UNIVERSITÀ DEGLI STUDI DI MILANO

Facoltà di Scienze Matematiche, Fisiche e Naturali

Corso di laurea in Fisica

Dinamica a tempi brevi di un gas su reticolo forzato fuori equilibrio

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PACS: 5.70.Jk, 64.60.-i

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Anno Accademico 2003-2004

Dedicata ai miei genitori

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Part I

Theory and simulation settings

Chapter 1

Introduction

This thesis is about the dynamics of the driven lattice gases. In particular, we studied the short - time dynamics. This thesis is organised as follows:

- The first chapter is a general introduction on non - equilibrium and driven lattice gases
- the second chapter regards the the critical dynamics on the field - theoretic point of view.
- the third chapter is more specifically on the short time scaling: known results are revised and an extension to non - equilibrium systems is attempted.
- the fourth chapter is about the simulations we performed to verify short time scaling.
- the fifth and last chapter summarise the results obtained and revise problems and perspectives.

All the plots and the tables are placed at the end of the thesis.

1.1 Non-equilibrium systems: general highlights

In nature most of the phenomena we could observe involve so many degrees of freedom that is impossible to give solutions accounting for all of them. Fortunately, at the beginning of 20^{th} century a fundamental set of ideas emerged and became what now is called statistical mechanics. Anyway, up to recent times, nearly the totality of studies focused on equilibrium statistical

mechanics. To simplify, equilibrium statistical mechanics is founded on the framework given by Gibbs: a microscopic Hamiltonian is given, so, in the canonical ensemble, the stationary distribution over configuration space is known via the Boltzmann factor: $e^{-\beta\mathcal{H}}$. The problems that springs from this approach are difficult, yet at least the general framework is clear and very powerful methods have been developed to give sensible approximations.

For non - equilibrium the above considerations are not true, and a general approach similar to equilibrium statistical mechanics is still missing, despite many efforts. It is thereafter sensible to turn our attention to simple non - equilibrium models that could be analysed with standard techniques as, say, field theory or the master equation.

The actual state of things in non - equilibrium statistical mechanics is in a way similar to another situation that occurred in equilibrium statistical mechanics: at the beginning the study of critical phenomena had little foundations and relied much on simple models, like the Ising model. Afterwards more general techniques were introduced leading to deeper understanding of the problem. The hope is that something similar would occur for non - equilibrium systems

1.2 Lattice gas models: general ideas

One of the first models that has been introduced following the program outlined in the previous section is the so - called driven lattice gas (DLG). It has been proposed by Katz, Lebowits and Spohn at the beginning of the eighties, partially motivated by the physics of solid electrolytes. Nowadays it is also referred as the infinite drive lattice gas (IDLG), directly as the KLS model or even the standard (non-equilibrium) model. It consists of a ordinary Ising lattice gas with attractive interactions in contact with a thermal bath, with a fundamental modification. The particles no more hop to nearest neighbour empty sites with a rate only specified by the internal energy, but a uniform driving field is added, so that the jumping rate are biased in a direction. Other conditions are fundamental to define precisely this model: periodic boundary conditions is surely the most relevant. In fact these conditions allow the system to satisfy translation invariance, making it far easier to study. Despite its conceptual simplicity, the model has not been solved exactly, even if a solution to very similar model has been found.

A first important highlight of the system could be gleaned even from the schematic discussion we are following: there is an energy flux through the system. The energy is supplied to the system through the field and is lost to the system through its coupling with the thermal reservoir.

A second important highlight of IDLG is the presence of a non - equilibrium steady state: the probability distribution associated with the system admits a limit distribution $P(t = \infty, C) = P_{eq}(C)$. It is not easy to demonstrate such a state exists in general nor for the particular case studied. Such a state is however always observed in simulations. When the system settles down in a non - equilibrium steady state, the energy flux through the system is stationary, so this state is in a way privileged and analogies with general non-equilibrium systems are more likely to be found. In fact recently some progress has been made in this direction.

To end this section we will mention variations to the standard model. As a fact, the the IDLG model is very difficult to realise in practice. A common criticism to it is that despite its simplicity and its connection to super-ionic conductors, no working experiments could be arranged because they would require very large electric fields. A way to go around such difficulties is to modify the model. We could retain the field but change its uniformity. This could be done in two main ways: we could impose a AC driving field instead of a uniform one, or we could suppose the field to be random. The first model presents additional complications such as a chaotic dynamic or even the absence of a steady state. The other, instead, is a much more tractable and it is known as the randomly driven lattice gas (RDLG). Many more variations of these models exists, for a general overview see [30].

1.3 Non - equilibrium: breaking of detailed balance

Let us first revise the concepts that lies at the base of equilibrium statistical mechanics. To discuss those concepts, let us first individuate the degrees of freedom of the system. Those could also be called configurations and indicated with C. Let we suppose that those configurations are discrete, like in the case of an Ising system on a lattice.

We also suppose that a good definition of the internal energy exists, via a function that maps the configuration space $S(C)$ on $\mathbb{R}$ and that we indicate with the Hamiltonian $\mathcal{H}(C)$. Now, if the system is *isolated* and the total energy is fixed at E, Boltzmann postulated that the distribution of probability is *uniform* within the configuration of the same energy:

$$P_{eq,i}(C) = N_i \, \delta[\mathcal{H}(C) - E] \tag{1.1}$$

First of all, the concept of temperature must be introduced and then following a standard procedure, one arrives at the so - called Gibbs measure

of probability. This probability distribution is valid for a system in contact with a very large (in theory infinite) thermal bath at temperature T:

$$P_{eq,G}(C) = N_G \exp -\beta \mathcal{H}(C) \tag{1.2}$$

where $\beta = 1/k_b T$ and k_b is the well - known Boltzmann constant. In the derivation of (1.1) some other hypothesis on the nature of the physical dynamics should be made.For example, if we stick to classical mechanics of particles, we could imagine a Newtonian dynamics is defined at microscopic level. This dynamics is expected to explore the configuration space in such a way, that, regardless the details of the dynamics and the initial conditions, after very large times, the probability distribution (1.1) is reached.

We could ignore this problem and postulate (1.1), if we want to study *equilibrium* statistical mechanics. We are however interested in the non - equilibrium statistical mechanics, so we surely need some more information on the dynamics. We postulate in general a **time dependent** distribution $P(C,t)$. The simplest context to study the problem of the dynamics is the Master Equation in continuous time:

$$\partial_t P(C,t) = \sum_{C'} \{W[C' \to C]P(C',t) - W[C \to C']P(C,t)\} \tag{1.3}$$

The W's are called *rates* of transition. They describe the probability of going from configuration C to configuration C'. It is clear that equation (1.3) is a global balance equation. It could be obtained by assuming that the probability in the system is described by a Markovian chain. Equation (1.3) is linear by assumption, and usually only rates W independent from t are used. This equation describes non - equilibrium as well as equilibrium systems, since it is completely general. We will now explain the differences between non - equilibrium and equilibrium. We could define a current of probability:

$$J(C \to t') = W[C' \to C]P(C',t) - W[C \to C']P(C,t) \tag{1.4}$$

Up to now we have assumed for the rates W to be time independent. In the case of equilibrium systems, we must ensure that when the limit $t \to \infty$ is taken, the time dependent probability distribution $P(C,t)$ must reach its equilibrium value $P_{eq}(C)$. If the system is in contact with a thermal bath, we know equilibrium distribution is given by (1.2). In the equilibrium case, there could always be found rates which satisfy detailed balance, that is:

$$W[C' \to C]P_{eq}(C') = W[C \to C']P_{eq}(C) \tag{1.5}$$

A few remarks about relation (1.5): first of all, in the case $P_{eq}(C)$ is the Gibbs distribution (1.2), condition (1.5) is more conveniently written as:

$$w[C, C'] = \frac{W[C \to C']}{W[C' \to C]} = e^{\beta(\mathcal{H}(C') - \mathcal{H}(C))} \tag{1.6}$$

We could now give a definition of what it is intended as non - equilibrium: a non - equilibrium system (that could be described by a Master equation) have transition rates that **violate** detailed balance. As we see, the relation (1.5) is extremely useful in equilibrium cases, especially in Monte Carlo simulations. It is less frequently noted that (1.5) implies for the current (1.4) to vanish at steady state (that is in the limit $t \to \infty$) for **any** configurations C, C'. Despite its utility in equilibrium systems, (1.5) does not help at all when we come to discuss non - equilibrium. In non - equilibrium, we are not in general able to calculate the static (or steady state) distribution. We know such a distribution must exist, in fact writing equation (1.3) as:

$$\partial_t P(C, t) = \mathbf{L} P(C, t) \tag{1.7}$$

$\mathbf{L}$ is a general operator on the configuration space, and we immediately see that it has a *left* eigenvector $v(C) = 1$ associated with eigenvalue 0. This is a consequence of the total conservation law implied by (1.3). It much more complicated to demonstrate that this state is unique. We could simply postulate that exists a unique steady state distribution and call it $P^*(C)$. As already stated, this distribution is in general unknown. A more direct method to verify if a dynamical system displays detailed balance is to consider:

$$\begin{aligned} W[C_1 \to C_2]\, W[C_2 \to C_3] ... W[C_n \to C_1] = \\ W[C_2 \to C_1]\, W[C_3 \to C_2] ... W[C_1 \to C_n] \end{aligned} \tag{1.8}$$

where $\{C_1 ... C_n\}$ are **all possible cycles**. In ref. [10] has been demonstrated that condition (1.8) is equivalent to (1.5) It is clear that this condition is quite difficult to verify in case detailed balance holds. It could be extremely handy in the non - equilibrium case, because we only need to find a cycle in which equivalence (1.8) fails.

1.4 The models

In this section we will analyse with much greater depth the models presented previously. The starting point is the Ising lattice gas: each point in the lattice carry a $n_i = 0, 1$ variable that represents the site being empty or occupied. we will also use the more usual Ising representation: it is easy to

switch from one to another using the standard transformation $s_i = 2n_i - 1$ The usual Ising lattice gas could defined for simplicity on a sub-lattice Λ of the hyper-cubic lattice $\mathbb{Z}^d$. The Hamiltonian that governs a lattice gas is the usual Ising Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle \in \Lambda} n_i n_j \tag{1.9}$$

were $< i, j >$ denotes nearest - neighbour sites. A positive J results in an attractive (ferromagnetic) interaction, negative J in a repulsive (anti-ferromagnetic) interaction. One of the problems regarding a possible experimental realization of the IDLG model is that most of the super-ionic conductors have a repulsive interaction between particles, while the standard model has an attractive one.

To be simple, and also because "real" particles are not expected to vanish and be created at low energies, we restrict ourselves at the canonical ensemble, so that the total number of particles is conserved. This is implemented by a global constraint:

$$\rho = \sum_{i \in \Lambda} n_i \tag{1.10}$$

with ρ fixed. Most studies focuses on a half - filled lattice: $\rho = \frac{1}{2}$. The constraint (1.10) should be always respected, so we choose a Kawasaki dynamics [21] in order to conserve the total number of particles. We also restrict to nearest - neighbours particle hopping. No big differences are expected considering longer finite jumps. We finally introduce the driving field $\vec{E}$. For simplicity, we assume it points in one direction of the lattice. Its effect is then to favour or suppress the jumps alongside or against the direction chosen. This could be written adding a term to the rate w in equation (1.6) so that it becomes:

$$W[C \to C'] = w\left[\beta(\Delta\mathcal{H} + lE)\right] \tag{1.11}$$

where l is:

$$l = (-1, 0, +1) \quad \text{(against,trasversal,along) } \vec{E} \tag{1.12}$$

Here we see how the periodic boundary conditions plays a fundamental role in violating detailed balance. If we impose closed boundary conditions, the field in (1.11) simply acts as "gravitational" constant field and induces a accumulation of particles on a side of the lattice. The field could be included in the Hamiltonian, so that:

$$\mathcal{H}' = \mathcal{H} - U(E) \tag{1.13}$$

where U is a potential term that takes into account the presence of field E. In other words we could not find cycles that violates the equality (1.8). Instead

if we impose periodic boundary conditions in the direction of the field[1], we could easily produce examples that violate (1.8). Think to a configuration with a single strip full of particles. Suppose the boundaries of the strip are perfectly smooth, so that there are no holes in them. Moreover, suppose a single particle is taken from the strip. Also impose $T = 0$, so that we have no transverse fluctuations. The particle (or the hole in the strip) could make a cycle in the direction of the field with non - zero probability that we call P_c. If we suppose that the rate w is monotonic in its argument, the probability for the particle to accomplish the backward cycle is in general less than P_c Thus, the field term in (1.11) violates explicitly detailed balance. In the argument, the equality (1.8) proved to be extremely useful. In general, we could no more individuate a global Hamiltonian to describe the behaviour of a DLG, and the Gibbs partition function is no more the limit distribution. We also warn that other conditions, different from classical periodic boundary conditions, could be imposed to obtain the same result. The effects on the limit (non - equilibrium) distribution are surprising.

We will now pass to RDLG. Most of the considerations made above are valid also for this model. Here the field is no more deterministically known; it obeys to a probability distribution function, for example the bimodal:

$$P(E) = \delta(E - E_0) + \delta(E + E_0) \tag{1.14}$$

In other words, chosen a direction in the hyper-cubic lattice, the field points with probability 1/2 downwards and with probability 1/2 upwards. This model, as argued in [29] is completely equivalent to a bi - temperature model.

A final remark: in this formalism it is easy to implement an infinite driving field; it is sufficient always to accept (reject) the jumps along (against) the field. More details will be given in 4.

1.5 Phase transitions in driven lattice gases

We first need to define what a phase transition is in a driven lattice gas. In magnetic systems, phase transition is usually associated with the breaking of a fundamental symmetry. In Ising systems, for example, a symmetry of the Hamiltonian is related to *spin flips*: if we invert all the spins, the Hamiltonian is exactly the same. Let us first analyse what generally happens in a system with discrete symmetry like the spin flip symmetry we already described. To be concrete, we will analyse the Ising model, since it is equivalent to the

[1] it is largely irrelevant to impose periodic boundary conditions also in the direction perpendicular to the field. It is usually preferred to do so because the resultant system is completely invariant by translations

lattice gas. Suppose in fact a lattice gas model with Hamiltonian (1.9) and in the *grancanonical* ensemble. Its partition function is of the form:

$$\mathcal{Z}(\mu) = \sum_C \exp\beta \left(J \sum_{\langle i,j\rangle \in \Lambda} n_i n_j - \mu \sum_i n_i \right) \tag{1.15}$$

here μ is a factor that governs the exchange of particles between the reservoir and the system and it is called the chemical potential. We called C a possible configuration for all the variables of occupation n_i We immediately see that with the canonical transformation $s_i = 2n_i - 1$, we obtain the Ising magnetical Hamiltonian:

$$-\mathcal{H}_I = \frac{J}{4} \sum_{\langle i,j\rangle \in \Lambda} s_i s_j + \frac{(dJ - \mu)}{2} \sum_i s_i \tag{1.16}$$

If we restrict to the canonical ensemble, we immediately obtain the partition function for a Ising magnetic model without magnetic field:

$$\mathcal{Z} = \sum_C \exp\beta(J \sum_{\langle i,j\rangle \in \Lambda} s_i s_j) \tag{1.17}$$

We remember that canonical and grancanonical ensembles coincides in the thermodynamic limit. If C_s and $C'_{s'}$ are two spin configurations we could then re - write (1.17) using the **transfer matrix**, indicated with $\mathbf{T}$, as:

$$\mathcal{Z} = \sum_C \langle C_s \left|\mathbf{T}\right| C'_{s'} \rangle \langle C'_{s'} \left|\mathbf{T}\right| C''_{s''} \rangle \ldots \tag{1.18}$$

We could give an explicit expression for this matrix in a multi dimensional Ising model:

$$< C_s|\mathbf{T}|C'_{s'} >= \exp\left[\beta \left(J \sum_{\langle i,j\rangle \in \Lambda} s_i s'_j \right) + \frac{1}{2} \sum_{i,j \in \Lambda} (s_i s_j + s'_i s'_j) \right] \tag{1.19}$$

We could also indicate the spin flip operator with $\mathbf{P}$. The two operators commute so they could be diagonalised together (remember we are in a finite sub-lattice Λ of $\mathbb{Z}^d$, so the operators are simply matrices). Since $\mathbf{P}^2 = \mathbf{1}$, $\mathbf{P}$ has two eigenvalues ± 1. The matrix $\mathbf{T}$ has positive components. Thus, we could apply the the results for this class of matrices and state that the the largest eigenvalue of $\mathbf{T}$ has the corresponding eigenvector $|0\rangle$ with all positive components. Since the operator $\mathbf{P}$ is diagonalised with $\mathbf{T}$, it could not change the basis of eigenoperators. Thus:

$$\mathbf{P}\left|0\right\rangle = \left|0\right\rangle \tag{1.20}$$

In the high - temperature limit, the transfer matrix becomes constant in all its components and thus all the eigenvalues vanish save the largest. It is immediate to see that in this limit the transfer matrix becomes a projector onto $|0\rangle$. In the low temperature limit, instead, only the states with lowest energy give a important contribution to the transfer matrix. Those are the ones with all spins aligned. We understand that there are *two* of those states, and we indicate them with $|+\rangle$ and $|-\rangle$. Since the temperature is low but finite, their component spins could not be completely aligned. A interesting and simple property of those states is that:

$$\frac{\langle +|\,\mathbf{T}\,|-\rangle}{\langle +|\mathbf{T}|+\rangle} \propto e^{-\beta J L^{d-1}} \tag{1.21}$$

We see that in the infinite volume limit and $d > 1$, the two states are degenerate (and orthogonal) even at finite temperature. Such a degeneracy could be lifted by a small magnetic field. The system orders in the direction of the field. Then, in the limit of zero field, the order is maintained. It is trivial to find a parameter that measure this type of order:

$$m = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} s_i \tag{1.22}$$

The temperature at which the spontaneous magnetisation appears is called *critical temperature*. A final important remark on phase transition in the Ising model is that it is *second - order*: the order parameter is continuous in temperature, while the correlation length diverges at T_c. As we will see, of the two characteristics the second is the most important. We now pass to the lattice gas. When $E = 0$, as already explained, the lattice gas is completely equivalent to a Ising model. Many properties are exactly the same. A difference here is important: the two degenerate states $|+\rangle$ and $|-\rangle$ must strictly *coexist*. This is a trivial consequence of the constraint (1.10). The phase transition could be understood as a liquid - gas transition: as shown in fig (1.1), domains rich in particles and domains rich in holes start to separate at critical temperature; in the low temperature phase, instead, the domains are completely separated. In other words, the ground state is a linear combination of the two $|+\rangle$ and $|-\rangle$ states:

$$|Gs\rangle = 1/2\ |+\rangle + 1/2\ |+\rangle \tag{1.23}$$

a trivial consequence is that order parameter (1.22) is always zero. Much is known on the Ising phase transition, especially in two dimensions. The

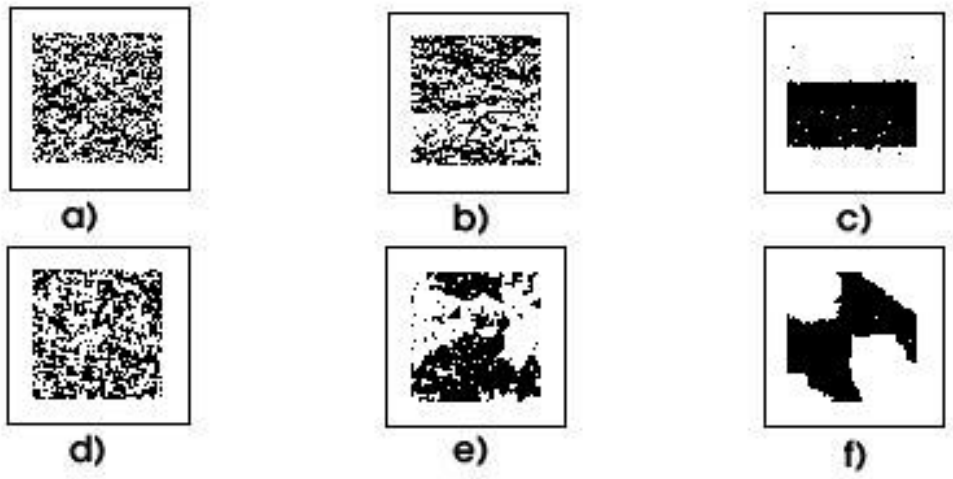

Figure 1.1: Some typical configurations as result of a simulation on a 50x50 lattice. Figures a) b) c) refer to the infinite - field driven lattice gas, while figures d) e) f) are the results for the Ising lattice gas. Figures a) and d) are the result of a simulation at $T = 2T_c$, b) and e) shows the result at T_c while c) and e) are at $T = 0.5T_c$. Notice that the strip of high density in figure c) is parallel to the field.

critical temperature is $J/(k_bT_O) = 1/2\ln(1+\sqrt{2})$[2], known as the Onsager temperature [28].

The question that naturally arises is: does this behaviour get modified if we switch on the field? Most of the studies up to now published focuses on the determination of the critical properties of the driven lattice gases. We remark that a "phase transition", as the one we explained in this section is difficult to translate in non - equilibrium. In particular, since we do not know the limit distribution, a well - founded discussion based on the transfer matrix as the one we presented, is still missing. However, the simulations shows the transition is of liquid - gas type as we argued and could be seen from (1.1). If the half filling is maintained, the driven lattice gas undergoes a second order phase transition. However,for example in $d = 2$ we have a shift of the critical temperature: a recent estimate [5], gives $J/(k_bT_c) = 0.312694$, that is about $1,40$ times the Onsager temperature. As discussed in [30] this could seem a bit weird at first, all in all we are supplying energy to the system, so how could be that the system "freezes" at higher temperature? This question could not be answered here, we re-address the interested reader to

[2]from now on, we will indicate temperatures and betas respectively in units of J/k_b and k_b/J respectively

literature [30]. Apart from this intriguing aspect, the simulations up to now have discovered many features of phase transition in the driven lattice gases. A first important feature is that stable high - density domains perpendicular to the field has never been observed. This means we could introduce a order parameter:

$$m_{DLG} = \frac{1}{|\Lambda|} \left\langle \left| s_{(k_\perp = 1, k_\parallel = 0)} \right| \right\rangle \tag{1.24}$$

where we have introduced the Fourier transform of the spin:

$$s_{\mathbf{k}} = \sum_{\mathbf{x} \in \Lambda} e^{i\mathbf{x}\mathbf{k}} s_{\mathbf{x}} \tag{1.25}$$

we remember that in a finite systems the momenta are discrete, so the the $(k_\perp = 1, k_\parallel = 0)$ notation in (1.24) indicates the first non - zero moment (in parallel to field direction).

Chapter 2

Critical dynamics

This chapter is dedicated to critical dynamics. In the Introduction, we mentioned many results that could be obtained in the framework of critical dynamics. This chapter is not meant to be exhaustive, since the argument is vast and complicated. Our aim is to review the main concepts and introduce some tools that are going to be useful in next chapters.

2.1 Langevin and Fokker - Plank equation

The usual and easier starting point for discussing dynamics in critical systems is the Langevin equation. The Langevin equation is the simplest and more tractable stochastic equation. Those class of equations are used to describe physical systems where the **noise** become important. The Langevin equation is:

$$\frac{d}{dt}q_i(t) = -\frac{1}{2}f_i(\mathbf{q}(t)) + \nu_i(t) \tag{2.1}$$

here $\mathbf{q}$ is a trajectory in $\mathbb{R}^d$, while ν is the noise term. It is a collection of stochastic functions, that have a functional distribution of probability $[d\rho(\nu)]$. In our case we consider explicitly the Gaussian distribution of probability:

$$[d\rho(\nu)] = [d\nu] \exp\left[-\frac{1}{2\Omega}\int dt \sum_i \nu_i^2(t)\right] \tag{2.2}$$

this seems very theoretic, but suppose a (classical) particle with very small mass in a liquid (or gas). We could reasonably suppose that such a particle is subjected to classical forces, that could be described by the function $f(\mathbf{q})$ and stochastic (random) forces due to thermal dynamics of the underlying liquid atoms. If the mass of the particle is very small and the dumping in

the liquid is strong, we could neglect any second - order derivative term and we obtain (2.1).

Strictly speaking, the derivative term in equation (2.1) is not well - defined. In fact it could be demonstrated (see [39]) that:

$$|q(t+\varepsilon) - q(t)| = O(\sqrt{\varepsilon}) \quad \text{when} \quad \varepsilon \to 0 \tag{2.3}$$

we see that the function $\mathbf{q}(t)$ is not differentiable, but only continuous in t. We could interpret equation (2.1) as purely symbolic.

We would like to obtain an equation for the probability distribution of q. The Langevin equation in general is solvable in few cases [1] so we seek to obtain a relation directly for the probability of $\mathbf{q}$ without passing through the solution of (2.1). A method that could be employed is to obtain the Fokker - Plank equation. We could get to a distribution of probability $P(q_i, t, \mathbf{q_0}, t_0)$ by fixing the initial conditions for (2.1) so that $\mathbf{q}(t_0) = \mathbf{q_0}$ and by symbolically writing:

$$P(q_i, t, \mathbf{q_0}, t_0) = \left\langle \prod_i \delta[q_i(t) - q_i] \right\rangle_\nu \tag{2.4}$$

Here we indicated the average over the noise with $<>_\nu$. The vector q_i has nothing to do with the functions $q_i(t)$. Without entering the particulars, from (2.4), we could write

$$\partial_t P(q_i, t, \mathbf{q_0}, t_0) = \frac{1}{2}\partial_{q,i}\left(\Omega \partial_{q,i} P(\mathbf{q}, t) + f(\mathbf{q}) P(\mathbf{q}, t)\right) \tag{2.5}$$

that is called the Fokker - Plank equation. It is always satisfied at any time $t > t_0$ We could also write (2.5) in a "Hamiltonian" fashion. Fist of all, we could make use of a bra and ket notation:

$$P(q_i, t, \mathbf{q_0}, t_0) = \langle \mathbf{q} | P(t, t_0) | \mathbf{q_0} \rangle \tag{2.6}$$

This representation could be derived from a semi-group property of the distribution (2.4) and from time translational invariance. We then immediately come to:

$$\partial_t P(t, t_0) = -H\, P(t, t_0) \tag{2.7}$$

The function H is called the Fokker - Plank Hamiltonian. From equation (2.5), we obtain for H

$$H = \mathbf{p}^2 + \mathbf{p}\dot{f}(\mathbf{q}) \tag{2.8}$$

We used a Fourier transform notation, where $\mathbf{p}$ is the moment associated with the vector $\mathbf{q}$, the space variable of distribution (2.4).

[1] For example, if the force $f(\mathbf{q})$ is linear in $\mathbf{q}$.

2.2 Dynamical field theory

In this section we wish to generalise the useful results of section (2.1). Instead of a $\mathbf{q}(t)$, we would introduce a field $\phi(x,t)$. We consider only scalar fields, however a generalisation to vector fields is straightforward. The choice of restricting to scalar fields is partly motivated by simplicity of notations, partly by the consideration that the models we wish to study are simple spin models so scalar field is completely adequate. In general $x \in \mathbb{R}^d$. We could write a field stochastic Langevin equation as:

$$\partial_t \phi(x,t) + \frac{1}{2}\,\Omega F_\beta[\phi(x,t)] = \nu(x,t) \tag{2.9a}$$

where the noise field ν has distribution

$$[d\rho(\nu)] = [d\nu(x,t)] \exp\left[-\frac{1}{2\Omega}\int dx\,dt \sum_i \nu^2(x,t)\right] \tag{2.9b}$$

In (2.9a) F is a generic functional. We will see what happens in particular cases, in which we specify this functional better. The results of 2.1 immediately generalise for a dynamic field, so the Fokker - Plank equation could be written as:

$$\partial_t P(t,t_0,\phi(x)) = -H\,P(t,t_0) \tag{2.10}$$

Here H reads:

$$H = -\frac{\Omega}{2}\int d^dx \frac{\delta}{\delta\phi}\left[\frac{\delta}{\delta\phi} + F[\phi]\right]$$

The Fokker - Plank formalism is quite precise and could be very useful when discussing general properties, especially in the large time limit. However, it is founded on differential equations and the renormalization of equations is not very well known. Another possible formalism could be constructed starting from Langevin equation (2.9a). This formalism is based on functionals and thus is very useful when discussing renormalization.

To implement such a formalism, we should make contact with the usual objects that are used in field theory when discussing renormalization. The simplest of those objects are correlation functions. Since we are in a dynamical context, where the time variable plays a role, we call them **dynamical correlation functions**. They could be formally calculated from a generating functional:

$$\mathcal{Z}[J] = \left\langle \exp\left[\int d^dx\,dt J(x,t)\phi(x,t)\right]\right\rangle_\nu \tag{2.11}$$

Note that the field ϕ is the dynamical variable in the Langevin equation (2.9a), so that we should know the dependence of ϕ from ν to explicitly calculate (2.11). Some progress could be made inserting in (2.11) a identity involving the functional δ:

$$\int [d\phi][d\rho(\nu)] K \delta \left[\frac{d\phi}{dt} + \frac{1}{2} F[\phi] - \nu \right] \exp \left[\int d^d x \, dt J(x,t) \phi(x,t) \right] \tag{2.12}$$

The term K in (2.12) is a Jacobian that is needed to take into account the constraint (2.9a).In a system with a finite number of degrees of freedom, indeed, the term K is a determinant. We need a good definition for it in the functional framework. A possibility is to exploit the matrix identity $detM \propto \exp\left(tr\left[\ln M\right]\right)$. The trace operation is well defined even for operators in functional spaces. After expanding in power series, we arrive to:

$$K \propto \frac{\Omega}{4} \int d^d x \, dt \frac{\delta F[\phi(x,t)]}{\delta \phi(x',t)}|_{x'=x} \tag{2.13}$$

In formula (2.13) we used a regularisation of the Heaviside step function θ, so that $\theta(0) = 1/2$. Putting all this together, from (2.12) it could be obtained the dynamic functional:

$$\begin{aligned} \mathcal{Z}[J] = \int [d\tilde{\phi}][d\phi] \exp \left[- \int d^d x dt \tilde{\phi} \left(\frac{d\phi}{dt} + \Omega \left(F[\phi] + \tilde{s}^2 \right) \right) \right] K \\ \exp \left[\int d^d x \, dt J(x,t) \phi(x,t) \right] \end{aligned} \tag{2.14}$$

Note that the noise term disappeared. In place of the noise, we find the new field $\tilde{\phi}$, that is known as Martin - Siggia - Rose response field [27]. This field arises from the Fourier transform representation of functional δ in (2.12). The renormalization procedure of (2.14) is quite complex. We will not completely refer it here, given its difficulty. In what follows, when needed we will introduce specific references to this procedure. For a much more specific and complete treatment could be found in [11].

Let us make a further remarks: another possible formulation of the dynamic functional exists, namely the supersymmetric formulation. Despite of its elegance and beauty, the supersymmetric formalism is very difficult to use in perturbative calculations. In particular, a supersymmetric approach to lattice gas models have been tempted [14], but despite some interesting results, no perturbative results have been obtained. For a through treatment of supersymmetric field theory in the dynamics see [39].

2.3 Scaling laws

Before entering in the particular case of dynamical scaling laws, let me explain what precisely scaling laws are. In the nearby of a phase transition point, typically near the critical temperature, some thermodynamic functions displays striking features. Many of those functions are typical of equilibrium systems, like the specific heat. However, many other quantities could be defined in both equilibrium and non - equilibrium systems. An example that will be very frequent are the multi - point correlation functions. In general, the critical behaviour is described using the generalised homogeneous functions: suppose a function with n variables:

$$O(x_1, ..., x_n) = \lambda^{x_O} f_O(x_1 \lambda^{e_1}, ...x_n \lambda^{e_n})$$

the exponents $e_1, ..., e_n$ are usually called *critical exponents*. The function written above is also said to display scaling properties, and could also be called scaling function, because the rescaling of the variables $x_1, ..., x_n$ results in a rescaling of the function O. We will use scaling function that usually depends on the momenta $\vec{k}$(or the space variable), on time t and on the deviation from critical temperature τ. Other quantities, like the magnetisation or the susceptibility, depends only on the τ and t

$$O(\vec{k}, t, \tau) = \lambda^{x_O} O(\vec{k}\lambda, t\lambda^{-z}, \tau\lambda^{\frac{1}{\nu}}) \tag{2.15}$$

Let me comment about the physical sense of such a scaling form: we could think of λ as the scale we are looking our physical system at; imagine now to increase the scale, we expect that the fast momenta disappear: this accounts for the first variable in (2.15). More interesting is the part regarding τ: we see that increasing λ, we go towards $\pm\infty$, depending on the initial sign of τ. Quantities that display such a behaviour are for example:

- the magnetisation m, characterised by the exponent $x_m = -\beta$. In infinite system, the scaling law for the magnetisation is valid only if $\tau < 0$.
- the correlation length ξ, that diverges at criticality with exponent ν
- the susceptibility, which is described by the exponent γ
- the specific heat, with exponent α

All this observables must be **long range** observables, so $k \to 0$.

Those exponents could be explained and (up to a certain degree) calculated in the framework of renormalization group . The models considered

admits a natural definition for some of the quantities listed above, but not all. To be specific, we have a good definition of:

- an order parameter, that could be also called a "magnetisation"
- a susceptibility
- the two - point function
- a correlation length

Other quantities similar to the specific heat could also be studied, but their definition is far more difficult to give, due to the absence of the Gibbs partition function.

In the preceding section, we discussed the dynamic field theories. We now enter more specifically in the main argument of the chapter, the critical dynamics scaling laws. We caution that as before, many calculations and demonstrations will be omitted. Before discussing particular models in the framework of dynamics, we revise some other general concepts. In particular, we want to emphasise the differences between static critical phenomena and dynamical critical phenomena. We could re - write the Langevin equation (2.9a) as:

$$\partial_t \phi(x,t) + \frac{\beta}{2}\,\Omega \frac{\delta \mathcal{H}}{\delta \phi} - F^*[\phi(x,t)] = \nu(x,t) \tag{2.16}$$

here $\beta = 1/k_b T$ as usual. It could be seen that only if the "streaming" term F^* satisfies equation:

$$\partial_\phi \left[F^*[\phi] e^{-\beta \mathcal{H}(\phi)} \right] = 0 \tag{2.17}$$

the corresponding equilibrium distribution is the well - known Boltzmann function $e^{-\beta \mathcal{H}}$. The most relevant consequences are that:

1. since the equation (2.17) does not identify uniquely the streaming term F^*, the Hamiltonians at equilibrium $\mathcal{H}$ could not completely describe the dynamics. In particular, there are infinite dynamical universality classes for each static universality class.

2. if the streaming term does not satisfy equation (2.17), the limit distribution for the corresponding Langevin equation (2.9a) is *not* the Gibbs partition function. The Langevin equation could then describe non - equilibrium models.

We will now proceed to define quantities that could be studied in a scaling analysis. The simplest ones are the dynamical correlation functions:

$$\langle \phi(x_1,t_1) ... \phi(x_n,t_n) \rangle_\nu = \frac{\delta}{\delta J(x_1,t1)} ... \frac{\delta}{\delta J(x_n,t_n)} \mathcal{Z}(J) \tag{2.18}$$

From the correlation function generating functional (2.14), that appears in (2.18), we could indeed pass to the generating functional of connected correlation functions and to proper vertices generating functional in the standard way: For the connected correlation functions generating functional we find:

$$\mathcal{G}[J] = \log Z[J] \tag{2.19}$$

while for proper vertices:

$$\Gamma[\varphi] = \int d^d x\, dt J(x,t)\varphi(x,t) - \mathcal{G}[J] \quad \text{with} \quad \varphi(x,t) = \frac{\delta \mathcal{G}[J]}{\delta J(x,t)} \tag{2.20}$$

deriving those functionals produce respectively the connected correlation functions and the one - point irreducible correlation functions.

Let us now return to Langevin equation (2.16). There is another interesting quantity to be studied in dynamical systems. Suppose a small perturbation is added to the Hamiltonian in equation (2.16):

$$\mathcal{H}' = \mathcal{H}[\phi] - \int dt\, d^d x h(x,t) O[\phi] \tag{2.21}$$

we are in particular interested to perturbations linear in the field itself:

$$\mathcal{H}' = \mathcal{H}[\phi] - \int dt\, d^d x h(x,t) \phi(x,t) \tag{2.22}$$

we call the following quantity a general response function:

$$R(t_0, t_1, ...t_n) = \frac{\delta}{\delta h(t_0)} \left\langle \phi(x_1,t_1)...\phi(x_n,t_n) \right\rangle_\nu \tag{2.23}$$

this quantity has a straightforward physical meaning, since it "measures" the effect a perturbation at t_0 has on a correlation function. A fundamental property is causality:

$$R(t_0, t_1, ...t_n) = 0 \quad \text{if} \quad t_i < t_0 \quad 1 \leq i \leq n \tag{2.24}$$

We could also define:

$$\begin{aligned} \mathcal{Z}[J, \tilde{J}] = \int [d\tilde{\phi}][d\phi] \exp\left[- \int d^d x dt \tilde{\phi} \left(\frac{d\phi}{dt} + \Omega \left(F[\phi] + \tilde{s}^2 \right) \right) \right] K \\ \exp\left[\int d^d x\, dt J(x,t)\phi(x,t) + \tilde{J}(x,t)\tilde{\phi}(x,t) \right] \end{aligned} \tag{2.25}$$

We notice that if the perturbation is of the form (2.22), the corresponding response function is very easy to calculate, since the field h couples with $\tilde{s}$, so that

$$R_l(t_0, t_1, ...t_n) = \left\langle \tilde{\phi}(x_0, t_0)\phi(x_1, t_1)...\phi(x_n, t_n) \right\rangle_\nu$$

A final remark: in what follows we will ignore the Jacobian term K in dynamic functional (2.25). This is correct if using dimensional regularisation. Indeed, from (2.13) follows, if the functional F is local as it is usually the case:

$$K \propto \delta^d(0)$$

and such terms immediately vanish in dimensional regularisation. The K is best taken into account in a supersymmetric treatment, where it is essential. Since most studies of the dynamics uses dimensional regularisation, we will directly ignore (2.13). We now refer the classification of dynamical models as first proposed by Hohenberg and Halperin [15].

2.3.1 Model A

The simplest model that could be analysed in the framework of dynamical critical models is known as model A. It is a purely dissipative model: in other words, the model's Langevin equation is written as:

$$\partial_t \phi(x,t) = \frac{\beta}{2}\,\Omega \frac{\delta \mathcal{H}}{\delta \phi(x,t)} + \nu(n,t) \tag{2.26}$$

as usual, the noise has distribution (2.9b). Being in the framework of critical phenomena, it is natural to choose the Ginzburg - Landau Hamiltonian:

$$\mathcal{H} = \int d^d x dt \left[\phi(x,t)\Delta_x \phi(x,t) + \tau\phi^2(x,t) + g\phi^4(x,t)\right] \tag{2.27}$$

notice that τ is the reduced temperature $(1 - T/T_c)$, while g is a coupling constant. From the previous discussion we get to the generating functional for correlation functions:

$$\begin{aligned} \mathcal{Z}[J,\tilde{J}] = \int [d\tilde{\phi}][d\phi] \exp\left[-\int d^d x dt \tilde{\phi}\left(\frac{d\phi}{dt} + \Omega\left(\frac{\delta\mathcal{H}}{\delta\phi(x,t)} + \tilde{s}^2\right)\right)\right] \\ \exp\left[\int d^d x\, dt J(x,t)\phi(x,t) + \tilde{J}(x,t)\tilde{\phi}(x,t)\right] \end{aligned} \tag{2.28}$$

A important property of this functional is that it reduces to the Gibbs distribution (with Hamiltonian (2.27)), in the infinite time limit. Before showing the results of renormalization group analysis to clarify the scaling laws of this

model, we restrict to Gaussian level. In particular, we calculate explicitly the correlator:

$$\langle \phi(\mathbf{k},t)\phi(\mathbf{k},t)\rangle_{conn} = \frac{\Omega^2}{\omega^2 + \Omega^2/4(\mathbf{k}^2+\tau)^2} \tag{2.29}$$

Even at this level, a important feature of critical dynamics emerges clearly; since the correlator should be a homogeneous function, we see that we must impose:

$$\omega = k^z \quad \text{with} \quad z = 2 \tag{2.30}$$

z is called the dynamical exponent, and at Gaussian level $z = 2$ for Model A. We could also calculate the response function:

$$\left\langle \phi(\mathbf{k},t)\tilde{\phi}(\mathbf{k},t)\right\rangle_{conn} = \frac{\Omega}{\omega + \Omega/2(\mathbf{k}^2+\tau)} \tag{2.31}$$

The frequency (the same is true for real time) is in a sense a special direction, in which the length scales differently. A similar situation occurs even in the static limit for the driven lattice gases, as we will see, and it is known as strong anisotropy. When renormalization procedure takes places, the dynamical exponent z is also modified. The most important feature of Model A is that the exponent z is is also independent from other (static) exponents. We now report and comment the scaling laws for Model A. In the critical domain a general connected correlation function with n fields of type ϕ and $\tilde{n}$ fields of type $\tilde{\phi}$ scales as:

$$\mathcal{G}_{n,\tilde{n}}(\mathbf{k}_i, w_i, \tau) = \lambda^{(2-\eta)n/2+(-2+\tilde{\eta})\tilde{n}/2-z}\mathcal{G}_{n,\tilde{n}}(\mathbf{k}_i\lambda, w_i\lambda^z, \tau\lambda^{1/\nu}) \tag{2.32}$$

We indicated with η and $\tilde{\eta}$ the anomalous dimensions of fields ϕ and $\tilde{\phi}$, respectively. All other exponents have their usual meaning. The exponent z could be calculated as $z = 2+\eta/2-\tilde{\eta}/2$. We notice that since the exponent z arises from the renormalization of the Ω parameter, that is needed since the correlator (2.29) generates divergences at $w \neq 0$. To make clearer significance of exponent z , we start considering the simplest possible correlation function, the two point correlation function.The theory of Model A is translationally invariant both in time and in space. The two point correlation function in could be written as (for simplicity):

$$\mathcal{G}_{2,0}(-\mathbf{k}, w, -w, \tau) = \mathcal{G}_{2,0}(\mathbf{k}, w, \tau) \tag{2.33}$$

From (2.32) we could derive immediately:

$$\mathcal{G}_{2,0}(\mathbf{k}, w, \tau) = \lambda^{(2-\eta)-z}\mathcal{G}_{n,\tilde{n}}(\mathbf{k}\lambda, w\lambda^z, \tau\lambda^{1/\nu}) \tag{2.34}$$

if we consider the deviation from critical temperature and define $\xi = \tau^{-\nu} = \lambda$, where ξ is the correlation length, we obtain (we switch to real time representation for a clearer output):

$$\mathcal{G}_{2,0}(\mathbf{k}, t, \tau) = \xi^{(2-\eta)} F_{2,0}(\mathbf{k}\xi, t\xi^{-z}) \tag{2.35}$$

where $t = (t_1 - t_2)$. We see that a correlation time appears, that scales like $\tau^{-\nu z}$. Other interesting results could be obtained: if we set $t = 0$ in (2.35), we recover the static results. Let me point out that this is possible because we are considering *complete* invariance in time translations (we are imposing the initial conditions at $t = -\infty$). Another consideration regards the susceptibility: in equilibrium systems, its definition is simple:

$$\chi(t, \tau) = \lim_{\mathbf{k}\to 0} \mathcal{G}_{2,0}(\mathbf{k}, t, \tau) \tag{2.36}$$

It is immediate to see that when approaching the critical temperature (from above) quantity (2.36) diverges like when $t = 0$:

$$\chi \propto \tau^{-\gamma} \quad \text{where} \quad \gamma = \nu(2 - \eta) \tag{2.37}$$

Fist of all we notice that such a quantity has a non - trivial dynamical behaviour at the critical temperature:

$$\chi \propto t^{(2-\eta)/z} \tag{2.38}$$

We also point out that exists another possible definition of the susceptibility, called *dynamical susceptibility*

$$\tilde{\chi}(t, \tau) = \lim_{\mathbf{k}\to 0} \mathcal{G}_{1,1}(t, \mathbf{k}, \tau) \tag{2.39}$$

In model A, the two definitions are not equivalent, since for dynamical susceptibility we obtain the scaling law:

$$\tilde{\chi}(t, \tau) = \tau^{-\tilde{\gamma}} \tilde{F}_{1,1}(t\xi^{-z}, \mathbf{k}\xi) \quad \text{where} \quad \gamma = \nu(-\eta - \tilde{\eta})/2 \tag{2.40}$$

2.3.2 Model B

The Model B is quite similar for many aspects to Model A. The only difference is that order parameter (the field ϕ in field theoretic language) is conserved. We have:

$$\frac{d}{dt} \int d^d x \, \phi(x, t) = 0 \tag{2.41}$$

At first it seems a minor modification of Model A. Such a modification has however deep consequences. In fact, the Langevin equation is very different from the one in Model A:

$$\partial_t \phi(x,t) = \nabla \left(\frac{\beta}{2} \Omega \nabla \frac{\delta \mathcal{H}}{\delta \phi(x,t)} + \nu(n,t) \right) \tag{2.42}$$

The Hamiltonian is the same as in Model A, i.e. the Ginzburg - Landau Hamiltonian (2.27). We also get a modification of the noise distribution function, that is Gaussian with variance:

$$\langle \nabla_x \nu(x,t) \nabla_{x'} \nu(x',t') \rangle = \delta(t-t') \Delta_x \delta(x-x') \tag{2.43}$$

At tree level the correlator gets modified by the conserved quantity:

$$\langle \phi(\mathbf{k},t) \phi(\mathbf{k},t) \rangle_{conn} = \frac{\Omega^2 \mathbf{k}^2}{\omega^2 + \Omega^2 / 4 \mathbf{k}^4 (\mathbf{k}^2 + \tau)^4} \tag{2.44}$$

Here to have homogeneity we need to impose $z = 4$ at Gaussian level. We also get a different renormalization theory: the correlator (2.44) does no more generate divergences when $w \neq 0$. All the divergences are contained in the equilibrium correlator, that is equal to the one in Model A. No new renormalization is required for the parameter Ω. As a consequence, we obtain for z:

$$z = 4 - \eta \tag{2.45}$$

2.4 Other models

There are many other dynamical models. As it is now clearer, the conservation of quantities like the angular momentum or the energy produces other fundamental modifications to the dynamics. In particular, we consider first energy conservation. This is known as Model C. We first couple a second field e to the field ϕ^2 in the Ginzburg - Landau Hamiltonian:

$$\begin{aligned} \mathcal{H} = \int d^d x dt \, \Big[& \phi(x,t) \Delta_x \phi(x,t) + \tau \phi^2(x,t) + g \phi^4(x,t) + \\ & \frac{1}{2} v e(x,t) \phi^2(x,t) + \frac{1}{2} e^4(x,t) \Big] \end{aligned} \tag{2.46}$$

This is justified since the ϕ^2 term is the most relevant, and thus could be considered as the standard energy operator. Then, a careful analysis should

be required to see the relevance of the field e in Hamiltonian (2.46). Two Langevin equations could then be written:

$$\partial_t \phi(x,t) = -\frac{\Omega}{2}\frac{\delta \mathcal{H}}{\delta \phi(x,t)} + \nu(x,t) \tag{2.47a}$$

$$\partial_t \phi(x,t) = \frac{\Omega'}{2}\nabla\left(\nabla\frac{\delta \mathcal{H}}{\delta e(x,t)} + \nu'(x,t)\right) \tag{2.47b}$$

the noises variances are:

$$\langle \nabla_x \nu(x,t)\nabla_{x'}\nu(x',t')\rangle = \delta(t-t')\Delta_x\delta(x-x') \tag{2.48a}$$

$$\langle \nu'(x,t)\nu'(x',t')\rangle = \delta(t-t')\delta(x-x') \tag{2.48b}$$

We will not analyse very profoundly Model C, since it has nothing to do with our discussion. We also cite Model E, that is a modification of Model C. The real scalar field in Model C must be substituted with a complex field and the Langevin equations read:

$$\partial_t \phi(x,t) = -\frac{\Omega}{2}\frac{\delta \mathcal{H}}{\delta \phi(x,t)} - is\frac{\delta \mathcal{H}}{\delta e(x,t)} + \nu(x,t) \tag{2.49a}$$

$$\begin{aligned}\partial_t \phi(x,t) =& \frac{\Omega'}{2}\nabla\left(\nabla\frac{\delta \mathcal{H}}{\delta e(x,t)} + \nu'(x,t)\right) + \\ & is\left[\phi^*\frac{\delta \mathcal{H}}{\delta \phi(x,t)} - \phi\frac{\delta \mathcal{H}}{\delta \phi^*(x,t)}\right]\end{aligned} \tag{2.49b}$$

2.5 Field theory for driven lattice gas models

In what follows we describe the field theoretic point of view in lattice gas models. We will first derive the Langevin equations that should describe driven lattice gases. Then we analyse the Gaussian structure of such equations. In the end, we turn to the study of critical behaviour, the main aim of this chapter.

2.5.1 Derivation of the equations

We would like to obtain field theories to describe our lattice gas models at mesoscopic level. In other words, we would like to build a theory that describe the **long range** behaviour of observables. We would like to pass from a microscopic spin variable s_x with $x \in \Lambda$ to a mesoscopic field $s(x)$, integrating out some of the initial degrees of freedom of the system. Such a procedure

is called *coarse graining* First of all, the case of $E = 0$ is well - known. It is possible to find a rigorous procedure to obtain the Model B Langevin equation (2.42) from the microscopic dynamics of the Ising lattice gas. We could not give an account of such a procedure here. Despite many efforts, this has not yet generalised to non - equilibrium systems, not even in the subclass of driven lattice gases. A easier way to obtain a Langevin Equation is to follow the ideas Landau and others developed to explain critical phenomena [39]. We could write down the easier, yet most general, Langevin Equation that respects the same symmetries of the models considered. The following symmetries are easily found to hold for the KLS model:

- translational symmetry in both space and time
- a combination of field reversal ($\vec{E} \leftrightarrow -\vec{E}$) and charge conjugation ($n_j \leftrightarrow 1 - n_j$) (or particle - hole exchange)
- a combination of field reversal and reflection of the coordinate parallel to the field ($x_\parallel \leftrightarrow -x_\parallel$)

In the case of IDLG, those symmetries should be taken into account when trying to write down a Langevin equation describing the essential properties of this driven lattice gas model. The starting point is the Model B Langevin equation (2.42). A first obvious modification is to include a term that respects all the symmetries and is proportional to the driving field (in his mesoscopic version, that we call ϵ):

$$\mathbf{J} = \mathbf{J}_0 + J_E \quad \text{where} \quad J_E = (1 - s^2)\epsilon \tag{2.50}$$

we introduced here a spin *field* $s(x,t)$. A important observation is the following: the field ϵ depends from E, but the dependence is not clear, since a viable coarse graining procedure has to be found yet. Some authors [13] even believe that the field ϵ go to zero when E is taken to infinity and the temperature tends to the critical temperature. The term (2.50) is not the only needed to obtain a viable field theory for IDLG. We expect the field to induce a anisotropy between its direction of application and the direction perpendicular to it. Also, the anisotropy is needed to render the field theory renormalizable, but this is a technical argument. Putting all together, we arrive at:

$$\begin{aligned}\partial_t s(\mathbf{x},t) = \Omega\Big\{ &\left[(\tau_\perp - \nabla_\perp^2)\nabla_\perp^2 + (\tau_\parallel - \partial_\parallel^2)\partial_\parallel^2 - 2\alpha\partial_\parallel^2\nabla_\perp^2\right] s(\mathbf{x},t) \\ &+ \frac{u}{3!}\left(\nabla_\perp^2 + \kappa\partial_\parallel^2\right) s^3(x,t) + \epsilon\partial_\parallel s^2(x,t)\Big\} - \left[\vec{\nabla}_\perp\vec{\nu}(\mathbf{x},t) + \partial_\parallel\nu(\mathbf{x},t)\right]\end{aligned} \tag{2.51a}$$

and the (Gaussian) noise ratify the following relations :

$$\langle \nabla_\perp \nu(x,t) \nabla'_\perp(x',t') \rangle = n_\perp \nabla_\perp^2 + n_\parallel \partial_\parallel^2 \delta(x-x')\delta(t-t') \tag{2.51b}$$

In general it is true that:

$$\frac{n_\perp}{n_\parallel} \neq \frac{\tau_\perp}{\tau_\parallel} \tag{2.52}$$

For RDLG, charge conjugation and reflection of the coordinate along the field hold *separately* from field reversal. So a different Langevin Equation is found:

$$\begin{aligned} \partial_t s(\mathbf{x},t) = \Omega \Big\{ & \left[(\tau_\perp - \alpha_\perp \nabla_\perp^2)\nabla_\perp^2 + (\tau_\parallel - \alpha_\parallel \partial_\parallel^2)\partial_\parallel^2 - 2\alpha \partial_\parallel^2 \nabla_\perp^2 \right] s(\mathbf{x},t) \\ & + \frac{u}{3!} \left(\nabla_\perp^2 + \kappa \partial_\parallel \right) s^3(x,t) \Big\} - \left[\vec{\nabla} \vec{\nu}(x,t) + \partial_\parallel \mu(x,t) \right] \end{aligned} \tag{2.53}$$

Again, the noise is described by the Gaussian functional distribution (2.51b).

2.5.2 Gaussian theory: algebraic power law decay far from criticality

We notice that at Gaussian level the two Langevin equations (2.51a) and (2.53) are exactly the same. We start our analysis from that Gaussian part, that is quite simple to handle. With the techniques developed in (2.2), we could write down the following generating functional:

$$\begin{aligned} J[s,\tilde{s}] = \int d^d x\, dt\, \tilde{s}(\mathbf{x},t) \partial_t s(\mathbf{x},t) + \tilde{s}(\mathbf{x},t) \Omega \left[(\tau_\perp - \alpha_\perp \nabla_\perp^2)\nabla_\perp^2 + \right. \\ \left. (\tau_\parallel - \alpha_\parallel \partial_\parallel^2)\partial_\parallel^2 - 2\alpha \partial_\parallel^2 \nabla_\perp^2 \right] s(\mathbf{x},t) - \Omega \left(n_\perp \nabla_\perp^2 + n_\parallel \partial_\parallel^2 \right) \tilde{s}^2(\mathbf{x},t) \end{aligned} \tag{2.54}$$

This functional is supposed to approximate significantly the DLG's behaviour in the high temperature phase, far from T_c. It is very interesting because it explains one of the striking features of the field driven lattice models. We could write the correlator in Fourier transform:

$$S(\mathbf{k},\omega) = \mathcal{G}_{2,0}(\mathbf{k},\omega) = \frac{n_\perp k_\perp^2 + n_\parallel k_\parallel^2}{\omega^2 + \Xi^2} \tag{2.55a}$$

where the factor Ξ reads:

$$\Xi = \Omega \left(\tau_\perp k_\perp^2 + \alpha_\perp k_\perp^4 + \tau_\parallel k_\parallel^2 + \alpha_\parallel k_\parallel^4 + \alpha k_\perp^2 k_\parallel^2 \right) \tag{2.55b}$$

We are interested in the static $t = 0$ limit, that reads:

$$S(\mathbf{k},0) = \int \frac{d\omega}{2\pi} S(\mathbf{k},\omega) = \frac{\mathbf{k}\mathbb{N}\mathbf{k}}{2\mathbf{k}\mathbb{D}\mathbf{k}} \tag{2.56}$$

We have discarded the k^4 terms in (2.55b), since we are interested in the long range limit behaviour of S. The two matrices $\mathbb{D}$ and $\mathbb{N}$ are diagonal, but not multiples of the identity. Moreover, from (2.52), we conclude that no further simplification is possible in (2.56). We could now backtrasform the static structure factor, so that after some calculations we find:

$$G(\mathbf{x},0) \sim \int \frac{d\omega}{2\pi} \frac{\mathbf{k}\mathbb{N}\mathbf{k}}{2\mathbf{k}\mathbb{D}\mathbf{k}} = \frac{\mathbf{x}\hat{\mathbb{M}}\mathbf{x}}{|\mathbf{x}|^d} \tag{2.57}$$

We have introduced the matrix

$$\mathbb{M} = \mathbb{D}^{-1/2}\mathbb{N}\mathbb{D}^{-1/2} \tag{2.58}$$

and we defined $\hat{\mathbb{M}}$ its traceless part. The result is that the structure factor decays *algebraically* even far from the correlation function. The "normal" definition of correlation length based on the exponential decay of a correlation function here fails. We remark that is still possible to obtain an exponential decay, but we must average the function S over the solid angle before backtrasforming in real space.

2.5.3 Critical behaviour

We now turn to critical behaviour for the driven lattice gases. Since we are interested in the critical behaviour, we could simplify considerably the two equations (2.51a) and (2.53), to get the functionals

$$J[s,\tilde{s}] = \Omega \int dt\, d^d x\, \{\tilde{s}[\lambda^{-1}\partial_t + \nabla_\perp^2\left(\nabla_\perp^2 - \tau_\perp\right) - \tau_\parallel \partial_\parallel^2]s + \epsilon(\partial_\parallel \tilde{s})s^2 - \tilde{s}\nabla_\perp^2\tilde{s}\} \qquad \text{(IDLG)} \tag{2.59a}$$

$$J_c[s,\tilde{s}] = \Omega \int dt\, d^d x\, \{\tilde{s}[\lambda^{-1}\partial_t + \nabla_\perp^2\left(\nabla_\perp^2 - \tau_\perp\right) - \tau_\parallel \partial_\parallel^2]s + u(\tilde{s}\nabla_\perp^2 s^3) - \tilde{s}\nabla_\perp^2\tilde{s}\} \qquad \text{(RDLG)} \tag{2.59b}$$

Note that in functional (2.59a) we discarded the irrelevant operator $\tilde{s}\nabla_\perp^2 s^3$. such operator is dangerously irrelevant, and should be properly handled through insertions in general n - point functions. Those equations seem similar, nevertheless a great difference is encountered when a ε expansion [34], in the framework of renormalization group [32], is set up. An important remark concerning functional (2.59a) is that other symmetries, apart from the original ones, could be found. Those are:

- $s \to \alpha s \quad \tilde{s} \to \alpha \tilde{s} \quad x_\parallel \to \alpha^{-2} x_\parallel \quad \tau_\parallel \to \alpha^{-4} \tau_\parallel \quad \epsilon \to \alpha^{-3} \epsilon$
- $s \to \beta s \quad \tilde{s} \to \beta^{-1} \tilde{s} \quad \epsilon \to \beta^{-1} \epsilon$
- the "Galilean" symmetry, i.e. $x_\parallel \to x_\parallel + 2\lambda \epsilon a t \quad s \to s + a \quad \tilde{s} \to \tilde{s}$

We could now turn to a scaling analysis that is necessary near the critical point. The Gaussian theory is no more sufficient near this point, since there are fluctuations on all the scales. To obtain sensible results the renormalization group must be used. The "galilean" symmetry in equation (2.59a) allows exact results to be obtained [8, 16]. In case of functional (2.59b), this powerful symmetry no longer is respected, so the renormalization group procedure is only valid up to a second order in ε. Further details could be found in refs. [30]. For RDLG a good reference is also [29]. Consider the IDLG first. We will only quote the most important results here: from solving the differential renormalization group equations, we obtain for the vertex functions:

$$\begin{aligned} \Gamma_{n,\tilde{n}}(k_\parallel, \vec{k_\perp}, t, \tau, u, L_\parallel, L) = \lambda^{-(d+\Delta+\frac{d+2+\Delta}{2}+\frac{d-2+\Delta}{2})} \\ \Gamma_{n,\tilde{n}}(\lambda^{1+\Delta} k_\parallel, \vec{k_\perp}\lambda, t\lambda^{-4}, \tau\lambda^2, \lambda^{-2\sigma} u) \end{aligned} \tag{2.60}$$

Here the exponents in (2.60) could be calculated exactly. The independent exponent reads:

$$\beta = 1/2; \quad \nu_\perp = 1/2; \quad \eta = 0; \quad \tilde{\eta} = 0; \quad z_\perp = 4 \quad \Delta = \frac{8-d}{3} \tag{2.61}$$

For RDLG, a similar result hold:

$$\begin{aligned} \Gamma_{n,\tilde{n}}(k_\parallel, \vec{k_\perp}, t, \tau, u, L_\parallel, L) = \lambda^{d+1-\frac{n}{2}(d-1+\eta/2)-\frac{\tilde{n}}{2}(d+3+3/2\eta)} \\ \Gamma_{n,\tilde{n}}(\lambda^{1+\Delta} k_\parallel, \vec{k_\perp}\lambda, t\lambda^{-4}, \tau\lambda^2, \lambda^{-2\sigma} u) \end{aligned} \tag{2.62}$$

We only have second order in ε results for the RDLG. We remember that $\varepsilon = d_c - d$ where d_c is the upper critical dimension. The independent exponents here reads:

$$\eta = \frac{4}{243}\varepsilon^2 \tag{2.63}$$

$$\nu_\perp = \frac{1}{2} + \frac{\varepsilon}{12} + \frac{\varepsilon^2}{18}\left[\frac{67}{108} + \ln\frac{2}{\sqrt{2}}\right] \tag{2.64}$$

Some observations are in order: in both cases 2 is the lower critical dimension, so we could expect logarithmic corrections to power law. Those corrections where not studied up to now and they are believed to be small. Hence we could neglect them. Another important point is the possible presence of non

- perturbative corrections, again no studies demonstrates their importance. A final remark: much disagreement still remains on the universality class of the IDLG. Recently, there has been proposals that sustain the idea that both the IDLG and the RDLG belong to the same universality class. The argument is that the field term (2.50) goes exactly to zero when the driving field goes to infinity. In that case, of course, the two dynamic functional (2.59a) and (2.59b) will coincide.

In next chapter we will see what are the main consequences of scaling forms are. In particular we will explain better the strongly anisotropic character of (2.62) and (2.60)

2.5.4 Strong anisotropy

The scaling forms (2.62) and (2.60) have a very particular structure: in fact it is found that momenta (the same is true for lengths) scales as

$$k_{\|} \sim k_{\perp}^{1+\Delta} \tag{2.65}$$

even some equilibrium systems display such a property [12], and there are simple models to explain it [9]. Critical dynamical systems , that we explained in section also are an example of such a behaviour: we could take time as a special "space" variable, so that z could be written as [9]:

$$z = 1 + z' \tag{2.66}$$

Strong anisotropy gives rise to interesting problems, see [6], and it complicates a bit the exponent description for both IDLG and RDLG. In fact, many different exponents exists. we will mention here only the ones that are useful to our discussion and ignore the others. The fundamental scaling law is related to the two - point function, that is well defined in both field theory approach and in simulations. Remember that an since two different field exist, s and $\tilde{s}$, two different functions are important, the so called correlation function $< ss >$ and the response function $< s\tilde{s} >$. Apart to some brief hints, we will deal exclusively with the correlation function. It is usually preferred to use the Fourier transform of the correlation function, also called the structure factor and denoted as S, that we already used. Form the scaling forms we derive

$$S(k_{\perp}, k_{\|}, \tau, t) = \lambda^{-2+\eta} S(k_{\perp}\lambda^{-1}, k_{\|}\lambda^{-(1+\Delta)}, \tau\lambda^{1/\nu}, \lambda^{z}t) \tag{2.67}$$

Using the standard substitution $\lambda = \tau^{\nu}$ notice that *two* correlation length appear now so two different ν's are found:

$$\xi_{\perp} = \tau^{\nu_{\perp}} \qquad \xi_{\|} = \tau^{\nu_{\|}} \qquad \nu_{\|} = (1+\Delta)\nu_{\perp} \tag{2.68}$$

The same procedure could be carried on substituting λ with $t^{1/z_b ot}$ so that two z's could be found, as related to critical slowing down:

$$z_{\|} = (1+\Delta) z_{\perp} \tag{2.69}$$

If we restrict to critical ($\tau = 0$) behaviour, the static($t = 0$) structure factor reads:

$$S(k_{\perp}, k_{\|}) = k_{\perp}^{-2+\eta_{\perp}} S\left(\frac{k_{\|}}{k_{\perp}^{1+\Delta}}\right) \quad \text{where} \quad \eta_{\perp} = \eta \tag{2.70a}$$

$$S(k_{\perp}, k_{\|}) = k_{\perp}^{-2+\eta_{\|}} S\left(\frac{k_{\perp}}{k_{\|}^{1/(1+\Delta)}}\right) \quad \text{where} \quad \eta_{\|} = \frac{\eta + 2\Delta}{1+\Delta} \tag{2.70b}$$

Some other consequences are important: when transforming from the momenta representation to real space, we need to take into account strong anisotropy: the volume term $d^{d-1} K_{\perp} dk_{\|}$ scales as $d + \Delta$, as a consequence of (2.65). So for example the 2 - point static function in real space scales as (at criticality)

$$G(r_{\perp}, r_{\|}) = r_{\perp}^{-2+\eta_{\perp}} G\left(\frac{r_{\|}}{r_{\perp}^{1+\Delta}}\right) \quad \text{where} \quad \eta'_{\perp} = \eta + \Delta \tag{2.71a}$$

$$G(r_{\perp}, r_{\|}) = r_{\perp}^{-2+\eta'_{\|}} G\left(\frac{r_{\perp}}{r_{\|}^{1/(1+\Delta)}}\right) \quad \text{where} \quad \eta'_{\|} = \frac{\eta - \Delta(d-3)}{1+\Delta} \tag{2.71b}$$

In previous scaling laws we called $r_{\perp} = |x_{\perp}|$ and $r_{\|} = \left|x_{|}\right|$. We now turn to the susceptibility. We have seen that in dynamic critical phenomena two possible definitions of the susceptibility exist, namely the static and the dynamic susceptibility. Due to strong anisotropy, we could also distinguish between a parallel and a transverse static susceptibility:

$$\chi_{\perp} = S\left(k_{\perp} \to 0, k_{\|} = 0\right) \tag{2.72a}$$

$$\chi_{\|} = S\left(k_{\perp} = 0, k_{\|} \to 0\right) \tag{2.72b}$$

If we *assume* those limits are finite and non - vanishing, we obtain:

$$\chi_{\perp,\|} \sim \tau^{-\gamma_{\perp,\|}} \qquad \gamma_{\perp,\|} = \nu(2-\eta) \tag{2.73}$$

However this assumption fails for IDLG, and the exponent $\gamma_{\|} = 0$ Finally a important remark about the the magnetisation, that should obey to:

$$m \sim (-\tau)^{-\beta} \qquad \tau < 0 \quad \text{as} \quad -\tau \to 0 \tag{2.74}$$

Usually one could exploit the fact that $S \sim s^2(k)$, so that with the definition (1.24), the order parameter exponent should obey to:

$$\beta = 1/2\nu(d + \Delta - 2 + \eta) \tag{2.75}$$

This law is not verified in IDLG, where $\beta = 1/2$.

2.5.5 Finite size scaling in strongly anisotropic systems

It is now clear the role simulations should have: to discriminate between different field theories and to test the previsions made with them. However, the maximum size a lattice usually reach in computer simulations is about 10^4 sites. Under very broad assumptions, in such a system no phase transitions could be found and all thermodynamic quantities are analytic. To make sensible comparisons between the data of such a simulation and results from field theories we need to know how the thermodynamic limit is reached.

In equilibrium a standard technique could be applied,called finite - size scaling [3]. The main idea of finite size scaling is that a diverging (at T_c) observable that obeys to scaling law (2.15) should instead scale as:

$$O(\vec{k}, t, \tau) = \lambda^{x_O/\nu} O(\vec{k}\lambda, t\lambda^{-z}, \tau\lambda^{\frac{1}{\nu}}, L\lambda^{-1}) \tag{2.76}$$

where L is a typical size of the system considered. Now it is clearer how results have to be extracted from simulations: we could simply use lattices of different sizes, and then apply the scaling form (2.76). We can not give a full motivation for such a behaviour, but it is quite intuitive if one bears in mind the main idea of renormalization group. This idea is to re-scale a space (or momenta) variable of the system, in order to understand how an observable changes under that flow. It is quite clear that being L a length, it should scale by definition as in (2.76).

The last statement should however be revised accurately and is not always true. Violations of this simple behaviour have been observed [4]. Even the models we are considering display a quite curious behaviour: we remember that a system with strong anisotropy is characterised by *two* lengths:

$$\xi_{\parallel} \sim \xi_{\perp}^{1+\Delta} \tag{2.77}$$

This is a natural consequence of (2.68). We caution that the scaling law (2.76) is in general valid because the correlation length in a finite size system reach at maximum the length of the "box" in which the system is enclosed. If we vary more than one parameter at the same time in (2.76), the results could not be the ones we expect. In strongly anisotropic systems, where two correlations lengths that scale differently are considered, one must be even more cautious. In general, it seems natural to consider:

$$L_{\parallel} \sim L_{\perp}^{1+\Delta} \tag{2.78}$$

This way the finite size correlation lengths should reach the right limit when $L_{\parallel}$ is taken to infinity. The finite size scaling law for a anisotropic system must be modified:

$$O(\vec{k}, t, \tau) = \lambda^{x_O/\nu} O(\vec{k}\lambda, t\lambda^{-z}, \tau\lambda^{\frac{1}{\nu}}, L_{\perp}\lambda^{-1}, L_{\parallel}\lambda^{-(1+\Delta)}) \tag{2.79}$$

The question is: what happens if one uses lattices in which the relation (2.78) does not hold? In general, in fact, one could choose lattices that respect:

$$L_{\parallel} \sim L_{\perp}^{1+\Delta+\delta} \tag{2.80}$$

Recently, some progress have been made on this point [6], but much debate is still on concerning the interpretation of IDLG data. The problem is that Δ could not be known *in advance*, so in doing the simulations fixing Δ could bias the final result.

Chapter 3

Short-time scaling

Short - time scaling is a method, introduced more than a decade ago by Jannsen and Schnittmann [18], to extract critical exponent analysing the evolution of a system from a certain state towards equilibrium. It was first developed in the equilibrium framework, but generalisations now exists. The advantages of this technique are well known, namely that very short simulations are needed to obtain critical exponents, while equilibrium (non equilibrium steady - state in our case) or longer - time methods usually are very demanding in computer time. Another important feature of short - time methods is the presence of new physics in some systems, like Model A. This means a new independent exponent could be found. However, this is not always the case and we could also obtain the "old"exponents using this new technique.

3.1 A qualitative look at dynamical properties

The dynamical properties of a system are a topic of general interest in statistical physics. It has became slowly more and more clear that such properties are important to complete the vision on critical phenomena. Universality, in particular, could be discussed in the framework of dynamical criticality. Moreover, even when studying static phenomena, dynamics is very important in a practical sense, because it gives an estimation on what is the typical relaxation time: this information has deep practical consequences in simulations. In quite recent times, a wealth of interesting properties has been discovered in dynamical context; a side effect is that the matter has became increasingly complicated. To help the reader in distinguishing the various time regimes, we tried to summarise most of the ideas that emerged trough

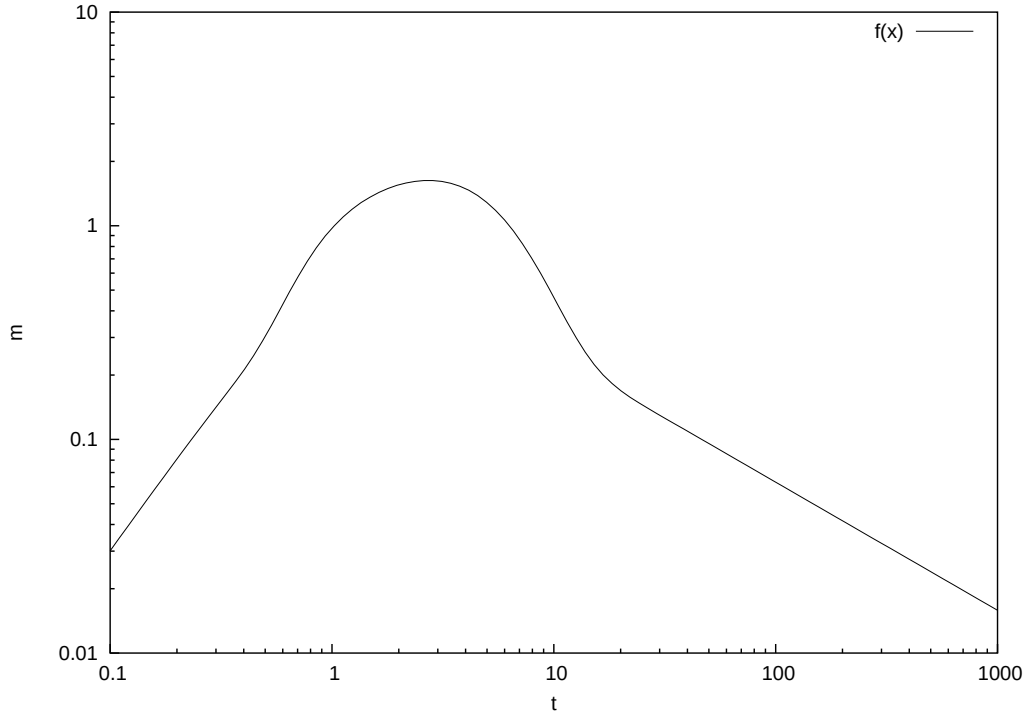

Figure 3.1: A qualitative plot of the magnetisation as function of t. Notice the critical initial increase followed by a transition to the more usual non-linear relaxation.

the years. Let me start with the following figure (3.1), that includes all the typical regimes a system undergoes through dynamics. We caution the reader the figure is only qualitative. Since we are studying magnetic systems, or similar, we will describe here the behaviour of the fundamental quantity, the magnetisation.

After a system is prepared in some state (this point is important, I will return on this later on), the system evolves for a very brief time in a microscopic fashion. This means that only a microscopic theory could in general make sensible predictions on this regime. The microscopical time scale, as it is called, it is usually very short and does not influence in general later behaviour.

After a macroscopically short time, the system settles down in the so - called short time scaling. It is the main argument of this chapter, so we will not speak diffusely of it now. It is characterised by a power law behaviour and it is influenced by the initial magnetisation.

The system then enters in a transition regime, that is not very studied nor well understood. After that, we find another critical non liner scaling

behaviour; it is characterised by the power law:

$$m \sim t^{-\beta/\nu z}$$

even if a regime like this it is known to exist and is well documented in literature, it is not so clear when it starts and how is influenced by the other regimes. To end my discussion, we will mention the last time regime: it is exponential and characterised by the presence of the correlation length ξ. When we are at the critical temperature, such a regime should disappear, so that the nonlinear relaxation extends to equilibrium.

In our description, we did not mention the effects of finite size on the dynamics. It seems that should not affects the short time behaviour, even if in some cases this could no more be true. Finite size surely modify the longer time behaviour, so that a exponential decay is now expected even at the critical temperature. Finite size dynamics is far richer and could not be explained in detail a good reference is [17]

3.2 The general idea

The basic idea of short - time is that a new region of critical behaviour exists, and that it lies in the vicinity of zero time. What does this precisely mean? Suppose to prepare a magnetic system (i.e.. an Ising system) with a certain magnetisation m_0 and at very high temperature. In this initial state correlations are practically zero, and we also need an external magnetic field to sustain the non - zero magnetisation. Now suppose we perform a rapid quench at temperature T_c, switching off the magnetic field as well. How does the magnetisation behaves under such a quench? In a infinite system, it obeys to the scaling law at criticality [18]:

$$m(t, m_0) = t^{\theta'} f_m(t^{\theta'+\beta/\nu z} m_0) \tag{3.1}$$

A similar scaling form hold for the structure factor:

$$S(t, t', q) = q^{-2+\eta} \left(\frac{t}{t'}\right)^{\theta-1} f_S(q\xi, q^z t) \tag{3.2}$$

A magnetic system like the one described before could be well described in its critical properties by the Ginzburg - Landau Hamiltonian:

$$\mathcal{H}[s] = \int d^d x \frac{1}{2} \left\{ (\nabla s)^2 + \tau s^2 + \frac{g}{6} s^4 \right\} \tag{3.3}$$

If no conservation law hold, and the dynamic is purely relaxational, this Hamiltonian is associated with the Model A dynamic functional [15]:

$$\mathcal{J}[s,\tilde{s}] = \int dt \int d^d x \tilde{s} \left[\dot{s} + \lambda \frac{\delta \mathcal{H}}{\delta s} - \lambda \tilde{s} \right] \tag{3.4}$$

However, we need to include the initial conditions, and to have a mean over them. We modify the functional:

$$\begin{aligned} &\mathcal{J}_i[s,\tilde{s}] = \mathcal{J}[s,\tilde{s}] + \mathcal{H}_i[s_0] \qquad \text{where} \quad \mathcal{H}_i[s_0] = \tau_0^{-1} \int d^d x s_0(x) \\ &\quad \text{and} \quad s_0(x) = s(x, t=0) \end{aligned} \tag{3.5}$$

This functional is the starting point of a complex analysis, based on the renormalization group, that demonstrates the scaling law could be obtained at least perturbatively. Some details are given in the following, but for a more precise and rigorous treatise see [18]

As we have seen, a new exponent θ exists, and enter all the expressions above. The exponent θ', instead, could be obtained from:

$$\theta' = 2 - \eta - \theta \tag{3.6}$$

Let us notice that if θ' is greater than 0, a rather counter - intuitive and peculiar behaviour shows up: the magnetisation **increases** near zero time; then, after a transient, it sets down in the critical relaxation at long time characterised by the exponent $\beta/z\nu$. This ideas are obtained through a quite complicated renormalization group calculation, that we could not entirely revise here. Let us only sketch the procedure to obtain the results we exposed before.

From general renormalization theory, we recognise the importance of primarily divergent vertex functions (or connected correlation function): they are the objects to renormalised in order to obtain a well - defined theory. In particular, we are not discussing a completely translationally invariant theory, so we must study the correlation function directly, instead of discussing the vertex functions. There are many other subtleties in renormalization theory, that we will not discuss here. Suppose we could cure the divergences of the theory only taking care of the primarily divergent functions. We wish to discuss their short time behaviour. As it will be clear in what follows, in this limit the degree of divergence of correlation functions is **reduced** by z, because we are expanding the fields using a short distance expansion [9, 33]. Two cases are possible: if all the primitive divergences disappears in short - time limit (i.e., all superficial degrees of divergences becomes negative as a

consequence of this limit), the theory becomes in a sense trivial, in the sense that no new exponents are found in short time regime.

If the primitive divergence could not be completely absorbed by the short - time limit, we need to reconsider the whole procedure. This case is very interesting, but also more complicated. In particular, a new divergence is found on the temporal surface $t = 0$, so that we must set up a renormalization group analysis for general connected correlators:

$$\mathcal{G}_{n,\tilde{n},\tilde{m}}(x_i, t, \tau) = \left\langle \prod_{i=1}^{n} \prod_{j=1}^{\tilde{n}} \prod_{k=1}^{\tilde{m}} s(x_i, t_i)\tilde{s}(x_j, t_j)\tilde{s}(x_k, t = 0) \right\rangle_{conn} \tag{3.7}$$

This function contains also m fields of type $\tilde{s}$ at zero time. A relation hold between the two fields $\tilde{s}$ and s:

$$\partial_t s(x, t) \sim \tilde{s}(x, t = 0) \tag{3.8}$$

This relation is true in weak sense, only if inserted in Green's functions. The details of this calculation are long and involved, so we will not refer them here. The result is:

$$\mathcal{G}_{n,\tilde{n},\tilde{m}}(x_i, t_i, \tau) = \lambda^{nD_1+(\tilde{n}+\tilde{m})D_2+\tilde{m}\,\eta_0}\, G(x_i\lambda, t_i\lambda^{-z}, \tau\lambda^{1/\nu}) \tag{3.9}$$

where we defined:

$$D_1 = [s] - \eta \tag{3.10a}$$

$$D_2 = [\tilde{s}] - \eta \tag{3.10b}$$

The relation (3.9) is valid when all couplings have been set to their critical values. We indicated as usual η and $\tilde{\eta}$ the anomalous dimensions of the fields s and $\tilde{s}$ The new exponent η_0 that appears indicates the anomalous dimension of $\tilde{s}_0$. This is the real novelty of short time scaling. We could now expand the fields at short time:

$$s(\mathbf{x}, t) = \sigma(t)\partial_t s(t)|_{t=0} \tag{3.11a}$$

$$\tilde{s}(\mathbf{x}, t) = \tilde{\sigma}(t)\tilde{s}(t = 0) \tag{3.11b}$$

Introducing the expansions (3.11a) in (3.9) we could obtain a scaling expression for the fields σ and $\tilde{\sigma}$, if we remember that $t \sim \lambda^{-z}$:

$$\sigma(t, \tau) = \lambda^{-z-\eta_0}\, \sigma(t\lambda^z, \tau\lambda^{-1/\nu}) \tag{3.12a}$$

$$\tilde{\sigma}(t, \tau) = \lambda^{-\eta_0}\, \sigma(t\lambda^z, \tau\lambda^{-1/\nu}) \tag{3.12b}$$

From those expressions is is quite easy to arrive to result (3.2). Instead much more work is needed to obtain the scaling law for the magnetisation (3.1), because also the initial field a must be taken into account. Moreover, WT identities for various operator insertion must be derived.

To get numerical results out of this general setting, we should use perturbation expansion and renormalization group explicitly. For Model A the framework is quite well established: an ϵ - expansion has been found [18], some $O(\infty)$ exact calculation has been carried on to show extensions to finite systems are possible [17,31] and many simulations corroborate analytical results. Many analytical extensions of this methods to different systems now exist, [2,26,36], but most authors concentrated on simulations [24,25,37].

3.3 Extension to finite systems

An extension of previously discussed ideas to finite systems has been made, see refs. [17,31]; the main ideas contained in this section are derived from there. This point is fundamental for our discussion, because we would like to obtain sensible comparisons between simulations and field theory. Indeed, some cautions must be held when extending this results to strongly anisotropic systems. Here we will present a discussion of phenomenological short - time finite - size scaling. A justification of these results are beyond the scope of this work. The basic scaling law for an observable is:

$$\mathcal{O}(t,\tau,L,m_0) = \lambda^{\gamma_\mathcal{O}/\nu} f_\mathcal{O}(\lambda^{-z}t, \lambda^{1/\nu}\tau, \lambda^{-1}L, \lambda^{x_O}m_0) \tag{3.13}$$

for now we will retain the exponent x_O, that characterises the short - time scaling behaviour for the magnetisation, to show the effect of initial magnetisation on time scales. To extract the critical behaviour we could set $\lambda = t^{1/z}$, so that:

$$\mathcal{O}(t,\tau,L,m_0) = t^{\gamma_\mathcal{O}/\nu z} g_\mathcal{O}(t^{1/\nu z}\tau, t^{-1/z}L, t^{x_O/z}m_0) \tag{3.14}$$

We could re - write this as:

$$\mathcal{O}(t,\tau,L,m_0) = t^{\gamma_\mathcal{O}/\nu z} g_\mathcal{O}(t/t_\tau, t/t_L, t/t_i) \tag{3.15}$$

this scaling form individuates different *time scales*, each associated with a particular scaling variable. The scales are:

$$t_\tau \sim \tau^{-1/\nu z}$$

$$t_L \sim L^z$$

$$t_i \sim (m_i)^{-z/\nu}$$

Some comments are in order: the first length is associated with the correlation length in bulk systems (remember that $\xi \sim \tau^{-\nu}$, and we could also write $t_\tau \sim \xi^{1/z}$, while the second length is typical of finite systems and the third is related with initial behaviour of the magnetisation. The most important consequences of this scaling form are:

- if m goes to zero, the time scale t_i goes to infinity.
- at criticality, t_τ goes to infinity too, and the relaxational properties are only due to finite - size effects.

To extract further information from finite - size scaling laws typical of such systems the following observation is fundamental: nearby $t = 0$ the spins are usually not correlated, or the correlation is very short -ranged. In other words, we could suppose the system is in a very high temperature zone. So, we are expected to know how the thermodynamic limit is reached in this region. Suppose that

$$\mathcal{O} \sim L^{-r}$$

It is obvious, from :

$$t \to 0 \qquad \text{is equivalent to} \qquad L^z \to \infty \tag{3.16}$$

If we know how the thermodynamic limit is reached (that is, if we know how $\mathcal{O}$ behaves as $L \to \infty$) we could collect all the ideas exposed and write down an important scaling form:

$$\mathcal{O}(t, \tau, L, m_0) \sim t^{r/z+\gamma_\mathcal{O}/\nu z} L^{-r} g^*_\mathcal{O}(t^{1/\nu z}\tau, t^{x_O/z} m_0) \tag{3.17}$$

The scaling function g^* is the bulk scaling function; we could re - obtain the section results noting that when $t \to 0$ the magnetisation must approximate the initial value m_i, so that for the magnetisation the following rule hold:

$$\lim_{\theta_i \to 0} g^*_m(\theta_\tau, \theta_i) = g^{**}_m(\theta_\tau)\, \theta_i^{x/z} \tag{3.18}$$

When all those relations are taken into account, we get a initial increase exponent for finite systems magnetisation:

$$c_i = \frac{r_m + x - \beta/\nu}{z} \tag{3.19}$$

Note that some caution must be held when taking the limits as in equation and: in real cases, the limit $t \to 0$ could not be reached exactly due to the presence of a *microscopical time scale* t_{mic}, under which the scaling form may not be true any more. As a general prescription, is better to use big lattices in simulations: due to the the presence of t_{mic}, the relation could fail when too small lattices are used.

3.3.1 The case of strongly anisotropic systems

To include strong anisotropy in this framework a natural extension of scaling law is needed. As before, we will start from a general operator $\mathcal{O}$. The simplest extension of equation reads:

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, m_0) = \lambda^{\gamma_\mathcal{O}/\nu_\perp} f_\mathcal{O}(\lambda^{-z_\perp} t, \lambda^{1/\nu_\perp}\tau, \lambda^{-\nu_\parallel/\nu_\perp} L_\parallel, \lambda^{-1} L_\perp, \lambda^{x_i} m_i) \tag{3.20}$$

As in the preceding section, we could set $\lambda = t^{1/z_\perp}$, and individuate the fundamental lengths. We get to:

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, m_0) = t^{\gamma_\mathcal{O}/\nu_\perp z_\perp} g_\mathcal{O}(t/t_\tau, t/t_{L_\parallel}, t/t_{L_\perp}, t/t_i) \tag{3.21}$$

Now *two time lengths* $t_{L_\perp} = L^{z_\perp}$ and $t_{L_\parallel} = L^{z_\parallel}$ are important, following the basic idea of strong anisotropy. We could also write the lengths as, simply using the definition of the form factor $S_\Delta = \frac{L_\parallel^{1/(1+\Delta)}}{L_\perp}$:

$$t_{L_\parallel} \sim S_\Delta t_{L_\perp} \tag{3.22}$$

For simplicity, let me denote $t_{L_\perp} = t_L$, so that we could re - write the scaling law as:

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, m_0) = t^{\gamma_\mathcal{O}/\nu_\perp z_\perp} g_\mathcal{O}(t/t_\tau, t/S_\Delta t_L, t/t_L, t/t_i) \tag{3.23}$$

If we need to find the usual exponents like β, ν or z, we must rule out the exponent x_i. This could be done preparing the system at $t = 0$ with zero magnetisation. Even if the exponent x_i is negative so that initial magnetisation is irrelevant, a non - zero initial magnetisation could yield corrections to scaling. Setting at zero m_0, we obtain:

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, 0) = t^{\gamma_\mathcal{O}/\nu_\perp z_\perp} G_\mathcal{O}(t/t_\tau, S_\Delta, t/t_{L_\perp}, 0) \tag{3.24}$$

At this point, we could proceed as before, provided we know how the operators we are interested in reach the thermodynamic limit in the short - time scaling region. It is quite obvious to suppose power law behaviour, in general different for $L_\parallel$ and $L_\perp$ (or $L_\parallel$ and S_Δ):

$$\mathcal{O}_i \sim L_\perp^{-x} S_\Delta^{-y}$$

As before, we have a scaling function independent from the sizes, that could be identified with the bulk scaling function:

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, 0) = t^{x/z_\perp + \gamma_\mathcal{O}/\nu_\perp z_\perp} L_\perp^{-x} S_\Delta^{-y} G^{**}_\mathcal{O}(t/t_\tau) \tag{3.25}$$

In the previous equation we have used the fundamental hypothesis of finite - size scaling and we have called:

$$\lim_{L_\perp, L_\parallel \to \infty, S_\Delta = c} G_{\mathcal{O}}(t/t_\tau, S_\Delta t_{L_\perp}, t/t_{L_\perp}, 0) = g^{**}_{\mathcal{O}}(t/t_\tau)$$

The limit above is the right one to be taken, however, at contrary with what discussed in the Introduction, we do not have here the problem of the identification of the right Δ. In fact, we are discussing the limit $t \to 0$. Let us emphasise that the time scales are independent from the particular shape we choose for lattices in simulations. We already made clear in previous chapters that the limit $t \to 0$ is equivalent at criticality to the limit $L \to \infty$. As we will see, this is confirmed by simulations, where we found fundamental quantities are largely independent from S_Δ at short times. From equation it is directly derived an important power law at criticality ($\tau = 0$):

$$\mathcal{O}(t, \tau, L_\parallel, L_\perp, 0) = t^{x/z_\perp + \gamma_{\mathcal{O}}/\nu_\perp z_\perp} L_\perp^{-x} S_\Delta^{-y} g^{**}_{\mathcal{O}}(0) \tag{3.26}$$

As a direct consequence, the short - time initial exponent for a general operator $\mathcal{O}$ has been found, it is:

$$c_{\mathcal{O}} = x/z_\perp + \gamma_{\mathcal{O}}/\nu_\perp z_\perp \tag{3.27}$$

A caution is important: in all this section we used *phenomenological* scaling, no deeper motivation has been given for the scaling laws studied. Next chapter should try to give more precise results based on field theory.

3.4 Field theory and short time scaling

We do not have yet any results about short-time behaviour in field theories that describe driven lattice gases. Let me first point out that for both models we are studying, the time variable follow the relation

$$t \sim \lambda^{4-\eta} \tag{3.28}$$

the only correlation function which has a primitive divergence in JSLC theory is:

$$\Gamma_{1,1}(\mathbf{k}_\perp, t) \sim \mathbf{k}_\perp^2 \tag{3.29}$$

for RDLG, we should consider also insertions of the operator $\tau_\perp \nabla_\perp^2 \tilde{s}s$ so that instead, we have:

$$\Gamma_{1,1}(\mathbf{k}_\perp, t) \sim \mathbf{k}_\perp^4 \tag{3.30a}$$

$$\Gamma_{1,3}(\mathbf{k}_\perp, t) \sim \mathbf{k}_\perp^2 \tag{3.30b}$$

$$\Gamma_{1,1,1}(\mathbf{k}_\perp, t) \sim \mathbf{k}_\perp^2 \tag{3.30c}$$

the $\Gamma_{1,1,1}$ is a function with one $\tau_\perp \nabla_\perp^2 \tilde{s}s$ insertion . It is still an unclear point if the RDLG displays a short time singularity. Imposing zero order parameter at zero time cancels any contribution coming from the anomalous dimension of the field $\tilde{s}$ at zero time. With this caution the short time scaling of both models are in a way trivial, since the fields do not have an anomalous dimension at $t = 0$.

Now we will discuss explicitly the short time scaling of multi - points correlation functions for driven lattice gases. For IDLG, we have a scaling law for any one - point irreducible n-$\tilde{n}$ - point function [5] [8, 16], that we already mentioned (2.60): this scaling law takes into account the presence of a dangerously irrelevant operator, a peculiarity of the JSLC theory. A similar law holds for RDLG (2.62). We restrict our analysis to correlation functions i.e. 2 - point functions where all fields are of s type. From expressions (2.62) and (2.60) we find following standard procedures (see for example [39, Chapter 7]):

$$\begin{aligned}\mathcal{G}_2(k_\parallel, \vec{k_\perp}, t, \tau, L_\parallel, L) = \lambda^{2-\eta} \\ \mathcal{G}_2(\lambda^{1+\Delta} k_\parallel, \vec{k_\perp}\lambda, t\lambda^{-z_\perp}, \tau\lambda^{1/\nu_\perp}, \lambda^{-1-\Delta} L_\parallel, \lambda^{-1} L_\perp)\end{aligned} \tag{3.31}$$

This expression is valid for both the infinite drive model and the random drive model, since $\eta = 0$ for the former. Note that we neglected the dangerously irrelevant operator for IDLG. This is justified, as we will see later. We can use (3.11a), provided that for power counting reasons we get for the field $\sigma(t, \tau, L_\parallel, L_\perp)$:

$$\sigma(t, \tau, L_\parallel, L_\perp) = \lambda^z \sigma(t\lambda^{-z}, \tau\lambda^{1/\nu_\perp}, L_\parallel \lambda^{-1-\Delta}, L_\perp \lambda^{-1}) \tag{3.32}$$

Now we could exploit the scaling laws for σ, setting $\lambda = t^{1/z}$, we attain to the simple result:

$$\sigma(t, \tau, L_\parallel, L_\perp) = t f_\sigma(\tau t^{1/\nu_\perp z_\perp}, L_\parallel t^{-1/z_\parallel}, L t^{-1/z_\perp}) \tag{3.33}$$

At criticality ($\tau = 0$) and if we are sufficiently near $t = 0$, both the temperature and the finite size effects are very small. In fact, we see that taking the limit $t \to 0$ is equivalent to:

$$\lim_{x,y\to\infty} f(x, y) = \lim_{x,y\to\infty} \sigma(1, x, y) \tag{3.34}$$

From (3.34), it is obvious to conclude that such a limit is equivalent in taking the thermodynamic limit, with $L_\perp, L_\parallel \to \infty$ and $L_\perp \sim L_\parallel^{-1-\Delta}$

$$\sigma(t, \tau, L_\parallel, L_\perp) \sim t \tag{3.35}$$

This is important, since we have directly verified that scaling law (3.31) have a good Taylor expansion, at least to first - order. We could then conclude:

$$\begin{aligned}\mathcal{G}_2(k_\parallel, \vec{k_\perp}, t, \tau, L_\parallel, L) = \lambda^{2-\eta-z} t \\ f_2^*(\lambda^{1+\Delta} k_\parallel, \vec{k_\perp}\lambda, \tau = 0, \lambda^{-1-\Delta} L_\parallel, \lambda^{-1} L_\perp)\end{aligned} \tag{3.36}$$

We have just expanded (3.31) at first order and neglected any finite term at zero time. Indeed, we could also obtain from that any correlation function with no $\tilde{s}$ fields, should respect:

$$\lim_{t\to 0} G_n(\vec{k_n}, t, \tau, L_\parallel, L_\perp) = 0 \tag{3.37}$$

since it is immediate to calculate explicitly those functions when the order parameter is zero at zero time:

$$\mathcal{G}_n(\vec{k_n}, t, \tau, L_\parallel, L_\perp) \sim \left\langle \sigma s_0(\vec{k_1}) ... s_0(\vec{k_{n-1}}) \right\rangle = 0 \tag{3.38}$$

for any n, if $\tau_0^{-1} = 0$ and $m_0 = 0$ We caution, however, that expansion (3.11a) is only the *leading* short distance expansion, so that corrections to the form (3.31) are possible. Those corrections are nevertheless small. Restricting the momenta to transverse one, and setting $\lambda = L_\perp$ our main result is (at criticality):

$$\mathcal{G}_2(\vec{k_\perp}, t, \tau, L_\parallel, L) = t L_\perp^{-2} f_2^*(\vec{k_\perp} L_\perp, S_\Delta) \tag{3.39}$$

The important part of this result is that it is *independent from the theory considered*, recalling that $z = 4 - \eta$. This means for correlation function does not exists any difference between RDLG and IDLG (in the short time limit). Regarding the infinite drive model, as seen in A.1 of this thesis, we could confirm the result (3.39) in a clearer and simpler way. Using directly the JSLC theory, we obtain for the correlation function at equal time, restricting to momenta transversal to the field E:

$$\mathcal{G}_{2,\perp}(k, t) = \frac{1 - e^{-2\Omega k^2 (k^2 + r) t}}{k^2 + r} \tag{3.40}$$

here we called r the parameter that control the correlation length, to avoid confusion with the more used τ, here representing the deviation from the critical temperature. Some comments are in order: first of all we are on a lattice, so the procedure of coarse - graining give rise to a normalisation constant:

$$\mathcal{G}_{2,\perp}(k, t) = Z \frac{1 - e^{-2\Omega k^2 (k^2 + r) t}}{k^2 + r} \tag{3.41}$$

moreover, given the exponents and the general setting, we know that $r \sim \tau$, to be preciser (here $w = (\beta - \beta_c)/\beta_c$

$$r = bw + O(w^2)$$

We immediately obtain the following expression in the short time limit.

$$\mathcal{G}_{2,\perp}(k,t) = 2Z\Omega k^2 t \tag{3.42}$$

We emphasise that such expression is an **exact** result. Moreover, since the leading behaviour is given by (3.41), we could neglect the irrelevant operator, as done in [5]. Such operator could only give correction to scaling behaviour. In particular, we could expand the form (2.60), setting $\lambda = L_\perp$:

$$G_{2,\perp}(0,\vec{k_\perp},t,\tau,u,L_\parallel,L) = L_\perp^2 f_2(\vec{k_\perp}L_\perp, tL^{-4}, \tau L_\perp^2, S_\Delta)[1+O(uL^{-2\sigma})] \tag{3.43}$$

This solves the dependence on the irrelevant operator. We could also obtain a corrected correlator at short time:

$$\mathcal{G}_{2,\perp}(0,\vec{k_\perp},t,\tau,L_\parallel,L) = L_\perp^{-2} t f_2^*(0,\vec{k_\perp}L_\perp,\tau L_\perp^2,S_\Delta)[1+O(uL^{-2\sigma})] \tag{3.44}$$

Another important property is that the scaling law (3.42) is independent from τ. This means we should get (3.42) at *any temperature* in the critical domain.

Now we will proceed to the four points correlation function. This is an important quantity, because we expect to see differences between the two theories considered for this particular quantity. At short time, as we will remark once more, those differences are very small or does not exist at all. As usual, we will consider the 4-s correlation function at vanishing parallel momentum, that is:

$$\mathcal{G}_{4,\perp}(\vec{k_\perp}) = \left\langle s(-\vec{k_\perp})s(-\vec{k_\perp})s(\vec{k_\perp})s(\vec{k_\perp})\right\rangle$$

For IDLG, we derived in the Appendix A.4 the following result at criticality:

$$\mathcal{G}_{4,\perp} \sim L^{-8}t^4 \tag{3.45}$$

It is an interesting result, that could be obtained from the following reasoning. In [5] a scaling form that takes into account the dangerously irrelevant operator, that has naive dimension $2\sigma = 2(d-2)/3$, has been found. From the general expression for proper vertices (2.60), we obtain:

$$\mathcal{G}_{4,0}(\{(0,q_\perp)\},t,\tau,u;L_\parallel,L_\perp) = \lambda^{d+4+\Delta} \mathcal{G}_{4,0}(\{(0,\lambda q_\perp)\},t\lambda^{-4},\lambda^2\tau,\lambda^{-2\sigma}u;\lambda^{-1-\Delta}L_\parallel,\lambda^{-1}L_\perp). \tag{3.46}$$

We can now notice that G_4 is the same order of magnitude as u, so that we obtain **in all dimensions**:

$$\mathcal{G}_{4,\perp}(\mathbf{k},t) = uL_\perp^8 f_4(k_\perp L_\perp, tL^{-4}, \tau L_\perp^2, S_\Delta)[1 + O(uL^{-2\sigma})] \tag{3.47}$$

If we *assume* that at short time the behaviour of this function is of order t^4, we could obtain easily the result (3.45) at criticality. We caution that for RDLG nothing similar have been derived, we however think that the four point function is very suppressed in this model too.

3.5 Longer time results

In the preceding sections, we showed how finite - size and short time techniques could be combined to get interesting and general results. As a fact, however, short - time scaling has little impact in the discussion on the exponents typical of IDLG or RDLG. In the Introduction, we mentioned the debate on the universality classes of the physical models considered. Measuring in a *univocal* way such exponents is clearly one of the greatest challenges in the field. As could be gleaned from table, short time scaling does not solve the problem, at least in dimension $d = 2$. The main exponents that could be measured in a simulation are equal in both cases, or varies very little, so that high precision data is needed to distinguish the cases. Moreover, the only observable that seems to be suitable for some serious confrontations has some confusing properties.

After such results has been confirmed by both theory and simulations, we turned naturally to longer time simulations. We believe those simulations could give clearer results. Remember that, as shown in , a quite simple *explicit* form for the susceptibility and the magnetisation exist:

The quantities of interest are the a - dimensional ones, so that for example Also Binder' cumulants, as discussed in , could be useful in discerning the differences between existing theories. In a purely Gaussian theory, the Binder cumulant (4.10) should be zero. This quantity is whence interesting to test the theory against Gaussian predictions. Moreover, a difference is expected in long - time behaviour of such a quantity in the two models considered if a different continuum model describe the long - range behaviour of observables. Since JSLC theory predicts a Gaussian behaviour for them, we should have that the Binder cumulant goes to zero in the thermodynamic limit. Instead, for the random model, the continuum description is quite different, and the Binder parameter (4.10) is expected to be associated with a **relevant** quantity, so it should diverge in the thermodynamic limit.

Another important test that could be done with longer time simulations regards the scaling law (3.39). The short time behaviour mixes the various time scales, so that a through test of (3.39) is in general impossible. Longer time simulations could highlight better the finite size scaling for the quantities considered. In particular, using (3.39) a precise determination of the dynamical exponent z is possible by data collapse for example. Moreover, as explained in [6], a data collapse is expected for a correlation length like (4.7) only if the right Δ is chosen. Such an analysis could be carried on much the better in static simulations, because we expect the non - equilibrium steady state ha maximal anisotropic properties; nevertheless a dynamical study could reveal how such an anisotropy is built up.

Chapter 4

Simulation study

4.1 Definition of observables

In this section we will discuss briefly what observables we chose and why. Let me start from the order parameter. Since the system could be seen as a magnetic one, as we explained in the Introduction, the obvious choice for order parameter is the magnetisation, defined as:

$$M = \frac{1}{L^2} \sum_{\vec{x} \in \Lambda} s_{\vec{x}} \tag{4.1}$$

Remember we are on a (finite) lattice of dimension $d = 2$, and Λ represents the set of points in this lattice. As usual, s is a spin and $\vec{x}$ represents the coordinate of a point in the lattice. As argued before,such a choice for the order parameter is impossible, because the model must be interpreted as a driven lattice gas, so that the total number of particles must be conserved. Moreover, the model is also implemented with a half - filled lattice. So:

$$M = 0$$

We could anyway turn to Fourier transforms of the local magnetisation, so the longest range observable that could be taken is:

$$M\left(k_\perp = \frac{2\pi}{L_\perp}\right) = \frac{1}{L_\perp L_\parallel} \left\langle \left| \sum_{x_\perp \in \Lambda} e^{-ik_\perp x_\perp} \sum_{x_\parallel \in \Lambda} s_{x_\perp, x_\parallel} \right| \right\rangle \tag{4.2}$$

This choice is indeed justified by the simple consideration that no longitudinal order has been seen up to now in simulations (see for example [20]), and that the ordered state is believed to be a single strip parallel to the direction of the field, as in figure 1.1.

Recently this well established framework has been enriched by yet another feature: the possible presence of a different phase in both studied models [38]. This phase is a "stringy" one: instead of having a single strip parallel to the field there are several. In this framework the observable 4.2 for the magnetisation is not the right one. In view of this and other considerations, Albano and Saracco [1] proposed another order parameter:

$$OP = \sum_{x_\perp \in \Lambda} \left\langle \left| \sum_{x_\parallel \in \Lambda} s_{\vec{x}} \right| \right\rangle \tag{4.3}$$

This quantity is of course independent from what phase is the "right" one, but let me remark that this very one property confuses the critical behaviour analysis. The short time analysis is probably independent from what state is reached at stationarity, but there are reasons to believe that longer time simulations, especially starting from a ordered configuration, are strongly dependent the presence of a stringy phase. Another possible definition for the order parameter is [22]:

$$M\left(k_\perp = \frac{2\pi}{L_\perp}\right) = \sin(\frac{2\pi}{L_\perp}) \left\langle \left| \frac{1}{L_\parallel} \sum_{x_\perp \in \Lambda} e^{-ik_\perp x_\perp} \sum_{x_\parallel \in \Lambda} s_{x_\perp, x_\parallel} \right| \right\rangle \tag{4.4}$$

Order parameter (4.4) has anyway the same finite size scaling of order parameter (4.2), it only takes in account better the lattice effects on small lattices. In chapter we spoke of the two - point correlation function. Its definition in the lattice formalism is:

$$S(\vec{k}) = \frac{1}{L_\perp L_\parallel} \left\langle \left| \sum_{\vec{x} \in \Lambda} e^{-i\vec{k}\vec{x}} s_{\vec{x}} \right|^2 \right\rangle \tag{4.5}$$

we directly used the structure factor, instead of the (real space) correlation function. Another quantity we studied is the so - called susceptibility: from the previous discussion of the order parameter, it is easy to understand the following definition of susceptibility:

$$\chi(k_\perp = \frac{2\pi}{L_\perp}) = \frac{1}{L_\perp L_\parallel} \left\langle \left| \sum_{x_\perp \in \Lambda} e^{-ik_\perp x_\perp} \sum_{x_\parallel \in \Lambda} s_{x_\perp, x_\parallel} \right|^2 \right\rangle \tag{4.6}$$

This is in close connection with the structure factor as in (4.5). A quantity of interest is also the correlation length; it was demonstrated in [4] that:

$$\xi_{\perp,13} = \sqrt{\frac{1}{\sin^2(2\pi/L_\perp) - \sin^2(2\pi/L_\perp)} \left(\frac{S(2\pi/L_\perp)}{S(6\pi/L_\perp)} - 1 \right)} \tag{4.7}$$

is a good definition for the correlation length in finite systems. In [5] this correlation length has been used to obtain directly the exponent ν in a static (long time) simulation for IDLG. Extending such a definition to dynamical simulations is not as easy as it could seem: as we showed explicitly in 4.2,a quantity like (4.7) in the limit $t \to 0$, is complex, since its square is negative. These difficulties are only for very short time, while for longer time the observable (4.7) is well defined. Even at short time, a modification of (4.7) could be useful

$$\mu_{\perp,13} = \frac{S_\perp(2\pi/L_\perp)}{S_\perp(6\pi/L_\perp)} - \frac{1}{9} \tag{4.8}$$

Notice that in the short time scaling limit, the two observables are closely correlated:

$$\mu_{\perp,13} \sim \frac{\xi^2_{\perp,13}}{L^2_\perp} - \frac{8}{9} \tag{4.9}$$

We also define an important parameter, the so called Binder's parameter. We already spoke of this parameter in preceding sections 3.5. It is defined via appropriate fourth moments of the magnetisation (4.4):

$$g = 2 - \frac{\langle M^4(2\pi/L_\perp)\rangle}{\langle M^2(2\pi/L_\perp)\rangle^2} \tag{4.10}$$

this parameter is related with the four - point function, as we explain in 4.2.

4.2 Results from theory: a summary

In this section we will summarise all the results obtained in previous chapters. Now that the observables have been defined, we could make contact with general results from field theory and the results we expect from simulation. Let me start from the two point function, for which we have a result valid for all t. We see that the moments are quantised, so that $k = 2\pi/L_\perp$ and we obtain from (4.5) and (3.42):

$$S\left(n\frac{2}{\pi L_\perp}\right) = Z\,t\,(2\pi n)^2 L_\perp^{-2} \quad \text{and} \quad \chi = Z\,t\,(2\pi)^2 L_\perp^{-2} \tag{4.11}$$

In the formula above we used the definition (4.6) of the susceptibility. We also have a definition for the correlation length. This definition, as already stated suffers some problems at very small times. In fact, assuming the form (3.41) for the structure factor, we obtain, for $L_\perp = L$ big enough:

$$\xi^2 = \frac{L^2}{32\pi^2}\left[\frac{G\left(\frac{2\pi}{L},t\right)}{G\left(\frac{6\pi}{L},t\right)} - 1\right] = \frac{L^2}{32\pi^2}\left[\frac{1-e^{-(2\pi/L)^4 t}}{1-e^{-(6\pi/L)^4 t}} - 1\right] \tag{4.12}$$

when $t \to 0$ we conclude:

$$\xi^2 \sim -\frac{L^2}{36\pi^2} + L^2 \frac{5}{9} 2\lambda \frac{t}{L^4} 4\pi^2 \tag{4.13}$$

In the same approximation, we could expand the correlation length in a Taylor series. We caution, however, that corrections could be important, especially in the crossover regime between long - time behaviour and short time. In order to study the 4 - point function, we have defined a Binder cumulant. It could be expressed in terms of:

$$g(t, \tau; L_\perp, L_\parallel) = -\frac{G_{4,conn}(-\mathbf{k}_\perp, -\mathbf{k}_\perp, \mathbf{k}_\perp, \mathbf{k}_\perp, t, L_\perp, L_\parallel)}{V \left[G_2(\mathbf{k}_\perp, t, L_\perp, L_\parallel)\right]^2} \tag{4.14}$$

In view of this and (4.10), we obtain:

$$\begin{aligned} g(t, \tau; L_\parallel, L_\perp) &= \frac{u L_\perp^{5-d}}{L_\parallel} f_4(\tau L_\perp^2, S_\Delta)(4\pi^2 + \tau L_\perp^2)^2 [1 + O(u L_\perp^{-2\sigma})] \\ &= u L_\perp^{-2\sigma} f_g(\tau L_\perp^2, S_\Delta)[1 + O(u L_\perp^{-2\sigma})], \end{aligned} \tag{4.15}$$

In spite of the existence of a dangerously irrelevant operator the Binder parameter should vanish in the thermodynamic limit. In $d = 2$ we expect logarithmic corrections to this behaviour. As already noted in [5], the Binder parameter vanishes because the zero mode is not present in our theory. In the $t \to 0$, we could use the relation (3.45) for the short - time expansion of the four - point correlation function.

$$g \sim L_\perp^{-4} t^2 \tag{4.16}$$

Since we are preparing the system at zero magnetisation, and the Binder parameter is expected to vanish at zero time in leading behaviour, a natural conclusion is that in the short - time regime the Binder parameter vanishes, with very small corrections.

At last we will turn to the the magnetisation. In it has been demonstrated that the magnetisation intended as the mean value of the field s has the distribution (A.27). Remembering the definition of the susceptibility(4.6) we have:

$$m^2 = \frac{\chi}{2\pi L_\parallel L_\perp^{d-1}} \tag{4.17}$$

A simple calculation yield:

$$m(t, L_\perp, L_\parallel) = \frac{(2Z\Omega t)^{1/2}}{L_\perp^{(d+1)/2} L_\parallel 1/2} \tag{4.18}$$

In ref. [5] has been proposed a generalisation of distribution (A.27):

$$P(\psi) = N \exp[-V(|\psi|^2)]\, d\psi d\psi^*. \tag{4.19}$$

and has been made the hypothesis that it could be sensible to expand V in its first terms, so that:

$$V(|\psi|^2) \approx a|\psi|^2 + b|\psi|^4, \tag{4.20}$$

considering the general n^{th} moment of such a distribution, that is indicated as $M_n(z)$, we have as a consequence:

$$\begin{aligned} g &= 2 - \frac{M_4(z)}{M_2(z)^2} \\ X &\equiv L_\perp^{d-1} L_\parallel \frac{m^2}{\chi_\perp} = \frac{M_1(z)^2}{M_2(z)}, \end{aligned} \tag{4.21}$$

where $z = b/a^2$. Thus, X is implicitly a function of the Binder parameter, as shown in Eq. (4.21). Since the Binder cumulant is practically zero at short time, we see that expression (4.17) is verified very precisely at short time.A justification of equation 4.19 has not been given at short time, but we believe such an equation is completely general.

A final comment on the order parameter (4.3) termed here OP, proposed by Albano and Saracco [1]. This order parameter could not be referred simply to transversal quantities. It could be demonstrated that it neglects the transversal correlation function, even if it detects the transversal order. It does not distinguish a multi - stripped configuration from a single strip configuration. Given those motivations, we could not follow the standard route for deriving a short - time expansion. We turn to phenomenological scaling. We notice that this order parameter scales as:

$$OP \sim L_\parallel \tag{4.22}$$

in a region where the distribution of the spins along a line parallel to the field is approximately Gaussian. During short time evolution, we expect such a condition to be verified, since the transversal dynamics is relatively slow, and we prepare the system in a way that follows the binomial distribution. If we also suppose OP does not depend on $L_\perp$ in the same regime, we could get from expression (4.22):

$$OP(t, \tau = 0, L_\parallel, L_\perp) \sim t^{1/2z_\parallel - 2\beta/\nu_\perp z_\perp} L_\parallel^{1/2} \tag{4.23}$$

From where we derive:

$$c_{OP} = 1/2z_\parallel - 2\beta/\nu_\parallel z_\parallel \quad \text{because} \quad \nu_\parallel z_\parallel = \nu_\perp z_\perp \tag{4.24}$$

this is also the result of Albano and Saracco [1]. Substituting the exponents of IDLG in the JSLC description we find in $d = 2$:

$$c_{OP} = \frac{5-d}{24} = 1/8 = 0.125$$

Instead, if we use the exponents for RDLG (2.63,2.64):

$$c_{OP} \approx 0.118$$

We also remark that the exponent c_{OP} could be expressed in the RDLG case only in terms of η. We must use expression (2.64) for β, while we also remember that $\Delta = 1 - \eta/2$:

$$c_{OP} = \frac{3-d-\eta}{2(4-\eta)} = \frac{-1-d+z_\perp}{2z_\perp} \sim \frac{3-d}{8} - \eta\frac{1}{4} \tag{4.25}$$

The exponents c_{OP} for IDLG and RDLG in two dimensions are very similar, but in $d = 3$ the difference should be bigger: while it vanishes for RDLG, it should be 1/12 for IDLG. A small table of the measurable exponents at short time follows.

c_{OP}	c_χ	c_{M_l}	
≈ 0.118	1	0.5	(RDLG)
0.125	1	0.5	(IDLG)

4.3 Simulation settings

Simulation were performed both for the random model (RDLG) and for the infinite - drive model (IDLG). we used the dynamics defined in with Metropolis rate, since we set

$$w(\Delta\mathcal{H}) = \min(1, \exp(-\beta\Delta\mathcal{H} + l\vec{E})) \tag{4.26}$$

The starting point is a Kawasaki dynamics, so that the total number of particles is conserved. This is easily done in simulations, using a particle - hole formalism. In practice, each "jump" is an exchange between a particle (represented by a 1 in an array of boolean variables) and a hole (a 0 in the same array). To be precise,in the case of infinite drive this dynamic is implemented as follows :

- one of the sites were the jump could take place is randomly chosen (a particle must be nearest neighbour of a hole or vice versa)
- if the jump is parallel to the field, it always takes place, provided the versus of the jump is the same of the versus of the field. The only difference with RDLG is that a additional randomness is provided, so that we must have another random number, in order to decide if the particle could jump backwards.
- if the jump is transversal to the field, the energy is evaluated, and the formula (4.26) is used to see if the jump could take place.

A fast way of evaluating the shift in energy is used, using tables and considering only neighbours. we performed various simulations to study the short - time behaviour of the observables defined in. My aim was to see if the results of these simulations are consistent with the field - theoretical description and the general scaling laws as defined in previous chapters.

Short - time scaling, as argued previously, is mainly useful for extracting critical exponents using quite little computer resources. Beside, we performed also simulations that runs for longer times. Remember that time scales in models like ours as $L_{\perp}^{4}$. Moreover, in computer simulations, the fundamental measure unit is the sweep: it is composed of a number of steps equal to the size of the lattice. Given those ideas, we get to the conclusion that using big lattices and exploring the long time limit we need a fast implementation on the machine we are using. Those needs could be full-fitted with a sufficiently fast algorithm. The one we used is based on multi - spin technique. This technique is quite simple to explain in principle: instead of using a boolean to represent the spin in each point of the lattice, one could use a integer or even another type which has in general N_{multi} bits; this uses best memory, disks and CPUs of modern computers, that have 32 - bits . Moreover, it allows to evolve N_{multi} systems contemporaneously. Luckily, this algorithm and the consequent computer program was available: we had only to modify them to fit into short - time scaling and dynamics in general. As already discussed in the algorithm is based on the Parisi - Rapuano congruential pseudo - random number generator:

$$a_n = (a_{n-24} + a_{n-55}) XOR a_{n-61} \tag{4.27}$$

here a_n is a 32 - bits shift register.

Since we used a multi - spin coding, we could naively expect a linear increase in the speed of the algorithm with the size of N_{multi} however we found, as in [5], that the speed is not monotonic in the number of systems coded.

A good compromise is using $N_{multi} = 128$, because it does not require excessive disk and memory. To have significant statistics when studying the systems evolving from random configurations we mediated over 12800 different systems at each sample. This means we repeated a run 100 times, having care of changing the seed of the random number generator at each run, so that independent time evolutions could be found.

4.4 The importance of sharp preparation

An important feature that distinguishes simulations performed in equilibrium (or non - equilibrium steady states) from those studying the dynamics of a system is the importance of initial conditions. As it is clear from previous discussions, one could exploit some features of the dependence from initial conditions to derive new scaling laws. The way the system is prepared at zero time becomes crucial. It has been argued in previous chapters that the right way to get precise results is to fix the magnetisation at zero at time zero. In our system the "magnetisation" could be identified with, so we have various ways to fix it at zero. Anyway, other definitions are also possible, as for example (4.3). In order ensure zero magnetisation at zero time with both (4.3) and (4.4) definitions, we fixed at zero the magnetisation of all the spin lines parallel to the external field. Writing this in a clearer way:

$$\sum_{x_\parallel \in \Lambda} s_{x_\parallel x_\perp} = 0 \qquad \text{for all} \quad x_\perp \in \{1...L_\perp\} \tag{4.28}$$

Note that this is not a statistical equivalence: no mean is involved, equation (4.28) is valid **for all the possible configurations** we prepare (at zero time). A natural consequence of this is that all possible moments of (4.3) and (4.4) are zero, together with all the observables perpendicular to the field. In a sense, the system is **sharply prepared** in perpendicular direction. To completely full-fit at zero time the condition of sharp preparation, we should say something on the parallel direction: remember that we must ensure $\tau_0^{-1} = 0$. Please remark that such a condition must be true in the mesoscopic sense, and it is clearly impossible to be verified in a microscopic sense. What matters is that the (macroscopic) observables we use are zero at zero time. This is not strictly necessary, since as was explained in , the coupling τ_0^{-1} is irrelevant in the renormalization group sense. To have statistically independent initial configurations could in any case be handy, since we get better statistics and we could exploit at maximum the multi - spin capacities of our program. This condition could be ensured by introduc-

ing some randomness in initial condition: for example we fixed the column magnetisation at zero with a very simple algorithm:

- each line along the field is deterministically half - filled (for example, the even places have spin up, the odd ones have spin down)
- a site at random is chosen, and the corresponding spin is exchanged with the one in the first position of the line.
- the same as before, but now the chosen spin is exchanged with the second one of the line.
- the algorithm is cycled since all spins in the line has been interchanged

This procedure ensures that no correlations could be find between spins (up to the extent this is possible using a pseudo - random number generator). Of course, all transversal quantities vanishes at zero time as a consequence of such a preparation.

4.5 Simulation results discussion

In a effort in being as clear as possible, we listed all results in table (A.1). As already explained, we wish to clarify various ideas that have been exposed in recent works over the short time scaling behaviour of the driven lattice gases. We performed various simulations, in the settings described in sect , to get a better understanding of the ideas underlying such methods. We then turned on longer time simulations, trying to confirm the results described in (4.2).

A important notice: in all plots we used the format $L_{\|}xL_{\perp}$ when referring to the size of the various lattices.

4.5.1 General results at short time

As a first step, we investigated the short time behaviour of the quantities defined in 4.1. The first aim of our simulation was to replicate the data presented in [1]. Their main claim is that dynamic functional (2.59b) describe the behaviour of both IDLG and RDLG. Moreover they present simulations for other models, concluding they all belong to the same universality class. They use both short and long time results to corroborate their claim. For now we will ignore the long time dynamics, and concentrate on the short time. First of all they do not prepare the system sharply and do not set the magnetisation to zero at time zero. This could result in small corrections,

that are however important (A.1). They prepare the system at random at zero time, so that their order parameter (4.3) at zero time behaves as:

$$OP(t=0) \sim L_{\|}^{-0.5}$$

We prepared the system following their methodology and in the more correct $OP(t=0)=0$ way, already described in 4.4. Plot (A.6) clarify an important issue: Albano and Saracco claim to be able to find the critical temperature by monitoring the deviations from power law behaviour at very short time. Such a method clearly fails if the system is not sharply prepared, since corrections given by the deviation from critical temperature mixes with corrections to be traced back to a non sharp preparation. In fact, as it is clear from (A.6), the effects of temperature on OP are quite important even at short time.

The table (A.1) describes simulations at the critical temperature for both RDLG and IDLG. To verify the assumption that at short time the quantity (4.3) is independent from the S_Δ and depends only on $L_\|$. This is well verified so far, since we notice a very good data collapse for all the figures. For critical temperatures we used the estimate [7] for IDLG and [7] for RDLG. This estimate is slightly different from Albano and Saracco one. To be completer, we also performed simulations using Albano and Saracco estimate. Using such an estimate we find small deviation from power law behaviour, at . Using the data (A.2) we could be tempted to assume that RDLG and IDLG are indeed in the same universality class, but the right description is given by the dynamic functional (2.59a). Such assumption is clearly too optimistic. We have no exhaustive idea on the corrections to IDLG for OP at short time , and a second order in ϵ expansion for RDLG has limited precision in calculating the right short time scaling exponent for OP.

We discussed in 3.4 the short - time behaviour of the KLS and the RKLS model. The main result was that the dynamical exponent for the observables defined in 4.1 are the same, apart a a possible small difference using OP (4.3). Such a result is substantially confirmed by our simulations. The results for IDLG of order parameter (4.4) are very near the theoretical value of 0.5, while for RDLG we get bigger deviations. It is still unclear why those differences held, but they are however quite small. Another notice to underline the importance of sharp preparation: we know order parameter (4.4) obeys to scaling law (4.18). If we prepare the system in Albano and Saracco way, we run into problems: the order parameter (4.4), scales as at zero time:

$$M_l(t=0) \sim L_{\|}^{-0.5} L_{\perp}^{-0.5} \tag{4.29}$$

This is very different form the "right" scaling form, as obtained in equation (4.18). The order parameter then shows a cross-over between (4.29) and

(4.18) as displayed in figure (A.1). This confuses scaling properties, and results in a much worse evaluation of c_m. Albano and Saracco claimed that parameter in (4.29) is not "good" for describing the short time properties of driven lattice gas models. They probably refer to this cross - over, but they give no serious motivation to their claim.

Another interesting test we did in short - time regime is on the formula (4.11), that posses a generalisation at all times for IDLG. In the short time limit is perfectly verified *both for IDLG and RDLG* (see figure A.4). This confirms the almost Gaussian nature of transverse fluctuations in IDLG, but also cast a shadow on the validity of short time as a method of investigating critical properties of system considered. As for the Binder parameter (4.10), we see that it is practically zero at short time in both IDLG and RDLG. Its behaviour confirms the form (4.16).

Another important result we obtained is about the exponent x_0. We verified explicitly that such exponent does not exists or only gives corrections to scaling, for IDLG. Since we have not performed yet simulations for RDLG at non - zero initial magnetisation, we have no data for RDLG. This is completely consistent with dimensional analysis.It is not completely necessary to set $m = 0$, but as we argued in precedence, it enhance precision.

4.5.2 Results at longer time

This section is dedicated to the results at longer times. First of all we verified the long time predictions of the JSLC theory for the susceptibility and the the first moments of the structure factor S. The results are quite clear: in figures (A.7), (A.9) and (A.8), we see that normalising time with a factor $L_\perp^{-4}$, we get a very good agreement with the values predicted by JSLC. In fact the normalisation constant for time is $\sin^4(2 * \pi/L_\perp)$, in agreement with the value of $z_\perp = 4$. At this level we did not measure any difference between the two models in the dynamic exponent z since the data collapse is substantially the same for both RDLG and IDLG.

The results are in quite good agreement with the formula (4.11) for the susceptibility. It should be noticed that, as shown in figure (A.8), the quantity $S(6\pi/L)$ is not in perfect agreement with the form (4.11). A little peak in this quantity is seen at intermediate times: we suspect such a peak is connected with corrections to S, but we did not came up with any coherent explanation of such a behaviour. A The correlation length deserves further consideration: as discussed in [6], a good correlation length should display the "right" finite - size scaling behaviour if the correct Δ is used in the initial choice of lattices. We used two different set of lattices: both with $S_\Delta = 0.2$, but with two different values of Δ: two with $\Delta = 1$ and two with $\Delta = 2$. The

results are summarised in figure (A.9): we see a not perfect data collapse using $\Delta = 1$ in IDLG and a very good collapse using $\Delta = 2$, while the contrary is true for RDLG. In the case of IDLG, we see from figure (A.9) that $(\xi/L)^2$ apparently tends to its infinite - time value $(1/2\pi)^2$We also observe a great difference between the two sets in the absolute value of the correlation length at long time: it is not clear why those differences held, since this seems to imply that the thermodynamic limit for the correlation length is dependent from S_Delta. Such a conclusion is very strange and a further investigation of the severe slowing down regime could instead suggest that such a difference is typical only of meta-stable states. We also studied the Binder parameter (4.10), that has practically no structure at short time. At longer time we observe a acceptable data collapse of $gL_\perp^{0.45}$, so that

$$g \sim L_\perp^{-0.45} \tag{4.30}$$

This is consistent with reference [5], but we remark that quantity (4.10) displays a quite unclear time behaviour, and a better statistics is probably needed.

4.5.3 Simulations starting from a ordered state

We already mentioned the possibility of simulations starting from a ordered configurations. It is not difficult to prepare such a state, since we just prepare the system in a single big high density strip where all the particles are, and a empty strip where all the holes are.There are some studies on this argument [19, 23, 35], albeit they are much less than those on short time scaling from disordered configurations. Such a preparation could be easily produced with a computer. Albano and Saracco [1] tried to extract critical exponents from simulations starting from a ordered preparation. We tried as well to obtain their results, but we our efforts failed quite soon. In figure (A.12) are reported the plots of the two order parameters (4.4) and (4.3) as function of time: they should obey to the scaling law:

$$m(t) \sim t^{-\beta/\nu_\perp z_\perp} \tag{4.31}$$

We remark that we were not able to spot the critical relaxation zone. At critical temperature such zone should extend down to the non - equilibrium steady state, save for finite - size effects. Those are completely unclear. We also remark that we found a good data collapse between different lattices sizes using t/L^2 that is very strange. The data collapse is however clear only at very **short time**. We remark that at very short time the behaviour is not very clear and much caution is needed, at last. Notice from figure (A.12) we

used lattices with the same form factor S_Δ but with two different class of Δ's. We see that we have collapse only between the data **with the same Δ** .

Chapter 5

Conclusions

5.1 What has been achieved?

The conclusions of this work are quite interesting even if not everything is perfectly clear. In particular, we extended the results of Albano and Saracco [1] to an orderer parameter (4.4) different from OP (4.3) and also for the susceptibility (4.6). The results are in good agreement with the "Gaussian" theory developed in A.1. In particular the exponents reported in table 4.2 have been measured and found to be very near the theoretical values. A better agreement could be found using bigger lattices so that the finite size effects are further reduced. Also the scaling laws reported in 4.2 have been tested with data collapse: the results are perfectly in agreement with theoretical predictions as plots (A.2) and (A.3) clearly shows. We also comment on some other tests that have been done: first we see from (A.9) that at long time in lattices with $S_2 = 0.2$ the correlation function (4.7) tends to the value $1/2\pi$ in agreement with [5]. We also see that the long time limit of susceptibility is in agreement with the a value $Z = 3.64$, where Z is the coarse graining constant in equation 3.41. We also performed fits at short time for the susceptibility, and we obtained $\Omega = 0.063$, assuming $Z = 3.64$ This is in agreement with longer time behaviour of $\xi_{1,3}$, as figure (A.9) shows. We notice that at longer times the behaviour of the susceptibility and the correlation length is in good agreement with the theory for IDLG.

5.2 Problems and open questions

This thesis' aim was to clarify and expand present results on **dynamical** aspects of the IDLG and RDLG models. This aim is however far from being reached. Many points clearly awaits further and deeper investigation. First

of all, the hypothesis of [38] on the existence of a new phase in both models must be checked. We think the best way to do that is a through finite size scaling analysis on *the static* behaviour. A dynamical analysis could then carried on. We think an advancement on this side could explain the quite peculiar behaviour of observable 4.3.

If we stick closer to the main argument of this work, many other quantities different from the ones defined in 4.1 awaits to measured or calculated. One of the most interesting is the three - point correlation function. Such a quantity is directly related to the forcing field or his coarse - grained version, so is a very good indicator of the differences in the models considered. Another interesting quantity is the **parallel** structure factor. Its definition is completely equivalent to 4.5. The parallel correlation length could also be studied and measured.

The correlation length, as exposed in 4.7, suffers from being not well - defined under a certain time. This is surely another interesting problem: as many of the observables defined in this thesis, it is not completely suitable for short - time analysis. A challenging perspective is surely to define and measure more proper observables.

On the interpretation front, some effort should be dedicated to a better understanding of short -time dynamics, especially for the RDLG. As we already noticed in 3.4, one of the diverging vertex of RDLG is divergent (albeit only logarithmically) also at short time. This probably needs further investigation. Another important investigation is about the logarithmic corrections in IDLG. Such a study also have an importance for static simulations.

On the simulation ground, a first need is in better statistics and more data, to obtain better results especially at intermediate time scale. Another problem of the dynamical methods is that they rely much on determinations of the critical temperature that must be obtained with other methods. A clear and objective method to dynamically obtain the critical temperature is another interesting problem to be pursued further.

To end this chapter, we remember that another possible way to obtain critical exponents is through simulation starting from **ordered** configurations. It is extremely important to verify if a regime of nonlinear relaxation (4.31) exists. In particular, remains to be understood the finite - size scaling in this regime.

Appendix A

JSLC theory: some explicit calculations

A.1 Exact Gaussian theory for IDLG

Let me start my analysis from the dynamic functional for IDLG. For now we neglect the dangerously irrelevant operator. So,this is:

$$J[s,\tilde{s}] = \mu \int dt\, d^d x\, \tilde{s}[\lambda^{-1}\partial_t + \nabla_\perp^2\big(\nabla_\perp^2 - \tau_\perp\big) - \tau_\parallel \partial_\parallel^2]s + \epsilon(\partial_\parallel \tilde{s})s^2 + \tilde{s}\nabla_\perp^2\tilde{s} \quad \text{(A.1)}$$

Here we used the standard notation of Schmittmann and Zia. Of course, the non - perpendicular terms have a $k_\parallel$ leg attached, so that, setting $k_\parallel = 0$, the following functional could be obtained:

$$J[s,\tilde{s}] = \Omega \int dt\, d^d x\, \tilde{s}[\lambda^{-1}\partial_t + \nabla^2\big(\nabla^2 - \tau_\perp\big)]s + \tilde{s}\nabla^2\tilde{s} \qquad \text{(A.2)}$$

we have set, as from now on, $\nabla_\perp = \nabla$, $k_\perp = k$ and $\tau_\perp = \tau$; moreover $d - 1 = d$ for simplicity. we recognise this functional as the Gaussian part of the dynamic functional for Model B. we now have to discuss initial conditions so that we must define the integration extremes for t. Considering initial conditions at $t = 0$:

$$J[s,\tilde{s}] = \mu \int_0^\infty dt \int d^d x\, \tilde{s}[\lambda^{-1}\partial_t + \nabla^2\big(\nabla^2 - \tau_\perp\big)]s + \tilde{s}\nabla^2\tilde{s} \qquad \text{(A.3)}$$

A average is needed also on initial conditions, so we must add a part to the dynamic functional:

$$Z[s,\tilde{s}] = \int [s, i\tilde{s}] e^{-J[s,\tilde{s}] - H_0[s_0]} \delta\Big(\int s_0(x)\Big)\delta\Big(\int a(x)\Big) \qquad \text{(A.4)}$$

where

$$H_0[s_0] = \frac{\tau_0}{2}\int dx\,(s_0(x) - a(x))^2 \qquad s_0(x) = s(x, t = 0) \tag{A.5}$$

Note that we included parts that ensure $\int d^dx\, s_0(x) = 0$ and $\int d^dx\, a(x) = 0$. However, these parts will not enter directly in the calculations.In fact, they only modify the zero - momentum behaviour of the theory. From now on, they will be neglected. we will call shortly $G = J + H_0$. we seek to calculate the generating functional of connected functions, that reads:

$$W[h, \tilde{h}] = ln \int [s, i\tilde{s}] e^{-G[s,\tilde{s}] + \int_0^\infty dt \int d^dx h(x,t) s(\mathbf{x},t) + \tilde{h}(x,t)\tilde{s}(\mathbf{x},t)} \tag{A.6}$$

this could be done by solving the variational equations:

$$[-\partial_t + k^2(k^2 + \tau)]\tilde{s_c}(\mathbf{k}, t) = h(k, t) \tag{A.7a}$$

$$[\partial_t + k^2(k^2 + \tau)]s_c(\mathbf{k}, t) - 2k^2\mu\tilde{s_c}(\mathbf{k}, t) = \tilde{h}(k, t) \tag{A.7b}$$

with initial conditions

$$\tilde{s}(k, t = \infty) = 0 \tag{A.7c}$$

$$s_0(k) - a(k) = \tau_0^{-1}\tilde{s}(k, 0) \tag{A.7d}$$

we wrote these equations in a mixed momentum - time representation that in order to show in a easier way how they can be solved. Writing explicitly the solutions:

$$\tilde{s_c}(\mathbf{k}, t) = \int_0^\infty dt'\, e^{-\lambda k^2(k^2+\tau)(t'-t)} h(k, t') \tag{A.8a}$$

$$\begin{aligned} s_c(\mathbf{k}, t) = \int_0^\infty dt' \Bigg[& e^{-\lambda k^2(k^2+\tau)(t-t')}\tilde{h}(k, t') \\ & + \frac{e^{-\lambda k^2(k^2+\tau)|t'-t|} - (1 - \tau_0^{-1}(k^2 + \tau))e^{-\lambda k^2(k^2+\tau)(t'+t)}}{k^2 + \tau}\Bigg] \\ & + a(k)e^{-\lambda k^2(k^2+\tau)t} \end{aligned} \tag{A.8b}$$

we now have to expand around these solutions and integrate, using the normalisation condition $W[0, 0] = 0$ we obtain for W

$$\begin{aligned} W[h, \tilde{h}] = & \frac{1}{2}\int_0^\infty dt \int d^dx h(x, t)s_c(x, t) + \tilde{h}(x, t)\tilde{s_c}(x, t) \\ & + a(x, t)\delta(t)\tilde{s_c}(x, t) \end{aligned} \tag{A.9}$$

We could express the solutions in terms of kernels, so that the functional written above transforms to:

$$W[h,\tilde{h}] = \left(h, G\tilde{h} + \frac{1}{2}Ch\right) + \int_0^\infty dt \int d^dk\, h(-k,t)G(k,t,0)a(k) \quad \text{(A.10)}$$

Where not explicitly written, we used a short - hand notation for integration over t and k. we have called $G(k,t,t')$ the propagator and $C(k,t,t')$ the correlator. Notice that from the expression for the generating functional $W[h,\tilde{h}]$ it could be also argued that the 2 - point function $< s(-k,t')s(k,t) >$ does not depend on the initial conditions. Explicitly, the correlator and the propagator are:

$$G(k,t,t') = \theta(t-t')e^{-\lambda k^2(k^2+\tau)(t-t')} \quad \text{(A.11a)}$$

$$C(k,t,t') = \frac{e^{-\lambda k^2(k^2+\tau)|t'-t|} - (1-\tau_0^{-1}(k^2+\tau))e^{-\lambda k^2(k^2+\tau)(t'+t)}}{k^2+\tau} \quad \text{(A.11b)}$$

Let me now comment about the finite - size scaling for the system considered. The fundamental observation is that with periodic boundary conditions, the only difference is in the quantisation of momenta. So, we could solve as well the equations and the results are the same. The difference arises in momenta, that are related to the lengths as $k = \frac{2\pi}{L}n$, where $n \in \mathcal{N}$.

Another feature of the JSLC theory is that it derives from a coarse - graining of a master equation defined on a lattice, so it is reasonable to assume that:

$$< s(-k,t')s(k,t) >_{latt} = Z < s(-k,t')s(k,t) > = Z\, C(k,t,t') \quad \text{(A.12)}$$

we are interested in the equal time correlation function, that reads

$$C(k,t,t') = Z\,\frac{1-(1-\tau_0^{-1}(k^2+\tau))e^{-\lambda k^2(k^2+\tau)2t}}{k^2+\tau} \quad \text{(A.13)}$$

It could be further simplified assuming $\tau_0^{-1} = 0$

$$C(k,t,t') = Z\,\frac{1-e^{-\lambda k^2(k^2+\tau)2t}}{k^2+\tau} \quad \text{(A.14)}$$

A.2 Corrections to the correlator

In previous chapters, we started the analysis of two - point correlation function stating that we dropped an irrelevant operator from the expression of the dynamic functional of the JSLC theory. However, this operator is dangerously irrelevant and could not be so easily discarded. Moreover, it is better to

have an idea on the subleading corrections to scaling. The irrelevant operator is:

$$\tilde{s}(\mathbf{x},t)\nabla_\perp^2\tilde{s}^3(x,t) \tag{A.15}$$

Under renormalization group transformations, it mixes with other operators of the same or smaller naive dimension. Here we will consider only the ones that give corrections at tree level in the two - point function. They are:

$$A_1 \quad = \tilde{s}(\mathbf{x},t)\Delta_\perp^3 s(\mathbf{x},t) \tag{A.16}$$

$$A_2 \quad = \tilde{s}(\mathbf{x},t)\Delta_\perp \partial_t s(\mathbf{x},t) \tag{A.17}$$

We are interested in the two - point function at zero parallel momentum, so at tree level all the other operators that involve a derivative with respects to $x_\parallel$ are identically zero when inserted in this two - point function. In this section, to simplificate we will consider only systems sharply prepared, that is, $\tau_0^{-1} = 0$ and at criticality, so that $\tau = 0$. Moreover, we will drop all the a - dimensional parameters. The response function and the correlator at Gaussian level is given by the expressions:

$$\tilde{R}_0(\mathbf{k},t,t') = \theta(t-t')e^{(k_\perp^4+k_\parallel^2)(t-t')} \tag{A.18a}$$

$$\tilde{C}_0(\mathbf{k},t,t') = \frac{e^{(k_\perp^4+k_\parallel^2)|t'-t|} - e^{(k_\perp^4+k_\parallel^2)(t+t')}}{k_\perp^4} \tag{A.18b}$$

Those expressions could be immediately obtained using the methods explained in A.1, with the Gaussian part of dynamic functional A.1. If we set $k_{||} = 0$, we recover the expressions in A.1. We would like to calculate the corrections to the same time perpendicular correlator. Considering only operators A.16, we see that if we seek to calculate the corrections to correlator (A.11b) at tree level, we must evaluate the following

$$D_{A1} = u_1 k^6 \int_0^\infty dt'\, \tilde{C}_0((k,0),t,t')\tilde{R}_0((k,0),t,t') \tag{A.19a}$$

$$D_{A2} = u_2 k^2 \int_0^\infty dt'\, \partial_{t_i}\tilde{C}_0((k,0),t,t')\tilde{R}_0((k,0),t,t') = -2g_2 k^6 \int_0^\infty dt'\, R(t,t_i,k)C(t,t_i,k) \tag{A.19b}$$

Carrying on the calculation, it is easy to obtain:

$$D_{tree} = D_{A1} + D_{A2} = e^{-2k^4 t}\left(e^{2k^4 t} - 1 - 2k^4 t\right)(u_1 - 2u_2) \tag{A.20}$$

We notice at first that at short time such correction has no effect. To see why, it is sufficient to expand the result at lowest possible order. We obtain:

$$D_{tree} \sim (u_1 - 2u_2)\, k^8 t^2 + O(t^3) \tag{A.21}$$

At short time we indeed obtain a order t^2 correction. Notice that its amplitude is also extremely small (order L^{-8}). In practice such correction is negligible at short time. In the infinite time limit, D_{tree} is subleading with respect to the bare correlator, which scales as L^2, since it only generates corrections of order $u_1 - 2u_2$. We remark that this calculation stops at tree level and thus could not be considered very predictive.

A.3 Langevin equation and short time

There is another interesting way to obtain the results of A.1. Instead of using dynamic functionals, the Langevin equation could be directly studied. The initial reasoning of A.1 is exactly the same. We consider the field $s(k_\perp, k_\parallel = 0) = \varphi(k)$. For this field we could write down the following Langevin equation:

$$\partial_t \varphi(k,t) = -\frac{\Omega}{2} k^2 (k^2 + \tau)\varphi(k,t) + ik\nu(t) \tag{A.22}$$

This Langevin equation is simply the Gaussian part of the Langevin Equation of Model B (2.42), recast in momentum representation. The initial conditions are simply given by $\varphi(k,0) = \varphi_0(k)$, if we consider sharp initial conditions. The Langevin equation (A.22) is simply a linear Langevin equation and could be solved exactly:

$$\varphi(k,t) = \varphi_0(k) e^{-\Omega k^2 (k^2+\tau)} + ik \int_0^t e^{-\Omega k^2 (k^2+\tau)(t-t')} \nu(t') \tag{A.23}$$

From this solution is immediate to find the average of φ:

$$\langle \varphi(k,t) \rangle_\nu = \varphi_0(k) e^{-\Omega k^2 (k^2+\tau)} \tag{A.24}$$

It is also very simple to find the covariance of the φ distribution:

$$\left\langle [\varphi(k,t) - \langle \varphi(k,t) \rangle]^2 \right\rangle_\nu = \frac{1 - e^{-2\Omega k^2 (k^2+\tau) t}}{(k^2 + \tau)} \tag{A.25}$$

This is also the result for the field theoretic calculation of A.1. We now argue that the linearity of equation (A.22) also implies that the distribution for φ is also a Gaussian. This could also be checked calculating higher order momenta. The final result is that a distribution for the random variable φ

has been found:

$$P\left(\varphi(k),t\right)=\left[\frac{\pi}{k^2+\tau}\left(1-e^{-2\Omega k^2(k^2+\tau)t}\right)\right]^{-1/2} \exp\left[\frac{(k^2+\tau)\left(\varphi(k)-\varphi_0(k)e^{-\Omega k^2(k^2+\tau)}\right)}{1-e^{-2\Omega k^2(k^2+\tau)t}}\right] \tag{A.26}$$

The result is exactly the same at short time, and since we are studying a coarse grained theory, we must include the Z term as already done in [5]. If we remember the definitions of the susceptibility, we have for $k=\frac{2\pi}{L_\perp}$:

$$P\left(\psi,t\right)=\left(\pi\,\chi\right)^{-1/2}\exp\left[\frac{L_\parallel L_\perp^{d-1}\left(\psi-\psi_0 e^{-\Omega k^2(k^2+\tau)t}\right)^2}{\chi}\right] \tag{A.27}$$

where we called $\psi=\varphi\left(k=2\pi/L_\perp\right)$. We notice a volume term appears as a consequence of the definitions (4.6) and (4.4). When setting $\psi_0=0$, we obtain the result (4.17). Such result is then verified at all times.

A.4 The four point function in short time limit

In this appendix we extend the results of Appendix N in [5]. In [5] have been demonstrated that corrections to the four - point function comes only from the irrelevant operator A.15. The reasoning was based on causality and the topology of the Feynmann graphs involved. We could simply follow the procedure without changes. We wish to compute the four-point "static" correlation function $\mathcal{G}(\mathbf{p},\mathbf{p},-\mathbf{p},-\mathbf{p},t)$. With static we intend the four point function with all the fields evaluated at the same time. We also impose the parallel momentum to vanish. We define $\mathbf{P}=(0,\mathbf{p}_\perp)$. The tree-level insertion is: The tree-level insertion of B_3 into the four-point correlation functions with all fields s taken at the same value of t (we can set $t=0$) and with momenta $\mathbf{P}=(0,\mathbf{p}_\perp)$ is given by

$$\mathcal{D}_{TL}=4u_3\mathbf{p}_\perp^2\int_0^{+\infty}dt_i\,R(t-t_i,\mathbf{P})C^3(t-t_i,\mathbf{P}), \tag{A.28}$$

From expressions (A.18a) and (A.18) we could simplify the integral (A.28). When evaluating it at short time, we could simply substitute C and R with

their short distance expansion:

$$\mathcal{D}_{TL} = 4u_3\mathbf{p}_\perp^2 \int_0^t dt_i\, R(t-t_i,\mathbf{P})C^3(t-t_i,\mathbf{P}) \sim u_3\mathbf{p}_\perp^{-8}t^4 \tag{A.29}$$

We see that the tree level the value of the correlation function is extremely small. At one loop there are two diagrams whose contributions can be written in the form

$$L_1(t_1,t_2,t_3;\mathbf{p}_\perp) = \\ -g^2\frac{1}{V}\sum_{(q_\parallel,\mathbf{q}_\perp)} \mathbf{q}_\parallel^{1,2}\mathbf{q}_\parallel^{1,3} R(t_1-t_2,\mathbf{q}^{1,2})R(t_1-t_3,\mathbf{q}^{1,3})C(t_2-t_3,\mathbf{q}^{2,3}), \tag{A.30}$$

$$L_2(t_1,t_2,t_3;\mathbf{p}_\perp) = \\ -g^2\frac{1}{V}\sum_{(q_\parallel,\mathbf{q}_\perp)} \mathbf{q}_\parallel^{1,2}\mathbf{q}_\parallel^{2,3} R(t_1-t_2,\mathbf{q}^{1,2})R(t_2-t_3,\mathbf{q}^{2,3})C(t_1-t_3,\mathbf{q}^{1,3}), \tag{A.31}$$

where $\sum'$ runs over the whole momentum space *except* the zero mode $(0,\mathbf{0})$ *and* $(0,\pm\mathbf{p}_\perp)$, and

$$\mathbf{q}^{1,2} = (q_\parallel, \mathbf{q}_\perp + \mathbf{p}_\perp), \tag{A.32}$$

$$\mathbf{q}^{1,3} = (-q_\parallel, -\mathbf{q}_\perp + \mathbf{p}_\perp), \tag{A.33}$$

$$\mathbf{q}^{2,3} = (q_\parallel, \mathbf{q}_\perp). \tag{A.34}$$

At short time, those contributions have no short time singularity, given the absence of the zero mode. The final result is given by $T = T_1 + T_2$, where T_i is:

$$T_i(\mathbf{p}_\perp) = \gamma_i u_3 \mathbf{p}_\perp^2 \int_0^t dt_1 dt_2 dt_3 L_i(t_1,t_2,t_3;\mathbf{p}_\perp) R(t-t_1,\mathbf{P}) \prod_{k=1}^{3} C(t-t_k,\mathbf{P}), \tag{A.35}$$

$\gamma_1 = 4!/2$, $\gamma_2 = 2\gamma_1$ are combinatorial factors.We remark that this contribution should be *zero* at $t = 0$. As before we could estimate this integral at short-time. The sums do not have singularities at short time, thus it should be regular functions in their temporal arguments. If they have a finite value for zero temporal argument, the short time estimate should be:

$$T(\mathbf{p}_\perp) \sim \frac{\mathbf{p}_\perp^8}{V} \tag{A.36}$$

At $d = 5$ and for $S = L_\perp^2/L_\parallel$ fixed, so that $V \sim L^{d+1}$ we immediately obtain a subleading contribution with respect to the tree level evaluation.

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Part II

Plots and tables

Notations

We used some particular notations in these plots to enhance the graphical output.

- where not specifically stated, we consider $L = L_{\perp}$
- in all plots legendas, we always indicate the size of the lattice with the notation $(L_{||}\mathrm{x}L_{\perp})$
- the quantities plotted in each plot are always indicated at the left size of the plot, where the scale is.

IDLG β=0.312694				
$L_\parallel$	$L_\perp$	c_{OP}	c_{M_L}	c_χ
120	120	0.1255 (2)	0.4917 (7)	0.9861 (6)
120	90	0.1241 (2)	0.4912 (6)	0.9857 (7)
125	50	0.1247 (1)	0.4872 (3)	0.9725 (5)
216	60	0.1238 (2)	0.4895 (5)	0.9763 (7)
240	120	0.1225 (1)	0.4951 (5)	0.9793 (8)
240	90	0.1229 (1)	0.4885 (3)	0.9821 (5)
262	32	0.1207 (3)	0.4817 (2)	0.9491 (18)
512	40	0.1224 (2)	0.4868 (3)	0.9758 (6)
IDLG β=0.3125 (AS estimate)				
$L_\parallel$	$L_\perp$	c_{OP}	c_{M_L}	c_χ
125	50	0.1240 (2)	0.4956 (4)	0.9847 (7)
216	60	0.1250 (1)	0.4927 (2)	0.9871 (7)
262	32	0.1224 (3)	0.4842 (6)	0.9631 (7)
512	40	0.1239 (2)	0.4856 (3)	0.9712 (6)
RDLG β=0.3172				
$L_\parallel$	$L_\perp$	c_{OP}	c_{M_L}	c_χ
125	50	0.1259 (2)	0.4951 (4)	0.9791 (8)
216	60	0.1242 (1)	0.4930 (4)	0.9945 (1)
262	32	0.1207 (3)	0.4849 (3)	0.9528 (11)
512	40	0.1224 (2)	0.4861 (3)	0.9800 (6)
RDLG β=0.3160 (AS estimate)				
$L_\parallel$	$L_\perp$	c_{OP}	c_{M_L}	c_χ
125	50	0.1209 (2)	0.4961 (7)	0.9878 (9)
216	60	0.1223 (1)	0.4980 (9)	0.9701 (7)
262	32	0.1189 (2)	0.4869 (4)	0.9421 (13)
512	40	0.1206 (2)	0.4851 (2)	0.9681 (17)

Table A.1: The main results of our simulations at short time. The difference with theoretical values are small, but the precision of the fit challenge the field theorethic methods. Here AS indicates Albano and Saracco [1]

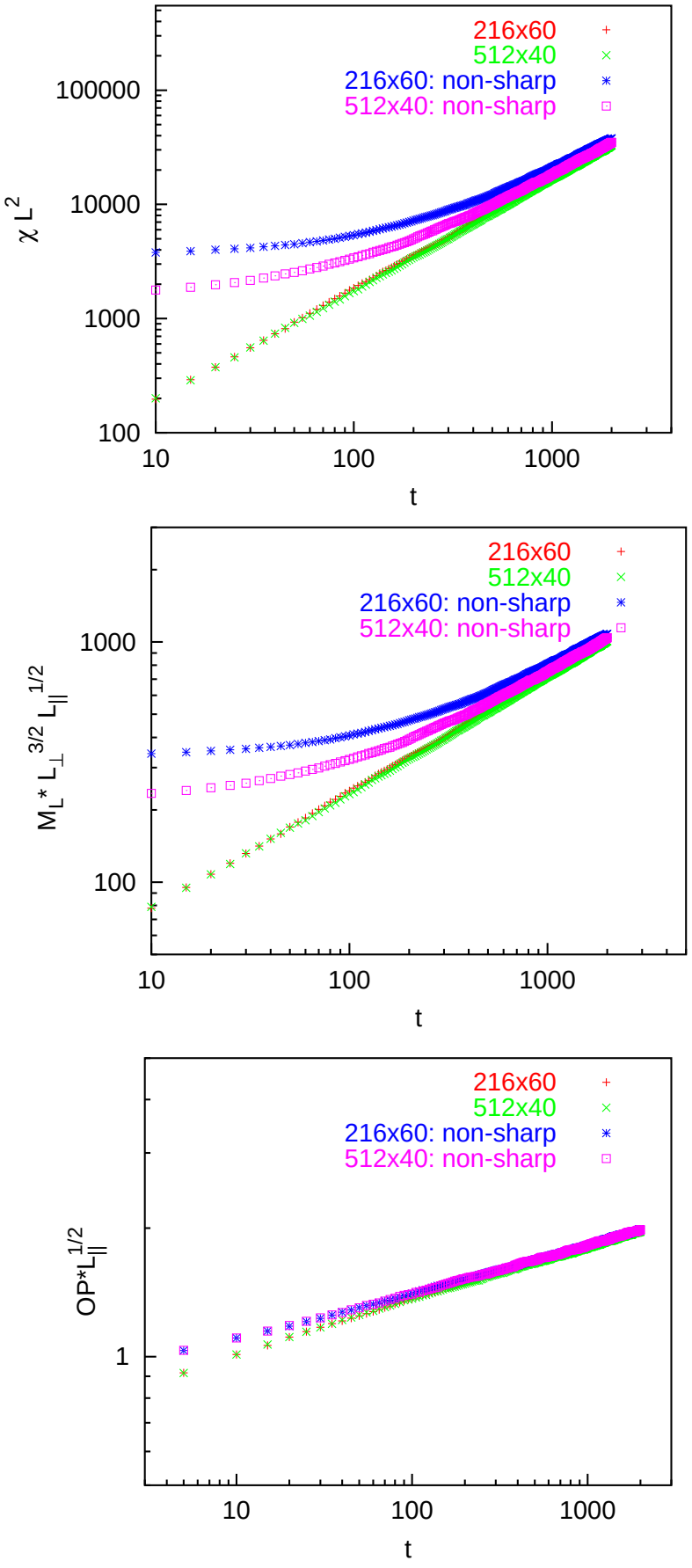


Figure A.1: This figure confronts data with our preparation at zero time and the preparation of Albano and Saracco. The simulations have been performed at $\beta_c = 0.312694$ for the IDLG. Note that the corrections are small but they persist for a long time

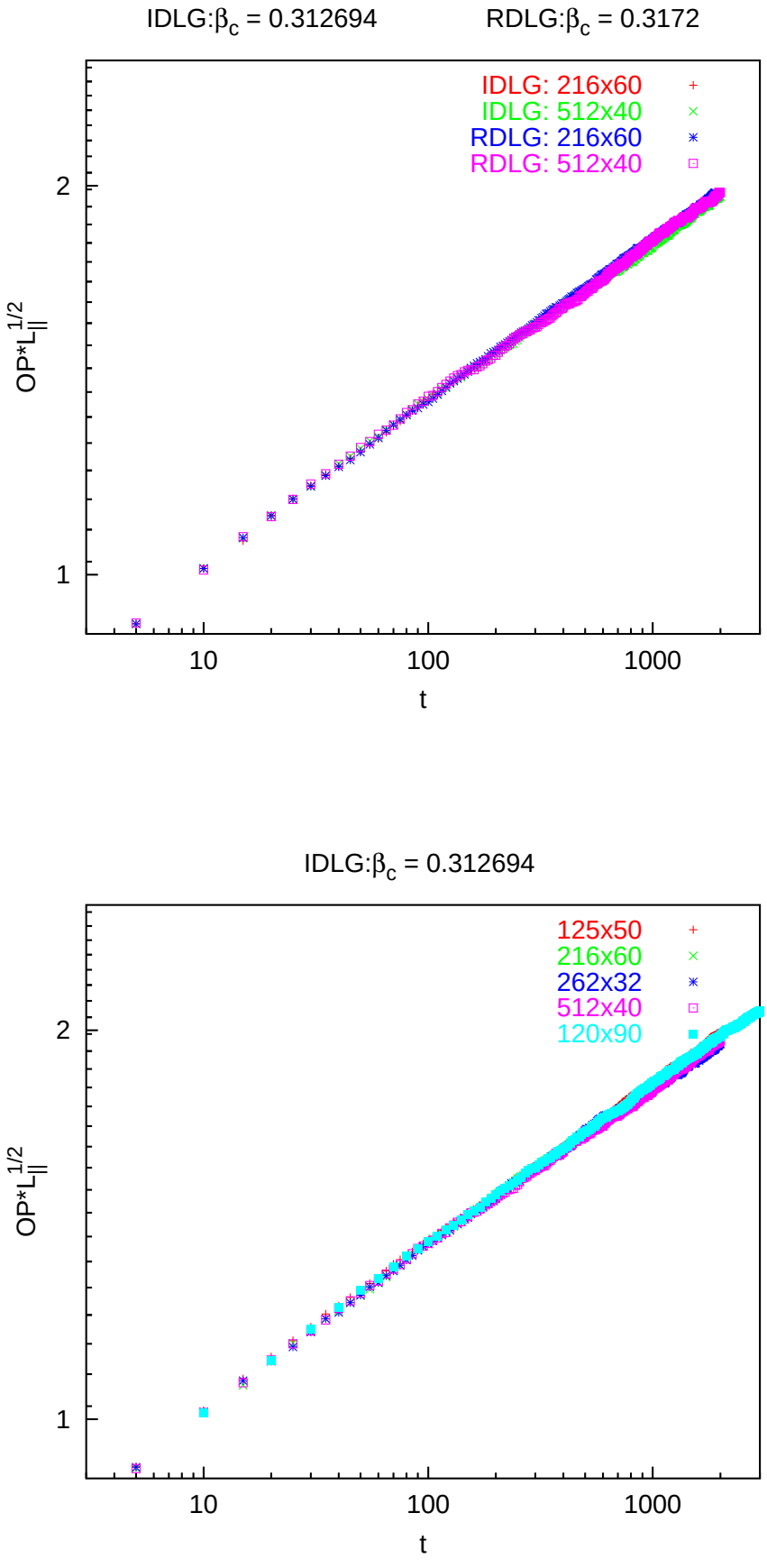


Figure A.2: At the top we confront data from IDLG and from RDLG for the OP order parameter. We find the data from the two models are practically the same. Also, we report at the bottom more data for the IDLG, showing the scaling law holds for many different values of $L_\perp$ and S_Δ

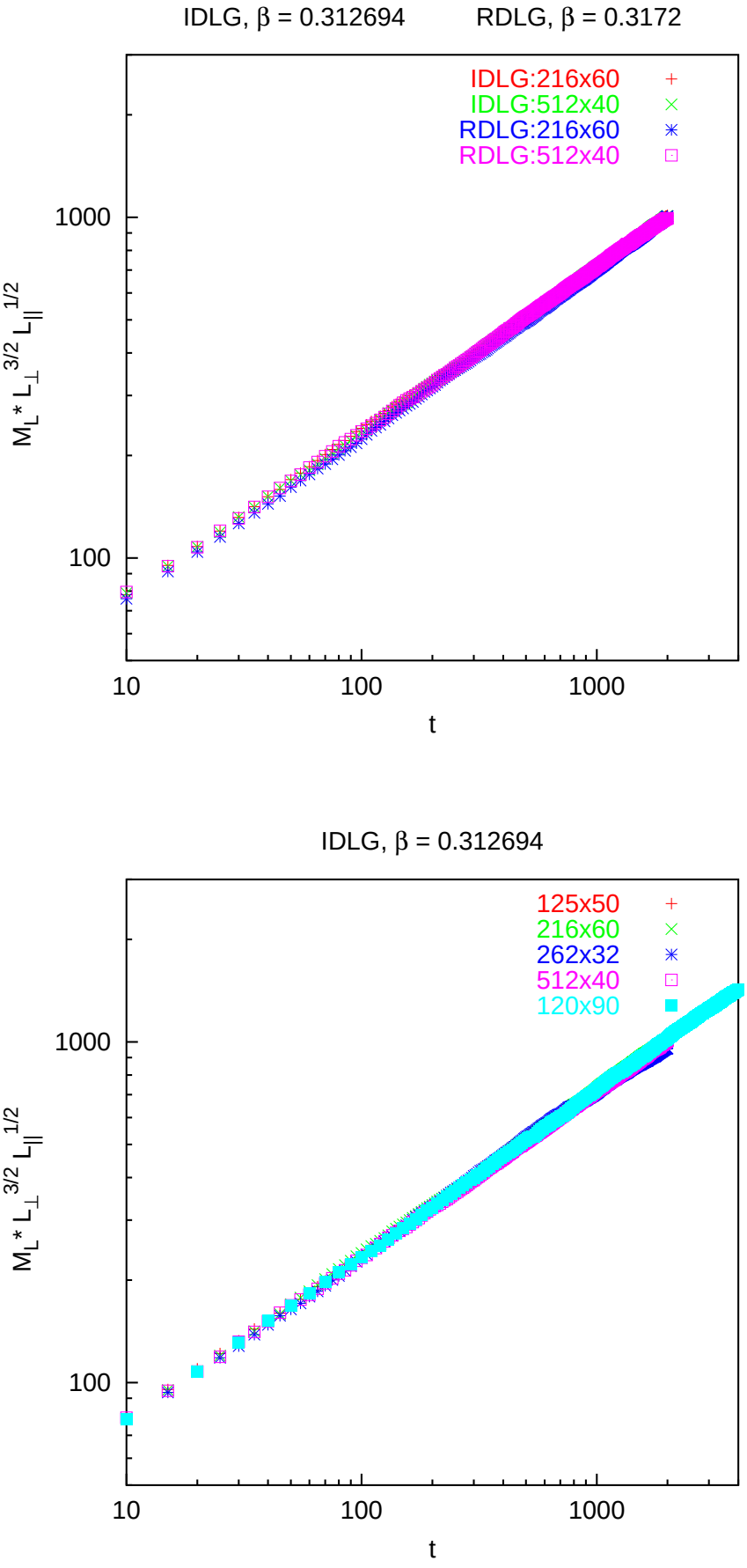


Figure A.3: The same as figure before. Here we plot the short time behaviour of Leung order parameter (see ())

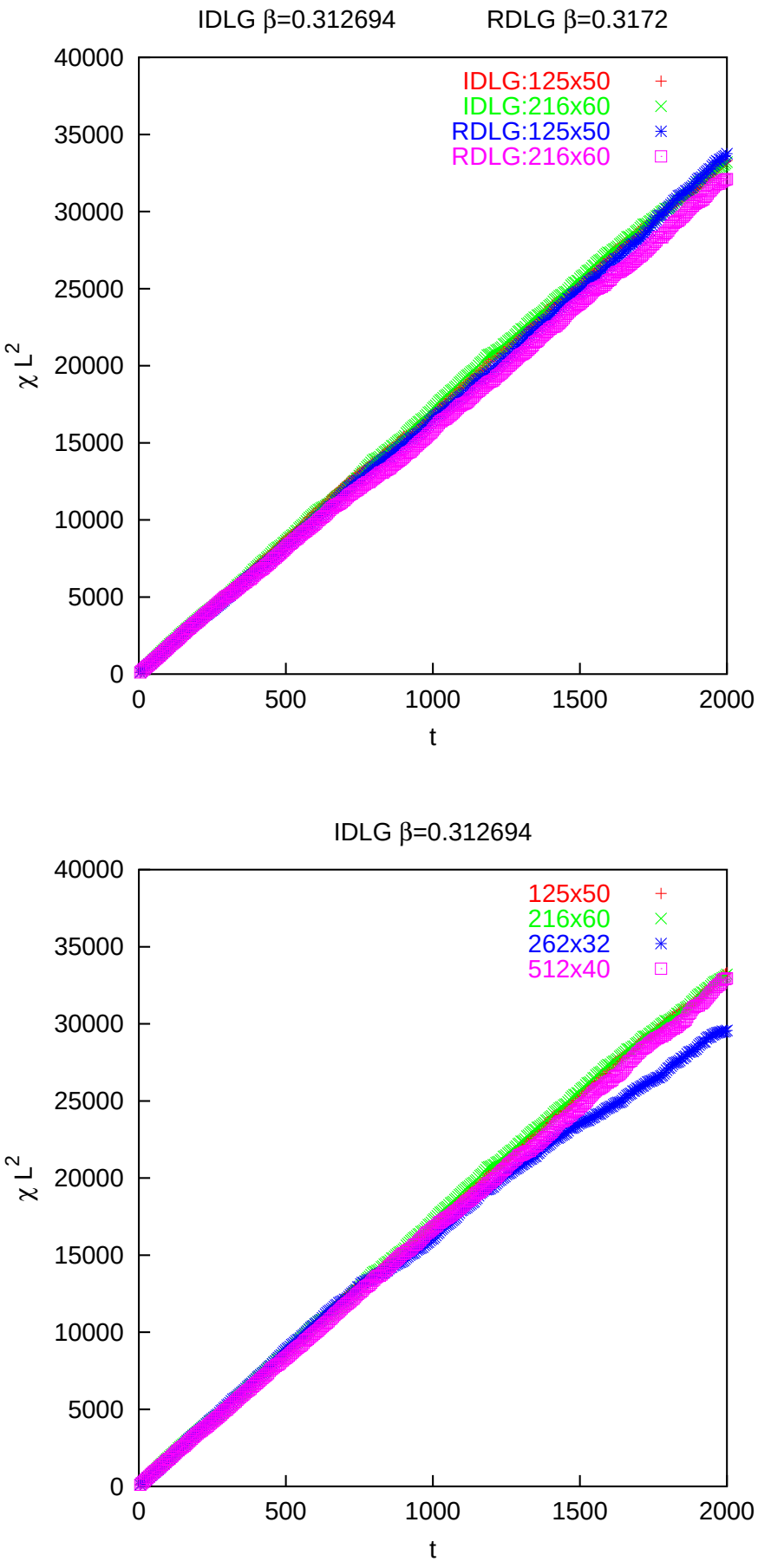


Figure A.4: Here is a plot a the susceptibility. Notice the linear behaviour. The smallest lattice simulation data (262x32) starts deviating from the linearity at about $t = 1500$

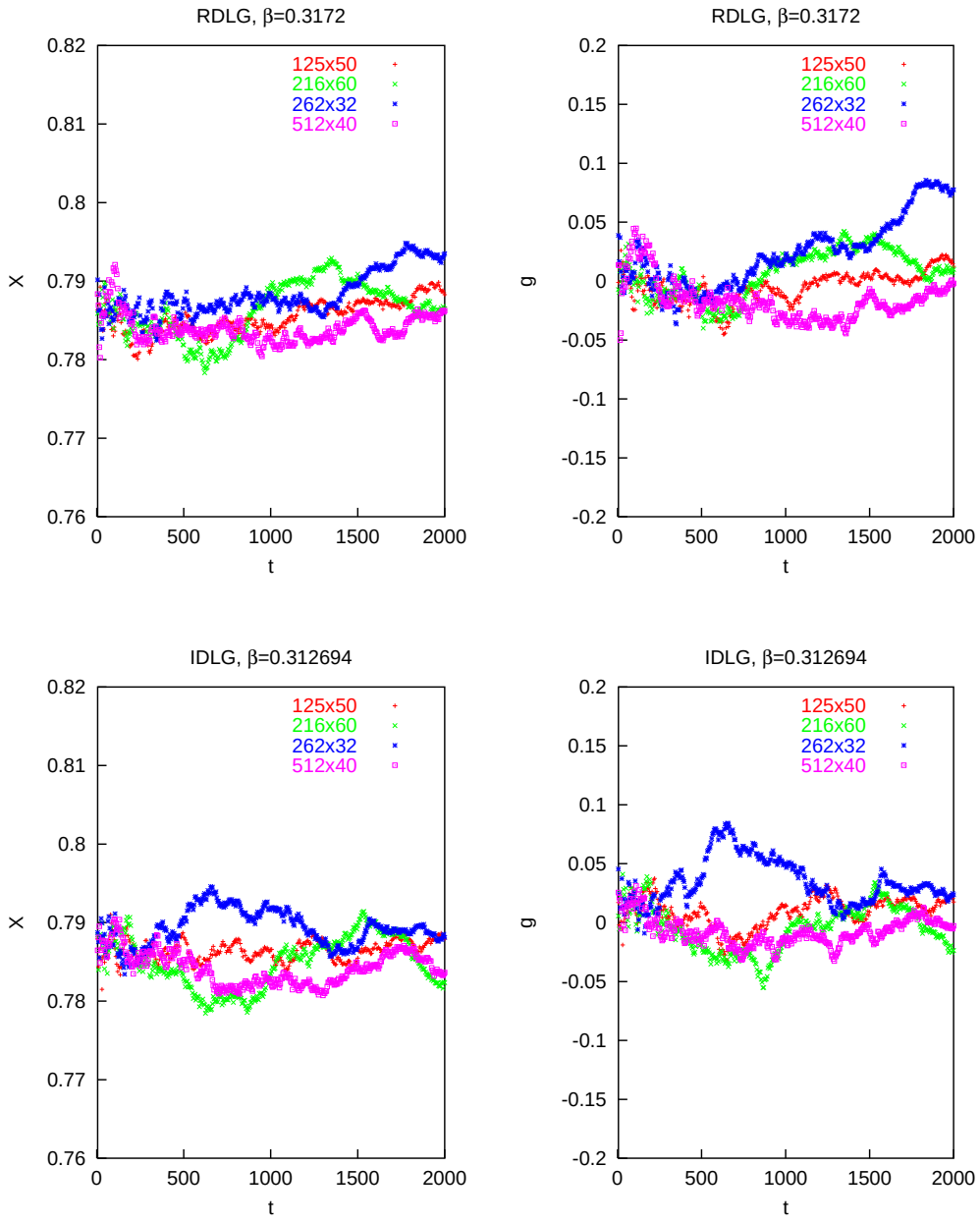


Figure A.5: This plot shows both the Binder parameter g and X. We see that the values of these parameters at short time are practically the Gaussian ones. We remember that if the Gaussian effective theory is verified, $X = \pi/4 \approx 0.7854$ and $g = 0$

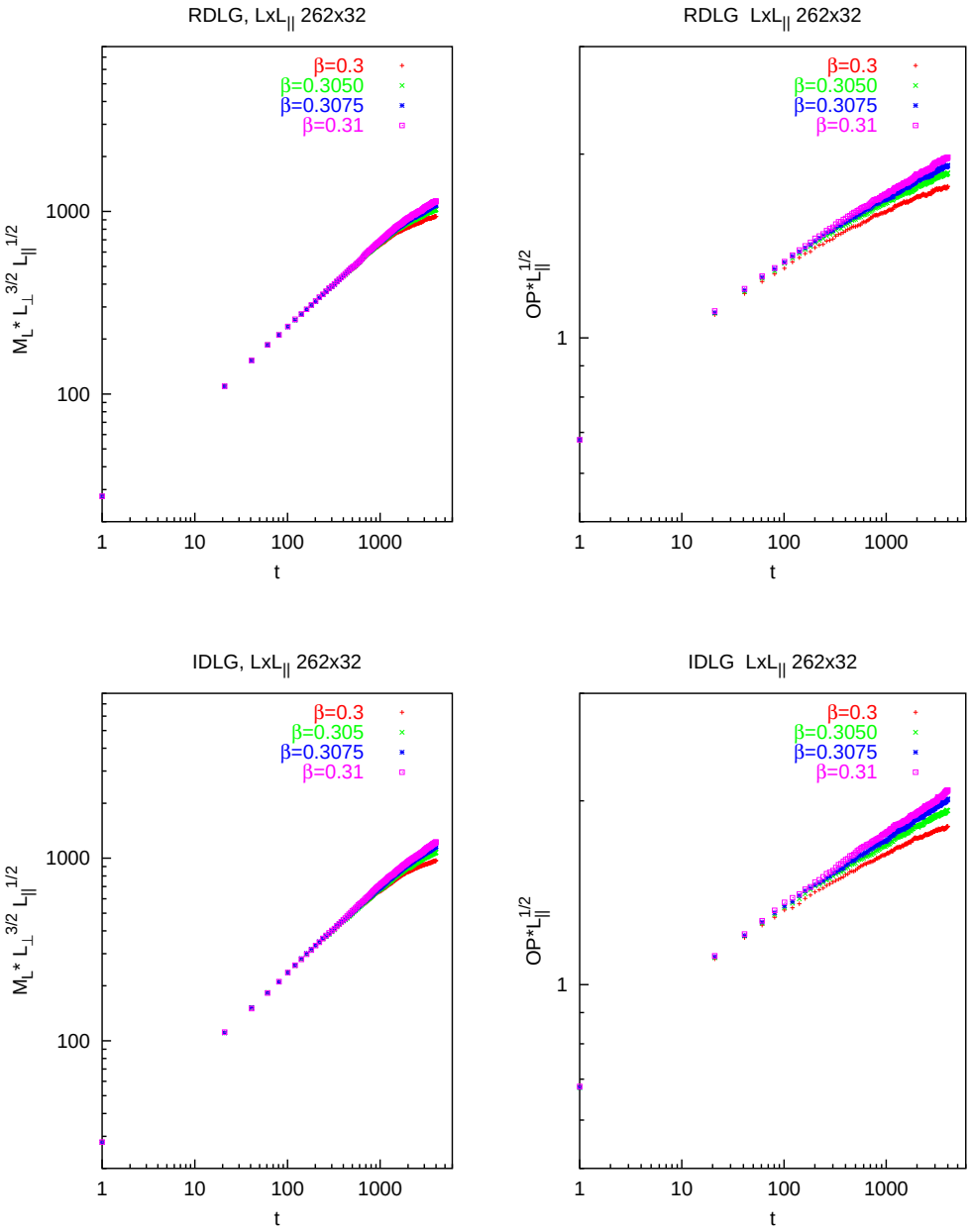
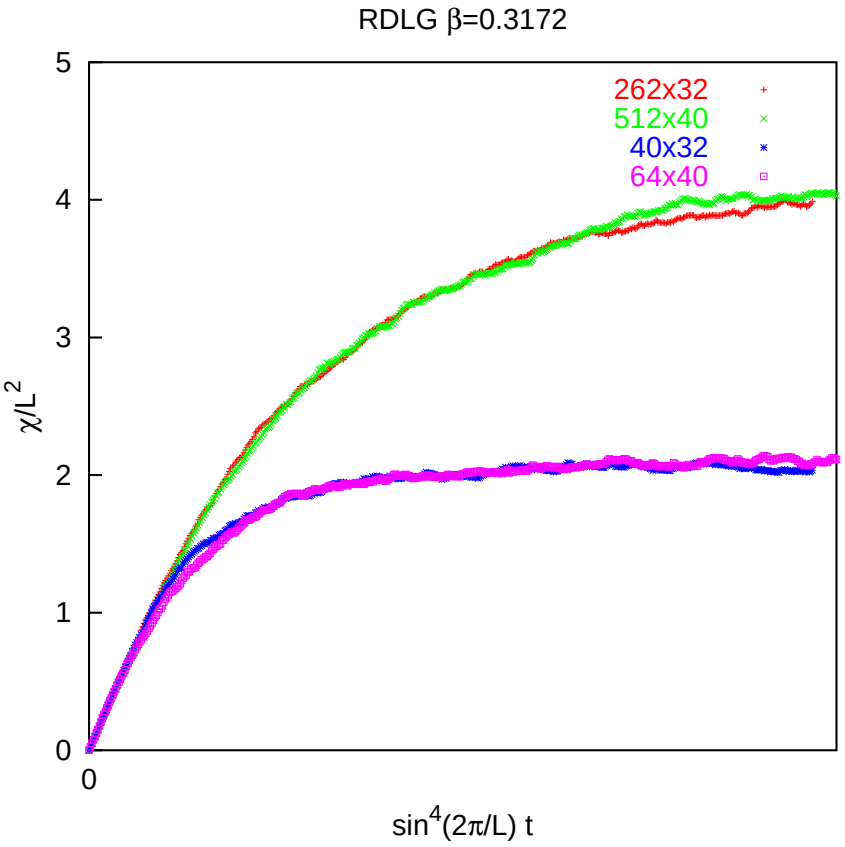


Figure A.6: This figure shows the temperature effets on both the IDLG and the RDLG: we considered both the OP order parameter and the Leung order parameter. We notice that OP is much more temperature dependent even at very short times. As previded by theory, instead, Leung's order parameter is practically independent from t.



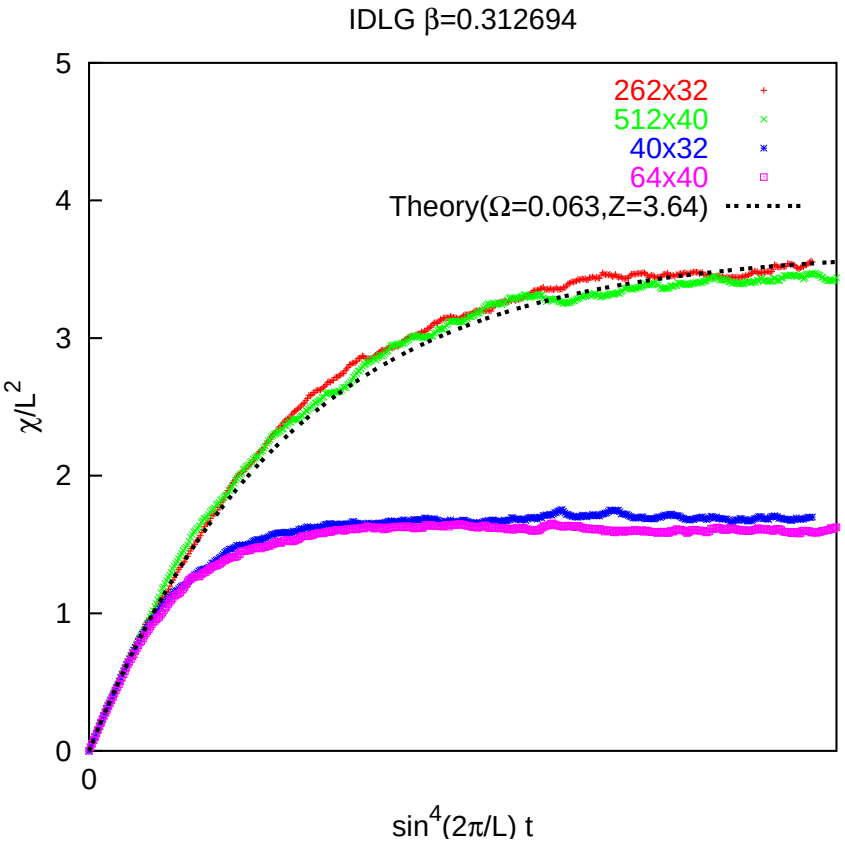
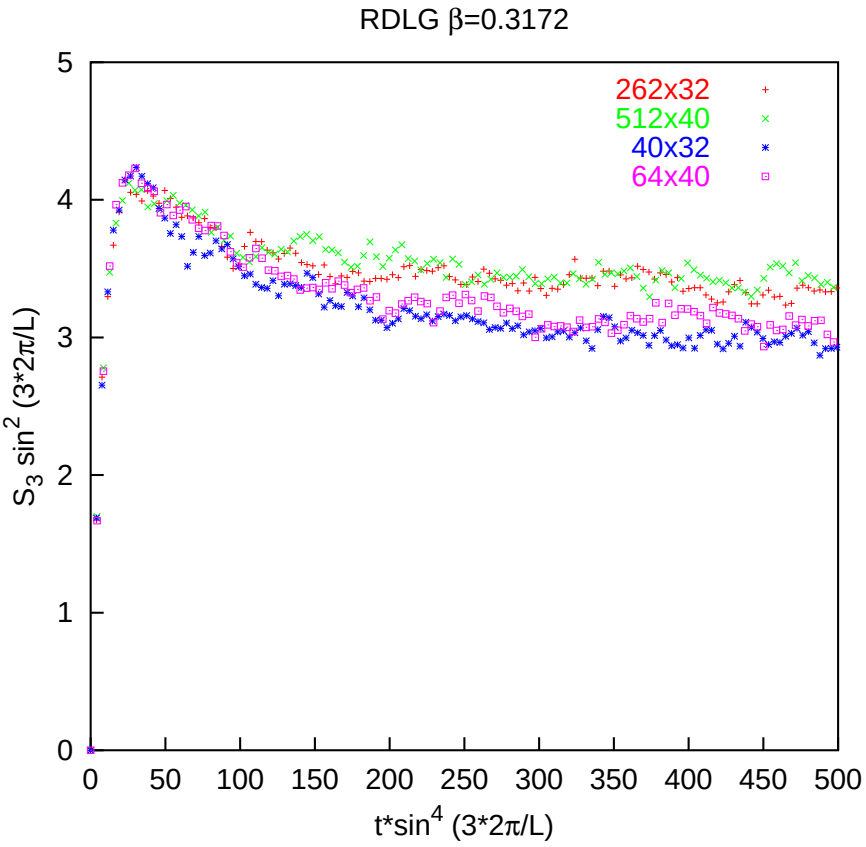


Figure A.7: The susceptivity at long time shows differences between the RDLG and the IDLG. The IDLG effective Gaussian theory still holds, but when varying the S_Δ, we interestingly have a different behaviour



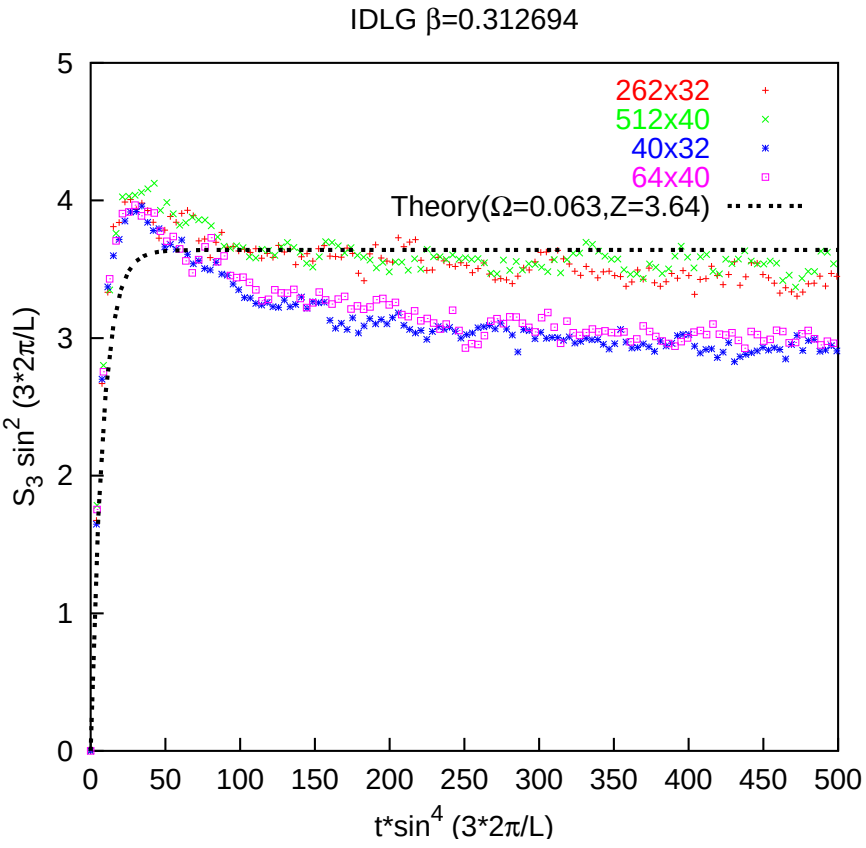
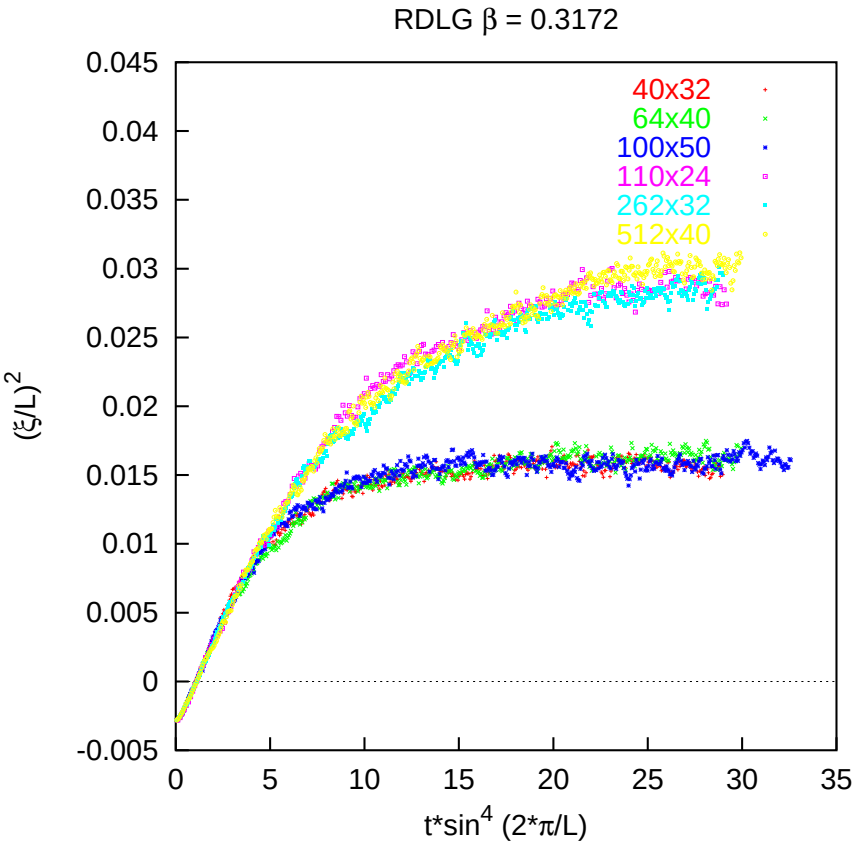


Figure A.8: We called here S_3 the function $S\left(32\pi/L_\perp, 0\right)$) We notice a small deviation from Gaussian theory. The small peak at about $t/L^4 \approx 50$ is probably originated by corrections to "mean field" theory.



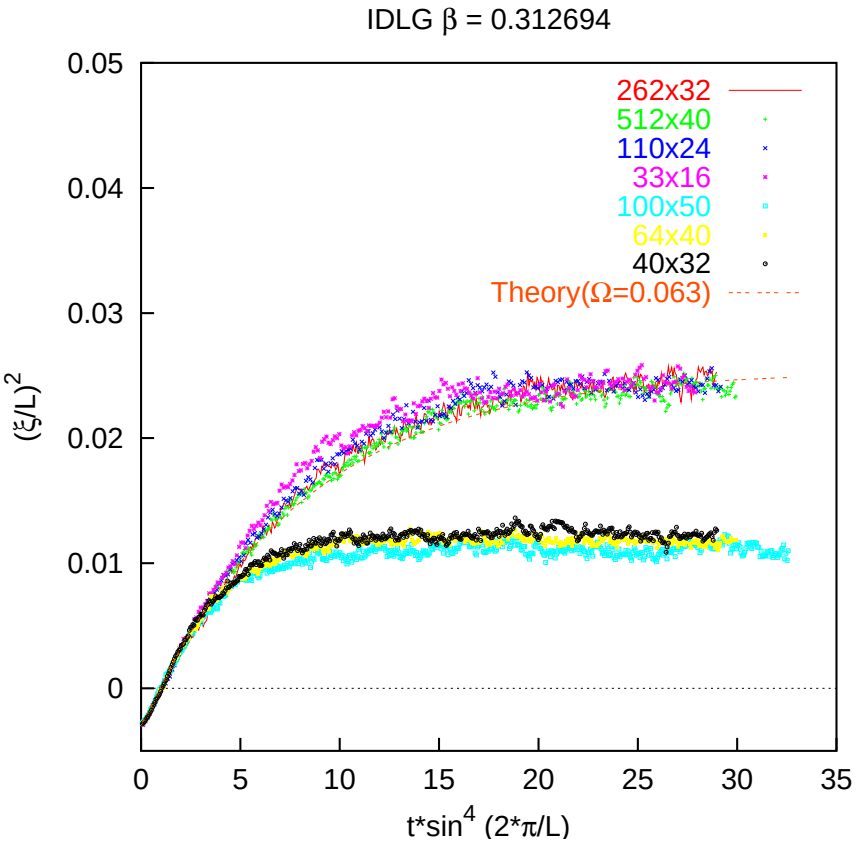
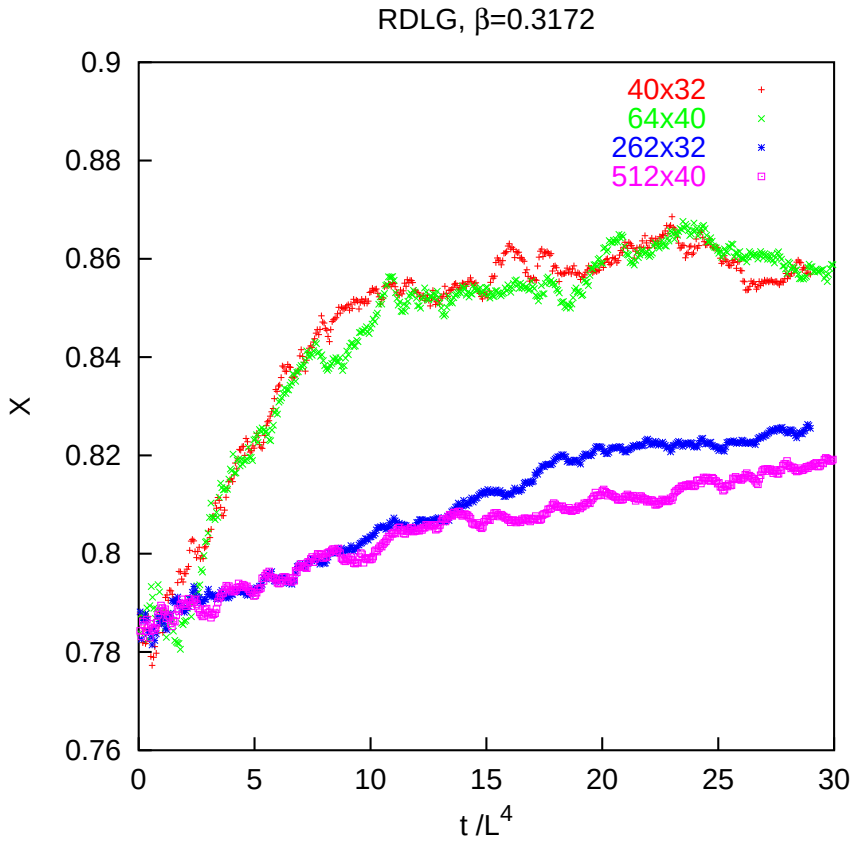


Figure A.9: The square of the correlation length plot is characterised by its good agreement with the Gaussian theory. We also notice that no trace of the small deviation in S_3.



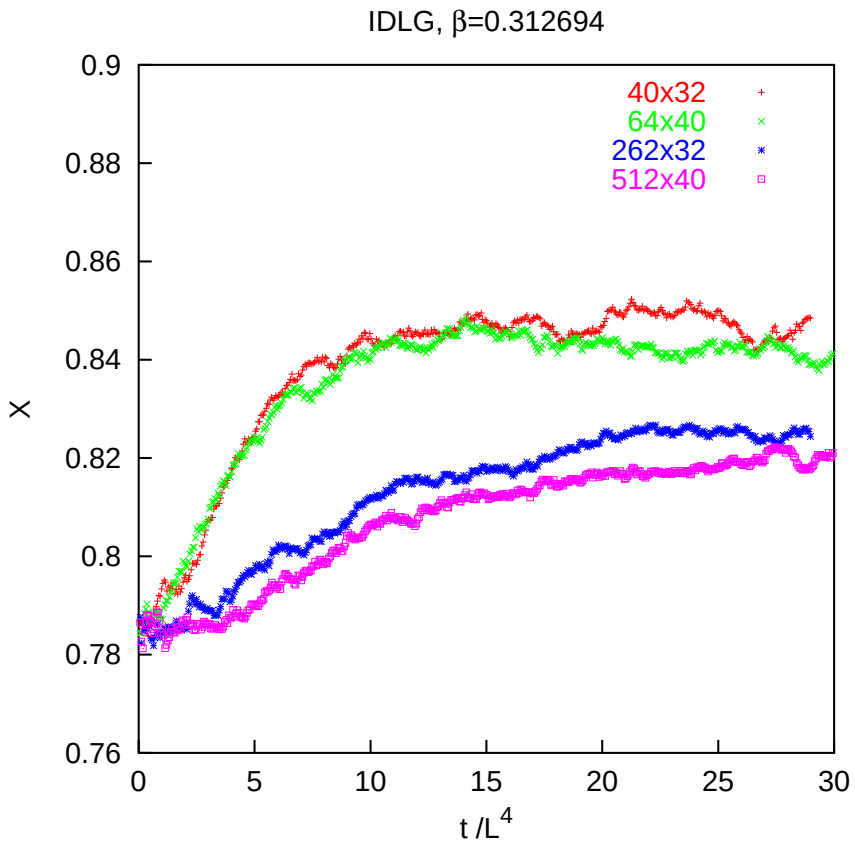
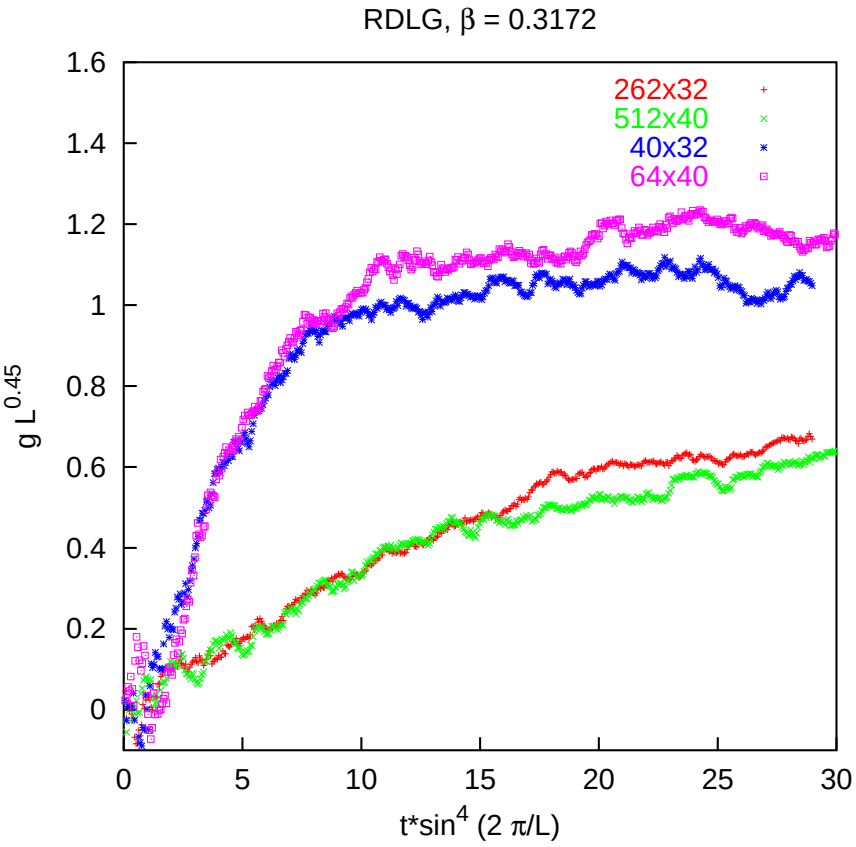


Figure A.10: At long time the X is no more gaussian.



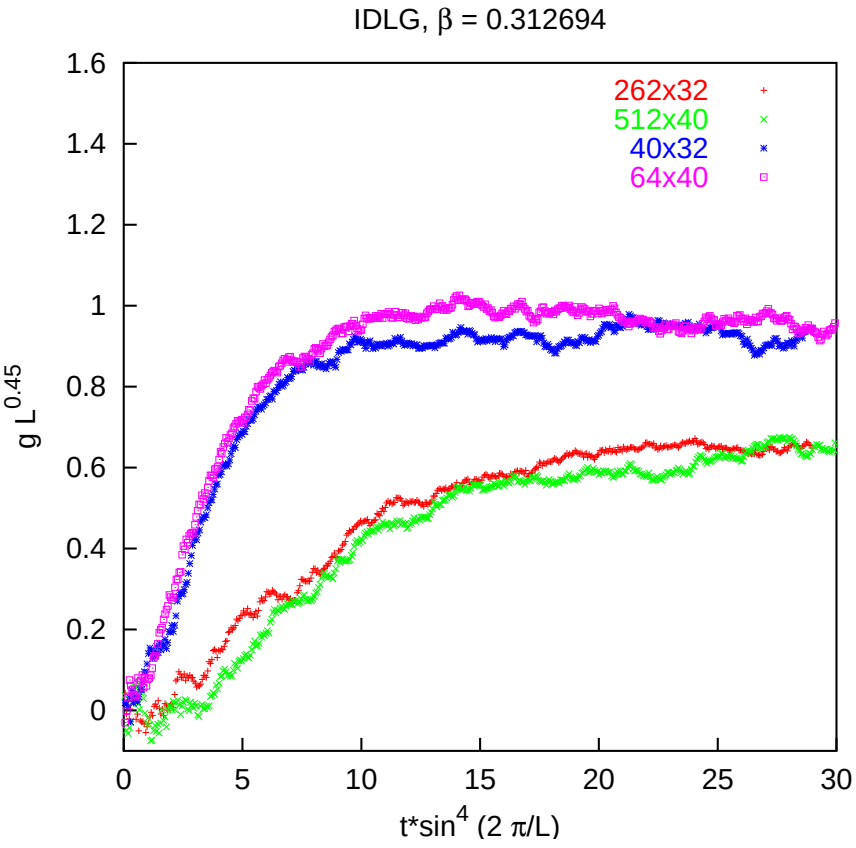


Figure A.11: Here we see the Binder parameter in his longer time evolution: we notice the acceptable data collapse with $z = 4$ and a scaling factor of $L^{0.45}$

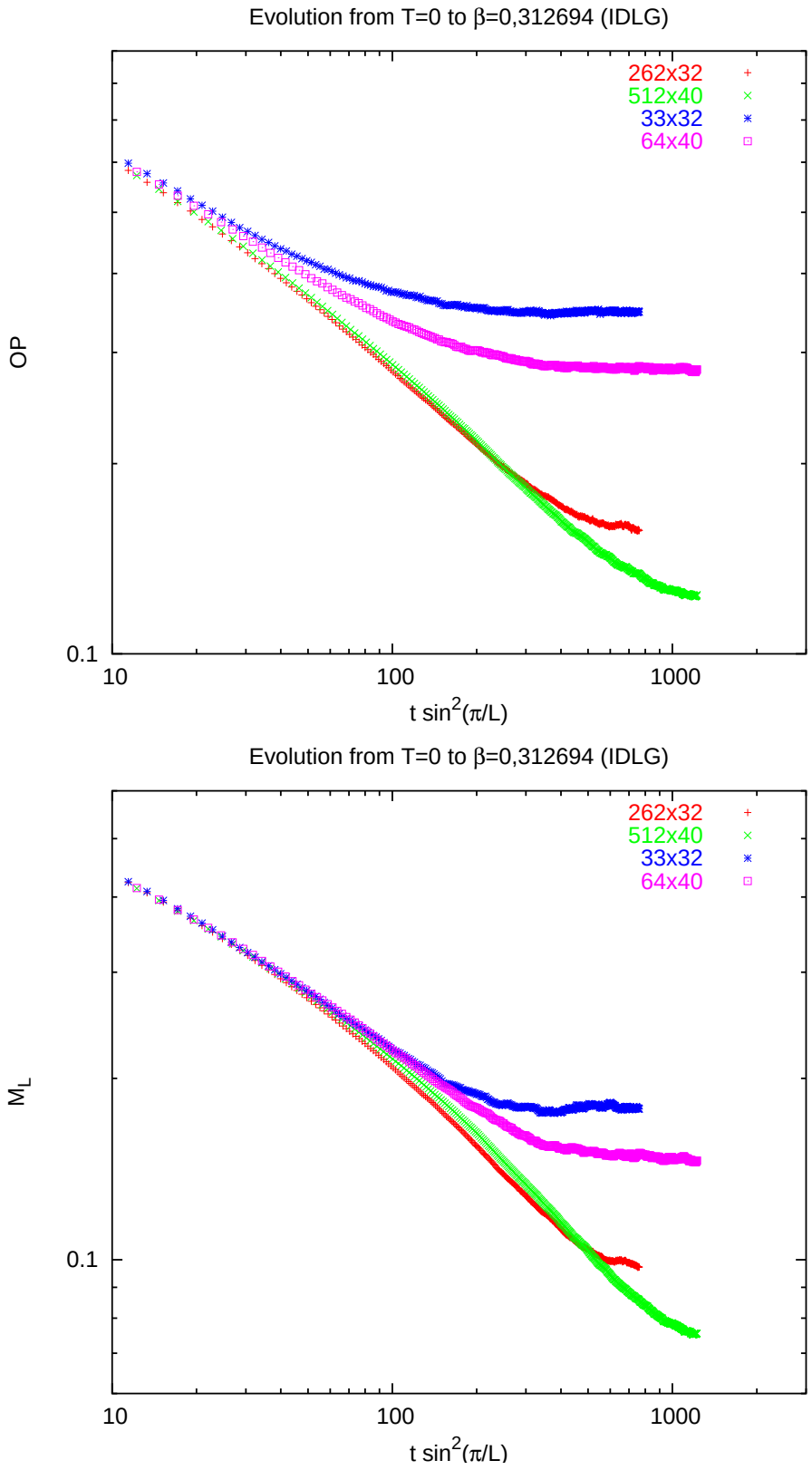


Figure A.12: The order parameters OP and M_L in a simulation starting from an ordered configuration.