(x,y) = \sum_\mu \frac{1}{a^2}\left\{U_\mu(x)\big[T_\mu\big]_{xy} + \big[T_\mu^\dagger\big]_{xy} U_\mu^\dagger(y) - 2\right\} \tag{A.9}$$

To expand lattice functions in the Fourier basis, we introduce the momentum representation in a single dimension (the multidimensional generalization is straightforward)

$$f(p) = a\sum_x f(x)e^{-ipx}, \qquad f(x) = \int_{BZ}\frac{\mathrm{d}p}{2\pi} f(p)e^{ipx} \tag{A.10}$$

where the integration variables span the first Brillouin zone:

$$\int_{BZ}\frac{\mathrm{d}p}{2\pi} \equiv \int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}p}{2\pi} \tag{A.11}$$

Therefore, the delta functions can be represented on the lattice as

$$\delta(p-q) = \frac{a}{2\pi}\sum_x e^{-ix(p-q)}, \qquad \frac{\delta_{xy}}{a} = \int_{BZ}\frac{\mathrm{d}p}{2\pi}\, e^{ip(x-y)} \tag{A.12}$$

For a matrix A the convention is

$$A(p,q) = a^2\sum_{x,y} A_{xy}e^{-i(px+qy)}, \qquad A_{xy} = \int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}q}{2\pi} A(p,q)e^{i(px+qy)} \tag{A.13}$$

Using these formulas it's easy to proof that, for any two operators A, B and functions f, g, the following convolution relations hold:

$$\mathrm{tr}\, A = \sum_x A_{xx} = \int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{1}{a}A(p,-p) \tag{A.14}$$

$$\begin{aligned}(AB)_{xy} &= \sum_z A_{xz}B_{zy} \\ &= \int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}q}{2\pi}\, e^{i(xp+yq)}\int_{BZ}\frac{\mathrm{d}s}{2\pi}\frac{1}{a}A(p,s)B(-s,q)\end{aligned} \tag{A.15}$$

$$Af(x) = \sum_y A_{xy}f(y) = \int_{BZ}\frac{\mathrm{d}p}{2\pi}e^{ipx}\int_{BZ}\frac{\mathrm{d}q}{2\pi}\frac{1}{a}A(p,q)f(-q) \tag{A.16}$$

$$\sum_x f(x)g(x) = \int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{1}{a}f(p)g(-p) \tag{A.17}$$

$$\sum_z A_{xz}f(z)B_{zy} = \int_{BZ}\frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}q}{2\pi}e^{i(px+qy)}\int_{BZ}\frac{\mathrm{d}r}{2\pi}\frac{\mathrm{d}t}{2\pi}\frac{1}{a}A(p,r)f(-r+t)B(-t,q) \tag{A.18}$$

and so on.

A.2 Berezin Integrals

To define a path integral for fermions it is customary to work with anticommuting ("Grassmannian") variables. This construction is standard and can be found in virtually every textbooks about QFT. We report here the main points for practicality and to establish conventions. Let θ_k, $\theta_k^\dagger$, $k = 1, \cdots, N$, be two (independent, in spite of the $\dagger$) sets of anticommuting symbols, that is

$$\{\theta_k, \theta_j\} = 0, \qquad \{\theta_k^\dagger, \theta_j^\dagger\} = 0 \qquad \{\theta_k, \theta_j^\dagger\} = 0 \tag{A.19}$$

An integral over these symbols is defined by the requests

$$\int d\theta_k \, 1 = 0, \qquad \int d\theta_k \, \theta_k = 1, \qquad \int d\theta_k^\dagger \, 1 = 0, \qquad \int d\theta_k^\dagger \, \theta_k^\dagger = 1 \tag{A.20}$$

This means that

$$\int d\theta_k \equiv \frac{\partial}{\partial \theta_k} \qquad \int d\theta_k^\dagger \equiv \frac{\partial}{\partial \theta_k^\dagger} \tag{A.21}$$

The product measures are simply

$$d\theta \equiv d\theta_1 \cdots d\theta_N, \quad d\theta^\dagger \equiv d\theta_N^\dagger \cdots d\theta_1^\dagger \quad \Longrightarrow \quad d\theta^\dagger d\theta \equiv \prod_{k=1}^N d\theta_k^\dagger d\theta_k \tag{A.22}$$

Using these definitions, it is easy two show that the following formula for Gaussian integrals holds:

$$\int d\theta^\dagger d\theta \, e^{-\theta^\dagger A \theta + \eta^\dagger \theta + \theta^\dagger \eta} = \det A \, e^{\eta^\dagger A^{-1} \eta} \tag{A.23}$$

A.3 Canonical coherent states

Let $\mathscr{F} = \bigotimes_{k=1}^N \mathscr{H}_k$ be a Fock space defined as a direct product of single-particle fermionic Hilbert spaces. The annihilation and creation operators obey canonical anticommutation relations

$$\left\{\hat{a}_k, \hat{a}_l\right\} = 0 \quad \left\{\hat{a}_k^\dagger, \hat{a}_l^\dagger\right\} = 0 \quad \left\{\hat{a}_k, \hat{a}_l^\dagger\right\} = \delta_{kl} \qquad k, l = 1, \cdots, N \tag{A.24}$$

For each sector, the vacuum state is defined by

$$\hat{a}_k \left|0\right\rangle_k = 0 \tag{A.25}$$

and therefore the vacuum state in $\mathscr{F}$ is

$$\left|0\right\rangle = \bigotimes_{k=1}^N \left|0\right\rangle_k \tag{A.26}$$

Applying the creation operators, we obtain

$$|k_1 \cdots k_p\rangle = \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p} |0\rangle \qquad p = 1, \cdots, N \tag{A.27}$$

These states span the entire Fock space: they form an orthonormal basis

$$\langle k_1 \cdots k_p | l_1 \cdots l_q\rangle = \delta_{pq} \sum_{\text{perm } \pi} (-)^\pi \delta_{k_1 \pi(l_1)} \cdots \delta_{k_p \pi(l_p)} \tag{A.28}$$

and the resolution of unity reads

$$\sum_{p=0}^{N} \frac{1}{p!} \sum_{k_1, \cdots, k_p} |k_1 \cdots k_p\rangle\langle k_1 \cdots k_p| = \mathbb{I} \tag{A.29}$$

The factor $1/p!$ takes account of the indistinguishability of the particles. A generic state $|\psi\rangle$ can be written as

$$|\psi\rangle = \psi(\hat{a}^\dagger) |0\rangle \qquad \psi(\hat{a}^\dagger) \equiv \sum_{p=0}^{N} \frac{1}{p!} \psi_{k_1 \cdots k_p} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p} \tag{A.30}$$

where the coefficients $\psi_{k_1 \cdots k_p}$ are totally antisymmetric. An arbitrary operator $\hat{A}$ can be written as

$$\hat{A} = \sum_{p,q} \frac{1}{p!q!} A_{k_1 \cdots k_p l_1 \cdots l_q} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p} \hat{a}_{l_q} \cdots \hat{a}_{l_1} \tag{A.31}$$

normal ordered.

The *canonical coherent states* are defined by

$$|\theta\rangle = e^{-\sum_k \theta_k \hat{a}^\dagger_k} |0\rangle = \prod_k e^{-\theta_k \hat{a}^\dagger_k} |0\rangle = \prod_k \left(1 - \theta_k \hat{a}^\dagger_k\right) |0\rangle \tag{A.32}$$

with θ_k Grassmannian variables such that

$$\theta_k \theta_l = -\theta_l \theta_k, \quad \theta^\dagger_k \theta_l = -\theta_l \theta^\dagger_k, \quad \theta_k \hat{a}_l = -\hat{a}_l \theta_k, \quad \theta_k \hat{a}^\dagger_l = -\hat{a}^\dagger_l \theta_k \tag{A.33}$$

(the variables θ_k and $\theta^\dagger_k$ are independent for the purpose of constructing the algebra). The state $|\theta\rangle$ is an eigenstate of $\hat{a}_k$, with eigenvalues θ_k:

$$\hat{a}_k |\theta\rangle = \theta_k |\theta\rangle \tag{A.34}$$

Indeed,

$$\begin{aligned} \hat{a}_k |\theta\rangle &= \hat{a}_k \prod_j \left(1 - \theta_j \hat{a}^\dagger_j\right) |0\rangle = \left[\prod_{j \neq k} \left(1 - \theta_j \hat{a}^\dagger_j\right)\right] \hat{a}_k \left(1 - \theta_k \hat{a}^\dagger_k\right) |0\rangle \\ &= \left[\prod_{j \neq k} \left(1 - \theta_j \hat{a}^\dagger_j\right)\right] \theta_k |0\rangle = \theta_k \left[\prod_j \left(1 - \theta_j \hat{a}^\dagger_j\right)\right] |0\rangle = \theta_k |\theta\rangle \end{aligned} \tag{A.35}$$

where, in the last step, we can reintroduce at no cost the factor $\left(1-\theta_k\hat{a}_k^\dagger\right)$ in the product because $(\theta_k)^2=0$. The inner product between two coherent states $|\theta\rangle$ and $|\eta\rangle$ is

$$\begin{aligned}\langle\theta|\eta\rangle &= \langle 0|\left(1-\hat{a}_1\theta_1^\dagger\right)\cdots\left(1-\hat{a}_N\theta_N^\dagger\right)\left(1-\eta_N\hat{a}_N^\dagger\right)\cdots\left(1-\eta_1\hat{a}_1^\dagger\right)|0\rangle \\ &= \prod_k\left(1+\theta_k^\dagger\eta_k\right)=\exp\left(\sum_k\theta_k^\dagger\eta_k\right)\equiv\exp\left(\theta^\dagger\eta\right)\end{aligned} \tag{A.36}$$

Using the formulas collected in section A.2 for the integrals over Grassmannian variables, we find the resolution of unity

$$\hat{\mathbb{I}}=\int d\theta^\dagger d\theta\,\frac{|\theta\rangle\langle\theta|}{\langle\theta|\theta\rangle} \tag{A.37}$$

Indeed, for a single variable ($N=1$),

$$\begin{aligned}\int d\theta^\dagger d\theta\,\frac{|\theta\rangle\langle\theta|}{\langle\theta|\theta\rangle} &= \int d\theta^\dagger d\theta\, e^{-\theta^\dagger\theta}\left(1-\theta\hat{a}^\dagger\right)|0\rangle\langle 0|\left(1-\hat{a}\theta^\dagger\right) \\ &= \int d\theta^\dagger d\theta\,\left(1-\theta^\dagger\theta\right)\left(1-\theta\hat{a}^\dagger\right)|0\rangle\langle 0|\left(1-\hat{a}\theta^\dagger\right) \\ &= \int d\theta^\dagger d\theta\,\left(-\theta^\dagger\theta\,|0\rangle\langle 0|+\theta\,|1\rangle\langle 1|\,\theta^\dagger\right) \\ &= \int d\theta^\dagger d\theta\,\theta\theta^\dagger\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)=|0\rangle\langle 0|+|1\rangle\langle 1|=\mathbb{I}\end{aligned} \tag{A.38}$$

(only the bilinear terms in θ, $\theta^\dagger$ survive in the Berezin integral), and so on for the entire Fock space. Using (A.37) a generic state can be written as

$$|\psi\rangle=\int d\theta^\dagger d\theta\, e^{-\theta^\dagger\theta}\psi(\theta^\dagger)\,|\theta\rangle \tag{A.39}$$

with

$$\psi(\theta^\dagger)\equiv\langle\theta|\psi\rangle=\sum_{p=0}^{N}\frac{1}{p!}\psi_{k_1\cdots k_p}\theta_{k_1}^\dagger\cdots\theta_{k_p}^\dagger \tag{A.40}$$

as it results from (A.30) and (A.34). Using (A.31), the expression for the representative of a generic operator follows:

$$A(\theta^\dagger,\eta)\equiv\langle\theta|\hat{A}|\eta\rangle=\exp\left(\theta^\dagger\eta\right)\sum_{p,q}\frac{1}{p!q!}A_{k_1\cdots k_p l_1\cdots l_q}\theta_{k_1}^\dagger\cdots\theta_{k_p}^\dagger\eta_{l_q}\cdots\eta_{l_1} \tag{A.41}$$

and so

$$A\psi(\theta^\dagger)\equiv\langle\theta|\hat{A}|\psi\rangle=\int d\eta^\dagger d\eta\, e^{-\eta^\dagger\eta}A(\theta^\dagger,\eta)\psi(\eta^\dagger) \tag{A.42}$$

$$AB(\theta^\dagger,\eta)\equiv\langle\theta|\hat{A}\hat{B}|\eta\rangle=\int d\eta'^\dagger d\eta'\, e^{-\eta'^\dagger\eta'}A(\theta^\dagger,\eta')B(\eta'^\dagger,\eta) \tag{A.43}$$

Suppose an operator $\hat{O}$ can be written in the special form

$$\hat{O} = \exp\Big(\sum_{k,l} \hat{a}_k^\dagger M_{kl} \hat{a}_l\Big) \tag{A.44}$$

Then, its representative is

$$O(\theta^\dagger, \eta) \equiv \langle\theta|\hat{O}|\eta\rangle = \exp\Big[\sum_{k,l} \theta_k^\dagger (e^M)_{kl} \eta_l\Big] \tag{A.45}$$

(see [31], Appendix C). Combining this formula with (A.43) it follows that, if two operators are in the form

$$\hat{O}_1 = \exp\Big(\sum_{k,l} \hat{a}_k^\dagger M_{kl} \hat{a}_l\Big) \qquad \hat{O}_2 = \exp\Big(\sum_{k,l} \hat{a}_k^\dagger N_{kl} \hat{a}_l\Big) \tag{A.46}$$

then

$$O_1 O_2(\theta^\dagger, \eta) = \exp\Big[\sum_{k,l} \theta_k^\dagger (e^M e^N)_{kl} \eta_l\Big] \tag{A.47}$$

Finally, the trace of an even operator $\hat{A}$ (such as it commutes with any Grassmannian variable, $\hat{A}\theta_k = \theta_k \hat{A}$) can be written as

$$\mathrm{tr}\, \hat{A} = \int \mathrm{d}\theta^\dagger \mathrm{d}\theta \, e^{-\theta^\dagger\theta} A(\theta^\dagger, -\theta) \tag{A.48}$$

If two kind of fermions (namely, particles and antiparticles) are admitted, the Fock space can be constructed from the set of canonical creation and annihilation operators

$$\big\{\hat{u}_i^\dagger, \hat{u}_j\big\} = \big\{\hat{v}_i^\dagger, \hat{v}_j\big\} = \delta_{ij}, \quad \big\{\hat{u}_i, \hat{u}_j\big\} = \big\{\hat{v}_i, \hat{v}_j\big\} = \big\{\hat{u}_i, \hat{v}_j\big\} = \big\{\hat{u}_i^\dagger, \hat{v}_j\big\} = 0 \tag{A.49}$$

where $u_i^\dagger$ and $v_i^\dagger$ create, respectively, a particle and an antiparticle in the state i. Canonical coherent states can be defined in the same way:

$$|\rho, \sigma\rangle = \exp\left(-\sum_k \rho_k \hat{u}_k^\dagger - \sum_k \sigma_k \hat{v}_k^\dagger\right)|0\rangle \tag{A.50}$$

and the resolution of unity reads as

$$\hat{\mathbb{I}} = \int \prod_k \mathrm{d}\rho_k^\dagger \mathrm{d}\rho_k \mathrm{d}\sigma_k^\dagger \mathrm{d}\sigma_k \, e^{-\sum_k \rho_k^\dagger \rho_k - \sum_k \sigma_k^\dagger \sigma_k} |\rho, \sigma\rangle\langle\rho, \sigma| \tag{A.51}$$

Appendix B

Fermion bilinear representation of the algebra of GL(n)

Let $\{ E^{(ij)} \mid i,j = 1,\cdots,n \}$ be a basis of $n \times n$ matrices of the algebra that generates the general linear group GL(n). A suitable choice is

$$[E^{(ij)}]_{kl} \equiv E^{(ij)}_{kl} = \delta_{ik}\delta_{jl} \tag{B.1}$$

so that

$$\begin{aligned} \left[E^{(ij)}, E^{(kl)}\right]_{ab} &= \sum_{c=1}^{n} [\delta_{ia}\delta_{jc}\delta_{kc}\delta_{lb} - \delta_{ka}\delta_{lc}\delta_{ic}\delta_{jb}] \\ &= \delta_{jk}E^{(il)}_{ab} - \delta_{il}E^{(kj)}_{ab} \end{aligned} \tag{B.2}$$

as it should be. Indeed, a generic matrix can then be written as

$$A_{ij} = \sum_{k,l} A_{kl} E^{(kl)}_{ij} \tag{B.3}$$

and

$$[A,B]_{ij} = \sum_{k,l} [A,B]_{kl} E^{(kl)}_{ij} \tag{B.4}$$

Let $\hat{\psi}_i$, $\hat{\psi}^\dagger_i$ be n couples of fermion canonical operators, such that

$$\{\hat{\psi}_i, \hat{\psi}_j\} = \{\hat{\psi}^\dagger_i, \hat{\psi}^\dagger_j\} = 0 \qquad \{\hat{\psi}^\dagger_i, \hat{\psi}_j\} = \delta_{ij} \tag{B.5}$$

Fermion bilinear operators are defined by

$$L_{ij} = \hat{\psi}^\dagger_i \hat{\psi}_j \tag{B.6}$$

They satisfy the same commutation relations of the $E^{(ij)}$:

$$\begin{aligned} [L_{ij}, L_{kl}] &= \hat{\psi}^\dagger_i \hat{\psi}_j \hat{\psi}^\dagger_k \hat{\psi}_l - \hat{\psi}^\dagger_k \hat{\psi}_l \hat{\psi}^\dagger_i \hat{\psi}_j \\ &= \delta_{jk} \hat{\psi}^\dagger_i \hat{\psi}_l - \hat{\psi}^\dagger_i \hat{\psi}^\dagger_k \hat{\psi}_j \hat{\psi}_l - \delta_{il} \hat{\psi}^\dagger_k \hat{\psi}_j + \hat{\psi}^\dagger_k \hat{\psi}^\dagger_i \hat{\psi}_l \hat{\psi}_j \\ &= \delta_{jk} L_{il} - \delta_{il} L_{kj} \end{aligned} \tag{B.7}$$

As a consequence, they provide a representation of GL(n):

$$A = \sum_{k,l} A_{kl} E^{(kl)} = \sum_{k,l} A_{kl} L_{kl} \tag{B.8}$$

A generic product of matrices in GL(n) can be expressed as

$$\exp[A] \exp[B] = \exp[f(A, B)] \tag{B.9}$$

where A and B are the corresponding elements of the algebra and

$$f(A, B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}\big([A, [A, B]] - [B, [A, B]]\big) + \cdots \tag{B.10}$$

is given by the *Baker-Campbell-Hausdorff* formula. Considering that $f(A, B)$ can be written in terms of the matrices A, B and their commutators, using (B.4) and (B.8) it follows easily that also

$$\begin{aligned} & \exp[AE] \exp[BE] = \exp[f(A, B)E] \\ \Longrightarrow \quad & \exp[AL] \exp[BL] = \exp[f(A, B)L] \end{aligned} \tag{B.11}$$

Appendix C

Saddle point for fermion doublers

In a free theory with Wilson fermions in d space-time dimensions the matrices $\mathcal{B}$, $\mathcal{M}$ and $\mathcal{N}$ are given by

$$\mathcal{B}(q) = 1 - 2\kappa \sum_{j=1}^{d-1} \cos aq_j = 1 - 2\kappa(d-1) + \kappa \sum_{j=1}^{d-1} (\widehat{aq}_j)^2 \tag{C.1}$$

$$\mathcal{N}(q) = -\frac{2\kappa}{\mathcal{B}} \sum_{j=1}^{d-1} \overline{aq}_j \sigma_j \tag{C.2}$$

$$e^{2\mathcal{M}}(q) = \frac{\mathcal{B}}{2\kappa}(q) = 1 + am + \frac{1}{2} \sum_{j=1}^{d-1} (\widehat{aq}_j)^2 \tag{C.3}$$

where we used the notations

$$\begin{aligned} \widehat{aq}_\mu &= 2 \sin \frac{aq_\mu}{2} \\ \overline{aq}_\mu &= \sin ap_\mu \end{aligned} \tag{C.4}$$

and

$$i\sigma_j = P_- \gamma_j P_+ \tag{C.5}$$

In a free theory the equation (2.47) naturally holds, so

$$\bar{\mathcal{F}} = \frac{\mathcal{N}}{2\mathcal{N}^2} \left(-\mathcal{Y} \pm \sqrt{\mathcal{Y}^2 + 4\mathcal{N}^2} \right) = \frac{\mathcal{N}}{\sqrt{\mathcal{N}^2}} \left(-\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}} \pm \sqrt{\left(\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}} \right)^2 + 1} \right) \tag{C.6}$$

Using the property of the Pauli σ matrices

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I} \tag{C.7}$$

we have

$$\mathcal{N}^2(q) = \left(\frac{2\kappa}{\mathcal{B}}\right)^2 \sum_{j=1}^{d-1} \left(\overline{aq}_j\right)^2 \tag{C.8}$$

and so

$$\frac{\mathcal{N}}{\sqrt{\mathcal{N}^2}}(q) = -\frac{\sum_j \overline{aq}_j \sigma_j}{\sqrt{\sum_j \left(\overline{aq}_j\right)^2}} \tag{C.9}$$

From the definition (2.48) we get

$$\mathcal{Y}(q) = 1 - \left(\frac{2\kappa}{\mathcal{B}}\right)^2 \left[1 + \sum_j \left(\overline{aq}_j\right)^2\right] \tag{C.10}$$

$$\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}} = \frac{1 - \left(\frac{2\kappa}{\mathcal{B}}\right)^2 \left[1 + \sum_j \left(\overline{aq}_j\right)^2\right]}{2\frac{2\kappa}{\mathcal{B}}\sqrt{\sum_j \left(\overline{aq}_j\right)^2}} \tag{C.11}$$

$$\left(\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}}\right)^2 + 1 = \frac{\left[1 - \left(\frac{2\kappa}{\mathcal{B}}\right)^2\right]^2 + \left(\frac{2\kappa}{\mathcal{B}}\right)^4 \left[\sum_j \left(\overline{aq}_j\right)^2\right]^2 + 2\left[1 + \left(\frac{2\kappa}{\mathcal{B}}\right)^2\right] \left(\frac{2\kappa}{\mathcal{B}}\right)^2 \sum_j \left(\overline{aq}_j\right)^2}{4\left(\frac{2\kappa}{\mathcal{B}}\right)^2 \sum_j \left(\overline{aq}_j\right)^2} \tag{C.12}$$

Suppose now that, for any value of the lattice spacing a, $|ap_j| \ll 1$ for any j. This means that, *before* the continuum limit, we are close to the physical pole $p_j = 0$, the one we would like to see also in the continuum theory. Then

$$\overline{aq}_j = aq_j + O(a^3) \tag{C.13}$$

$$\frac{2\kappa}{\mathcal{B}} = 1 - ma + O(a^2) \tag{C.14}$$

$$\frac{\mathcal{N}}{\sqrt{\mathcal{N}^2}} = -\frac{\sum_j q_j \sigma_j}{\sqrt{\sum_j q_j^2}} + O(a^2) = -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{|\mathbf{q}|} + O(a^2) \tag{C.15}$$

$$\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}} = \frac{1 - (1 - 2ma)}{2(1 - ma)a|\mathbf{q}|} + O(a) = \frac{m}{|\mathbf{q}|} + O(a) \tag{C.16}$$

and so

$$\begin{aligned}
\bar{\mathcal{F}} &= -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{|\mathbf{q}|} \left(-\frac{m}{|\mathbf{q}|} + \sqrt{1 + \frac{m^2}{\mathbf{q}^2}}\right) \\
&= -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{\mathbf{q}^2} \left(-m + E_q\right) \\
&= -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{\mathbf{q}^2} \frac{E_q^2 - m^2}{E_q + m} \\
&= -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{E_q + m}
\end{aligned} \tag{C.17}$$

That's exactly what we expected for the Foldy-Wouthuysen transformation that diagonalize the free Dirac Hamiltonian (see [8]).

Let's move near a spurious pole: there is now at least one value for j for which

$$ap_j = \pi - a\epsilon_j \qquad \text{with } |a\epsilon_j| \ll 1 \tag{C.18}$$

Obviously

$$\overline{ap}_j = \sin(\pi - a\epsilon_j) = \sin(a\epsilon_j) = \overline{a\epsilon}_j \tag{C.19}$$

but

$$\widehat{ap}_j = 2\sin\left(\frac{\pi}{2} - \frac{a\epsilon_j}{2}\right) = 2\cos\frac{a\epsilon_j}{2} \simeq 2\left[1 - \frac{1}{2}\left(\frac{a\epsilon_j}{2}\right)^2\right] \tag{C.20}$$

Introducing the counting variable

$$n_j = \begin{cases} 1 & \text{if } ap_j \simeq \pi \\ 0 & \text{if } ap_j \simeq 0 \end{cases} \tag{C.21}$$

then

$$\frac{\mathcal{B}}{2\kappa} = 1 + ma + \sum_j n_j + O(a^2) \tag{C.22}$$

$$\frac{2\kappa}{\mathcal{B}} = \frac{1}{1+\sum_j n_j} - \frac{ma}{\left(1+\sum_j n_j\right)^2} + O(a^2) \tag{C.23}$$

$$\frac{\mathcal{Y}}{2\sqrt{\mathcal{N}^2}} = \frac{1-\left(1+\sum_j n_j\right)^{-2}}{2\left(1+\sum_j n_j\right)^{-1} a|\mathbf{q}|} + O(a) = \frac{M}{|\mathbf{q}|} + O(a) \tag{C.24}$$

with

$$M = \frac{1-\left(1+\sum_j n_j\right)^{-2}}{2\left(1+\sum_j n_j\right)^{-1} a} = \frac{\left(1+\sum_j n_j\right)^2 - 1}{2\left(1+\sum_j n_j\right) a} \tag{C.25}$$

This "mass" factor has the same role as m in the previous calculation, so

$$\bar{\mathcal{F}} = -\frac{\mathbf{q}\cdot\boldsymbol{\sigma}}{E_q + M} \tag{C.26}$$

However, this time

$$M \sim \frac{1}{a} \tag{C.27}$$

and in the continuum limit

$$\bar{\mathcal{F}} \to 0 \tag{C.28}$$

In the Wilson formulation, the spurious poles take a divergent mass term that, when $a \to 0$, makes them decouple from the physical phenomena!

Appendix D

Numerical methods

The numerical methods we use to evaluate the solution of the saddle point equation (3.41) are well explained in reference [16]; we report and elaborate them here for completeness. The first step is to obtain a compact interval of integration making the change of variable

$$x = \arctan p \qquad \Longrightarrow \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{D.1}$$

so that the functional of θ in (3.41) becomes

$$f_x(\theta) = -m \sin\theta(x) + \tan x \cos\theta(x) - \frac{\gamma}{2} \int_{-\pi/2}^{\pi/2} \mathrm{d}x' \, \frac{\sin[\theta(x) - \theta(x')]}{\cos^2(x')(\tan x - \tan x')^2} \tag{D.2}$$

Then, to evaluate the integral we can *discretize* the interval in N evenly spaced sample points, so that

$$x \longrightarrow x_n = \frac{n\pi}{N} - \frac{\pi}{2} \qquad n = 0, 1, \cdots, N \tag{D.3}$$

Between two adjacent points there is the interval

$$\mathrm{d}x \longrightarrow \Delta x = x_{n+1} - x_n = \frac{\pi}{N} \tag{D.4}$$

In this way, the functional becomes the ordinary function

$$f_n(\boldsymbol{\theta}) = -m \sin\theta_n + \tan x_n \cos\theta_n - \frac{\gamma \Delta x}{2} \sum_{\substack{n'=0 \\ n' \neq n}}^{N} \frac{\sin[\theta_n - \theta_{n'}]}{\cos^2(x_{n'})(\tan x_n - \tan x_{n'})^2} \tag{D.5}$$

where $\boldsymbol{\theta}$ is a $(N{+}1)$-dimensional vector with components $\theta_n \equiv \theta(x_n)$. The sum is performed to avoid the point where the denominator is not defined. Therefore, the integral equation (3.41) becomes the non linear system

$$\mathbf{f}(\boldsymbol{\theta}) = 0 \tag{D.6}$$

where $\mathbf{f}$ is the vector with components f_n. To evaluate it, we use the Newton method: starting from an arbitrary "solution" $\bar{\boldsymbol{\theta}}$, an improvement $\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}$ is obtained solving, with a standard numerical procedure (we adopt an LU decomposition followed by the Gaussian elimination), the linear system for the increment

$$J(\bar{\boldsymbol{\theta}})\delta\boldsymbol{\theta} = -\mathbf{f}(\boldsymbol{\theta}) \tag{D.7}$$

This equation can be derived from

$$\mathbf{f}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) = \mathbf{f}(\bar{\boldsymbol{\theta}}) + J(\bar{\boldsymbol{\theta}})\delta\boldsymbol{\theta} + O\big(\delta\boldsymbol{\theta}^2\big) \tag{D.8}$$

imposing that, in the incremented point, the relation $\mathbf{f}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) = 0$ holds. From this last equation we can see that

$$[J]_{mn} = \frac{\partial f_m}{\partial \theta_n} \tag{D.9}$$

is the Jacobian matrix. Iterating the procedure until $\delta\boldsymbol{\theta}$ is less than some ϵ, we get the solution to the target precision. Choosing properly the starting point $\bar{\boldsymbol{\theta}}$ for the iteration, the convergence is not an issue: using the value from the free theory $\gamma = 0$ we obtain the desired precision in few iterations.

Appendix E

Wilson fermions and chiral anomaly

E.1 Fermion doubling and Dirac-Wilson action

In a continuum d-dimensional space-time, the action for an Euclidean field theory of free fermions is

$$S_0 = \int \mathrm{d}^d x \, \bar{\psi}(x)(\not{\partial} + m)\psi(x) \tag{E.1}$$

The fermion fields ψ, $\bar{\psi}$ belong to the fundamental representation of the gauge (colour) group $\mathrm{SU}(N_c)$. This action can be discretized on a hypercubic lattice with spacing a (in order, for example, to regularize the interacting theory): just replace the derivative in (E.1) with appropriate finite-differences of fields defined on nearby lattice sites. Using (A.4), a possible discretized version is

$$S_0 = a^d \sum_{x \in (a\mathbb{Z})^d} \left[\bar{\psi}(x) \sum_{\mu=1}^{d} \gamma_\mu \frac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a} + m\bar{\psi}(x)\psi(x) \right] \tag{E.2}$$

Diagonalizing this expression in Fourier space we get, for the quadratic operator in it,

$$\mathcal{D}^{-1}(p) = m + i \sum_\mu \gamma_\mu \frac{\sin a p_\mu}{a} \tag{E.3}$$

where, because of the discrete nature of the variables x, the conjugate momenta take value in the first Brillouin zone

$$-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}; \qquad \mu = 1, \cdots, d \tag{E.4}$$

The quark propagator is then

$$\mathcal{D}(p) = \frac{m - i\sum_\mu \gamma_\mu \frac{\sin ap_\mu}{a}}{m^2 + \left(\sum_\mu \frac{\sin ap_\mu}{a}\right)^2} \tag{E.5}$$

which, in continuum limit, correctly tends to

$$\mathcal{D}(p) \quad \underset{a\to 0}{\longrightarrow} \quad \frac{m - i\not{p}}{m^2 + p^2} \tag{E.6}$$

In addition, the (E.2) action has the same symmetry properties of the action (E.1) (if we exclude, obviously, the space-time symmetries, replaced by the symmetries of the hypercubic lattice). Indeed, it remains unchanged under the global transformation of the fields

$$\psi \longrightarrow e^{-i\alpha\Xi}\psi \qquad \bar{\psi} \longrightarrow \bar{\psi}e^{i\alpha\Xi} \tag{E.7}$$

with Ξ an Hermitian matrix in gauge indices, as well as, in the zero-mass limit, under the *chiral rotation*

$$\psi \longrightarrow e^{-i\alpha\gamma_5}\psi \qquad \bar{\psi} \longrightarrow \bar{\psi}e^{-i\alpha\gamma_5} \tag{E.8}$$

(and also under chiral transformations for which γ_5 at the exponent multiplies an Hermitian matrix in gauge indices).

However, the propagator in (E.5) has a serious problem: its poles provide the mass-shell conditions and so the particle content of the theory, but, in the zero-mass limit, the equation

$$\sum_\mu \frac{\sin ap_\mu}{a} = 0 \tag{E.9}$$

admits 2^d different solutions for p in the first Brillouin zone. Each component p_μ, con $\mu = 1, \cdots, d$, can be 0 or π/a, then the solutions are

$$\Pi^A = \{(0, 0, \cdots, 0), (\pi/a, 0, \cdots, 0), \cdots, (\pi/a, \pi/a, \cdots, \pi/a)\} \tag{E.10}$$

with $A = 1, \cdots, 2^d$. This means that, discretizing in a naive way a theory of a single free fermion, we get a theory with 2^d different particles in the spectrum!

A possible way out to evade the doubling problem, proposed by K. Wilson, consists in adding to the action a term of the type

$$S_0^{(r)} = -a^d \sum_{x\in(a\mathbb{Z})^d} \left[\bar{\psi}(x) \sum_{\mu=1}^{d} r\, \frac{\psi(x + a\hat{\mu}) + \psi(x - a\hat{\mu}) - 2\psi(x)}{2a}\right] \tag{E.11}$$

As we can see from (A.6), this term is proportional to $a\partial^2\psi$ and so it tends to 0 in the continuum limit, leaving the action unchanged. We obtain

$$\begin{aligned} S_{0W} = a^d \sum_{x\in(a\mathbb{Z})^d} \Bigg\{ \bar\psi(x) \sum_{\mu=1}^{d} \Bigg[\gamma_\mu \frac{\psi(x+a\hat\mu)-\psi(x-a\hat\mu)}{2a} \\ - r\,\frac{\psi(x+a\hat\mu)+\psi(x-a\hat\mu)}{2a} \Bigg] + \left(m+\frac{rd}{a}\right)\bar\psi(x)\psi(x) \Bigg\} \end{aligned} \tag{E.12}$$

From this the propagator

$$\begin{aligned} \mathcal{D}_W^{-1}(p) &= \left(m+\frac{rd}{a}\right) - r\sum_\mu \frac{\cos ap_\mu}{a} + i\sum_\mu \gamma_\mu \frac{\sin ap_\mu}{a} \\ \Longrightarrow\ \mathcal{D}_W(p) &= \frac{M(p) - i\sum_\mu \gamma_\mu \frac{\sin ap_\mu}{a}}{M(p)^2 + \left(\sum_\mu \frac{\sin ap_\mu}{a}\right)^2} \end{aligned} \tag{E.13}$$

follows, with

$$M(p) = \left(m+\frac{rd}{a}\right) - r\sum_\mu \frac{\cos ap_\mu}{a} = m - \frac{r}{a}\sum_\mu (\cos ap_\mu - 1) \tag{E.14}$$

If one (or more) component of p is π/a, $M(p)$ diverges for $a \to 0$: the Wilson term gives an infinite mass to the fermion doublers in the continuum limit, so that they are produced in no physical process, even at perturbative level. Grouping terms,

$$\begin{aligned} S_{0W} = a^d \sum_{x\in(a\mathbb{Z})^d} \Bigg\{ \left(m+\frac{rd}{a}\right)\bar\psi(x)\psi(x) \\ - \frac{1}{a}\bar\psi(x)\sum_{\mu=1}^{d}\left[\frac{r-\gamma_\mu}{2}\psi(x+a\hat\mu) + \frac{r+\gamma_\mu}{2}\psi(x-a\hat\mu)\right] \Bigg\} \end{aligned} \tag{E.15}$$

The role of the operators $(r\pm\gamma_\mu)/2$ is particularly clear for $r=1$: they are *projectors* in spin space.

To make this action invariant under the gauge transformation

$$\psi(x) \to \Omega(x)\psi(x); \qquad \bar\psi(x) \to \bar\psi(x)\Omega^\dagger(x) \tag{E.16}$$

we have to insert a parallel transporter along lattice links that transforms as

$$U(x, x+a\hat\mu) \to \Omega(x)U(x, x+a\hat\mu)\Omega^\dagger(x+a\hat\mu) \tag{E.17}$$

From

$$\begin{aligned} &U^\dagger(x, x+a\hat{\mu}) \to \Omega(x+a\hat{\mu})U^\dagger(x, x+a\hat{\mu})\Omega^\dagger(x) \\ &\implies U^\dagger(x, x+a\hat{\mu}) = U(x+a\hat{\mu}, x) \end{aligned} \tag{E.18}$$

and imposing the lattice translational invariance, we get

$$\begin{aligned} \sum_{x\in(a\mathbb{Z})^d} \bar{\psi}(x)U(x, x-a\hat{\mu})\psi(x-a\hat{\mu}) &= \sum_{x\in(a\mathbb{Z})^d} \bar{\psi}(x+a\hat{\mu})U(x+a\hat{\mu}, x)\psi(x) \\ &= \sum_{x\in(a\mathbb{Z})^d} \bar{\psi}(x+a\hat{\mu})U^\dagger(x, x+a\hat{\mu})\psi(x) \end{aligned} \tag{E.19}$$

We will use the shorthand

$$U_\mu(x) \equiv U(x, x+a\hat{\mu}) \tag{E.20}$$

Introducing the anti-Hermitian, algebra valued, gauge field[1]

$$A_\mu(x) = ig\sum_{a=1}^{N^2-1} A_\mu^a \Theta^a \tag{E.21}$$

we have

$$U_\mu(x) = e^{aA_\mu(x)} \tag{E.22}$$

The field strength is given by

$$F_{\mu\nu}(x) = \partial_\mu^{(+)} A_\nu(x) - \partial_\nu^{(+)} A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{E.23}$$

In this way we find the *Dirac-Wilson action*

$$\begin{aligned} S_W = a^d \sum_{x\in(a\mathbb{Z})^d} \Bigg\{ &\left(m + \frac{rd}{a}\right)\bar{\psi}(x)\psi(x) \\ - \sum_{\mu=1}^{d} &\left[\bar{\psi}(x)\frac{r-\gamma_\mu}{2a}U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})\frac{r+\gamma_\mu}{2a}U_\mu^\dagger(x)\psi(x)\right]\Bigg\} \end{aligned} \tag{E.24}$$

E.2 Axial current on the lattice

The Dirac-Wilson action, insofar as it allows to eliminate duplicates from the spectrum and thus to obtain the correct continuum theory, explicitly breaks the invariance under chiral rotation, even in the zero-mass limit: the contributions proportional to the parameter r do not contain any γ matrix

[1] For our present convenience, in this chapter we use a convention different from (1.11).

and thus do not allow, exploiting the anti-commuting property with γ_5, to simplify the transformations of ψ and $\bar{\psi}$ in the bilinear. This is a consequence of the more general Nielsen-Ninomiya theorem, which denies the possibility of finding a theory, satisfying very natural physical demands, at the same time chirally symmetric and without duplicates: although the transformation (E.8) leaves the classical action unchanged, the corresponding conservation law is broken by quantum correction. It is expected that the terms that explicitly break the chiral symmetry are responsible for this *anomaly*.

Consider the partition function

$$\mathcal{Z} = \int \left[\mathrm{d}\psi\, \mathrm{d}\bar{\psi}\right] e^{-S_W[\psi,\bar{\psi},U]} \tag{E.25}$$

with

$$\left[\mathrm{d}\psi\, \mathrm{d}\bar{\psi}\right] = \prod_{x\in(a\mathbb{Z})^d} \mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x) \tag{E.26}$$

The local version of the chiral rotation (E.8) consists in the change of variables

$$\begin{aligned} \psi(x) &\longrightarrow \psi'(x) = e^{-i\alpha(x)\gamma_5}\psi(x) \simeq (1 - i\alpha(x)\gamma_5)\psi(x) \\ \bar{\psi}(x) &\longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\alpha(x)\gamma_5} \simeq \bar{\psi}(x)(1 - i\alpha(x)\gamma_5) \end{aligned} \tag{E.27}$$

in the partition function integral. As the integration variables are “mute”, it must be

$$\int \left[\mathrm{d}\psi\, \mathrm{d}\bar{\psi}\right] e^{-S_W[\psi,\bar{\psi},U]} = \int \left[\mathrm{d}\psi'\, \mathrm{d}\bar{\psi}'\right] e^{-S_W[\psi',\bar{\psi}',U]} \tag{E.28}$$

On the lattice the integration measure is well defined, since it is a discrete product, so the Jacobian of the transformation (E.27) equals to 1 (since $\mathrm{tr}\,\gamma_5 = 0$), and so

$$\left[\mathrm{d}\psi\, \mathrm{d}\bar{\psi}\right] = \left[\mathrm{d}\psi'\, \mathrm{d}\bar{\psi}'\right] \tag{E.29}$$

The variation of the integral is due exclusively to the variation of the action:

- the m term varies accordingly

$$\delta_\alpha S_W^{(m)} = a^d \sum_{x\in(a\mathbb{Z})^d} \left[-i\alpha(x)\right] 2m\bar{\psi}(x)\gamma_5\psi(x) \tag{E.30}$$

- the r one accordingly

$$\begin{aligned} \delta_\alpha S_W^{(r)} = a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} \left(-\frac{r}{2a}\right) \big[\\ - i\alpha(x)\bar{\psi}(x)\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) - i\alpha(x+a\hat{\mu})\bar{\psi}(x)U_\mu(x)\gamma_5\psi(x+a\hat{\mu}) \\ - i\alpha(x+a\hat{\mu})\bar{\psi}(x+a\hat{\mu})\gamma_5 U_\mu^\dagger(x)\psi(x) - i\alpha(x)\bar{\psi}(x+a\hat{\mu})U_\mu^\dagger(x)\gamma_5\psi(x) \\ + 4i\alpha(x)\bar{\psi}(x)\gamma_5\psi(x)\big] \end{aligned}$$

Translating $x \to x - a\hat{\mu}$ in the second e third addends we can make everything proportional to $\alpha(x)$, so

$$\delta_\alpha S_W^{(r)} = a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} [-i\alpha(x)] \left(-\frac{r}{2a}\right) [\\ \bar{\psi}(x)\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x-a\hat{\mu})U_\mu(x-a\hat{\mu})\gamma_5\psi(x) \\ + \bar{\psi}(x)\gamma_5 U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})U_\mu^\dagger(x)\gamma_5\psi(x) \\ - 4\bar{\psi}(x)\gamma_5\psi(x)]$$

In short,

$$\delta_\alpha S_W^{(r)} = a^d \sum_{x\in(a\mathbb{Z})^d} [-\alpha(x)] X_r(x) \tag{E.31}$$

with

$$X_r(x) \equiv \\ -\frac{r}{2a}\sum_{\mu=1}^{d} \bar{\psi}(x) i\gamma_5 \left[U_\mu(x)\psi(x+a\hat{\mu}) + U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) - 2\psi(x)\right] \\ -\frac{r}{2a}\sum_{\mu=1}^{d} \left[\bar{\psi}(x-a\hat{\mu})U_\mu(x-a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})U_\mu^\dagger(x) - 2\bar{\psi}(x)\right] i\gamma_5\psi(x) \tag{E.32}$$

- finally, the term with the γ matrices varies according to

$$\delta_\alpha S_W^{(\gamma)} = a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} \frac{1}{2a} [- i\alpha(x)\bar{\psi}(x)\gamma_5\gamma_\mu U_\mu(x)\psi(x+a\hat{\mu}) \\ - i\alpha(x+a\hat{\mu})\bar{\psi}(x)\gamma_\mu\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) \\ + i\alpha(x+a\hat{\mu})\bar{\psi}(x+a\hat{\mu})\gamma_5\gamma_\mu U^\dagger(x)\psi(x) \\ + i\alpha(x)\bar{\psi}(x+a\hat{\mu})\gamma_\mu\gamma_5 U^\dagger(x)\psi(x)]$$

Translating where necessary and using $\{\gamma_\mu, \gamma_5\} = 0$,

$$\delta_\alpha S_W^{(\gamma)} = a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} \frac{1}{2a} [-i\alpha(x)][- \bar{\psi}(x)\gamma_\mu\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) \\ + \bar{\psi}(x-a\hat{\mu})\gamma_\mu\gamma_5 U_\mu(x-a\hat{\mu})\psi(x) \\ + \bar{\psi}(x)\gamma_\mu\gamma_5 U^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) \\ - \bar{\psi}(x+a\hat{\mu})\gamma_\mu\gamma_5 U^\dagger(x)\psi(x)]$$

and so

$$\begin{aligned}\delta_\alpha S_W^{(\gamma)} &= -a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} [-\alpha(x)] \frac{J_\mu^5(x) - J_\mu^5(x-a\hat{\mu})}{a} \\ &= -a^d \sum_{x\in(a\mathbb{Z})^d} \sum_{\mu=1}^{d} [-\alpha(x)] \nabla_\mu^{(-)} J_\mu^5(x)\end{aligned} \tag{E.33}$$

with

$$J_\mu^5(x) = \frac{1}{2}\left[\bar{\psi}(x) i\gamma_\mu\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu}) i\gamma_\mu\gamma_5 U_\mu^\dagger(x)\psi(x)\right] \tag{E.34}$$

Ultimately, imposing the invariance (E.28) of the partition function, and thus that the linear terms in $\alpha(x)$ in the RHS are null, we found

$$\langle \sum_{\mu=1}^{d} \nabla_\mu^- J_\mu^5(x)\rangle = 2mi\,\langle \bar{\psi}(x)\gamma_5\psi(x)\rangle + \langle X_r(x)\rangle \tag{E.35}$$

E.3 Continuum limit

Introduction

To prove that equation (E.35) reproduce the well known ABJ anomaly in the continuum limit we have to show that[2]

$$\langle X_r(x)\rangle \underset{a\to 0}{\longrightarrow} \frac{i}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma} \tag{E.36}$$

The limit procedure is not a trivial operation, since it requires to expand $\langle X_r(x)\rangle$ in a formal series of operators, that must be truncated with a *power-counting* argument. We took the following discussion from [19], with the proper clarifications in [30]. We found the idea to apply the power-counting theorem [28] (well explained in [21]) to our case in [1]. An alternative discussion, mathematically more consistent, is in [13].

Kerler series

The first step is to evaluate explicitly the expression on the lattice. To find the expectation values in (E.35) we use the generating functional formalism. To this end, we rewrite the action (E.24) as a quadratic form:

$$S_W = a^4 \sum_{x\in(a\mathbb{Z})^4} \sum_{y\in(a\mathbb{Z})^4} \bar{\psi}(x)\big(\not{D} - W + M\big)(x,y)\,\psi(y) \tag{E.37}$$

[2] In the sequel we take $d=4$, as QCD_4 is the most natural setting to study this problem.

where

$$
\begin{aligned}
\not{D}(x,y) &= \sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(x)\delta_{x+a\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x,y+a\hat{\mu}}}{2a} = \sum_{\mu=1}^{4} \gamma_\mu D_\mu(x,y) \\
W(x,y) &= r\sum_{\mu=1}^{4} \frac{U_\mu(x)\delta_{x+a\hat{\mu},y} + U_\mu^\dagger(y)\delta_{x,y+a\hat{\mu}} - 2\delta_{x,y}}{2a} = \sum_{\mu=1}^{4} W_\mu(x,y) \\
M(x,y) &= m\delta_{x,y}
\end{aligned}
\tag{E.38}
$$

The moment-generating function is

$$
Z[\eta,\bar{\eta}] = \int \left[\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\right] \exp\left[-(\bar{\psi},(\not{D} - W + M)\psi) + (\bar{\eta},\psi) + (\bar{\psi},\eta)\right] \tag{E.39}
$$

where we used the notation

$$
(A,B) = a^4 \sum_{x\in(a\mathbb{Z})^4} A(x)B(x) \qquad (MC)(x) = \sum_{y\in(a\mathbb{Z})^4} M(x,y)C(y) \tag{E.40}
$$

Evaluating the Gaussian integral we get

$$
Z[\eta,\bar{\eta}] = Z\exp\left[(\bar{\eta},(\not{D} - W + M)^{-1}\eta)\right] \tag{E.41}
$$

with $Z = Z[0,0]$ given by (E.25). The correlation functions can be obtained deriving this expression with respect to the sources. For example, the two-points function is

$$
\begin{aligned}
[G(x_1,x_2)]_{\alpha i}{}^{\beta j} &\equiv a^4 \langle \psi_{\alpha i}(x_1)\bar{\psi}^{\beta j}(x_2)\rangle \\
&= \frac{a^{-4}}{Z}\left(-\frac{\delta}{\delta\bar{\eta}^{\alpha i}(x_1)}\right)\left(\frac{\delta}{\delta\eta_{\beta j}(x_2)}\right) Z[\eta,\bar{\eta}]\bigg|_{\substack{\eta=0\\ \bar{\eta}=0}} \\
&= \{[(\not{D} - W + M)^{-1}](x_1,x_2)\}_{\alpha i}{}^{\beta j}
\end{aligned}
\tag{E.42}
$$

with α, β spin indices, i, j colour ones. A typical term in $\langle X_r(x)\rangle$ is

$$
\begin{aligned}
-\frac{r}{2a}\langle\bar{\psi}(x)i\gamma_5 U_\mu(x)\psi(x+a\hat{\mu})\rangle &= -\frac{r}{2a}\langle\bar{\psi}^{\alpha i}(x)(i\gamma_5)_\alpha{}^\beta U_\mu(x)_i{}^j\psi_{\beta j}(x+a\hat{\mu})\rangle \\
&= \frac{r}{2a}(i\gamma_5)_\alpha{}^\beta\langle\psi_{\beta j}(x+a\hat{\mu})\bar{\psi}^{\alpha i}(x)\rangle\, U_\mu(x)_i{}^j \\
&= \frac{r}{2a}(i\gamma_5)_\alpha{}^\beta[G(x+a\hat{\mu},x)]_{\beta j}{}^{\alpha i}U_\mu(x)_i{}^j\frac{1}{a^4} \\
&= \frac{r}{2a}\,\mathrm{tr}\, i\gamma_5 G(x+a\hat{\mu},x)U_\mu(x)\frac{1}{a^4}
\end{aligned}
$$

and so

$$
\begin{aligned}
&a^4 \langle X_r(x)\rangle \\
&= \frac{r}{2a} \operatorname{tr} \sum_{\mu=1}^{4} i\gamma_5 \big[G(x+a\hat{\mu},x)U_\mu(x) + G(x-a\hat{\mu},x)U_\mu^\dagger(x-a\hat{\mu}) - 2G(x,x) + \\
&\qquad\qquad G(x,x-a\hat{\mu})U_\mu(x-a\hat{\mu}) + G(x,x+a\hat{\mu})U_\mu^\dagger(x) - 2G(x,x)\big]
\end{aligned}
$$

Using

$$
\begin{aligned}
&(GW)(x,y) \\
&\quad= \sum_{z\in(a\mathbb{Z})^4} G(x,z)W(z,y) \\
&\quad= \sum_{z\in(a\mathbb{Z})^4} G(x,z)\frac{r}{2a}\sum_{\mu=1}^{4}[U_\mu(z)\delta_{z+a\hat{\mu},y} + U_\mu^\dagger(y)\delta_{z,y+a\hat{\mu}} - 2\delta_{z,y}] \\
&\quad= \frac{r}{2a}\sum_{\mu=1}^{4}[G(x,y-a\hat{\mu})U_\mu(y-a\hat{\mu}) + G(x,y+a\hat{\mu})U_\mu^\dagger(y) - 2G(x,y)]
\end{aligned}
$$

and the analogue for $(WG)(x,y)$, we arrive at the compact expression

$$
\langle X_r(x)\rangle = \frac{i}{a^4}\operatorname{tr}[\gamma_5(GW+WG)(x,x)] \tag{E.43}
$$

The two-points function G is a matrix inverse (indeed, it is the propagator): with a bit of work, we will represent it as a formal Neumann series, that is the generalization of the geometric series for an algebra of operators. Firstly, we note that

$$
\begin{aligned}
G &= (\not{D} - W + M)^{-1} \\
&= (\not{D} + W - M)(\not{D} + W - M)^{-1}(\not{D} - W + M)^{-1} \\
&= (\not{D} + W - M)[(\not{D} - W + M)(\not{D} + W - M)]^{-1} \\
&= (\not{D} + W - M)\left[\not{D}^2 + [\not{D}, W] - W^2 + 2MW - M^2\right]^{-1}
\end{aligned} \tag{E.44}
$$

where we used the fact that M commutes with everything because it's proportional to the identity matrix. For convenience, we define

$$
\begin{aligned}
\Sigma &= \sum_{\mu,\nu}\frac{1}{4}[D_\mu, D_\nu][\gamma_\mu,\gamma_\nu] \\
\Gamma &= \sum_{\mu,\nu}\gamma_\mu[D_\mu, W_\nu] = [\not{D}, W] \\
V &= \Sigma + \Gamma
\end{aligned} \tag{E.45}
$$

The commutators in Σ and Γ can be evaluated directly, using the definitions (E.38):

$$
\begin{aligned}
[D_\mu, D_\nu](x,y) = \frac{1}{4}\Big[& F^{\mathrm{I}}_{\mu\nu}(x)\delta_{x+a\hat{\mu}+a\hat{\nu},y} + F^{\mathrm{II}}_{\mu\nu}(x-a\hat{\nu})\delta_{x+a\hat{\mu}-a\hat{\nu},y} + \\
& F^{\mathrm{III}}_{\mu\nu}(x-a\hat{\mu})\delta_{x+a\hat{\nu}-a\hat{\mu},y} + F^{\mathrm{IV}}_{\mu\nu}(x-a\hat{\mu}-a\hat{\nu})\delta_{x-a\hat{\mu}-a\hat{\nu},y}\Big] \\
[D_\mu, W_\nu](x,y) = \frac{r}{4}\Big[& F^{\mathrm{I}}_{\mu\nu}(x)\delta_{x+a\hat{\mu}+a\hat{\nu},y} - F^{\mathrm{II}}_{\mu\nu}(x-a\hat{\nu})\delta_{x+a\hat{\mu}-a\hat{\nu},y} + \\
& F^{\mathrm{III}}_{\mu\nu}(x-a\hat{\mu})\delta_{x+a\hat{\nu}-a\hat{\mu},y} - F^{\mathrm{IV}}_{\mu\nu}(x-a\hat{\mu}-a\hat{\nu})\delta_{x-a\hat{\mu}-a\hat{\nu},y}\Big]
\end{aligned}
\tag{E.46}
$$

where

$$
\begin{aligned}
F^{\mathrm{I}}_{\mu\nu}(x) &= [U_\mu(x)U_\nu(x+a\hat{\mu}) - U_\nu(x)U_\mu(x+a\hat{\nu})]/a^2 \\
F^{\mathrm{II}}_{\mu\nu}(x) &= [U^\dagger_\nu(x)U_\mu(x) - U_\mu(x+a\hat{\nu})U^\dagger_\nu(x+a\hat{\mu})]/a^2 \\
F^{\mathrm{III}}_{\mu\nu}(x) &= [U_\nu(x+a\hat{\mu})U^\dagger_\mu(x+a\hat{\nu}) - U^\dagger_\mu(x)U_\nu(x)]/a^2 \\
F^{\mathrm{IV}}_{\mu\nu}(x) &= [U^\dagger_\mu(x+a\hat{\nu})U^\dagger_\nu(x) - U^\dagger_\nu(x+a\hat{\mu})U^\dagger_\mu(x)]/a^2
\end{aligned}
\tag{E.47}
$$

All these quantities are indistinguishable, in the continuum limit, from the field strength (E.23): for example,

$$
\begin{aligned}
a^2 F^{\mathrm{I}}_{\mu\nu}(x) &= [1+aA_\mu(x)][1+aA_\nu(x+a\hat{\mu})] \\
&\quad - [1+aA_\nu(x)][1+aA_\mu(x+a\hat{\nu})] + o(a^2) \\
&= aA_\mu(x) + aA_\nu(x+a\hat{\mu}) + a^2 A_\mu(x)A_\nu(x+a\hat{\mu}) \\
&\quad - aA_\nu(x) - aA_\mu(x+a\hat{\nu}) - a^2 A_\nu(x)A_\mu(x+a\hat{\nu}) + o(a^2) \\
&= a^2\{\nabla^{(+)}_\mu A_\nu(x) - \nabla^{(+)}_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]\} + o(a^2)
\end{aligned}
$$

Let's go back to the two-points function G: using (as we can see from the defining properties of the Clifford algebra γ matrices)

$$
\not{D}^2 = \sum_{\mu,\nu}\left(\frac{1}{4}\{D_\mu, D_\nu\}\{\gamma_\mu,\gamma_\nu\} + \frac{1}{4}[D_\mu, D_\nu][\gamma_\mu,\gamma_\nu]\right) = D^2 + \Sigma \tag{E.48}
$$

and grouping in the second bracket of (E.44) all the terms with no γ matrix, we find

$$
\begin{aligned}
&\left[\not{D}^2 + [W,\not{D}] - W^2 + 2MW - M^2\right] \\
&= \left[1 + (\Sigma+\Gamma)\left(D^2 - W^2 + 2MW - M^2\right)^{-1}\right]\left(D^2 - W^2 + 2MW - M^2\right)
\end{aligned}
$$

and so, taking the inverse,

$$\begin{aligned}&\left[\not{D}^2 + [W, \not{D}] - W^2 + 2MW - M^2\right]^{-1} \\ &\quad = \left(D^2 - W^2 + 2MW - M^2\right)^{-1}\left[1 + V\left(D^2 - W^2 + 2MW - M^2\right)^{-1}\right]^{-1}\end{aligned} \tag{E.49}$$

Defining the quantity

$$\mathcal{G} = \left(D^2 - W^2 + 2MW - M^2\right)^{-1} \tag{E.50}$$

the result for the two-points function is

$$G = (\not{D} + W - M)\mathcal{G}(1 + V\mathcal{G})^{-1} \tag{E.51}$$

The last bracket can be represented as a geometric series (Neumann series) and, neglecting considerations about convergence, can be written as

$$\frac{1}{1 + V\mathcal{G}} = \sum_{n=0}^{+\infty}(-V\mathcal{G})^n \tag{E.52}$$

and so

$$G = (\not{D} + W - M)\left(\mathcal{G} - \mathcal{G}V\mathcal{G} + \mathcal{G}V\mathcal{G}V\mathcal{G} \pm \cdots\right) \tag{E.53}$$

The trace (E.43) is thus of the type

$$\begin{aligned}&\frac{i}{a^4}\,\mathrm{tr}[\gamma_5(GW)] \\ &\quad = \frac{i}{a^4}\,\mathrm{tr}\{\gamma_5[(\not{D} + W - M)\left(\mathcal{G} - \mathcal{G}V\mathcal{G} + \mathcal{G}V\mathcal{G}V\mathcal{G} \pm \cdots\right)W](x,x)\}\end{aligned} \tag{E.54}$$

Since only the terms with at least four γ matrices contracted with γ_5 survive the trace and since Σ and Γ have, respectively, two and one of them, the first addend that does not cancel is the one quadratic in $V = \Sigma + \Gamma$ (and cubic in $\mathcal{G}$):

$$\begin{aligned}&\frac{i}{a^4}\,\mathrm{tr}\{\gamma_5[(\not{D} + W - M)\mathcal{G}V\mathcal{G}V\mathcal{G}W \pm \cdots](x,x)\} \\ &= \frac{i}{a^4}\,\mathrm{tr}\{\gamma_5[(W - M)\mathcal{G}\Sigma\mathcal{G}\Sigma\mathcal{G}W + \not{D}\mathcal{G}\Gamma\mathcal{G}\Sigma\mathcal{G}W + \not{D}\mathcal{G}\Sigma\mathcal{G}\Gamma\mathcal{G}W \pm \cdots](x,x)\}\end{aligned} \tag{E.55}$$

Power-counting

To understand where the series truncation occurs in the continuum limit we need to establish some sort of hierarchy, for $a \to 0$, among the operators we defined. Their product can be written in Fourier representation as an integral in the momentum space (E.4) (*lattice Feynman integral*). One might think that an integral of this type is converging in the continuum limit if the limit of the integrand is absolutely convergent, according to the dominant convergence theorem. However, this is not the case, and it is easy to find counterexamples: the fact is that the lattice integration spans a compact space (the first Brillouin zone) that implements a momentum cut-off, while the continuum limit drastically modifies the situation, as it brings the cut-off to infinity. It is therefore clear where the trouble arises: in addition to controlling convergence for small a, we must, at the same time, take in account the behaviour of the integrand for large momenta. This is the idea behind the power-counting theorem [28], which generalizes the usual method for calculating the superficial degree of divergence for continuum integrals. A lattice Feynman integral is of the type

$$I_F(q) = \int_{\mathcal{B}} \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 p_n}{(2\pi)^4} \frac{\mathcal{V}(p,q;m,a)}{\mathcal{C}(p,q;m,a)} \tag{E.56}$$

The numerator has the properties:

i) there is an integer ω and a smooth function F such that

$$\mathcal{V}(p,q;m,a) = a^{-\omega} F(ap, aq; am) \tag{E.57}$$

F is a periodic function in ap_i with period 2π and a polynomial in m;

ii) the continuum limit

$$P(p,q;m) = \lim_{a\to 0} \mathcal{V}(p,q;m,a) \tag{E.58}$$

exists.

The denominator is a generic product of operators, of the type

$$\mathcal{C}(p,q;m,a) = \prod_i \mathcal{C}_i(k_i;m,a) \tag{E.59}$$

with k_i linear combinations of the internal momenta p and the external ones q. $\mathcal{C}_i$ must have the following properties:

i) there is a smooth function Q_i such that

$$\mathcal{C}_i(k_i;m,a) = a^{-2} Q_i(ak_i; am) \tag{E.60}$$

ii) The continuum limit exists and

$$\lim_{a\to 0} \mathcal{C}_i(k_i; m, a) = k_i^2 + m_i^2 \tag{E.61}$$

with m_i linear combinations of the mass parameters m;

iii) There are two positive constants a_0 and A such that

$$|\mathcal{C}_i(k_i; m, a)| \geq A \left| \sum_\mu \left(\frac{2}{a} \sin \frac{ak_{i\mu}}{2} \right)^2 + m_i^2 \right| \qquad \forall\, a < a_0 \tag{E.62}$$

There are further conditions for the linear combinations k_i, but they are quite technical in nature and are trivially satisfied in the present case, where there is only one internal momentum: in the sequel, the "1-loop version" of the theorem will be enough. We define the *degree* of $\mathcal{V}$, and design it with $\deg \mathcal{V}$, the integer number n appearing in the asymptotic expression

$$\mathcal{V}(\lambda p, q; m, a/\lambda) \underset{\lambda\to\infty}{\longrightarrow} K\lambda^n + O(\lambda^{n-1}) \qquad K \neq 0 \tag{E.63}$$

(and analogously for $\mathcal{C}$). Obviously,

$$\deg a = -1 \qquad \deg p = 1 \tag{E.64}$$

The role of the parameter λ is to control at the same time, when $\lambda \to \infty$, the continuum limit and the high momenta behaviour of $\mathcal{V}$. The degree of the integral is

$$\deg I_F(q) = 4 + \deg \mathcal{V} - \deg \mathcal{C} \tag{E.65}$$

Now, the Reisz theorem states that the continuum limit of I_F exists if $\deg I_F < 0$ and

$$\lim_{a\to 0} I_F(q) = \int \mathrm{d}^4k\, \frac{P(k, q; m)}{\prod_i (k_i^2 + m_i^2)} \tag{E.66}$$

In order to apply the theorem to the series (E.55) we only need to write the operators in it in momentum space and then evaluating their degree. Let's start from the operators of the free theory, where $A_\mu(x) = 0$ e $U_\mu(x) = \mathbb{I}_{N\times N}$. We find

$$D_{0\mu}(p) = \frac{i \sin ap_\mu}{a} \tag{E.67}$$

$$W_{0\mu}(p) = \frac{r}{a} (\cos ap_\mu - 1) \tag{E.68}$$

$$\begin{aligned} \mathcal{G}_0(p) &= \frac{-a^2}{\sum_\mu \sin^2 ap_\mu + [r \sum_\mu (\cos ap_\mu - 1) - am]^2} \\ &= \frac{-a^2}{\sum_\mu \sin^2 ap_\mu + r^2 \left[\sum_\mu (\cos ap_\mu - 1)\right]^2} + O(a) \end{aligned} \tag{E.69}$$

and so

$$\deg D_0 = \deg W_0 = 1 \qquad \deg \mathcal{G}_0 = -2 \tag{E.70}$$

We can also evaluate the momentum space version of the operators Σ and Γ at the lowest order in a, where all the $F_{\mu\nu}^{\mathrm{I},\cdots,\mathrm{IV}}$ appearing in (E.46) are at the same point and equal to the $F_{\mu\nu}$ defined in (E.23). It is

$$\begin{aligned}
\Sigma(p) &= \sum_{\mu,\nu} \frac{1}{16}[\gamma_\mu,\gamma_\nu][F_{\mu\nu}+O(a)]\left[e^{ia(p_\mu+p_\nu)}+e^{ia(p_\mu-p_\nu)}+e^{ia(p_\nu-p_\mu)}+e^{-ia(p_\mu+p_\nu)}\right] \\
&= \sum_{\mu,\nu} \frac{1}{4}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\cos ap_\mu \cdot \cos ap_\nu + O(a) \qquad\qquad (\text{E.71}) \\
\Gamma(p) &= \sum_{\mu,\nu} \frac{r}{4}\gamma_\mu[F_{\mu\nu}+O(a)]\left[e^{ia(p_\mu+p_\nu)}-e^{ia(p_\mu-p_\nu)}+e^{-ia(p_\mu-p_\nu)}-e^{-ia(p_\mu+p_\nu)}\right] \\
&= r\sum_{\mu,\nu} \gamma_\mu F_{\mu\nu}\cos ap_\mu \cdot i\sin ap_\nu + O(a^2) \qquad\qquad (\text{E.72})
\end{aligned}$$

and so

$$\deg \Sigma = \deg \Gamma = 0 \tag{E.73}$$

We have now all the ingredients to cut off the series (E.55). The highest degree term in it is

$$\begin{aligned}
&a^{-4}\left(\Sigma^2 W_0^2 \mathcal{G}_0^3 + 2\Gamma\Sigma \not{D}_0 W_0 \mathcal{G}_0^3\right) \\
&\qquad \longrightarrow \quad \deg a^{-4}\left(\Sigma^2 W_0^2 \mathcal{G}_0^3 + 2\not{D}_0 \Gamma\Sigma W_0 \mathcal{G}_0^3\right) = 4+2-6 = 0
\end{aligned}$$

Indeed, in the interacting theory the operators D_0, W_0, ... are corrected by terms D_1, W_1, ... that depend on increasing powers of a (since $U_\mu = \mathbb{I} + aA_\mu + \cdots$) and so have a lower degree; the same is true for the successive addends in the Kerler series, as they depend on increasing powers of $\mathcal{G}$ (and so of a^2). A 0-degree term is said to be *marginal*: it is at the boundary between the negative-degree terms and the positive-degree ones. In the lower degree (*irrelevant*) terms we can take the continuum limit under the integral sign, via the power-counting theorem we have just mentioned, but in this case at least on power of a is left in the numerator, so the integrand tends to 0: the only terms that do not cancel in the series (E.55) when $a \to 0$ are the marginal ones!

Spinorial structure

Using the expressions (E.67), (E.68), (E.69), (E.71) and (E.72) it's easy to obtain

$$\operatorname{tr} a^{-4}\gamma_5\Sigma^2 W_0^2 \mathcal{G}_0^3 \underset{a\to 0}{\longrightarrow} -\sum_{\substack{\mu,\nu\\ \rho,\sigma}} \frac{1}{16} 4 \operatorname{tr}\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma F_{\mu\nu}F_{\rho\sigma}\right)$$
$$\cdot\, r^2 a^4 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\cos ap_\mu \cos ap_\nu \cos ap_\rho \cos ap_\sigma \left[\sum_\kappa (\cos ap_\kappa - 1)\right]^2}{\left\{\sum_\kappa \sin^2 ap_\kappa + r^2\left[\sum_\kappa(\cos ap_\kappa - 1)\right]^2\right\}^3}$$

Using the property of the γ matrices

$$\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4\epsilon_{\mu\nu\rho\sigma} \tag{E.74}$$

we found (all the indices μ, ν, ρ, σ must be different to give a non null contribution in the trace, so the cosines product is always over 4 out of 4 different indices and can be factorized)

$$\lim_{a\to 0} \operatorname{tr} a^{-4}\gamma_5\Sigma^2 W_0^2\mathcal{G}_0^3 = -I_\Sigma \sum_{\substack{\mu,\nu\\ \rho,\sigma}} \epsilon_{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu}F_{\rho\sigma} \tag{E.75}$$

with

$$I_\Sigma = r^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \prod_{\lambda=1}^{4} \cos p_\lambda \frac{\left[\sum_\kappa(\cos p_\kappa - 1)\right]^2}{\left\{\sum_\kappa \sin^2 p_\kappa + r^2\left[\sum_\kappa(\cos p_\kappa - 1)\right]^2\right\}^3} \tag{E.76}$$

We also rescaled the momenta according to $p_\mu \to p_\mu/a$ (the integration intervals span now from $-\pi$ to π). The term with Γ requires a little more work:

$$\operatorname{tr} 2a^{-4}\gamma_5 \not{D}_0 \Gamma\Sigma W_0 \mathcal{G}_0^3 \underset{a\to 0}{\longrightarrow} -2 \sum_{\substack{\mu_1,\mu_2\\ \mu_3,\mu_4}} \sum_\lambda \frac{1}{4} 2 \operatorname{tr}\left(\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4} F_{\mu_2\lambda}F_{\mu_3\mu_4}\right)$$
$$\cdot\, r^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i\sin p_{\mu_1} \cos p_{\mu_2} i \sin p_\lambda \cos p_{\mu_3} \cos p_{\mu_4} \sum_\kappa(\cos p_\kappa - 1)}{\left\{\sum_\kappa \sin^2 p_\kappa + r^2\left[\sum_\kappa(\cos p_\kappa - 1)\right]^2\right\}^3}$$

The index λ must be different from μ_2 because of the antisymmetry of $F_{\mu_2\lambda}$. Furthermore, the trace of the gamma matrices is proportional to the totally antisymmetric symbol $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$, but when λ is μ_3 or μ_4 the product $F_{\mu_2\lambda}F_{\mu_3\mu_4}$ (in the colour trace) becomes symmetric under the exchange of two indices: the contraction is null. For example, if $\lambda = \mu_4$,

$$\operatorname{tr}(F_{\mu_2\mu_4}F_{\mu_3\mu_4}) \underset{\mu_2\leftrightarrow\mu_3}{\longrightarrow} \operatorname{tr}(F_{\mu_3\mu_4}F_{\mu_2\mu_4}) = \operatorname{tr}(F_{\mu_2\mu_4}F_{\mu_3\mu_4})$$

We can thus insert a $\delta_{\mu_1\lambda}$ factor in the sum without changing the result, obtaining

$$\operatorname{tr} 2a^{-4}\gamma_5 \not{D}_0 \Gamma\Sigma W_0 \mathcal{G}_0^3 \underset{a\to 0}{\longrightarrow} -4 \sum_{\substack{\mu_1,\mu_2\\ \mu_3,\mu_4}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} \operatorname{tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4})$$
$$\cdot\, r^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\sin^2 p_{\mu_1}\cos p_{\mu_2}\cos p_{\mu_3}\cos p_{\mu_4}\sum_\kappa(\cos p_\kappa - 1)}{\left\{\sum_\kappa \sin^2 p_\kappa + r^2\left[\sum_\kappa(\cos p_\kappa - 1)\right]^2\right\}^3}$$

With μ_1 fixed, the others sums are of the type

$$-4 I_{\mu_1} \sum_{\substack{\mu_2\\ \mu_3,\mu_4}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} \operatorname{tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4})$$

with

$$I_{\mu_1} = r^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\sin^2 p_{\mu_1}}{\cos p_{\mu_1}} \prod_{\lambda=1}^{4} \cos p_\lambda \frac{\sum_\kappa(\cos p_\kappa - 1)}{\left\{\sum_\kappa \sin^2 p_\kappa + r^2\left[\sum_\kappa(\cos p_\kappa - 1)\right]^2\right\}^3}$$

Exploiting the antisymmetry of F we can write

$$F_{\mu_1\mu_2}F_{\mu_3\mu_4} = \frac{1}{4}(F_{\mu_1\mu_2} - F_{\mu_2\mu_1})(F_{\mu_3\mu_4} - F_{\mu_4\mu_3})$$
$$= \frac{1}{4}\sum_{\substack{\mu,\nu\\ \rho,\sigma}}(\delta_{\mu\mu_1}\delta_{\nu\mu_2} - \delta_{\mu\mu_2}\delta_{\nu\mu_1})(\delta_{\rho\mu_3}\delta_{\sigma\mu_4} - \delta_{\rho\mu_4}\delta_{\sigma\mu_3})F_{\mu\nu}F_{\rho\sigma}$$

and so

$$\sum_{\substack{\mu_2\\ \mu_3,\mu_4}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} \operatorname{tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4}) = \frac{1}{2}\sum_{\substack{\mu,\nu\\ \rho,\sigma}}(\delta_{\mu\mu_1}\epsilon_{\mu_1\nu\rho\sigma} + \delta_{\nu\mu_1}\epsilon_{\mu\mu_1\rho\sigma}) \operatorname{tr}(F_{\mu\nu}F_{\rho\sigma})$$

Using the cyclicity of the trace, we can swap the indices $(\mu,\nu) \leftrightarrow (\rho,\sigma)$ without changing the result. Taking their symmetric linear combination we get

$$\sum_{\substack{\mu_2\\ \mu_3,\mu_4}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} \operatorname{tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4})$$
$$= \frac{1}{4}\sum_{\substack{\mu,\nu\\ \rho,\sigma}}(\delta_{\mu\mu_1}\epsilon_{\mu_1\nu\rho\sigma} + \delta_{\nu\mu_1}\epsilon_{\mu\mu_1\rho\sigma} + \delta_{\rho\mu_1}\epsilon_{\mu\nu\mu_1\sigma} + \delta_{\sigma\mu_1}\epsilon_{\mu\nu\rho\mu_1}) \operatorname{tr}(F_{\mu\nu}F_{\rho\sigma})$$

The indices that are set equal by the δ factors can be confused, so we obtain

$$\sum_{\substack{\mu_2\\ \mu_3,\mu_4}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} \operatorname{tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4}) = \frac{1}{4}\sum_{\substack{\mu,\nu\\ \rho,\sigma}}(\delta_{\mu\mu_1} + \delta_{\nu\mu_1} + \delta_{\rho\mu_1} + \delta_{\sigma\mu_1})\epsilon_{\mu\nu\rho\sigma} \operatorname{tr}(F_{\mu\nu}F_{\rho\sigma}) \tag{E.77}$$

We can now perform the sum over μ_1: since all the indices μ, ν, ρ, σ are different, we find

$$\begin{aligned}\sum_{\mu_1}(\delta_{\mu\mu_1}+\delta_{\nu\mu_1}+\delta_{\rho\mu_1}+\delta_{\sigma\mu_1})\frac{\sin^2 p_{\mu_1}}{\cos p_{\mu_1}} &= \frac{\sin^2 p_\mu}{\cos p_\mu}+\frac{\sin^2 p_\nu}{\cos p_\nu}+\frac{\sin^2 p_\rho}{\cos p_\rho}+\frac{\sin^2 p_\sigma}{\cos p_\sigma}\\ &= \sum_{\tau=1}^{4}\frac{\sin^2 p_\tau}{\cos p_\tau}\end{aligned}$$

and so, factorizing,

$$\lim_{a\to 0}\operatorname{tr} 2a^{-4}\gamma_5 \not{D}_0\Gamma\Sigma W_0\mathcal{G}_0^3 = -I_\Gamma\sum_{\substack{\mu,\nu\\ \rho,\sigma}}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr} F_{\mu\nu}F_{\rho\sigma} \tag{E.78}$$

with

$$I_\Gamma = r^2\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\prod_{\lambda=1}^{4}\cos p_\lambda\frac{\sum_\tau\frac{\sin^2 p_\tau}{\cos p_\tau}\sum_\kappa(\cos p_\kappa-1)}{\left\{\sum_\kappa\sin^2 p_\kappa+r^2\left[\sum_\kappa(\cos p_\kappa-1)\right]^2\right\}^3} \tag{E.79}$$

Finally, introducing also a factor of 2 from the (WG) term, the equation (E.43) becomes, in the continuum limit,

$$\lim_{a\to 0}\langle X_r(x)\rangle = -2iI_4\sum_{\substack{\mu,\nu\\ \rho,\sigma}}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr} F_{\mu\nu}F_{\rho\sigma} \tag{E.80}$$

with (the subscript stand for the dimensionality of the space)

$$I_4 = r^2\int_{-\pi}^{\pi}\frac{\mathrm{d}^4 p}{(2\pi)^4}\prod_{\lambda=1}^{4}\cos p_\lambda\frac{\left[\sum_\kappa(\cos p_\kappa-1)\right]^2+\sum_\tau\frac{\sin^2 p_\tau}{\cos p_\tau}\sum_\kappa(\cos p_\kappa-1)}{\left\{\sum_\kappa\sin^2 p_\kappa+r^2\left[\sum_\kappa(\cos p_\kappa-1)\right]^2\right\}^3} \tag{E.81}$$

Evaluation of the coefficient

To demonstrate equation (E.36) we still need to verify that

$$I_4 = -\frac{1}{32\pi^2} \tag{E.82}$$

We will show that, although the integrand has a dependence on the parameter r, the integral does not and has the expected value. The result is in the original work by Karsten and Smit [18], who first evaluated, in the perturbative approach, the continuum limit of the chiral anomaly in the Wilson's theory. The result is generalized to I_d in [30]. We have

$$I_d = r^2\int_{-\pi}^{\pi}\frac{\mathrm{d}^d p}{(2\pi)^d}\prod_{\lambda=1}^{d}\cos p_\lambda\frac{\left[\sum_\kappa(\cos p_\kappa-1)\right]^2+\sum_\tau\frac{\sin^2 p_\tau}{\cos p_\tau}\sum_\kappa(\cos p_\kappa-1)}{\left\{\sum_\kappa\sin^2 p_\kappa+r^2\left[\sum_\kappa(\cos p_\kappa-1)\right]^2\right\}^{1+d/2}} \tag{E.83}$$

where all the sums run now from 1 to d. We use the definition

$$\mathscr{D} = \sum_\kappa \sin^2 p_\kappa + r^2 \left[\sum_\kappa (\cos p_\kappa - 1)\right]^2 \tag{E.84}$$

for the combination in the curly brackets in the denominator. Since

$$\begin{aligned}\frac{\partial}{\partial p_\tau}\mathscr{D}^{-d/2} &= -\frac{d}{2}\mathscr{D}^{-1-d/2}\left[2\sin p_\tau \cos p_\tau - 2r^2 \sin p_\tau \sum_\kappa (\cos p_\kappa - 1)\right] \\ \Longrightarrow\ r^2 \sin p_\tau \sum_\kappa (\cos p_\kappa - 1) &= \frac{1}{d}\mathscr{D}^{1+d/2}\frac{\partial}{\partial p_\tau}\mathscr{D}^{-d/2} + \sin p_\tau \cos p_\tau\end{aligned}$$

the identity

$$r^2 \sum_\tau \frac{\sin^2 p_\tau}{\cos p_\tau}\sum_\kappa (\cos p_\kappa - 1) = \frac{1}{d}\mathscr{D}^{1+d/2}\sum_\tau \tan p_\tau \frac{\partial}{\partial p_\tau}\mathscr{D}^{-d/2} + \sum_\tau \sin^2 p_\tau$$

holds, and so the numerator can be written as

$$\begin{aligned}r^2\left[\sum_\kappa (\cos p_\kappa - 1)\right]^2 + r^2 \sum_\tau \frac{\sin^2 p_\tau}{\cos p_\tau}\sum_\kappa (\cos p_\kappa - 1)& \\ = \mathscr{D} + \frac{1}{d}\mathscr{D}^{1+d/2}\sum_\tau \tan p_\tau \frac{\partial}{\partial p_\tau}\mathscr{D}^{-d/2}&\end{aligned} \tag{E.85}$$

The fraction becomes

$$\frac{\mathscr{D} + \frac{1}{d}\mathscr{D}^{1+d/2}\sum_\tau \tan p_\tau \frac{\partial}{\partial p_\tau}\mathscr{D}^{-d/2}}{\mathscr{D}^{1+d/2}} = \left(1 + \frac{1}{d}\sum_\tau \tan p_\tau \frac{\partial}{\partial p_\tau}\right)\mathscr{D}^{-d/2} \tag{E.86}$$

and so

$$\begin{aligned}I_d = \int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d}\prod_{\lambda=1}^{d}\cos p_\lambda& \\ \cdot \left(1 + \frac{1}{d}\sum_\tau \tan p_\tau \frac{\partial}{\partial p_\tau}\right)\left\{\sum_\kappa \sin^2 p_\kappa + r^2\left[\sum_\kappa (\cos p_\kappa - 1)\right]^2\right\}^{-d/2}&\end{aligned} \tag{E.87}$$

That's an integral over a period of a periodic function: for each direction μ we can translate the integration interval

$$[-\pi, \pi] \quad \longrightarrow \quad [-\pi/2, \pi/2] \cup [\pi/2, 3\pi/2]$$

On these intervals the sine is a strictly monotonic function, so we can change variables of integration according to

$$p_\mu \longrightarrow s_\mu = \sin p_\mu; \qquad \mathrm{d}p_\mu \longrightarrow \mathrm{d}s_\mu = \cos p_\mu \, \mathrm{d}p_\mu$$

On $[-\pi/2, \pi/2]$ the sine is strictly increasing and the cosine is positive, on $[\pi/2, 3\pi/2]$ it's the opposite, so I_d becomes the sum of 2^d integrals on $[-1,1]^d$ that differ only for a total sign and for the sign of the cosines:

$$I_d = \sum_{A=1}^{2^d} \prod_{\lambda=1}^{d} \epsilon_\lambda^A \cdot \int_{-1}^{1} \frac{\mathrm{d}^d s}{(2\pi)^d} \left(1 + \frac{1}{d} \sum_\tau s_\tau \frac{\partial}{\partial s_\tau}\right) \left\{ \sum_\kappa s_\kappa^2 + r^2 \left[\sum_\kappa \left(\epsilon_\kappa^A \sqrt{1 - s_\kappa^2} - 1 \right) \right]^2 \right\}^{-d/2} \tag{E.88}$$

where ϵ_μ^A is one of the d-vector

$$\epsilon_\mu^A = (\pm 1, \pm 1, \cdots, \pm 1) \qquad A = 1, \cdots, 2^d \tag{E.89}$$

(the A index selects one of the 2^d hypercubes in which the integration region is now split). Passing in spherical coordinates, we define the radial coordinate

$$\sigma^2 = \sum_\kappa s_\kappa^2 \tag{E.90}$$

Using the vectorial identity

$$\sum_\tau s_\tau \frac{\partial}{\partial s_\tau} = \sigma \frac{\partial}{\partial \sigma} \tag{E.91}$$

its easy to get

$$\begin{aligned} I_d &= \sum_{A=1}^{2^d} \prod_{\lambda=1}^{d} \epsilon_\lambda^A \\ &\quad \cdot \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \int_0^{\bar\sigma(\Omega)} \mathrm{d}\sigma\, \sigma^{d-1} \left(1 + \frac{1}{d}\sigma \frac{\partial}{\partial \sigma}\right) \left\{ \sigma^2 + r^2 \left[\sum_\kappa \left(\epsilon_\kappa^A \sqrt{1 - s_\kappa^2} - 1 \right) \right]^2 \right\}^{-d/2} \\ &= \sum_{A=1}^{2^d} \prod_{\lambda=1}^{d} \epsilon_\lambda^A \\ &\quad \cdot \frac{1}{d} \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \int_0^{\bar\sigma(\Omega)} \mathrm{d}\sigma \frac{\partial}{\partial \sigma} \left\{ \sigma^d \left\{ \sigma^2 + r^2 \left[\sum_\kappa \left(\epsilon_\kappa^A \sqrt{1 - s_\kappa^2} - 1 \right) \right]^2 \right\}^{-d/2} \right\} \end{aligned} \tag{E.92}$$

where now the variables s_k are expressed in spherical coordinates, such as

$$\begin{aligned} s_1 &= \sigma \cos\phi_1 \\ s_2 &= \sigma \sin\phi_1 \cos\phi_2 \\ &\vdots \\ s_{d-1} &= \sigma \sin\phi_1 \cdots \sin\phi_{d-2} \cos\phi_{d-1} \\ s_d &= \sigma \sin\phi_1 \cdots \sin\phi_{d-2} \sin\phi_{d-1} \end{aligned}$$

The integration region was a hypercube with edge 2; from the relations

$$\begin{array}{ccc} -1 \le s_1 \le 1 & & -1 \le \sigma \cos\phi_1 \le 1 \\ -1 \le s_2 \le 1 & & -1 \le \sigma \sin\phi_1 \cos\phi_2 \le 1 \\ \vdots & \Longrightarrow & \vdots \\ -1 \le s_{d-1} \le 1 & & -1 \le \sigma \sin\phi_1 \cdots \sin\phi_{d-2} \cos\phi_{d-1} \le 1 \\ -1 \le s_d \le 1 & & -1 \le \sigma \sin\phi_1 \cdots \sin\phi_{d-2} \sin\phi_{d-1} \le 1 \end{array}$$

we get an expression for the limit of integration $\bar\sigma$ as a function of the angular variables:

$$\bar\sigma(\Omega) = \min\left\{ \frac{1}{|\cos\phi_1|}, \frac{1}{|\sin\phi_1 \cos\phi_2|}, \cdots, \frac{1}{|\sin\phi_1 \cdots \sin\phi_{d-1}|} \right\} \tag{E.93}$$

The denominator in (E.92) is a sum of quadratic terms: it is null only when each one of them is null. This can happen only when $\epsilon^A_\kappa = 1$ in any direction κ, in a neighbourhood of $\sigma = 0$: for all the others $2^d - 1$ cases the integrand is regular and the radial integral can be solved. Assign $A = 1$ to the vector with all positive components, $\epsilon^1_\mu = (+1, \cdots, +1)$; the regular integrals are given by

$$\begin{aligned} &\frac{1}{d} \sum_{A=2}^{2^d} \prod_{\lambda=1}^{d} \epsilon^A_\lambda \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\, \sigma^d \left\{ \sigma^2 + r^2 \left[\sum_\kappa \left(\epsilon^A_\kappa \sqrt{1 - s^2_\kappa} - 1 \right) \right]^2 \right\}^{-d/2} \Bigg|_{\sigma=0}^{\sigma=\bar\sigma(\Omega)} \\ &= \frac{1}{d} \sum_{A=2}^{2^d} \prod_{\lambda=1}^{d} \epsilon^A_\lambda \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\, \bar\sigma^d \left\{ \bar\sigma^2 + r^2 \left[\sum_\kappa \left(\epsilon^A_\kappa \sqrt{1 - s^2_\kappa|_{\sigma=\bar\sigma}} - 1 \right) \right]^2 \right\}^{-d/2} \end{aligned}$$

The angular region of integration can be split in d subregions where each of the possibilities (E.93) holds for the limit of integration $\bar\sigma$. This process select for each region a direction τ such that the integral does not depend anymore on ϵ^A_τ, because the square root that this factor multiplies is null.[3] The sum over A is made of pairs of addends which differ only for the sign of the component ϵ^A_τ and so, since we can factorize the angular integral, only for a total sign, because of the factor $\prod_{\lambda=1}^d \epsilon^A_\lambda$: the pairs cancel out and the sum is null. This cancellation can not be complete, because the sum over $A \neq 1$ is restricted to an odd number of terms; however, the integration is

[3] For example, in the subregion where $\bar\sigma = 1/|\cos\phi_1|$,

$$\epsilon^A_1 \sqrt{1 - s^2_1|_{\sigma=\bar\sigma}} = \epsilon^A_1 \sqrt{1 - \frac{1}{|\cos\phi_1|^2} \cos^2\phi_1} = 0$$

and thus the integral does not depend on ϵ^A_1.

possible, in the improper sense, also in the sector $A = 1$, excluding a ball $\mathcal{B}_\delta(0)$ with radius δ centered in the origin, where the integrand diverges. The term evaluated at the upper limit of integration is of the same type of those for $A \neq 1$ and gives the missing addend to complete the cancellation. Of the original integral remains only the term evaluated at the lower limit of integration, with $\sigma = \delta$ in the sector $A = 1$, to be calculated in the limit $\delta \to 0^+$:

$$I_d = -\lim_{\delta\to 0^+} \frac{1}{d} \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\, \delta^d \left\{ \delta^2 + r^2 \left[\sum_\kappa \left(\sqrt{1 - s_\kappa^2|_{\sigma=\delta}} - 1 \right) \right]^2 \right\}^{-d/2} \tag{E.94}$$

It is

$$\lim_{\delta\to 0^+} \left[\sum_\kappa \left(\sqrt{1 - s_\kappa^2|_{\sigma=\delta}} - 1 \right) \right]^2 = \lim_{\delta\to 0^+} \left[\sum_\kappa \left(-\frac{1}{2}\, s_\kappa^2|_{\sigma=\delta} \right) \right]^2 = \lim_{\delta\to 0^+} \left(-\frac{\delta^2}{2} \right)^2$$

so the limit is 1 and the result

$$I_d = -\frac{1}{d} \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \tag{E.95}$$

does not depend on r. That's the end of the proof.

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