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QUENCHES IN QUANTUM CHAINS WITH $\mathbb{Z}_n$ SYMMETRY

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Introduction

Consider an isolated quantum system composed of many particles and ruled by an Hamiltonian H. Suppose to prepare the system in a state $|\psi\rangle$ which is not an eigenstate of H. This state may be, for definiteness, the ground state of another Hamiltonian H_0. Consider then the time evolution of this state

$$|\psi(t)\rangle = e^{-iHt}|\psi\rangle$$

This very simple setup is basically the most studied protocol of a so called quantum quench problem. Its solution is a non trivial task, since it requires the knowledge of all the eigenstates of H, together with their overlaps on the initial state $|\psi\rangle$. Also its experimental realization has been prohibitive until a few years ago, especially for many body quantum systems whose Hamiltonian was simple enough to be solved, in order to compare theory with experiments.
However crucial experimental advances have been achieved in recent times and to realize in the real world the setup of above is not impossible nowaday. These experimental techniques mainly include a high control of ultra cold atom gases and optical lattices, over than the possibility to tune to very high precision the interaction between the constituents of the system. This is the very motivation which led in the last decade to a wide theoretical effort aimed to describe this so far unexplored branch of physics.
Moreover this problem has its own relevance also from a pure theoretical point of view. Indeed the usual treatment of isolated quantum systems at equilibrium is to consider them as described by a state which is a termal mixture characterized by a density matrix

$$\rho = \sum_{|E_n - E| < \Delta} |E_n\rangle\langle E_n| \tag{1}$$

where Δ fixes an energy shell specifying all the eigenstates $|E_n\rangle$ of the Hamiltonian having nonzero overlap with the initial state. This is done in complete analogy with classical statistical mechanics, but it overlooks the fact that it is very difficult to imagine this mixed state as a result of unitary evolution, since the latter maps pure states into pure states. A possible way out to this issue is to suppose that as long as local observables are measured the equilibrium state of the system, which is supposed to be the outcome of an out of equilibrium setup as the one discussed above, will look like the mixture (1)

$$\lim_{t\to\infty} \langle\psi(t)|O(x)|\psi(t)\rangle = \mathrm{Tr}\,(\rho O(x))$$

as the rest of the system acts as a thermal bath for that subsystem in which O is measured. Of course the thermal bath must be very large, so that the thermodynamic limit has alway to be taken. The picture has recently emerged from the extensive theoretical and experimental investigation of this problem is that the situation might be more complicated than this. Indeed an initial state is in general identified not only by the quantum numbers with respect to the Hamiltonian if the latter are degenerate. Then is not easy to justify with quantum mechanics how this state can loose memory of the quantum numbers relative to the other operators conserved during the dynamics. In fact the result of quenching an isolated quantum system is observed both theoretically and experimentally

not always to produce a thermal mixture like (1).
In the following all these issues will be examined in depth and the focus will be soon moved to the one-dimensional case. This mainly because the few many body quantum systems for which the dyanmics can be solved exactly live basically always in a one-dimensional space and moreover they seem to be the ones for which the equilibration result is more interesting. Actually it will turn out that these two facts are related. More specifically the simplest prototype of quantum models in one dimension with many degrees of freedom will be considered. This is a quantum chain whose Hilbert space is factorized as

$$\mathcal{H} = \bigotimes_{a=1}^{N} V_a$$

where a label the sites of the chain and V_a is a finite-dimensional Hilbert space localized on the site a. Despite their simplicity and their being not so close to reality, these models can provide a very rich background in which to study a quench problem. In fact it is not even trivial to solve their dynamics and a numerical treatment is sometimes necessary. The latter still being a real challenge because of the exponential growth of the Hilbert space with the number of constituents of the system.

The work is structured as follows. The first chapter is an introduction to quench problems in quantum systems, aimed at giving a general picture of the current theoretical understanding of this topic. In the second chapter the powerful field theory approach to the physics of critical phenomena is discussed, in view of the fact that the model under investigation undergoes a quantum phase transition driven by the only parameter from which the Hamiltonian depends. The chapter ends with the application of this framework to the quench problem. In the third chapter the model is introduced. The exact solution of the $q = 2$ case is discussed in detail and the difficulties in obtaining the complete spectrum in the general case are explained. Then a perturbative analysis of the spectrum is performed and the quantum critical point's universality class of the model is reviewed. In the fourth chapter the outcome of the numerical analysis of the dynamics is presented and compared with the analytic solution of $q = 2$ case. Then an attempt in the direction of understanding the formal structure of the numerically computed time dependent quantities is made. Finally the field theory predicitions previously discussed are compared with the lattice results.

Chapter 1

Quantum relaxation and thermalization

In this chapter the notion of quantum quench will be introduced, together with the main reasons why this topic has caught soo much attention in the last decade. In section 1 the idea of "quenched" regime of the nonequilibrium dynamics will be compared to the so called adiabatic evolution induced by an hamiltonian which varies slowly in time. In section 2 the concept of integrability at both the classical and quantum level will be briefly discussed, along with the reason why this is at the foundation of statistical mechanics. In section 3 and 4 the present understanding of the dynamics following a quench will be sketched together with the main related open problems. Finally in section 5 the most famous and recent experiments on cold atoms will be shortly described and their outcome reported.
Through this chapter some topics which would require a more precise treatment will be mentioned. The most of them will be discussed in more details in the following chapters. On the other hand many basic topics such as the formal framework of quantum and classical statistical mechanical, ensambles theory, the density matrix formalism ecc. will not be reviewed.

1.1 What is a quantum quench?

Consider a quantum system governed by an hamiltonian H and prepared in one of its eigenstate $\psi_{n,0}$ at time $t = 0$. Then make H time-dependent and ask what happens to the initial state. As the hamiltonian is changing, being its hermiticity preserved, it will have an istantaneous ortonormal and complete basis of eigenvectors

$$H_t\,\psi_{n,t} = E_{n,t}\,\psi_{n,t} \qquad \langle\psi_{n,t}|\psi_{m,t}\rangle = \delta_{n,m} \qquad \sum_n |\psi_{n,t}\rangle\langle\psi_{n,t}| = \mathbb{1} \qquad \forall t \tag{1.1}$$

Integrating the time-dependent Schroedinger equation starting from $t = 0$ then yields

$$i\dot{\psi}_{n,t} = H_t\,\psi_{n,t} = E_{n,t}\,\psi_{n,t} \qquad \Rightarrow \qquad \psi_{n,t}(t) = e^{-i\int_0^t \mathrm{d}t' E_{n,t'}}\psi_{n,t}$$

Thanks to (1.1), the initial state $\psi_{n,0}$ may be expanded and evolved on the insantaneous eigenbasis

$$\psi_{n,0}(t) = \sum_m c_{m,t}\, e^{-i\int_0^t \mathrm{d}t' E_{n,t'}}\psi_{m,t} \qquad c_{m,0} = \delta_{n,m}$$

Substituting this expression in (1.1)

$$\sum_m e^{-i\int_0^t \mathrm{d}t' E_{m,t'}} i(\dot{c}_{m,t}\psi_{m,t} - iE_{m,t}c_{m,t} + c_{m,t}\dot{\psi}_{m,t}) = \sum_m e^{-i\int_0^t \mathrm{d}t' E_{m,t'}} H_t\,\psi_{m,t} = \sum_m e^{-i\int_0^t \mathrm{d}t' E_{m,t'}} E_{m,t}\,\psi_{m,t}$$

$$\Rightarrow \qquad \sum_m e^{-i\int_0^t \mathrm{d}t' E_{m,t'}} \dot{c}_{m,t}\psi_{m,t} = -\sum_m e^{-i\int_0^t \mathrm{d}t' E_{m,t'}} c_{m,t}\dot{\psi}_{m,t}$$

$$\Rightarrow \qquad \dot{c}_{\ell,t} = -\sum_m e^{-i\int_0^t \mathrm{d}t'(E_{m,t'}-E_{\ell,t'})} c_{m,t}\langle\psi_{\ell,t}|\dot{\psi}_{m,t}\rangle \tag{1.2}$$

Then differetianting (1.1) and projecting on $\psi_{m,t}$

$$\langle\psi_{\ell,t}|\dot{H}_t|\psi_{m,t}\rangle + E_{\ell,t}\langle\psi_{\ell,t}|\dot{\psi}_{m,t}\rangle = \dot{E}_{m,t}\delta_{\ell,n} + E_{m,t}\langle\psi_{\ell,t}|\dot{\psi}_{m,t}\rangle$$

For $m \neq \ell$ this equation provides an expression for $\langle\psi_{\ell,t}|\dot{\psi}_{m,t}\rangle$, which upon substitution in (1.2) yields

$$\dot{c}_{m,t} = -c_{m,t}\langle\psi_{m,t}|\dot{\psi}_{m,t}\rangle - \sum_\ell e^{-i\int_0^t \mathrm{d}t'(E_{\ell,t'}-E_{m,t'})} c_{\ell,t}\frac{\langle\psi_{m,t}|\dot{H}_t|\psi_{\ell,t}\rangle}{E_{\ell,t}-E_{m,t}}$$

Now when the second term in the r.h.s is negligible, i.e.

$$\left|\langle\psi_{m,t}|\dot{H}_t|\psi_{\ell,t}\rangle\right| \ll |E_{m,t}-E_{\ell,t}| \qquad\qquad \forall m,\ell \tag{1.3}$$

the time-dependent coefficient $c_{m,t}$ evolves according to

$$\dot{c}_{m,t} = -\langle\psi_{m,t}|\dot{\psi}_{m,t}\rangle\, c_{m,t} \qquad \Rightarrow \qquad c_{m,t} = \exp\left(-\int_0^t \mathrm{d}t'\langle\psi_{m,t'}|\dot{\psi}_{m,t'}\rangle\right) c_{m,0} = e^{i\alpha(t)}\delta_{m,n}$$

Moreover $e^{i\alpha(t)}$ is a pure phase since $\langle\psi_{m,t}|\dot{\psi}_{m,t}\rangle$ is purely imaginary

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_{m,t}|\psi_{m,t}\rangle = \langle\psi_{m,t}|\dot{\psi}_{m,t}\rangle + \overline{\langle\psi_{m,t}|\dot{\psi}_{m,t}\rangle}$$

so the time evolution of the initial state is given by

$$\psi_{n,0}(t) = e^{i\alpha(t)}e^{-i\int_0^t \mathrm{d}t' E_{n,t'}}\psi_{n,0} \tag{1.4}$$

This proves the so called adiabatic theorem: a quantum system remains in its istantaneous eigenstate $\psi_{n,t}$ if the hamiltonian varies slowly with respect to the gap between the $E_{n,t}$ and the rest of the energy spectrum, i.e. if (1.3) holds.
A quantum quench is exactly the situation in which the adiabatic approximation cannot be applied and level crossing occurs. This may be caused by the fact that the rate of change of the hamiltonian $\dot{H}$ is not small or by the absence of a gap between the initial eigenstate considered and its neghbouring ones. In both cases there is no general theoretical framework useful to describe the out of equilibrium dynamics of the system, as the usual time-dependent perturbation theory cannot be applied since the change in the hamiltonian is definitely not small. Then what happens after the rapid change of the hamiltonian has to be studied case by case. The most important questions one may be interested in: what is the long time limit of the dynamics? Does the system reach an equilibrium state? How can this state be characterized?
The answers to these questions are not only important because they give a meaning to the usual statistical treatment of closed quantum systems with many degrees of freedom, but also because in recent years crucial advances in atomic physics, quantum optics and other fields made possible to probe experimentally this out of equilibrium dynamics, and possibly the steady state reached in the long time limit, in systems described to very high accuracy by relatively simple models in which the analytical treatment of the quench problem is not prohibitive and theoretical predictions of the experimental outcome are possible.

Note finally that the standard quench protocol used in theoretical calculations and numerical simulations is that of a "sudden" quench, in which the system is prepared in an eigenstate, usually the ground state, of an hamitonian which depends on certain parameters $H(\{g_i\})$ at time $t = 0$, then some parameters of the hamiltonian are changed and the initial state is evolved with this new hamiltonian. By the way it is worth noting that also the dependence on the initial state and on the rate of change in time of the hamiltonian have been studied by many authors, e.g. [2, 10], also considering the possibility to take as initial conidition a thermal mixture

$$\hat{\rho} = \sum_n e^{-\beta E_n} |E_n\rangle\langle E_n| \tag{1.5}$$

at a certain finite temperature $\beta = 1/k_B T$. The ground state, which is pure, coincide with this thermal mixture at zero temperature.

1.2 Classical vs quantum integrability

In the last century statistical mechanics has proven to be a powerful tool to describe systems with a very large number of degrees of freedom, exploiting the so called ensambles theory. For classical isolated systems the microscopic justification of the use of the microcanonical ensamble is based on the assumption that the dynamics of this large number of particles is ergodic: the trajectories of the microscopic degrees of freedom of the system span uniformly the whole phase space. This implies that one is allowed to replace time averages of observables $O(X)$ defined on the phase space, where X is a $2DN$-dimensional point for a D-dimensional system of N particles, with averages over the phase space weighted with a uniform probability distribution: once selected an initial conidition X_0 whose energy is $E = H(X_0)$ and time translational invariance is assumed, one has

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, O(X(t)) = \int \mathrm{d}X O(X) \rho_{mc}(E, \Delta) \tag{1.6}$$

where ρ_{mc} is a constant function with support on a shell of width Δ around the $E = \text{const}$ hypersuperfaces in the phase space. Note that the trajectory on the l.h.s. depends on the initial condition $X(0) = X_0$, while the r.h.s. is independent of the latter. This leads to the just mentioned microcanonical ensamble, and eventually to the more general ensamble theories. Then the concept of thermalization arises, which may be considered, classically, a synonym of ergodicity and it is basically the statement that (1.6) holds. In quantum mechanics one would like to write down directly (1.6), with a density matrix

$$\hat{\rho}_{mc}(E, \Delta) = \frac{1}{Z_{E,\Delta}} \sum_{|E_n - E| < \Delta} |E_n\rangle\langle E_n| \tag{1.7}$$

in place of the classical Liouville density, and pretending that it describes a quantum isolated system when the number of its constituents is large. The motivation of this step would be, a part from a formal analogy with the classical case, the will to carry directly to quantum mechanics the successful idea according to which many-body systems at equilibrium may be described, on average, by a small number of macroscopic observables. Unfortunately this assumption is not easly justified at a formal level when the quantum system is isolated. In fact if it was an open system one would trace away the part of the Hilbert space relative to the environment resulting in a mixed state which may assume a thermal form like (1.7), but if there is nothing to trace way and the whole Hilbert space is the one in which the initial pure state lives, unitarity of temporal evolution prevents the system to reach a mixed state like (1.7) in the long time limit. Some trials in the direction of the solution to this problem will be described in the next section.

The notion of ergodicity is deeply related to the one of integrability, which is perfectly well defined in a classical system of n degrees of freedom. The latter is said to be integrable (in the Liouville sense) if it possesses a set of n functional independent integrals of motion $\{I_k\}$ commuting with each other, i.e. with zero Poisson brackets. If this happens to be true the equation of motion can be sistematically integrated and the phase space can be mapped via a canonical transformation in a n-dimensional torus identified by the n constant values $\{I_k\}$, depending on the initial condition, and spanned by n angles $\{\varphi_k\}$ (action-angle coordinates). In these coordinates the equation of motion are very simple and reads $\dot{I}_k = 0$, $\dot{\varphi}_k = \omega_k$, with ω_k real constants depending on the initial coniditon. Since the torus represents a very little portion of the whole constant-energy hypersurface of the phase space, which depends on the initial values of the actions I_k, in this case (1.6) cannot hold. Integrability is then a sufficient condition in order for (1.6) not to hold, but not necessary. Indeed a very celebrated theorem named after Kolmogorov, Arnold and Moser rules the behavoir of this torus when integrability is broken, stating basically that non-ergodicity is stable under small perturbations and that there is a sharp threshold in the strength of the integrability-breaking perturbation above which the motion becomes ergodic.
This being said let's now turn to the quantum case and consider a system with hamiltonian H. The first guess for a definition of quantum integrability leads to the conclusion that every quantum system is integrable. Indeed first of all the time evolution can be worked out sistematically for any initial state solving the eigenvalue problem for H

$$|\psi(t)\rangle = \sum_{k=1}^{\dim \mathcal{H}} |\langle E_k|\psi\rangle| e^{i\mathrm{Arg}(\langle E_k|\psi\rangle) - iE_n t} =: \sum_{k=1}^{\dim \mathcal{H}} I_k e^{i\varphi_k(t)} \tag{1.8}$$

so it seems that Liouville's integrability condition is not required in the quantum case. Overlooking this detail and pursuing the anology with the classical case, the last equation gives all the natural analogues with the classical system: the constant actions and the angles are the real-valued functionals defined on $\mathcal{H}$ given by

$$I_k[\psi] = |\langle E_k|\psi\rangle| \qquad\qquad \varphi_k[\psi](t) = \mathrm{Arg}\left(\langle E_k|\psi\rangle\right) - E_n t \tag{1.9}$$

These functionals also satisfies the same equations of motions of the action-angle coordinates mentioned in the classical case. The problem remaining to solve is that in this way, as anticipated, all quantum systems would be integrable. One may appeal to the fact that the I_ks chosen as above are not operators, for which there would be a more natural notion of commutativity. Then a second guess could be to simply require the existence of $n = \dim \mathcal{H}$ commuting operators which commute also with H. Once again this is a trivial requirement since it is enough to take the projectors on the energy eigenstates $I_k = |E_k\rangle\langle E_k|$, which are nothing but the operators related to the functionals (1.9). One soon realizes that a definition of quantum integrability is not easy to get and some of the classical analogies have to be lift. Nevertheless undoubtedly integrability must have to do with the existence of a certain number of conserved charges which scales with the number of degrees of freedom of the system, but what this number should be and which constraints should be put on these conserved charges in order to make quantum integrability non trivial is still unclear in general. This remains nowaday an open problem, although many attempts have been made (see for example [4] for a review), and it is worth noticing that there are many quantum models commonly accepted to be integrable in spite of the lack of a strict definition for their class.
To sum up, the notion of ergodicity and its relation to integrability is very well understood in classical mechanics, and leads to a quite rigorous justification[1] of the formalism used in statistical mechanics. In contrast in quantum statistical physics thermalization has not the same solid foundation in the

[1] It is worth noting that the only classical system rigorously proven to be ergodic is the hard sphere gas

dynamics of the elementary constituents of an isolated quantum system and it may be that some other mechanism than ergodicity has to be used to legitimize its success.

1.3 Non-integrability and thermalization

The main reason why it is difficult to deal with quantum ergodicity is the linearity of Schroedinger equation: classical chaos arises from the non-linearity of Hamilton's equations while in quantum mechanics any notion of non-linearity is absent. This is nothing but the very reason why it is difficult to assert in general which quantum mechanical systems are non-integrable. In fact the general time-evolution of a state can be written as in (1.8) precisely because of the superposition principle.
To get a little bit deeper in the problem let's translate the r.h.s of (1.6) in the quantum language, fixing the initial condition as an initial state $|\psi\rangle$ in the Hilbert space

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\langle\psi(t)|\hat{O}|\psi(t)\rangle = \sum_{m,n}^{\dim\mathcal{H}} \bar{c}_m c_n\langle E_m|\hat{O}|E_n\rangle \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\, e^{-i(E_m-E_n)t} = \sum_{n}^{\dim\mathcal{H}} |c_n|^2\langle E_n|\hat{O}|E_n\rangle \tag{1.10}$$

where the last step follows from a stationary phase approximation. The last member of this equation can be written as

$$\langle\hat{O}\rangle = \mathrm{Tr}\left(\hat{\rho}\,\hat{O}\right) \qquad \hat{\rho}_{diag} = \sum_n |c_n|^2 |E_n\rangle\langle E_n| \qquad c_n = \langle E_n|\psi\rangle \tag{1.11}$$

the density matrix above identifies the so called diagonal ensamble and its equivalence with the quantum microcanonical density matrix (1.7) is not straightforward at all, as in this case $\hat{\rho}_{diag}$ explictly depends on the initial state via the coefficients c_n. Assuming that the initial state lies in the energy shell selected via $\hat{\rho}_{mc}$, i.e. $c_n = 0$ if $|E_n - E| > \Delta$, the only way to make the two ensambles equal at the level of density matrices would be to require that $|c_n|^2 = \text{const}$, which of course cannot be possible for every initial state. Some different mechanism has to be invoked to explain the equality of the two ensambles and the most appealing possibility is called eigenstate thermalization hypotesis (ETH), first proposed in [6] and [7]. It states that the equality of the two ensambles should hold first of all only for some set of observables $\{\hat{O}_k\}$ and second only once their expectation values are considered

$$\mathrm{Tr}(\hat{\rho}_{mc}\hat{O}_k) = \frac{1}{Z_{E,\Delta}}\sum_{|E_n-E|<\Delta}\langle E_n|\hat{O}|E_n\rangle = \sum_n^{\dim\mathcal{H}} |c_n|^2\langle E_n|\hat{O}|E_n\rangle = \mathrm{Tr}(\hat{\rho}_{diag}\hat{O}_k)$$

The the ETH is basically the requirement that the matrix elements of these observables are constant on the energy shell and equal to the microcanonical average

$$\mathrm{Tr}(\hat{\rho}_{diag}\hat{O}_k) = \sum_n^{\dim\mathcal{H}} |c_n|^2\langle E_n|\hat{O}|E_n\rangle \simeq \mathrm{Tr}(\hat{\rho}_{mc}\hat{O}_k)\sum_n^{\dim\mathcal{H}} |c_n|^2 = \mathrm{Tr}(\hat{\rho}_{mc}\hat{O}_k)$$

In this light thermalization in quantum many-body systems arises in a completly different way with respect to its classical counterpart. In the latter case the initial state seems to have nothing to do with its thermal limit a part from sharing the same energy, while in the former the thermal state is already present in the initial one and only the dephasing process due to time average (1.10) is needed in order to disclose its thermal nature, which manifests itself once the appropriate observables are considered. This explains the name "eigenstate thermalization".
All these requirements reproduce the desired equality but for which class of observables they should

hold is not specified, of course they cannot be all the selfadjoint operators defined on the Hilbert space. Probably the most unanimous feature these observables should share is locality, i.e. they should be written as sums of operators which act non trivially only on a small subsystem of $\mathcal{H}$, e.g. for a lattice system they should be different from the identity only on few nearest neighbour sites

$$\hat{O} = \sum_a o_a \sim \int \mathrm{d}x\, o(x)$$

This of course requires the quite natural assumption that $\mathcal{H}$ factorizes in a tensor product of local Hilbert spaces, which, although its naturalness, leads to the loss of generality on the model under consideration. The assumption of locality is moved by the physical intuition that quantum mechanical averages of observables might reach an equilibrium value in the long time limit if the rest of the system in which the local term acts trivially behaves as the "thermal" bath which makes the time-dependent expectation values to relax, in this way even if time evolution is unitary and the density matrix of the whole system must remain that of a pure state for all times, when the rest of the system is traced away what remains may be a termal state[2]. Rephrased in terms of the density matrix of the system this means that although the limit

$$\lim_{t\to\infty} \hat{\rho}(t) = \lim_{t\to\infty} e^{-iHt}|\psi\rangle\langle\psi|e^{iHt}$$

cannot exist if $|\psi\rangle$ is a pure state, when measuring local observables in a subsystem A of the whole system what matters is the reduced density matrix on A and this might admit a mixed state as limit

$$\lim_{t\to\infty} \mathrm{Tr}_{A^c}\hat{\rho}(t) = \hat{\rho}_{\mathrm{mixed}}$$

At present there is no known analytical proof of ETH in some large class of models, moreover it has been argued not to be a necessary condition for thermalization [8]. Nevertheless it has been successfully verified numerically for various interacting systems in which it was supposed to hold [10, 12].

1.4 Integrability and equilibration

Up to this point no reference was made to integrability, as no mention to system which do not thermalize. In fact integrability, whatever it means, plays an essential role in studying the dynamics after a quantum quench, characterizing the nature of the steady state reached by the system in the long time limit.
The picture emerged in the last decade is that there is a class of systems for which not all local observables thermalize. From the theoretical side this problem dates back to the 70s, when the dynamics of some observables were computed in probably the simplest 1-dimensional quantum mechanical model defined on a lattice, namely the XY model [9], which is a spin model defined on the Hilbert space $\mathcal{H} = \otimes_{a=1}^{N}\mathbb{C}^2$ whose hamiltonian is

$$H = -\sum_{a=1}^{N}\left(\frac{1+\gamma}{2}\sigma_a^x\sigma_{a+1}^x + \frac{1-\gamma}{2}\sigma_a^y\sigma_{a+1}^y + h\sigma_a^z\right) \tag{1.12}$$

In recent years this issue has came up again, amazingly from experiments [17], and a general conjecture has been proposed in order to explain theoretically what distinguish systems which do thermalize from systems which do not [10].

[2] This is exactly the mechanism used to justify the arising of mixed states when the quantum system is open

In short the conjecture states that as long as local observables are considered they will always reach a stationary expectation value, but when the model which describes the system is integrable these expectation values cannot be obtained as averages over a quantum mechanical thermal ensamble[3] but over a generalized Gibbs ensamble described by the "relevant" conserved charges which characterize the initial state, whose density matrix would be given by

$$\hat{\rho}_{GGE} = \frac{e^{-\sum_k \lambda_k \hat{I}_k}}{Z} \qquad\qquad \lambda_k = \langle\psi|\hat{I}_k|\psi\rangle \tag{1.13}$$

with the Lagrange multipliers λ_k, typical of a variational ensamble, fixed by the expectation values of the conserved charges on the initial state $|\psi\rangle$. To distinguish this process from thermalization it has been given the name of equilibration. This conjecture motivates the need of a clear characterization of quantum integrable systems and it also recovers part of the analogy with the classical case, in that, even if the thermalization mechanism may be different, integrability is still a sufficient condition for non-thermalization. The reason for this is that the conservation laws constrain the dynamics in such a way that the asymptotic final state cannot be caracterized only by its initial energy, but also by the conserved quantum numbers related to these conserved charges.
However this conjecture suffers from the same amibiguites one runs into in defining quantum integrability, since the set of conserved charges is not univocal. Nonetheless it might be used to shed light on the properties these constants of motion shoud exhibit. Once again it seems that locality plays a role [15], but this time local does not refer to coordinate space but it is defined with respect to some other basis which identifies the fundamental excitations of the system, alias its quasiparticles. More on this will be discussed in the next chapters, when concrete examples will be at hand, roughly speaking in an integrable model the hamiltonian may be written as

$$H = \sum_k \varepsilon_k \, A_k^\dagger A_k$$

where the creation and annihilation operators act on the ground state of the system and produce excitations. Note first that this form of H does not mean these quasiparticles are free, in that they may interacts through non trivial commutation relations of these operators[4] and second that the summation contains much less terms than the dimension of the Hilbert space, since the structure of the latter is that of a Fock space of multiparticles states

$$\mathcal{H} = \sum_n \mathcal{H}_n \qquad\qquad A_{k_1} \dots A_{k_n}|0\rangle \in \mathcal{H}_n$$

where the vacuum $|0\rangle$ is as usal the one annihilated by all the A_k. Note also that this way of writing H is not guaranteed by the fact that these quasiparticles exist, but some extra requirements are necessary, namely these quasiparticles should mantain their identity upon scattering between each other, which in turn is ensured by the complete factorization of many-body scattering amplitudes into 2-body scattering processes. In this picture the conserved charges are simply the occupation number of each single-particle eigenmode $I_k = A_k^\dagger A_k$, so for what above they are much less than the dimension of the Hilbert space, and a local operator is such that it can be written as a product of few of these A and $A^\dagger$.
The validity of the GGE conjecture has been verified analitically in a wide variety of models [10, 13], and more recently also experimentally [14], but it has also been observed that there are some observables whose long time average is in fact not described by the GGE, the reason being these

[3] Namely the microcanonical ensamble, which of course in the thermodynamic limit gives the same prediction as the canonical one, whose density matrix operator is (1.5)

[4] In this very heuristic argument it has been implicitly assumed that the system lives in 1-dimensional space

observables are non local in the quasiparticles basis. However it has also been noticed that there are some models in which the GGE constructed from the conserved charges one would expect to identify the initial state completly fails in describing the long time expectation value of any observable, the reason being rooted in the non trivial quasiparticles structure of the model [16]. It still remains an open problem how to choose the constants of motion which characterize the equilibrium state and if it is always possible to describe, locally, the latter in terms of a mixed state density matrix of the form (1.13).

1.5 Experimental overview

Ultracold atomic gases provied the ideal framework to probe the out of equilibrium dynamics of closed quantum systems, since they can be considered to very high accuracy isolated from the environment. Moreover the possibility to use optical lattices to realize particular geometries, together with the development of many techniques to tune the interaction between ultracold atoms, made it possible to realize accurately specific quantum hamiltonians whose analytic treatment is feasible.
Probably the most known experiment realized in the last decade consists in confining $2 \cdot 10^5$ bosons (atoms of ^{87}Rb) in an array composed of some thousands of 1-dimension harmonic traps, each containing from 40 to 250 atoms [17]. This was realized via a 2D optical lattice, which creates the tubes, together with a crossed dipole trap necessary to keep the atomes confined in a specific part of the tubes. Each "tube" can be considered a 1-dimensional system of bosons interacting with a contact 2-body term which depends on the density of bosons per tube and subjected to a 1-body almost harmonic potential provided by the optical lattice.
The hamiltonian of this experimental setup is very well described by the so called Lieb-Liniger model plus harmonic potential, in which the Hilbert space is the Fock space composed of N-particles level $L^2(\mathbb{R}^N)$ and its hamiltoniam in second quantization form is

$$H = \int \mathrm{d}x \left(-\psi^\dagger(x) \frac{\partial^2}{\partial x^2} \psi(x) + 2c\, \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) + \omega^2 x^2 \psi^\dagger(x)\psi(x) \right) \tag{1.14}$$

The 2-body interaction can be tuned varying the number of bosons in each tube and the harmonic trap via changing the frequency of the ligth of the optical lattice. The Lieb-Liniger model without harmonic potential is integrable, via the so called Bethe ansatz [18]. This technique will not be discussed in details in this work. In short it is divided into three steps which closely resemble the quantum mechanical treatment of free particles: in the first one makes the ansatz that the N-body energy eigenfunctions are factorized in the form

$$\psi_{k_1,\dots,k_N}(x_1,\dots,x_N) = \sum_\sigma A\left(k_{\sigma(1)},\dots,k_{\sigma(N)}\right) \prod_{a=1}^{N} e^{i x_a k_{\sigma(a)}} \qquad x_1 < x_2 < \dots < x_N \tag{1.15}$$

where the sum runs over all permutations of the quantum numbers k_i in order to account for bosons statistics, in the second the amplitudes $A(k_1,\dots,k_n)$ are determined, other than the overall wave function normalization, via substitution of (1.15) into the stationary Schroedinger equation, in the third periodic boundary conditions are imposed on the wavefunction leading to Bethe equations, whose solutions are the sets of wavevectors $\{k_i\}$ identifying the true eigenstates. This machinery allows in principle to write down all the energy eigenfunctions together with their corresponding eigenvalues, from which would follow the whole dynamics from any initial state. Unfortunately to solve Bethe equations and to compute the overlaps of the initial state with the eigenfunctions might be a highly non trivial task, nevertheless models on which this procedure works are said to be integrable[5], since it

[5] It is worth noting that this is sometimes belevied to be also a necessary conditions for integrability

is possible, in principle, to write down their dynamics analytically at least in a implicit form. Finally note that this technique has no generalization to more than 1 dimension, which is the only feature common to all solvable models in quantum mechanics up to now. Actually the 1-dimensionality of the system, being deeply related to its integrability, is believed to be a necessary condition for non thermalization.
Coming back to the cold atoms experiment, the rubidium atoms are initially trapped and not allowed to oscillate along the direction transverse to the tubes via the crossed dipole trap. The non-equilibrium initial state is prepared through grating pulses of light in the direction parallel to the tube, splitting the atoms into two bunches, let's call $|A\rangle$ this initial state. The trap is then released and the atoms are let free to evolve for a time varying between 10^{-2} and 10^{-1} s, then an absorption image is taken to get the instantaneous position of the atoms for different times. The evolution of the systems nicely resembles that of a, quantum, Newton's cradle.
Doing the same measurement for each of the thousands of tubes, it is extracted exactly the expectation value of the density operator

$$\langle A(t)|\psi^\dagger(x)\psi(x)|A(t)\rangle$$

and what has been observed is that this does not relax to a gaussian distribution, which is the one expected from a thermal ensamble of bosons in an harmonic trap. The authors of [17] compared these results for the spatial distribution of the atoms with the corresponding one of a gas of bosons initially prepared in the ground states. Keeping into account the loss of some atoms ocurring at each complete oscillation of the cradle, they observe in the latter case a nearly gaussian distribution even for short times after the crossed trap is released. Despite the fact that (1.14) is a perturbation of the Lieb-Liniger integrable hamiltonian in addition to many anharmonicity effects caused by the experimental setup, it seems that if this system is prepared in an out of equilibrium state it does not equilibrate to a thermal ensamble. This also suggests a certain stability of non-thermalization when an integrable hamiltonian is perturbed via a non-integrable term.
This was just one of the many experiments aiming to probe the out of equilibrium dynamics of closed quantum systems. Not only the properties of the final stationary state have been investigated, but also the dynamics in approaching equilibrium and the effects of the finite size of the system [19, 20]. In the latter regard an issue not before mentioned which also has attracted some attention is to find, if there are, universal features in the relaxation process. This will be discussed from a theoretical point of view in chapter 4.

Chapter 2

Phase transitions, critical phenomena and conformal field theory

This chapter is a digression on the physics of critical phenomena and the powerful field theory techniques commonly used to deal with them. The subject is so wide that a detailed discussion of all the topics proposed below is impossible to achieve in few pages. For this reason what follows is unavoidably a summary which will be made as far as possible selfconsistent.
In section 1 the concept of phase transition will be introduced, both at the classical and quantum level, and a formal bridge will be built between these two. In section 2 field theory will be introduced as a natural mathematical framework to study critical phenomena and the arising of conformal invariance will be motivated. This will lead to conformally invariant field theories: the conformal group will be briefly discussed in general dimensionality and rapidly specialized to $D = 2$. Then the treatment of 2-dimensional CFTs will be addressed in general and the most simple CFTs classified. This long section will end with a comparison between quantities one can measure on the lattice and the field theory predictions at the critical point. Finally, in section 3, the very recent applications of all this machinery to the quench problem and its results will be presented.

2.1 Phase transitions and universal critical behavior

2.1.1 Classical phase transitions

Materials in nature can be found in different forms, or phases. The transition between one phase and another is driven by some parameter, e.g. temperature, pressure or magnetic field, and is characterized by discontinuities in the thermodynamic quantities. Very often the transition is between an ordered and a disordered phase and in this situation an order parameter m can be constructed, which is nonzero in the ordered phase and zero in the disordered one. Let's focus on the example in which the parameter driving the transition is the temperature, in this case the ordered and disordered phases are usually at low and high temperature respectively. Phase transitions in which an order parameter can be defined fall into two distinct classes: first order and continuous phase transitions. The former are the most common in nature but also the most boring one. They are characterized by phase coexistence, discontinuity in the order parameter and release of latent heat at the transition point. The reason why they are boring resides in the fact that they strongly depend on the microscopic details of the system in which they occur. The latter instead are much more interesting, the order parameter vanishes continuously at the transition point T_c

$$m \sim m_0(T_c - T)^\beta \qquad T \to T_c^-$$

and many other thermodynamic quantities show non analytic behavior, chracterized by some other critical exponents like β. They are interesting because the system displays critical behavior when crossing the transition point which is independent on the microscopic details, and for this reason is called universal. Criticality basically means that the system behaves as it were composed of only one indivisible block. This can be modelled identifying a length ξ, representing the size of the block in which the system may be divided in order to have a good effective description of it, which diverges[1] at the critical point with its own critical exponent

$$\xi \sim \xi_0^{\pm} \, |T - T_c|^{-\nu} \qquad T \to T_c^{\pm}$$

If this happens of course the microscopic details, which can be thought of as the geometry and the spacing of the underlying lattice used to build a model of the system under investigation, are irrelevant and what matters are just few numbers characterizing the non analyticity of the thermodynamic observables.
Capability of statistical mechanics in explaining the physics at a critical point was questioned for long time after its birth. After all the partition function, from which one can compute all thermodynamic quantities, is a sum of Boltzmann weights, which are analytic functions. As long as the sum is finite there is no hope to get some non analytic result, but if the thermodynamic limit is taken something weird may happen. This is what indeed Onsager discovered solving the Ising model on a 2-dimensional square lattice in 1944. He showed that non analyticity was indeed encoded in the theory and the critical exponents can be read off directly from his calculations. From that time the study of universal critical behavior in nature made giants strides ahead and it was soon realized that universality classes are mainly identified by the dimensionality of the system and by the symmetry it enjoys. From the knowledge of the symmetry group of a system a huge amount of information is immediately at hand and symmetry also plays a crucial role in identifying phases: the disordered one is the most symmetric and a typical configuration of the lattice variables will be as symmetric as the Hamiltonian, the ordered one instead will be less symmetric than the Hamiltonian because of spontaneous symmetry breaking. An example of this will be given below.

2.1.2 Quantum phase transitions

In the previous section the focus was on transitions driven by temperature and characterized by an ordered phase at low temperature. As the temperature is increased the order is progressively destroyed by thermal fluctuations until the order parameter vanishes and the transition occurs. But there is another kind of fluctuations which might drive a transition from order to disorder, namely quantum fluctuations. Quantum phase transitions occur in quantum systems at zero temperature, since what changes between the two phases is the nature of the ground state. There non-analyticity appears in the ground state energy while crossing the separation between the two phases. To clarify what this means consider the simplest possible example of a spin chain of N sites in which the Hilbert space is $\mathcal{H} = \bigotimes_{a=1}^{N} \mathbb{C}^2$ and the Hamiltonian is

$$H = -J \sum_{a=1}^{N} \left(\sigma_a^z \sigma_{a+1}^z + g \sigma_a^x \right) \qquad J, g > 0 \tag{2.1}$$

where σ_a^i is a Pauli matrix acting trivially on all the sites except the site a and periodic boundary conditions are imposed on the spins $\sigma_{N+1} = \sigma_1$. This is the so called quantum Ising model in a transverse field, the transverse field being the σ^x term, which is a field acting in a transverse direciton with respect to the "interaction direction" z. The reason for the name Ising model will

[1] In the thermodynamic limit in which the system size is infinite

become clear in the next section. The symmetry group under which H is invariant is $\mathbb{Z}_2$ and it is generated by the operator $\bigotimes_a \sigma_a^x$.
The first term in the Hamiltonian favours the alignment of all the spins in the same direction along the z-axis, either $|\uparrow\rangle$ or $|\downarrow\rangle$, while the second tries to put all the single-site states in the eigenstate of σ^x relative to the eigenvalue 1, let's denote it $|\rightarrow\rangle$. It is immediately clear that the role of the temperature is now played by the transverse field coupling g. In the limit $g \to 0$ the ground state is degenerate and one can take any combination of $|\uparrow\uparrow ... \uparrow\rangle$ and $|\uparrow\uparrow ... \uparrow\rangle$, forcing the system to choose one of these two in some ways[2]. The $\mathbb{Z}_2$ symmetry will be spontaneously broken and the ground state will no more be invariant. Instead for $g \to \infty$ the ground state is non degenerate and given by $|\rightarrow\rightarrow ... \rightarrow\rangle$. It follows that the properties of the ground state change drastically between these two perturbative regimes. However this is not a proof of the non analyticity of its energy and even if there were a critical point $g = g_c$ in between is not obvious where it should be located exactly. The answer turns out to be very simple, as the exact solution in the next chapter will show the transition point is $g_c = 1$ and the ground state energy (3.35) is in fact a non-analytic function of g near g_c, when $N \to \infty$. But this is not the end of the story, since of course it is possible to define the order parameter, which in the thermodynamic limit behaves as

$$m = \lim_{N\to\infty} \langle 0|\sigma_a^z|0\rangle = \begin{cases} m_0(g_c - g)^\beta & \text{if } g \to g_c^- \\ 0 & \text{if } g > g_c \end{cases}$$

the correlation length, which diverges at the critical point

$$\langle 0|\sigma_a^z \sigma_b^z|0\rangle = m^2 + e^{-|a-b|/\xi} \qquad \xi \sim \xi_0^{\pm} |g - g_c|^{-\nu}$$

and most importantly β and ν turn out to be exactly the same as the ones of the classical Ising model on a 2-dimensional lattice. This seems to indicate a correspondence between the two models, which holds in fact for many (D+1)-dimensional classical statistical theories on a lattice and D-dimensional quantum chains.

2.1.3 QC mapping

Specializing to $D = 1$, in particular to a quantum spin chain, the precise mapping mentioned in the last section brings the partition function of a quantum spin chain into the partition function of a classical 2-dimensional anisotropic lattice model defined on a cylinder or on a torus if on the chain are imposed open or periodic boundary conditions respectively. This is nothing but the well known result that the quantum partition function at temperature β of a quantum field theory in D dimension can be written as the path integral of an euclidean field theory in $D + 1$ dimensions, whose fields are periodic, with period β, in the imaginary time direction. Field theory in this case arises once the spatial lattice spacing is sent to zero on quantum D-dimensional lattice, that is a chain when $D = 1$. Consider then the Hamiltonian (2.1). Its partition function at finite temperature β reads

$$\mathcal{Z}_\beta = \mathrm{Tr}(e^{-\beta H}) = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \langle \sigma_1 \cdots \sigma_N | e^{-\beta H} | \sigma_1 \cdots \sigma_N \rangle$$

where the states on which the trace is taken are chosen to be product of eigenstates of σ^z, i.e.

$$\sigma^z|\sigma\rangle = \sigma|\sigma\rangle \qquad \sigma = \pm 1$$

Inserting now M completeness relations on this basis and using Trotter's formula

$$e^{A+B} \sim (e^{A/M} e^{B/M})^M \qquad M \to \infty$$

[2] For example imposing other boundary conditions or inserting a small longitudinal term $h \sum_a \sigma_a^z$, $|h| \ll 1$ in H

the quantum partition function can be written, in this limit, as

$$\mathcal{Z}_\beta \sim \sum_{\{\sigma_a\}} \sum_{\{\sigma_{a,1}\}} \cdots \sum_{\{\sigma_{a,M}\}} e^{\frac{\beta J}{M}\sum_{a,b}^{N,M}\sigma_{a,b}\sigma_{a+1,b}} \langle \sigma_1 \cdots \sigma_N | e^{\frac{\beta J g}{M}\sum_{a=1}^{N}\sigma_a^x} | \sigma_{1,1} \cdots \sigma_{N,1}\rangle \cdots$$

$$\cdots \langle \sigma_{1,M} \cdots \sigma_{N,M} | e^{\frac{\beta J g}{M}\sum_{a=1}^{N}\sigma_a^x} | \sigma_1 \cdots \sigma_N \rangle$$

While the diagonal part of H on this basis was trivial, the non diagonal part can be computed observing that the exponential of σ^x can be reexponentiated thanks to

$$\exp\left(\frac{\beta J g}{M}\sigma_a^x\right)_{\sigma,\sigma'} = \begin{pmatrix} \cosh\frac{\beta J g}{M} & \sinh\frac{\beta J g}{M} \\ \sinh\frac{\beta J g}{M} & \cosh\frac{\beta J g}{M} \end{pmatrix}_{\sigma,\sigma'} = e^{J_\tau \sigma\sigma' + A}$$

$$J_\tau = \log\sqrt{\coth\frac{\beta J g}{M}} \qquad A = \log\sqrt{\frac{1}{2}\sinh\frac{2\beta J g}{M}}$$

Note that this step is possible only because $e^{\alpha\sigma^x}$ is a matrix having equal diagonal and off-diagonal entries and that the transverse field term in (2.1) produces a spin-spin interaction in the new direction introduced by this "Trotterization" process. Inserting these expressions in $\mathcal{Z}_\beta$ and exploiting periodic boundary conditions on the chain

$$\mathcal{Z}_\beta = \lim_{M\to\infty} \sum_{\{\sigma\}} e^{-\bar{H}_M(\sigma) + MNA}$$

$$\bar{H}_M(\sigma) = -\sum_{a,b}^{N,M}\left(\frac{\beta J}{M}\sigma_{a,b}\,\sigma_{a+1,b} + J_\tau \sigma_{a,b}\,\sigma_{a,b+1}\right) \tag{2.2}$$

which is, as promised, the Hamiltonian of a 2-dimensional anisotropic Ising model on a cylinder (or torus) having only M spins in the imaginary time direction. It is now customary to define $\delta\tau := \beta/M$ and to keep the vertical[3] length of this $2D$ lattice fixed to β while sending the "time" spacing $\delta\tau$ to zero. In this way one gets a correspondence between expectation values evolved in imaginary, continuous, time in the quantum theory at finite temperature and expectation values in the classical lattice. Going through the same steps as before one can prove [22]

$$\frac{\mathrm{Tr}\left(e^{-\beta H}\mathscr{T} A(\tau_1)B(\tau_2)\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)} = \lim_{\delta\tau\to 0}\frac{\sum_{\{\sigma\}} A_{\tau_1}(\sigma)B_{\tau_2}(\sigma)e^{-\bar{H}_{\delta\tau}(\sigma)}}{\sum_{\{\sigma\}} e^{-\bar{H}_{\delta\tau}(\sigma)}} = \langle A_{\tau_1} B_{\tau_2}\rangle \tag{2.3}$$

where $\mathscr{T}$ denotes the τ-ordered product, which always arises when writing a "path integral" n-point function in terms of a "canonical one". A and B are two operators initially defined on the Hilbert space of the chain and $A(\tau) := e^{H\tau}Ae^{-H\tau}$ is the quantum operator evolved in imaginary time. In the r.h.s. of (2.3) the operators on the classical lattice are calculated on the corresponding (imaginary time) slices. Notice that the constant factor which diverges in the limit $M\to\infty$ does not appear when calculating expectation values on the classical lattice and that upon sending $\beta\to 0$ the classical lattice becomes infinite in both directions and the trace on the l.h.s. becomes the expectation value on the ground state. The map at zero quantum temperature will be the one considered in the following. From (2.3) follows an important general fact: the imaginary time 2-point function at zero temperature of an operator O defined on the chain can be written inserting a completeness relation on the basis of the eigenstates of the quantum Hamiltonian H

$$\langle 0|O(\tau)O|0\rangle = \sum_n \langle 0|O(\tau)|n\rangle\langle n|O|0\rangle = \sum_n e^{-(E_n-E_0)\tau}\,|\langle n|O|0\rangle|^2 \underset{\tau\to\infty}{\sim} e^{-\tau\Delta}\,|\langle n_0|O|0\rangle|^2$$

[3] Thinking the initial quantum chain horizontal

where n_0 is the lowest energy eigenstate which is coupled to the ground state by the operator O. If the $|n_0\rangle$ is the first excited state and the spectrum of H is thougth in terms of particles, as will be the case for (2.1), then $\Delta = E_1 - E_0$ is nothing but the rest mass of that particle. So one concludes that for such an operator the correlation length is equal to the inverse mass gap of the theory, in other words at the critical point ($\xi = \infty$) the energy gap goes to 0. This fact is very general and it is the first hint that critical phenomena are described by theories of massless particles.
Up to now it was shown that there is a correspondence between quantum and classical models which has been explicitly constructed for the quantum Hamiltonian (2.1). Moreover it was claimed that this quantum system undergoes a quantum phase transition at the critical value $g = 1$, but no direct relation between the classical and the quantum transitions was exhibited. This relation is in fact quite easy to get: first let's move to quantum zero temperature $\beta = \infty$ and in addition keep $\delta\tau = \beta/M =: K$ fixed. In this way (2.2) is again the Hamiltonian of a 2-dimensional anisotropic classical Ising model, but this time J_τ is not diverging and it may be set equal to a finite coupling L. So one has the two relations

$$K = \delta\tau = \frac{\beta}{M} \qquad\qquad L := J_\tau = \log\left(\coth(\delta\tau g)\right) = \frac{1}{2}\log\left(\coth(Kg)\right)$$

Upon setting $g = 1$ into the zero temperature quantum Hamiltonian, i.e. placing H on the quantum phase transition point, one obtains from this relation

$$e^{2L} = \coth K \qquad\Rightarrow\qquad \sinh(2L)\sinh(2K) = 1$$

This is the famous relation for the critical couplings of the classical 2-dimensional Ising model. This relation between quantum and classical phase transitions, as already stressed above, does not stop to the location of the critical point but it persists in the critical behavior exhibited by the classical and the quantum models, which have the same critical exponents and they are thus members of the same universality class. The interesting fact is that the correspondence between quantum and classical critical points is not one to one, indeed there is no freedom in the quantum model once $g = 1$. In the classical model instead one can remain at the critical point varying the anisotropy parameter K/L. This is the first manifestation of universality: the anisotropy, which is a microscopic detail, is irrelevant for the critical behavior of the system.
To conclude this section let's complete the correspondence between classical and quantum models comparing first order and continuous phase transitions in the classical and quantum case. First note that in the QC-mapping the quantum propagator $e^{-\delta\tau H}$ corresponds to the transfer matrix T of the classical lattice model, so that their eigenvalues will be related by $\lambda_n = e^{-\delta\tau E_n}$, where n runs from 0 to $2^N - 1$ in the case of (2.1). Denoting N_s the number of sites of the 2-dimensional classical lattice, its partition function reads

$$Z = \mathrm{Tr}\left(T^{N_s}\right) = \sum_n \lambda_n^{N_s} \underset{N\to\infty}{\sim} \lambda_0^{N_s}$$

The last step is what one usually does in calculating the free energy per site as $f = -\log\lambda_0/N_s$ and it is indeed justified by the fact that the thermodynamic limit should be taken before looking if there is a phase transition. However it obscures the difference between the non critical behavior of a first order transition and the critical behavior of a continuous one. Indeed to make the free energy non-analytic the partition function must vanish at some critical temperature T_c and this can happen either if only the first eigenvalue vanishes or if an infinite number (for $N_s \to \infty$) of them do so. The first situation corresponds to a level crossing in the lowest excited states of the quantum Hamiltonian H, which leads to a discontinuity in the entropy $s \propto \partial_T f$ i.e. disengagement of latent heat. This is not a critical phenomenon since it involves only the lowest two energy eigenvalues of H. In the second case instead not only the gap of H is closing and a particle of the theory is becoming massless, but also an infinite number of eigenstates are collapsing on the ground state. This is the collective behavior typical of criticality as sketched in the previous section.

2.2 Conformal field theories

It should be clear at this point that critical phenomenon basically means infinite correlation length in the system and independence from the microscopic details, from which universality follows. However $\xi = \infty$ means nothing in physics, since being a length it has to be compared with a length. Thinking of an underlying lattice what has to be infinite in order for the system to exhibit collective behavior is the ratio ξ/a where a is the lattice spacing[4]. This implies that at whatever scale the system is observed it will look the same, since there will be no glimps of a microscopic structure for a huge range of length scales down to a. This leads to the notion of scale invariance of the physics at the critical point and to the idea that statistical field theory can be used to get rid of all the irrelevant microscopic degrees of freedom, alias the high energy modes, to make calculations much simpler than on the lattice, still being able to extract predictions on the universal behavior which unite all the systems differing only at the microscopic scale. For example a system of spins $\sigma_{i_1,...,i_D}$ on a D-dimensional lattice becomes a field theory simply by replacing

$$\{i_1, ..., i_D\} \in \mathbb{Z}^D \;\longrightarrow\; (x_1, ..., x_D) \in \mathbb{R}^D \qquad \sum_{i_1,...,i_D} \;\longrightarrow\; \frac{1}{a^D} \int \mathrm{d}x_1, ..., \mathrm{d}x_D$$

which means taking the so called continuum limit of the lattice theory. Then the partition function of statistichal mechanics becomes a path integral over the field configuration on a continuous space

$$Z = \sum_{\{\sigma\}} e^{-\beta H(\{\sigma\})} \sim \int \mathcal{D}\phi \, e^{-\mathcal{A}[\phi]}$$

together with any expectation value of observables defined on the lattice

$$\langle O \rangle = \frac{1}{Z} \sum_{\{\sigma\}} O(\{\sigma\}) e^{-\beta H(\{\sigma\})} = \int \mathcal{D}\phi \, O[\phi] \, e^{-\mathcal{A}[\phi]}$$

This defines a statistical field theory, also called euclidean quantum field theory, since can be obtained from a QFT placing an imaginary unit in front of the real time. This theory will be relativistic invariant, since the invariance of the lattice under discrete translations and rotations becomes the full Poincaré invariance (in imaginary time) in the continuum limit. Once the field theory is given the notion of scale invariance becomes formally much more practical with respect to the lattice. One supposes that the field theory describing the physics of the critical point is ruled by an action $\mathcal{A}^*$ which is invariant under rescaling of coordinates $x \to \lambda x$, $\lambda > 0$ and that there exists a set of scaling fields $\{\phi_i\}$ which are well-behaved under this rescaling

$$\phi_i(x) \;\longrightarrow\; \phi_i'(x) = \lambda^{\eta_i} \phi_i(\lambda x) \tag{2.4}$$

Here the scaling dimension η_i is conventionally defined in such a way that $[\phi_i] = [\text{length}]^{-\eta_i}$. These fields are required to form a basis for all the composite operators one may put into the action in order to move away from the critical point, for example moving a little the temperature from the critical value T_c in the real system. There a field theory description of the lattice is expected to be valid since the correlation length is still very large

$$\mathcal{A} = \mathcal{A}^* + \sum_i g_i \int \mathrm{d}^D x \, \phi_i(x) \tag{2.5}$$

[4] In any of the two directions, since, as last section showed, a quantum problem in imaginary time can always be mapped into a classical lattice

The scaling fields are then classified in terms of their dimension η. They are said to be relevant, irrelevant or marginal if their scaling dimension is smaller, greater or equal to the dimensionality D of the lattice. The relevant fields will be the ones important for the physics at large distance and they will encode the universal behavior which field theory is supposed to capture. However this is possibile only if the critical exponents are directly related to their scaling dimension. In fact these relations can be simply obtained from the scaling hypotesis (2.4) and from dimensional analysis on the action. For example suppose one of the relevant fields in (2.5) is the one associated to the temperature, let's call it ϕ_ε. Then its parameter $g_\varepsilon \propto (T - T_C)$ will have the dimension of a length to the power of $\eta_\varepsilon - D$, as follows from the fact that the action $\mathcal{A}$ must be dimensionless. But near the critical point the only dimensionful quantity provided by the field theory is the correlation length ξ. So one concludes that

$$g_\varepsilon \sim \xi^{\eta_\varepsilon - D} \qquad \Rightarrow \qquad \xi \sim g_\varepsilon^{1/\eta_\varepsilon - D} \sim (T - T_c)^{1/\eta_\varepsilon - D} =: (T - T_c)^{-\nu} \qquad \Rightarrow \qquad \nu = \frac{1}{D - \eta_\varepsilon} \quad (2.6)$$

In the same way it turns out that all critical exponents characterizing the behavior of observables approaching the critical point can be obtained from the set of dimensions $\{\eta_i\}$ of the relevants field of the field theory near the critical point. Note that these dimensions may not be the canonical ones obtained by dimensional analysis of the action. Indeed is a well known fact in quantum field theory that interaction might affect the scaling properties of its operator content in such a way that local operators acquire anomalous dimensions different from the engineering ones.
All these ideas date back to the 60s, when Kadanoff and Wilson introduced the Renormalization Group (RG) in low energy physics. Their work was for sure an enormous breakthrough towards the understanding of universality, but the quantitative results achievable from it were still either numerical or perturbative. RG was capable of providing critical exponents from field theory through relations like (2.6), but the only mathematical tool available at that time to get the scaling dimensions $\{\eta_i\}$ was perturbation theory, which does not allow to obtain exact results as well as being unreliable in many cases. Moreover it is not easy in general to grasp the action $\mathcal{A}^*$ which should describe the scaling limit of the lattice model and in fact Wilson's perturbative expansion used as small parameter the dimensionality of the system $\varepsilon = D - D_C$, where D_C is a certain critical dimension, called upper critical dimension, in which the critical action is known. The knowledge of the symmetries the theory enjoys of course helps in constructing $\mathcal{A}^*$, together with all the scaling operators, starting from few fundamental objects living in some representation of the symmetry group. Nonetheless this is usually not enough and one has to find some other way to gets quantitative predictions from field theory. It would be nice if there were no need of an action formulation and the scaling limit of lattice correlation functions could be computed directly from other principles. This is indeed the case if one takes seriously one of the other intuitions about critical phenomena formalized by Wilson, namely the operator product expansion (OPE). Since near the critical point the system is the same if observed at many different scales, one can look at it "from very far" while measuring the same physics. But from long distances it is impossible to distinguish between a complicated distribution of sources and a sum over multipole-like charges placed in the middle of this distribution. Then it is natural to think that in a field theory which pretends to be scale invariant any product of operators at different points may be written as a, possibly infinite, sum of local operators evaluated at just one point. Then exploiting translational and rotational invariance the OPE takes the form[5]

[5] Hereafter all the expectation values taken in the field theory are meant in the path integral formalism

$$\langle ... \rangle = \int \mathcal{D}\phi \, (...) \, e^{-\mathcal{A}[\phi]}$$

$$\langle \dots \phi_i(x)\,\phi_j(y) \dots \rangle = \sum_k \frac{c_{ij}^k}{|x-y|^{\eta_i+\eta_j-\eta_k}} \langle \dots \phi_k(y) \dots \rangle \tag{2.7}$$

This equations defines the so called algebra of local fields[6]. Starting from it and making the most of scale invariance, it turns out that, in the very special 2-dimensional case, the task set above is in fact realizable and expectation values of observables on the lattice can be computed directly in the field theory, without even knowing the explicit form of $\mathcal{A}^*$. The main steps one has to go through to compute all the correlators in the field theory are basically two. The first is to observe that scale invariance implies that 1-point functions of any scaling field vanish[7] and constrains the form of their 2-point functions (and in fact also 3-point functions) to be

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{A_{i,j}}{|x-y|^{\eta_i+\eta_j}} \tag{2.8}$$

Then conformal invariance, which will soon be defined, implies that $A_{i,j} = \delta_{i,j}$. The second is to find a way to get the structure constant c_{ij}^k in (2.7) in order to use this equation as kind of non-perturbative Wick theorem to compute all the correlators of the theory. Of course this would in principle work in any dimension and the reason why it is realizable in $D = 2$ will be explored below.
In the following sections, in order to avoid dealing with operatorial identities which require a definition of the appropriate order for the operators, equations like (2.7) will be written leaving implicit that the product of the operators is meant once the expectation value is taken.

2.2.1 The conformal group

The conformal group acting on the D-dimensional euclidean space is defined as the class of transformations leaving invariant the euclidean quadratic form up to rescaling. Taking a general infinitesimal transformation this means

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x) \qquad \delta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \sum_{\mu=0}^{D-1} (\mathrm{d}x^\mu)^2 = \lambda \sum_{\mu=0}^{D-1} (\mathrm{d}\tilde{x}^\mu)^2 = \lambda\,\delta_{\mu\nu}\,\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu$$

The condition on the right above is equivalent to the so called conformal Killing equation for the infinitesimal displacement $\xi^\mu(x)$

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}\,\delta_{\mu\nu}\,\partial\cdot\xi \tag{2.9}$$

This equation is satisfied by the following infinitesimal transformations

$$\xi^\mu = \varepsilon^\mu \qquad \xi^\mu(x) = \omega^\mu{}_\nu x^\nu \qquad \xi^\mu(x) = \varepsilon x^\mu \qquad \xi^\mu(x) = b^\mu x^2 - 2(b\cdot x)x^\mu \tag{2.10}$$

where ω is antisymmetric matrix. The first two are translations and rotations and they generate the full Poincaré group. The third is the, still linear, rescaling transformation which should leave invariant the critical action $\mathcal{A}^*$. Finally the fourth is a set of non linear transformations called special conformal transformations (SCT). Counting the number of free parameters one discovers that the generators of the conformal group are $(D+1)(D+2)/2$ and in fact when acting on the Euclidean

[6] From a technical point of view these expansion can be carried out in any QFT. What conformal invariance ensures is that this formal series is convergent in a whole neighborhood of $|x-y| = 0$ and resumming it one is able to get the exact non perturbative correlator on the l.h.s.

[7] It follows from

$$\langle \phi_i(x)\rangle_{\mathbb{R}^D} = \lambda^{\eta_i} \langle \phi_i(\lambda x)\rangle_{\lambda\mathbb{R}^D} = \lambda^{\eta_i} \langle \phi_i(x)\rangle_{\mathbb{R}^D} \qquad \forall\lambda$$

space it is isomorphic to $SO(D+1,1)$, since it can also be represented on the $(D+2)$-dimensional Minkowski space through appropriate combination of the generators of the Lorentz group in $D+2$ dimensions. The full algebra of the D-dimensional conformal group will not be presented and this synthetic exposition is going to move rapidly to $D=2$.
In order to find representations of the conformal group in terms of fields the only necessary ingredient is the energy-momentum tensor $T_{\mu\nu}$. It usually arises in field theory as the vector of Noether currents associated to translational symmetry, but it might be defined as the response of the action under a general infinitesimal transformation

$$\delta x^\mu = \xi^\mu(x) \qquad \Rightarrow \qquad \delta A \propto \int \mathrm{d}^D x \, T_{\mu\nu} \, \partial^\mu \xi^\nu \tag{2.11}$$

It immediately follows from this definition that the invariance of the theory under the first three transformations in (2.10) implies, in the same order, that the energy-momentum tensor is conserved ($\partial_\mu T^{\mu\nu} = 0$), symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and traceless ($\delta^{\mu\nu} T_{\mu\nu} = T^\mu{}_\mu = 0$). Using these facts and inserting the infinitesimal SCT in the definition above one gets

$$\delta A \propto \int \mathrm{d}^D x \, T_{\mu\nu} \, (2b^\nu x^\mu - 2b^\mu x^\nu - 2(b\cdot x)\delta^{\mu\nu})$$

but the first two terms cancel because $T_{\mu\nu}$ is symmetric, while the latter is zero thanks to rescaling invariance. So if the theory is scale invariant in addition to Poincaré invariant, it is automatically invariant under the whole conformal group. This result goes under the name of Polyakov's theorem and it is the reason why from the physical requirement of invariance under rescaling a wider symmetry arises and it follows only from the mathematics.
This symmetry becomes even wider when one specializes to the case of $D=2$, indeed (2.9) says that the two components of the infinitesimal transformation $\xi^\mu(x)$ are the real and imaginary part of a complex analytic function, since the former are nothing but the Cauchy-Riemann equations

$$\partial_0 \xi_1 = -\partial_1 \xi_0 \qquad\qquad \partial_0 \xi_0 = \partial_1 \xi_1$$

So that switching to complex variables $z = x^0 + ix^1$ these equations implies that the function $\xi = \xi^0 + i\xi^1$ is a conformal transformation if it statisfies the Cauchy-Riemann equations, i.e. $\partial_{\bar{z}} \xi = 0$. This means that when $D=2$ the conformal group is generated, infinitesimally, by all the analytic functions. But since any of such functions may be Laurent expanded with infinite coefficients, one concludes there is an infinite number of generators. However note that the Killing equation (2.9) is a local condition, since it is written for infinitesimal transformations. This means that in general a conformal transformation satisfying (2.9) will not be globally well-defined, as is the case for analytic functions with cuts. This in turn implies that the $2D$ action $\mathcal{A}^*$ will not be invariant under all analytic maps $\tilde{z} = z + \xi(z)$, but only under the finite dimensional conformal group of global transformations, which in complex notation and in finite form is given by all the Möbius maps

$$\tilde{z} = \frac{az+b}{cz+d} \qquad ad - bc = 1 \qquad a, b, c \in \mathbb{C}$$

This group has dimension $6 = (D+1)(D+2)/2|_{D=2}$ since its transformations depend on 3 complex parameters, as it should. Nonetheless it is possible to find fields which are well behaved under any analytic map, these are called primary fields and are the central objects above which the construction of an exact solutions of the CFT is possible.
Note that instead of taking the infinitesimal transformation $\xi = x^0 + ix^1$ one can consider the antiholomorphic displacement $\bar{\xi} = \xi^0 - i\xi^1$, satisfying $\partial_z \bar{\xi} = 0$ as a consequence of (2.9). This different choice may be used to obtain more general representation of the conformal group in terms of fields which depend on z and $\bar{z}$ as they were independent variables. Then set $\bar{z}$ equal to the complex conjugate on z only at the end of the day.

2.2.2 2-dimensional CFTs

The breakthrough in the construction of the exact solution of many $2D$ conformal field theories came from a very celebrated article written by Belavin, Polyakov and Zamolodchikov (BPZ)[26]. They were able to find sets of operators $\{\phi_k\}$ for which the OPE algebra (2.7) closes, identifying in such a way the full operator content of the field theory. Moreover this set, still being infinite, can be characterized by only a finite number of operator, i.e. the primary fields. Hereafter complex coordinates $(z, \bar{z})$ will be used, in which the euclidean metric tensor $g_{\mu\nu}$ becomes off-diagonal

$$\begin{cases} z = x^0 + ix^1 \\ \bar{z} = x^0 - ix^1 \end{cases} \qquad \Rightarrow \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0 \qquad g_{z\bar{z}} = g_{\bar{z}z} = 2$$

and the energy momentum tensor components will be denoted

$$T_{zz} =: T \qquad T_{\bar{z}\bar{z}} =: \bar{T} \qquad 4T_{z\bar{z}} = g^{\mu\nu}T_{\mu\nu} =: \Theta$$

so that its conservation law reads

$$\partial_\mu T^{\mu\nu} = 0 \qquad \Rightarrow \qquad \begin{cases} \partial_{\bar{z}} T + \dfrac{1}{4}\partial_z \Theta = 0 \\ \partial_z \bar{T} + \dfrac{1}{4}\partial_{\bar{z}} \Theta = 0 \end{cases}$$

This implies that when $\Theta = 0$, i.e. when the theory is scale invariant, the non zero components of $T_{\mu\nu}$ become holomorphic and antiholomorphic functions.
In order to find representations of the conformal group in terms of fields one has first of all to find an abstract expression for the action of the generators of the conformal group on a generic field, in such a way to find their commutation relations. Note that a representation in terms of functions is easly obtainable Laurent expanding the general infinitesimal transformation

$$\tilde{z} = z + \xi(z) = z + \sum_{n=-\infty}^{\infty} c_n z^n \qquad \Rightarrow \qquad f(z + \xi(z)) = f(z) + \xi(z)\partial_z f(z) + \mathcal{O}\left(\xi^2\right)$$

then setting all but one the group parameter c_n to zero

$$f(z + \xi) = f(z) + c_n z^n \partial_z f(z) =: f(z) - c_n \left(L_{n-1} f\right)(z) \qquad \Rightarrow \qquad L_n = -z^{n+1}\frac{\partial}{\partial z}$$

These differential operators satisfy the commutation relations

$$\left[L_n, L_m\right] = (n - m)L_{n+m} \tag{2.12}$$

which define the so called Witt algebra. Note that the operators $\{L_{-1}, L_0, L_1\}$ form a subalgebra which is in fact the group of the six Möbius transformations, representing the two SCTs, the rotation and dilatation and the two translations respectively. This subalgebra is isomorphic to $sl_2(\mathbb{C})$. In the case of fields the situation is different, since quantum, alias statistical, fluctuations produce an anomaly in these relations. To see this one has first of all to realize that the operator generating all the conformal tansformations is the energy momentum tensor, more precisely the infinite number of generators are its Laurent components

$$T(z) =: \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad \bar{T}(\bar{z}) =: \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} \tag{2.13}$$

These relations can be inverted using $\oint_C \mathrm{d}z/z^{n-m+1} = 2\pi i\,\delta_{n,m}$ to get the L_n in terms of $T(z)$

$$L_n = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, z^{n+1}T(z) \qquad\qquad \bar{L}_n = \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\, \bar{z}^{n+1}\bar{T}(\bar{z}) \tag{2.14}$$

where C is a circle around the origin of the complex plane. Not surprisingly it turns out that $T(z)$ and $\bar{T}(\bar{z})$ commute, so that the same holds for the L_n and $\bar{L}_n$. This comes from the hypotesis that a path integral formulation of the theory exists, even if it is never necessary to write down the action explicitly, and from the definition of the energy-momentum tensor (2.11). Indeed from these two facts one gets, after a bit of work, that the variation of any bilocal[8] operator $O(z,\bar{z})$ under infinitesimal conformal transformations $z \to z + \xi(z)$ or $\bar{z} \to \bar{z} + \bar{\xi}(\bar{z})$ reads

$$\delta_\xi O(z,\bar{z}) = \frac{1}{2\pi i}\oint_C \mathrm{d}w\, \xi(w)T(w)O(z,\bar{z}) \qquad\qquad \delta_{\bar{\xi}} O(z,\bar{z}) = \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{w}\, \bar{\xi}(\bar{w})\bar{T}(\bar{w})O(z,\bar{z}) \tag{2.15}$$

This relations mark the splitting between the holomorphic and antiholomorphic sectors of fields, which bears much resemblance to the splitting of the right and left moving solutions of the wave equations: they are independent as long as no particular boundary condition is imposed. The definition of the generators (2.14) together with (2.15) are enough to get the L_n's and $\bar{L}_n$'s commutation relations if one supposes to know the OPE of $T(z)$ with itself, which is equivalent via (2.15) to the transformation law of T under the conformal group. The latter turns out to be

$$T(z) \longrightarrow T'(w) = T(z(w))\left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2 + \frac{c}{12}\big\{z(w), w\big\} \tag{2.16}$$

where $\big\{w(z), z\big\} = w'''/w' - \frac{3}{2}\,(w''/w')^2$ is called Schwarzian derivative, the apex denoting the derivative w.r.t. z. The pure number c is called central charge, it is in general nonzero and can be spotted only once the energy-momentum tensor is known. The reason why it has been given this name will become clear in few lines. Upon putting (2.16) in infinitesimal form, substituting it in (2.15) and inverting that relation with Cauchy theorem, the OPE of $T(z)$ with itself reads

$$T(w)T(z) = \frac{c/2}{(w-z)^4} + \frac{2\,T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \text{ Regular terms} \tag{2.17}$$

The other way around one can argue that this is the only OPE consistent with dimensional analysis. Indeed the energy momentum tensor must have dimension 2 and cannot get anomalous corrections since it is a conserved current. Then supposing the theory does not contain operators with negative dimension[9], the most singular term in the OPE of $T(z)$ with itself must be of the form above. One can then exclude a third order pole by symmetry arguments and fix the other terms once again by dimensional analysis. Finally from (2.15) one gets (2.16).
Using (2.17) together with (2.14), the commutation relations of the conformal group generators in the field theory are obtained

$$\big[L_n, L_m\big] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \tag{2.18}$$

This defines the Virasoro algebra, which is sometimes referred to as the central extension of the Witt algebra (2.12), since it contains a new operator which commutes with all the others, namely the central charge c. This explains both the name central and charge: it lives in the center of the algebra and it is a new Noether's charge associated to the conformal symmetry. Of course the same

[8] Only because z and $\bar{z}$ are still considered independent
[9] This will be justified later

commutation relations are satisfied by the generators of the antianalytic sector.
Now is the time to define the primary fields, they are a very special subclass of scaling fields which transform covariantly under any analytic function, not only the rescaling transformation as the fields in (2.4)

$$\phi_i(z,\bar{z}) \;\longrightarrow\; \phi_i'(w,\bar{w}) = \left(\frac{\partial z}{\partial w}\right)^{\Delta_i} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{\Delta}_i} \phi_i(z(w),\bar{z}(\bar{w})) \tag{2.19}$$

Here Δ_i is called conformal weight and different weights are allowed on the two sectors, being them independent. Note that considering the rescaling $z(w) = \lambda w$, with λ real, and taking momentarily for $\bar{z}$ the complex conjugate of z, one gets the scaling dimension in terms of the two conformal weights

$$\phi_i'(w,\bar{w}) = \lambda^{\Delta_i+\bar{\Delta}_i}\phi_i(\lambda w,\lambda\bar{w}) \qquad \Rightarrow \qquad \eta = \Delta_i + \bar{\Delta}_i$$

Note also that if the two weights are different the field gets a non trivial transformation law under the rotation $z = e^{i\theta}w$, since (2.19) implies

$$\phi_i(z,\bar{z}) \;\longrightarrow\; \phi_i'(w,\bar{w}) = e^{i(\Delta_i-\bar{\Delta}_i)\theta}\phi_i(e^{i\theta}w, e^{-i\theta}\bar{w}) =: e^{is\,\theta}\phi_i(e^{i\theta}w, e^{-i\theta}\bar{w})$$

The quantity s is called conformal spin and has nothing to do with the usual spin. Henceforth only the analytic sector will be considered and the dependence on $\bar{z}$ of the fields will not be written explicitly, given that all the transformations will be performed only on $\bar{z}$. The construction for the antianalytic sector goes exactly the same way. Consider then (2.19) in infinitesimal form, i.e.

$$\phi_i(z) \;\longrightarrow\; \phi_i'(w) = (1+\partial_w\xi)^{\Delta_i}\,\phi_i(w+\xi) \sim \phi_i(w) + (\Delta_i\,\partial_w\xi + \xi\,\partial_w)\phi_i(w)$$

as before this transformation law is enough to get the OPE of this primary field with the energy-momentum tensor, yielding

$$T(w)\phi_i(z) = \frac{\Delta_i\phi(z)}{(w-z)^2} + \frac{\partial_z\phi_i(z)}{w-z} + \text{ Regular terms} \tag{2.20}$$

Setting $z = 0$ in this expression, replacing $T(w)$ with (2.13) and integrating around the origin upon multiplying by w one gets that all but the first term on the r.h.s. drop out and what remain is

$$L_0\,\phi_i(0) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}\oint_C \mathrm{d}w \frac{L_n}{w^{n+1}}\phi_i(0) = \frac{1}{2\pi i}\oint_C \mathrm{d}w \frac{\Delta_i\phi_i(0)}{w} = \Delta_i\phi_i(0) \tag{2.21}$$

Moreover in the same way it is easy to show for a primary field that $L_n\,\phi_i(0) = 0$ if $n > 0$. So it seems that in some sense the primary fields are eigenstate of L_0, with eigenvalue their conformal weight, and they are annihilated by all the L_ns with positive n. These statements can be made more precise moving to the Hilbert space of the theory, which might be quantized via foliations of the euclidean space in spheres centered at the origin, in contrast to the usual quantization procedure in which the Minkowski spacetime is foliated in $t =$ const hypersurfaces. Instead of propagating states between the leaves via the the time-evolution generated by the Hamiltonian, one has to use the dilatation to move from one sphere to another and these are generated by L_0. So by analogy one may build an operators-states correspondence defining the states at the origin

$$|\phi_i\rangle = \phi_i(0)|0\rangle \tag{2.22}$$

and then propagate them all other the Hilbert space. The "vacuum" state might be identified by the requirement of being invariant under the $sl_2(\mathbb{C})$ algebra generated by L_0 and $L_{\pm1}$, i.e. the global

conformal transformations, and possibly also under the rest of the, local, symmetries enjoyed by the theory. But this is not possible[10] and one is happy with the following definition of the vacuum

$$L_n|0\rangle = 0 \qquad \forall n \geq -1 \tag{2.23}$$

For what said above if $\phi_i(0)$ is a primary field then

$$L_0|\phi_i\rangle = \Delta_i|\phi_i\rangle \qquad\qquad L_n|\phi_i\rangle = 0 \quad \forall n > 0$$

Now using this definition of the Hilbert space of theory it is possible to start the algebraic construction of the representations of (2.18). They are lowest weight representations (LWRs) constructed above the states corresponding to primary fields, which will be labelled by the eigenvalue of L_0, i.e. their conformal weight: $|\phi_i\rangle =: |\Delta_i\rangle$. Using (2.18) it is easy to prove that the L_{-n} with $n > 0$ has the effect of raising the eigenvalue of L_0

$$L_0 L_{-n_1} \dots L_{-n_k}|\Delta_i\rangle = (\Delta_i + n_1 + \dots + n_k)|\Delta_i\rangle \qquad \Rightarrow \qquad L_{-n_1} \dots L_{-n_k}|\Delta_i\rangle = |\Delta_i + n_1 + \dots + n_k\rangle \tag{2.24}$$

This makes the LWR an infinite tower of levels identified by their eigenvalue w.r.t. L_0. All the states forming this representation other than $|\Delta_i\rangle$ are called descendants of the primary field ϕ_i and they correspond to some scaling field in the theory. Starting from the 0-th level, which contains only the lowest weight, the n-th level will contain as many states as the number of partitions of an integer $P(n)$. This structure is called Verma module and each Verma module provides a representation of (2.18). At this point there are no restrictions on c and on the conformal weight Δ of the primary field which generates the LWR, and each LWR is labelled by these two numbers. The restrictions on c and Δ comes from a very physical requirement, namely the absence of negative normed states in the representation spaces. For instance, assuming to know that the proper definition of the scalar product between states implies that $L_n^\dagger = L_{-n}$, using (2.18) and (2.24) to compute the norm of the state $L_{-n}|\Delta\rangle$ at level n one has

$$\|L_{-n}|\Delta\rangle\|^2 = \langle\Delta|L_n L_{-n}|\Delta\rangle = \langle\Delta|\big[L_n, L_{-n}\big]|\Delta\rangle = 2\Delta + \frac{c}{12} n(n^2-1) \geq 0$$

This implies for $n \to \infty$ that $c \geq 0$, while for $n = 1$ that $\Delta \geq 0$. A more careful analysis of the nonnegativity requirement on the theory, which involves many astonishing mathematical results as the computation of the determinant of the $P(n) \times P(n)$ Gram matrix at level n (Kac formula), shows that a representation (Δ, c) of (2.18) has no negative normed states if either $c \geq 1$ and $\Delta \geq 0$ or

$$c = 1 - \frac{6}{p(p+1)} \qquad \Delta_{r,s} = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)} \qquad p = 3, 4, 5, \dots \qquad \begin{matrix} r = 1, 2, \dots, p \\ s = 1, 2, \dots, p+1 \end{matrix} \tag{2.25}$$

The operators corresponding to the weights $\Delta_{r,s}$ for fixed c are usually organized in a matrix called Kac table. Explicit exemples of these tables will be given in the next chapter.
All what above gives a hint that something special is happening when $c \in (0, 1]$, but the punchline comes realizing that in this very special interval, in which both the central charge and the conformal weights are quantized, the representations obtained are reducible[11]. Their reducibility shows up by means of particular states appearing at some level of the Verma module, the level depending on the value of c and Δ chosen. These states are called degenerate fields and they satisfy (2.24) as they were another LW inside the representation. This would provide an invariant subspace inside the Verma module, making that particular representation reducible. The only way to save the irreducibility is

[10] The reason being that $L_n|0\rangle = 0$ for all n implies that $|0\rangle$ is the trivial state
[11] While for $c > 1$ they are irreducible

to set these states to zero. But since they are nothing but special linear combination of states at the n-th level, this condition would read

$$\left(L_{-n} + \cdots + L_{-2}L_{-1}^{n-2} + L_{-1}^{n}\right)|\Delta\rangle = \left(L_{-n} + \cdots + L_{-2}L_{-1}^{n-2} + L_{-1}^{n}\right)\phi_{\Delta}(0)|0\rangle = 0 \tag{2.26}$$

This equation translates into an operatorial identity for the primary field at the bottom of the Verma module, which can be put in the form of a linear differential equation for correlators containing only primary fields. Indeed from (2.20) and (2.14) one can show that for primary fields

$$\langle (L_{-k}\phi)(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle = -\sum_{i=1}^{n}\left[\frac{(1-k)\Delta_i}{(z_i-z)^k} + \frac{\partial_{z_i}}{(z_i-z)^{k-1}}\right]\langle\phi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle \tag{2.27}$$

So not only it is possible to find correlators of primary fields solving a differential equation, but also correlators of all descendant fields can be obtained by the ones of the primaries applying linear differential operators. Moreover it is possible to impose the consistency of the OPE (2.7) by inserting different primaries in (2.27), so different Verma modules, having degenerate fields at different levels. This leads in the end to the determination of the structure constants of the OPE algebra and to the remarkable result that the algebra closes with only a finite number of primaries, namely the ones corresponding to a fixed central charge and to all the conformal weights $\Delta_{r,s}$ in (2.25) allowed for that p. So for fixed p one has a conformally invariant field theory, with its full operator content, which is commonly denoted $\mathcal{M}_p$. These models for $p = 3, 4, \dots$ are called the unitary minimal series. In all what above the antianalytic part of the conformal algebra has been completely disregarded since its treatment produces exactly the same results.

The only thing which remains to discuss is how to combine the analytic and antianalytic sectors obtaining a set of operators $\{(\Delta, \bar{\Delta})\}$ closing the OPE (2.7). First it turns out that in order the theory to be parity invariant the central charge of the two sectors must be the same, so it is not possible to combine analytic and antianalytic fields from different Kac tables. Then, for fixed c, the appropriate combination may not require to consider the whole Kac table $\Delta_{r,s}$. Moreover in a physical theory whose scaling limit is conformal some operators may occur more than once. One way to find constraints on the allowed combinations of the two sectors is to study the differential equations satisfied by the fields correlators. These differential equations are not of the first order, since the differential operators (2.27) are not, and therefore they have many independent solutions. Moreover the correlators as function of z and $\bar{z}$ satisfy exactly the same differential equation with z or $\bar{z}$. This basically implies that given, for instance, a 4-point correlator $G(z, \bar{z})$, which thanks to conformal invariance depends only on one complex variable, it satisfies the two differential equations of order M

$$\mathcal{D}_z G(z,\bar{z}) = \mathcal{D}_{\bar{z}} G(z,\bar{z}) = 0 \qquad \Rightarrow \qquad G(z,\bar{z}) = \sum_{a,b}^{M} \alpha_{a,b}\, I_a(z) I_b(\bar{z})$$

Here M, as can be seen from (2.26), is the first level of the Verma module in which a degenerate state appears, while $\{I_a\}$ is a set of independent solutions of $\mathcal{D}$. The coefficients $\alpha_{a,b}$ form the so called monodromy matrix, since they are found imposing the correlator to be single valued around a particular set of points of the complex plane, which are the singular points of the differential operator $\mathcal{D}$. This in the end leads directly to the correlators of the primary operators of the CFT on the plane with left and right part appropriately combined. However it does not keep into account of any physical restriction on the combinations allowed.

A physical restriction comes moving the CFT on a different geometry, this is not a problem since, as will be shown in an explicit example, conformal invariance allow easly to relates correlators in different geometries. In a nutshell the restriction comes after imposing that the partition function Z

of the CFT on a torus does not depend on the way the plane is mapped into the torus. It turns out that Z, which is a sum over all the states of theory[12], can be written as a sum of characters of the Verma modules occurring in the theory

$$Z = \sum_{\Delta,\bar{\Delta}} N_{\Delta,\bar{\Delta}} \chi_\Delta(q)\chi_{\bar{\Delta}}(\bar{q}) \qquad \chi_\Delta(q) := \mathrm{Tr}\left(q^{L_0 - c/24}\right) = q^{\Delta - c/24} \sum_{n=0}^{\infty} d_\Delta(n) q^n$$

where $q = e^{2\pi i \tau}$, τ being the modular parameter of the torus[13], and $N_{\Delta,\bar{\Delta}}$ is the number of time the representation $(\Delta, \bar{\Delta})$ occurs in the CFT. So the requirement of invariance of Z written in different but equivalent toroidal geometries can be rephrased in the invariance of the characters under the modular group[14], which are the maps leaving invariant the modular parameter. The characters turns out to transform linearly under the modular group acting on the complex plane and this allows to write down an explicit and manageable condition in order to get what are the characters allowed to appear in the trace. This idea was first proposed in [32] and then all the CFTs corresponding to modular invariant partition functions were classified in [33].

2.2.3 Extended symmetries

The whole construction of the minimal models in the previous section made no references to the symmetry of the system they should describe. Symmetry is in fact a fundamental ingredient in characterizing universal behavior of critical phenomena and together with the dimensionality of the system it almost completely identifies the universality class. Since now the dimension is fixed to $D = 2$ one should expect that, somewhere in the complete conformal operators set, some hints on the internal symmetry enjoyed by the system may be spotted.
Indeed it turns out that all the CFTs whose primary operators are obtained by coupling the holomorphic and antiholomorphic sectors of the minimal model $\mathcal{M}_p$ diagonally ($\Delta = \bar{\Delta}$) describe the $(p-1)$-critical[15] point of system with $\mathbb{Z}_2$ symmetry. Nevertheless it is possible to extend the "target space symmetry" provided by the conformal group without leaving the Kac tables (2.25) but coupling the two sectors non diagonally. This happens because in some modular invariant CFTs there are operators representing new conserved charges of the theory in addition to the energy momentum tensor. These charges can be decomposed similarly to (2.14) and their components do not commute. Moreover they have non trivial commutation relations with the L_n, extending in such a way the Virasoro algebra (2.18). Furthermore these commutation relations might not describe an ordinary Lie algebra, but more complicated mathematical structures.
Even if the detailed structure of this extended algebras will not be discussed, it is worth noting that extending the Virasoro algebra leads to an important improvement on the BPZ solution of the CFTs in the minimal series, since it allows to identify degenerate fields for conformal field theories whose central charge is greater than 1 and thus not contained in the minimal series (2.25). Setting to zero these degenerate fields permits as before to compute all n-point functions of the CFT and to solve completely the theory.

[12] These states are, via the CFT's operator-state correspondence on the plane, all the primary fields with their descendants

[13] A torus is characterized by the two non parallel sides α_1 and α_2 of a parallelogram which upon identification give the former. The modular parameter is defined as $\tau := \alpha_1/\alpha_2$

[14] The modular group is the discrete subgroup of the Möbius group previously introduced, generated by the two transformation $\mathcal{S} : \tau \mapsto -1/\tau$ and $\mathcal{T} : \tau \mapsto \tau + 1$

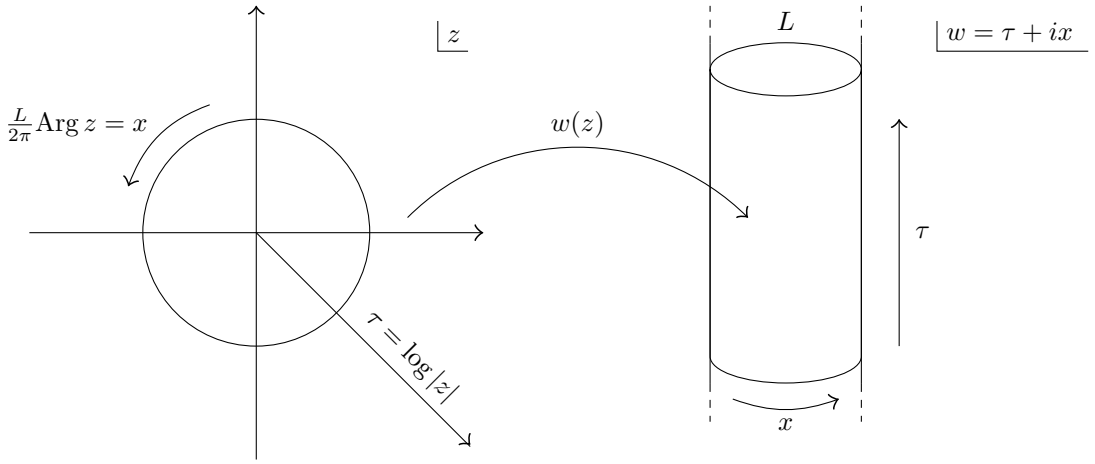
[15] A p-critical point is a continuous transition point whose transition is driven by p parameters, i.e. which is reachable by tuning p parameters, like the temperature, in the model

2.2.4 The central charge on the lattice

What comes out from the last section is that many complete solutions of 2-dimensional conformal field theories exist, but if they correspond to the critical point of some particular lattice model is not obvious at all. From the whole set of operators in a given CFT one can pick up the relevant ones, compute critical exponents from their anomalous dimensions and then try to match them with some lattice theory. The problem in this procedure is that the known exact solutions of lattice models are just a bunch, while here there is an infinite series of exact solutions. In addition to this, for fixed central charge there may exist many different allowed combinations of the two sectors of the conformal algebra. In spite of this complications many CFTs from the minimal series (2.25) have been found to correspond at least to one lattice theory. This was made possible by the solution, esentially contemporary to the work of BPZ, of an infinite series of lattice models [27], whose critical exponents were shown to be in one to one correspondence with the scaling dimension of the relevant operators of the Kac tables of the unitary minimal series [28]. Note however that universality pretends that the lattice model exhibiting the same universal behavior at the critical point, and thus corresponding to the same conformal field theory in the scaling limit, is not unique. This, indeed, turns out to be the case and many other lattice models whose exact solution is not known were shown, numerically, to have the same critical behavior of some model in (2.25).
One of the most powerful features of a theory which is conformally invariant is the possibility to bring results obtained in a simple geometry as could be the plane into a much more complicated one simply using the known transformation properties of the operators under a general analytic map. The only complication in these procedure is to find, if it exists, the conformal transformation which maps one geometry into the other. All the discussion above was for a field theory living on a plane, and this was supposed when the coordinate z on the complex plane were considered. A field theory on the plane may be the scaling limit of some classical theory on an infinite lattice or, via the QC-mapping, of some infinite quantum chain. However different geometries may represent field theories for finite systems and even the thermodynamic of quantum systems.
The most useful geometry is the cylinder, which is obtained from the plane via the analytic map

$$w(z) = \frac{L}{2\pi} \log z$$

where z are the points on the plane while w spans the cylinder whose time sections are circles of length L. The importance of this map is twofold, first it allows to get the euclidean QFT of a finite quantum system with periodic boundary conditions setting $w = \tau + ix$ and second it permits to obtain the thermodynamics of an infinite quantum system at temperature β setting $w = x + i\tau$ and $L = \beta$.



Let's compute the Hamiltonian on the cylinder in the first case, using the transformation law of the energy-momentum tensor (2.16) and the expressions (2.14) for the generators on the plane

$$\begin{aligned}\hat{H} &= \frac{1}{2\pi}\int_0^L \hat{T}^{\mathrm{cyl}}_{tt}(x,t)\mathrm{d}x = \frac{1}{2\pi}\int_0^L \left(\hat{T}^{\mathrm{cyl}}(x+it) + \hat{\bar{T}}^{\mathrm{cyl}}(x-it)\right)\mathrm{d}x \\ &= \frac{1}{2\pi i}\left(\int \hat{T}^{\mathrm{cyl}}(w)\mathrm{d}w - \int \hat{\bar{T}}^{\mathrm{cyl}}(\bar{w})\mathrm{d}\bar{w}\right) = \frac{1}{2\pi i}\frac{2\pi}{L}\left(\oint_C z\,\hat{T}^{\mathrm{pl}}(z)\mathrm{d}z + \oint_C \bar{z}\,\hat{\bar{T}}^{\mathrm{pl}}(\bar{z})\mathrm{d}\bar{z} - i\pi\frac{c}{6}\right) = \\ &= \frac{2\pi}{L}\left(\hat{L}_0 + \hat{\bar{L}}_0 - \frac{c}{12}\right)\end{aligned}$$

Here the hat on the operators has been restored. Note that in the last step $c = \bar{c}$ was assumed. This is one of the most important formulas provided by CFT, it relates the spectrum of the Hamiltonian on the cylinder to the conformal weights of the operator content of the theory and amazingly it can be verified also on lattice models at their critical point, which are supposed to be described by a CFT. So on the lattice the eigenvalues of the quantum Hamiltonian should be of the form

$$E(\Delta,\bar{\Delta}) = \frac{2\pi}{L}\left(\Delta + \bar{\Delta} - \frac{c}{12}\right)$$

up to corrections of the order a/L given by the finite lattice spacing. In particular from the assumption that the scaling limit of the lattice theory contains no operator with negative weights follows that the ground state energy is $E_{GS} = -\pi c/6L$, since of course the identity operator is always present and its conformal weights are both zero. This would provide a way to measure the central charge via finite size scaling of the ground state energy, the only problem with this being that in all the above treatment of relativistic field theories the velocity of light v was set to 1 from the beginning. In general this is a non trivial number which depends on the normalization of the lattice Hamiltonian and to get it one should know the exact solution at the critical point on the lattice. By dimensional analysis[16] this dimensionful factor shows up in the Hamiltonian as

$$\hat{H} = \frac{2\pi v}{L}\left(\hat{L}_0 + \hat{\bar{L}}_0 - \frac{c}{12}\right) \qquad \Rightarrow \qquad E_{GS} = -\frac{\pi c v}{6L} \tag{2.28}$$

This form of the scaling of the ground state energy will be verified in the next chapter analytically for the lattice Hamiltonian (2.1), whose spectrum can be computed exactly for finite number of spins, and numerically tested for the ferromagnetic 3-states and 4-states Potts chain, to be introduced in the next chapter. These results were obtained independently in [29] and [30].
There is however another way to spot the central charge measuring quantities on the lattice and this does not require the knowledge of the relativistic velocity v. The quantity one should compute is the entanglement entropy of the ground state, which is the Von Neumann entropy of the reduced density matrix of a subsystem A of the whole system $A \cup B$, i.e.

$$S_A = -\mathrm{Tr}\,\hat{\rho}_A \log \hat{\rho}_A \qquad\qquad \hat{\rho}_A = \mathrm{Tr}_B\,\hat{\rho}$$

Overlooking all the wide field related to the study of this quantity, the entenglement entropy will be used here just as a means to get the central charge of a critical lattice theory. In a pretty recent work [31] it has been shown that in a quantum system of length L at its critical point, with periodic boundary conditions, and bipartite in two subsystems A and B whose length is ℓ and $L-\ell$, S_A reads

$$S_A(\ell) = \frac{c}{3}\log\left(\frac{\pi L}{a}\sin\frac{\pi\ell}{L}\right) + \mathrm{const} \tag{2.29}$$

[16] Remember that $\hbar = 1$

This formula is derived from CFT and a is a dimensionful ultraviolet cutoff, which is customarily interpreted as the lattice spacing. The proof of this formula consists in writing a quantity related to S_A by simple differentiation as a two point function of two weird local fields on the simple geometry of the plane. Then exploiting conformal invariance of the critical point it is proved that these fields behave as they were primary fields of a known dimension. In this way their two point function is immediately known from (2.8) and by derivation they get the result.
Equation (2.29) can be used in two ways to measure the central charge on the lattice. The first is to fix the length of the chain, compute $S_A(\ell)$ varying ℓ and try to fit directly (2.29). The second is to compute $S_A(L/2)$ and fit it varying the length L of the chain. As will be shown in the next chapter, both these methods give very good results even for very short chains.

2.3 Quenches to the critical point

All the machinery set up in the previous section can be used also to study the dynamics of a system after a sudden quench[17] in which the final Hamiltonian is at its critical point. This non stationary problem can be translated in the field theory language into the problem of finding correlation functions into a field theory with boundary. Since this field theory with boundary is conformally invariant (BCFT) the problem can be solved under particular assumptions on the boundary conditions, which in the end translates in the geometry in which the theory itself is defined. Given that conformal field theory is built to describe universal features of critical phenomena, it is natural to expect that the results for the quench dynamics will be independent on the underlying lattice and will depend only on the numbers characterizing the universality class of the model under investigation, i.e. the conformal weights of the primary operators relevant in the field theory description.

2.3.1 Boundary conformal field theories

In dealing with a field theory with boundary one has to choose some boundary conditions to be imposed on the fields. These conditions may be inserted directly into the path integral adding boundary terms to the action. However in CFT this is not possible in general since a Lagrangian formulation not always exists. What instead is always defined is the energy momentum tensor, so these boundary conditions might be imposed on the components of $T_{\mu\nu}$. The obvious BCFT to consider first is the one living in the upper half plane (UHP), having the real axis as the boundary. A natural requirement is to impose there is no momentum flow across the boundary and this translates into the fact that there is only one independent component of $T_{\mu\nu}$, since now T and $\bar{T}$ are related by real analyticity $T(z) = \bar{T}(\bar{z})$. This condition is referred to as conformally invariant, since it is in fact invariant under conformal transformation, over than being physically sensible. This makes the left and right sectors no more independent, in such a way that all the treatment of the whole plane sketched in the previous section has to be modified. Nonetheless it is still possible to compute correlators in the bounded geometry in a way very similar to the method of image charges in electrostatics [34].
In the following only one point functions of scaling operators will be needed and to compute them in the bounded geometry of the UHP it is enough to appeal to the invariance under translation along the imaginary direction. This constrains one point functions to depend only on $\mathrm{Im}z$, where z is a point on the UHP, then the scaling properties of the field fix its expectation value to be

$$\langle \phi_i(z) \rangle_{\mathrm{UHP}} = \frac{A_{BC}^{\phi_i}}{(2\mathrm{Im}z)^{\Delta_i}} \tag{2.30}$$

[17] In a sudden quench the Hamiltonian is abruptly changed from H_0 to H and the system was initially prepared in the ground state of H_0

where $A_{BC}^{\phi_i}$ is a non trivial constant depending both on the field and on the boundary conditions. Indeed the latter are not completely fixed by the requirement of real analyticity of the non trivial component of $T_{\mu\nu}$ and in order to do so some physical insight is needed. Much like to what has been done to sew together the holomorphic and antiholomorphic sectors of the plane via mapping the latter into the torus, in this case the UHP can be mapped into the anulus and the partition function of the BCFT can be physically understood as representing a system which is infinite in one, say vertical, direction and has some boundary condition on the first and last rows. For instance in the Ising model one can impose the spins of the extremal rows to be fixed to $\pm$ or to be unconstrained. These rows can be seen in the BCFT as homogeneous field configurations propagating in imaginary time. They can then be represented as boundary states $|B\rangle$, on which the real analyticity of T must be imposed. It turns out that, like the modular invariant partition functions on the torus, these boundary states can be classified and a microscopical interpretation for most of them can be found in many BCFTs [35].

2.3.2 Quench dynamics in BCFT

To apply these ideas to the quench problem one has to deal mainly with two issues. The first is that the conformally invariant boundary states of above are not the initial states of the problem since there are very few of them, hence they cannot represent the ground states of the different possible initial Hamiltonians one may consider. Moreover these states are actually non renormalizable, being homogeneous field configurations, then to avoid infinities one has in any event to put a regulator on them. The second is that in order to predict the real time dynamics an analytic continuation of euclidean time is necessary. Disregarding momentarily the former, in order to compute time dependent expectation values like

$$\langle\psi_0|O(x,t)|\psi_0\rangle = \langle\psi_0|e^{iHt}O(x)e^{-iHt}|\psi_0\rangle$$

where x varies over the length of the system which is considered infinite, one could start from the euclidean correlator

$$\langle B|e^{-H\tau_2}O(x)e^{-H\tau_1}|B\rangle \tag{2.31}$$

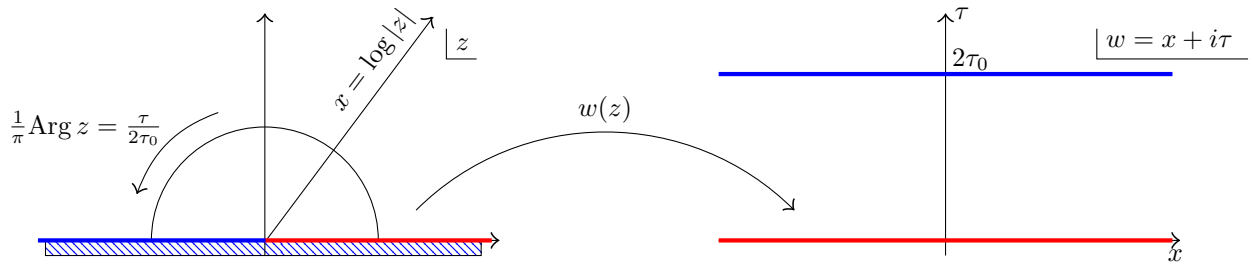
which is nicely the BCFT expectation value on a infinite slab geometry with equal conformally invariant boundary conditions imposed on both the sides of the slab, and the operator O inserted at a distance τ_1 from the lower side and τ_2 from the upper side. Of course the mapping of the expectation value (2.31) into a boundary field theory expectation value works even if the boundary condition is not conformally invariant, but in order to use (2.30) the special initial state $|B\rangle$ must be chosen. Now the problem with the analytic continuation , $\tau_1 = it$, $\tau_2 = -it$, is that the width of the slab $\tau_1 + \tau_2$ shrinks to zero. In order to avoid this in [36] was proposed to modify the initial state as

$$|\psi_0\rangle = e^{-\tau_0 H}|B\rangle \tag{2.32}$$

in such a way that on the one hand the width of the slab is kept finite to $2\tau_0$ and on the other the initial state is modified w.r.t. the conformally invariant one, making the boundary state normalizable. Of course τ_0 is completely undetermined and will be dependent on the microscopic details both of the physical model and of the real initial state. Moreover at this level is not even clear if (2.32) can really represent a field configuration corresponding to the usual initial state considered in a global quench, i.e. the eigenstate of a certain Hamiltonian H_0. However from boundary renormalization group arguments [36], it can be argued that not only this is the case, but also that τ_0 is expected to be of the order of the iniverse mass gap of the initial, non critical, Hamiltonian $\tau_0 \propto m_0^{-1}$.
Once all this assumptions have been accepted, it is easy to compute one point functions of primary

operators in the slab geometry via mapping (2.30) from the UHP. The analytic map which does the job is

$$w(z) = \frac{2\tau_0}{\pi} \log z \qquad \Rightarrow \qquad z = e^{\pi w/2\tau_0} = e^{\pi x/2\tau_0} e^{i\pi\tau/2\tau_0} \tag{2.33}$$



Now using (2.30) together with the transformation law (2.19) for primary fields, which thanks to real analyticity in the UHP becomes

$$\phi_i^{\text{slab}}(w) = \left|\frac{\partial z}{\partial w}\right|^{2\Delta_i} \phi_i(z(w)) = \left|\frac{\partial z}{\partial w}\right|^{\eta_i} \phi_i(z(w))$$

one obtains that the one point function of a primary field in the strip reads

$$\langle \phi_i^{\text{slab}}(w) \rangle = A_{BC}^{\phi_i} \left(\frac{\pi}{4\tau_0 \sin(\pi\tau/2\tau_0)} \right)^{\eta_i} \tag{2.34}$$

Now from the domain of the map (2.33) in the picture it is clear that in order to put the primary in the middle of the strip one has to set $\tau = \tau_0$ and then to analytically continue to real time $\tau = \tau_0 + it$. In this way the one point function becomes

$$\langle \phi_i^{\text{slab}}(t) \rangle = A_{BC}^{\phi_i} \left(\frac{\pi}{4\tau_0 \cosh(\pi t/2\tau_0)} \right)^{\eta_i}$$

So for time larger than τ_0 the BCFT prediction is that one point functions of primary operators decay exponentially, with a lifetime

$$t_{\text{rel}}^i = \frac{2\tau_0}{\eta_i \pi} \tag{2.35}$$

depending both on the scaling dimensions of the primary and on the non universal parameter τ_0. Note also that the information on the initial state is encoded both in τ_0 and in the constant $A_{BC}^{\phi_i}$. It might happen this constant to be zero and in fact this will be the case for the order parameter when the quench starts from the disordered phase. Note also that, in contrast to the power law behavior of correlation functions in critical phenomena at equilibrium, the CFT time evolution has the form of an exponential. This is due to the map from the UHP to the slab: in the UHP the expectation values are power laws, but the conformal map to the slab sets, upon analytic continuation, $\mathrm{Im} z \propto e^{-\pi t/2\tau_0}$. This behavior, as will be shown in the last chapter, is easly seen on the lattice for observables whose continuum limit is a primary operator in the CFT.

The expectation value of the Hamiltonian per unit length of the strip in the BCFT represents in the quench problem the energy density, w.r.t. the ground state value, acquired by the system after the quench. This is nothing but the expectation value per unit length of the new Hamiltonian H on the initial, non stationary, ground state $|GS_0\rangle$ of H_0, with the ground state energy E_{GS} of H subtracted off

$$\frac{\langle H_{CFT} \rangle_{\text{slab}}}{L} = \frac{\langle GS_0|H|GS_0\rangle - E_{GS}}{L}$$

Of course the r.h.s. is conserved during the quench dynamics, but the same occurs for the l.h.s., being constant along the τ direction. This expectation value in the BCFT on a strip of width $2\tau_0$ with conformally invariant boundary conditions has been computed in [29] and it can actually be obtained slightly modifying the calculation done in section 2.2.4 for the map to the cylinder, the result is

$$\frac{\langle H_{CFT} \rangle}{L} = \frac{\pi c}{24(2\tau_0)^2}$$

where this time, in contrast to what happened in (2.28), no dimensionful factor arises.
To conclude this section a comment on the range of applicability of this field theory methods is in order. As discussed at the beginning of this chapter a field theory description of a physical system is valid in the vicinity of a critical point, where the correlation length diverges, the microscopic details are inessential and the high energy structure of the spectrum does not condition the physics observed. For this reason the fact that field theory can be used to treat a global quench is not to be taken for granted, since the quench injects into the system a huge amount of energy, thus populating also the high energy modes of the Hamiltonian and undermining the reliability of these predictions. Nevertheless, as will be shown in chapter 4, the agreement with the results obtained on the lattice is much better than expected.

Chapter 3

The q-states Potts chain

In this chapter the quantum models which will be studied in the rest of this work, together with its out of equilibrium dynamics, is defined. In section 1 the quantum Hamiltonian will be given and its phases and the related order parameters will be introduced. In section 2 the QC-mapping of the model will be discussed with some observations on its possible applications and a very brief review of its classical 2-dimensional counterpart. In section 3 the description of the quantum Hamiltonian in terms of quasiparticles will be treated perturbatively and compared to numerical simulations. In section 4 the exact solutions of the simplest of these models, namely the quantum Ising chain in a transverse field already introduced in the previous chapter, will be discussed in some detail and its scaling limit near the critical point will be explicitly taken and compared to what discussed in chapter 2. In section 5 will be explained why the exact solutions of all these models is not achievable in general and the conformal field theory describing the critical point of the $q = 3$ case will be introduced, along with a brief mention of the $q = 4$ universality class. In section 5 the central charge c and the velocity of light v will be numerically measured directly on the lattice with the techniques discussed in chapter 3.

3.1 The quantum Hamiltonian

The quantum Hamiltonian of the q-states Potts chain was first considered in [40] in order to study the critical behavior of the classical q-Potts model via Renormalization Group techniques, it reads

$$H = -J \sum_{a=1}^{N} \sum_{\ell=1}^{q-1} \sigma_a^\ell \sigma_{a+1}^{q-\ell} - Jg \sum_{a=1}^{N} \sum_{\ell=1}^{q-1} \tau_a^\ell \tag{3.1}$$

where the matrices σ and τ satisfy the algebra

$$\sigma^q = 1 = \tau^q \qquad \sigma^\dagger = \sigma^{q-1} \qquad \tau^\dagger = \tau^{q-1} \qquad \sigma\tau = \omega\tau\sigma \tag{3.2}$$

and they might be represented as

$$\sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{q-1} \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \omega = e^{i\frac{2\pi}{q}} \tag{3.3}$$

H acts on the Hilbert space $\mathcal{H} = \bigotimes_{a=1}^{N} \mathbb{C}^q$ and periodic boundary conditions $\sigma_{L+1}^i = \sigma_1^i$ are assumed. In the following only the ferromagnetic case ($J > 0$) will be considered and the overall factor J will

be set to one.
The Hamiltonian is symmetric under any permutation of the components of the q-dimensional one-site vectors, but in order to decompose the Hilbert space in q different sectors in which H is block-diagonal it is sufficient to consider only the cyclic $\mathbb{Z}_q$ group, which is generated by the operator

$$\omega^{\mathcal{Q}} = \prod_{a=1}^{N} \tau_a \qquad\qquad \left(\omega^{\mathcal{Q}}\right)^{\dagger} H \omega^{\mathcal{Q}} = \omega^{-\mathcal{Q}} H \omega^{\mathcal{Q}} = H \tag{3.4}$$

Its eigenvalue are ω^Q where $Q = 0, 1, .., q-1$. Since H commutes with $\omega^{\mathcal{Q}}$ the energy eigenvalues can be labelled by the value of this discrete conserved charge and the Hilbert space breaks as $\mathcal{H} = \bigoplus_{Q=0}^{q-1} \mathcal{H}_Q$. In addition to this internal symmetry, H is also invariant under translation of the spin label a thanks to periodic boundary condition. If $e^{i\mathcal{P}}$ is the operator shifting the spins on the chain, i.e. $e^{i\mathcal{P}}|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_N\rangle = |v_N\rangle \otimes |v_1\rangle \otimes \cdots \otimes |v_{N-1}\rangle$, then its eigenvalues are $P = 2\pi n/N$, $n = 0, 1, \ldots, N-1$ and may be used to specify further the energy eigenstates.
This system undergoes a quantum phase transition at zero temperature at the critical value $g_c = 1$, as can be seen observing that g_c is a self-dual point for H, under the duality transformation

$$\mu_b^k = \prod_{a<b} \tau_a^{q-k} \qquad \nu_b^k = \sigma_{b+\frac{1}{2}}^k \sigma_{b-\frac{1}{2}}^{q-k} \qquad b \in \mathbb{Z} + \frac{1}{2} \tag{3.5}$$

Here the operators μ_b and ν_b are defined on the links between the initial chain sites. Indeed using the algebra satisfied by both (σ, τ) and (μ, ν), i.e. $\sigma^q = \tau^q = \mu^q = \nu^q = 1$, one has

$$\sum_{a=1}^{N} \sum_{\ell=1}^{q-1} \sigma_a^\ell \sigma_{a+1}^{q-\ell} + g \sum_{a=1}^{N} \sum_{\ell=1}^{q-1} \tau_a^\ell = g \left(\sum_{b=\frac{1}{2}}^{N-\frac{1}{2}} \sum_{\ell=1}^{q-1} \mu_a^\ell \mu_{a+1}^{q-\ell} + \frac{1}{g} \sum_{b=\frac{1}{2}}^{N-\frac{1}{2}} \sum_{\ell=1}^{q-1} \nu_b^\ell \right)$$

so that the Hamiltonian is invariant in form when $g = 1$ and it satisfies an important relation which maps its spectrum in the ferromagnetic phase ($g < 1$) into its spectrum in the paramagnetic phase ($g > 1$)[1], i.e.

$$H_g = g H_{1/g} \qquad\qquad \Rightarrow \qquad\qquad E(g) = g E\left(\frac{1}{g}\right) \tag{3.6}$$

where E is any eigenvalue H.
There are q order parameters associated to this phase transition, which, once normalized so that they are 0 in the disordered phase ($g > 1$) and 1 in the ordered phase ($g < 1$), can be written in term of the matrices (3.3) as

$$m^\alpha = \frac{1}{q-1} \left(\sum_{k=0}^{q-1} \sigma^k \omega^{-k\alpha} - 1 \right) \qquad\qquad \alpha = 0, 1, \ldots, q-1 \tag{3.7}$$

but only $q-1$ of them are independent since their sum is 0 in both phases.

[1] It is not properly true that this map is one to one, since for topological reason which will be discussed later some levels which are present in the paramagnetic phase are absent in the ferromagnetic one, moreover the degeneracy of the energy levels is not the same

3.2 QC mapping

Going through the same steps as in chapter 2 the quantum Hamiltonian (3.1) can be Trotterized to give the classical q-states Potts model. Indeed the partition function at finite temperature β of the quantum chain is

$$\mathcal{Z}_\beta = \mathrm{Tr}(e^{-\beta H}) = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \langle \sigma_1 \cdots \sigma_N | e^{-\beta H} | \sigma_1 \cdots \sigma_N \rangle$$

where now the states on which the trace is done are the tensor product of the single site basis $\{|\sigma\rangle\}_{\sigma=0}^{q-1}$ composed by the eigenstate of the matrix σ, which in the representation adopted is nothing but the canonical basis

$$\sigma|\sigma\rangle = \omega^\sigma |\sigma\rangle = e^{i\frac{2\pi\sigma}{q}}|\sigma\rangle \qquad \sigma = 0, 1, \ldots, q-1$$

Inserting as before M completeness relations on this basis and using Trotter's formula formula

$$e^{A+B} \sim (e^{A/M} e^{B/M})^M \qquad M \to \infty \tag{3.8}$$

and the quantum partition function becomes

$$\mathcal{Z}_\beta \sim \sum_{\{\sigma_a\}} \sum_{\{\sigma_a^1\}} \cdots \sum_{\{\sigma_a^M\}} e^{\frac{\beta}{M}\sum_{a,b}^{N,M}\sum_{\ell=1}^{q-1} e^{i\frac{2\pi l}{q}(\sigma_{a,b}-\sigma_{a+1,b})}} \langle \sigma_1 \cdots \sigma_N | e^{\frac{\beta g}{M}\sum_{a=1}^{N}\sum_{\ell=1}^{q-1}\tau_a^\ell} | \sigma_{1,1} \cdots \sigma_{N,1} \rangle \cdots$$

$$\ldots \langle \sigma_{1,M} \cdots \sigma_{N,M} | e^{\frac{\beta g}{M}\sum_{a=1}^{N}\sum_{\ell=1}^{q-1}\tau_a^\ell} | \sigma_1 \cdots \sigma_N \rangle$$

The diagonal part is readly evaluated using

$$\sum_{\ell=1}^{q-1} e^{i\frac{2\pi l}{q}(\sigma_a - \sigma_{a+1})} = q\delta(\sigma_a, \sigma_{a+1}) - 1$$

The non diagonal part is slightly more complicated, but still can be exponentiated for general q and it is of the right form to be reexponentiated to a local term

$$\exp\left(\frac{\beta g}{M}\sum_{\ell=1}^{q-1}\tau_a^\ell\right)_{\sigma,\sigma'} = \exp\left(\frac{\beta g}{M}\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \vdots \\ \vdots & 1 & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}\right)_{\sigma,\sigma'} = \begin{pmatrix} \alpha & \gamma & \cdots & \gamma \\ \gamma & \alpha & \gamma & \vdots \\ \vdots & \gamma & \ddots & \gamma \\ \gamma & \cdots & \gamma & \alpha \end{pmatrix}_{\sigma,\sigma'} = e^{\delta(\sigma,\sigma')\log(\alpha/\gamma)+\log\gamma}$$

$$J_\tau := \log\frac{\alpha}{\gamma} = \log\left(\frac{q-1+e^{q\beta g/M}}{e^{q\beta g/M}-1}\right) \qquad \gamma = \frac{-1+e^{q\beta g/M}}{qe^{\beta g/M}} \tag{3.9}$$

Thanks to these expressions and periodic boundary conditions the quantum partition function $\mathcal{Z}_\beta$ on the chain reads

$$\mathcal{Z}_\beta = \lim_{M\to\infty}\sum_{\{\sigma\}} e^{-\bar{H}_M(\sigma)+MNA_M} \tag{3.10}$$

$$\bar{H}_M(\sigma) = -\sum_{a,b}^{N,M}\left[\frac{\beta q}{M}\delta(\sigma_{a,b},\sigma_{a+1,b}) + J_\tau\,\delta(\sigma_{a,b},\sigma_{a,b+1})\right] \qquad A = \log\left(\frac{-1+e^{q\beta g/M}}{qe^{\beta(g+1)/M}}\right)$$

which is the Hamiltonian of a classical Potts model on a $N \times M$ lattice with different coupling on the two directions. Also in this case if the limit $\beta \to \infty$ is taken, keeping fixed the ratio $\beta q/M =: K$ the

transverse coupling J_τ is finite and upon setting the quantum coupling g to its critical value $g_c = 1$ one has the critical classical model

$$J_\tau|_{g=1} = L = \log\left(\frac{q-1+e^K}{e^K-1}\right) \qquad \Rightarrow \qquad (e^K-1)(e^L-1) = q \tag{3.11}$$

The latter is indeed the equation for the critical temperature of the anistropic q-states Potts model on a square lattice [38].

3.2.1 Quantum ground state energy and classical free energy at criticality

As already stressed in chapter 2 the QC-mapping at the critical point is not one to one. The anisotropy parameter K/L is free on the classical side. Mathematically this freedom is introduced when the relation

$$\lim_{\beta,M\to\infty} \frac{\beta q}{M} =: K$$

is imposed. Forgetting temporarily of this issue let's pause a moment on equation (3.10) and look at what would happen if the limit $\beta \to \infty$ were taken on the l.h.s.: assuming the ground state energy of the quantum Hamiltonian not to be degenerate one would be left with

$$\lim_{\beta\to\infty} \mathcal{Z}_\beta = e^{-\beta E_0}$$

where E_0 is expected to be an extensive quantity with the number N of sites of the chain. On the r.h.s., instead, in order to obtain the classical partition function one has to send $M \to \infty$ immediately to avoid calculating all the corrections to Trotter's decomposition (3.8), which is obviously an impossible task. Just then one is allowed to take the limit $\beta \to \infty$ to get insight on what happens to the l.h.s.. Let's now do the tricky step of assuming

$$\lim_{\beta\to\infty}\lim_{M\to\infty} = \lim_{\beta,M\to\infty,\,\frac{\beta q}{M}=K} \tag{3.12}$$

and fix the arbitrary anistropy parameter $K/L = 1$. In this way the classical lattice model together with its partition function are chosen. Place it at the critical point setting $g = 1$, take the logarithm and divide by β both sides, then use (3.11) in the isotropic case. This yields

$$E_0(q) = -\frac{NM}{\beta}\left[A_M + \frac{1}{MN}\log\left(\sum_{\{\sigma\}} e^{-\frac{\beta q}{M}\bar{H}_M(\sigma)}\right)\right] = -\frac{Nq}{K_c}\left[\log\left(\frac{-1+e^{K_c}}{qe^{2K_c/q}}\right) + \frac{\log Z_{N_\mathrm{s}}(K_c)}{N_\mathrm{s}}\right] \tag{3.13}$$

where $N_\mathrm{s} = NM$ is the number of sites of the square lattice and $f_{N_\mathrm{s}} = -\log Z_{N_\mathrm{s}}(K_c)/N_s\beta$ is the free energy per site of the classical system at its critical point, which is known for any value of q [38], and for $q = 2, 3$ reads

$$\lim_{N_\mathrm{s}\to\infty}\frac{1}{N_\mathrm{s}}\log Z_{N_\mathrm{s}}(K_c) = \frac{1}{2}\log q + 2\phi(K_c)$$

$$\phi(K_c) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{t}\frac{\sinh(\pi-\mu)t\sinh 2\gamma t}{\sinh\pi t\cosh\mu t} \qquad 2\cos\mu = \sqrt{q} \qquad \frac{\sin\gamma}{\sin(\mu-\gamma)} = 1$$

For $q = 4$ the expressions is different, but still known (see [38]). (3.13) relates the ground state energy of the quantum chain to the free energy of a classical model, which has been arbitrarly chosen among the infinitely many ones provided by the QC-mapping. Substituting this expression in (3.13) along with the explicit expression for the critical temperature $K_c = \log(1+\sqrt{q})$ one finally gets

$$\varepsilon_{\mathrm{vac}} := \frac{E_0(q)}{N} = 2\left(1 - \frac{2q\,\phi(K_c)}{\log\left(1+\sqrt{q}\right)}\right) \tag{3.14}$$

Evaluating numerically[2] this expression for $q = 2, 3, 4$, with a different expression for $\phi(K_c)$ in the latter case, and comparing with the ground state energy obtained from numerical simulation on the quantum chain (with the most important finite size correction appropriately removed) one gets the following results[3]

q	ε_{vac} **from** (3.14)	ε_{vac} **from chain**
2	-1.324	$-4/\pi \simeq$ -1.273
3	-2.540	$-(6\sqrt{3}+4\pi)/3\pi \simeq$ -2.436
4	-3.703	-3.546

So formula (3.14) seems to provide a quite good approximation to the true ground state energies of the quantum chains, even if it is derived from the questionable assumption (3.12) and fixing arbitrarly the gauge freedom provided by universality via the QC mapping.

3.2.2 The classical q-states Potts model

The classical q-states Potts model was first proposed by Cyril Domb to his student Renfrey Potts as a natural generalization of the Ising model. Instead of considering the classical lattice spin variable confined on a line, Domb invented a model in which the spin can point in q different directions on a plane, equally spaced by an angle $2\pi/q$. The Hamiltonian he proposed has became known with the name planar Potts model

$$H_{\text{cl}}(\sigma) = -J \sum_{\langle i,j \rangle} \cos \frac{2\pi}{q} (\sigma_i - \sigma_j) \qquad \sigma_i = 0, 1, \ldots, q \tag{3.15}$$

Potts [39] first solved this model in one dimension and this is easly achieved by usual methods as the transfer matrix or the recursive method [23]. As in the case of the Ising model no phase transition at finite temperature occurs in this case. Then he moved to the 2-dimensional case and he was able to locate, by duality argument, the position of a phase transition for $q = 2, 3, 4$, without being able to extend this result for $q > 4$. In order to do so he wrote down a new Hamiltonian which is nowaday known as the standard Potts model

$$H_{\text{cl}}(\sigma) = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) \tag{3.16}$$

This model is equivalent to (3.15) if $q = 2, 3$ and the location of its transition point in $D = 2$ is given by (3.11) with $K = L$ for any q. The partition function on a square lattice was computed in [42] at the transition point for all q, while an expression for any temperature is currently not known. Nevertheless in [42] the important result that the transition becomes of first order for $q > 4$ in $D = 2$ was obtained, whereas a mean field analysis for general dimensionality indicates that this is so for $q > 2$, although this is reliable only if $D \geq 4$, as all mean field calculations. So there is a line $q_c(D)$ above which the transition becomes of first order and the know values of this curve are $q_c(2) = 4$ and $q_c(4) = 2$. In a recent work [46] the value $q_c(3) = 2,62$ was obtained from numerical simulations. Another important result holding for general topology of the lattice $\mathcal{L}$ which is worth mentioning is the possibility to analytically continue the partition function for q complex [43], writing it as

$$Z(q) = \sum_{\{\sigma\}} e^{\beta J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j)} = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} [1 + v\delta(\sigma_i, \sigma_j)] = \sum_{\mathcal{G}} q^C v^{N_b}$$

[2] By the way they perhaps may be evaluated also analitically
[3] More on the exact values for the ground state energy for $q = 2, 3$ later

where $v = e^{\beta J} - 1$, the graps $\mathcal{G}$ is composed of N_b edges and its vertexes are the lattice sites. A certain edge belongs to $\mathcal{G}$ if the two spins at the ends assume the same colour. C is the number of connected components of $\mathcal{G}$, counting the lattice sites with no edges as a single component. The analytical continuation of $Z(q)$ is linked to many other interesting problems. The Kirchhoff problem on a infinite grid of resistors for $q = 0$ and the study of percolation for $q = 1$. Moreover the $\lim_{J\to-\infty} Z(q)$ gives the chromatic polynomial of the lattice $\mathcal{L}$, whose zeros for integer q are related to the problem of colouring $\mathcal{L}$ in such a way that no adjacent sites have the same colour: if $Z(q) = 0$ for some integer q then q colours are not enough.
For $D = 2$ the critical behavior manifests for $q \leq 4$ and the critical exponents were first conjectured in [44] and the conjecture numerically tested in [45]. For the order parameter and the correlation length these are

$$\nu_q = \frac{2\,t_q}{3(t_q - 1)} \qquad \beta_q = \frac{(t_q - 1)(t_q + 3)}{8 t_q (t_q + 1)} \nu_q \qquad 2\sin\left(\frac{\pi}{2}\frac{t_q - 1}{t_q + 1}\right) = \sqrt{q} \tag{3.17}$$

yielding $\nu = 1, 5/6, 2/3$ and $\beta = 1/8, 1/9, 1/12$ for $q = 2, 3, 4$ respectively. In fact these numbers correspond to 3 minimal models $\mathcal{M}_p$ of the minimal series (2.25) with $p = t_q$, i.e. $p = 3, 5, \infty$ and central charge $c = 1/2, 4/5, 1$. This will be discussed in more detail in the next sections.

3.3 Quasiparticles description

As many other quantum mechanical systems with many degrees of freedom the q-states Potts chain can be described in terms of quasi-particles which are not local with respect to the sites of the chain, neither are plane waves with a trivial dispersion relation. As will be shown in the next section, when $q = 2$ the transformatin which maps the local operators σ_a to the ones which make the Hamiltonian diagonal can be written down explictly together with the exact eigenvalues for all g. For general q instead the Hamiltonian is not exactly diagonalizable (for reasons which will become clear later), but still is possible to get a perturbative picture of what quasi-particles represent and what is their dispersion relation at first order. It is not easy for general g to visualize quasiparticles in terms of spin operators, but in the limiting cases $g \ll 1$ and $g \gg 1$ they can be interpreted as the lowest energy excitations upon the trivial ground state(s).
For $g \ll 1$ the ground state is q-fold degenerate and it is nothing but the ground state of the first term in (3.1), which is the state with all spins aligned to one of the q directions, i.e. the tensor product of one-site states with all but one component equal to zero:

$$|0\rangle_\mu = \prod_{a=1}^{N} |\mu\rangle_a \qquad \mu = 1, 2, ..., q \tag{3.18}$$

These states are in fact not eigenstates of the $\mathbb{Z}_q$ symmetry generator (3.4) since $\omega^{\mathcal{Q}}|0\rangle_\mu = |0\rangle_{\mu+1}$, but taking the linear combination

$$|0\rangle_Q = \frac{1}{\sqrt{q}} \sum_{Q=0}^{q-1} \omega^{-Q\mu} |0\rangle_\mu \qquad Q = 0, 1, \ldots, q-1 \tag{3.19}$$

yelds the ground states of the q sectors defined in the first section. They are all relative to the $P = 0$ eigenvalue of the momentum operator $\mathcal{P}$ and they break the S_q symmetry since they are not invariant under permutations of the one-site vectors components. Above each of these ground states there is a spectrum which may be interpreted in terms of quasiparticles at first order. In order to understand more easly the nature of the ordered-phase quasiparticles, let's first disregard periodic boundary

conditions and consider the ground states (3.18), pick up one of them, let's say $\tilde{\mu}$. Quasiparticles are then domain walls which separate a string of spins aligned to the direction $\tilde{\mu}$ and a string of spins aligned to one of the other $q-1$ directions. Therefore there are $q-1$ different excitations and they are the local states of the form

$$|\tilde{\mu},\nu;b\rangle = \prod_{a=1}^{b} |\tilde{\mu}\rangle_a \otimes \prod_{a=b+1}^{N} |\nu\rangle_a \qquad \nu = 1,2,...,q \qquad \nu \neq \tilde{\mu} \qquad b = 1,2,...,N \tag{3.20}$$

As long as the transverse field term in (3.1) is neglected the ground state energy is

$$E_{GS}^{(0)} = {}_{\tilde{\mu}}\langle 0|H_0|0\rangle_{\tilde{\mu}} = -\sum_{a=1}^{N} {}_a\langle\tilde{\mu}| \sum_{\ell=1}^{q-1} \sigma_a^{\ell}\sigma_{a+1}^{q-\ell} |\tilde{\mu}\rangle_a = -\sum_{a=1}^{N}\sum_{\ell=1}^{q-1} e^{i\tilde{\mu}\frac{2\pi}{q}\ell} e^{i\tilde{\mu}\frac{2\pi}{q}(q-\ell)} = -N(q-1)$$

and all the excited one-particle states (3.20) have the same energy $E_1^{(0)} = E_{GS}^{(0)} + q$. Treating now the transverse field term in (3.1) as a small perturbation, at first order in g one has that the ground state energy is unchanged since

$$E_{GS}^{(0)} = {}_{\tilde{\mu}}\langle 0|H_p|0\rangle_{\tilde{\mu}} = -g\sum_{a=1}^{N} {}_a\langle\tilde{\mu}| \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \vdots \\ \vdots & 1 & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} |\tilde{\mu}\rangle_a =: -g\sum_{a=1}^{N} {}_a\langle\tilde{\mu}|M_a|\tilde{\mu}\rangle_a = 0$$

while the states (3.20) are not eigenstates of the perturbation H_p and one has to diagonalize H_p in that degenerate subspace in order to find the first order correction to the eigenvalue E_1, which will partially lift the degeneracy. This task is easly achieved exploiting translational invariance of H_p, which makes it diagonal on the basis generated by the Fourier transform of the local vectors (3.20), i.e.

$$|\nu;k\rangle := |\tilde{\mu},\nu;k\rangle = \frac{1}{\sqrt{N}}\sum_{a=1}^{N} e^{ika}|\tilde{\mu},\nu;a\rangle \qquad \nu = 1,2,...,q \quad \nu \neq \tilde{\mu} \qquad k = \frac{2\pi n}{N} \quad n = 0,...,N-1 \tag{3.21}$$

The perturbation is indeed diagonal in this new basis, since its matrix elements are given by

$$\begin{aligned}
\langle\nu;k|H_p|\nu';k'\rangle &= -\frac{g}{N}\sum_{a=1}^{N}\sum_{b,c=1}^{N} e^{i(k'c-kb)}\langle\nu;b|M_a|\nu';c\rangle = \\
&= -\delta_{\nu,\nu'}\frac{g}{N}\sum_{a=1}^{N}\sum_{b=1}^{N} e^{i(k'-k)b}\left(e^{ik'}\langle\nu;b|M_a|\nu;b+1\rangle + e^{-ik'}\langle\nu;b|M_a|\nu;b-1\rangle\right) = \\
&= -\delta_{\nu,\nu'}\frac{g}{N}\sum_{a=1}^{N} e^{i(k'-k)a}\left(e^{ik'} + e^{-ik'}\right) = -\delta_{\nu,\nu'}\delta_{k,k'}2g\cos k = \delta_{\nu,\nu'}\delta_{k,k'}E_{1,\nu}^{(1)}(k)
\end{aligned}$$

So the first order eigenvector of $H_0 + H_p$ are (3.21) and their energy is

$$E_\nu(k) = E_{1,\nu}^{(0)} + E_{1,\nu}^{(1)}(k) =: E_{GS}^{(0)} + \varepsilon^\nu(k) = E_{GS}^{(0)} + q\left(1 - g\frac{2}{q}\cos k\right) \tag{3.22}$$

All the perturbative eigenstates considered above were not eigenstates of the $\mathbb{Z}_q$ generator $\omega^{\mathcal{Q}}$. Regarding the ground states this is not a problem since the first order correction $E_{GS}^{(1)} = 0$, then in

particular independent of μ or, switching to (3.19), of Q. Moreover it has been argued [52] that this degeneracy of the ground states persists to all order in g. For what concerns the 1-particle eigenstates instead, one can take appropriate linear combinations of the states $|\mu,\nu;k\rangle$ in (3.21) to make eigenstates of $\omega^{\mathcal{Q}}$ observing that $\omega^{\mathcal{Q}}|\mu,\nu;k\rangle = |\mu+1,\nu+1;k\rangle$, so that

$$|\lambda;k\rangle_Q = \sum_{\mu=1}^{q} \omega^{-\mu Q}|\mu,\mu+\lambda;k\rangle \qquad\qquad \omega^{\mathcal{Q}}|\lambda;k\rangle_Q = \omega^{Q}|\nu;k\rangle_Q$$

and they are of course eigenstates of the momentum operator $\mathcal{P}$ with eigenvalue k and of the Hamiltonian with eigenvalue (3.22).
All what above completely disregards periodic boundary conditions, which imply it is not possible to have a single excitation state in the ferromagnetic phase since a single domain wall cannot be created on a circle, i.e. the one-particle level is absent when $g < 1$. The first excited state is then composed of two domain walls and the perturbative treatment of the two-particles level is much more complicated and will not be discussed. It is natural to ask whether the 2-domain walls eigenvalues might be written as

$$E_2(k_1,\lambda_1;k_2,\lambda_2) = \varepsilon^{\lambda_1}(k_1) + \varepsilon^{\lambda_2}(k_2) + E_{GS}$$

which basically means that the quasiparticles are not interacting. For the case $q = 2$ this turns out to be true for all $g < 1$, with the additional constraint $k_1 \neq k_2$, reflecting the fermionic nature of $q = 2$ free quasiparticles. That the same relation holds for $q > 2$ is unlikely, even if it may be well verified perturbatively, since if so the Hamiltonian would be presumably diagonalizable exactly. Unluckily this is not the case, as will be discussed in the next sections. Nevertheless some efforts in the literature were aimed at proving that the whole spectrum can be written as a sum of single-particle energies [53]. Another interesting issue to address is if the degeneracy of the q sectors persists up to $g = 1$ or if it is broken by the perturbation as soon as g is switched on. In [52] it was claimed that the former is the case, so that there are q equal spectra built above the q degenerate vacua labelled by the eigenvalues of $\mathcal{Q}$. Once again this statement is verified by the $q = 2$ exact solution, although this degeneracy is quite accidental since it cannot be attributed to the S_q invariance of the Hamiltonian, which already explains the $(q-1)$ degeneracy of the quasiparticles dispersion relation.
When $g \gg 1$ the perturbation theory works with the roles of H_0 and H_p interchanged and everything is simpler. Quasiparticles are now excitations above the non degenerate state given by the ground state of the transverse field, which is the product state of the lowest eigenvalue eigenstate of $-M_a$, i.e. the vector, properly normalized, with all the components equal to 1, relative to the eigenvalue $(1-q)$. This will be denoted $|\lambda_0\rangle$. The other eigenvalue of $-M_a$ is 1 and it is $(q-1)$-fold degenerate, the corresponding eigenvectors will be denoted $|\lambda\rangle$, with $\lambda = 1,2,...,q-1$. So the ground state of H_0 this time is given by

$$|0\rangle = \prod_{a=1}^{N} |\lambda_0\rangle \tag{3.23}$$

as before it is labelled by $P = 0$, but this time it lies only in the $Q = 0$ sector since $\tau|\lambda_0\rangle = |\lambda_0\rangle$. Note that this is not a perturbative statement, since if nothing strange happens, like a quantum phase transition, it is not possible that the ground state changes eigenvalue with respect to a symmetry generator of the Hamiltonian. Then down to $g = 1$ the ground state will lie in the $Q = 0$ sector. The first excited level, which as long as the perturbation is switched off has dimesion $N(q-1)$, is obtained inserting in $|0\rangle$ any of the $q-1$ states $|\lambda\rangle$ on any of the N site, i.e.

$$|\lambda;b\rangle = \prod_{a=1}^{b-1} |\lambda_0\rangle_a \otimes |\lambda\rangle_b \otimes \prod_{a=b+1}^{N} |\lambda_0\rangle_a \qquad \lambda = 1,2,...,q-1 \qquad b = 1,2,...,N \tag{3.24}$$

So this time elementary excitations are "spin flips" (the word spin now refers to an arrow aligned to the direction $(1,1,1...,1)$) whose energy is $E_1^{(0)} = E_{GS}^{(0)} + gq$, being $E_{GS}^{(0)} = -g(q-1)N$ the unperturbed ground state energy. The rest of the analysis goes exactly as for the one-domain wall subspace of the ferromagnetic phase: the ground state energy gets no order $1/g$ corrections, while the degeneracy of the first level (3.24) is partially removed by the perturbation, since the states

$$|\lambda;k\rangle = \frac{1}{\sqrt{N}}\sum_{a=1}^{N} e^{ika}|\lambda;a\rangle \qquad \lambda = 1,2,..,q-1 \qquad k = \frac{2\pi n}{N} \qquad n = 0,1,...,N-1 \quad (3.25)$$

are eigenstates of H_p and have different eigenvalues labeled by k, i.e.

$$\langle\lambda;k|H_p|\lambda';k'\rangle = -\delta_{\lambda,\lambda'}\delta_{k,k'}2\cos k = \delta_{\lambda,\lambda'}\delta_{k,k'}E_{1,\lambda}^{(1)}(k)$$

The quasiparticles dispersion relation can then be read from

$$E_\lambda(k) = E_{1,\lambda}^{(0)} + E_{1,\lambda}^{(1)}(k) =: E_{GS}^{(0)} + \varepsilon^\lambda(k) = E_{GS}^{(0)} + qg\left(1 - \frac{1}{g}\frac{2}{q}\cos k\right)$$

Of course (3.25) are eigenstates of $\mathcal{P}$ with eigenvalue k and for fixed k the $(q-1)$-degenerate subspace contains all the $Q \neq 0$ sectors and the states $|\lambda\rangle$ can be chosen to be eigenstates of $\mathcal{Q}$ with eigenvalue $\lambda = Q$ in the following way

$$|\lambda\rangle = \frac{1}{\sqrt{q}}\sum_{\mu=1}^{q} e^{i\lambda\mu\frac{2\pi}{q}}|e_\mu\rangle \qquad \lambda = 1,2,...,q-1$$

To sum up quasiparticles in the q-state Potts chain are understood in completly different ways in the two perturbative regimes $g \ll 1$ and $g \gg 1$, in both cases there are $(q-1)$ excitations with the same dispersion relation at first order in perturbation theory. They are domain walls in the ferromagnetic phase and spin flip in the paramagnetic one, which must be combined in the form of waves to diagonalize the perturbation at first order. Their dispersion relation reads

$$\varepsilon(k) = \begin{cases} q\left(1 - g\,\dfrac{2}{q}\cos k\right) & \text{if} \quad g \ll 1 \\ qg\left(1 - \dfrac{1}{g}\dfrac{2}{q}\cos k\right) & \text{if} \quad g \gg 1 \end{cases} \quad (3.26)$$

and it satisfies $\varepsilon(1/g) = g\,\varepsilon(g)$ as anticipated. This must remain true for all orders in perturbation theory because of the duality invariance (3.6) of the Hamiltonian, although this duality is not a one to one map between n-particle eigenstates in the two phases and in fact the one-domain wall level does not exist when periodic boundary conditions are assumed. Moreover, as a consequence of the fact that part of the full symmetry enjoyed by H has been disregarded, it is likely that the $q-1$ quasiparticles degeneracy persists for all g.
It is also possible to get sistematically higher order corrections to both the ground state energy and the first excited state, the second order calculation will be performed for the paramegnetic phase in appendix A and the result is

$$E_{GS}^{(2)} = -\frac{1}{g}\frac{N(q-1)}{2q}x \qquad E_1^{(2)}(k) = -\frac{1}{g}\left[\frac{N(q-1)}{2q} + \frac{2}{q}\left((q-2)\cos k + \frac{\cos 2k}{2}\right)\right] \quad (3.27)$$

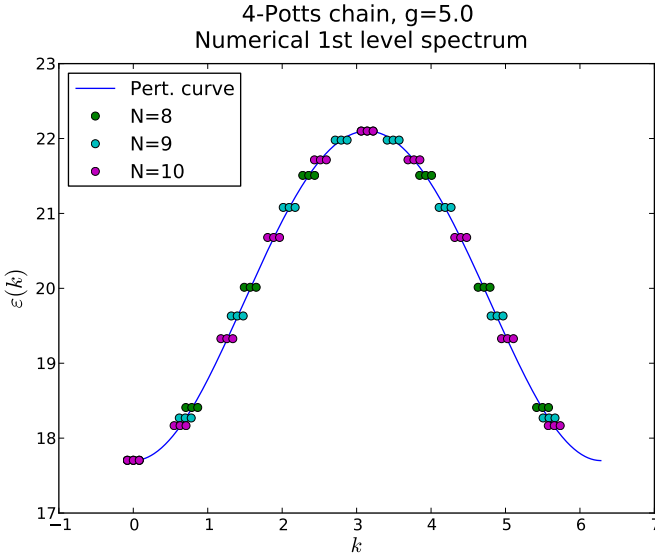
where the extensive part in the first excited state correction is just the second order ground state correction, so that the second order dispersion relation is given by

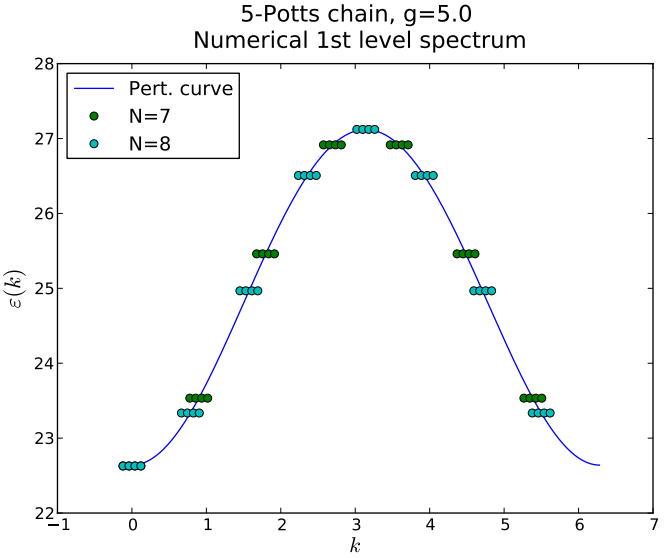
$$\varepsilon(k) = qg\left[1 - \frac{1}{g}\frac{2}{q}\cos k - \frac{1}{g^2}\frac{2}{q^2}\left((q-2)\cos k + \frac{\cos 2k}{2} + \frac{q-3}{2}\right)\right] \quad (3.28)$$

and the first excited state still posseses a $q-1$ degeneracy. This expression holds for $g \gg 1$ and the ferromagnetic one is given by (3.6) when the corresponding eigenvalue exists.
Let's now try to have a look on the whole structure of the spectrum. In the paramagnetic phase the ground state is non degenerate, the first excited state is a band of N eigenvalues[4] labelled by $k = 2\pi n/N$ each with degeneracy $q-1$, the second excited state contains 2 spin flips and its energy eigenvalues must be worked out perturbatively[5] but the total number of states contained in this band is $(q-1)^2 N(N-1)/2$ and so on, up to the p-th bands containing p spin flips and $\binom{N}{p}(q-1)^p$ eigenstates. In this way the total number of state for finite N is correctly given by

$$\sum_{p=0}^{N} \binom{N}{p} (q-1)^p = q^N$$

As long as g is large enough and the 1-particle band is well separated from the 2-particle level it is possible to check numerically the accuracy of (3.28), simply taking the first lowest $1 + N(q-1)$ eigenvalues of H, discarding the ground state (which agrees with (3.27) with a relative error of 10^{-4} when $g \geq 5$) and matching the others with $\varepsilon(2\pi n/N)$, $n = 0, 1, ..., N-1$. The result for $q = 4, 5$ and different values of N is shown below, confirming both the correctness of (3.28) and the persistence of the $q-1$ degeneracy of each k-mode.





In the graphs above the 3 and 4 points corresponding to the same energy are plotted with a small separation just to show their degeneracy, but they correspond to the same wave vector k.
In the ferromagnetic phase there are q degenerate ground states and the 1-particle level is absent. This is observable numerically, since setting g small enough and discarding the first q, essentially equal[6], eigenvalues of H (which agrees with (3.27) once (3.6) is used) one gets a first excited state with energy almost exactly equal to $2\,\varepsilon(0)$ with $\varepsilon(k)$ given by (3.28) and (3.6). Note also that in this case it is more difficult to count the number of eigenstates contained in a band with a fixed number of domain walls: the 0-domain walls level is composed of the q degenerate ground states, the 2-domain walls level contains $q(q-1)N(N-1)/2$ states, the 3-domain walls level contains $q(q-1)(q-2)\binom{N}{3}$ states, but then is non trivial to count in how many ways the different q colours may be placed in each of the $\binom{N}{4}$ configurations with 4-domain walls. Nonetheless each one of the q sectors labelled by the eigenvalues of $\mathcal{Q}$ has to contain q^{N-1} eigenstates.

[4] Which becomes continuous for infinite N
[5] See for example [48] for a brief discussion
[6] Their relative energy difference is $\geq 10^{-10}$ even for $N = 8$ for any q

3.4 The quantum Ising chain in a transverse field

In the case $q = 2$ the Hamiltonian in (3.1) describes the ferromagnetic quantum Ising chain in a transverse field

$$H = -\sum_{a=1}^{N} \sigma_a^z \sigma_{a+1}^z - g \sum_{a=1}^{N} \sigma_a^x$$

here $\omega = (-1)$ and the $\mathbb{Z}_2$ symmetry generator will be written $(-1)^{\mathcal{F}}$, where $\mathcal{F}$ is an operator having a very physical interpretation. This model is exactly solvable in the sense that its spectrum can be computed explicitly together with its eigenstates thanks to the Jordan-Wigner transformation which maps the spin operators given by the Pauli matrices in a set of hermitian anticommuting operators

$$a_j = \prod_{\ell=1}^{j-1} \sigma_\ell^x \sigma_j^z \qquad b_j = i a_j \sigma_j^x \qquad\qquad \sigma_j^x = -i a_j b_j \qquad \sigma_j^z = \prod_{\ell=1}^{j-1} (-i a_\ell b_\ell) a_j \tag{3.29}$$

$$\{a_j, a_\ell\} = \{b_j, b_\ell\} = 2\delta_{j,\ell} \qquad\qquad \{a_j, b_\ell\} = 0 \tag{3.30}$$

These relations define the so called Majorana fermions, they are hermitian matrices satisfying $a^2 = b^2 = 1$. Defining the complex matrices $c_j = (a_j + i b_j)/2$ one obtains the usual Dirac fermions satisfying

$$\{c_j, c_\ell^\dagger\} = \delta_{j,\ell} \qquad\qquad \{c_j, c_\ell\} = \{c_j^\dagger, c_\ell^\dagger\} = 0 \tag{3.31}$$

which in turn implies $c^2 = (c^\dagger)^2 = 0$. The Hamiltonian is quadratic in these new fermionic variables

$$H = -gN + \sum_{j=1}^{N} \left(2g\, c_j^\dagger c_j - c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger + c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1} \right) + (-1)^{\mathcal{F}} \left(c_N^\dagger c_1 - c_N c_1^\dagger + c_N^\dagger c_1^\dagger - c_N c_1 \right) \tag{3.32}$$

here $\mathcal{F} = \sum_{j=1}^{N} c_j^\dagger c_j$ is the number operator, which counts the total number of fermions on the chain, so the $\mathbb{Z}_2$ conserved quantum number is the parity of the number of fermions. The boundary term in H is due to the periodic boundary conditions imposed on the spins, it comes out because the last addend in the interaction sum in H is

$$\sigma_N^z \sigma_{N+1}^z = \sigma_N^z \sigma_1^z = \prod_{\ell=1}^{N-1} \left(1 - 2 c_l^\dagger c_l\right) \left(c_N + c_N^\dagger\right) \left(c_1 + c_1^\dagger\right) = (-1)^{\mathcal{F}} \left(c_N - c_N^\dagger\right) \left(c_1 + c_1^\dagger\right)$$

which is exactly the last term in (3.32). However the Jordan-Wigner transformation (3.29) is not well defined for $j = N + 1$ and for this reason the boundary condition on the cs are ambiguous. In order to lift the last term and extend the sum up to N one chooses antiperiodic boundary coniditons on the sector with even fermion number, also called Neveu-Schwarz sector, and periodic boundary condition on the sector with odd fermion number, also called Ramond sector. In this way $c_{N+1} = (-1)^{F+1} c_1$. This choice ensures translational invariance, so (3.32) lends itself to be diagonalized in Fourier space, of course paying attention to the different boundary conditions in the two sectors

$$c_j = \frac{e^{i\pi/4}}{\sqrt{N}} \sum_k e^{ikj} c_k \qquad\qquad \begin{cases} k_n^0 = \dfrac{2\pi}{N}\left(n + \dfrac{1}{2}\right) & \text{if } F = 0 \\ k_n^1 = \dfrac{2\pi n}{N} & \text{if } F = 1 \end{cases} \qquad\qquad n = 0, 1, ..., N-1$$

where the overall factor is purely conventional and only ensures the appearance of no imaginary units. Now the analysis proceeds in two different ways in the two sectors. The Hamiltonian in momentum

space reads

$$H = \begin{cases} -gN + \sum_{n=0}^{N-1} \left[2\Big(g - \cos k_n^0\Big) c_n^\dagger c_n + \sin k_n^0 \left(c_{-n-1}^\dagger c_n^\dagger - c_{-n-1} c_n \right) \right] & \text{if } F = 0 \\ -gN + \sum_{n=0}^{N-1} \left[2\Big(g - \cos k_n^1\Big) c_n^\dagger c_n + \sin k_n^1 \left(c_{-n}^\dagger c_n^\dagger - c_{-n} c_n \right) \right] & \text{if } F = 1 \end{cases}$$

where $c_n = c_{k_n}$. The strategy at this point is to perform a rotation in momentum space, i.e. a Bogoliubov transformation of the form

$$d_n = \cos \frac{\theta_{k_n^0}}{2} c_n - \sin \frac{\theta_{k_n^0}}{2} c_{-n-1}^\dagger \qquad\qquad d_{-n-1}^\dagger = \sin \frac{\theta_{k_n^0}}{2} c_n + \cos \frac{\theta_{k_n^0}}{2} c_{-n-1}^\dagger$$

and similarly in the $F = 1$ sector. Choosing

$$\tan \theta_{k_n} = \frac{\sin k_n}{g - \cos k_n}$$

the Hamiltonian becomes diagonal in both sectors, with the important difference that in the odd one ($F = 1$) the mode with $k_n = 0$ is present and for this mode $\theta_{k_n} = 0$, i.e. it does not transform under the rotation. Its contribution to the Hamiltonian must be singled out

$$H = \begin{cases} \sum_{n=0}^{N-1} \varepsilon(k_n^0) \left(d_n^\dagger d_n - \frac{1}{2} \right) & \text{if } F = 0 \\ 2(g-1) \left(d_0^\dagger d_0 - \frac{1}{2} \right) + \sum_{n=1}^{N-1} \varepsilon(k_n^1) \left(d_n^\dagger d_n - \frac{1}{2} \right) & \text{if } F = 1 \end{cases} \tag{3.33}$$

where

$$\varepsilon(k) = 2\sqrt{1 + g^2 - 2g \cos k} \tag{3.34}$$

Note that this dispersion relation satisfies $\varepsilon(g) = g\,\varepsilon(1/g)$ and agrees with (3.28) when expanded to second order in g, indeed

$$2\sqrt{g^2 + 1 - 2g \cos k} \sim 2 \left(1 - g \cos k + \frac{g^2}{2} \sin^2 k + \mathcal{O}(g^3) \right)$$

The ground state of the whole Hamiltonian always lies in the even sector and it is the state with no fermions, defined by

$$d_p |0\rangle = 0 \quad \forall p$$

Its energy is given by (3.33) in the $F = 0$ sector, with all the ds equal to 0, i.e.

$$E_{GS} = -\frac{1}{2} \sum_{n=0}^{N-1} \varepsilon(k_n^0) \underset{N\to\infty}{\sim} -\frac{N}{2\pi} \int_0^{2\pi} \mathrm{d}k \sqrt{1 + g^2 - 2g \cos k} \tag{3.35}$$

Note that at the critical point $g = 1$ the finite size scaling of the ground state energy can be written down explictly

$$E_{GS} = -\sum_{n=0}^{N-1} 2 \sin \left[\frac{\pi}{N} \left(n + \frac{1}{2} \right) \right] = \frac{2}{\sin\left(\frac{\pi}{2N}\right)} \underset{N\to\infty}{\sim} -\frac{4}{\pi} N - \frac{\pi}{6N} + \mathcal{O}\left(\frac{1}{N^3} \right) = \varepsilon_{vac} N - \frac{\pi c\, v}{6N} \tag{3.36}$$

but since (3.34) when $g = 1$ yields

$$\varepsilon(p) = 4 \sin \frac{p}{2} \sim 2p = vp \qquad\qquad p \to 0$$

one reads that the quasiparticles velocity at criticality is $v = 2$. The central charge of the model can then be obtained directly from the lattice in this simple example, with the result $c = 1/2$. This is the central charge of a free fermionic field theory, as will be shown a the end of this section.
Coming back to the energy spectrum for finite N, since when $g > 1$ the "zero mode" corresponding to $k = 0$ in the odd sector raises the energy, the state containing the particle with zero momenta jumps in the 1-particle level. As a consequence in the paramagnetic phase the spectrum of the two sectors simply merge to give p-particles bands, each with $\binom{N}{p}$ different eigenvalues, none of which is degenerate. When p is even the state lies in the $F = 0$ sector, while when p is odd it lies in the $F = 1$ sector. The p-particles level will have an energy given by

$$E_p(k_1, ..., k_p) = \varepsilon(k_1) + ... + \varepsilon(k_p) \qquad k_1 \neq k_2 \neq ... \neq k_p$$

$$\begin{cases} k_i = \dfrac{2\pi}{N}\left(n + \dfrac{1}{2}\right) & \text{if } p \text{ even} \\ k_i = \dfrac{2\pi n}{N} & \text{if } p \text{ odd} \end{cases}$$

When $g < 1$ instead the zero mode lowers the energy and in the odd sector there is a state which contains a "particle" with negative energy and the eigenvalue of this state is

$$E^1_{GS} = (g-1) - \frac{1}{2}\sum_{n=1}^{N-1} \varepsilon(k^1_n) = -\frac{1}{2}\sum_{n=0}^{N-1} \varepsilon(k^1_n) \underset{N\to\infty}{\sim} E_{GS}$$

Then in the odd sector there is now a state which is allowed by the parity of the fermion number and in the thermodynamic limit becomes degenerate with the ground state. Moreover one can add excitations on this state in such a way that the number of fermions remains odd and the excited state is degenerate, in the thermodynamic limit, with one of the excited states of the even sector. In this way all energy levels becomes degenerate in the ferromagnetic phase. As mentioned in the previous section, this degeneracy is quite unexpected. It has been explained very recently in terms of conserved charges associated to the two edges of the chain when open boundary conditions are assumed [56, 57], but its deep origin and relation to the ordered phase has not yet been completly understood.
As already anticipated in chapter 2, the ground state energy (3.35) is not analytic at the critical point, moreover in the thermodynamic limit an infinite number of eigenstates collapse to the ground state when $g = 1$. These are all the multiparticle states in which all the fermions have momentum k infinitesimally close to $k = 0$. This is the signal of a second order phase transition. In other words the spectrum is gapped and the quasiparticle's mass goes to zero at the quantum critical point, with an exponent $\nu = 1$ which corresponds to the critical exponent of the correlation length of the 2-dimensional classical Ising model

$$\Delta = E_1 - E_0 = \varepsilon(0) = |g - 1| \sim |g - g_c|^\nu \qquad g \to g_c$$

It can be also shown [49] that the order parameter expectation value defined as[7]

$$m^2 := \lim_{a\to\infty} \langle 0|\sigma^z_0 \sigma^z_a|0\rangle \qquad\qquad m \sim J^{-\frac{1}{8}}(1-g)^{\frac{1}{8}} \qquad g \to 1^-$$

has exactly the same behavior as the order parameter of the classical 2-dimensional Ising model universality class. Here the overall factor J has been restored for later convenience.

[7] The ground state has to be chosen in such a way to break the $\mathbb{Z}_2$ symmetry, otherwise the expectation value would vanish

3.4.1 The scaling limit and the critical point

When the QC mapping was discussed in chapter 2 the Trotterization process was used to build another direction representing imaginary time. This direction was initially discretized with spacing $\delta\tau$ and then the so called imaginary time scaling limit was taken in which

$$\delta\tau \to 0\,, \quad M \to \infty \qquad \beta = M\delta\tau \quad \text{fixed} \tag{3.37}$$

where M is the number of intervals of lenght $\delta\tau$. However the quantum problem has from the beginning a natural real time dynamics governed by the Heisenberg equation

$$i\frac{\mathrm{d}O}{\mathrm{d}t} = \big[O, H\big] \tag{3.38}$$

and here the time is already continuous. But the chain is discrete, so the space scaling limit can be taken

$$a \to 0\,, \quad |g - g_c| \to 0\,, \quad J \to \infty \qquad \Delta = 2J(g_c - g)\,, \quad v = 2Ja \quad \text{fixed}$$

This automatically assumes the vicinity of the critical point, since there a field theory description is supposed to be valid. Let's start from (3.32) restoring the dimensionful factor J in front. Disregarding the boundary term and the overall constant and performing the substitutions

$$e^{i\pi/4}\psi(x_j) \longrightarrow \frac{1}{\sqrt{a}}c_j \qquad \sum_j \longrightarrow a\int_0^L \mathrm{d}x$$

implies that the anticommutation relations (3.31) become

$$\big\{c_j, c_\ell^\dagger\big\} = \delta_{j,\ell} \longrightarrow \big\{\psi(x), \psi^\dagger(y)\big\} = \delta(x-y)$$

where $\delta(x-y)$ is the dimensionful Dirca delta function. Then expanding

$$\psi(x)\psi(x+a) \simeq \psi(x)\left(\psi(x) + a\partial_x\psi(x)\right) = a\psi(x)\partial_x\psi(x) \tag{3.39}$$

together with the other terms, the Hamiltonian becomes

$$H = \int \mathrm{d}x \Big[-\frac{iv}{2}\Big(\psi^\dagger(x)\partial_x\psi^\dagger(x) - \psi(x)\partial_x\psi(x)\Big) - \Delta\psi^\dagger(x)\psi(x)\Big] \tag{3.40}$$

where two terms have been dropped since they vanish upon integration by parts. This Hamiltonian is the one of a free massive Majorana field, indeed defining the right and left moving component of a spinor as

$$\Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = \begin{pmatrix}\psi - \psi^\dagger \\ \psi + \psi^\dagger\end{pmatrix}$$

they represent only two degrees of freedom for the fermion field. The equations of motion follow from Heisenberg equation (3.38)

$$\left(\partial_t - v\partial_x\right)\psi_L(t,x) = i\Delta\psi_R(t,x) \qquad \left(\partial_t + v\partial_x\right)\psi_R(t,x) = i\Delta\psi_L(t,x)$$

which upon switching to imaginary time $t = i\tau$ and defining $z = \tau + ix$ become

$$\partial_z\psi_L(z,\bar z) = -2\Delta\psi_R(z,\bar z) \qquad \partial_{\bar z}\psi_R(z,\bar z) = -2\Delta\psi_L(z,\bar z)$$

These two equations decouple when $\Delta = 0$, i.e. at the critical point, and the the left and right components of the Majorana spinor becomes holomorphic and antiholomorphic fields

$$\psi_L(z,\bar z) = \bar\psi(\bar z) \qquad \psi_R(z,\bar z) = \psi(z)$$

This is exactly the same decoupling observed for the analytic and antianalytic sectors of the conformal group and indeed the hamitlonian (3.40) is conformally invariant at the phase transition point. The quantum Ising chain is in fact a very special case in which the critical Hamiltonian can be written down explicitly in a whole neighborhood of the critical point and its energy momentum tensor computed to obtain the central charge directly from the field theory. The equations of motion may be derived from the euclidean action

$$\mathcal{A} = \int \mathrm{d}^2 z \left(-\psi \partial_{\bar{z}} \psi - \bar{\psi} \partial_z \bar{\psi} + 2\Delta \bar{\psi} \psi \right) = \mathcal{A}^* + 2\Delta \int \mathrm{d}^2 z \, \bar{\psi} \psi \tag{3.41}$$

so the conformally invariant action describing the critical point is the one of a free Majorana field.

CFT of a Majorana fermion field

Let's compute the central charge of this CFT directly from the fields. First one needs the energy-momentum tensor, which can be obtained via the definition (2.11). Setting $\Delta = 0$ in (3.41) and fixing the overall normalization in (2.11) to 1 one has

$$T_{zz}(z) = T(z) = -\psi \partial_z \psi \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z}) = -\bar{\psi} \partial_{\bar{z}} \bar{\psi} \qquad T_{z\bar{z}} = 0$$

To compute the central charge the propagators of ψ and $\bar{\psi}$ are necessary, they reads

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{2(z-w)} \qquad \langle \bar{\psi}(\bar{z}) \bar{\psi}(\bar{w}) \rangle = \frac{1}{2(\bar{z}-\bar{w})}$$

Then taking the expectation value of (2.17) only the first term in the OPE remains and c can be read from the overall factor. Using Wick theorem, with the energy-momentum tensor properly regularized in order to avoid contractions of operators at the same point, one obtains

$$\begin{aligned} \langle T(z) T(w) \rangle &= \langle \psi(z) \partial_z \psi(z) \, \psi(w) \partial_w \psi(w) \rangle = \\ &= -\frac{1}{4} \left[\frac{1}{z-w} \partial_z \partial_w \left(\frac{1}{z-w} \right) + \partial_w \left(\frac{1}{z-w} \right) \partial_z \left(\frac{1}{z-w} \right) \right] = \frac{1/4}{(z-w)^4} \end{aligned}$$

Then once again one reads that $c = 1/2$ and of course $\bar{c} = c$ since both the $\bar{T}$ and the antiholomorphic propagator are the same with $\bar{z}$ in place of z. Moreover, since the theory is free, ψ and $\bar{\psi}$ can acquire no anomalous dimension and their scaling dimension must be the canonical one, readly obtained by dimensional analysis on the action. By the way it can also be computed from the definition (2.19) and it turns out that they are primary fields with conformal weight $(1/2, 0)$ and $(0, 1/2)$ respectively.

At this point is quite clear that the scaling limit of this lattice theory at the critical point is described by the $\mathcal{M}_3$ minimal model CFT whose Kac table is given by

$$\Delta_{r,s} = \begin{pmatrix} \frac{1}{2} & \frac{1}{16} & 0 \\ 0 & \frac{1}{16} & \frac{1}{2} \end{pmatrix}$$

The scaling limit of the lattice observables can be obtained via diagonal coupling of the left and right sectors, these are

$$(0,0) = \mathbb{1} \qquad \left(\frac{1}{16}, \frac{1}{16} \right) =: \sigma \qquad \left(\frac{1}{2}, \frac{1}{2} \right) =: \varepsilon$$

and their scaling dimensions are respectively

$$\eta_{\mathbb{1}} = 0 \qquad \eta_\sigma = \frac{1}{8} \qquad \eta_\varepsilon = 1 \tag{3.42}$$

The first is the trivial identity matrix on the lattice, the second is the order parameter σ^z and the third is usually called energy density, it should represents, in the quantum chain, the scaling limit of the operator

$$\varepsilon_a = \sigma_a^z \sigma_{a+1}^z - \frac{1}{2}\left(\sigma_a^x + \sigma_{a+1}^x\right) \tag{3.43}$$

The identification of the order parameter σ^z with the primary field σ is quite mandatory, while the fact that the field ε is the scaling limit of the operator above is not so trivial. Indeed from (3.29) one could be tempted to write

$$\sigma_j^x = -ib_j a_j = (c_j - c_j^\dagger)(c_j + c_j^\dagger) \sim (\psi^\dagger - \psi)(\psi^\dagger + \psi) = \bar{\psi}\psi$$

but looking more carefully at the scaling limit (3.40) one realizes that the field $\bar{\psi}\psi$ gets a contribution also from the first term in the lattice Hamiltonian. Then to find the appropriate combination of lattice matrices yielding the CFT's energy density is not enough to pick up an operator which is invariant under $\mathbb{Z}_2$, but also another transformation property has to be checked, namely the duality transformation (3.5). In fact it turns out that the energy density field ε is odd under this transformation, so one has to take the appropriate combination of one-site lattice matrices ε_a which more than statisfy

$$(-1)^{\mathcal{F}} \varepsilon_a (-1)^{\mathcal{F}} = \left(\prod_{b=1}^{N} \sigma_b^x\right) \varepsilon_a \left(\prod_{c=1}^{N} \sigma_c^x\right) = \varepsilon_a$$

has to change sign when (3.5) is applied. This leads to the operator (3.43).
From the scaling dimensions the critical exponents ν and β are obtained

$$\nu = \frac{1}{2 - \eta_\varepsilon} = 1 \qquad\qquad \beta = \eta_\sigma \nu = \frac{1}{8}$$

Note that the operators of the CFT, as the lattice observables, can be classified in terms of representation of the symmetry group $\mathcal{G}$, which in this case is $\mathbb{Z}_2$. Quite generally the order parameter is the most relevant of the primaries which do carry charge under $\mathcal{G}$, while the energy density is the most relevant operator invariant under the symmetry of the model. These symmetry properties of the primaries can be inferred by the knowledge of the complete OPE they satisfy.
Since one of the possible representations of the scaling limit of the action near the critical point in terms of a fundamental field ψ is (3.41), these operators should be expressible in terms of ψ. An easy guess for the energy density ε is of course $\varepsilon(z, \bar{z}) = \bar{\psi}(\bar{z})\psi(z)$. Indeed the conformal weights matches and moreover it is the $\mathbb{Z}_2$-invariant field driving the action away from the critical one

$$\mathcal{A} = \mathcal{A}^* + 2\Delta \int \mathrm{d}^2 z\, \bar{\psi}(\bar{z})\psi(z) = \mathcal{A}^* + 4J(g_c - g) \int \mathrm{d}^2 z\, \varepsilon(z, \bar{z})$$

Regarding σ instead, its expression in terms of the Jordan-Wigner fermions ψ and $\bar{\psi}$ is non trivial, the reason being that the relation between the two is in local. This goes back to (3.29), from which can be seen that the order parameter σ_a^z is a product of fermions on all the sites before a.
To conclude this section it is worth stressing that the CFT describing this quantum chain at the critical point is free, in addition to massless, the latter of course being a feature of all CFTs. This fact greatly simplifies the matching of objects in the field theory with lattice, i.e. physical, observables. This feature of the conformal critical point is not at all guaranteed and in general all the CFTs in the minimal series are strongly interacting.

3.5 The 3-states quantum Potts chain

For $q = 3$ equation (3.1) becomes the Hamiltonian limit[8] of the classical 3-states Potts model

$$H = -\sum_{a=1}^{N} \left(\sigma_a \sigma_{a+1}^\dagger + \sigma_a^\dagger \sigma_{a+1}\right) - g \sum_{a=1}^{N} \left(\tau_a + \tau_a^\dagger\right)$$

The internal symmetry group leaving H invariant is S_3, which may be usefully decomposed as $\mathbb{Z}_3 \otimes \mathbb{Z}_2$, the generator of $\mathbb{Z}_3$ being (3.4), with eigenvalues $Q = 0, \pm 1$. The generator of $\mathbb{Z}_2$ may be taken as the product of one-site matrices exchanging two of the three components of the single-site vectors, let's stay the second and the third, i.e.

$$\mathcal{C} = \prod_{a=1}^{N} c_a \qquad\qquad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

In this case there are two quasi particles, whose 2-degeneracy can be explained via the fact that both $\mathcal{C}$ and $\mathcal{Q}$ commute with H, but they do not commute with each other. Indeed $\mathcal{C}$ maps the $Q = 1$ into the $Q = -1$ sector since $\mathcal{Q}\mathcal{C} = \mathcal{C}\mathcal{Q}^\dagger$.
Although it exists a generalization of the Jordan-Wigner mapping, named after Fradkin and Kadanoff [51], the degrees of freedom in which the model is mapped to are not fermionic since they do not anticommute, rather they acquire a phase $\omega = e^{i2\pi/3}$ upon exchange. These kind of quasiparticle excitations are called parafermions and the map reads

$$a_j = \prod_{k=1}^{j-1} \tau_k \sigma_j \qquad b_j = \omega a_j \tau_j \qquad \tau_j = \omega^{-1} a_j^\dagger b_j \qquad \sigma_j = \prod_{k=1}^{j-1} \bar{\omega} a_k b_k^\dagger a_j \tag{3.44}$$

Note that, in contrast to the Majorana Fermions (3.29), the as and bs are not hermitian. Using the algebra satisfied by σ and τ

$$\sigma\tau = \omega\tau\sigma \qquad \sigma^3 = \tau^3 = 1$$

it is easy to find the parafermionic commutation relations

$$\begin{aligned} a_j a_\ell &= \omega^{\mathrm{sign}(\ell-j)} a_\ell a_j & b_j b_\ell &= \omega^{\mathrm{sign}(\ell-j)} b_\ell b_j & a_j b_\ell &= \omega^{\mathrm{sign}(\ell-j)} b_\ell a_j \\ a_j a_\ell^\dagger &= \omega^{\mathrm{sign}(j-\ell)} a_\ell^\dagger a_j & b_j b_\ell^\dagger &= \omega^{\mathrm{sign}(j-\ell)} b_\ell^\dagger b_j & a_j b_\ell^\dagger &= \omega^{\mathrm{sign}(j-\ell)} b_\ell^\dagger a_j \end{aligned}$$

and to observe that, like τ and σ, these variables do not square to one but

$$a_j^3 = a_j a_j^\dagger = 1 \qquad b_j^3 = b_j b_j^\dagger = 1 \qquad a_j b_j = \omega b_j a_j \qquad b_j a_j^\dagger = \omega a_j^\dagger b_j$$

The Hamiltonian in terms of these new variables takes the form

$$H = -\sum_{j=1}^{N} \left[(\bar{\omega} b_j^\dagger a_{j+1} + \omega a_{j+1}^\dagger b_j) + g(\omega b_j a_j^\dagger + \bar{\omega} a_j b_j^\dagger)\right] - \omega^{-\mathcal{Q}} b_N a_1^\dagger - a_1 b_N^\dagger \omega^{\mathcal{Q}}$$

Due to the fact that the commutation relations of these operators depend on the site, it is not possibile to go through the same steps as in the Ising chain to diagonalize H, so the exact spectrum of this model is not known for all g. Nevertheless, as for the classical 3-Potts model, this quantum model can be solved at the critical point. Its spectrum has been computed using Bethe Ansatz techniques

[8] The Hamiltonian limit of a classical lattice model basically is the inverse QC-mapping discussed in section 3.2

[54], in a form pretty different from the one briefly discussed in chapter 1. The result is that its eigenvalues when $g = 1$ and $N \to \infty$ can be understood in terms of massless quasiparticles with dispersion relation

$$\varepsilon(p) = 3\sqrt{3}\sin\frac{p}{2} \sim \frac{3\sqrt{3}}{2}p = vp \qquad\qquad p \to 0 \tag{3.45}$$

above a ground state energy whose exact value is given by

$$\varepsilon_{\text{vac}} = \lim_{N\to\infty}\frac{E_{GS}}{N} = -\frac{2\sqrt{3}}{\pi} - \frac{4}{3} \simeq 2.436 \tag{3.46}$$

Moreover the model has been tackled pertubatively in the paramagnetic phase up to very high order [53], aiming to test its critical properties analytically from perturbation theory. For example the critical exponents of the order parameter and of the mass gap

$$\langle 0|m|0\rangle \underset{g\to 1^-}{\sim} m_0(g-1)^{\frac{1}{9}} \qquad\qquad \varepsilon(0) \underset{g\to 1^+}{\sim} \Delta_0(1-g)^{\frac{5}{6}}$$

have been found in agreement with perturbation theory.

3.5.1 Critical point

Although in this case it is not possible to take directly the space continuum limit of the quantum Hamiltonian, its critical point is still described by a CFT. This fact is in general a non trivial statement: it is not enough to have infinite spacial correlation length for two point functions along the chain to get a conformal critical point, but also a linear dispersion relation for the low energy excitations is needed. Actually this is verified by the exact dispersion relation (3.45). Only in this case it is ensured that the quantum field theory which comes out from the limit in which the distance between the sites on the chain goes to zero will be relativistic invariant. The CFT corresponding to the 3-states Potts model was first spotted in [58], simply by symmetry arguments the author made an ansatz for the OPE, from which consintency relations between the conformal weights of the primaries were extracted. In the end he was able to write down the whole Kac table of the minimal model. It was then realized that the scaling limit of this model corresponds to a non diagonal modular invariant realization of the minimal model $\mathcal{M}_5$, whose central charge and Kac table are given by

$$c = \frac{4}{5} \qquad\qquad \Delta_{r,s} = \begin{pmatrix} 3 & \frac{13}{8} & \frac{2}{3} & \frac{1}{8} & 0 \\ \frac{7}{5} & \frac{21}{40} & \frac{1}{15} & \frac{1}{40} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{40} & \frac{1}{15} & \frac{21}{40} & \frac{7}{5} \\ 0 & \frac{1}{8} & \frac{2}{3} & \frac{13}{8} & 3 \end{pmatrix}$$

The full operator content of the CFT which comes out from the non diagonal combination of the left and right Virasoro algebras turns out to be quite complicated. Since $\Delta \neq \bar{\Delta}$ many of this operators have a non trivial conformal spin $s = \Delta - \bar{\Delta}$ and two of them represent new conserved currents generating an algebraic structure which extends the Virasoro algebra (2.18) called $\mathcal{W}$ algebra [55]. Not all these primaries represents trivial observables on the lattice but a rather complete correspondence between local matrices on the chain and operators in the CFT has been established in [59]. The most relevant ones, apart from the identity, are two spins operator carrying S_3 charge and the analogue of the Ising energy density which is instead invariant under the internal symmetry group

$$(0,0) = \mathbb{1} \qquad \left(\frac{1}{15},\frac{1}{15}\right) =: \sigma \qquad \left(\frac{1}{15},\frac{1}{15}\right) =: \sigma^\dagger \qquad \left(\frac{2}{5},\frac{2}{5}\right) =: \varepsilon$$

The scaling dimensions of these operators are respectively

$$\eta_{\mathbb{1}} = 0 \qquad \eta_\sigma = \frac{2}{15} \qquad \eta_\varepsilon = \frac{4}{5}$$

so that the critical exponents ν and β are

$$\nu = \frac{1}{2 - \eta_\varepsilon} = \frac{5}{6} \qquad \beta = \eta_\sigma \nu = \frac{1}{9}$$

the latter being the same for both the order parameters. Then among the representations of (2.18) arising in this CFT some of them appear twice. Actually they are not, strictly speaking, the same field since they would be distinguished by their quantum numbers w.r.t. the conserved charges extending the Virasoro algebra. The local fields σ and $\sigma^\dagger$ can be identified with the lattice matrices σ_a and $\sigma_a^\dagger$, which are related to the two independent order parameters in (3.7) by

$$m_a^1 = \frac{1}{2}\left(\sigma_a + \sigma_a^\dagger\right) = \mathrm{Re}\,\sigma_a \qquad m_a^2 = \frac{1}{2}\left(\bar\omega\sigma_a + \omega\sigma_a^\dagger\right) = \mathrm{Re}\,(\bar\omega\sigma_a)$$

Their critical behavior will then be the same, being the same for σ and $\sigma^\dagger$. Regarding the energy density ε instead, it corresponds to the combination of lattice matrices

$$\varepsilon_a = \sigma_a\sigma_a^\dagger + \sigma_a^\dagger\sigma_a - \frac{1}{2}\left(\tau_a + \tau_a^\dagger + \tau_{a+1} + \tau_{a+1}^\dagger\right) \tag{3.47}$$

for the same reasons as for the Ising chain: it has to be invariant under the whole symmetry group S_3, so under both $\omega^{\mathcal{Q}}$ and $\mathcal{C}$, and it must change sign under the duality transformation (3.5), which represents an additional symmetry of the Hamiltonian at the critical point.

3.6 The critical point for $1 \leq q \leq 4$

For general q the quantum Hamiltonian (3.1) has not been studied extensively in the literature apart from numerical analysis aiming to prove that the transition for $q > 4$ becomes of the first order and to characterize first order quantum phase transitions [40, 61]. The map into parafermions (3.44) still works and leads to a generalization of the $\mathbb{Z}_3$ parafermions considered in the previous section, but as before it is of no use to diagonalize H.
Nevertheless the scaling limit of the theory for $q \leq 4$ still gives a conformal field theory, as can be seen studying the finite size scaling of the ground state energy or the behavior of the entanglement entropy, which display the typical features of a conformal point when $g = 1$.
The central charge of these CFTs is given by [62]

$$c = 1 - \frac{6}{t_q(t_q+1)} \qquad 2\sin\left(\frac{\pi}{2}\frac{t_q - 1}{t_q + 1}\right) = \sqrt{q} \tag{3.48}$$

and the primary operators corresponding to the order parameter σ and the energy density ε are those operators with $\Delta = \bar\Delta$ obtained from the Kac table provided by the central charge above setting $(r, s) = ((t_q - 1)/2, (t_q + 1)/2)$ and $(r, s) = (2, 1)$ respectively. From these conformal weights follow the critical exponents presented before in (3.17). For $q = 4$ they are

$$\left(\frac{1}{16}, \frac{1}{16}\right) =: \sigma \qquad \left(\frac{1}{4}, \frac{1}{4}\right) =: \varepsilon$$

from which one gets the critical exponents

$$\nu = \frac{1}{2 - \eta_\varepsilon} = \frac{2}{3} \qquad \beta = \eta_\sigma \nu = \frac{1}{8}$$

As for $q = 2, 3$ the operator σ does not come alone, since it will be part of a triplet of primaries, with the same scaling dimensions, belonging to a 3-dimensional irreducible representation of S_4. These fields can be put in one to one correspondence with the independent order parameters of the model, which once again will exhibit the same critical behavior. The matrix on the chain corresponding to the local field σ is of course σ_a, while the energy density ε can be constructed as before from S_4 invariant one-site operators, changing sign under duality.
Note that (3.48) is valid down to $q = 1$ through the analytic continuation of the classical partition function discussed in section 3.2.2, which wierdly has central charge $c = 0$, not necessarily implying that the CFT is trivial. However the 1-Potts model does not admit a quantum Hamiltonian formulation and will not be discussed.

3.7 Central charge and light velocity at the critical point

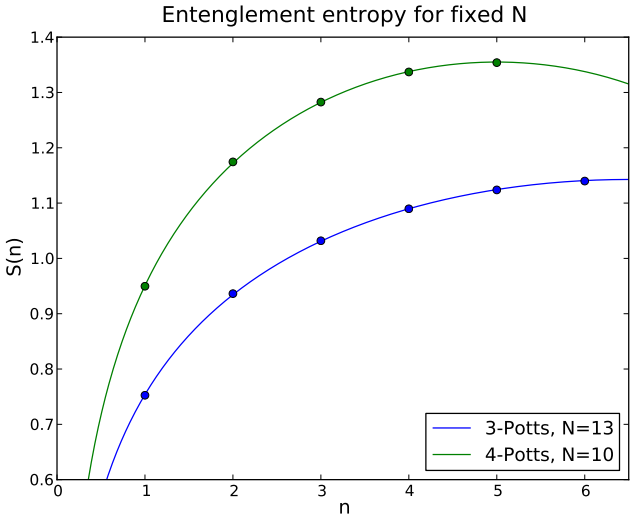
In order to check what said above about the CFTs of the critical points of the q-states Potts model for $q = 3, 4$, a measure of the central charge using the techniques outlined in chapter 2 is now performed. Calculating numerically the entanglement entropy the central charge is extracted and then from finite size scaling of the ground state energy the velocity of light is obtained.
The procedure has been already discussed in the previous chapter, so below are reported the numerical results and some plots to show the goodness of the fits. More information about the code used to simulate the chains numerically will be given in the next chapter.
Measuring the entanglement entropy for fixed number of sites ($N = 13, 10$ for $q = 3, 4$ respectively) and fitting (2.29) with $\ell/L = n/N$, with n the number of spins considered for the subsystem not traced away, varying n between 1 and $N/2$ one obtains

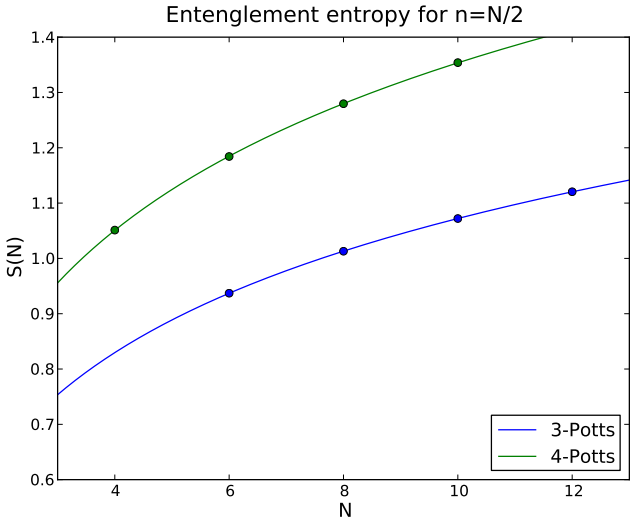
$$S(n) = \frac{c}{3} \log\left(\pi N \sin\frac{\pi n}{N}\right) + a \qquad \begin{array}{lll} c = 0.8(1) & a = 0.75(6) & q = 3 \\ c = 1.0(3) & a = 0.95(6) & q = 4 \end{array}$$

Whereas measuring the entanglement entropy for $n = N/2$ varying N and fitting once again (2.29) the results are

$$S(N) = \frac{c}{3} \log(\pi N) + a \qquad \begin{array}{lll} c = 0.79(4) & a = 0.76(6) & q = 3 \\ c = 0.99(1) & a = 0.97(0) & q = 4 \end{array} \tag{3.49}$$

Below are plotted the numerical data on the fitted curves, the agreement with the exact values discussed in the previous section is very good, even if the chains are relatively small.





For what regards the dimensionful factor representing the conversion from time to space appearing in the formula for the energy spectrum of a CFT on a cylinder, it can be estimated calculating the ground state energy, which will contain an extensive part proportional to the number of sites on the chain, already discussed in section 3.2.1. This non universal term depends on the particular implementation of the universality class on the lattice and represents the ground state energy per site in the thermodynamic limit. In the case of periodic boundary condition on the chain the cylinder CFT description shall be applied and the most important correction due to the finite size of the chain can be read from (2.28) to be of order $1/N$, since $L = aN$ with the lattice spacing $a = 1$. Then, in order to get the velocity of light v in the CFT, once the central charge is known, one can compute numerically the ground state energy varying N and fitting it via

$$E_{GS}(N) = \varepsilon_{\text{vac}} - \frac{\pi c v}{6N}$$

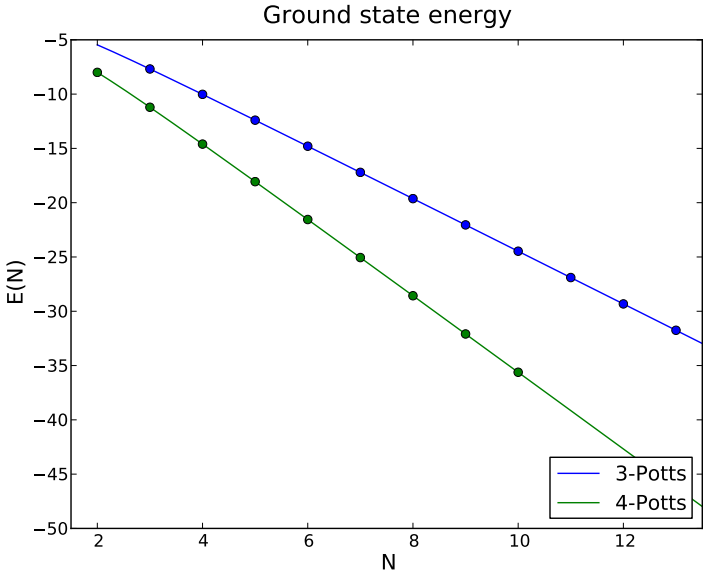
Observe that this formula holds short of corrections of order $1/N^2$. In fact in the case of the Ising chain from (3.36) one can see that the order $1/N^2$ corrections vanishes and the next non trivial term is of order $1/N^3$. The same seems to hold for $q = 3, 4$, indeed the fit has been performed via

$$E_{\text{GS}}(N) = \varepsilon_{\text{vac}} N - \frac{\pi c v}{6N} + \frac{a}{N^2} + \frac{b}{N^3}$$

and the results obtained for the coefficient a are perfectly compatible with zero:

$$\begin{array}{lllll} \varepsilon_{\text{vac}} = -2.4359(9) & v = 2.59(9) & a = 0.00(4) & b = -0.38(1) & q = 3 \\ \varepsilon_{\text{vac}} = -3.545(1) & v = 3.1(6) & a = 0.00(9) & b = -0.67(6) & q = 4 \end{array} \tag{3.50}$$

Note that the value of ε_{vac} and v for $q = 3$ are in perfect agreement with the exact results reported in (3.45) and (3.46). Below are shown the numerical curves for N varying between 3 and 13 for the 3-Potts chain and between 2 and 10 for the 4-Potts chain.



Chapter 4

Quench dynamics of the quantum q-states Potts chain

In this chapter will be presented the numerical results obtained from simulations of the dynamics after a global quench on the Hamiltonian (3.1) for $q = 2, 3$ and a partial analysis for $q = 4$. The $q = 2$ case corresponds to the quantum Ising chain in a transverse field and has already been extensively studied analytically and numerically in the literature[1]. Some of the analytical results for this model are reobtained and compared with numerical data, while some analytical conjecture are proposed for $q = 3, 4$. The focus will be on one point functions of local observables, since the heaviness of the simulations does not allow to get sensible results for two point functions.
The chapter is structured as follows: in section 1 the quench protocol, together with the quantities which will be studied and the numerical details will be explained, in section 2 the guideline according to which the results will be interpreted is presented and in section 3 the results are discussed. At the end of each subsection of section 3 quenches to the quantum critical point are analyzed in light of the CFT results discussed in chapter 2. Finally section 5 contains the conclusions of the work.

4.1 Quench protocol

The setup of the system for the quench is the usual one for a global quench in quantum chains: the initial state $|\psi_{g_0}\rangle$ is taken to be the ground state of the Hamiltonian (3.1) with some coupling g_0, then the Hamiltonian of the system is changed to a new one having a coupling g instead of g_0 and the dynamics is then governed by this new time evolution generator.
In what follows the ground state of the Hamiltonian H_g will be referred to as $|\psi_g\rangle$. In section 4.3 the following quantities will be analyzed:

- the energy per unit length of the post-quench Hamiltonian with respect to its ground state

$$\Delta E_g(g_0) := \frac{\langle\psi_{g_0}|H_g|\psi_{g_0}\rangle - \langle\psi_g|H_g|\psi_g\rangle}{N} \tag{4.1}$$

- the time dependence of the Loschmidt echo

$$\mathcal{L}(t) := |\langle\psi_{g_0}|e^{-iH_g t}|\psi_{g_0}\rangle|^2 \tag{4.2}$$

- the time dependent one point function of the transverse field

$$\langle M_a(t)\rangle = \langle\psi_{g_0}|e^{iH_g t}\, M_a\, e^{-iH_g t}|\psi_{g_0}\rangle$$

[1] The references will be given through the chapter

- the time dependent one point function of the order parameter

$$\langle m_a(t)\rangle = \langle\psi_{g_0}|e^{iH_g t}\, m_a\, e^{-iH_g t}|\psi_{g_0}\rangle$$

Here M_a is the one-site transverse field acting non trivially only on the site a whith the single site matrix given in (3.9) and the same for m, with the single site matrix defined in (3.7), choosing $\alpha = 0$. By the way the result is independent of α thanks to the S_q symmetry of the q-states Potts chain.

4.1.1 Numerical details

The code has been written in Python 2.7 and run on a 2.4 GHz quad core machine with 8 GB of RAM. It relies on exact diagonalization of sparse hermitian matrices provided by the python library SciPy. The maximum number of spins achievable with these performances is $N = 20, 12, 9$ respectively for the Ising, 3-Potts and 4-Potts chain full time evolution, necessary to compute the time dependence of the Loschmidt echo and of the one point functions, while $N = 21, 13, 10$ when just the ground state was needed in computing the energy density of the post-quench Hamiltonian.
The main limit on N is given by data storage: in order to allocate the $q^N \times q^N$ sparse Hamiltonian matrix the all available RAM memory was necessary with $N = 21$ for the Ising chain, $N = 13$ and $N = 10$ for the 3-Potts and 4-Potts chains, while to compute a useful finite interval time evolution in a sensible computational time the number of spins must be reduced by one.
The ground state $|\psi_{g_0}\rangle$ is obtained once again using a SciPy's method suitable for sparse hermitian matrices that employs Lanzcos's algorithm, which works iteratively starting from a guess for the eigenvector required. In the case of the order parameter time evolution from an initial ferromagnetic state the problem of breaking the symmetry to have nonzero initial value of $\langle m\rangle$ arises. Trying to make this symmetry breaking spontaneuos through a "magnetize" Lanzcos's eigenvector guess which did the job leads to a very slow convergence of the algorithm, by the way to a state which in most of the cases has zero mean of the order parameter as the critical point is approached from below. So it's better in this case to get $|\psi_{g_0}\rangle$ by adding an explicit symmetry breaking term in the initial Hamiltonian

$$H_{g_0} \longrightarrow H_{g_0} - h\sum_{a=1}^{N} m_a$$

in this way the convergence is much faster and the results are the same and independent of h as long as $0.1 \leq h \leq 0.001$ and $|g - g_0| \gtrsim 0.2$ and as long as short times are considered. By the way this timescale is the one of interest in order to observe the dynamics of the thermodynamic limit, as will be discussed in the next section. In fact the presence of h has the advantage of making time dependent averages more stable for a longer period of time, as will be discussed in detail in section 4.3.2. When g_0 approaches g one has to keep h the smallest possibile in order to make the observables stationary and the post-quench Hamiltonian's energy density zero as they should. Of course one could avoid this problem by calculating the two-point function for large distance between the two points and then exploiting cluster decomposition[2]

$$\lim_{|a-b|\to\infty}\langle m_a(t)m_b(t)\rangle = \langle m(t)\rangle^2$$

but in this case this trick is not applicable because of the shortness of the simulated chain.
Finally periodic boundary conditions are employed because they make the convergence of the computed quantities to the thermodynamic limit's ones much faster than open boundary conditions. The consequence of this choice on the spectrum of the Hamiltonian have been discussed in section 3.3.

[2] This should be verified a posteriori, since it is not obvious that the cluster decomposition property holds at all time, although this is true for the order parameter two point function of the Ising chain [65]

4.2 Quench dynamics as propagation of quasiparticles

As discussed in section 1.1 a quench process is exactly the opposite of an adiabatic process, in which the initial state evolves slowly as the perturbation is switched on, remaining in the ground state of the istantaneous Hamiltonian. But what happens if the perturbation is switched on suddenly? Is there a way to write down the initial state $|\psi_{g_0}\rangle$ in terms of eigenstates of H_g to make it possible to write down its time evolution in a closed form? Undoubtely an abrupt change of the Hamiltonian will make the would-have-been ground state a superposition of excited states of H_g, whose energy with the respect to the new ground state is exactly (4.1). But the main question is how does this energy reorganize in terms of excited states of H_g?
Of course there are many possibilities, but a natural guess is that if both H_{g_0} and H_g are translational invariant all the eigenstates of the two can be chosen as eigenstates of the momentum, and in particular $|\psi_{g_0}\rangle$ can be written as a superposition of momentum eigenstates which are eigenstates of H_g. Moreover since the quench conserves momentum all these eigenstates of H_g will have zero momentum eigenvalue, since the same holds for $|\psi_{g_0}\rangle$. This leads to an expression of the form

$$|\psi_{g_0}\rangle = \sum_{n=0}^{\infty} \sum_{p_1 \dots p_n} K_{p_1 \dots p_n} |p_1, -p_1, \dots, p_n, -p_{-n}\rangle = \sum_{n=0}^{\infty} \sum_{p_1 \dots p_n} K_{p_1 \dots p_n} a^\dagger_{p_1} a^\dagger_{-p_1} \dots a^\dagger_{p_n} a^\dagger_{-p_n} |\psi_g\rangle \quad (4.3)$$

that is a superposition of multiparticles states composed of particles pairs with opposite momenta created above the ground states of the new Hamiltonian. Each multiparticle state in the linear combination is then weighted by a non trivial amplitude $K_{p_1 \dots p_n}$. Of course this way of writing $|\psi_{g_0}\rangle$ does not solve the problem at all, since it requires the knowledge of the structure of the Fock space of both Hamiltonians, if such a structure exists. Interestingly this picture of the initial state as a sea of particles pairs has been first proposed in [36], when the BCFT prediction of the time dependence of two point functions suggested the interpretation of the quench dynamics as the propagation of particles pairs and has apparently[3] nothing to do with the obscure initial state (2.32) considered in the BCFT approach to the quench problem.
Although at this level the way of writing the initial state (4.3) seems mere formalism, let's press on supposing that the amplitude factorizes as

$$K_{p_1 \dots p_n} = \frac{{K_p}^n}{n!} \qquad \Rightarrow \qquad |\psi_{g_0}\rangle = \sum_{n=0}^{\infty} \frac{\left(\sum_p K_p a^\dagger_p a^\dagger_{-p}\right)^n}{n!} |\psi_g\rangle = \exp\left(\sum_p K_p a^\dagger_p a^\dagger_{-p}\right) |\psi_g\rangle \quad (4.4)$$

This very suggestive form of the quench initial state has a long story. It first appeared in [63] as integrability preserving boundary condition in integrable[4] field theories with a boundary, and for this reason is usually called boundary state, and then as initial state after quenching the mass of a free field [64]. In general this form of the initial state arises when the set of creation and annihilation operators building up the Fock space of H_g and H_{g_0} are related by a Bogoliubov transformation, as the one considered in section 3.4 to diagonalize the Ising chain. Indeed this is exactly the case for the Ising chain's initial state for the quench: since the operator ds and $d^\dagger$s diagonalizing both H_g and H_{g_0} are related to the cs and $c^\dagger$s by a Bogoliubov transformation with angles θ^g and θ^{g_0}, and given that a Bogoliubov transformation is just a rotation, there must be a rotation relating the ds and $d^\dagger$s of H_g and H_{g_0}. In fact in [65] it was shown that the ground state $|\psi_{g_0}\rangle$ of the Ising chain can

[3] Actually this is not the case, but the link between the two will not be discussed in detail

[4] Integrable field theories are basically quantum field theory in which some conservation laws allow one to prove that all the n-particles scattering amplitued factorize into 4-particles amplitudes which can be computed exactly for all the particles in the theory

be written in terms of $|\psi_g\rangle$ as a boundary state of the form (4.4), with a particular K_p computable from the Bogoliubov angles

$$|\psi_{g_0}\rangle = \mathcal{N} \exp\left(\sum_{p>0} K_p d^\dagger_{-p} d^\dagger_p\right) |\psi_g\rangle \qquad\qquad K_p = i \tan\left(\frac{\Delta\theta_p}{2}\right) \tag{4.5}$$

where $\Delta\theta_p = \theta^g_p - \theta^{g_0}_p$ is the difference between the Bogoliubov angles used to diagonalize the pre-quench and post-quench Hamiltonians[5]

$$e^{i\theta^g_p} = \frac{g - e^{ip}}{\sqrt{1 + g^2 - 2g\cos p}} \qquad \Rightarrow \qquad K_p = \frac{i(g - g_0)\sin p}{\frac{1}{4}\varepsilon_g(p)\varepsilon_{g_0}(p) + 1 + gg_0 - (g + g_0)\cos p} \tag{4.6}$$

and the d_ks annihilate the vacuum $|\psi_g\rangle$.
Once one has this state much straightforward algebra follows and this is considered in appendix B. The most important consequence of this form of the initial state is that its time evolution can be formally computed in a very simple closed form. Indeed using (B.7) with $\lambda_p = -i\varepsilon_g(p)t$ and noting that, due to the fact that this time all ps are included in the sum and that $\varepsilon_g(-p)d^\dagger_p d^\dagger_{-p} = \varepsilon_g(p)d^\dagger_{-p}d^\dagger_p$, a factor of 2 in the exponent arises, the time evolved initial state reads

$$|\psi_{g_0}(t)\rangle = e^{-iH_g t}|\psi_{g_0}\rangle = \mathcal{N} \exp\left(\sum_{p>0} e^{-2i\varepsilon_g(p)t} K_p d^\dagger_{-p} d^\dagger_p\right) |\psi_g\rangle \tag{4.7}$$

Moreover using (B.9) a physical interpretation of the function K_p is immediately at hand

$$\langle\psi_{g_0}|n^g_p|\psi_{g_0}\rangle = \langle\psi_{g_0}|\left(d^g_p\right)^\dagger d^g_p|\psi_{g_0}\rangle = \frac{|K_p|^2}{1 + |K_p|^2}$$

where the superscript g refers to the fact the fermion occupation number is related to the post-quench Hamiltonian. So $|K_p|^2$ is directly related to the density of quasiparticles per unit of momentum seen by the modes of H_g in the initial state. Moreover when $|K_p|^2 \ll 1$ it represents exactly the quasiparticles density since in that case $\langle n^g_p\rangle \simeq |K_p|^2$.
The idea that the post-quench initial state can be seen as filled of pairs of excitations of the new Hamiltonian H_g turns out to be very usefull not only for practical computational reasons, but also for heuristic arguments aimed to explain the behavior of time dependent quantities when the system has finite size, as will be discussed in section 4.3.2.
For later use let's take the scaling limit of the boundary state in (4.5). Upon restoring the lattice spacing a via defining the physical momentum

$$p_{\text{phys}} = \frac{p}{a} \in (-\infty, \infty)$$

and lifting immediately the subscript, the Bogoliubov angle reads

$$\theta^g_p = \arctan\left(\frac{\sin ap}{\cos ap - g}\right) \underset{a\to 0}{\sim} \arctan\left(\frac{ap}{1 - g}\right) = \arctan\left(\frac{vp}{\Delta}\right)$$

where $v = 2a$ is the relativistic velocity of Ising chain free fermions, while $\Delta = \varepsilon(0)$ is the mass gap, which must be kept fixed in the scaling limit. Note that J has not been restored, but, as discussed in section 3.4.1, is needed only to keep finite the velocity v.
For a quench starting from g_0 and ending at the critical point the boundary state then becomes

$$K_p = i \tan\left[\frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{vp}{\Delta_0}\right)\right] = i \tan\left(\frac{1}{2}\arctan\frac{\Delta_0}{vp}\right) \tag{4.8}$$

[5] In this chapter, differently from section 3.4, the factor of $\sqrt{i}$ will not be put in front of the Fourier transform of the c modes, with the difference that the Bogoliubov transformation will not properly be a rotation since the angle becomes imaginary

4.3 Results

In this section the numerical results will be presented and compared with the exact lattice formulas provided by [65] for the quantum Ising chain, which allows one to compute almost all the quantities before mentioned. Moreover the effect of the finite size of the chain on the time dependence of one point functions will be considered in detail, studying the structure of revivals of the initial state. Starting from the Ising chain exact formulas and using the perturbative results for general q obtained in section 3.3, some conjectures for the analytic expressions of the Loschmidt echo and the transverse magnetization one point function will be proposed and compared with numerical data. The quite fair agreement with the numerics supports the hypotesis that the quench dynamics in the perturbative regime of the q states Potts chain is, also at a formal level, very similar to the Ising chain's one. The order parameter one point function is the only one for which an exact expression for all times is not available either in the Ising chain and will be discussed in the last section. At the end of each section particular attention will be devoted to quenches ending to the quantum critical point, for which the more general CFT approach of [36] can be used to predict the behavior of observables just in terms of scaling dimensions of primary operators. As discussed in chapter 2 the quench dynamics is translated, after a Wick rotation, into the problem of calculating expectation values of primary operators in a conformal field theory defined on a slab. This BCFT then corresponds to the scaling limit of the quench dynamics of the chain and its geometry contains an undetermined parameter τ_0, called extrapolation time, which signals the lack of some information on the microscopic degrees of freedom of the system and mathematically quantifies how much the boundary conditions used in [36] to compute n-point functions are far from the conformally invariant ones. This parameter is expected to be of the order of the inverse mass gap of the Hamiltonian before the quench H_{g_0}, then it can be compared to the dependence of the mass gap computed perturbatively when the initial Hamiltonian is far from the critical point and to the critical behavior expected from conformal field theory when g_0 is near to 1.

4.3.1 Energy density after the quench

As discussed in section 4.2 the initial state of the quench problem is in general a superposition of excited states of the Hamiltonian H_g governing the evolution of the system in time. These excited states may have a very high energy w.r.t. the ground state of H_g and summing up all these contributions results in a quantity which is extensive in the volume of the system, thus leading to a finite energy density per unit length. In the case $q = 2$ the post-quench Hamiltonian of the Ising chain can be written as

$$H_g = \sum_p \varepsilon_g(p) d_p^\dagger d_p + E_0 \qquad\qquad \varepsilon_g(p) = 2\sqrt{1 + g^2 - 2g\cos p}$$

in computing (4.1) the constant E_0, which is exactly the second term in the numerator, cancels out and one is left with

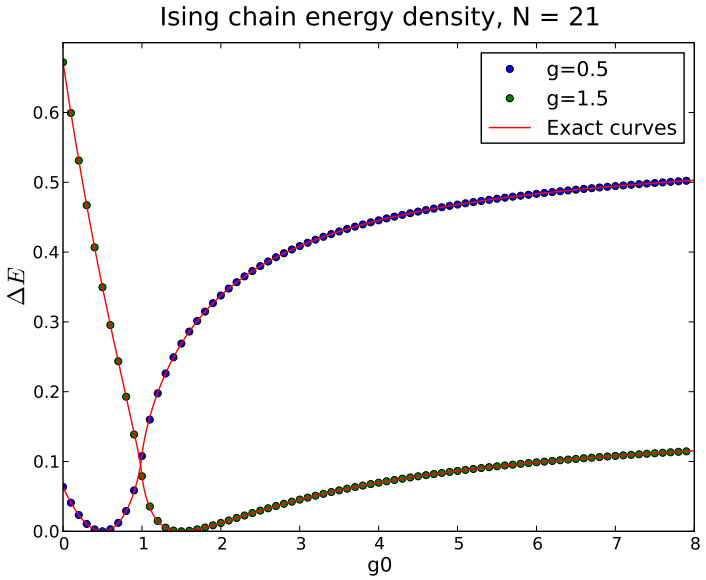
$$\Delta E_g(g_0) = \frac{\langle \psi_{g_0}|H|\psi_{g_0}\rangle - \langle \psi_g|H_g|\psi_g\rangle}{N} = \frac{1}{N}\sum_p 2\sqrt{1 + g^2 - 2g\cos p}\,\langle \psi_{g_0}|d_p^\dagger d_p|\psi_{g_0}\rangle \tag{4.9}$$

Note that the this quantity does not depend on time as it should, since the new Hamiltonian H_g is conserved after the quench. Formally this comes from the fact that the expectation value of the post-quench number operator $n_p = d_p^\dagger d_p$, given in (B.9), involves the absolute square of the kernel of the time dependent boundary state (4.7), so that the time dependent phase cancels. Indeed using the expression for the boundary state (4.5) and (B.9), the energy of the new Hamiltonian with respect

to its ground state is

$$\Delta E_g(g_0) = \frac{1}{N}\sum_p \varepsilon_g(p)\frac{|K_p|^2}{1+|K_p|^2} \underset{N\to\infty}{\sim} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}p\, \varepsilon_g(p)\frac{1}{2}\left[1-\frac{4(gg_0+1-(g+g_0)\cos p)}{\varepsilon_g(p)\varepsilon_{g_0}(p)}\right] \quad (4.10)$$

One can verify numerically that this expression is in fact correct for any value of g and g_0, as can be seen below where it is plotted for fixed $g = 0.5$ and $g = 1.5$ as a function of g_0 against numerical data for $N = 21$, so the corrections due to the finite size of the chain must be very small.



From (4.10) one can verify that the vacuum energy is always finite in the limit $g_0 \to \infty$, this is of course physically sensible since a quench with $g_0 = \infty$ means simply to start from a state which is the ground state of (minus) the transverse field. Even in the more general case of the q-states chain it is easy to get this ground state, since, as already discussed in section 3.3 to setup perturbation theory, it is factorized in ground states $|\psi\rangle_a$ of the one-site matrices

$$-M_a = -\sum_{\ell=1}^{q-1}\tau_a^\ell = -\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \vdots \\ \vdots & 1 & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \qquad -M_a|\psi_\infty\rangle_a = -M_a\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = -(q-1)|\psi_\infty\rangle_a \quad (4.11)$$

An interesting fact is that thanks to the duality symmetry possessed by the critical Hamiltonian, one has $\Delta E_{g=1}(g_0) = \Delta E_{g=1}(1/g_0)$, indeed for quenches to the critical point (4.10) simplifies to

$$\Delta E_{g=1}(g_0) = \frac{4}{\pi}\left[1-\int_0^1 \mathrm{d}x\,\sqrt{1-x^2}\Big/\sqrt{1-\left(\frac{2\sqrt{g_0}\,x}{1+g_0}\right)^2}\right] = \frac{4}{\pi}\left[1-\frac{1+g_0}{2\sqrt{g_0}}E\left(\frac{2\sqrt{g_0}}{1+g_0};\frac{1+g_0}{2\sqrt{g_0}}\right)\right] \quad (4.12)$$

where $E(a,b)$ is the incomplete elliptic integral of the second kind

$$E(a;b) = \int_0^b \mathrm{d}s\,\frac{\sqrt{1-a^2s^2}}{\sqrt{1-s^2}}$$

and the ratio $\sqrt{g_0}/(1+g_0)$ is invariant under $g_0 \to 1/g_0$. The value of the gap at the two extremal points is

$$\Delta E_{g=1}(0) = \Delta E_{g=1}(\infty) = \frac{4}{\pi}\left(1 - \int_0^1 \mathrm{d}x\, \sqrt{1-x^2}\right) = \frac{4}{\pi}\left(1 - \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{8}{\pi} - 2\right) \simeq \frac{1}{2}\, 0.546479 \tag{4.13}$$

where the factor of $1/2$ represents the Ising model central charge for reason which will become clear in a moment.
The first thing to note at this point is that the invariance above mentioned is numerically evident also for quenches in the 3-states and 4-states Potts chains, this seems natural since the self-duality at the critical point holds for any q. Moreover the BCFT prediction discussed in section 2.3.2 for the energy density is

$$\Delta E_{g=1}(g_0) = \frac{\pi c}{24(2\tau_0)^2} \tag{4.14}$$

where τ_0, as discussed at the beginning of this section, is expected to be proportional to the inverse mass gap $\Delta^{-1}(g_0)$ of the initial Hamiltonian H_{g_0}, which in the Ising case is

$$\Delta(g_0) = \varepsilon_{g_0}(0) = 2\,(1-g_0) \tag{4.15}$$

while in the 3-states and 4-states Potts chain should be

$$\Delta(g_0) \underset{g_0 \to 1}{\sim} A(1-g_0)^{\nu} \qquad\qquad \nu_3 = \frac{5}{6} \qquad \nu_4 = \frac{2}{3}$$

where ν_3 and ν_4 are the critical exponents discussed at the end of chapter 3 and A is a non trivial constant. In fact trying to match this expression with the exact one for the Ising case (4.12), one immediately realizes that this is not possible with the simple expression (4.15) for $\tau_0(g_0)$. It still can be argued that this should hold only when the scaling limit of both the final and initial Hamiltonian is taken and indeed in the continuum limit and when $g = 1$, using (4.8) and (4.10) becomes

$$\Delta E_{g=1}(g_0) = \frac{1}{\pi}\int_0^\infty vp\, \frac{K_p^2}{1+K_p^2} = \frac{\Delta_0^2}{v\pi}\int_0^\infty x\,\mathrm{d}x \sin^2\left(\frac{1}{2}\arctan\frac{1}{x}\right)$$

so that one gets $\Delta E_{g=1}(g_0) \sim \Delta_0^2$, but unfortunately the result is not finite since the remaining integral diverges. Then the extrapolation time cannot be determined[6] according to (4.15). This means that switching to the field theory without introducing any cutoff leads to a divergent energy density of quasiparticles after the quench.
Nonetheless the central charge seems to appear in front of the asymptotic values $\Delta E_{g=1}(0) = \Delta E_{g=1}(\infty)$ in both the 3 and 4-states Potts chains. First of all note that it is possible to isolate part of the finite-size corrections to expression (4.1) at $g = 1$. Indeed the second term in the numerator is nothing but the ground state energy of the critical Hamiltonian, which contains an extensive term representing the vacuum energy $H_{g=1}$ and a $O(1/N)$ correction proportional to the central charge times the relativistic velocity of the quasi particles low-energy spectrum

$$\langle\psi_g|H_g|\psi_g\rangle|_{g=1} = \varepsilon_{\mathrm{vac}} N - \frac{\pi c\, v}{6N}$$

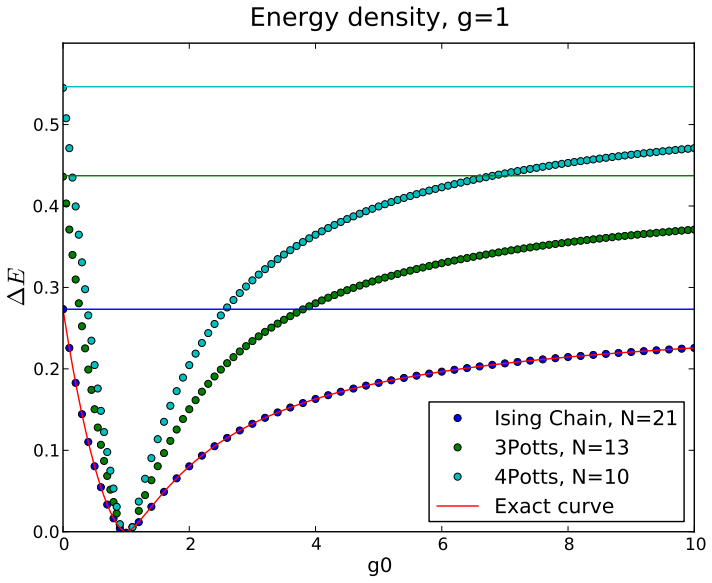
These corrections has been discussed in section 3.7 and v and $\varepsilon_{\mathrm{vac}}$ have been numerically estimated and compared with the exact results available for $q = 2, 3$. Note also that there are no other finite-size corrections to $\Delta E_{g=1}(\infty)$, since the first term in the numerator can be written down explictly given that the ground state $|\psi_\infty\rangle$ is the tensor product of the states in (4.11) and the first term in the

[6] Even if this divergence might be appropriately regularized

Hamiltonian (3.1) has zero expectation value on such a state because the σ_a matrices are traceless, so that one is left with

$$\Delta E_{g=1}(\infty) = \frac{1}{N}\left(\langle\psi_\infty|H_1|\psi_\infty\rangle - \langle\psi_1|H_1|\psi_1\rangle\right) =$$
$$= \frac{1}{N}\left(-\sum_{a=1}^{N} {}_a\langle\psi_\infty|M_a|\psi_\infty\rangle_a - \varepsilon_{\text{vac}} N + \frac{\pi c\, v}{6N}\right) = -(q-1+\varepsilon_{\text{vac}}) + \frac{\pi c\, v}{6N^2}$$

Subtracting off the $O(1/N^2)$ correction to numerical data for all g_0s one finds that the value of ΔE at the two extremal seems to be exactly the same as the Ising chain's one (4.13), but with the substitution of $1/2$ with the corresponding central charge. Below are plotted numerical data up to $g_0 = 10$ and it is easy to verify the powerlaw convergence for $g_0 \to \infty$ to the same value assumed in $g_0 = 0$.



The horizontal lines in the plot above are the values $c\,(8/\pi - 2)$, with $c = 0.5, 0.8, 1$, which are the central charges of the CFTs describing respectively the $q = 2, 3, 4$ cases, as discussed in chapter 3.
If the relations written up to this point were exact, one would have a nice expression for the vacuum energy density of the critical q-states Hamiltonian with $q = 2, 3, 4$, i.e.

$$\varepsilon_{\text{vac}} = -(q-1) - c\left(\frac{8}{\pi} - 2\right) \tag{4.16}$$

Unluckily this is not the case, since this expression is correct only for $q = 2$, as can be seen comparing the exact value (3.46) for ε_{vac} when $q = 3$ and the numerical value obtained for $q = 4$ in section 3.7. Interestingly the error is extremely small since (4.16) gives $\varepsilon_{\text{vac}} = -2,437, -3.546$, while the exact (or numerical) results are $\varepsilon_{\text{vac}} = -2,436, -3.545$ for $q = 3, 4$ respectively.

4.3.2 Loschmidt echo

In this section the probability that at time t the state of the system comes back to the initial one will be investigated. First in the case of finite systems, where the finiteness of the spectrum makes this return, or partial return, to happen infinitely many times during the evolution. Then the thermodynamic limit will be taken and this probability is computed in the Ising chain case and compared with the general q case.

Finite chains

Dynamical quantities in quantum quenches in finite systems are characterized mainly by three time scales, which will be denoted relaxation time T_{rel}, returning time T_{ret} and revival time T_{rev}. The relaxation time is simply how long the system takes to get (if it gets) a stationary state in which observables have stationary average, it is numerically evident that it is independent from the system size but it depends on the observable considered. The returning time is typical of the dynamics of every quantum system with a finite number of degrees of freedom, which implies (assuming discrete the energy spectrum) a finite number of energy eigenvalues. This in turn implies that every time dependent average of an observable on a state $|\psi\rangle$ is a quasi-periodic function, being a superposition of a finite number of periodic functions with possibly incommensurable frequencies

$$\langle\psi|O(t)|\psi\rangle = \sum_{n,m}^{L} e^{i(E_n - E_m)t}\, \overline{c_n} c_m \langle n|O|m\rangle$$

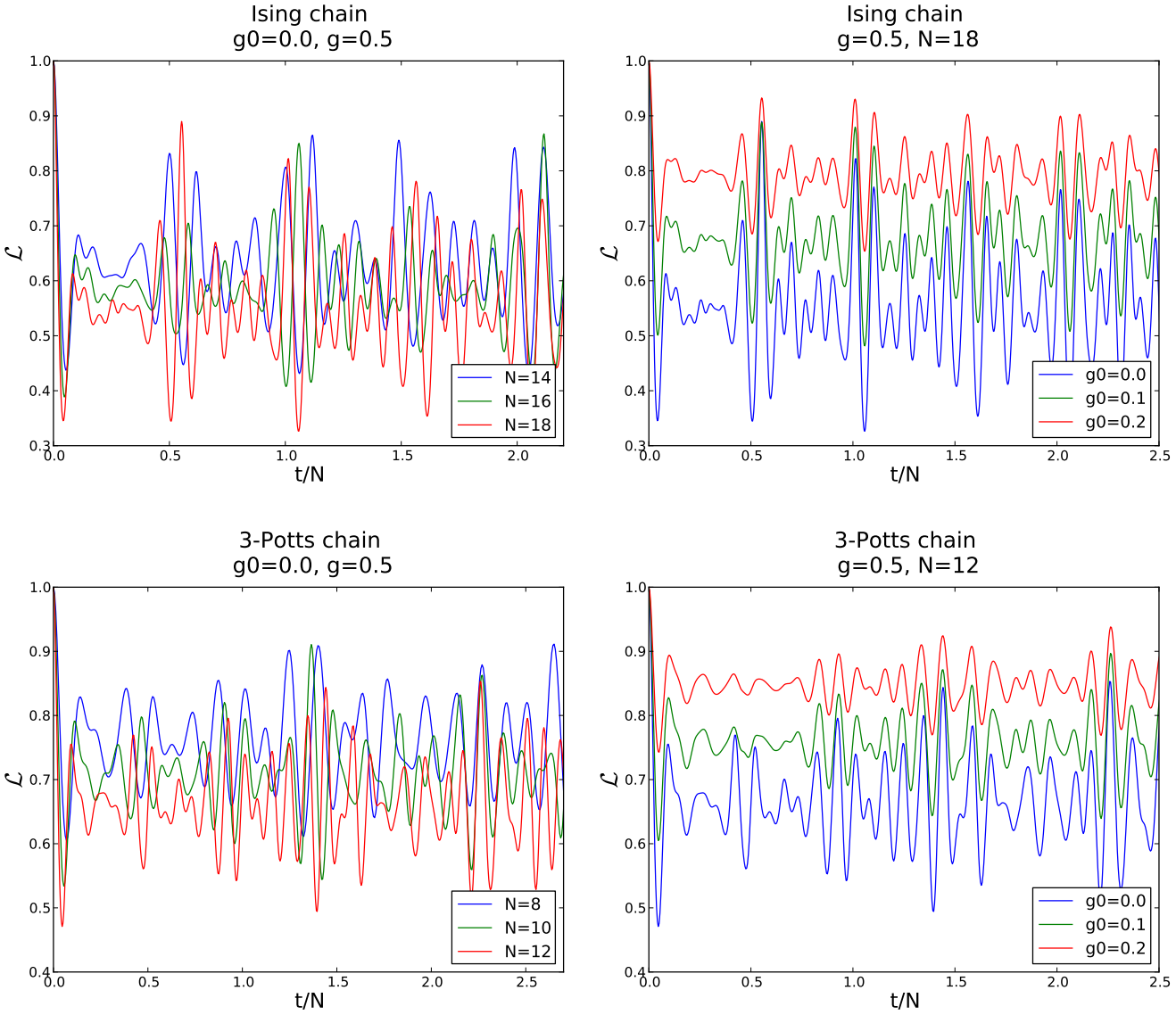
Here $|n\rangle$ are the Hamiltonian eigenstates and $c_n = \langle n|\psi\rangle$. The time T_{ret} up to which the value of the above average gets infinitesimally close to the initial one and the Loschmidt echo infinitesimally close to 1 is exponentially large with the system size, so that when the number of degrees of freedom goes to infinity the system will never reach its initial state. If one is interested in the dynamics of the thermodynamic limit, as in the case of quenches, this time scale is usually not a problem since even for modestly small systems simulated through exact diagonalization the inequality $T_{\text{rel}} \ll T_{\text{rec}}$ holds, so that one can in principle obtain at least the mean values reached in the large time limit by the observables of interest and check if the asymptotic state equilibrates or not and if the equilibrium state is thermal or not. Lastly T_{rev} is the time scale at which revivals occur, these are time intervals, much shorter than the separation between them, in which the Loschmidt echo and the averages of observables move away from the values to which they have relaxed. The problem with this time scale is that, more than being associated to the unwanted finite size of the system, it is of the order of system size. So what can happen with exact diagonalization is that the simulated system is too small to make $T_{\text{rev}} > T_{\text{rel}}$. In this case no stationary state's property of the infinite system can be observed numerically.

When the quasiparticles description of the quench dynamics discussed in section 4.2 applies one can get a rough estimate of the revival time from a very simple argument. Suppose that the quasiparticles have all the same velocities v and that at time $t = 0$ the quasiparticles in each pair start propagating in opposite directions. Then, because of periodic boundary conditions, they will meet again after a time $t = L/2v$, where L is the length of the system. If all the quasiparticles had exactly the same velocity, at that time a complete revival of the initial state would be observed, since all the pairs would be exactly in the same position as they were in the initial state. Of course this is not the case, since in general the dispersion relation is non linear, and one may expect a partial revival when the fastest pair meet again, yielding a revival time $t = L/2v_{\text{max}}$. This heuristic way of looking at revivals turns out to be in fact very accurate. Indeed the Loschmidt echo in the Ising chain has already been extensively studied by many authors [67, 68, 69] and the first revival time has been estimated in [69] in a more formal way exactly in terms of the maximum velocity of the quasiparticles excitations of the post-quench Hamiltonian

$$T_{\text{rev}} \simeq \frac{N}{2v_{\text{max}}} \qquad v_{\text{max}} = \operatorname*{Max}_k \left| \frac{\mathrm{d}\varepsilon(k)}{\mathrm{d}k} \right| \tag{4.17}$$

where N is the number of sites on the chain. This formula has been taken as a sign of universality in the quench process: T_{rev} is independent on the quench details and it can be simply exctracted from the post-quench quasiparticles dispersion relation. The first thing to do to check the validity

of (4.17) is to verify that $T_{\rm rev}$ is independent on the initial state. This should be true also for the 3 and 4 states Potts chains and this statement, at a qualitative level, does not require the knowledge of the quasiparticles dispersion relation. Below are shown the Loschmidt echos for quenches starting from $g_0 = 0.0$ to $g = 0.5$ for different values of N as a function of t/N on the left and for quenches starting from different values of g_0 to $g = 0.5$ for fixed $N = 18, 12$ for the Ising and 3-Potts chains respectively.

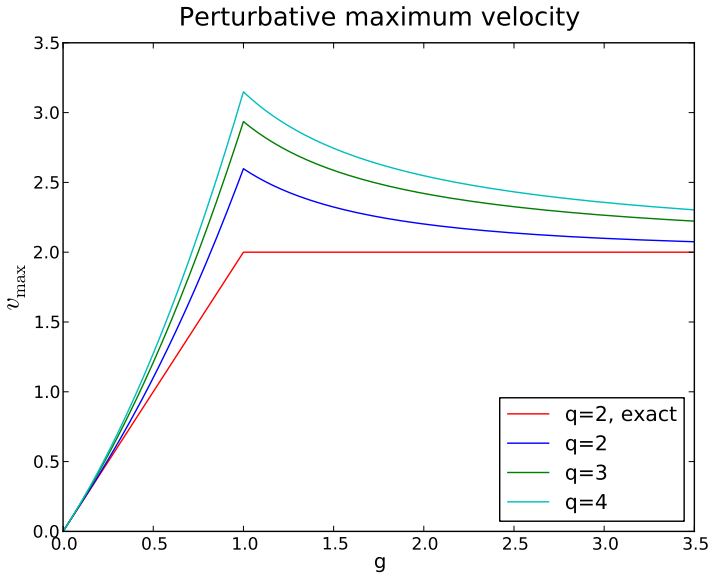


Even if revivals for the 3-Potts chain are less evident, in both cases it is clear from the plots on the left that the $T_{\rm rev}$ is proportional to N and from the plots on the right that it is independent on the initial coupling g_0. Moreover in both cases the first revival is at $t/N \simeq 0.5$ and indeed from the exact dispersion relation for the Ising chain (3.34) the maximum quasiparticles velocity is

$$v_{\rm max} = \max_k \left[\frac{2g \sin k}{\sqrt{1 + g^2 - 2g \cos k}} \right] = \begin{cases} 2g & \text{if} \quad g < 1 \\ 2 & \text{if} \quad g > 1 \end{cases}$$

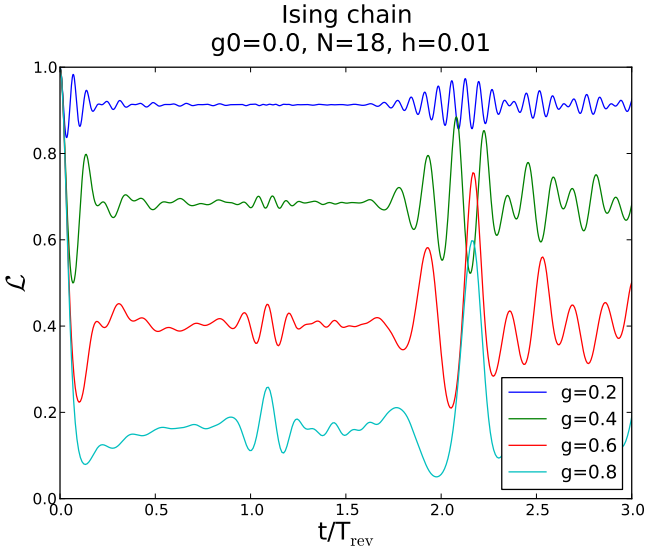
so that when $g = 0.5$ equation (4.17) predicts $T_{\rm rev} = 0.5$, as observed. In the 3-Potts case the same formula surely holds for $g \ll 1$ and $g \gg 1$, as apparent from the first order perturbative dispersion

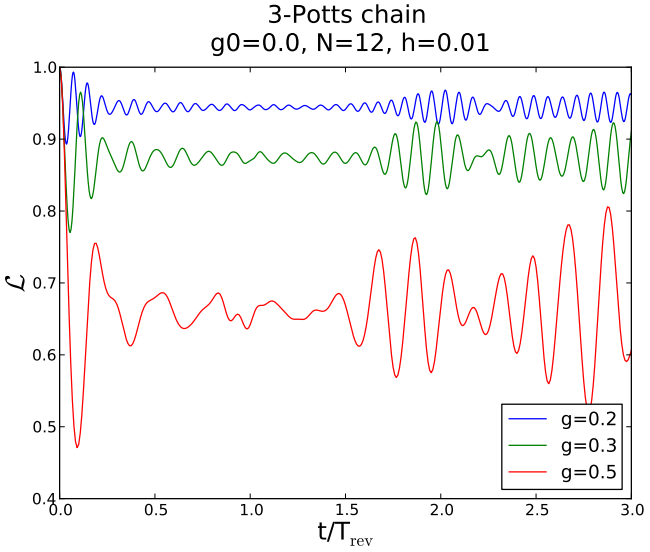
relation (3.26). This means that a peak in $\mathcal{L}$ should be observed at the same T_{rev} and this is true for the plot on the left. For the 3-Potts chain instead, on the right, it seems that the first revival is destroyed for some value of g_0, more on this issue later in this section. In fact $g = 0.5$ is not so smaller than 1 and indeed the first revival is not precisely at a multiple of T_{rev}, but it moves in the right position as g is decreased toward 0, as will be shown in the next plots. By the way, taking into account the second order correction to the dispersion relation (3.28), it is possible to correct the maximum quasiparticles velocity and to show that it starts depending on the final coupling g in both phases. The expression for v_{max} is not illuminating, the plot below shows its behavior varying g.



In any event (4.17) works quite well with the first order corrected v_{max} and it states that the first revival time is $T_{\text{rev}}/N = 0.5$ and at that time the Loschmidt echo detaches from the equilibrium value in both phases and in both chains.

It is interesting to observe that in the presence of a small longitudinal field in the Hamiltonian, which is necessary to make the order parameter nonzero at $t = 0$ as discussed in section 4.1.1, the first revival is almost completely destroyed when the initial coupling g_0 is deep in the ferromagnetic phase, while the revivals structure is unaffected in the paramagnetic phase, as is depicted below for both the chains when $g, g_0 < 1$.

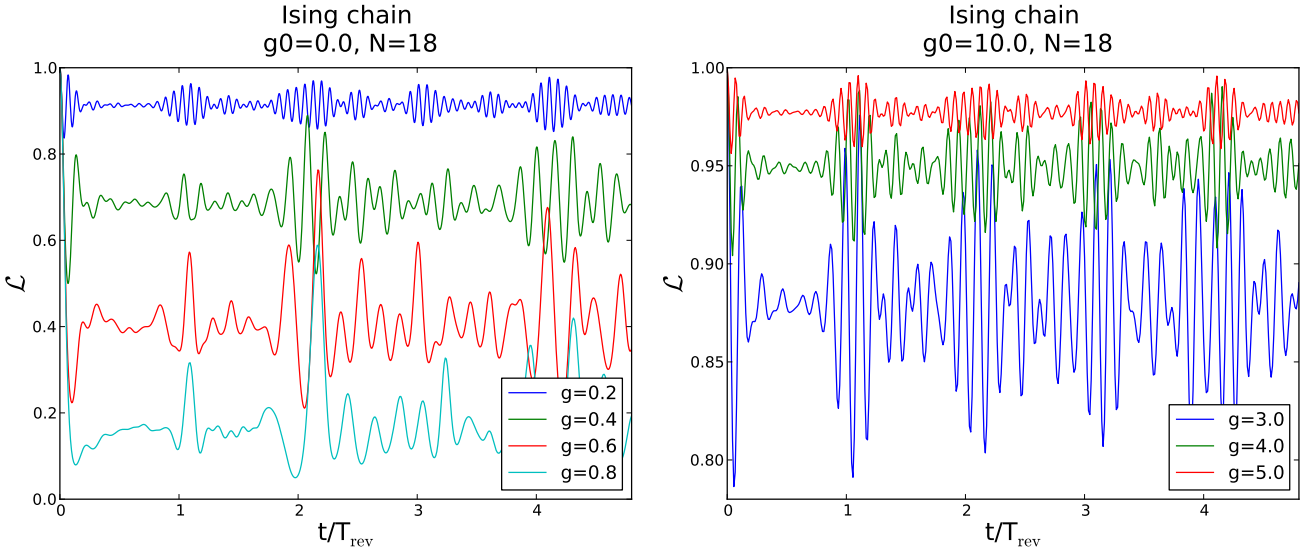




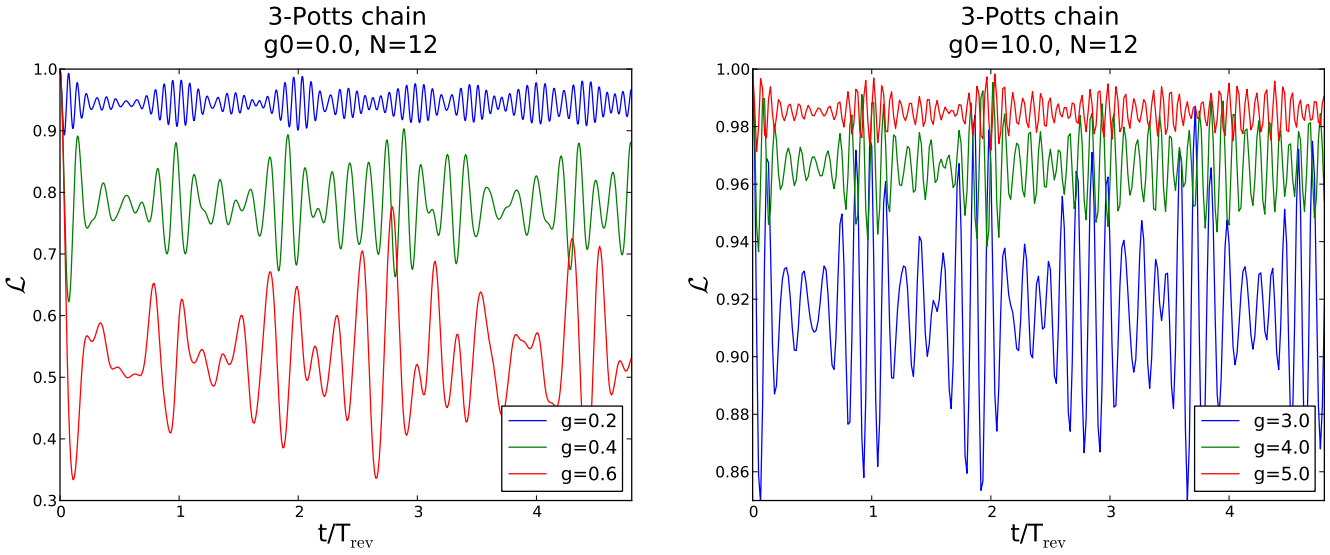
This may be linked to the fact that in the ferromagnetic phase the Hamiltonian does not have one-particle eigenstates when PBC are imposed, as discussed in section 3.3. In any event the difference between having or not a symmetry breaking term in the initial Hamiltonian is that when $h \neq 0$ the initial state is one of the "coloured" ground states $|0\rangle_\mu$[7] of H_{g_0}, with μ depending on which order parameter is inserted in H_{g_0}. If $h = 0$ instead, the initial state can be any linear combination of them. Since in the former case the first revival time is suppressed, this phenomenon must caused by the cross terms

$$ {}_\mu\langle 0|e^{-iH_g t}|0\rangle_\nu \qquad\qquad \mu \neq \nu \tag{4.18}$$

which are not present when a nonzero longitudinal field is inserted. In the 3-Potts case it is more likely that, although $h = 0$, the $|\psi_{g_0}\rangle$ chooses randomly to be one of the magnetized ground states $|0\rangle_\mu$, causing the first revival to disappear even with no symmetry breaking term in H_{g_0}.
This been said the second thing to check is that the first revival time goes as $1/g$ for quenches ending in the ferromagnetic phase and is constant for quenches ending in the paramagnetic phase. This is perfectly verified in the Ising chain. In the 3-Potts chain, in contrast, it holds as long as the final Hamiltonian is far enough from the critical point, as evidence that the exact dispersion relation for $q = 3$ gives a maximum excitations velocity which is not linear in g in the ferromagnetic phase and it is not constant in the paramagnetic regime. In fact, quantitatively, this can be explained in terms of the second order corrected $v_{\max}$ in the plot above, which makes T_{rev} smaller approaching $g = 1$. However it should be noticed that when $q = 2$ the exact dispersion relation shows that the first order result is better than the second order one in estimating $v_{\max}$, clearly the same is not true for $q = 3$. Below the plots starting from the same value of g_0 and ending at different values of g for fixed N, both in the ferromagnetic and in the paramagnetic phase.



[7] Here $|0\rangle_\mu$ refers to the ground state which is an eigenstate of σ_a with eigenvalue ω^μ, not necessarily for $g_0 = 0$ as was the case in section 3.3

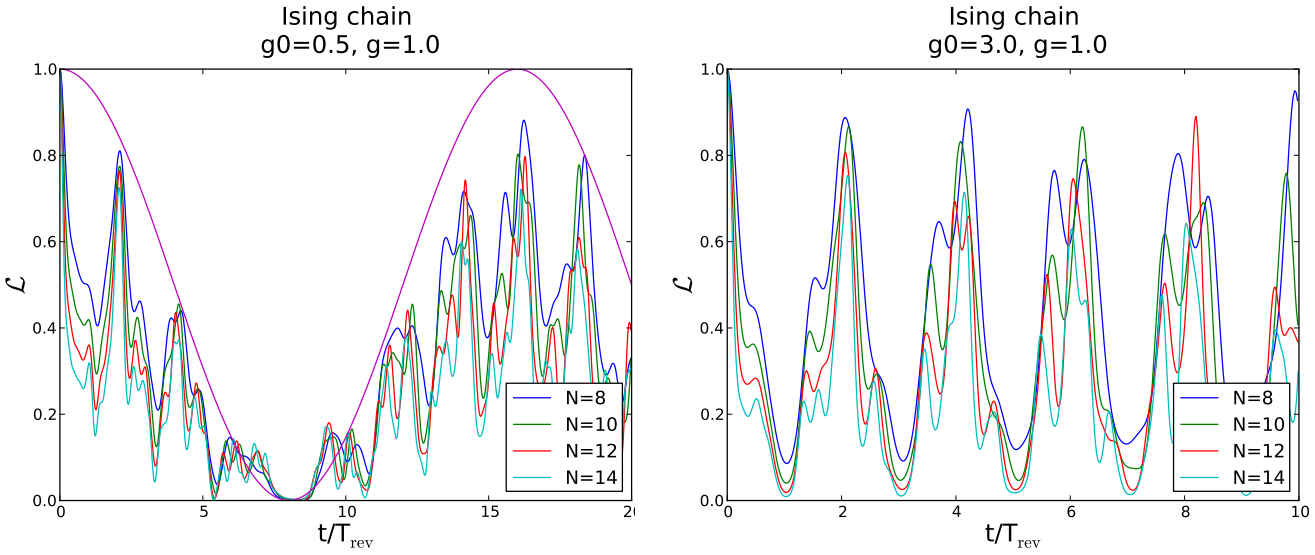


Note that revival amplitudes are proportional to the distance of the new coupling constant g from the old one g_0. Moreover it is quite evident looking at the time at which the fourth revival occurs that in the 3-Potts chain $\mathrm{T_{rev}}$ gets smaller than in the Ising chain as g approaches 1. This means that, as obtained quantitatively from the second order corrected dispersion relation, $v_{\max}$ is larger for $q = 3$ near the critical point. Finally it seems that the revivals structure is very similar in the two chains in the paramagnetic phase, being the revivals equally spaced in time. In the ferromagnetic phase instead there are hints of a different pattern.
It remains to discuss what happens at the revivals structure when the quench ends at the critical point $g = 1$. Pursuing the heuristic argument at the beginning of the section one may conclude that since at the critical point the quasiparticles become massless there is no dispersion in the relation between their energy and momentum $\varepsilon(p) = vp$. Then the revivals should become exact returns equally spaced in time. In fact the lattice adds a dispersion to the velocities, since actually $\varepsilon(p) = 2v \sin p/2$, and the revivals structure is not so simple. Nonetheless, as usual for quenches ending at the critical point, a CFT approach can be used and this issue has recently been addressed in [37], where explicit formulas were obtained for the Ising chain's revivals. There the quench problem of a finite system with PBC is mapped into a BCFT defined on a anulus and the Loschmidt echo of the chain corresponds to the (analytically continued) partition function of the BCFT. Revivals are understood as discrete points in time (multiples of $t = N/2v$, i.e. T_{rev} at the critical point) which can be reached applying to the anulus partition function the T modular transformation. In this case v is the relativistic velocity of the massless excitation governing the Hamiltonian at the critical point, which as already been discussed in section 3.7. The BCFT prediction for $\mathcal{L}$ in the Ising chain is the following:

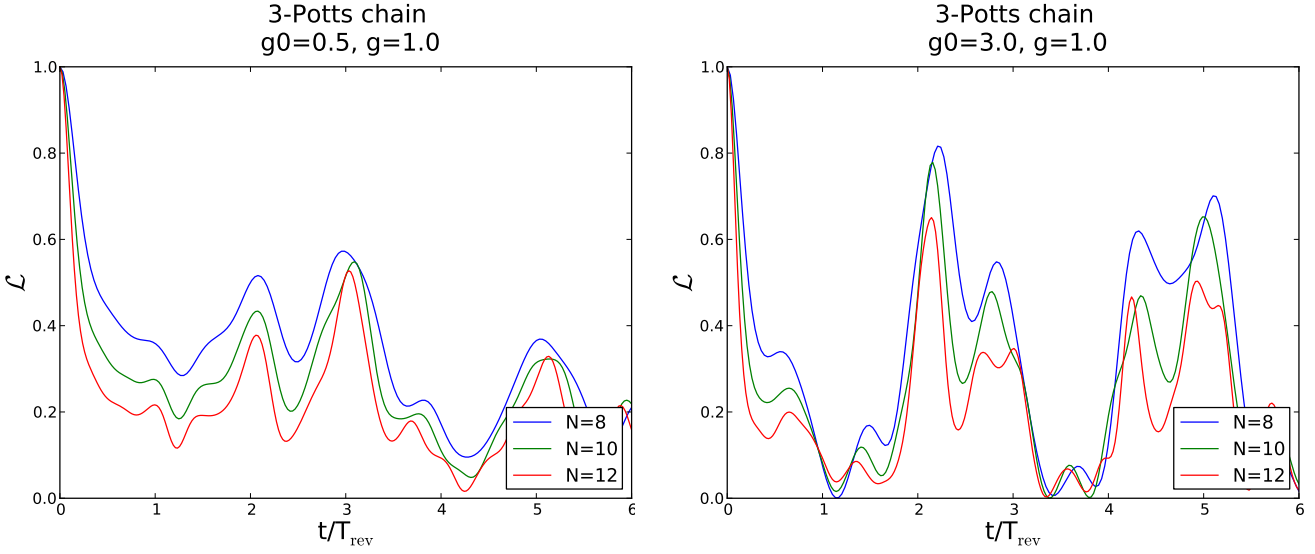
$$\mathcal{L}_{g_0<1}\left(nT_{\mathrm{rev}}\right) = \begin{cases} \dfrac{1}{4} & \text{if} \quad n \text{ odd} \\ \cos^2\left(\dfrac{\pi n}{16}\right) & \text{if} \quad n \text{ even} \end{cases} \qquad \mathcal{L}_{g_0>1}\left(nT_{\mathrm{rev}}\right) = \begin{cases} 0 & \text{if} \quad n \text{ odd} \\ 1 & \text{if} \quad n \text{ even} \end{cases}$$

Although this prediction does not account for the lattice and is only valid in the scaling limit of the finite system[8], it seems that it correctly captures the revivals structure even for N up to 16.

[8] So that to make it working the best one should look at a very long chain for very long time



An attempt of the same analysis as in [37] has not produced good results to be compared with the lattice ones. Below is shown the short time structure of the $\mathcal{L}$ for $q = 3$ when $g = 1$.



Note that the revival time $T_{\rm rev}$ is in this case obtained using the exact relativistic velocity for $q = 3$, given in (3.45) and also numerically obtained in section 3.7. The agreement with the quasiparticles picture is very good.
To conclude this section it is worth noting that very similar results as the above can be obtained for $q = 4$, although in this case the revivals structure is much less evident being the chain shorter. This fact makes real revivals much less distinguishable from the "noise".

Infinite chains

Let's now turn to the evaluation of the Loschmidt echo in the thermodynamic limit for the Ising chain, which once again can be computed exactly for all time together with its equilibration value (which is nothing but its limit for $t \to \infty$), thanks to the known exact expression of the energy spectrum (3.34) and of the boundary state (4.5). That fact that the limit for $t \to \infty$ exists once N is sent to infinity is non trivial at all and it shows that an infinite-size system may equilibrate to a steady state after a quench.

Using (B.8) together with the expression for the evolved initial state (4.7), the Loschmidt echo (4.2) reads

$$\mathcal{L}(t) = \left|\exp\left[\sum_{p>0}\log\left(\frac{1+e^{-2i\varepsilon_g(p)t}|K_p|^2}{1+|K_p|^2}\right)\right]\right|^2 \underset{N\to\infty}{\sim} \left|\exp\left[\frac{N}{2\pi}\int_0^\pi \mathrm{d}p\,\log\left(\frac{1+e^{-2i\varepsilon_g(p)t}|K_p|^2}{1+|K_p|^2}\right)\right]\right|^2 =$$

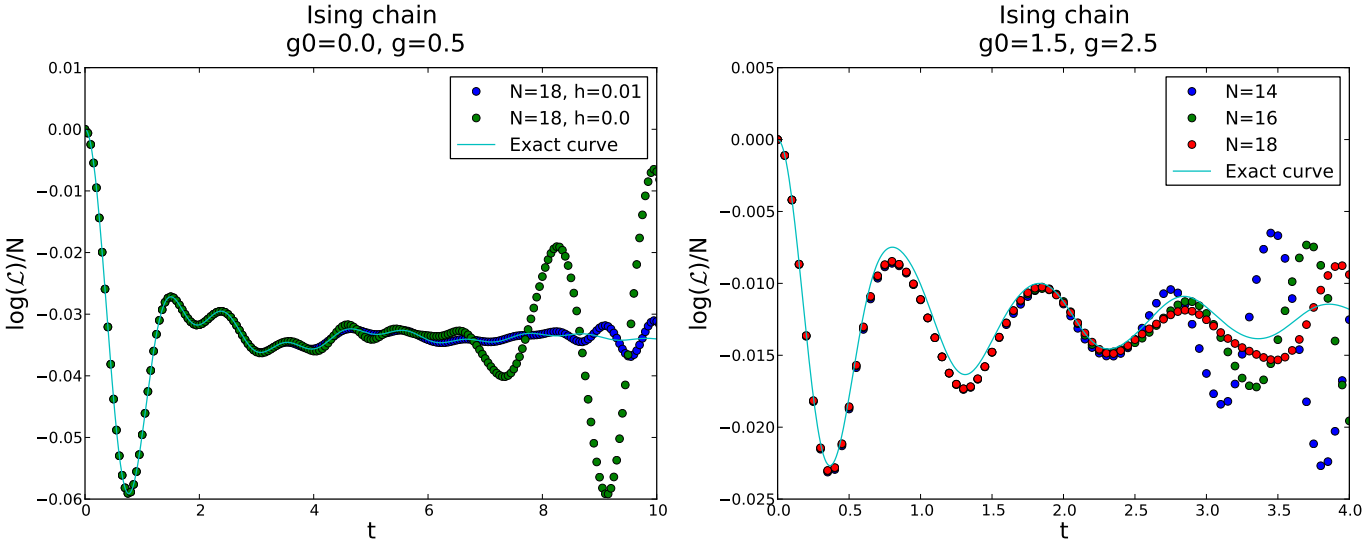
$$= \exp\left[\frac{N}{2\pi}\int_0^\pi \log\left|\frac{1+e^{-2i\varepsilon_g(p)t}|K_p|^2}{1+|K_p|^2}\right|^2\right] = \exp\left[-Nf(g_0,g,t)\right]$$

So $\mathcal{L}$ is exponentially suppressed with the system size N and it converges to 0 in the limit $N\to\infty$ for all time at which $f\neq 0$. The latter quantity is called logarithmic Lochmidt echo (LLE) per site. Inserting the expression for the boundary state (4.5) gives

$$f(g_0,g,t) := -\frac{\log\mathcal{L}(t)}{N} = -\frac{1}{2\pi}\int_0^\pi \mathrm{d}p\,\log\left(1-4\sin^2\left(\varepsilon_g(p)t\right)\frac{|K_p|^2}{(1+|K_p|^2)^2}\right) =$$

$$= -\frac{1}{2\pi}\int_0^\pi \mathrm{d}p\,\log\left(1-\sin^2\left(\varepsilon_g(p)t\right)\frac{(g-g_0)^2\sin^2 p}{(1+g_0^2-2g_0\cos p)(1+g^2-2g\cos p)}\right) \tag{4.19}$$

Below are plotted numerical data for different values of N for quenches in both phases and the expression above works also for quenches crossing the critical point.



The time at which numerical data start to detach from the exact curve is exactly T_{rev} for the plot on the right, as one can verify using (4.17) with $g=2.5$. In the plot on the left instead the different first revival times are compared depending on the non zero initial longitudinal field. Independently of how h is large (as long as it stays small) a non zero h in the initial state doubles T_{rev}, making the simulated dynamics the infinite N one for a doubled time window. Note that this holds also when the quench ends in the paramagnetic phase, given that $g_0<1$, while the same is not true for quenches with $g,g_0>1$. Since the same features are numerically evident also for general q, hereafter a nonzero h will be put in any initial state lieing in the ferromagnetic phase, in order to observe a longer dynamics which resembles the one of an infinite chain.
The limit for $t\to\infty$ of $f(g_0,g,t)$ can be obtained expanding the logarithm and using the Riemann-Lebesgue lemma on all the terms one obtains from the binomial expansion of $\sin^{2k}\left(\varepsilon_g(p)t\right)$, for each

k only one survives:

$$\lim_{t\to\infty} f(g_0,g,t) =: -\frac{1}{2\pi}\int_0^\pi \mathrm{d}p \log\left(1-\alpha(p)\sin^2(\varepsilon_g(p)t)\right) \underset{t\to\infty}{\sim}$$
$$\sim \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{4^k k}\binom{2k}{k}\int_0^\pi \mathrm{d}p(\alpha(p))^k = -\frac{1}{\pi}\int_0^\pi \mathrm{d}p \log\left(\frac{1+\sqrt{1-\alpha(p)}}{2}\right) =$$
$$= -\frac{1}{\pi}\int_0^\pi \mathrm{d}p \log\left[\frac{1}{2}\left(1+\frac{1+gg_0-(g+g_0)\cos p}{\sqrt{1+g_0^2-2g_0\cos p}\sqrt{1+g^2-2g\cos p}}\right)\right] \tag{4.20}$$

Where the last step follows from the expansion

$$\sum_{k=1}^{\infty}\frac{x^k}{4^k k}\binom{2k}{k} = -2\log\left(\frac{1+\sqrt{1-x}}{2}\right)$$

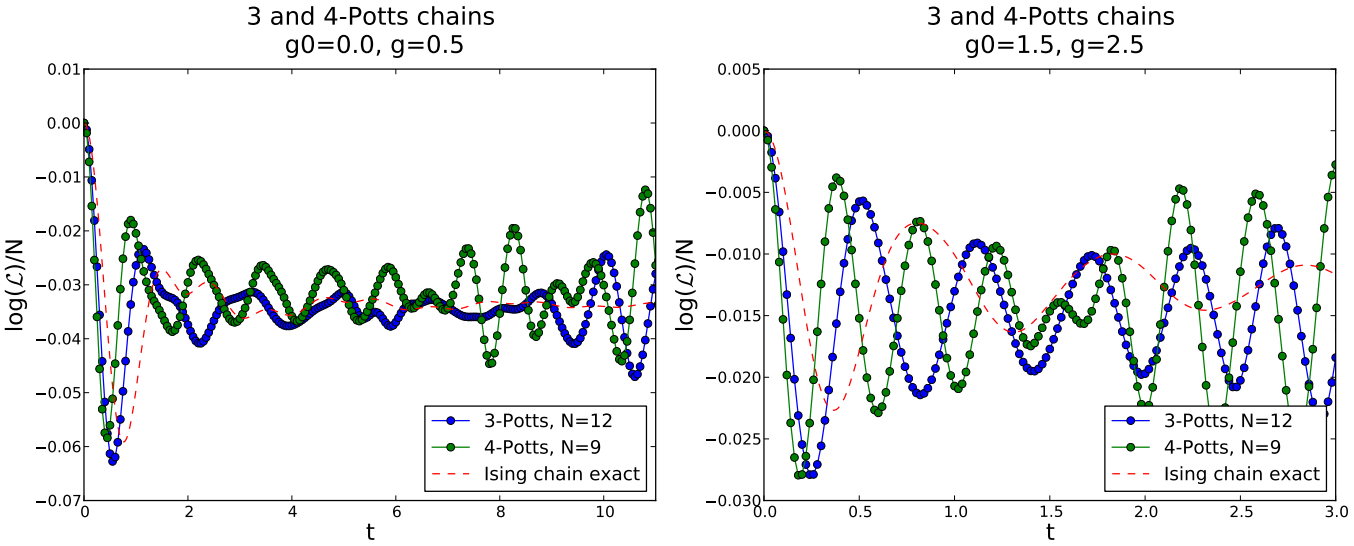
As already stressed, the existence of this limit is in general a non trivial statement, even if N goes to infinity. In fact, as will be shortly observed, there are some values of g for which the LLE does not converge to an equilibrium value. This then proves that in the thermodynamic limit equilibration is indeed possible.
For quenches ending at the critical point the LLE, as ΔE, is invariant under $g_0 \to 1/g_0$, it is non analytic for $g_0 \to 1$ and its asymptotic expansion for $g_0 \to 0$ is given by

$$\lim_{t\to\infty} f(g_0,t) \sim \left(\log 4 - \frac{4G}{\pi}\right) - \left(\frac{4}{\pi}-1\right) g_0$$

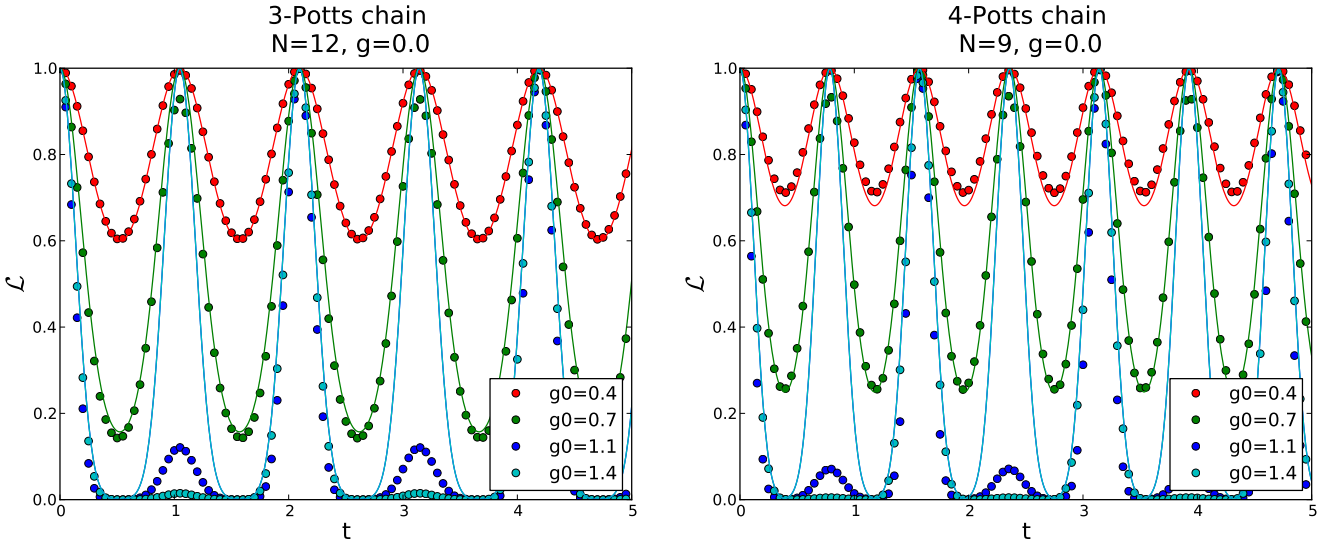
where G is Catalan's constant. These results were already present in [67]. Note that (4.20) holds as long as the final coupling constant g is different from 0. Indeed if $g = 0$ the final Hamiltonian is classical, T_{rev} goes to infinity and, as already noticed in [65], the dynamics is anomalous. This means that in general expectation values are constant or periodic and in the latter situation the limit for $t \to \infty$ does not exist as is the case for (4.19). The integral with $g = 0$ can be worked out and the result is

$$f(g_0,0,t) = \begin{cases} \log\left(\dfrac{1+\sqrt{1-g_0^2\sin^2 2t}}{2}\right) & \text{if} \quad g_0 < 1 \\ \log\left(\dfrac{1+|\cos 2t|}{2}\right) & \text{if} \quad g_0 > 1 \end{cases} \tag{4.21}$$

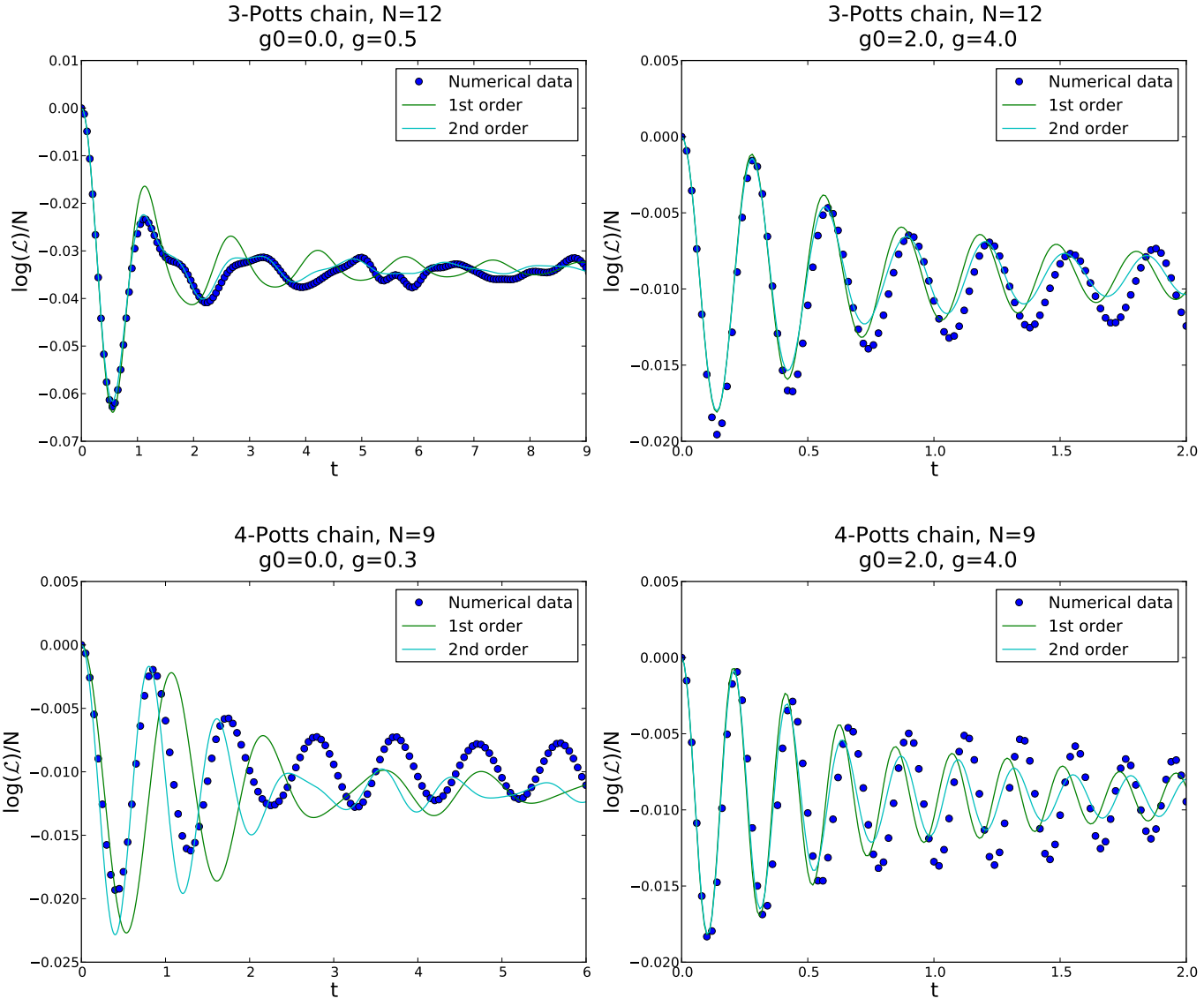
So $f(g_0,0,t)$ is independent of g_0 if the quench starts in the paramagnetic phase and this is of course perfectly verified numerically. Moreover $\mathcal{L} = 1$ at $t_k = (k+1)\pi/2$, i.e. there is a periodic complete revival of the initial state.
For the 3 and 4-states Potts chains the structure of $\mathcal{L}$ is of course the same: it is exponentially suppressed with the system size, the LLE per site seems to converge to a universal N-independent function up to the revival time $T_{\mathrm{rev}} \sim N$ (with the same agreement with (4.17) if one uses (3.26) in calculating $v_{\max}$ staying far away from the critical point) and so on. Moreover the value of the limit of the LLE seems to be independent of q. Below are shown the LLE graphs of the same quenches plotted above for the Ising chain for the 3 and 4 states Potts chains with $N = 12, 9$ respectively.



Note that the revival times are different in the two case because of the different legths of the two chains: the blue curve is the "universal" one up to $t \lesssim 2 \cdot 12/4 \cdot 0.5 = 12$ for the plot on the left and $t \lesssim 12/4 = 3$ for the plot on the right, while the green one stops to be the infinite N curve at $t \lesssim 2 \cdot 4.5 = 9$ on the left and $t \lesssim 2.25$ on the right. The factor of 2 for quenches in the ferromagnetic phase is there thanks to a non zero h in the initial state. The fact that T_{rev} is actually an upper bound is due to the non exact $v_{\max}$, which for $g = 0.5$ is most likely greater than 1. The shape of the LLE seems very similar to the Ising chain's one, but there is a glimpse of a different (dumped) periodicity for short times. This different periodicity becomes evident when quenches ending at $g = 0$ are considered: as before the LLE becomes exactly periodic, but with frequencies which are $3/2$ and $4/2$ the Ising chain's one. Although this similiraty there is also an important difference which helps in shedding light on the analytic expression of the LLE, namely the latter is no more independent from g_0 for quenches ending in the paramagnetic phase. The different periodicity is easly explained if one assumes that $f(g_0, g, t)$ is still a function of $\sin(\varepsilon_g(p)t)$, so that when $g = 0$, using (3.26) which is now exact, one gets a function of $\sin(qt)$. It turns out numerically that (4.21) works perfectly for quenches with $g_0 \gtrsim 0.5$, while it doesn't work when g_0 is greater than this value, the reason being that the odd peaks of the Loschmidt echo have an amplitude which goes rapidly to zero as g_0 is increased. This is depicted in the following plots.



The solid curves in the plots represent (4.21) with qt in place of $2t$; note that the amplitudes of odd revivals goes to zero more rapidly in the 4-Potts chain than in the 3-Potts one.
Taking more seriously the fact that it is enough to replace the quasiparticles dispersion relation in (4.19) to get the right behavior of the LLE, one gets results which are surprisingly in good agreement with numerical data, at least for $q = 3$, and they also improve correcting the q-Potts dispersion relation from first to second order. Below are shown some plots for quenches deep in the ferromagnetic and paramagnetic phase, both for $q = 3$ and $q = 4$.



It is worth noting that this naive ansatz for the expression of $\mathcal{L}(t)$ does not work when the quench crosses the transition point.
The relatively good agreement of this analytic expression with the real one signals that also for general q the quench dynamics might be ruled by the propagation of quasiparticles in a way possibly very similar to the Ising chain case. Of course it is hard to imagine how the initial state can be written in terms of pairs of quasiparticles, provided that these quasiparticles exist, for all the value of the coupling g. It is enough to think at the exact solution of the $q = 2$ case which, despite its simplicity, exhibits a highly non trivial change of basis which maps the coordinate Hilbert space to the Fock space of multiparticles states, to realize that in the case of $q > 2$ it is quite unlikely that the

initial state has always the form (4.5). Nevertheless this agreement in the perturbative regime may indicate that, far away from the critical point, the q-Potts quasiparticles have very similar properties to the Jordan-Wigner fermions of the Ising chain and that the initial state might be in fact written as a superposition of pairs with opposite momenta. The detailed shape of the quasiparticles density $|K_p|^2$ is not important as long as the latter is small, as is the case when perturbation theory applies. Note finally that, given that the quasiparticles picture is correct also for the q-Potts chain and that the time dependence of observables is an oscillatory function of the form

$$\langle O(t)\rangle = \int \mathrm{d}p f\left(\varepsilon(p)t\right)$$

the knowledge of the perturbative dispersion relation is expected to give good results only for short times, since the perturbative error gets larger as time is increased. However, if adding higher order corrections to $\varepsilon(p)$ improved the agreement with numerical data this would be an indication of the fact that the limit value of the LLE is the same for all q, since it does not depend on the frequency of the oscillations of the LLE.

4.3.3 Transverse field

The transverse field is the operator in the Hamiltonian which leads to disorder as g is increased, making the ground state multicoloured. In the general q case its expectation value at $t = 0$ varies monotonically between 0 when $g_0 = 0$ and $q-1$ when $g_0 = \infty$, with this limit values readly obtained from the perturbative ground states considered in section 3.3. In the Ising chain it has been branded as a non ergodic operator since its equilibrium value after a quench strongly depends on the initial state. The same feature is, not surprisingly, observed also in the general q chain.
In the Ising chain the time dependent average of the transverse field σ^x can also be evaluated exactly, in the thermodynamic limit, without too much effort. Indeed inverting (3.29), Fourier transforming and using the Bogoliubov rotation (4.6) which links the c-modes to the d-modes which make the Hamiltonian diagonal, the average transverse field per site is

$$\begin{aligned}\frac{1}{N}\sum_a^N \sigma_a^x &= \frac{1}{N}\sum_a^N \left(1-2c_a^\dagger c_a\right) = \frac{1}{N}\sum_p^N \left(1-2c_p^\dagger c_p\right) = \\ &= \frac{1}{N}\sum_p \left[1-2\left(\cos\frac{\theta_p^g}{2}\,d_p^\dagger - i\sin\frac{\theta_p^g}{2}\,d_{-p}\right)\left(\cos\frac{\theta_p^g}{2}\,d_p + i\sin\frac{\theta_p^g}{2}\,d_{-p}^\dagger\right)\right]\end{aligned}$$

Note that the c and d modes are implictly the ones of the post-quench Hamiltonian, that's why the Bogoliubov angle refers to the coupling g. This is done because they are going to be sandwiched between the boundary state (4.5) and their expectation value on that state is easy to compute via the formulas obtained in appendix B. Using the expressions for the evolved boundary state (4.7) to take the expectation value of the transverse field, then using (B.9), (B.10) and taking the thermodynamic limit in the end one has

$$\begin{aligned}\langle\frac{1}{N}\sum_a^N \sigma_a^x(t)\rangle &= \frac{1}{N}\sum_p^N \left[\cos\theta_p^g - 2\left(\cos^2\frac{\theta_p^g}{2}\langle n_p\rangle - \sin^2\frac{\theta_p^g}{2}\langle n_{-p}\rangle + \frac{i}{2}\sin\theta_p^g\langle(d_p^\dagger d_{-p}^\dagger - d_{-p}d_p)(t)\rangle\right)\right] = \\ &= \frac{1}{N}\sum_p^N \left[\cos\theta_p^g\left(1-\frac{2|K_p|^2}{1+|K_p|^2}\right) + 2\sin\theta_p^g\,\frac{\mathrm{Im}\left(K_p\,e^{-2i\varepsilon_g(p)t}\right)}{1+|K_p|^2}\right] \underset{N\to\infty}{\sim} \\ &\sim \frac{4}{\pi}\int_0^{2\pi}\mathrm{d}p\,\frac{(gg_0+1-(g+g_0)\cos p)\,(g-\cos p)-(g-g_0)\sin^2 p\cos\left(2\varepsilon_g(p)t\right)}{\left(\varepsilon_g(p)\right)^2\varepsilon_{g_0}(p)}\end{aligned} \tag{4.22}$$

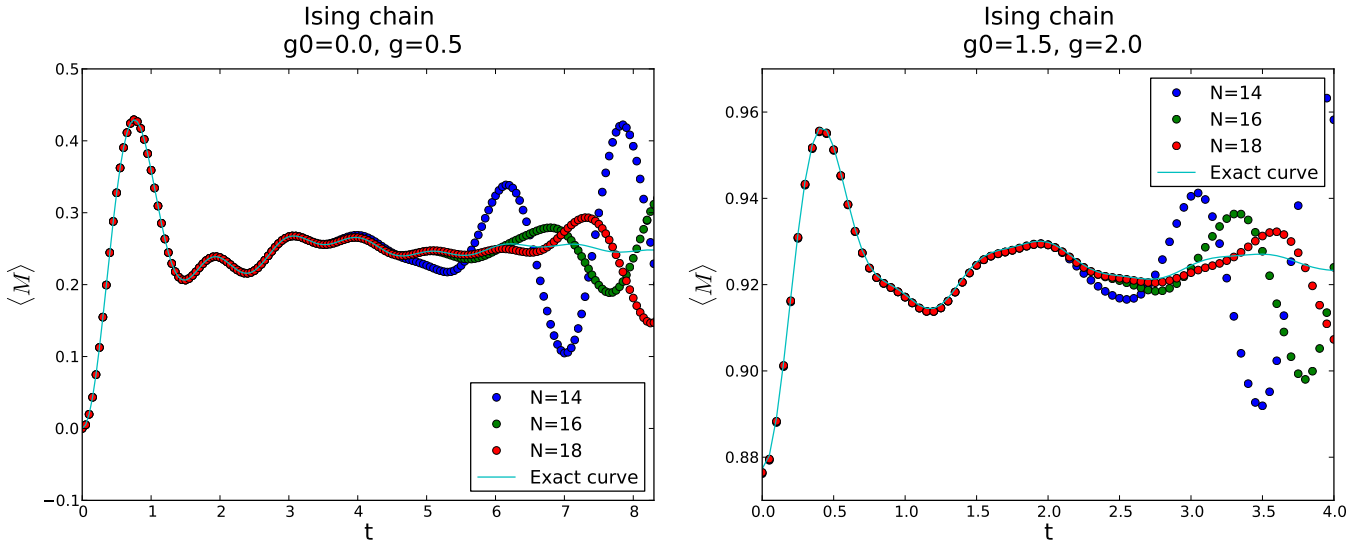
where in the first line $n_p = d_p^\dagger d_p$ and its average is time-independent since it involves only the absolute square of the boundary state's kernel, so that the time-dependent phase disappears. The last step follows from the substitution of the explicit expressions for K_p (4.5) and for the bogoliubov angle (4.6). Note that thanks to translation invariance both of H_g and of the initial state, ensured by PBC also when N is finite, one has

$$\langle M_a(t)\rangle = \langle \sigma_a^x(t)\rangle = \frac{1}{N}\sum_{a=1}^{N}\langle \sigma_a^x(t)\rangle =: \langle M(t)\rangle$$

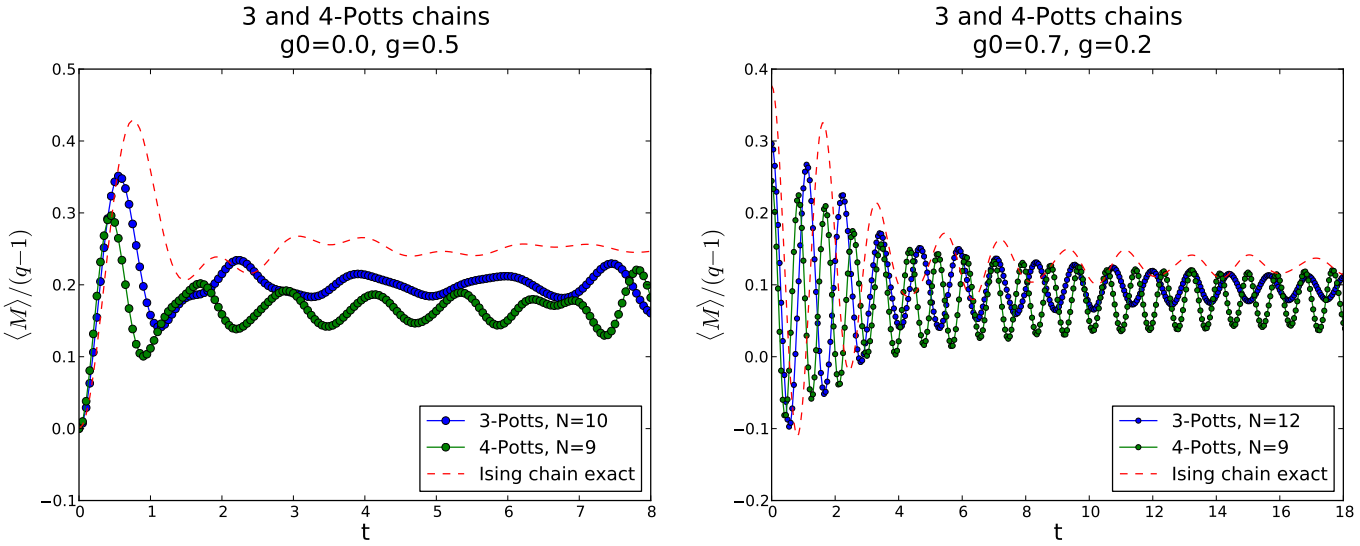
and this of course holds for any one point functions. Observe that although the initial value of M is always positive since it is positive its expectation value on the ground state $|\psi_{g_0}\rangle$ for any g_0, the same is not true for its dynamical expectation value, which might change sign as is evident from (4.22). This time is easier to obtain the equilibrium expectation value than in the case of the LLE, since from the Riemann-Lebesgue lemma simply follows that the time-dependent term cancels out in the infinite time limit, so that

$$\lim_{t\to\infty}\langle M(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}p\, \frac{(gg_0 + 1 - (g+g_0)\cos p)\,(g-\cos p)}{(1+g^2-2g\cos p)\sqrt{1+g_0^2-2g_0\cos p}}$$

Note that from the value of the limit and the one at $t = 0$ follows that $\Delta M = \langle M(\infty)\rangle - \langle M(0)\rangle = A(g - g_0)$, with A a positive constant depending both on g and g_0. So the asymptotic value of $\langle M\rangle$ increases if the coupling is increased during the quench and viceversa. Once again the same feature is numerically observed in the 3 and 4 states Potts chains.
To show that formula (4.22) is in fact correct below are reported the usual plots for quenches in the two phases. By the way, as for the LLE, it holds for general g and g_0.



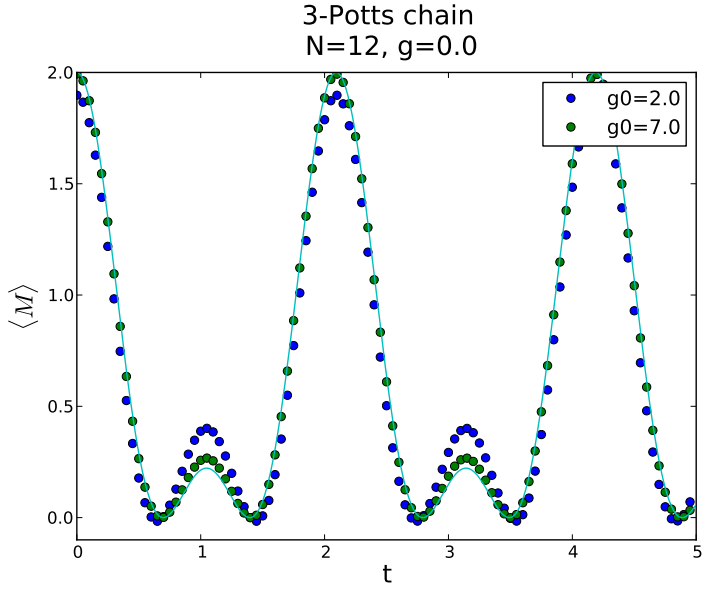
Since for general q the value of the transverse field varies between 0 and $q-1$, in order to compare the dynamics of M for different values of q it is useful to normalize them to 1. The results are the following: if $g_0 < 1$ it is easy to see from the numerics the asymptotic value of the rescaled one point function and it seems to decrease with q, moreover when the coupling is decreased $\langle M(t)\rangle$ exhibits evident oscillations while converging to its limit value; if instead $g_0 > 1$ it is difficult to grasp its asymptotic value and to compare it for different q. As $g \to \infty$ the dynamics for different q converge to the same one and eventually the transverse field becomes stationary, since it commutes with $H_{g=\infty}$. The situation for quenches in the ferromagnetic phase, which are the less trivial ones, is represented in the following plots.



When the final coupling is $g = 0$, as noted in the previous section for the LLE, the dynamics is not converging to equilibration and, as can be also noticed from (4.22) for the Ising chain, there are persistent oscillations in time. In the case of the Loschmidt echo a periodicity of $1/q$ was observed and this was given by the different dispersion relation. Moreover half of the peaks present in $\mathcal{L}$ were destroyed as the initial g_0 was increased. For the transverse field the situation is very similar, with the difference that the odd peaks are not completely destroyed, but they converge to a function which this time can be computed by brute force setting $g_0 = \infty$ and evolving with $H_{g=0}$ the factorized ground state (4.11). The result for $q = 3$ is

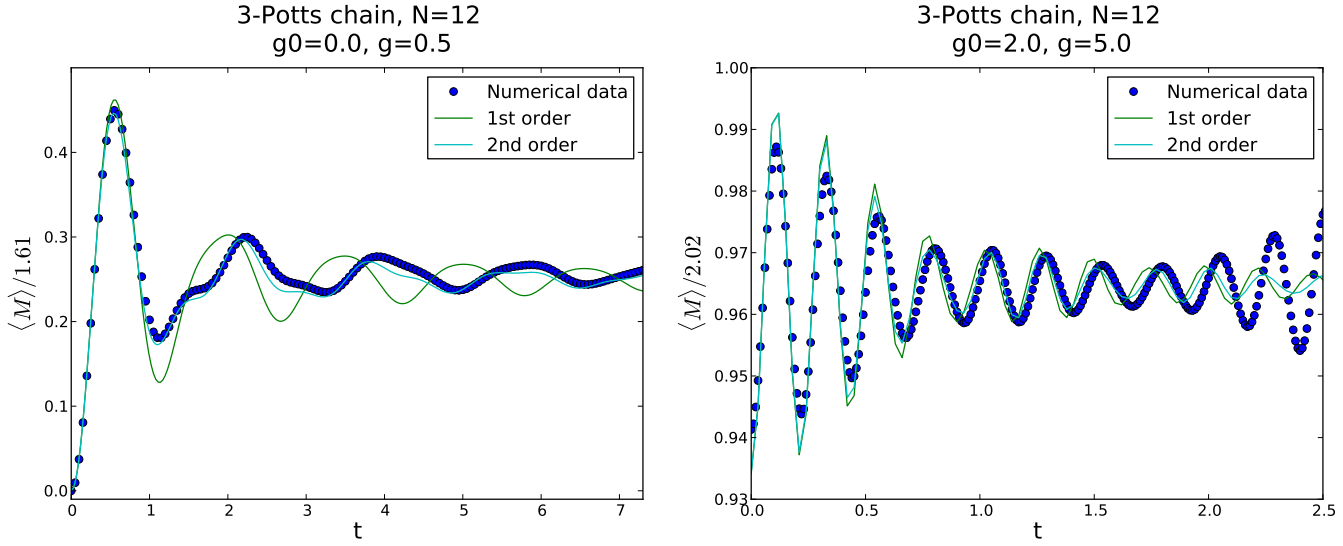
$$\langle M(t)\rangle_{g=0} \underset{g_0\to\infty}{\to} \frac{2}{9}\left(1 + 2\cos 3t\right)^2$$

The plot below shows that numerical data converge very rapidly to this function



As done for the LLE one can attempt to fit the data simply replacing the dispersion relation in (4.22) with the perturbative corrected one while using the same quasiparticles density K_p. Once again the result is surprisingly good. The only issue is that the expectation value of the magnetization has to

be rescaled appropriately in order to have the point exactly on the conjectured curve. This factor is not the trivial $q-1$ necessary to rescale the transverse fields for different q to the value assumed at $t=0$ when $g_0=\infty$. It seems instead to depend both on g and g_0. In the case of the 3-Potts chain for quenches in the ferromagnetic phase the recaling factor decreases as $\Delta g = |g-g_0|$ is increased varying between 1.4 when $\Delta g = 0.2$ and 1.7 when $\Delta = 0.6$, while in the paramagnetic phase it seems to be nearly constant to 2. For quenches crossing the transition point instead the conjectured formula does not work. Below it is shown the agreement with numerical data when $q=3$, with the corresponding rescaling factor reported.



Let's now consider what happens for quenches ending at the critical point. In order to apply the BCFT prediction of chapter 2 for the one point function of a primary operator (2.34), take the one-site matrix which on the chain should correspond to the scaling limit of the energy density. This matrix has been discussed in chapter 3 when dealing with the class of universality of the quantum critical point and its expression is given in (3.43) and (3.47) for the Ising and 3-Potts chains respectively. This local operator will now be denoted ϵ_a to avoid confusion with the dispersion relation ε. Its expectation value can be written for any q, thanks to PBC which ensure translational invariance, as

$$\langle \epsilon_a \rangle = -\frac{\langle H_g \rangle}{N} - (1+g)\langle M_a \rangle \tag{4.23}$$

So basically it corresponds just to the transverse field shifted by a constant value which does not depend on time, being the Hamiltonian conserved. Plugging in (4.22) and the expression for the energy (4.10) previously obtained[9], one can easly get its full time dependence in the Ising chain

$$\langle \epsilon(t) \rangle = \frac{4}{\pi}\int_0^{2\pi} \mathrm{d}p\, \frac{2(1-g)\cos^2\frac{p}{2}(gg_0+1-(g+g_0)\cos p)+(1+g)(g-g_0)\sin^2 p\cos(2\varepsilon_g(p)t)}{\varepsilon_{g_0}(p)\left(\varepsilon_g(p)\right)^2} \tag{4.24}$$

Then the constant shift in (4.23) simply sets to zero the asymptotic value of this operator when the quench ends at the critical point, since as $g=1$ the time independent term cancels while the rest of this expression goes to zero in the limit $t\to\infty$. This is reasonable, since ϵ_a is just the difference between the two terms in the Hamiltonian, which have the same importance at the critical point, being the transition a competition between the two. Of course the same holds if the quench starts from the critical point, leading to an initial zero value of the energy density ϵ. Not surprisingly also

[9] This time also the ground state energy must be added

for general q this operator has the same features observed in the Ising chain.
Actually the expression (4.24) for the time dependent one point function of the energy density on the Ising chain is not so clearly an exponential of the form (2.34) since, when $g = 1$ it reads

$$\langle \epsilon(t) \rangle = \frac{1-g_0}{\pi} \int_0^{2\pi} \mathrm{d}p \, \frac{\cos^2 \frac{p}{2} \cos\left(8t \sin \frac{p}{2}\right)}{\sqrt{(1-g_0)^2 + 4g_0 \sin^2 \frac{p}{2}}} \tag{4.25}$$

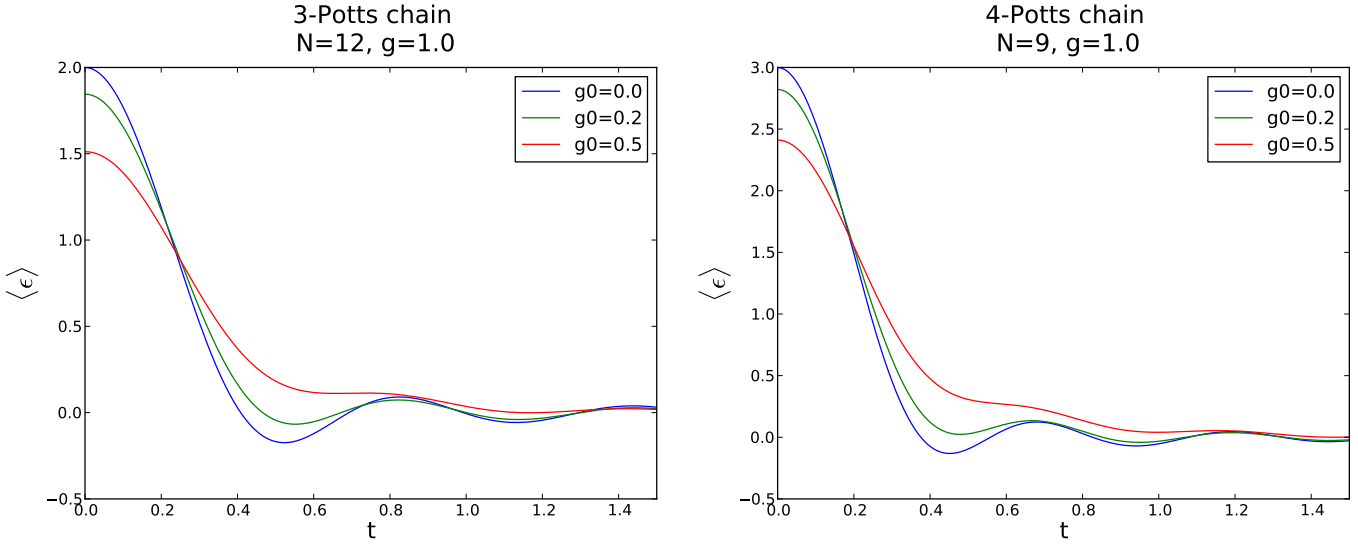
Note the usual symmetry $g_0 \to 1/g_0$ typical of quenches to the critical point, that this time holds for less than a change of sign, since the role of the longitudinal and transverse term is exchanged by duality. The expansion for $t \to \infty$ of this function has a leading contribution of order $1/t$, times an oscillatory function of t, then to recast the exponential behavior predicted by BCFT with that lattice expression is a non trivial task. As will be discussed in the next section this is not the case for the order parameter, whose exponential decay is noticeable also on the lattice[10]. Nonetheless taking the scaling limit of the lattice result (4.25) yields the wanted exponential behavior

$$\langle \epsilon(t) \rangle = \frac{2\Delta_0}{\pi} \int_0^{\infty} \mathrm{d}p \, \frac{\cos(2vpt)}{\sqrt{\Delta_0^2 + v^2 p^2}} = \frac{2}{\pi} \frac{\Delta_0}{v} K_0(2t\Delta_0) \underset{t\to\infty}{\sim} \sqrt{\frac{\Delta_0}{\pi t}} \frac{e^{-2t\Delta_0}}{v}$$

where K_0 is a modified Bessel function of the second kind. So, apart from the overall time-dependent factor, one gets a lifetime from which it is possible to extract the extrapolation time τ_0 which is the "free parameter" in the BCFT

$$\frac{1}{t^{\epsilon}_{\mathrm{rel}}} = 2\Delta_0 = \frac{\pi \eta_\epsilon}{2\tau_0} = \frac{\pi}{2\tau_0} \qquad \Rightarrow \qquad \tau_0 = \frac{\pi}{4\Delta_0} \tag{4.26}$$

Notwithstanding the above the energy density one point functions exhibit a shape resembling very much (2.34) times an oscillatory function. The latter can be attributed to lattice effects which are irrelevant in the scaling limit. Below are shown some examples for quenches starting from the ferromagnetic phase in both the 3 and 4 Potts chains.



The oscillatory prefactor does not allow to extract the lifetime of $\langle \varepsilon(t) \rangle$ via fitting data with a pure exponential function. This in turn does not permit to compare these lifetimes with the ones of

[10] And actually not only when the quench ends at the critical point

the corresponding order parameter one point function in order to check quantitatively the BCFT prediction on their ratio

$$\frac{t^{\sigma}_{\rm rel}}{t^{\epsilon}_{\rm rel}} = \frac{\eta_{\epsilon}}{\eta_{\sigma}}$$

However note that the ratio on the r.h.s. is equal to 8, 6, 4 for $q = 2, 3, 4$ respectively, as can be seen plugging in the anomalous dimensions given in chapter 2. Then the order parameter one point function should decay much more slowly than $\langle \varepsilon(t) \rangle$ and this will be in fact observed at the end of the next section.

4.3.4 Order parameter

The order parameter expectation value is the most important one point function to analyze, since it gives information on the symmetries enjoyed by the asymptotic steady state characterizing the system at equilibrium. A natural and important issue to address is whether the ensamble to which the system equilibrates in the limit $t \to \infty$ shares symmetries with the initial state or not. It might happen that even if the initial state breaks some symmetries of the Hamiltonian, these are restored in the long time limit.
The first thing to note is that if the quench starts from the paramagnetic phase the order parameter expectation value is zero for all times. This is due to the fact that, as discussed in section 3.3, the ground state lies in the $Q = 0$ sector and is then invariant under the $\mathbb{Z}_q$ transformation implemented by $\omega^{\mathcal{Q}}$. But also the Hamiltonian does not change under this transformation, then follows

$$\begin{aligned} \langle m^{\alpha}(t) \rangle &= \langle \psi_{g_0>1} | (-1)^{\mathcal{Q}} (-1)^{-\mathcal{Q}} e^{iH_g t} (-1)^{\mathcal{Q}} (-1)^{-\mathcal{Q}} m^{\alpha} (-1)^{\mathcal{Q}} (-1)^{-\mathcal{Q}} e^{-iH_g t} (-1)^{\mathcal{Q}} (-1)^{-\mathcal{Q}} | \psi_{g_0>1} \rangle = \\ &= \langle \psi_{g_0>1} | e^{iH_g t} m^{\alpha-1} e^{-iH_g t} | \psi_{g_0>1} \rangle = \langle m^{\alpha+1}(t) \rangle \end{aligned} \tag{4.27}$$

and since all the q order parameters sum to zero this implies that $\langle m^{\alpha}(t) \rangle = 0$ for all t and for all α. Then the non trivial case is when the initial Hamiltonian is ferromagnetic. If the final coupling $g > 1$ it is quite natural to expect that the order of the initial state will be destroyed by the disordered dynaamics and the symmetry restored. If instead $g < 1$ it is not obvious what should happen, one may think that there are some cases in which $\lim_{t\to\infty} \langle m(t) \rangle \neq 0$ and that the asymptotic steady state remain ordered as the initial one. Heuristically it can be argued that this is not the case since the energy (4.1) injected into the system by the quench process will always lead to disorder and to symmetry restoration. Indeed this is what happens in the quantum Ising chain, as proved analitically in [65]. Unfortunately the calculation of the order parameter one point function for $q = 2$ is not so easy to be briefly reproduced here, as was the case for the other time dependent quantities considered so far. The reason for this is that for quenches starting in the ferromagnetic phase the initial state has to be properly chosen in such a way to have nonzero initial expectation value of σ^z. Actually up to this point this issue was completely disregarded and in computing the energy density after the quench, the LLE and the transverse field expectation value the initial state was taken to be (4.5) without worrying about the fact that the sea of excitations pairs may have been taken from two different sectors. In fact the formulas obtained worked perfectly for general g and g_0 because taking any linear combination of the degenerate ground states as the initial state would have produced the same results. This follows from the fact that the observables considered up to this point, namely H and σ^x, do not couple the two sectors with $F = 0$ and $F = 1$ and moreover their expectation value is independent of F, then

$$\left(a\langle 0, F=0| + b\langle 0, F=1|\right) O(t) \left(a|0, F=0\rangle + b|0, F=1\rangle\right) = (a^2 + b^2)\langle 0, F|O(t)|0, F\rangle$$

In the case of the order parameter instead a similar argument to the one which leads to (4.27) implies that, for general q [11]

$$ {}_Q\langle 0|m(t)|0\rangle_Q = 0 \qquad\qquad \forall Q = 0, 1, \ldots, q-1 \tag{4.28}$$

Then in order to have a non trivial dynamics the ground state has to be chosen as the "magnetized" linear combination of the groundstates $|0\rangle_Q$, which yields the ground states $|0\rangle_\mu$. In the Ising chain case this linear combination is $|0,\pm\rangle = (|0, F=0\rangle \pm |0, F=1\rangle)/\sqrt{2}$, whose initial expectation values of the order parameter σ^z differ only by the sign. Once this initial state has been prepared the actual difficulty in computing $\langle\sigma^z(t)\rangle$ shows up when expressing σ_z in terms of the c modes, since the latter is a product (infinite in the thermodynamic limit) of c operators. It turns out [65] that this computation is not feasible in this way except for the case of small quasiparticles density after the quench, i.e. $|K_p|^2 \ll 1$, and only in the limit $t \to \infty$. Another way to get the exact result for all $g_0 < 1$ and all g (for large times) is to rely on cluster decomposition of the order paramater two point function, which can be expressed as the determinant of a particular matrix, the former being computable in the thermodynamic limit. The result for $g_0, g < 1$ is the following

$$ \langle\sigma^z(t)\rangle \underset{t\to\infty}{\sim} A(g, g_0)\, e^{-t/t^\sigma_{\rm rel}} \qquad\qquad \frac{1}{t^\sigma_{\rm rel}} = -\frac{1}{\pi}\int_0^\pi \mathrm{d}p\, \frac{\mathrm{d}\varepsilon_g}{\mathrm{d}p} \log\left(\frac{1-K_p^2}{1+K_p^2}\right) \tag{4.29}$$

For quenches ending in the paramagnetic phase the authors only conjectured that the result has the same form, with a different amplitude which also depends on time via an oscillatory function. The important conclusion coming out from this result is that the order parameter goes always to zero in the long time limit, meaning that the symmetry is always restored in the asymptotic steady state.
Intuition says that the same thing should happen for the q-states Potts quantum chains, since they are the most trivial generalization of the Ising one and they share many properties with it, the most important of which is the fact that their symmetry group is discrete. Moreover, as discussed in the previous sections, their dynamics is very similar to the Ising chain's one and it is enough to replace the exact dispersion relation of the $q = 2$ case with the perturbative corrected one[12] to get very good agreement with numerical data. Unluckily the small revival time caused by the shortness of the chain does not permit to grasp the asymptotic value reached by m for all the value of g_0 and g for which the dynamics is non trivial. Nevertheless there is strong evidence that the dynamics of the order parameter on the q states Potts chain is alike in all respects to the one of the Ising chain.
Let's start considering the two limiting case in which the final Hamiltonian has $g = 0$ and $g = \infty$. In the former the order parameter is stationary, since it commutes with $H_{g=0}$, then whichever is the initial coupling g_0, $\langle m\rangle$ will not depend on time. In the latter case instead, the exponential of H_g factorizes and the time evolution of m_a can be written down explictly as a matrix whose entries exhibits persistent oscillations, implying that no equilibration can occur. When $g_0 = 0$ the expectation value of this matrix on the initial symmetry breaking state is simply its first diagonal entry, and for general q this yields

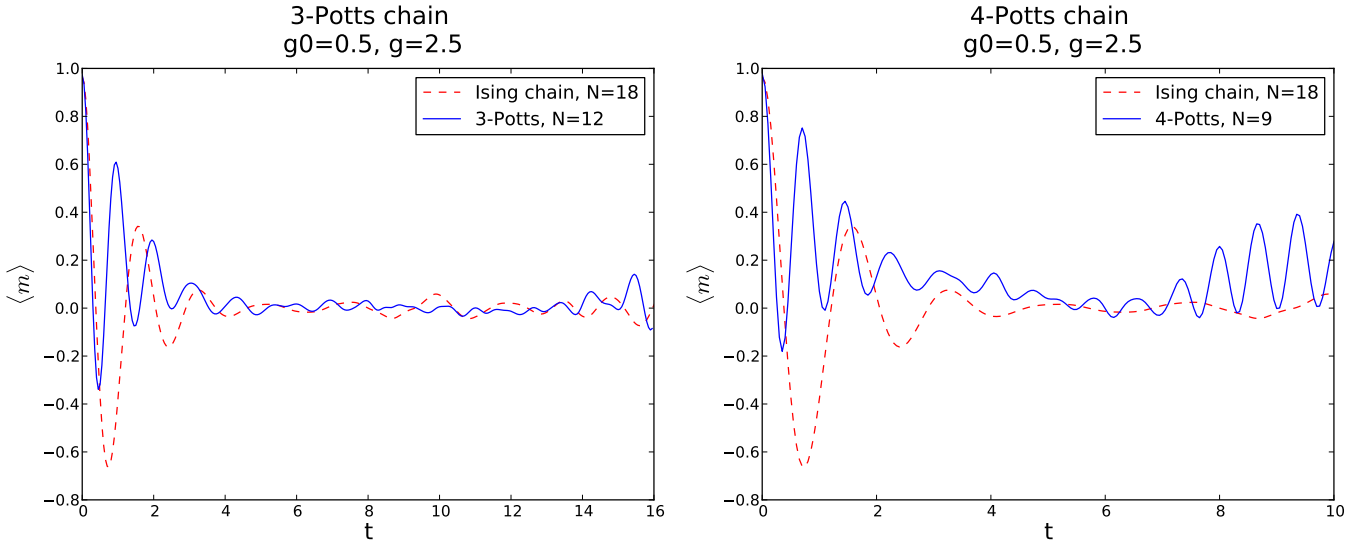
$$ \langle m(t)\rangle_{g_0=0} \underset{g\to\infty}{\sim} \frac{(q-2) + 2\cos(qgt)}{q} \tag{4.30}$$

When $g < \infty$ but still large this periodicity is clearly observed in simulations even if $g_0 > 0$ and the oscillations are exponentially suppressed. For any finite $g > 1$ and any initial coupling constant g_0 the expectation value of the order parameter relaxes to zero in a time much smaller than the revival time, which, for g large enough should be approximately $T_{\rm rev} \simeq N/v = N/2$, considering the fact

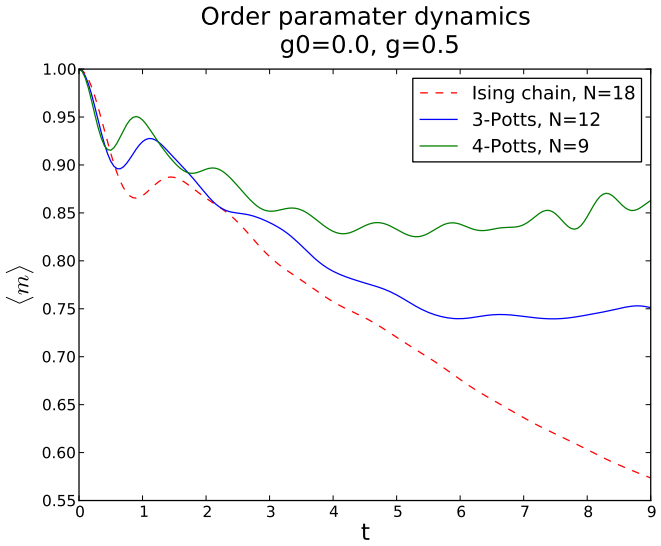
[11] As before for the ground state $|0\rangle_\mu$, here the state $|0\rangle_Q$ is the ground state lieing in the eigenspace relative to the eigenvalue Q of $\mathcal{Q}$ for a general $g_0 < 1$

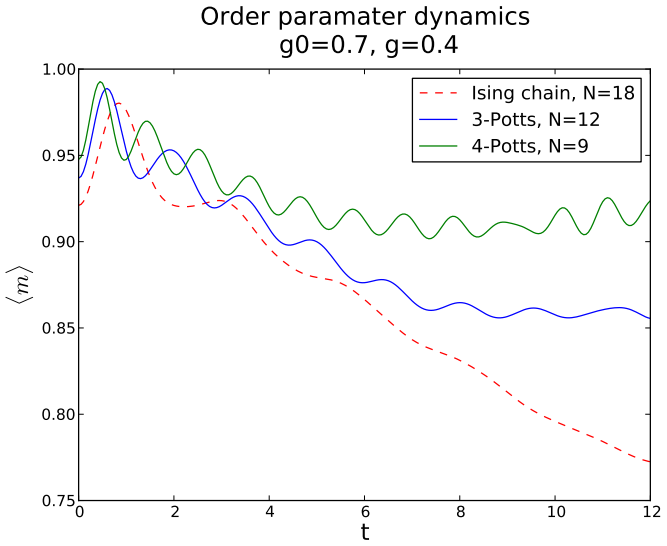
[12] This of course holds far away from the critical point, but for quenches ending at the critical point one can relies on the BCFT results whose comparison with numerical data will be discussed at the end of this section

that it is doubled when a longitudinal magnetic field is inserted in the initial state and that the first order $v_{\max}$ can be used since $g = 2.5$ is quite large. The curious fact is that for the order paramater dynamics when $g > 1$ this time window seems to be doubled again, so that $T_{\text{rev}} \simeq N$. However this effect could also be just an accidental suppression of the beginning of the first revival. This fact is shown below in a quench from $g_0 = 0.5$ to $g = 2.5$ for $q = 3$ and $q = 4$, comparing the outcome with the, still numerical, results for the Ising chain.



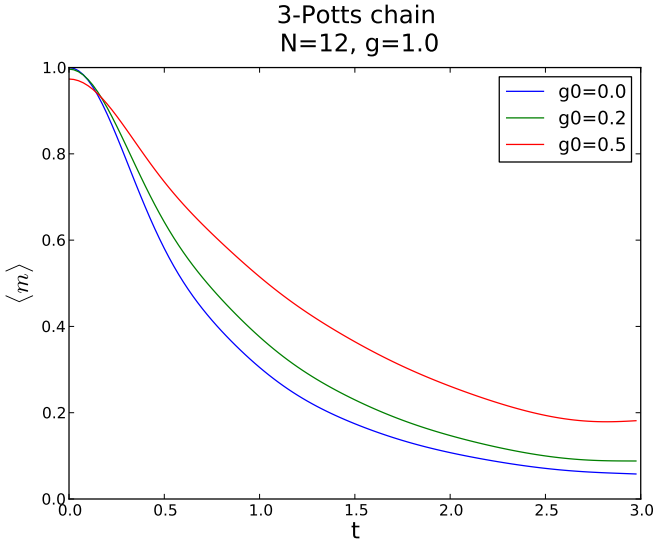
From what above it is quite clear that the equilibrium value of the order parameter is 0 when quenching to the paramagnetic phase. Moreover note that the amplitude of the exponentially suppressed oscillations decreases as q is increased, suggesting a greater stability of $\langle m(t) \rangle$ while apporaching equilibrium. Regarding the frequency it increases with q as expected from the rough estimate (4.30), which implies a frequency proportional to qg.
The problem with equilibration is when $g, g_0 < 1$ since then the relaxation process is very slow over than having a revival time which is one fourth of the $g > 1$ one. The fact that $T_{\text{rev}} \simeq N/2v$ although a longitudinal magnetic field is inserted in the initial state is related to the vanishing (4.28) of the diagonal matrix elements of m on the basis $\{|0\rangle_Q\}$ of the ground state eigenspace. This restores the off diagonal terms (4.18) which causes revivals at that time. However also the final Hamiltonian H_g must play a role in the destruction of revivals, since when $g > 1$ the initial state is the same but T_{rev} seems to be multiplied by 4. Notwithstanding these complications, the behavior of $\langle m(t) \rangle$ is exactly the same as the one observed in the Ising chain: if $g_0 > g$ the equilibration process is slower, $\langle m(t) \rangle$ initially increases for very short times and then starts decreasing oscillating, with a rate of decrease which is smaller the smaller is the final coupling g; if $g_0 < g$ instead the one point function decreases for short times still not being monotonic. In the latter case on the one hand the drop of $\langle m(t) \rangle$ is faster, but on the other T_{rev} is smaller so that it is still not possible to observe numerically the value to which the order parameter relaxes. Below are shown both the two cases just described.

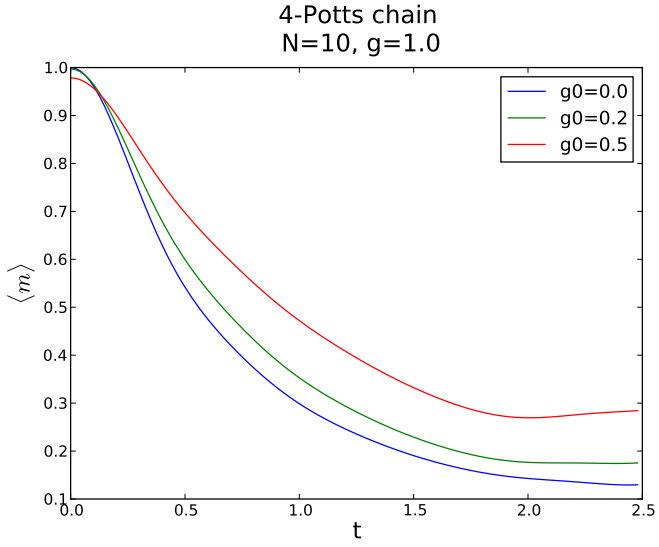




By the way note that, even if not explicitly shown above, the equilibrium value of $\langle m(t) \rangle$ is not apparent even for the Ising chain, since the revival time provided by a chain of length $N = 18$ [13] is still too small to lead the one point function to equilibrium. Only the exact result (4.29) proves its equilibration toward zero.

For quenches ending at the critical point it is impressive to see how the exponential decay (2.34) accurately describes the dynamics on the chain. In fact the expression obtained in [65] for the order paramater one point function in the Ising chain holds only for long times and does not provide information on the short time behavior of $\langle m(t) \rangle$, which is actually oscillatory for any value of g and g_0. The only exception is when $g = 1$, there oscillations stop completely and a pure exponential decay is observed. As for the energy density operator (4.23) this behavior should be observed for any $q \leq 4$ as the order parameter corresponds to the primary operator σ in the CFT. Indeed this is the case as it is clear from the plots below which show $\langle m(t) \rangle$ for quenches to the critical point varying the initial coupling g_0.





Note that the revival time is in agreement with the values (3.50) of the massless quasiparticles velocities of the critical point estimated from finite size scaling of the ground states energy in section

[13] And in fact also $N = 20$

3.7, since they give $T_{\text{rev}} = 12/2 \cdot 2.59 \simeq 2.3$ and $T_{\text{rev}} = 10/2 \cdot 3.1 \simeq 1.61$ for $q = 3$ and $q = 4$ respectively. Of course the numerical outcome for the Ising chain is analogue in all respects to the plots shown above and in fact the order parameter lifetime can be obtained from (4.29) yielding

$$\frac{1}{t^{\sigma}_{\text{rel}}} = -\frac{2}{\pi}\int_0^{\pi} \mathrm{d}p \cos\left(\frac{p}{2}\right) \log\left(\frac{(1+g_0)\sin\left(\frac{p}{2}\right)}{\sqrt{1+g_0^2-2g_0\cos p}}\right) = \frac{2}{\pi}\frac{1-g_0}{\sqrt{g_0}}\arctan\left(\frac{2\sqrt{g_0}}{1-g_0}\right) \tag{4.31}$$

So, as in the case of the energy density[14] of the post-quench excitations (4.12) and of the one point function of the operator ϵ (4.25), the lattice result for the order parameter lifetime cannot be matched perfectly with the BCFT prediction (2.35), since, although in this case the exponential decay of the one point function arises also on the chain, the lifetime is not proportional to the mass gap of the initial Hamiltonian for any value of g_0. However notice that

$$\frac{1}{t^{\sigma}_{\text{rel}}} \sim \begin{cases} \dfrac{4}{\pi} \simeq 1.2732 > 1 & g_0 \to 0 \\ (1-g_0) = \dfrac{\Delta_0}{2} & g_0 \to 1 \end{cases}$$

Then actually the lifetime does go as the mass gap of the initial Hamiltonian, given that the latter is close enough to the critical point. Moreover taking the scaling limit of the expression (4.31) for t^{σ}_{rel}, in the same way as for the energy density in section 4.3.1 and for $\langle\epsilon(t)\rangle$ in section 4.3.3, one has

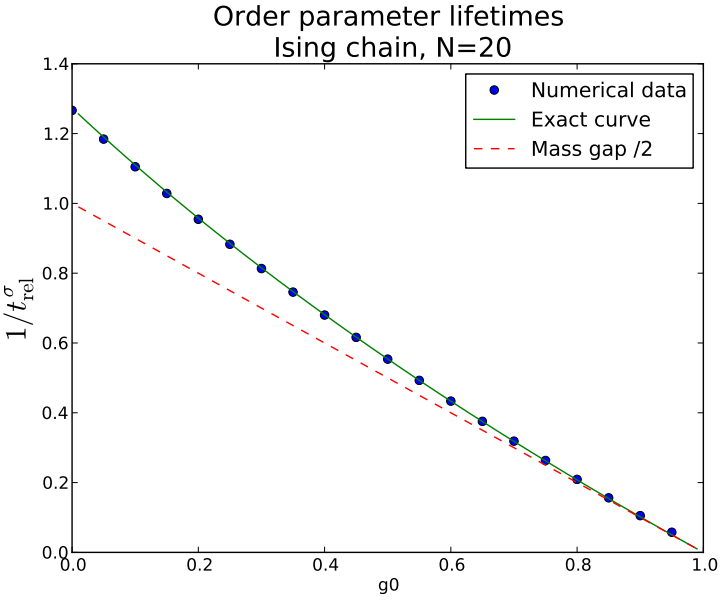
$$\frac{1}{t^{\sigma}_{\text{rel}}} = -\frac{v}{\pi}\int_0^{\infty} \mathrm{d}p \log\cos\left(\Delta\theta_p\right) = -\frac{v}{\pi}\int_0^{\infty} \log\left(\frac{vp}{\sqrt{\Delta_0^2+v^2p^2}}\right) = \frac{\Delta_0}{2}$$

Comparing this result with the (2.35) one can determine the extrapolation time, yielding

$$\tau_0 = \frac{\pi}{8\Delta_0} \tag{4.32}$$

where $\eta_\sigma = 1/8$ is the Ising chain order parameter's scaling dimension. Note that there is a disagreement of a factor of two between the extrapolation time determined from the one point function of σ^z in this way, and the one obtained from the one point function (4.26). Moreover the extrapolation length was determined also in [65] via matching the exact result for the spacetime scaling limit of the two-point function of σ^z on the chain and the BCFT two point function of the order parameter σ. The τ_0 extracted in that case agrees with (4.26), but not with (4.32). Although this seems to suggest a mistake in the calculation above, equation (4.31) is in fact correct, as shown in the plot below, in which the lifetimes of $\langle\sigma^z(t)\rangle$ are extracted by fitting numerical data for different g_0s via an exponential function $\langle\sigma^z(t)\rangle = A\exp(-t/t^{\sigma}_{\text{rel}})$.

[14] Not to be confused with the operator ϵ which has been previously referred to as the energy density operator



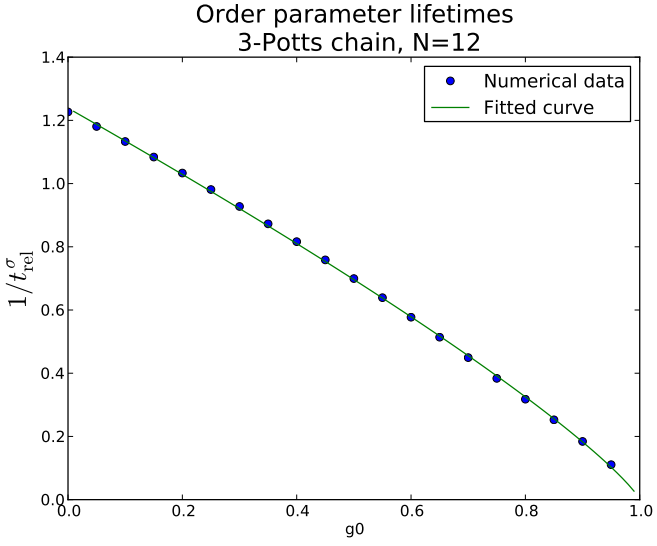
Although in the Ising chain $1/t^{\sigma}_{\rm rel}$ is not proportional to the mass gap for any value of g_0, it is interesting to observe that for $q = 3$ and $q = 4$ the order parameter lifetimes seems to be proportional exactly to $(1-g_0)^{-\nu}$, with ν the mass gap critical exponent of the corresponding chain. Indeed fitting once again numerical data for $\langle m(t)\rangle$ to extract the lifetemes and then fitting the lifetimes for $q = 3$ and $q = 4$ via

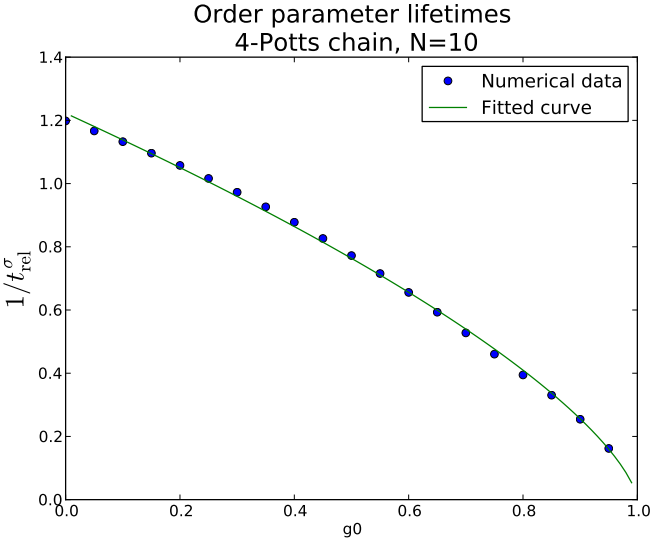
$$\frac{1}{t^{\sigma}_{\rm rel}} = A(1 - g_0)^{\nu} \tag{4.33}$$

one gets the following results

$$\begin{aligned} A &= 1.23(8) \qquad & \nu &= 0.8(3) \qquad & q &= 3 \\ A &= 1.2(2) \qquad & v &= 0.6(7) \qquad & q &= 4 \end{aligned}$$

which are curiosly in perfect agreement with the critical exponent for the mass gap discussed in chapter 2, i.e. $\nu_3 = 5/6 = 0.833$ and $\nu_4 = 2/3 = 0.666$. Below are shown the lifetimes extracted against the fitted curves





This fact is likely to happen by chance, since actually the mass gap of the $q = 3$ and $q = 4$ chains is not, as in the Ising chain, equal to (4.33) for all values of g_0. Indeed perturbation theory shows that for $g_0 \ll 1$ its value is $m(g_0) = q(1 - 2g_0/q)$, which is definitely not the small g_0 expansion of (4.33). However for $g_0 \lesssim 1$ it is a confirmation of the validity of the CFT results.
Let's now briefly resume the comparison, sketched at the end of section 4.3.3, of the lifetimes of the one point function of ϵ with the ones of the order parameter. Although it is not possible to do the same kind of analysis as above for $1/t^{\epsilon}_{\text{rel}}$ as a function of g_0, it can be seen that, trying to fit numerical data for $\langle \epsilon(t) \rangle$ with an exponential, the lifetimes for ϵ are, for fixed g_0, almost independent of q over than being definitely smaller that t^{σ}_{rel}, as it is clear comparing their decay toward zero depicted in this section and in section 4.3.3. Instead t^{σ}_{rel} decreases with q in the region $g_0 \lesssim 1$, then the ratio $t^{\sigma}_{\text{rel}}/t^{\epsilon}_{\text{rel}}$ decreases with q, as the ratio of the anomalous dimensions $\eta_{\varepsilon}/\eta_{\sigma}$. Unfortunately the former and the latter ratio do not coincide.

4.4 Conclusions

In the analysis carried out through this chapter the q-states Potts chain's dynamics after a quench, for $q \leq 4$, has shown itself to be very similar in many respects to the $q = 2$ case, which has been solved completely in [65]. Since not even the stationary problem is solvable for general q, it is very unlikely to be able to solve analytically the dynamics of this model. However the anology with the Ising chain gives important information on the structure of the dynamics toward equilibrium.
In fact both the analysis of the revivals of the initial state in the finite-size system and the perturbative treatment of the problem show a fairly solid interpretation of the dynamics in terms of quasiparticles propagation. Indeed the fact that revivals occur exactly at a time T_{rev} given by (4.17) is derived in [69] on a very peculiar assumption on the form of the initial state, namely (4.5). However the derivation of the expression for the revival time does not require the knowledge of the quasiparticles density $|K_p|^2$. The same is not true for the detailed time dependence of the other quantities considered, since they strongly depend on K_p. Nevertheless the fact that a simple substitution of the second order corrected dispersion relation gives so good agreement with numerical results for short times seems to indicate that, if it were possible to write the initial state for the general q case as (4.5), the kernel K_p may have a very similar form. Moreover when the quench crosses the transition point this naive substitution in the $q = 2$ exact formulas does not give acceptable results, even if the both g and g_0 are far from $g = 1$. This may be explained by the fact that a different quasiparticles density is necessary, since K_p depends both on the final and on the initial Hamiltonian. This function may be obtained supposing the Hamiltonian can be written as

$$H_g = \sum_p \varepsilon_g(p) a^{\dagger}_g(p) a_g(p) \tag{4.34}$$

and then finding some constraint relating the operator before and after the quench. This in complete analogy with the Ising chain case. Unfortunately it is not at all obvious that a similar form of H is sensible in the q states Potts chain, as it is not obvious how these operators should behave when exchanged. However if this were the case it would be possible to find, at least in the perturbative regimes, the function K_p.
Further work is necessary in order to characterize the steady state reached by the system in the long time limit. Actually for the Ising chain also this problem was completely solved in [66] for the kind of quenches considered in this work. The result is that the equilibrium values of local observables can be obtained as averages over a generalized Gibbs ensamble fixed by the expectation value of the occupation number $n_k = d^{\dagger}_k d_k$ on the initial state. The fact that the quasiparticles of the general q case seem to rule the dynamics far from the critical point in a way very similar to the fermions

diagonalizing the Ising chain might suggest that an analogue result holds for the q-state Potts chain. Nevertheless it is undoubtedly impossible to check it via the numerical approach employed in this work. More sophisticated numerical techniques may be used in order to simulate longer chains, thus pushing farther the revival time, and to study two point functions for a wide range of the distance between the two points. Another interesting issue related to the equilibrium state is the connection between this state and integrability. The q-Potts chain is strictly speaking non integrable for $q > 2$, since its spectrum cannot be computed exactly. Nevertheless the quasiparticles description, which does not imply integrability on the lattice, and the possibility to express the Hamiltonian as in (4.34), even without knowing the exact form of the dispersion relation and the detailed structure of the Hilbert space in terms of multiparticle states, may give a means to characterize the asymptotic steady state.

Finally the comparison carried out between numerical data from the chain and the results from CFT for quenches to the critical point has proved the latter to be qualitative good, although a quantitative interpretation has not been achieved in a satisfactory way. By the way this was expected from the fact that the outcome of a global quench is not efficiently described by a CFT because of the large amount of energy possesed by the initial state.

Appendix A

Second order perturbation theory

The second order calculation will be performed in the paramagnetic phase $g \gg 1$, in which the ground state and the first excited state are given in (3.23) and (3.25) respectively . From standard second order perturbation theory, the correction to the ℓ-th excited state reads

$$E_\ell^{(2)} = \sum_{n \neq l} \frac{|\langle n|H_p|l\rangle|^2}{E_\ell^{(0)} - E_n^{(0)}} = \frac{1}{qg} \sum_{n \neq l} \frac{|\langle n|H_p|l\rangle|^2}{\ell - n} \tag{A.1}$$

where in this case $\ell = 0, 1$ for the ground state and the first excited state respectively, the state $|n\rangle$ are the unperturbed, correctly rotated to make H_p diagonal in the first level degenerate subspace. Since H_p connects the non degenerate ground state $|0\rangle$ only to the second level with two spin flips, for the ground state correction (A.1) simplifies exploiting the completeness of the unperturbed eigenstates basis

$$E_{GS}^{(2)} = -\frac{1}{2qg} \sum_n \langle 0|H_p|n\rangle\langle n|H_p|0\rangle = -\frac{\langle 0|H_p^2|0\rangle}{2qg} = -\frac{\||H_p|0\rangle\|^2}{2qg} \tag{A.2}$$

while in the case of the first level this step is not allowed since, as will be shown below, H_p connects it both to the 2-nd and 3-rd level, so its second order correction is given by

$$E_{1,\lambda}^{(2)}(k) = -\frac{1}{qg} \sum_{a,b=1}^{N} \sum_{\alpha,\beta=1}^{q-1} |\langle (\alpha;a),(\beta;b)|H_p|\lambda;k\rangle|^2 - \frac{1}{2qg} \sum_{a,b,c=1}^{N} \sum_{\alpha,\beta,\gamma=1}^{q-1} |\langle (\alpha;a),(\beta;b),(\gamma;c)|H_p|\lambda;k\rangle|^2 \tag{A.3}$$

where the state $|\dots(\alpha;a)\dots\rangle$ has a spin flip of type α in the position a.
In order to evaluate the action of H_p on the ground state and on the first excited state it is useful to write it in a more convenient form as

$$H_p = -q \sum_{a=1}^{N} \sum_{\mu=1}^{q} P_a^\mu P_{a+1}^\mu \qquad P^\mu = |e_\mu\rangle\langle e_\mu| - \frac{1}{q} \qquad \mu = 1, 2, ..., q$$

where $|e_\mu\rangle$ is the canonical basis, and to choose the Fourier basis on each site as the ortonormal basis of eigenstates of the single term in H_0, i.e. the matrix before denoted M_a

$$|\lambda\rangle = \frac{1}{\sqrt{q}} \sum_{\mu=1}^{q} e^{i\lambda\mu\frac{2\pi}{q}} |e_\mu\rangle \qquad M|\lambda\rangle = -|\lambda\rangle \qquad \lambda = 1, 2, ..., q-1 \tag{A.4}$$

note that $|\lambda = 0\rangle = |\lambda_0\rangle$ is the eigenstate of M relative to the eigenvalue $q - 1$. The action of P_μ on $|\lambda\rangle$ then reads

$$P_\mu|\lambda\rangle = \frac{1}{\sqrt{q}} e^{i\mu\lambda\frac{2\pi}{q}} |e_\mu\rangle - \frac{1}{q}|\lambda\rangle \qquad \Rightarrow \qquad P_\mu|\lambda_0\rangle = \frac{1}{\sqrt{q}}|e_\mu\rangle - \frac{1}{q}|\lambda_0\rangle$$

So that the action of each two-sites term in H_p on a two site state $|\lambda_0\rangle \otimes |\lambda\rangle =: |\lambda_0\, \lambda\rangle$ reads

$$q\sum_\mu P^\mu P^\mu |\lambda_0\, \lambda\rangle = q\sum_\mu \left(\frac{1}{\sqrt{q}}|e_\mu\rangle - \frac{1}{q}|\lambda_0\rangle\right)\left(\frac{1}{\sqrt{q}}e^{i\mu\lambda\frac{2\pi}{q}}|e_\mu\rangle - \frac{1}{q}|\lambda\rangle\right) =$$
$$= \sum_\mu |e_\mu\, e_\mu\rangle - \frac{1}{\sqrt{q}}\sum_\mu \left(|e_\mu\, \lambda\rangle + e^{i\lambda\mu\frac{2\pi}{q}}|\lambda_0\, e_\mu\rangle\right) + |\lambda_0\, \lambda\rangle$$

then writing $|e_\mu\rangle$ in terms of $|\lambda_0\rangle$ and $|\lambda\rangle$ as well as using the ortonormality relation

$$\frac{1}{q}\sum_{\mu=1}^{q} e^{i\mu(\alpha-\beta)\frac{2\pi}{q}} = \delta_{\mathrm{mod}(\alpha-\beta,q),0}$$

one gets

$$q\sum_\mu P^\mu P^\mu |\lambda_0\, \lambda_0\rangle = \sum_{\alpha=1}^{q-1} |\alpha\ q-\alpha\rangle \tag{A.5}$$

$$q\sum_\mu P^\mu P^\mu |\lambda_0\, \lambda\rangle = |\lambda\, \lambda_0\rangle + \sum_{\alpha=1,\alpha\neq\lambda}^{q-1} |\alpha\ \mathrm{mod}(\lambda-\alpha,q)\rangle \qquad \lambda = 1,2,...,q-1 \tag{A.6}$$

Using now (A.2) the second order ground state energy correction is very simply evaluated as

$$E^{(2)}_{GS} = -\frac{1}{2qg}\left\|-\sum_a q\sum_\mu P^\mu_a P^\mu_{a+1}\bigotimes_b |\lambda_0\rangle_b\right\|^2 = -\frac{1}{2qg}\left\|\sum_a\sum_{\alpha=1}^{q-1}|(\alpha;a),(q-\alpha;a+1)\rangle\right\|^2 =$$
$$= -\frac{N}{2qg}\sum_{\alpha,\beta=1}^{q-1}\langle\alpha\, q-\alpha|\beta\, q-\beta\rangle = -\frac{N}{2qg}\sum_{\alpha=1}^{q-1}\langle\alpha\, q-\alpha|\alpha\, q-\alpha\rangle = -\frac{N}{2qg}\sum_{\alpha=1}^{q-1} = -\frac{N(q-1)}{2qg}$$

where the last two steps follow from the fact that the vectors (A.4) are ortonormal.
It is more complicated to evaluate the first level correction (A.3), since in this case one has to deal with some combinatorics. Let's treat the two terms separately. Inserting the expression for $|k;\lambda\rangle$ the first term reads

$$-\frac{1}{qg}\sum_{a,b=1}^{N}\sum_{\alpha,\beta=1}^{q-1}\left|\frac{1}{\sqrt{N}}\sum_c e^{ikc}\langle(\alpha;a),(\beta;b)|H_p|\lambda;c\rangle\right|^2 \tag{A.7}$$

For fixed state $|(\alpha;a),(\beta;b)\rangle$ in the sum outside the square modulus the only possibility to get a non-zero scalar product is when a term in H_p acts on a two-sites state of the string $|(\lambda,c)\rangle$ in which the spin flip $\lambda \neq \lambda_0$ is present, but since H_p creates a spin flip α on the left or on the right of λ according to (A.6) there are in fact two such terms in the sum over c and b must be equal to $a+1$, moreover $\beta = \mathrm{mod}(\lambda-\alpha,q)$ because of (A.6). So the first term becomes

$$-\frac{1}{qg}\frac{1}{N}\sum_{a=1}^{N}\sum_{\alpha=1}^{q-1}\left|e^{ika}+e^{ik(a+1)}\right|^2 = -\frac{4(q-1)}{qg}\cos^2\frac{k}{2}$$

The second term instead gets contributions from different kinds of states $|(\alpha;a),(\beta;b),(\gamma;c)\rangle$, but in any case the non trivial terms arise when a term in H_p acts on a two-sites state of the string $|(\lambda,c)\rangle$ in which the spin flip $\lambda \neq \lambda_0$ is not present, so now only (A.5) is needed. Of course one among α,β,γ

must be λ, while the other two must sum to q according to (A.5). After inserting again (3.24) for $|k;\lambda\rangle$ the second term is

$$-\frac{1}{2qg}\sum_{a,b,c=1}^{N}\sum_{\alpha,\beta,\gamma=1}^{q-1}\left|\frac{1}{\sqrt{N}}\sum_{d}e^{ikd}\langle(\alpha;a),(\beta;b),(\gamma,c)|H_p|\lambda;d\rangle\right|^2$$

Keeping into account (A.5) and assuming $\lambda \neq q-\lambda$, i.e $\lambda \neq q/2$, the are four kinds of states which give different contributions to the sum outside the modulus, these are states of the following form:

- $|(\alpha;a),(q-\alpha;a+1),(\lambda,b)\rangle$ with $b \neq a-1, a+2$
- $|(\lambda;a),(q-\lambda;a+1),(\lambda,a+2)\rangle$ and $|(q-\lambda;a),(\lambda;a+1),(\lambda,a-1)\rangle$
- $|(\lambda;a),(q-\lambda;a+1),(\lambda,a-1)\rangle$ and $|(q-\lambda;a),(\lambda;a+1),(\lambda,a+2)\rangle$
- $|(\alpha;a),(q-\alpha;a+1),(\lambda,b)\rangle$ with $b = a-1, a+2$ and $\alpha \neq \lambda$

The first of these states gives always a k-independent contribution, since every time it arises in the sum it gets non zero scalar product with only one term in the sum over d so one has only to count how many time this happens: α can be chosen among $q-1$ values, once this is done there are N sites where (λ,b) can appear and, for fixed b, $(N-4)$ places where to put the pair $(\alpha;a),(q-\alpha;a+1)$, so the total contribution is

$$-\frac{1}{2qg}\frac{1}{N}(q-1)N(N-4) = E_{GS}^{(2)} + \frac{2(q-1)}{2qg}$$

The second state in the list instead gets, for fixed a, non zero scalar product with two term in the sum over d, i.e. when $d=a$ and $d=a+2$, but there is no freedom in choosing the colour α of the state, since it must be equal to λ, so this class of states gives

$$-\frac{1}{2qg}\frac{1}{N}\sum_{a=1}^{N}\left|e^{ika}+e^{ik(a+2)}\right|^2 = -\frac{2}{qg}\cos^2 k$$

The third state gives again a constant contribution equal to 1 with a factor of $2N$ in front since the 2 strings $|\cdots\lambda,\lambda,q-\lambda\cdots\rangle$ and $|\cdots q-\lambda,\lambda,\lambda\cdots\rangle$ can be placed in N positions. Finally the fourth state gives once again 1 each time it appears, with a prefactor $2N(q-3)$ because the pair $\alpha, q-\alpha$ can be placed both on the left and on the right of λ, with α takable among $(q-3)$ values since λ and $q-\lambda$ are excluded. Summing up all this contributions the second term in (A.7) reads

$$E_{GS}^{(2)} - \frac{1}{qg}\left[(q-3)+2(q-2)\cos k+\cos 2k\right]$$

then adding the first term in the r.h.s. of (A.7), evaluated above, one gets (3.28). Note that in all what is above $\lambda \neq q-\lambda$ was assumed, when $\lambda = q/2$ the combinatorial argument is slightly different but in the end the result is the same. Note also that the case $q=2$ is much more simple than for general q, but (3.28) works also for $q=2$.

Appendix B

Boundary states

Consider the state

$$|B_K\rangle = \exp\left(\sum_{p>0} K_p a^\dagger_{-p} a^\dagger_p\right)|0\rangle = \prod_{p>0} e^{K_p a^\dagger_{-p} a^\dagger_p}|0\rangle \tag{B.1}$$

where the creation and annihilation operators satisty the anticommutation relations

$$a_p a^\dagger_q = -a^\dagger_q a_p + \delta_{p,q} \qquad a_p a_q = -a_q a_p \qquad a^\dagger_p a^\dagger_q = -a^\dagger_q a^\dagger_p \tag{B.2}$$

and the vacuum state $|0\rangle$ is determined via

$$a_p|0\rangle = 0 \qquad \forall p \tag{B.3}$$

Because of this relation one has $a_p^2 = 0 = (a^\dagger_p)^2 \quad \forall p$ so that

$$e^{K_p a^\dagger_{-p} a^\dagger_p} = 1 + K_p a^\dagger_{-p} a^\dagger_p$$

Note that the first equality in (B.1) is justified by the fact that the operators $a^\dagger_{-p} a^\dagger_p$ *commute* for all p. It follows that the boundary state's norm for fermions is easly evaluated as

$$\begin{aligned}
\langle B_K|B_K\rangle &= \langle 0|\prod_{p>0} e^{\overline{K}_p a_p a_{-p}} \prod_{q>0} e^{K_q a^\dagger_{-q} a^\dagger_q}|0\rangle = \langle 0|\prod_{p>0} e^{\overline{K}_p a_p a_{-p}} e^{K_p a^\dagger_{-p} a^\dagger_p}|0\rangle = \\
&= \prod_{p>0}\langle 0|\Big(1+\overline{K}_p a_p a_{-p}\Big)\Big(1+K_p a^\dagger_{-p} a^\dagger_p\Big)|0\rangle = \prod_{p>0}\Big(1+|K_p|^2\langle 0|a_p a_{-p} a^\dagger_{-p} a^\dagger_p|0\rangle\Big) = \\
&= \prod_{p>0}\Big(1+|K_p|^2\Big) = \exp\left[\log\prod_{p>0}\Big(1+|K_p|^2\Big)\right] = \exp\left[\sum_{p>0}\log\Big(1+|K_p|^2\Big)\right]
\end{aligned}$$

where the first step follows from the fact that the operators $a^\dagger_{-p} a^\dagger_p$ and their adjoints commute for different p and $\langle 0|a_p a_{-p} a^\dagger_{-p} a^\dagger_p|0\rangle = 1$ is given by (B.2), (B.3) and by the absence of $p = 0$ in all summations.
Note that going through the same steps it is easy to get the scalar product between different boundary states

$$\langle B_K|B_H\rangle = \exp\left[\sum_{p>0}\log\left(1+\overline{K}_p H_p\right)\right] \tag{B.4}$$

One could also be interested in the expectation value of some power of the number operator $n_q = a_q^\dagger a_q$ between boundary states like (B.1), to this aim is convenient to evaluate the generating function

$$F_K(\lambda_q) = \frac{\langle B_K| \exp\left(\sum_{q>0} \lambda_q n_q\right) |B_K\rangle}{\langle B_K|B_K\rangle} =: \langle \exp\left(\sum_{q>0} \lambda_q n_q\right)\rangle_K$$

so that all the n_q's momenta are given by

$$\langle n_{p_1} \dots n_{p_\ell}\rangle_K = \frac{\partial^\ell}{\partial\lambda_{p_1} \dots \partial\lambda_{p_\ell}} F_K(\lambda_p)\Big|_{\lambda_p=0} \tag{B.5}$$

To evaluate $F_k(\lambda_q)$ first note that from (B.2) one has

$$n_q a_{-p}^\dagger a_p^\dagger = a_{-p}^\dagger a_p^\dagger (n_q + \delta_{p,q} + \delta_{p,-q}) \tag{B.6}$$

but since all summation indexes are assumed to be positive, the last term gives always zero and this equation implies

$$\sum_{q>0} \lambda_q n_q \sum_{p>0} K_p a_{-p}^\dagger a_p^\dagger = \sum_{p>0} K_p a_{-p}^\dagger a_p^\dagger \left(\lambda_p + \sum_{q>0} \lambda_q n_q\right)$$

from which follows

$$\exp\left(\sum_{q>0} \lambda_q n_q\right) \sum_{p>0} K_p a_{-p}^\dagger a_p^\dagger = \sum_{p>0} K_p e^{\lambda_p} a_{-p}^\dagger a_p^\dagger \exp\left(\sum_{q>0} \lambda_q n_q\right)$$

and finally the action of the number operator's exponential on the boundary state is

$$\begin{aligned}\exp\left(\sum_{q>0} \lambda_q n_q\right) |B_K\rangle &= \exp\left(\sum_{q>0} \lambda_q n_q\right) \exp\left(\sum_{p>0} K_p a_{-p}^\dagger a_p^\dagger\right) |0\rangle = \\ &= \exp\left(\sum_{p>0} K_p e^{\lambda_p} a_{-p}^\dagger a_p^\dagger\right) \exp\left(\sum_{q>0} \lambda_q n_q\right) |0\rangle = |B_{Ke^\lambda}\rangle\end{aligned} \tag{B.7}$$

where the last step follow from the fact the n_q applied to the vacuum yields zero for all qs.
From (B.4) is now easy to evaluate the generating funcion $F(\lambda_q)$, which reads

$$F_K(\lambda_p) = \frac{\langle B_K|B_{Ke^\lambda}\rangle}{\langle B_K|B_K\rangle} = \prod_{p>0} \frac{1 + e^{\lambda_p}|K_p|^2}{1 + |K_p|^2} = \exp\left[\sum_{p>0} \log\left(\frac{1 + e^{\lambda_p}|K_p|^2}{1 + |K_p|^2}\right)\right] \tag{B.8}$$

Inserting this expression in (B.5) one gets the ℓth moment of the fermionic number operator

$$\langle n_{p_1} \dots n_{p_\ell}\rangle_K = \prod_{j=1}^{l} \frac{|K_{p_j}|^2}{1 + |K_{p_j}|^2} \tag{B.9}$$

To calculate instead the expectation value of a string of $a_{-q}^\dagger a_q^\dagger$ or their adjoint, one needs the following generating function

$$G_K(\lambda_q) = \frac{\langle B_K| \exp\left(\sum_{q>0} \lambda_q a_{-q}^\dagger a_q^\dagger\right) |B_K\rangle}{\langle B_K|B_K\rangle} = \frac{\langle B_K|B_{K+\lambda}\rangle}{\langle B_K|B_K\rangle} = \prod_{q>0} \frac{1 + \overline{K}_q(K_q + \lambda_q)}{1 + |K_p|^2}$$

or its complex conjugation. The ℓth moment is easly obtained by derivation

$$\langle a^\dagger_{-p_1} a^\dagger_{p_1} \dots a^\dagger_{-p_\ell} a^\dagger_{p_\ell} \rangle_K = \frac{\partial^\ell}{\partial\lambda_{p_1} \dots \partial\lambda_{p_\ell}} G_K(\lambda_p)\Big|_{\lambda_p=0} = \prod_{j=1}^{l} \frac{\overline{K}_{p_j}}{1+|K_{p_j}|^2} \tag{B.10}$$

As an example consider the total number of quasiparticle excitations created by a quench from the ferromagnetic phase $g_0 < 1$ to the critical point of an Ising chain. This can be readly calculated from (4.5) and (B.9), yielding

$$\begin{aligned} n(g_0) &:= \frac{1}{N}\langle\psi_{g_0}| \sum_k d^\dagger_k d_k |\psi_{g_0}\rangle = \frac{1}{N}\sum_k \frac{K_p^2}{1+K_p^2} \quad \underset{N\to\infty}{\sim} \\ &\sim \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}p \left[1 - \frac{(1+g_0)\sin\left(\frac{p}{2}\right)}{\sqrt{(1+g_0)^2 - 2g_0\cos p}}\right] = 1 - \frac{1+g_0}{\pi\sqrt{g_0}} \arcsin\left(\frac{2\sqrt{g_0}}{1+g_0}\right) \end{aligned}$$

so the density of quasiparticles is of course 0 if $g_0 = 1$, and, on the lattice, it tends to a finite valute for $g_0 \to 0$, i.e.

$$n(g_0) \underset{g_0\to 0}{\sim} 1 - \frac{2}{\pi} \simeq 0.36338$$

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