UNIVERSITÀ DEGLI STUDI DI MILANO

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Fermionic field theory for Trees and Forests on triangular lattice

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Agli
abitanti dello SPAZIO IN GENERALE
E a H.C. IN PARTICOLARE
È Dedicata Quest'Opera
Da un Umile Nativo della Flatlandia
Nella Speranza che,
Come egli fu Iniziato ai Misteri
Delle TRE Dimensioni
Avendone sino allora conosciute
SOLTANTO DUE
Così anche i Cittadini di quella Regione Celeste
Possano aspirare sempre più in alto
Ai Segreti delle QUATTRO O CINQUE O ADDIRITTURA
SEI Dimensioni
In tal modo contribuendo
All'Arricchimento dell'IMMAGINAZIONE
E al possibile Sviluppo
Della MODESTIA, qualità rarissima ed eccellente
Fra le Razze Superiori
Dell'UMANITÀ SOLIDA

Edwin A. Abbott - *Flatlandia* - 1882

a mia sorella Giulia

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Introduction

The methods of Quantum Field Theory have led to a huge step towards the understanding of the weak, strong and electromagnetic interactions of elementary particles. The same methods are also applicable to condensed matter physics and offer a fruitful approach to the study of phase transitions. The key of these success relies in the Renormalization Group procedures, since they provide a way to join together the physical quantities at different energy scales and give a qualitative description of the asymptotic behaviour at high or low energy.

This work of thesis takes place in the context of the theoretical physics in which the Field Theory formalism is an useful approach to a Statistical Mechanics problem.

We study the model of a fermionic field theory, that allows the description of statistical ensembles on a graph. In fact, it is a recent result [1] that it is possible for a graph to construct a particular Hamiltonian, in term of Grassmann variables, such that the related partition function coincides with the generating function of spanning forests on the graph, and thus give insights into its graph-theoretical and combinatoric characteristics.

Graph theory has many applications in differents fields of mathematics and physics, and one of its task is to supply the description of some peculiar objects of the graph, as the paths of minimal length, the cycles, the subgraphs named "trees", which are connected, and branch without having closed paths, and "forests", which are as above, but can have more connected components.

In particular, a connection between Graph Theory and Statistical Field Theory arise in the context of the Potts Model. The Potts model is a Statistical Mechanics model, that can be seen as the generalization of the Ising model in which the spin variable can assume q colors. A reformulation of the partition function expression (Fortuin-Kasteleyn) leads to an expansion in terms of spanning subgraphs, weighted both with local "thermodynamic" factors, and with global "topological" properties of the diagrams, such as number of connected components, or alternatively of independent cycles. This expansion is deeply related with the Tutte polynomial of a graph, introduced purely in the context of Graph Theory by the mathematic W.T. Tutte in the 50's.

The correspondence among our fermionic model and the q-state Potts model on a graph appears in the limit of the parameter q and the coupling v going to zero simultaneouly, with a fixed ratio w: in that case the partition function of the Potts

model is the partition function for the unrooted spanning forests of the graph, each connected component being weighted with a factor $t = 1/w$.

As we said above, the model that we study is able to express an interesting polynomial associated to a given graph trough a fermionic integral. The secret of this corrispondence relies in the matrix-tree theorem, this is a result due by Kirchhoff in 1847 [13], that links the number of spanning trees of a graph with the Laplacian matrix of such graph.

Furthermore with the introduction of a particular form of interaction, our fermionic Hamiltonian is such that its integral represents the number of all the possible unrooted forests of the graph, so it is the generating function of them.

We study that fermionic model for unrooted spanning forest, in the case in which the graph is a regular lattice, using the perturbative theory for small values of the multiplier which "counts" the number of connected components.

It is an interesting fact that our model is equivalent, in a perturbative approach, to the non-linear σ-model in the peculiar case of the analityc extension of the variable N, that is the number of vector components, to the value -1. So the analytic results of the two models are the same.

We have dealt with regular two-dimensional lattices. The non-linear σ-model is perturbatively renormalizable in two dimensions, as shown by E. Brézin, J. Zinn-Justin and J.C. Le Guillou, [9]; moreover, from the analysis of the beta function of the Renormalization Group, it is known to show asymptotic freedom behaviour. This means that, around the $T = 0$ fixed point (which now we know, for $N = -1$, to correspond to a free-fermion theory, with only one scalar fermion), the Renormalization flow in the parameter T for $T < 0$ is marginally repulsive, and flows toward the high-temperature fixed point. This is expected in general for $O(N)$ theory at $N < 2$ (and, conversely, to happen for $T > 0$ in the case $N > 2$, with the case $N = 2$ being the very special case leading to the Kosterlitz-Thouless transition of XY-model).

The asymptotic freedom is the peculiar ingredient of the quantum chromodynamics theory in four dimensions, that is the modern theory describing the quarks, which is nowadays still far form a full comprehension, in particular for what concerns the crucial phenomenon of confinement; so it is of big interest to study a (much simpler) related model that shows this same charcteristic.

For this reasons, we evaluated the beta function for the case of the square and triangular lattices, up to the third non-vanishing coefficient (i.e. the first non-universal one). We start from the square lattice as a benchmark of the new purely-fermionic theory, although we know that the calculus of the beta function for our model is equivalent to the one for the general-N non-linear σ-model, when specialized to $N = -1$, and in this case the calculus has been already performed up to four loops in diagrammatic expansion [2, 3].

We gave the basis for the perturbative expansion of the fermionic model, writing the Feynman rules and evaluating the diagrams involved in the calculation up to the desired order. After computation, we found a matching with the expected results.

Subsequently, we applied the procedure to the case of the triangular lattice. The perturbative expansion is essentially the same of the square lattice, but the integration procedure presents some new features. In particular, the geometrical characteristics of the triangular lattice, which lead to a not-so-trivial Brillouin zone, caused a certain new care in the manipulation of momentum-space integrations which allowed for analytic results. Essentially, formulations which preserve the invariance under discrete rotations of angle $2\pi n/6$ lead to unfeasible trigonometrics, while formulations which break this symmetry, leading to standard angular integrations, need an extra effort to exploit the underlying symmetry of the theory in terms of identities which allow to reduce the number of independent lattice integrals. Although a large number of such relations has been individuated (which are often more tricky and essentially new with respect to the square-lattice counterparts), the class of different contributions is fairly larger than the one arising from square lattice, or even hypercubic lattice in generic dimension.

A collaudated technique for the evaluation of lattice integrals involving propagators in momentum space has been developed by Lüscher and Weisz in the 90's, and is known as Coordinate Method. We generalized this procedure to the case of the triangular lattice. Non-trivial new features arised also at this point, but we can state that the generalized procedure now exists, both under the theoretical aspects, and in the form of a computer Mathematica program with relative front-end flexibility.

The results obtained for the triangular-lattice beta function also find application in the comparison with numerical data recently obtained by A.D. Sokal and collaborators [4] for the Potts model on a cylindric strip, in which the zeroes in the complex plane of the parameter w have been studied for different values of the strip width, with a transfer matrix technique.

The region of small w (that is, large t, i.e. the high-temperature regime in which almost all the trees are isolated points) is studied with good results, in particular a good numerical convergence of the locum of zeroes is achieved. On the other side, it is expected that, in the region of large real part of w (that is, small t, which is our perturbative regime), the locum of zeroes converge to a pair of complex-conjugate curves with horizontal asymptote, but this convergence is far more slow in this numerical context. It turns out that the shape of this curve can be deduced perturbatively in $t = 1/w$ from the expression of the beta function. This has been done with good results for the square lattice, while the analogue results were lacking on the triangular lattice.

1. Graph theory

1.1 Definitions

A graph is an ensemble of points and lines. The lines join pairs of points. In general the same pair of points may be joined by several lines, and a point may be joined to itself by a "loop" line, like in a ring. Nonetheless, for most of 'intrinsic' properties of graphs, these patological situations are irrelevant, and lead to notational confusion. For this reason, it is common to restrict to *simple* graphs, i.e. graphs without multiple lines for each pair of points and without rings. This restriction will be assumed in this work, unless differently specified. In this case, each line is univocally identified by an unordered pair of distinct points.

The large amount of applications of graph theory, in many fields of mathematics and physics, has led to a variety of different nomenclatures. In the following, it will be intended that "point" is synonimous of vertex, or site, and "edge" of bond, or link.

A graph G will be identified by the (finite) sets of its vertices $V(G)$ and edges $E(G) \subseteq V(G) \times V(G)$. Cardinality of finite sets will be denoted by $|\cdot|$. A subgraph S of G is a pair (V', E') with $V' \subseteq V$ and $E' \subseteq E$, such that all the vertices neighbouring edges in E' are in V'. It is *spanning* if $V' \equiv V$. The set $\mathcal{S}$ of spanning subgraphs is thus in natural bijection with the set of subsets of $E(G)$, and has a vector-space structure on $\mathbb{Z}_2$, where the sum is the symmetric difference of finite sets, $E_1 + E_2 := (E_1 \cup E_2) \setminus (E_1 \cap E_2)$.

An edge $e \in E$ that joins the vertices $v, v' \in V(G)$ is written as $e = (v, v')$, and vertices v and v' are said to be adjacent. Given $v, v' \in V(G)$, a path γ on G from v to v' of length ℓ is a sequence of edges $e_1, \ldots . e_\ell$ such that $e_1 = (v, v_1), e_2 = (v_1, v_2), \ldots, e_l = (v_{\ell-1}, v')$. A cycle is a closed path, i.e. a path such that $v \equiv v'$.

A graph is connected if every pair of vertices is joined by at least one path on G. We denote by $C(G)$ the number of connected components.

For each vertex of the graph is defined a coordination number $c(v)$, that is the number of edges converging on it. Naturally, for a given subgraph S, the coordination number of a given vertex v on S, $c_S(v)$, is defined as the number of edges in $E(S)$ converging on it.

A vector subspace of $\mathcal{S}$ closed under the sum defined above is the space $\mathcal{L}$ of subgraphs L (loops) such that $c_L(v)$ is even for each vertex v. This space has dimension $L(G)$, which is the number of *independent loops* in G. Loops such that contain a single connected component, and for which all the coordinations are either

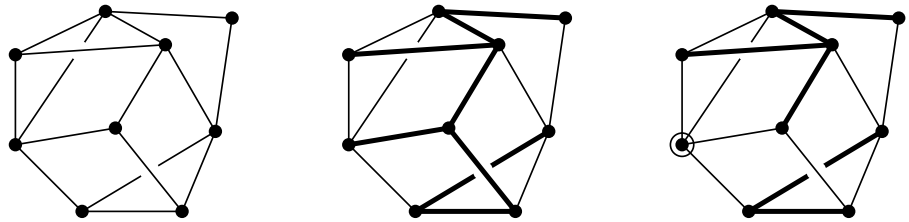


Figure 1.1 A graph, a spanning tree and a spanning forest (an isolated vertex is highlighted for clarity).

0 or 2, are the cycles defined above. It turns out that a basis for $\mathcal{L}$ can be found, such that all the basis elements are cycles.

For every graph is valid the Euler formula, which relates the number of vertices and edges to the number of connected components and independent loops:

$$|V| + L = |E| + C \tag{1.1}$$

A graph S such that $L(S) = 0$ is called a *forest*. In the case $C(S) = 1$ it is called a *tree*. For this case, Euler formula reduces to

$$|V| = |E| + 1 \; . \tag{1.2}$$

A planar graph is a graph with the special property that an embedding on a genus-0 surface (a sphere) can be found such that there are no edge crossings. For these graphs, a notion of duality can be introduced. The dual graph G^* of a graph G has vertices corresponding to the faces of G, (say, draw each vertex in a point inside the corresponding face), and, for every edge e of G, there is an edge e^* of G^* joining the two faces of G that contains e. The number of sides of a given face in the original graph G is equal to the coordination of the corresponding vertex on the dual graph.

This notion of duality has a natural extension to the set of spanning subgraphs. Given a spanning subgraph $S = (V, E')$ on G, its dual subgraph $S^* = (V^*, E'^*)$ is such that, for each pair of edge and dual-edge (e, e^*), either $e \in E'$ and $e^* \notin E'^*$, or $e \notin E'$ and $e^* \in E'^*$. With this definition, it is clear that a cycle on the original subgraph corresponds to a non-spanning connected component on the dual graph, and vice-versa, thus we have the following correspondences among subsets of spanning subgraphs, under the application of duality (D) on a graph:

$$\text{connected subgraphs} \quad \xrightarrow[D]{} \quad \text{subgraphs without loops} \tag{1.3}$$

$$\text{subgraphs without loops} \quad \xrightarrow[D]{} \quad \text{connected subgraphs} \tag{1.4}$$

Thus the set of spanning subgraphs of a given planar graph has the special property that its defining characteristic is self-dual:

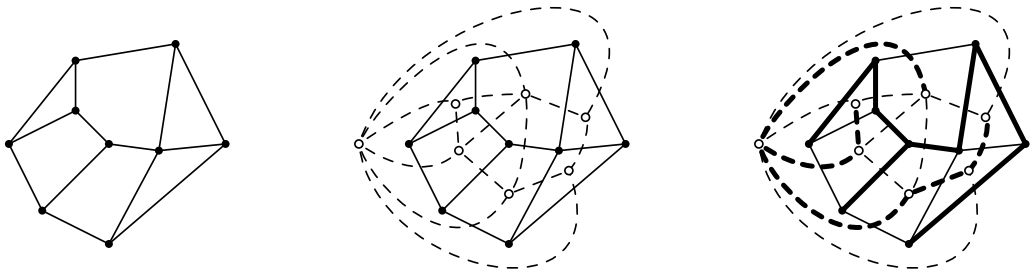


Figure 1.2 A planar graph, its planar dual, and a pair of dual spanning subtrees.

$$\text{spanning trees} = \left\{ \begin{array}{l} \text{connected} \\ \text{no loops} \end{array} \right. \quad \stackrel{\longrightarrow}{\scriptstyle D} \quad \left. \begin{array}{r} \text{no loops} \\ \text{connected} \end{array} \right\} = \text{spanning trees} \; . \tag{1.5}$$

Besides the property of being spanning is preserved going from G to its dual. It results the the number of spanning trees is the same for a graph and its dual graph, and each spanning tree is in one-to-one correspondence with a spanning tree on dual.

1.2 Algebraic graph theory

The study of the properties of a graph is strongly helped by an algebraic approach. The power of this tool relies on the fact that the *local description* of the graph (i.e. the information deriving from local properties such vertex coordinations, adjacency,...), it is easy to obtain informations about global properties, such as the number of loops, spanning trees, dimer coverings and so on (for a complete overview of algebraic graph theory see [6]).

For a given graph G with V vertices and E edges, we define the *adjacency matrix* as the $V \times V$ matrix A with entries

$$A_{ij} = \left\{ \begin{array}{ll} 1 & \text{if vertex } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{array} \right. \; .$$

It follows directly from its definition that A is a real symmetric matrix with zeros on the diagonal. It is interesting to study the spectral properties of A and we call the *spectrum* of graph G the set of eigenvalues of its adjacency matrix. From a spectral analysis we gain further information on the graph, for example, if we consider the characteristic polynomial of A

$$\chi(G,\lambda) = \lambda^V + b_1\lambda^{V-1} + b_2\lambda^{V-2} + b_3\lambda^{V-3} + \cdots + b_V \; , \tag{1.6}$$

the coefficients b_i can be interpreted as sums of principal minors of A, and so it is easy to prove that:

(i) b_1=0,

(*ii*) $-b_2$= number of edges of G,

(*iii*) $-b_3$= twice the number of triangles in G.

It is useful to introduce an *orientation* for the graph, i.e. for each edge (i, j) we choose a direction between the two possible: from i to j, or from j to i, so we can draw an arrow on every edge e.

Given an orientation for G, the *incidence matrix* D of G is the $V \times E$ matrix with entries

$$D_{ie} = \begin{cases} +1 & \text{if arrow } e \text{ starts from } i \\ -1 & \text{if arrow } e \text{ finishes in } i \\ 0 & \text{otherwise} \end{cases} .$$

For each vertex i we define the *coordination* c_i as the number of edges departing from it (independently from their orientation), and we collect all the coordinations of the graph in the diagonal $V \times V$ matrix Δ with $\Delta_{ii} = c_i$.

Then it can be proved that

$$DD^T = \Delta - A \tag{1.7}$$

and DD^T is independent of the orientation. We call this matrix $L = DD^T$ the unweighted *Laplacian matrix*, because it has the features of a Laplacian operator (that is, on a regular lattice it acts as the lattice Laplacian operator $-\nabla^2$). If we associate a weight $w_e \equiv w_{ij}$ to each edge $e = (i, j)$, then the Laplacian matrix L is written in this way

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_{k=1} w_{ik} & \text{if } i = j \end{cases} .$$

The sum of the elements in a row or a column is always equal to zero, so the matrix annihilates the vector with all entries equal to 1; since L has a null eigenvalue it follows that $\det L = 0$.

1.3 The matrix-tree theorem

The matrix-tree theorem is a powerful tool in combinatorial theory. It is an old result, already discovered by Kirchhoff in 1847 [13], first applied in theory of electric circuits, and that in recent times has found many applications in combinatorics and theoretical physics [14, 15].

In its simplest formulation, it says that any cofactor of the Laplacian matrix of the graph G gives the exact number of spanning trees of the graph. We appoint with Z_{ST} the generating function of the spanning trees and, if we assigned a weight w_e to each edge (as we made above), it is

$$Z_{ST} = \sum_{T \in \mathcal{T}} \prod_{e \in T} w_e \, , \tag{1.8}$$

where $\mathcal{T}$ is the ensemble of the spanning trees of G. Kirchhoff's theorem states that Z_{ST} is equal to a cofactor of L, i.e. to the determinant of the Laplacian matrix

after the removal of the i-th row and the j-th column (we denote this matrix by $L(i,j)$), multiplied by a factor $(-1)^{i+j}$, with no attention to which i and j we choose ($i,j = 1,\ldots,V$), since all the cofactor of the Laplacian take the same value (for a proof see [6]).

$$\begin{aligned} Z_{ST} &= \textit{any}\ \text{cofactor of}\ L \\ &= (-1)^{i+j} \det L(i,j)\ . \end{aligned} \tag{1.9}$$

In particular if we choose $i = j$ the cofactor is a principal minor of L and so it is

$$\det L(i) = \sum_{T\in\mathcal{T}} \prod_{e\in T} w_e\ . \tag{1.10}$$

We quote here two algebraic facts on the cofactor of a principal minor of a Laplacian matrix, which can be useful in derivations and applications of the Matrix-Tree theorem

$$\det L(i) = \frac{1}{V^2}\det(L+J) \tag{1.11}$$

$$= \frac{1}{V}\prod_{i=1}^{V-1}\lambda_i \tag{1.12}$$

where J is the $V\times V$ matrix with all the elements equal to 1, and is intended that the first omitted eigenvalue λ_0 is the null one. The first relation is called *Temperley formula*. Both these relations are proved in A.

1.4 Graph theory applications

The most famous example of a problem set on a weighted graph is the Travelling Salesman Problem (TSP): a salesman has to visit a set of towns joined by a network of roads, to each road is assigned a length (its weight); he is interested in finding the best itinerary, that one that passes from all the towns and that has the minimal length (weight). In spite of its simple formulation, this problem has not been solved completely; the number of possible itineraries grows so fast with the system size, that it is not possible to solve it on a large calculator; it is known to belong to the Non-deterministic Polynomial-time problems (NP problems), i.e. there is not an algorithm for it that gives a solution of the problem in a polynomial time, that means that there is not a solution, but it is only possible to check a feasible solution in a polynomial time. The study of NP problems is one of the most interesting challenges in Computer Science.

Other similar questions about enumeration of objects on a graph has been solved by algorithms; for the generations of random spanning trees with flat probability is possible to use the Wilson algorithm [31] that use a Loop-Erased Random Walk procedure. The enumeration of such quantities as spanning trees, minimal trees, maximal flow etc. has a big interest in all the kind of network system, a

classical application is to electrical networks, where the weights of the graph are the resistences of the electrical circuits; but today it is easy to find many other network system, starting from the Web, to application in biophysics problems and in economic models.

In Statistical Mechanics the lattice models, i.e. models on regular graphs, have been widely studied (for example: Ising model first of all and percolation); a graph description gives big advantages to formulate the problem.

Furthermore in chemistry, the setting on a graph allows to describe in an easier way the possible structural formulae of a chemical compound; if we consider the problem of enumerating the possible structures of hydrocarbon chain, the carbon atom is a vertex with coordination four and the hydrogen atom with coordination one; so for example the enumeration of the paraffin series (chains of carbon atoms that make only simple bonds and without cycles) reduces to the problem of enumerating the rooted trees with vertices with coordination four.

2. A model for trees and forests

2.1 Integrals with Grassmann variables

Now we want to introduce a new *language* to talk about a graph, that is the Grassmann variables, the advantages and the rich developments of this new approach are the essence of the trees and forests model and are going to be showed.

We put in appendix B the basic definitions and known results for a Grassmann algebra, from which we will start.

How it is well-known, the determinant of a matrix can be expressed by an integral in fermionic variables. For the case of Laplacian matrix, that has null determinant, we have

$$\int \prod_i d\psi_i d\bar{\psi}_i \; e^{\sum_{ij} \bar{\psi}_i L_{ij} \psi_j} = 0 \,. \tag{2.1}$$

Moreover we can use fermionic integrals to express the determinant of minors of L, and so the matrix-tree theorem shown before (1.10) becomes

$$\det L(0) = \int \mathcal{D}(\psi, \bar{\psi}) \; \bar{\psi}_0 \psi_0 \; e^{\bar{\psi} L \psi} = \sum_{T \in \mathcal{T}} \prod_{e \in T} w_e \,; \tag{2.2}$$

so we linked the fermionic formalism to a description of the graph; let us analyse the details of the fermionic integration to see what sort of objects it creates on a graph.

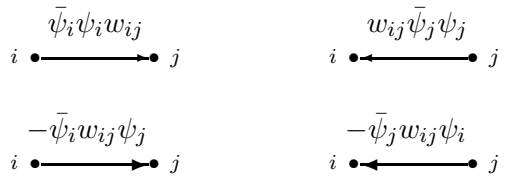
Fix vertex i and leave running j, expanding the exponential we have

$$\begin{aligned} e^{\sum_j \bar{\psi}_i L_{ij} \psi_j} &= 1 + \bar{\psi}_i L_{ii} \psi_i + \sum_{i \neq j} \bar{\psi}_i L_{ij} \psi_j \\ &= 1 + \left(\sum_k w_{ik} \right) \bar{\psi}_i \psi_i - \sum_{i \neq j} \bar{\psi}_i w_{ij} \psi_j \,. \end{aligned} \tag{2.3}$$

Now consider edge (ij): its weight w_{ij} appears in the Laplacian matrix in two kind of terms: on the diagonal elements ii and jj as $+w_{ij}\bar{\psi}_i\psi_i$ and $+w_{ij}\bar{\psi}_j\psi_j$ and on the off-diagonal elements ij and ji with $-\bar{\psi}_i w_{ij}\psi_j$ and $-\bar{\psi}_j w_{ij}\psi_i$.

So, the weight w_{ij} can appear as a factor in a Gibbs weight only from these four terms. We want to show with combinatorial arguments that the Grassmann algebra

related to the fields "sticked" to these four terms is such that only combinations describing spanning trees survive. At this aim, we choose a graphical representation for each of these terms:



The crucial fact is the Grassmann integration rule

$$\int (d\psi_i d\bar\psi_i)\ (a + b\bar\psi_i + c\psi_i + d\bar\psi_i\psi_i) = d \tag{2.4}$$

which forces the combinatorics of link occupation to put one and exactly one pair of Grassmann fields per vertex. Now, with the correspondence of the figure above, thin edges have a pair of fields in the tail vertex, while thick edges have one field in the head, and one conjugate field in the tail.

The constraint of having exactly one conjugate field per site forces every vertex (different from the root vertex 0) to be touched by exactly one arrow tail (and, for the root, by no arrow tails). Then, if a vertex is visted by the tail of a thich arrow, in order to complete the pair of Grassmann variables, it must be visited also by a thick-arrow head, and then it can be visited by an arbitrary number of thin-arrow heads. On the other side, if a vertex is visted by the tail of a thin arrow, or it is the root vertex, we already have a full pair of Grassmann variables, and it can be visited only by an arbitrary number of thin-arrow heads.

Thus, we conclude that thick arrows only come in closed self-avoiding circuits, while thin arrows makes a sort of arborescence, spanning the whole graph in a set of connected components. For each connected component, as for each vertex we have exactly one out-going arrow, there must be one "root" structure such that, for each vertex in the component, either it is in the root structure, or there is a single path which connect it to the root structure touching it only at the last vertex, and this path must be oriented toward the root. So, each component besides the root structure is a tree attached to the root.

The only allowed root structures are either the original root vertex 0, or a closed oriented cycle of thin arrows, or a closed oriented cycle of thick arrows. Here another crucial ingredient of Grassmann algebra play a role. Anticommutation of the variables implies that a given oriented cycle, when composed of thick or thin arrows, comes with a relative minus sign, due to the famous fermionic-loop phenomenon (indeed, in thin lines, the pairs $\bar\psi_i\psi_i$ acts as composite bosons, while in thick lines the two fields $\bar\psi_i$ and ψ_j are at all effects fermions). Consider for example a loop visiting in sequence the vertices $(1, 2, \dots, \ell)$, then we have

$$\begin{aligned}(-\bar\psi_1\psi_2 w_{12})(-\bar\psi_2\psi_3 w_{23})\cdots(-\bar\psi_\ell\psi_1 w_{\ell 1}) &= (-\bar\psi_1 w_{12})(\bar\psi_2\psi_2 w_{23})\cdots(\bar\psi_\ell\psi_\ell w_{\ell 1})\psi_1 \\ &= -(\bar\psi_1\psi_1 w_{12})(\bar\psi_2\psi_2 w_{23})\cdots(\bar\psi_\ell\psi_\ell w_{\ell 1})\end{aligned} \tag{2.5}$$

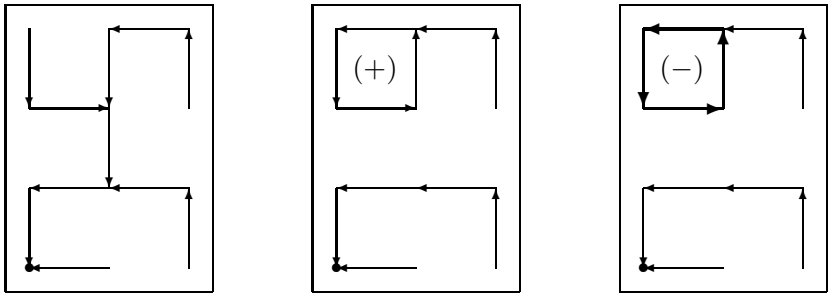


Figure 2.1 Three typical configurations on the 3×4 square lattice rooted at the bottom-left corner site. The first one, corresponding to a spanning tree, gives a contribution to the partition function, while the other two have contributions which cancel out because of fermionic loop cancellation.

So we have found that, if we remove from the ensemble of configurations these pairs which exactly cancel out, we are left only with configurations containing an unique component rooted in the root vertex 0. Also arrow orientation does not give any entropic contribution, as the choice of orientation is uniquely fixed by the constraint that all the paths going to the root should be oriented towards the root itself. This proves the Matrix-Tree theorem statement.

The reasoning easily extend to the case of more than one root given *a priori*. The fact which comes out is that the term $\mathcal{S}_T = \bar{\psi} L \psi$ of the action is a sort of generator of "spanning arborescences", which fill the part of the graph which has not been visited by other pairs of conjugate Grassmann fields.

Evaluating the expectation value of a set of *rooting* fields with the action $\mathcal{S}_T$ above, we can explore all the possible spanning-forest configurations compatible with this set of roots. The possibilities of rooting configurations are several: the simplest case is that one of a vertex-root

$$\int \mathcal{D}(\psi, \bar{\psi})\ \bar{\psi}_i \psi_i\ e^{\bar{\psi} L \psi} \tag{2.6}$$

already seen in (2.2) matrix tree theorem; we can choose also an edge-root

$$\int \mathcal{D}(\psi, \bar{\psi})\ \bar{\psi}_i \psi_j\ e^{\bar{\psi} L \psi} \tag{2.7}$$

or fix an ensemble of n vertices-root

$$\int \mathcal{D}(\psi, \bar{\psi})\ \left(\prod_{\alpha=1}^{n} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right)\ e^{\bar{\psi} L \psi}\ . \tag{2.8}$$

With this choice, we have to observe that every root is in one-to-one correspondence with a tree, because a tree cannot have more than one root; in fact if it were, there

would be a path joining two vertex-root, this path can be made only by f-links and the first and the last f-link near the vertex-root must be directed toward it (because the head takes no fields); but this means the the path changes its orientation going from one end to the other and this is not possible. So taking as rooting set a set of n vertices-root as in (2.8), instead of one rooted tree, we enumerate n rooted trees, i.e. a forest of n spanning trees (a n-forest).

From (B.17) we know also that

$$\int \mathcal{D}(\psi,\bar\psi)\ \left(\prod_{\alpha=1}^{n}\bar\psi_{i_\alpha}\psi_{i_\alpha}\right)\ e^{\bar\psi L\psi} \;=\; \det L(i_1,\dots,i_n) \tag{2.9}$$

$$\;=\; \sum_{F\in\mathcal{F}(i_1,\dots,i_n)}\ \prod_{e\in F} w_e \tag{2.10}$$

this is the "principal-minors matrix-tree theorem", $\mathcal{F}_{i_1,\dots,i_n}$ is the set of all the spanning forests F of n trees, rooted in $i_1,\dots,i_n$.

Going on, we can fix all strange objects as we like, for example in chemistry application, we construct some ring form objects that reproduce carbon compounds as benzene ring.

2.2 The generating function for unrooted spanning forests

Now we follow the derivation procedure proposed in [1], to arrive at an interesting fermionic model.

To a given a subgraph $\Gamma=(V_\Gamma,E_\Gamma)$ of G, we associate the operator

$$\mathcal{Q}_\Gamma=\Big(\prod_{e\in E_\Gamma} w_e\Big)\Big(\prod_{i\in V_\Gamma}\bar\psi_i\psi_i\Big). \tag{2.11}$$

Take a family $\mathbf{\Gamma}=\{\Gamma_1,\dots,\Gamma_\ell\}$ with $\ell\geq 0$ and consider the integral

$$\int \mathcal{D}(\psi,\bar\psi)\ \mathcal{Q}_{\Gamma_1}\cdots\mathcal{Q}_{\Gamma_\ell}\ e^{\bar\psi L\psi} \tag{2.12}$$

we proceed as follows:

- we note that the integral is non vanishing only if $V_{\Gamma_i}\cap V_{\Gamma_j}=\emptyset$ for all $i\neq j$;
- without the factor $\left(\prod_{e\in E_\Gamma} w_e\right)$ we already know that

$$\int \mathcal{D}(\psi,\bar\psi)\Big(\prod_{k=1}^{\ell}\prod_{i\in V_{\Gamma_k}}\bar\psi_i\psi_i\Big)e^{\bar\psi L\psi}=\sum_{F\in\mathcal{F}(\bigcup_k V_{\Gamma_k})}\ \prod_{e\in F} w_e \tag{2.13}$$

  i.e. it is a sum over all the possible forests rooted in $V_{\mathbf{\Gamma}}=\bigcup_{k=1}^{\ell} V_{\Gamma_k}$;

- these forests do not include the edges $E_{\mathbf{\Gamma}} = \bigcup_{k=1}^{\ell} E_{\Gamma_k}$, since otherwise they would have a path joining two root-vertex that is impossible (as we explained under the (2.8)) or one of the root vertices would be connected to itself by a loop edge in contrast with no-loop property;

- if we add to these forests the edges in $E_{\mathbf{\Gamma}}$, we connect together all the trees that were rooted in a same set V_{Γ_i}, and so the forest becomes a subgraph $H = \bigcup_{i=1}^{\ell} H_i$, collection of ℓ connected components, with the property that each component H_i contains a single Γ_i, no loops other than those lying entirely within Γ_i. We say in this case that Γ_i *marks* H_i ($\Gamma_i \prec H_i$). We call this new ensemble of subgraphs $\mathbf{\Gamma}$-*forests*;

- the addiction of the edges $E_{\mathbf{\Gamma}}$ has restored the factor $\left(\prod_{e\in E_\Gamma} w_e\right)$ to (2.13), so we can write

$$\int \mathcal{D}(\psi,\bar\psi)\ \mathcal{Q}_{\Gamma_1}\cdots\mathcal{Q}_{\Gamma_l}\ e^{\bar\psi L\psi} = \sum_{H\in\mathcal{F}_{\mathbf{\Gamma}}}\prod_{e\in H} w_e \tag{2.14}$$

  where H is an elements from the ensemble $\mathcal{F}_{\mathbf{\Gamma}}$ of the $\mathbf{\Gamma}$-forests.

Now we make a little bit of manipulation on (2.14): introduce a coupling costant t_{Γ_i} to each $\mathcal{Q}_{\Gamma_i}$, and use the identity $1 + t_{\Gamma_i}\mathcal{Q}_{\Gamma_i} = e^{t_{\Gamma_i}\mathcal{Q}_{\Gamma_i}}$ that also allows us to sum over all family $\mathbf{\Gamma}$ (i.e. over every value of its cardinality ℓ, $\forall i$ in the binomial we choose $t_{\Gamma_i}\mathcal{Q}_{\Gamma_i}$ or 1 respectively if Γ_i *is* or *is not* in the family), so we have

$$\int \mathcal{D}(\psi,\bar\psi)\ e^{\bar\psi L\psi+\sum_\Gamma t_\Gamma\mathcal{Q}_{\Gamma_i}} = \sum_{\mathbf{\Gamma}\text{ vertex-disjoint}}\left(\prod_{\Gamma\in\mathbf{\Gamma}} t_\Gamma\right)\sum_{H\in\mathcal{F}_{\mathbf{\Gamma}}}\prod_{e\in H} w_e \tag{2.15}$$

now we interchange, in the right hand side, the summations over $\mathbf{\Gamma}$ and H, so we obtain

$$\int \mathcal{D}(\psi,\bar\psi)\ e^{\bar\psi L\psi+\sum_\Gamma t_\Gamma\mathcal{Q}_{\Gamma_i}} = \sum_{\substack{H\text{ spanning}\subseteq G\\ H=(H_1,\dots,H_\ell)}}\left[\prod_i^{\ell} W(H_i)\right]\prod_{e\in H} w_e \tag{2.16}$$

with $W(H_i) = \sum_{\Gamma\prec H_i} t_\Gamma$, so now we sum before over H, that is a generic spanning ensemble of subgraphs of G, and then we check if its components are marked by the Γ in a $\mathbf{\Gamma}$-forest. We stress at this point the generality of this formula. In principle, one can sum over the ensemble of spanning subgraphs, with any choice of weights $W(H_i)$ per connected component. For general weights, it would be necessary a certain skill choice of the family $\mathbf{\Gamma}$ of diagrams, and of the weights t_Γ, such that the desired $W(H_i)$ is reproduced, but it can be seen that in many cases of interest this family is relatively simple, and contains only diagrams Γ of finite size, for combinatorial problems on graphs of arbitrary size. In particular, if all the components H_i are required to be trees, also all the Γ can be chosen to be trees.

The case in which the only allowed Γ is the single-vertex diagram ($\bullet$) leads to the enumeration of rooted spanning forests that we discussed above. The first simplest diagram Γ other than a single vertex is the one which contains an edge of the graph ($\bullet\!\!-\!\!\!-\!\!\bullet$). So we can analize this case in deeper detail, with the choice of weights

- if Γ is a single vertex ($\bullet$), it has weight t;
- if Γ is a dimer (two vertex linked by a single edge, $\bullet\!\!-\!\!\!-\!\!\bullet$), it has weight u;
- otherwise $t_\Gamma = 0$.

In this case the integral is

$$\int \mathcal{D}(\psi,\bar\psi)\ \exp\left[\bar\psi L\psi + t\sum_i \bar\psi_i\psi_i + u\sum_{\langle ij\rangle} w_{ij}\bar\psi_i\psi_i\bar\psi_j\psi_j\right]$$

$$= \sum_{\substack{F\in\mathcal{F}\\ F=(F_1,\dots,F_\ell)}} \left[\prod_i^\ell (t|V_{F_i}| + u|E_{F_i}|)\right]\prod_{e\in H} w_e \tag{2.17}$$

where the summation on the right hand side runs over the spanning forests F, and $|V_{F_i}|$, $|E_{F_i}|$ are respectively the number of vertices and edges in the tree F_i. In the case in which all the weights w_{ij} are real positive, the combinatorics on the right side involves only positive weights for each configuration of spanning forest, for the range of parameters $t \geq 0$ and $t+u \geq 0$. When $t+u$ is strictly larger than 0, large components F_i are asymptotically weighted with a factor which scales with their size, and thus the generating function can be reasonably expected to have critical properties in the same universality class of rooted spanning forests (that is, in the fermionic description, a pure mass insertion). A specially interesting case is the one $u=-t$. Indeed, because of Euler formula $V-E=1$, the weight simplifies in a factor t for each tree, the dependence from the number of vertices and edges in the components F_i disappears, and we obtain

$$\int \mathcal{D}(\psi,\bar\psi)\,\exp\left[\bar\psi L\psi + t\sum_i \bar\psi_i\psi_i - t\sum_{\langle ij\rangle} w_{ij}\bar\psi_i\psi_i\bar\psi_j\psi_j\right] = \sum_{\substack{F\in\mathcal{F}\\ F=(F_1,\dots,F_\ell)}} t^{\,\ell}\prod_{e\in H} w_e \tag{2.18}$$

this is the fermionic model which describes the generating function of *unrooted* spanning forests: although in the combinatorial derivation we made use of rootings on vertices and edges of the trees, because of the crucial cancellation $V-E=1$, we can interpret the result as a pure combinatorics of forests, each component being weighted with a factor t, independent of size parameters. Moreover, noticing that

$$\sum_{\langle ij\rangle} w_{ij}\bar\psi_i\psi_i\bar\psi_j\psi_j = -\frac{1}{2}\sum_{ij}\bar\psi_i\psi_i L_{ij}\bar\psi_j\psi_j \tag{2.19}$$

we can write also the interaction part of the Lagrangian in terms of the Laplacian matrix

$$\begin{aligned} Z_F &= \int \mathcal{D}(\psi,\bar\psi)\,\exp\Big[\bar\psi L\psi + t\Big(\sum_i \bar\psi_i\psi_i + \frac{1}{2}\sum_{ij}\bar\psi_i\psi_i L_{ij}\bar\psi_j\psi_j\Big)\Big] \\ &= \sum_{\substack{F\in\mathcal{F}\\ F=(F_1,\dots,F_\ell)}} t^\ell \prod_{e\in H} w_e\,. \end{aligned} \tag{2.20}$$

2.3 Correspondence with lattice non-linear σ model

The generating function for unrooted spanning forests, described in the previous section, has an interesting correspondence with the non-linear σ-model. In particular we show in this paragraph that the low-temperature perturbative expansion of the non-linear σ-model at $N=-1$, on a certain graph G, coincides with the generating function for spanning forests on graph G, with parameter $t=-T$, where T is the temperature.

For integer positive N, the non-linear σ-model on graph G is the theory described by $V=|V(G)|$ spin variables $\boldsymbol{\sigma}_i\in\mathbb{R}^N$, (i.e. a real vector with N components $\boldsymbol{\sigma}_i=\{\sigma_i^1,\dots,\sigma_i^N\}$), constrained to lie on the unit sphere ($\boldsymbol{\sigma}_i^2=1$), and the Hamiltonian couples first-neighbouring spins $\langle ij\rangle$ on the graph

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{T}\sum_{\langle ij\rangle} w_{ij}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j - 1) \tag{2.21}$$

If there were no constraints on the norm of the spins, the theory would be purely quadratic. The non-linearity introduced by the delta constraints is the crucial ingredient that produces interaction. In order to see this fact, we can consider a parametrization which solves the constaint: $\boldsymbol{\sigma}=(\epsilon\sqrt{1-T\boldsymbol{\pi}^2},\sqrt{T}\boldsymbol{\pi})$, with $\boldsymbol{\pi}\in \mathrm{D}_{N-1}(T^{-1/2})$, the disk of radius $T^{-1/2}$ in $\mathbb{R}^{N-1}$, and $\epsilon=\pm1$. The Jacobian of this transformation is

$$\prod_i \frac{1}{\sqrt{1-T\boldsymbol{\pi}_i^2}} \tag{2.22}$$

and so the new Hamiltonian is

$$\mathcal{H}'(\boldsymbol{\pi},\epsilon) = \mathcal{H}(\boldsymbol{\sigma}) + \frac{1}{2}\sum_i \log(1-T\boldsymbol{\pi}_i^2) \tag{2.23}$$

In a perturbation theory around the fully ordered phase $\boldsymbol{\sigma}_i=(1,0,0,\dots)$, we can take $\epsilon=+1$, neglect the constraint $\boldsymbol{\pi}\in\mathrm{D}$ and expand in powers of $\boldsymbol{\pi}$. We have

$$\mathcal{H}'(\boldsymbol{\pi}) = \frac{1}{2}\sum_{ij} L_{ij}\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_j - \frac{T}{2}\sum_i \boldsymbol{\pi}_i^2 - \frac{T}{4}\sum_{\langle ij\rangle} w_{ij}\boldsymbol{\pi}_i^2\boldsymbol{\pi}_j^2 + O(\boldsymbol{\pi}_i^4,\boldsymbol{\pi}_j^4)\,. \tag{2.24}$$

and the partition function is $Z=\int\prod_i d\boldsymbol{\pi}_i e^{-\mathcal{H}'}$.

This is the traditional perturbative approach for integer positive N. Our claim concerns the choice $N = -1$, so we should be able to define the model in analytic continuation in N, and perform a limit $N \to -1$. This task is particularly hard. Nonetheless, at the special values of integer non-positive N we can invoke the bosonic-fermionic correspondence, such that fermionic degrees of freedoms "counts" as -1, and argue that a σ-model in which the N-component sphere is replaced by a *supersphere* with one real (bosonic) and two Grassmann (fermionic) components could describe the $N = -1$ non-linear σ-model.

Define the spin variables

$$\boldsymbol{\sigma}_i = (\phi_i, \psi_i, \bar{\psi}_i) \tag{2.25}$$

with $\phi \in \mathbb{R}$ and $\bar{\psi}$, ψ pairs of Grassmann variables, and introduce the orthosymplectic bilinear form defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} \tag{2.26}$$

that is, given two spins $\boldsymbol{\sigma}_i$, $\boldsymbol{\sigma}_j$, the new "scalar product" is

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \phi_i \phi_j + t(\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j) \tag{2.27}$$

We can solve the constraint $1 = |\boldsymbol{\sigma}_i|^2 = \phi_i^2 + 2t\bar{\psi}_i\psi_i$ w.r.t. the bosonic variable, exploiting the nilpotency of the Grassmann variables, obtaining $\phi_i = 1 - t\bar{\psi}_i\psi_i = e^{-t\bar{\psi}_i\psi_i}$, and similarly for the Jacobian,

$$\prod_i \frac{1}{\sqrt{1 - 2t\bar{\psi}_i\psi_i}} = \exp \sum_i \left(-\frac{1}{2} \log(1 - 2t\bar{\psi}_i\psi_i) \right) = \exp \left(t \sum_i \bar{\psi}_i\psi_i \right) \tag{2.28}$$

Collecting all the contributions, we end up with the Hamiltonian

$$-\mathcal{H}(\psi, \bar{\psi}) = \bar{\psi} L \psi + t \sum_i \bar{\psi}_i\psi_i + \frac{t}{2} \sum_{ij} \bar{\psi}_i\psi_i L_{ij} \bar{\psi}_j\psi_j \tag{2.29}$$

which coincides with the one in formula (2.20). We remark that the only delicate point in the procedure has been the formal correspondence

$$\boldsymbol{\pi}_i \boldsymbol{\pi}_j \to \psi_i\bar{\psi}_j - \bar{\psi}_i\psi_j \ , \tag{2.30}$$

which directly leads to

$$\frac{1}{2} \sum_{ij} L_{ij} \boldsymbol{\pi}_i \boldsymbol{\pi}_j \quad \longrightarrow \quad - \sum_{ij} \bar{\psi}_i L_{ij} \psi_j \tag{2.31}$$

$$-\frac{T}{2} \sum_i \boldsymbol{\pi}_i^2 \quad \longrightarrow \quad +T \sum_i \bar{\psi}_i\psi_i \tag{2.32}$$

$$-\frac{T}{4} \sum_{\langle ij \rangle} w_{ij} \boldsymbol{\pi}_i^2 \boldsymbol{\pi}_j^2 \quad \longrightarrow \quad \frac{T}{2} \sum_{ij} \bar{\psi}_i\psi_i L_{ij} \bar{\psi}_j\psi_j \tag{2.33}$$

$$\mathcal{O}(\boldsymbol{\pi}_i^4, \boldsymbol{\pi}_j^4) \quad \longrightarrow \quad 0 \tag{2.34}$$

and so taking $T = -t$ we have that $\mathcal{H}'(\boldsymbol{\pi}) = \mathcal{H}(\psi, \bar{\psi})$. We remark the important inversion of the sign, the generating function for spanning forests at a positive value of parameter t corresponds to the antiferromagnetic model.

Now that we showed the equivalence between the two models, we can argue from the known results on the beta function for the non-linear σ model that our model of spanning forests is asymptotically free, i.e. under renormalization group in a positive neighbourhood of $t = 0$, the flow is marginally repulsive. This fact follows from the statement that at $N = -1$ (actually in all the range $N < 2$) the non-linear σ model is asymptotically free in a *negative* neighbourhood of the temperature $T = 0$, and that there is a crucial *minus sign* in the correspondence between the temperature parameter T of the non-linear σ model and the multiplier t of the spanning-forest generating function.

A further important consequence of this correspondence is the following: the perturbative expansion for the non-linear σ-model has already been performed on a square lattice up to four loops [2, 3], using the method that we will show in Chapter 6, so we are able to write the beta function up to four loop (we will define beta function and talk about the general formulaion of the Renormalization Group analisys in Chapter 5). Since this is equivalent to write the beta function for the forests model on a square lattice, we are able to write it in term of the t paramater of the action

$$\frac{\mathrm{d}t}{\mathrm{d}\ell} = \frac{3}{(2\pi)^2}t^2 - \frac{3}{(2\pi)^2}t^3 + \frac{c}{(2\pi)^3}t^4 + O(t^4) \tag{2.35}$$

where c is the first non universal coefficient that depends from the lattice, that for square lattice is equal to 2.34278457, as reported in [1].

3. The Potts model

3.1 The model

The Potts model is one of the possible generalizations of the Ising model: each spin, instead of being allowed to be directed up or down, can be denoted by one of q colors.

Let $G = (V, E)$ a graph, with V and E respectively the sets of vertices and edges; we label the vertices with a latin index $i = 1, \dots, V$ and we associate to each site i a 'spin' σ_i, which can take q values, with q a positive integer (for definiteness, one can think of $\sigma_i \in \mathbb{Z}_q = \{1, 2, \dots, q\}$, or in $\{e^{2\pi i n/q}\}_{n=0,\dots,q-1}$, or in $\{\text{red}, \text{blue}, \text{green}, \dots\}$ – remark that the ensemble of spin variable corresponds to the case $q = 2$, written in terms of "roots of unity").

For each pair $\langle ij \rangle$ of adjacent spins, we have a delta-function interaction $-J\delta(\sigma_i, \sigma_j)$ in the Hamiltonian, where J is the coupling costant. We write $\boldsymbol{\sigma} = \{\sigma_i\}_{i \in V}$ to denote the spin configuration of the system; the Hamiltonian of the Potts model on G is

$$\mathcal{H}(\boldsymbol{\sigma}) = -J \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j) \,. \tag{3.1}$$

If we introduce the costant

$$v = e^{\beta J} - 1 \,, \tag{3.2}$$

the partition function $Z = \sum_{\boldsymbol{\sigma}} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})}$ of the q-state Potts model on G with parameter v can be written as

$$Z_G(q, v) = \sum_{\boldsymbol{\sigma}} \prod_{\langle ij \rangle} \Big(1 + v\, \delta(\sigma_i, \sigma_j) \Big) \,. \tag{3.3}$$

The model is *ferromagnetic* if $J \geq 0$ ($v \geq 0$), in this case nearby spins with same color are energetically favorite, the model is *antiferromagnetic* if $J \in \{-\infty, 0\}$ ($-1 \leq v \leq 0$), and if $v \notin [-1, \infty)$ it is *unphysical*, i.e. the Boltzmann weight is no more a positive quantity, as we expect for a statistical mechanic model.

3.2 Fortuin-Kasteleyn representation

The fact that the interaction is via delta functions has strong implications on the combinatorial content of the model, which allows for a redefinition in which the

parameter q has a natural analytic continuation. This redefinition is called *Fortuin-Kasteleyn representation* of the Potts model. We look at the partition function in the form (3.3)

$$Z_G(q,v) = \sum_{\boldsymbol{\sigma}} \prod_{\langle ij \rangle} \Big(1 + v\,\delta(\sigma_i,\sigma_j)\Big) \;,$$

Each term of the sum is a product of $E = |E(G)|$ factors, one per edge; the factor for edge (ij) is either 1 or $v\,\delta(\sigma_i,\sigma_j)$, in this second case we place a mark on the edge. The 2^E choices of factors are in bijection with subsets of $E(G)$, and thus, in accord with definition in 1.1, with spanning subgraphs of G. Each subgraph takes a weight $v^{|E'|}$. Furthermore each connected component, for the effect of the delta function, is made of spins of the same color; so summing over allowed spin configurations gives a contribution q^C for each subgraphs with C components (including isolated vertices). So the Potts partition function can be related to sum over the spanning subgraphs of G

$$Z_G(q,v) = \sum_S q^{C(S)} v^{|E'(S)|} \tag{3.4}$$

where $C(S)$ is the number of connected components of the subgraph $S = (V, E')$. This is the Fortuin-Kasteleyn representation in the simplest case of couplings J all equals. The same reasoning can be repeated if we have a set $\{J_{ij}\}$ of coupling constants for the edges (ij), so also $v_{ij} = \exp(\beta J_{ij}) - 1$ becomes edge-dependent and writing $\mathbf{v} = \{v_e\}_{e\in E}$ for the configuration of parameters of the system, we obtain

$$Z_G(q,\mathbf{v}) = \sum_{E'\subseteq E} q^{C(E')} \prod_{e\in E'} v_e \;; \tag{3.5}$$

this subgraph expansion of the Potts model partition function was introduced in the late 1960s by Fortuin and Kasteleyn. It is useful to rewrite it using the Euler formula (1.1), that for each spanning subgraph $S = (V, E')$ is

$$|V| + L(S) = |E'(S)| + C(S) \;, \tag{3.6}$$

with $L(S)$ the number of its independent loops; so (3.5) is

$$Z_G(q,\mathbf{v}) = q^{|V|} \sum_{E'\subseteq E} q^{L(E')} \prod_{e\in E'} \frac{v_e}{q} \;. \tag{3.7}$$

This alternative statement is fruitful especially when the graph G is connected and planar. Indeed, in this case, as explained in Chapter 1, a dual graph G^* can be introduced in a natural way, such that duality also extends to a bijection among the spanning subgraphs of the two dual graphs. Pairs $(S = (V, E'), S^* = (V^*, E'^*))$ of spanning subgraphs in bijection are such that

$$V \equiv V(G) \qquad\qquad V^* \equiv V(G^*) \tag{3.8}$$

$$E' \cup (E'^*)^* = E(G) \qquad\qquad E' \cap (E'^*)^* = \emptyset \tag{3.9}$$

$$C(S) = L(S^*) \qquad\qquad L(S) = C(S^*) \tag{3.10}$$

that is, the vertex sets coincide with the ones of the corresponding graphs (spanning property), the edge sets are complementars, when the correspondence among edges and dual-edges is understood, and the number of connected components C and the cyclomatic number L (the number of independent loops) are exchanged.

Now, the partition function for a graph G at a certain value $\mathbf{v}$ of the couplings is in relation with the partition function for the dual graph G^* at a new set $\mathbf{v}'$ of couplings, indeed

$$Z_{G^*}(q, \mathbf{v}') = \sum_{S^* \subseteq G^*} q^{C(S^*)} \prod_{e^* \in E'^*} v'_e = \sum_{S \subseteq G} q^{L(S)} \prod_{e \notin E'} v'_e = q^{-|V|} \prod_{e \in E(G)} v'_e Z_G(q, \mathbf{v}) \tag{3.11}$$

where $v_e v'_e = q$ for each edge.

At the starting point, in the basic definition of the Potts model, q is a positive integer and v_e are real numbers in the interval $(-1, +\infty)$, but the Fortuin-Kasteleyn representation shows that the partition function $Z_G(q, v)$ for a graph with all couplings equal to v is in fact a polynomial in q and v. This allows to interpret q and v as taking arbitrary real or even complex values, and to study the phase diagram of the Potts model in the real (q, v)-plane or in complex (q, v)-space[1]; in mathematical language this is equivalent to study the complex zeros of the Tutte polynomial, indeed this polynomial and $Z_G(q, v)$ are essentially equivalent. The Tutte polynomial T_G is a polynomial in two variables, defined by

$$T_G(x, y) = \sum_{E' \subseteq E} (x-1)^{C(E')-C_0} (y-1)^{L(E')} \tag{3.12}$$

where with C_0 we denote the number of connected components of the original graph G and, as said above, with $C(E')$ the one of its subgraph $S = (V, E')$; Comparing with (3.4) we find that

$$Z_G(q, v) = T_G(1 + \tfrac{q}{v}, 1 + v) \left(\frac{q}{v}\right)^C v^{|V|} . \tag{3.13}$$

The Tutte polynomial has been widely studied in the context of algebraic properties of graphs, see for example [29].

From the location of the complex zeros of the partition function, according to the Yang-Lee theorem of phase transition, we gain information about the phase transition of the model [4].

3.3 The $q \to 0$ limit

Now that we have seen how to study the Potts model partition function $Z_G(q, v)$ with arbitrary and also non physical values for q and v, as q complex value or

[1] This is a particular result for Potts model with delta interaction, for Potts-Gauge model [28] the Fortuin-Kasteleyn representation can not be extended to analytic values of q. It is interesting to study the general class of models like Potts that admit an analytic extension of FK representation.

$v < -1$, we are allowed to deal with the $q \to 0$ limit. This limit brings back the Potts model to our generating function (2.20) for unrooted spanning forests. We perform a $q \to 0$ with fixed values $\mathbf{w} = \mathbf{v}/q$; from (3.7) we see that this selects out the subgraphs $S = (V, E') \subseteq G = (V, E)$ having the smallest possible number of independent loops. We therefore have

$$\lim_{q \to 0} q^{-|V|} Z_G(q, q\mathbf{w}) = F_G(\mathbf{w}) \,, \tag{3.14}$$

where

$$F_G(\mathbf{w}) = \sum_{\substack{E' \subseteq E \\ C(E')=0}} \prod_{e \in E'} w_e \tag{3.15}$$

is the generating polynomial of spanning forests, i.e. spanning subgraphs not containing any loop, with a weight w_e for each edge e in the forests. To see the correspondence with our generating function (2.20)

$$Z_F = \sum_{\substack{F \in \mathcal{F} \\ F=(F_1,\ldots,F_\ell)}} t^\ell \prod_{e \in H} w_e = \sum_{\substack{E' \subseteq E \\ C(E')=0}} t^\ell \prod_{e \in E'} w_e \tag{3.16}$$

we make the same passage made above (to pass from the for the Fortuin-Kasteleyn representation (3.5) to the (3.7)) exploiting the Euler formula, we have

$$Z_F = t^{|V|} \sum_{\substack{E' \subseteq E \\ C(E')=0}} \prod_{e \in H} \frac{w_e}{t} \,. \tag{3.17}$$

So the identification between the Potts model and the fermionic model is obtained trough

$$w_e^{Potts} = \frac{w_e}{t} \,. \tag{3.18}$$

Since we are going to develop our fermionic model on a lattice where we have for all the edges $w_e = 1$, we have the correspondence more easily trough

$$w_e^{Potts} = \frac{1}{t} \,. \tag{3.19}$$

3.4 Potts model phase diagram

On square lattice the Potts model has been widely studied and it is possible to draw a phase diagram for it, even though there are some aspects not fully clear. A crucial ingredient is the graph duality explained in the previous section, which for hte suare lattice, that is self-dual, gives important consequences (for example, it allows to identify directly with a simple reasoning the transition point, in the verified

conjecture that it coincides with the only fixed point of the duality convolution). We consider the phase diagram in (q, v)-plane, and we draw the curves:

$$v = \sqrt{q}\ , \tag{3.20}$$

$$v = -2 \pm \sqrt{4-q}\ , \tag{3.21}$$

on which Baxter [21] has found the expression for the free energy in closed form. The first one corresponds to the ferromagnetic phase-transition line, the second with + sign has been conjectured to correspond to antiferromagnetic phase-transition point [2]. So above the (3.20) line we have the ferromagnetic region, in the region bounded up by (3.20) and down by (3.21) with + sign we have a noncritical region, and under (3.21) with − sign we have the antiferromagnetic region. In the limit of $q \to 0$ with $w = v/q$ fixed, we distinguish the three regions around the point $(q, v) = (0, 0)$, parametrized by the slope $w = v/q$: all the values of w in the range $[-1/4 = w_0, +\infty]$ are in a noncritical region, while the ones in $[-\infty, -1/4]$ are in the antiferromagnetic region, and $w = +\infty$ is the ferromagnetic critical point.

A similar behaviour appears on the triangular lattice, but the antiferromagnetic phase-transition curve is not known. Again, two special curves has been found by the group of Baxter, on which the free energy is known in closed form

$$v^3 + 3v^2 - q = 0\ , \tag{3.22}$$

$$v = -1\ , \tag{3.23}$$

the first of them, in the half-plane $v > 0$, corresponds to the ferromagnetic phase-transition point. Unfortunately, the second one is not critical in general but it cointains the critical point for $q = 2$ and $q = 4$. On the other side, the antiferromagnetic critical curve is not known, and neither its existence in general is a confirmed fact (remark that, in the antiferromagnetic region, the triangular and the square lattice are deeply different, as the first one shows the important ingredient of frustration). Nonetheless, A.D. Sokal et al. propose the existence of a 'mystery' curve of fixed points that, like the one on the square lattice, goes from $(q, v) = (0, 0)$ in the quadrant $v < 0$ and $q > 0$. They also have strong numerical suggestions on the fact that this curve must exist at least in a right-neighbourhood of $(q, v) = (0, 0)$. Following their convention, the slope of this 'mystery' curve is called $w_0(tri)$, and as for the square lattice, in the limit $q \to 0$ with $w = v/q$ fixed, they distinguish the noncritical region for all the values $w_0(tri) \leq w \leq +\infty$, and an antiferromagnetic region for all the values $-\infty \leq w \leq w_0(tri)$. The results obtained in [4] by a transfer-matrix method, suggest that, both for square and triangular lattice, the $w = w_0$ is a *first-order critical point*, i.e. it is a first order transition point because it shows discontinuity of the first derivatives of the free energy, but also it presents a correlation length that diverges for $w \searrow w_0$ from above, and that is infinite for $w \leq w_0$.

[2] This conjecture agrees with other known results for the Potts model with $q = 2$ (Ising) $q = 3, 4$, see [4] for a complete overview of known results and conjectures for the Potts model phase diagram, also in terms of conformal field theory, and a wide source of references on it.

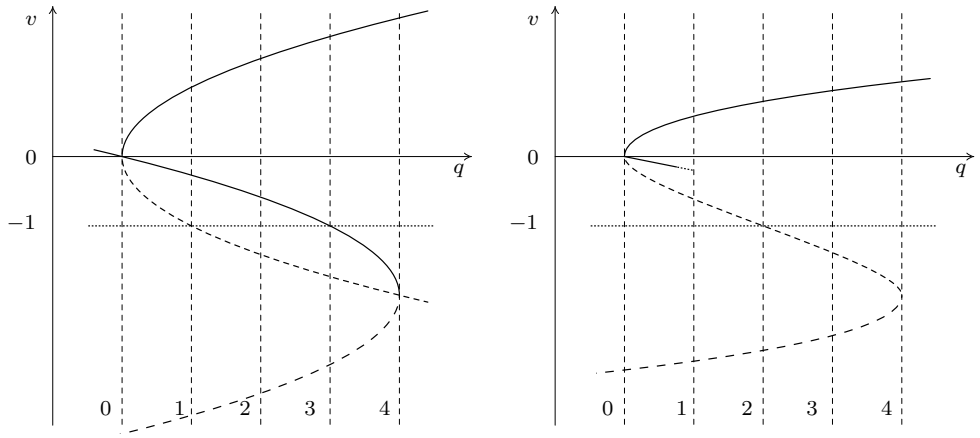


Figure 3.1 A schematic description of the Potts model phase diagram in the parameter space of real q and v, on the left for the square lattice, and on the right for the triangular lattice. The integer values of q are highlighted with the dashed gray vertical lines. The Baxter lines are plotted as bold lines, with different dashing depending on their physical meaning. The $v = 0$ real axix corresponds to infinite temperature. The dotted $v = -1$ line corresponds to the zero-temperature antiferromagnet (pure "colouring" problem). The solid parabolic curves are the ferromagnetic (up) and antiferromagnetic (down, for the square lattice) curves. The dashed lines are prosecutions of the Baxter lines, whose physical meaning is less clear. Remark, for the triangular-lattice figure, the line which starts at $q = v = 0$ ang goes down with slope approx. -0.1753, corresponding to $w_0(tri)$, whose prosecution is unknown.

Furthermore in [4] it is found for the square lattice $w_0 = -1/4$ as expected, and it is estimated on the triangular lattice $w_0(tri) = -0.1753 \pm 0.0002$.

4. Renormalization of the non-linear σ-model in two dimensions

4.1 The non-linear σ-model

How we saw in 2.3, the generating function for unrooted spanning forests can be put in relation with the partition function of the $N = -1$ lattice non-linear σ-model defined by the Hamiltonian

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{T}\sum_{\langle ij\rangle} w_{ij}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1) \tag{4.1}$$

where $\boldsymbol{\sigma}_i \in \mathbb{R}^N$ is a vector with N components $\boldsymbol{\sigma}_i = \{\sigma_i^1, \dots, \sigma_i^N\}$ constrained to lie on the unit sphere: $\boldsymbol{\sigma}_i^2 = 1$; i is the site index ($i = 1, \dots, V$), and the sum in (4.1) runs over all the pairs of first neighbours among the V sites of the graph.

Here we consider as graph a regular d-dimensional hypercubic lattice and so the sum becomes

$$\sum_{\langle ij\rangle} \to \sum_{i,\mu} \tag{4.2}$$

with $i = 1, \dots, V$ and $\mu = 1, 2, \dots, d$, so the first neighbours of site i will be $\{i \pm \hat{\mu}\}$, with $\hat{\mu}$ the versor in the μ direction. For $w_{i,i+\hat{\mu}} = 1$, in the continuum limit the Hamiltonian reduces to

$$\mathcal{H}(\boldsymbol{\sigma}) = \frac{1}{2T}\int d^d x\, \partial_\mu \boldsymbol{\sigma}(x) \cdot \partial_\mu \boldsymbol{\sigma}(x) \tag{4.3}$$

which is the widely-studied field theory in the continuum, counterpart of the non-linear σ-model in Statistical Mechanics. The action in standard notation is

$$\mathcal{S}(\boldsymbol{\sigma}) = \frac{1}{2g}\int d^d x (\partial_\mu \boldsymbol{\sigma}(x))^2 \tag{4.4}$$

where the coupling costant is $g = T$, and the Euclidean partition function is

$$Z(g) = \int \prod_x d\boldsymbol{\sigma}(x)\delta(\boldsymbol{\sigma}^2 - 1) e^{-\mathcal{S}(\boldsymbol{\sigma})}\,, \tag{4.5}$$

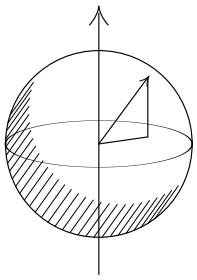


Figure 4.1 The decomposition of the σ-model field, for the case $N = 3$. The magnetic field is directed along the z axis.

the delta function sets the modulus of the N-component field

$$\boldsymbol{\sigma}^2(x) = 1 \; ; \tag{4.6}$$

from this quadratic form follows the name *non-linear* model.

In the continuum limit we have to adopt a continuum regularization, we shall use dimensional regularization. To satisfy the $O(N)$ invariant constraint it is usual to split the N-vector in a $(N-1)$ component field and in a one-component (or scalar) field

$$\boldsymbol{\sigma}(x) = \left\{ \begin{array}{l} \boldsymbol{\pi}(x) \\ \sigma(x) \end{array} \right.$$

with

$$\pi^\alpha = \phi^\alpha(x) \quad \alpha = 1, \ldots, N-1 \qquad \sigma(x) = \sqrt{1 - \boldsymbol{\pi}^2(x)}. \tag{4.7}$$

this particular choice is convenient for the renormalization procedure, as we will see in the next section.

Taking into account the Jacobian of the transformation, the model in terms of the new fields is

$$Z = \int \prod_x \frac{d\boldsymbol{\pi}(x)}{\sqrt{1-\boldsymbol{\pi}^2}} \, e^{-\mathcal{S}(\boldsymbol{\pi})} \tag{4.8}$$

$$\mathcal{S}(\boldsymbol{\pi}) = \frac{1}{2g} \int d^d x \left[(\partial_\mu \boldsymbol{\pi}(x))^2 + \frac{(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi})^2}{1 - \boldsymbol{\pi}^2(x)} \right] \; . \tag{4.9}$$

4.2 The renormalization method

4.2.1 Power counting and regularization

The first way to look if a theory is renormalizable, it is the power counting method, i.e. going to look the superficial degree of divergences of its Feynman diagrams. By power counting, it can be seen that the non-linear σ-model is super renormalizable

in 1 dimension, renormalizable in $d = 2$ and not renormalizable in $d > 2$; in fact the dimension $[\boldsymbol{\pi}]$ of the field is

$$[\boldsymbol{\pi}] = \frac{d-2}{2} \tag{4.10}$$

and so in $d = 2$, since the field is dimensionless, all the terms of the Lagrangian are of dimension 2 and the model is just renormalizable (that is neither super nor non-renormalizable). So from now on in this chapter, we will focus on the model in two dimensions, since it is the only one perturbatively renormalizable. We also make use of dimensional regularization, that is, we consider the theory at a generic dimension d, and reach the limit $d = 2$ (which suffers from divergences) from above. We stress this fact using the notation $d = 2 + \epsilon$.

Furthermore, in order to make a perturbative expansion, we have to take a classical minimum of the action (the vacuum of the theory) and expand around it, but, as a consequence of the $O(N)$ invariance, the action (4.4) has a continuous set of degenerate minima, and there exist $N - 1$ massless Goldstone modes, which also show up as divergences in the propagator in the $d = 2$ limit.

We solve both problems adding to the action an external magnetic field h, linearly coupled to the first component of the field $\boldsymbol{\sigma}$. The new action is

$$\mathcal{S}(\boldsymbol{\pi}, h) = \mathcal{S}(\boldsymbol{\pi}) - h \int d^2x \sqrt{1 - \boldsymbol{\pi}^2(x)} \tag{4.11}$$

where $\mathcal{S}(\boldsymbol{\pi})$ is the action in (4.9). As the new term in (4.11) gives a mass $h^{1/2}$ to each Goldstone mode π_i, we provide a genuine classical minimum of the action

$$\boldsymbol{\sigma}(x) = \begin{cases} \mathbf{0} \\ 1 \end{cases}$$

The classical expectations for transverse fields, $|\boldsymbol{\pi}(x)|$, are of order $\sqrt{g}$; this allows us to expand in powers of $\boldsymbol{\pi}^2$ the action. As a consequence, the term h also appears as a mass regularization of the diverging massless propagator ($p^{-2} \to (p^2 + h)^{-1}$), which caused the infrared divergences. On the other side, ultraviolet divergencies are regularized by our choice of dimensional regularization.

The apperance of these infrared divergences is expected from the statement of the Mermin-Wagner-Coleman theorem, which asserts that in $d = 2$ can not be found a local gauge-invariant order parameter and so $O(N)$ symmetry is never broken; in fact it is not possible to fix one minimum of the energy because the presence of Goldstone modes (i.e. our massless modes) makes easy to pass from one minimun to another.

4.2.2 Ward identities

Now that we regularized the model, we discuss the Ward identities expressing the consequence of the $O(N)$ symmetry for correlation functions.

Consider the generating functional of the $\boldsymbol{\pi}$ Green functions

$$Z[\boldsymbol{J},h]=\int\prod_x\frac{d\boldsymbol{\pi}(x)}{\sqrt{1-\boldsymbol{\pi}^2}}\,\exp\left[-\mathcal{S}(h)+\frac{1}{g}\int d^2x\;\boldsymbol{J}(x)\cdot\boldsymbol{\pi}(x)\right] \tag{4.12}$$

with

$$\mathcal{S}(\boldsymbol{\pi},h)=\frac{1}{g}\int d^dx\left[\frac{1}{2}(\partial_\mu\boldsymbol{\pi}(x))^2+\frac{1}{2}\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2}{1-\boldsymbol{\pi}^2(x)}-h(x)\sqrt{1-\boldsymbol{\pi}^2(x)}\right]\;, \tag{4.13}$$

here we took for a more general discussion $h(x)$, depending from position#. After the parametrization that we showed above (4.7)

$$\boldsymbol{\sigma}(x)=\begin{cases}\boldsymbol{\pi}(x)\\ \sigma(x)\end{cases}\;,$$

we can decompose the generators of the $O(N)$ algebra into the set of generators of $O(N-1)$ and the complementary set, these are respectively $\frac{(N-1)(N-2)}{2}$ and $N-1$ generators. The first ones act linearly on $\boldsymbol{\pi}$ and leave unchanged σ, so they get a particular infinitesimal rotation because it does not mix together $\boldsymbol{\pi}$ and σ components, moreover these transformations has only trivial consequences on the Ward identities. For these reasons we want to consider the second ones, that are the generators of the complementary set, they act non-linearly on $\boldsymbol{\pi}(x)$ as it is for a generic $O(N)$ rotation, if $\omega^\alpha,\ \alpha=1,\dots,N-1$ are the infinitesimal parameters, the infinitesimal transformation induced on the fields are

$$\delta\pi^\alpha(x)=\omega^\alpha\sigma(x)\;; \tag{4.14}$$

$$\delta\sigma(x)=-\boldsymbol{\omega}\cdot\boldsymbol{\pi}(x)\;. \tag{4.15}$$

So we perform a change of variables on the fields which corresponds to an infinitesimal rotation of the group. In such transformation new operators may be generated, here it is only $\sqrt{1-\boldsymbol{\pi}^2}$. So we have to add new sources for this new operator in the Lagrangian. Then we examine what is generated by this new operator under a group of transformation, and, if new operators arise, we add the corrisponding new sources. This has to be repeated until the system is closed under the group transformations. With the parametrization $(\boldsymbol{\pi},\sigma)$ chosen here, the system is already closed after one step, in fact $\sqrt{1-\boldsymbol{\pi}^2}$ generates nothing but $\boldsymbol{\pi}$.

The Ward identities are the conditions that we have to fix in order to keep, to all order of perturbation theory, the invariance of the generating functional $Z[\boldsymbol{J},h]$ under the transformations (4.14), (4.15). So we ask that

$$0=\delta Z[\boldsymbol{J},h]=\int d^dx\left[\frac{\delta Z[\boldsymbol{J},h]}{\delta\boldsymbol{\pi}(x)}\cdot\delta\boldsymbol{\pi}(x)+\frac{\delta Z[\boldsymbol{J},h]}{\delta\sigma(x)}\cdot\delta\sigma(x)\right]\;. \tag{4.16}$$

We know that our starting action without external sources (neither h nor $\boldsymbol{J}$) is already invariant under rotations, so we have to look at the new source terms. We

solve the functional derivatives $\frac{\delta Z[\boldsymbol{J},h]}{\delta \boldsymbol{\pi}(x)} = \boldsymbol{J}(x)$ and $\frac{\delta Z[\boldsymbol{J},h]}{\delta \sigma(x)} = h(x)$, we use the (4.14), (4.15) to write the infinitesimal variations of the fields, and finally we express $\boldsymbol{J}(x)$ and $\sigma(x)$ as functional derivatives of $Z[\boldsymbol{J},h]$, so we have from (4.16)

$$\begin{aligned} 0 &= \int d^d x [\boldsymbol{J}(x)\cdot\delta\boldsymbol{\pi}(x) + h(x)\ \delta\sigma(x)] \\ &= \sum_{\alpha}^{N-1} \omega^{\alpha} \int d^d x \left[J^{\alpha}(x) \frac{\delta Z[\boldsymbol{J},h]}{\delta h(x)} - h(x) \frac{\delta Z[\boldsymbol{J},h]}{\delta J^{\alpha}(x)} \right] \end{aligned} \tag{4.17}$$

This last equation express, for all α, the $N-1$ Ward identities for the generating functional $Z[\boldsymbol{J},h]$ of the correlation functions. We want to write the same identities for $\Gamma[\boldsymbol{\pi},h]$ the generating functional of the one particle irreducible (1PI) Green functions[1].

In order to do this, we know that

$$W[\boldsymbol{J},h] = g \log Z[\boldsymbol{J},h] \tag{4.18}$$

is the generating functional for the connected Green functions, and from (4.17) it satisfies the identities

$$\int d^d x \left[J^{\alpha}(x) \frac{\delta W[\boldsymbol{J},h]}{\delta h(x)} - h(x) \frac{\delta W[\boldsymbol{J},h]}{\delta J^{\alpha}(x)} \right] = 0 \qquad \alpha = 1,\dots,N-1\ . \tag{4.19}$$

The $\Gamma[\boldsymbol{\pi},h]$ functional is defined as the Legendre transformation of $W[\boldsymbol{J},h]$

$$\Gamma[\boldsymbol{\pi},h] = \int d^d x\, (\boldsymbol{J}(x)\cdot\boldsymbol{\pi}(x) - W[\boldsymbol{J},h]) \tag{4.20}$$

and so with

$$\pi^{\alpha} = \frac{\delta W[\boldsymbol{J},h]}{\delta J^{\alpha}(x)}\ . \tag{4.21}$$

Noticing that

$$\frac{\delta W[\boldsymbol{J},h]}{\delta h(x)} = -\frac{\Gamma[\boldsymbol{\pi},h]}{\delta h(x)} \qquad \text{and} \qquad J^{\alpha}(x) = \frac{\delta\Gamma[\boldsymbol{\pi},h]}{\delta\pi^{\alpha}(x)} \tag{4.22}$$

from (4.19) we obtain

$$\int d^d x \left[\frac{\delta\Gamma[\boldsymbol{\pi},h]}{\delta\pi^{\alpha}(x)} \frac{\delta\Gamma[\boldsymbol{\pi},h]}{\delta h(x)} + h(x)\pi^{\alpha}(x) \right] = 0 \qquad \alpha = 1,\dots,N-1\ , \tag{4.23}$$

these are the Ward identities that we are going to use.

[1] A Green function is *one-particle irreducible* or *1PI* if its Feynman diagram is connected and remains connected when any one internal line is cut.

4.2.3 Renormalization

We want to exploit the Ward identities in order to renormalize the theory. We proceed perturbatively, and so we make an expansion of Γ in terms of Γ_n, that contains only diagrams with just n loops

$$\Gamma[\boldsymbol{\pi}, h] = \sum_{n=0}^{\infty} \Gamma_n g^n \ , \tag{4.24}$$

$\Gamma_0 = \mathcal{S}(\boldsymbol{\pi}, h)$ is the original action without source terms $\boldsymbol{J}$ (it follows for definition (4.20)). For the $O(N)$ invariance of the model, we require to fulfill conditions (4.23): the case $n = 0$ is already satisfied, for the one-loop term the condition reads

$$\int d^d x \left[\frac{\delta \Gamma^{(0)}}{\delta \pi^\alpha(x)} \frac{\delta \Gamma^{(1)}}{\delta h(x)} + \frac{\delta \Gamma^{(1)}}{\delta \pi^\alpha(x)} \frac{\delta \Gamma^{(0)}}{\delta h(x)} \right] = 0 \ , \tag{4.25}$$

for all $\alpha = 1, \ldots, N-1$; from now on, we adopt the same notation of [9] writing this last expression as[2]

$$\Gamma^{(0)} * \Gamma^{(1)} = 0 \ . \tag{4.27}$$

At each order the Γ functional has a finite part and a divergent part, in the ultraviolet limit (i.e. in p-space when $p \to \infty$, where we see the superficially divergent diagrams) the divergent part is predominant and we have

$$\Gamma^{(0)} * \Gamma^{(1)}_{\text{div}} = 0 \ , \tag{4.28}$$

with this condition we are able to write the right counterterm that remove the divergences from the action $\mathcal{S}(\boldsymbol{\pi}, h)$ at tree level. So taking

$$\mathcal{S}^{(1)}_R(\boldsymbol{\pi}, h) = \mathcal{S}(\boldsymbol{\pi}, h) - \frac{1}{g} \Gamma^{(1)}_{\text{div}} \tag{4.29}$$

we constructed a one-loop renormalized action.

In the same way we proceed for the next orders, the procedure is inductive: if the action is renormalized until order $n-1$, renormalization at n loops is achieved from the proper Ward identity that it is

$$\Gamma^{(0)} * \Gamma^{(n)} = -(\Gamma^{(1)} * \Gamma^{(n-1)} + \Gamma^{(2)} * \Gamma^{(n-2)} + \ldots) \tag{4.30}$$

and in the limit of large momenta, we have

$$\Gamma^{(0)} * \Gamma^{(n)}_{\text{div}} = 0 \tag{4.31}$$

[2] We define an operator

$$\Gamma^{(k)} * \equiv \int \left(\frac{\delta \Gamma^{(k)}}{\delta \boldsymbol{\pi}} \frac{\delta}{\delta h} + \frac{\delta \Gamma^{(k)}}{\delta h} \frac{\delta}{\delta \boldsymbol{\pi}} \right) \ . \tag{4.26}$$

this equation allows us to find the right counterterms in order to make the perturbative expansion ultraviolet finite up to n loops. The model in two dimension has been studied, and in Ref.[9] it is shown that the theory can be renormalized with only two renormalization constants Z_g and Z_π and the other are written in term of them. So we define the renormalized quantities in the following way:

$$g \equiv \frac{\mu^{2-d}}{Z_g}\, g_R \,, \tag{4.32}$$

$$\boldsymbol{\pi}(x) \equiv Z_\pi^{1/2} \boldsymbol{\pi}_R(x) \,, \tag{4.33}$$

$$\sigma(x) \equiv Z_\pi^{1/2} \sigma_R(x) = \sqrt{1 - Z_\pi \boldsymbol{\pi}_R^2(x)} \,, \tag{4.34}$$

$$h \equiv \frac{Z_g}{Z_\pi^{1/2}} h_R \,. \tag{4.35}$$

Where μ is the renormalization scale and the factor $N_d = \frac{(4\pi)^{-\epsilon/2}}{\Gamma(1+\epsilon/2)}$ is introduced by dimensional regularization. The renormalized action is

$$\mathcal{S}_R(\boldsymbol{\pi}, h) = \frac{Z_\pi}{2 g_R Z_g} \int d^d x \left[(\partial_\mu \boldsymbol{\pi}_R(x))^2 + \frac{(\boldsymbol{\pi}_R \cdot \partial_\mu \boldsymbol{\pi}_R)^2}{1 - \boldsymbol{\pi}_R^2(x)} \right] - \frac{1}{g_R} \int d^d x h_R(x) \sigma_R(x) \,. \tag{4.36}$$

Furthermore from the condition of renormalization above (4.33), it follows the important relation between bare and renormalized 1PI n-points correlation functions

$$\Gamma_R^{(n)}(p_i; g_R, h_R) = Z_\pi^{n/2}(\mu, g, \epsilon) \Gamma^{(n)}(p_i; g, h, \epsilon) \tag{4.37}$$

where p_i with $i_1, \ldots, n$ are the external momenta.

4.2.4 Renormalization costant and $\overline{MS}$-scheme

We provided an UV cut-off using dimensional regularization and introducing the parameter $d = 2 + \epsilon$, so the UV divergent quantities may arise only in the limit $\epsilon \to 0$ and we can expand the renormalization constant in poles of $1/\epsilon$:

$$Z_\pi(g) = 1 + \sum_{n=0}^{\infty} \frac{\alpha_\pi^{(n)}(g)}{\epsilon^n} + \text{finite quantities when } \epsilon \to 0 \tag{4.38}$$

$$Z_g(g) = 1 + \sum_{n=0}^{\infty} \frac{\alpha_g^{(n)}(g)}{\epsilon^n} + \text{finite quantities when } \epsilon \to 0 \,. \tag{4.39}$$

The *minimal subtraction* scheme or MS is one of the possible schemes that can be implemented to remove the divergent parts and to make the renormalization, and that it is specially useful in dimensional regularization; it consists simply in removing the $(1/\epsilon)$ poles in potentially divergent quantities. In our case, since these $(1/\epsilon)$ poles are accompanied by powers of N_d, it is convenient to subtract these terms as well, this is said the *modified* minimal subtraction scheme or $\overline{MS}$.

5. The renormalization group

5.1 The study of phase transitions

In order to study the properties of a physical sistem, we have to look if it presents some critical behaviour, i.e. we have to study its phase transitions.

Given a model described by the Hamiltonian $\mathcal{H}$ and at the temperature T, if we consider a configuration ω of the system, we write $\mathcal{H}(\omega)$, and its Gibbs distribution is $e^{-\beta\mathcal{H}(\omega)}$ with $\beta = 1/T$; the partition function is

$$Z(\beta) = \sum_{\{\omega\}} e^{-\beta\mathcal{H}(\omega)} \tag{5.1}$$

and the free energy

$$F = -\frac{1}{\beta}\log Z(\beta)\ . \tag{5.2}$$

Investigating the analyticity of the free energy, we gain information about the phase transitions of the model. In general we said to have a phase transition of n-order, if one of the n-th derivatives of F is the first to be discontinuos; usually we encounter only transitions of *first* or *second*-order.

The first kind are said also *discontinuos*, because present discontinuities in thermodynamic quantities (first derivatives of the free energy); so at the critical point it is possible to have coexistence of different phases and the correlation length is generally finite (an example of first kind transition is the condensation of gas into a liquid).

The second kind, instead, are said also *continuos* beacause the physical quantities change continuosly; there can not be a mixed phase, since the correlation length $\xi \to \infty$, all the system approaches the critical region (two examples are: the liquid-gas critical point in a fluid and the paramagnetic to ferromagnetic region transition). This second kind are more interesting, because they may show spontaneus symmetry breaking (SSB).

In order to define a spontaneus symmetry breaking, we have to introduce the *order parameter* of the system. If we have an Hamiltonian $\mathcal{H}$ that is invariant under a group of symmetry G, we have

$$\mathcal{H}(g\,\omega) = \mathcal{H}(\omega) \qquad \forall g \in G\ , \tag{5.3}$$

(in which $g\omega$ is the imagine of the configuration ω under the action of the transformation g).

exponent	physical quantity	behaviour	classical value
α	C_h	$C_H \sim \lvert t\rvert^{\alpha}$	0
β	m	$m \sim (-t)^{\beta}$	1/2
δ	m	$m \sim h^{1/\delta}$	3
γ	ξ_T	$\xi \sim \lvert t\rvert^{-\gamma}$	1
ν	ξ	$\xi \sim \lvert t\rvert^{-\nu}$	1/2
η	$G(\vec{r})$ $\tilde{G}(\vec{k})$	$G(\vec{r}) \sim \lvert\vec{r}\rvert^{-(d-2+\eta)}$ $\tilde{G}(\vec{k}) \sim \lvert\vec{k}\rvert^{-2+\eta)}$	0

Table 5.1 mytable

We said that M is an *order parameter* if:

(*i*) M is an exstensive quantity;

(*ii*) M does not respect the symmetry of $\mathcal{H}$, i.e.

$$M(g\,\omega) = T(g)M(\omega) \qquad \forall g \in G\,. \tag{5.4}$$

with T a linear operator that is a representation of G.

If we have an order parameter, it breaks the symmetry of $\mathcal{H}$, and so we have a SSB. In the paramagnetic to ferromagnetic transition, choosing one of the two direction of magnetization, we have a SSB, in fact such thermodynamic state does not respect the invariance of the Hamiltonian under the spin flip (inversion of all the local magnetization). In the case of the critical point of liquid-gas transition we do not have SSB beacuse the Hamiltonian do not have a symmetry. From now on, if it is not said explicitly, we will refer to second order phase transition.

In order to describe a phase transition, it is common to look at the behaviours of the physical quantities in the surroundings of the critical temperature T_c, that is the temperature at which the transition occurs. So we define as

$$t \equiv \frac{T - T_c}{T_c} \tag{5.5}$$

the *reduced* temperature, and we find the *critical exponents*, i.e. for a quantity A the critical exponent α associated is the exponent that describes its behaviour near T_c in this way:

$$A \sim t^{\alpha}\,. \tag{5.6}$$

In the tabular are shown the classical exponents. Where the physical quantities are referred to a magnetic system: C_h is the specific heat at fixed magnetic field

h , m is the magnetization, ξ_T is the correlation length at fixed temperature, and $G(\vec{r})$, $\tilde{G}(\vec{k})$ are respectively the two point function and its fourier transform.

The study of phase transition benefits from the consolidated results of classical theories: the Van der Waals theory (1873), the Mean Field theory (firstly introducted by Weiss on 1907) and the Landau theory (1965). These classical theories provided the values for critical exponents, said so the classical exponents. Comparing these values with other results coming from experimental data or from analytic computations, we find that they are correct only in a particular range of the dimension d of the system; in fact the occurrence of the phase transitions in a system is strongly dependent on its dimension. To show this, we define two critical dimension: the *superior* one d_{sup}, and the *inferior* one d_{inf}. We have that for a d-dimensional system: if $d > d_{sup}$ the classical exponents are correct, if $d < d_{inf}$ there are no phase transitions. So we are interested in the region $d_{inf} < d < d_{sup}$ where we expect to find the occurrence of critical phenomena.

In order to gain further information about this region, we need an other kind of approach, the new idea is provided in 1974 by the Wilson and Kogut's work about the Renormalization Group analysis ([32]).

We sketch their new basic idea: near the transition point, $\xi \to \infty$ and so the degrees of freedom effectively interacting with each others is $\sim \xi^d$, so it also diverges and every sort of approximated procedure is impossible to handle; since we are interested in long distances behaviour, we make a *coarse graining*, i.e. we reduce the degree of freedom by integrating over short distances. The physical description is the same, but now is described in terms of new coarsed or renormalized variables; iterating this transformation we identify a flux of the Hamiltonian toward a point, that is the critical point of the model. Hystorically the Renormalization Group technique developes the idea of the scaling law, already suggested before by Kadanoff.

5.2 The general formulation

Consider a general critical ($T = T_c$, $\xi = \infty$) Hamiltonian $\mathcal{H}(\phi)$ depending on a set of fields $\phi(x)$, that are continuos variables on a continuos d-dimensional space. The Hamiltonian will in general depend on an infinite number of parameters or coupling constants. We assume that $\mathcal{H}$ can be expanded in powers of ϕ:

$$\mathcal{H}(\phi) = \sum_{n=0} \frac{1}{n!} \int d^d x_1 d^d x_2 \dots d^d x_n \mathcal{H}_n(x_1, x_2, \dots x_n)\phi(x_1)\dots\phi(x_n) \; . \qquad (5.7)$$

and that the Fourier transform of the functions $\mathcal{H}_n$ are regular at low momenta, i.e. the theory is regularized at short distance.

Considering the generating functional $W[\boldsymbol{J}] = 1/\beta \log Z[\boldsymbol{J}]$ of the connected correlation functions, we want to consider a n-points connected function $W^{(n)}(x_1, \dots, x_n)$ and look at its behaviour at long distances: we introduce a dilatation parameter λ and we evaluate $W^{(n)}(\lambda x_1, \dots, \lambda x_n)$ when λ becomes large.

We define the Hamiltonian $\mathcal{H}_\lambda(\phi)$, assuming that its connected correlation functions are:

$$W_\lambda^{(n)}(x_1,\ldots,x_n) = Z^{-n/2}(\lambda)\; W^{(n)}(\lambda x_1,\ldots,\lambda x_n) \tag{5.8}$$

with $Z(1) = 1$. In the case of models invariant under space translations, (5.8) in p-space becomes:

$$\tilde{W}_\lambda^{(n)}(p_1,\ldots,p_n) = Z^{-n/2}(\lambda)\lambda^{(1-n)d}\; \tilde{W}^{(n)}(p_1/\lambda,\ldots,p_n/\lambda)\;. \tag{5.9}$$

The mapping $\mathcal{H}(\phi) \to \mathcal{H}_\lambda(\phi)$ is a RG transformation. The coupling constants appearing in $\mathcal{H}_\lambda$ are now all explicit functions of λ. Let us assume that, when λ becomes large, the Hamiltonian $\mathcal{H}_\lambda(\phi)$ has a limit $\mathcal{H}^\star$, called the fixed point Hamiltonian. If such a fixed point exists in hamiltonian space, then the correlations functions $W_\lambda^{(n)}$ have corrisponding limits $W_\star^{(n)}$ and the (5.8) becomes:

$$W^{(n)}(\lambda x_1,\ldots,\lambda x_n) \underset{\lambda\to\infty}{\sim} Z^{n/2}(\lambda)\; W_\star^{(n)}(x_1,\ldots,x_n)\;. \tag{5.10}$$

If we consider a second scale parameter ρ, using (5.10) we can write:

$$W^{(n)}(\lambda\rho x_1,\ldots,\lambda\rho x_n) \underset{\lambda\to\infty}{\sim} Z^{n/2}(\lambda)\; W_\star^{(n)}(\rho x_1,\ldots,\rho x_n) \tag{5.11}$$

$$W^{(n)}(\lambda\rho x_1,\ldots,\lambda\rho x_n) \underset{\lambda\to\infty}{\sim} Z^{n/2}(\lambda\rho)\; W_\star^{(n)}(x_1,\ldots,x_n) \tag{5.12}$$

and from the equivalence of the two l.h.s. it follows

$$W_\star^{(n)}(\rho x_1,\ldots,\rho x_n) = Z_\star^{n/2}(\rho)\; W_\star^{(n)}(x_1,\ldots,x_n)\,, \tag{5.13}$$

with

$$Z_\star(\rho) = \lim_{\lambda\to\infty} \frac{Z(\lambda\rho)}{Z(\lambda)}\;. \tag{5.14}$$

The relation (5.13) says that $W^{(n)}$ is a homogeneus function [1] in n variables, and from this it follows s

$$Z_\star(\lambda) = \lambda^{-2d_\phi} \tag{5.16}$$

where d_ϕ is a positive number, called the dimension of the order parameter $\phi(x)$, and that is peculiar to the fixed point. Because of the positivity of d_ϕ, applying the (5.13) to the 2-points connected function, it results that it diverges at the fixed point Hamiltonian, and so that $\mathcal{H}^\star$ is necessarily critical.

[1] A function $f(x)$ is *homogeneus* in x if

$$f(\lambda x) = g(\lambda) f(x) \quad \forall\lambda\in\mathbb{R}\;; \tag{5.15}$$

from this it follows that $g(\lambda) = \lambda^p$, where p denotes the degree of homogeneity. In fact, taking another parameter μ, we know that $f(\lambda\mu x) = g(\lambda)f(\mu x) = g(\lambda)g(\mu)f(x)$, but also $f(\lambda\mu x) = g(\lambda\mu)f(x)$, so it is $g(\lambda)g(\mu) = g(\lambda\mu)$ and this is satisfied only if the function g has a power law behaviour.

From (5.14) it follows that also $Z(\lambda)$ has asymptotically a power law behaviour, so we rewrite the (5.10) as

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda\to\infty}{\sim} \lambda^{-nd_\phi}\, W_\star^{(n)}(x_1, \ldots, x_n)\,; \tag{5.17}$$

from this relation, we know that the critical behaviour of the correlation functions depends only on the fixed point Hamiltonian. In other words the correlation functions corresponding to all hamiltonians which flow, after RG transformations, into the same fixed point, have the same critical behaviour. This property is called universality and the space of critical hamiltonians is thus divided into universality classes.

In order to study this flow toward the fixed point Hamiltonian, we write the RG equations. We perform a small dilatation of the parameter λ until $\lambda(1+\delta\lambda/\lambda)$, we describe this transformation in differential form in terms of a mapping $\mathcal{R}$ of the space of critical hamiltonians into itself and a real function γ defined on the space of hamiltonians:

$$\lambda\frac{d}{d\lambda}\mathcal{H}_\lambda = \mathcal{R}[\mathcal{H}_\lambda]\,, \tag{5.18}$$

$$\lambda\frac{d}{d\lambda}\log Z(\lambda) = 2 - d - \gamma[\mathcal{H}_\lambda]\,. \tag{5.19}$$

In this way the fixed Hamiltonian will satisfies:

$$\mathcal{R}[\mathcal{H}^\star] = 0\,, \tag{5.20}$$

and according to (5.16), the dimension of the field ϕ is:

$$d_\phi = \frac{1}{2}(d - 2 + \gamma[\mathcal{H}_\lambda]). \tag{5.21}$$

Since we are interested in the study of the critical behaviour in the surroundings of the critical temperature, we assume that near T_c, i.e. close to $\mathcal{H}^\star$, we can linearize the RG equations above, assuming:

$$\mathcal{H}_\lambda = \mathcal{H}^\star + \Delta\mathcal{H}_\lambda\,, \tag{5.22}$$

$$\lambda\frac{d}{d\lambda}\Delta\mathcal{H}_\lambda = L^\star(\Delta\mathcal{H}_\lambda)\,, \tag{5.23}$$

where $L^\star$ is a linear operator independent of λ, acting on hamiltonian space. If $L^\star$ has a discrete set of eigenvalues l_i corresponding to a set of eigenoperators $\mathcal{O}_i$, then $\Delta\mathcal{H}_\lambda$ can be expanded on the $\mathcal{O}_i$:

$$\Delta\mathcal{H}_\lambda = \sum h_i(\lambda)\mathcal{O}_i\,, \tag{5.24}$$

and the trasformation (5.23) becomes:

$$\lambda\frac{d}{d\lambda}h_i(\lambda) = l_i h_i(\lambda)\,, \tag{5.25}$$

from which it follows

$$h_i(\lambda) = \lambda^{l_i} h_i(1) \ . \tag{5.26}$$

The eigenoperators $\mathcal{O}_i$ can be classified into three families:

- *relevant*, if the eigenvalue associated $l_i > 0$; the component of $\mathcal{H}_\lambda$ along a similar operator will grow with λ and so causes a relevant movement from $\mathcal{H}^\star$;
- *irrelevant*, if $l_i < 0$; the corrispondent component of $\mathcal{H}_\lambda$ will go to zero when λ grows;
- *marginal*, if $l_i = 0$; the corrispondent component of $\mathcal{H}_\lambda$ is stable in the linear approximation, to study its behaviour we have to expand up the further order, generically it is $\lambda \frac{d}{d\lambda} h_i(\lambda) \sim h_i^2$ and so $h_i(\lambda) \sim 1/\log(\lambda)$; so marginal operators usually take logarithmic approach to the critical point.

The flux toward the fixed point is determined by the relevant operators, that we expect to correspond to the significant physical quantities of the model (for example the temperature or the external magnetic field). If the Hamiltonian is described by a set of k parameters, and we have m relevant operators, these are like knobs to tune in order to arrive at the critical point; and so in the space of hamiltonian parameters we have a $(k-m)$-dimensional surface, the critical surface, in which from each point we start, iterating the $\mathcal{R}$ map we necessarily arrive at the fixed point.

5.3 Renormalization Group Equations for the non-linear σ-model

In chapter 4 we described the renormalization procedure for the non linear σ-model in $d = 2 + \epsilon$ dimension, through the Ward-Identities method. We report the renormalized action found there

$$\mathcal{S}_R(\boldsymbol{\pi}, h) = \frac{Z_\pi}{2 g_R Z_g} \int d^d x \left[(\partial_\mu \boldsymbol{\pi}_R)^2 + (\partial_\mu \sigma_R^2) \right] - \frac{1}{g_R} \int d^d x h_R(x) \sigma_R(x) \ .$$

and the renormalization conditions

$$\begin{aligned} g_R &\equiv \mu^{2-d} Z_g \, g \ , \\ \boldsymbol{\pi}_R(x) &\equiv Z_\pi^{1/2} \boldsymbol{\pi}(x) \ , \\ \sigma_R(x) &\equiv Z_\pi^{1/2} \sigma(x) = \sqrt{1 - Z_\pi \boldsymbol{\pi}^2(x)} \ , \\ h_R &\equiv \frac{Z_g}{Z_\pi^{1/2}} h \ . \end{aligned}$$

In the previous section 5.2 we talk about a different method of renormalization, through the rescaling of the energy scale, in order to investigate the behaviour of

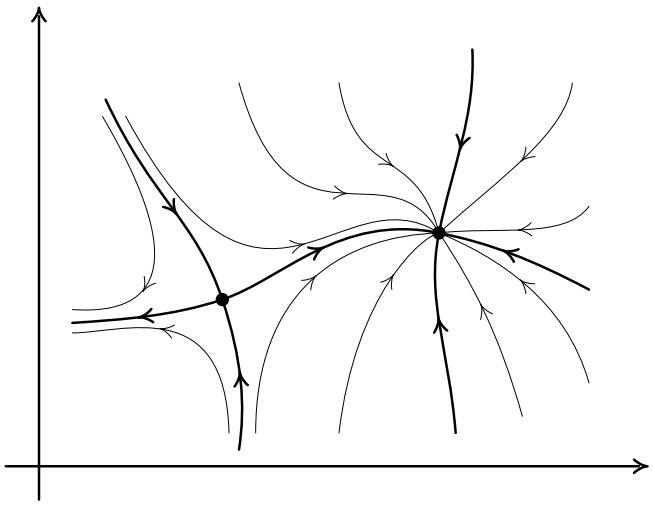


Figure 5.1 A typical Renormalization-group flow in a two-dimensional space of parameters. On the left, a fixed point with one relevant and one irrelevant operator; on the right, a fixed point with two irrelevant operators.

a field theory model with sliding coupling costants. At the light of this different formulation we want to apply the RG analysis to the non linear σ-model.

We have to consider a change of scale and for this reason we will use here, instead of dimensional regularization, an ultraviolet cut-off Λ, that acts as an inverse lattice spacing, fixing $|p| < \Lambda$. Furthermore to observe the dependence of the RG equations from the dimension, we mantain the parameter $d = 2 + \epsilon$.

In this regularization the relations (4.37) read:

$$\Gamma_R^{(n)}(p_i; t_R, h_R, \mu) = Z_\pi^{n/2}(\Lambda/\mu, t)\Gamma^{(n)}(p_i; t, h, \Lambda) \; . \tag{5.27}$$

In the ultraviolet limit $\Lambda \to \infty$ Γ_R has to be a finite quantity and to be cut-off independent, so we expect

$$\Lambda \frac{d\Gamma_R}{d\Lambda}\bigg|_{\mu, t_R} = 0 \tag{5.28}$$

with μ abd t_R fixed, or in terms of the bare constants

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{\mu, t_R} Z_\pi^{n/2}\left(\frac{\Lambda}{\mu}, t\right)\Gamma^{(n)}(p_i; t, h, \Lambda) = \mathcal{O}\left(\frac{\log^p(\Lambda)}{\Lambda^2}\right) \tag{5.29}$$

where p depends from the perturbative order. We introduce the functions β, γ and

ρ, that are dimensionless and are defined with the renormalization μ and t_R fixed,

$$\beta(t) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{\mu,t_R} t \,, \tag{5.30}$$

$$\gamma(t) = -\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{\mu,t_R} \log Z_\pi \,, \tag{5.31}$$

$$\rho(t) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{\mu,t_R} \log h \,. \tag{5.32}$$

By using them, we rewrite (5.29) as:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(t)\frac{\partial}{\partial t} - \frac{n}{2}\gamma(t) + \rho(t)h\frac{\partial}{\partial h}\right]\Gamma^{(n)}(p_i;\Lambda,g) = 0\,, \tag{5.33}$$

that is known as the Callan-Symanzik (CS) equation.

Using the renormalization condition $h_R = Z_g/\sqrt{Z_\pi}h$ we are able to express $\rho(t)$ in terms of $\gamma(t)$ and $\beta(t)$:

$$\rho(t) = \frac{1}{2}\gamma(t) + \frac{\beta(t)}{t} + \epsilon\,. \tag{5.34}$$

Applying the CS equation on $\Gamma^{(2)}$

$$\Gamma^{(2)}(p) = \frac{\Lambda^\epsilon}{t}(p^2+h) + \left[p^2 + \tfrac{1}{2}(N-1)h\right]\frac{1}{(2p)^d}\int_\Lambda \frac{d^dq}{q^2+h} + \mathcal{O}(t)\,, \tag{5.35}$$

and identifying the coefficients of p^2 and h, we derive two equations which determine $\beta(t)$ and $\gamma(t)$ at one-loop order

$$\beta(t) = \epsilon t - \frac{N-2}{2\pi}t^2 + \mathcal{O}(t^3, t^2\epsilon)\,, \tag{5.36}$$

$$\gamma(t) = \frac{N-1}{2\pi}t + \mathcal{O}(t^2, t\epsilon)\,. \tag{5.37}$$

Looking back at their definition, we know that these two functions describe the behaviour of Hamiltonian under a variation of the parameter Λ, i.e. the cut-off on the momenta; if Λ grows we analyse the ultra-violet limit, if Λ decreases the infra-red one.

For $\epsilon \neq 0$ the β function has two fixed points, a trivial one a $t = 0$ and a non trivial one, located at

$$t_c = \frac{2\pi\epsilon}{N-2}\,. \tag{5.38}$$

For $\epsilon > 0$ (when the model is perturbatively renormalizable) from the sign of the β function we see that t_c is an ultraviolet-stable fixed point, and $t = 0$ an ultraviolet instable fixed point (or infrared-stable). The point t_c represents the critical temperature, and when $\epsilon \to 0$ it approaches $t = 0$, in accord with the Mermin-Wagner theorem.

For $d = 2$ we have only the trivial zero fixed point and we have to distinguish among three cases:

- if $N = 2$ the beta function vanishes, this is the well-known case of the XY model, which exhibits a line of fixed point;

- if $N > 2$ the beta function is negative, so $t = 0$ is an ultraviolet-stable fixed point, i.e. at small distances the coupling costant becomes small, and it grows at large distances, this behaviour is called *asymptotic freedom*;

- if $N < 2$ the beta function is positive, so if we approach $t = 0$ as usual from the right, we have an ultraviolet instable fixed point, but if we approach from the left, considering negative temperature, we have an ultraviolet fixed point a $t = 0$ and so the model is asymptotically free.

A negative temperature means that we are considering a σ-model with negative spin coupling, in fact looking at the lattice Hamiltonian (4.1)

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{T}\sum_{\langle ij\rangle} w_{ij}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1) \tag{5.39}$$

if we take the weights $w_{ij} > 0$ we favourite the spins to be parallel to each other and so we construct a ferromagnetic model, instead choosing $w_{ij} < 0$ we realize an antiferromagnetic coupling; fixing $w_{i,i+e_\mu} = 1$, as we did in chapter 4, we restore the two possibilities choosing the correct sign of the temperature.

Resuming, in $d = 2$ we have two ways to find an asymptotically free model: a ferromagnetic model if $N > 2$, and an antiferromagnetic model if $N < 2$. So we shown that the non-linear σ-model in two dimensions is an important example of a field theory which is asymptotically free, that is its short distance behaviour is governed by the weak coupling fixed poin a $t = 0$; it is interesting to study this model, since this feature is believed to be shared with the quantum chromodynamics that is the modern theory of strong interactions.

At this point the statement reported at the end of Chapter 2 about the asymptotic freedom of the fermionic spanning-forest model is clear, since it is the case of $N = -1$ and negative temperature, it belongs to the third case explained above.

5.3.1 Universality of the first two terms of beta function

Finally we want to show here a general result about the beta function, since we will use it in the next chapter. Given a theory with coupling costant g the form of $\beta(g)$ depends on the model and on the regularization method utilized, for example for lattice theory, it is different for square or triangular lattice; nevertheless in the power series expansion of $\beta(g)$ this dependence appears only from the third term of the expansion and going on, the first two terms are instead universal and not model-dependent. This can be seen in the following way: suppose to have the same model regularized in two different schemes, so we call g and $\tilde{g}$ the coupling costants. In the RG procedure, the costant are running, so they depend from the renormalization scale μ in a way peculiar of the scheme, so we have g_μ and $\tilde{g}_\mu$. The

couplings do not depend from other parameters, since they are dimensionless; so it is possible to write $\tilde{g}_\mu = \tilde{g}(g_\mu)$, from this it follows that

$$\tilde{\beta}(\tilde{g_\mu}) \equiv \mu \frac{d}{d\mu} \tilde{g}_\mu = \frac{d\tilde{g}((g_\mu)}{dg_\mu} \beta(g_\mu) \tag{5.40}$$

Furthermore the two couplings start from the same model, so at first order they are equal and in series expansion

$$\tilde{g}(g) = g + ag^2 + O(g^3) \tag{5.41}$$

from which it follows

$$g = \tilde{g} - a\tilde{g}^2 + O(\tilde{g}^3) \; . \tag{5.42}$$

So we can write

$$\frac{d\tilde{g}}{dg} = 1 + 2ag + O(g^2) = 1 + 2a\tilde{g} + O(\tilde{g}^2) \; . \tag{5.43}$$

So if we write

$$\beta(g) = c_1 g^2 + c_2 g^3 + O(g^4) \tag{5.44}$$

in terms of $\tilde{g}$

$$\beta(\tilde{g}) = c_1 \tilde{g}^2 + (c_2 - ac_1)\tilde{g}^3 + O(\tilde{g}^4) \; , \tag{5.45}$$

so from (5.40)

$$\begin{aligned} \tilde{\beta}(\tilde{g}) &= (1 + 2a\tilde{g} + O(\tilde{g}^2))(c_1\tilde{g} + (c_2 - 2ac_1)\tilde{g}^2 + O(\tilde{g}^3)) \\ &= c_1\tilde{g} + c_2\tilde{g}^2 + O(\tilde{g}^3) \end{aligned} \tag{5.46}$$

and we obtain what said above: the form of $\tilde{\beta}(\tilde{g})$ in terms of $\tilde{g}$ for the first two term is the same of $\beta(g)$, it is clear that it is not true for the higher-order terms.

6. The calculus of the beta function

In the previous chapter we introduced the fundamental concept of the Renormalization Group and we defined the beta function. We also gave an hint of the important role that this function plays, especially for our model, of which we want to investigate the important property of asymptotic freedom. This chapter is mainly devoted to the description of the method that we used to compute the beta function for our spanning-forest model.

6.1 The method

We follow a procedure which already found application in [2] and [3]. For a lattice theory, i.e. a theory regularized by introducing a discretization of the coordinates space, in principle the beta function can be found by a direct lattice computation, which also provides a regularization. However, our lattice σ-model has a natural continuum counterpart, with action

$$\mathcal{S}(\boldsymbol{\pi}, h) = \beta \int d^2x \left[\frac{1}{2} (\partial_\mu \boldsymbol{\pi}(x))^2 + \frac{1}{2} \frac{(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi})^2}{1 - \boldsymbol{\pi}^2(x)} - h\sqrt{1 - \boldsymbol{\pi}^2(x)} \right] , \qquad (6.1)$$

which has been already widely investigated. In particular, via a dimensional regularization, Brézin and Hikami [26] already performed the calculation up to three loops.

A general theorem of Renormalization states that the n-loop beta function within a certain regularization scheme can be deduced from the knowledge of the beta function in any other scheme, at the same perturbative order, and of the renormalization constants in the desired scheme, up to order $n-1$. So, a possible procedure, which we will indeed follow in this work, is to relate the beta function on the square and triangular lattice to the continuum results of Brézin and Hikami via the calculation of the two renormalization constants of the non-linear σ-model, denoted by Z_β and Z_π.

More in detail, in our case we have to compare our lattice theory with the continuum theory renormalized in [26] using $\overline{MS}$-scheme (Minimal Subtraction modified) and in dimensional regularization. The starting point is the relation

$$\Gamma^{(n)}_{latt}(p_1, \cdots, p_n; \beta, h; 1/a) = Z_\pi^{n/2} \Gamma^{(n)}_{\overline{MS}}(p_1, \cdots, p_n; Z_\beta^{-1}\beta, Z_\beta Z_\pi^{-1/2} h; \mu) \qquad (6.2)$$

where a and μ are respectively the lattice spacing and the scale of renormalization for the continuum, while $p_1,\ldots,p_n$ are the external momenta.

This is the same relation (5.27) already seen in the previous chapter, here it states the general result about the possibility of leading back a scheme of renormalization to another one, up to the definition of the proper renormalization constants. Here we consider the lattice theory (denoted by subscript *latt*) as the bare theory and we compare it with the continuum theory renormalized in the $\overline{MS}$-scheme (denoted by subscript $\overline{MS}$).

We recall that we defined the renormalized fields and coupling through

$$\beta \equiv \frac{\mu^{2-d}}{Z_\beta}\,\beta_R\,, \tag{6.3}$$

$$\boldsymbol{\pi}(x) \equiv Z_\pi^{1/2}\boldsymbol{\pi}_R(x)\,, \tag{6.4}$$

$$\sigma(x) \equiv Z_\pi^{1/2}\sigma_R(x) = \sqrt{1 - Z_\pi \boldsymbol{\pi}_R^2(x)}\,, \tag{6.5}$$

$$h \equiv \frac{Z_\beta}{Z_\pi^{1/2}} h_R\,. \tag{6.6}$$

For both the regularized theories, the invariance under the Renormalization Group leads to the corresponding Callan-Symanzyk equations:

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Gamma^{(n)}_{\overline{MS}} = 0\,; \qquad\qquad -\frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{(n)}_{latt} = 0\,; \tag{6.7}$$

where we added a minus sign for the lattice equation, because when $a \to 0$ we are making a RG flux toward short distances behaviour, that has the reversed sign respect to the $\mu \to \infty$ limit made for the continuum theory. We also define in this way the beta and γ functions (in the previous chapter the beta function was indeed denoted by the primitive notation β, while here, in order to avoid confusion with the coupling costant, and in accord with the literature on the subject, we denote the beta function as $W(\beta)$):

$$W^{\overline{MS}}(\beta_R) \equiv \mu\frac{\mathrm{d}\beta_R}{\mathrm{d}\mu}\bigg|_\beta \qquad\qquad \gamma^{\overline{MS}}(\beta) \equiv \mu\frac{\mathrm{dlog}Z_\pi(\beta_R)}{\mathrm{d}\mu}\bigg|_\beta \tag{6.8}$$

$$W^{latt}(\beta) \equiv -a\frac{\mathrm{d}\beta}{\mathrm{d}a}\bigg|_{\beta_R} \qquad\qquad \gamma^{latt}(\beta) \equiv a\frac{\mathrm{dlog}Z_\pi(\beta)}{\mathrm{d}a}\bigg|_{\beta_R} \tag{6.9}$$

where we note that in the continuum theory we keep the bare constant fixed (β) and instead on the lattice we take fixed the renormalized constant (β_R) and we go looking at the variation of the lattice spacing a in order to keep unchanged the renormalized constants.

In the Callan-Symanzik equation for the lattice theory

$$\begin{aligned} 0 = -a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{(n)}_{latt} = \Bigg[&- a\frac{\partial}{\partial a} + W^{latt}(\beta)\frac{\partial}{\partial\beta^{-1}} \\ &- \frac{n}{2}\gamma^{latt}(\beta) + \left(\frac{1}{2}\gamma^{latt}(\beta) + \beta W^{latt}(\beta)\right) h\frac{\partial}{\partial h}\Bigg]\Gamma^{(n)}_{latt}\,, \end{aligned} \tag{6.10}$$

using the condition (6.2), we are able to join together the beta and gamma function on the lattice to those in $\overline{MS}$-scheme, in fact we find

$$W^{\overline{MS}}(Z_\beta^{-1}\beta) = \left(Z_\beta + \frac{1}{\beta}\frac{\partial Z_\beta}{\partial \beta^{-1}}\right) W^{latt}(\beta) \tag{6.11}$$

$$\gamma^{\overline{MS}}(Z_\beta^{-1}\beta) = \gamma^{latt}(\beta) - \frac{1}{Z_\pi}\frac{\partial Z_\pi}{\partial \beta^{-1}} W^{latt}(\beta) \tag{6.12}$$

The first of them is the important relation that allows us to express the coefficients of the beta function on the lattice in terms of the coefficients of the continuum theory.

6.2 The beta function series expansion

Given the beta function for the non-linear σ-model with N the number of vector components, we expand it in power of the coupling costant $1/\beta$ in a generic *scheme* of regularization

$$W^{scheme}(\beta) = -\frac{w_0}{\beta^2} - \frac{w_1}{\beta^3} - \frac{w_2^{scheme}}{\beta^4} - \frac{w_3^{scheme}}{\beta^5} + O(\beta^{-6}) \; ; \tag{6.13}$$

the first two coefficients have not the superscript *scheme* because they are universal (for a proof of this see the end of the previous chapter), they come from the calculation respectively at one and two loops (the term from order zero vanishes in two dimensions); explicitly they are given by

$$w_0 = \frac{N-2}{2\pi} \; , \tag{6.14}$$

$$w_1 = \frac{N-2}{(2\pi)^2} \; ; \tag{6.15}$$

all the other terms are scheme-dependent; the w_n^{scheme} coefficient is associated with $1/\beta^{n+2}$ term of series expansion and correspond to a computation at $(n+1)$ loops. We report here the known results in $\overline{MS}$-scheme ([26], or see [2],[3] for other references)

$$w_2^{\overline{MS}} = \frac{1}{4}\frac{N^2-4}{(2\pi)^3} \; , \tag{6.16}$$

$$w_3^{\overline{MS}} = \frac{N-2}{(2\pi)^4}\left(-\frac{1}{12}(N^2-22N+34) + \frac{3}{2}\zeta(3)(N-3)\right) \; , \tag{6.17}$$

where ζ is the Riemann Zeta function and $\zeta(3) \approx 1.2020569$. We also expand in $1/\beta$ the two renormalization constants

$$Z_\beta = Z_\beta^{(0)} + \frac{Z_\beta^{(1)}}{\beta} + \frac{Z_\beta^{(2)}}{\beta^2} + \frac{Z_\beta^{(3)}}{\beta^3} + O(\beta^{-4}) \tag{6.18}$$

$$Z_\pi = Z_\pi^{(0)} + \frac{Z_\pi^{(1)}}{\beta} + \frac{Z_\pi^{(2)}}{\beta^2} + \frac{Z_\pi^{(3)}}{\beta^3} + O(\beta^{-4}) \tag{6.19}$$

With the above conventions on the series expansions, now we look at (6.11) and we rewrite it as:

$$W^{latt}(\beta) = \frac{W^{\overline{MS}}(Z_\beta^{-1}\beta)}{Z_\beta + \frac{1}{\beta}\frac{\partial Z_\beta}{\partial \beta^{-1}}}\,; \tag{6.20}$$

from this equation it can be seen that the coefficient of order n of the expansion of W^{latt} (i.e. w^{latt}_{n-2}) can be evaluated as long as one knows the coefficients of $W^{\overline{MS}}$ up the same order (i.e. $w_1^{\overline{MS}}$, $w_2^{\overline{MS}}$, ..., $w_{n-2}^{\overline{MS}}$) and performs the computation on the lattice of the constants Z_β and Z_π up order $n-1$.[1] So we can argue the general result:

$$w^{latt}_{n-1} = w^{latt}_{(n-\text{loop})} = F\left(\{w_i^{\overline{MS}}\}_{i=\{0,1,\cdots,n-1\}}; \{Z_\beta^{(j)}\}_{j=\{0,1,\cdots,n-1\}}\right)\,. \tag{6.21}$$

For example, for the first scheme-dependent coefficient w_2^{latt}, from (6.20) we find

$$w_2^{latt} = w_0\left((Z_\beta^{(1)})^2 - Z_\beta^{(2)}\right) + w_1 Z_\beta^{(1)} + w_2^{\overline{MS}}\,. \tag{6.22}$$

6.3 Evaluation of the constants of renormalization

In order to obtain the perturbative expansion of the constants Z_β and Z_π, we use relation (6.2) for the two-point function 1PI. We proceed as follows: we compute $\Gamma^{(2)}_{latt}$ at $n-1$ loops and, from the knowledge of $\Gamma^{(2)}_{\overline{MS}}$ at the same order, and the requirement of validity of (6.2), we find Z_β and Z_π at $n-1$ loops.

The continuum theory

For the continuum theory we consider the expansion

$$\Gamma^{(2)}_{\overline{MS}} = \beta(p^2+h) + \Pi^{(1)}_{\overline{MS}} + \frac{\Pi^{(2)}_{\overline{MS}}}{\beta} + \frac{\Pi^{(3)}_{\overline{MS}}}{\beta^2} + \dots\,; \tag{6.23}$$

we report the already known two-loop results [2, 3] in the case of $N=-1$

$$\Pi^{(1)}_{\overline{MS}} = \frac{1}{4\pi}(p^2-h)\log\frac{h}{\mu^2} \tag{6.24a}$$

$$\begin{aligned}\Pi^{(2)}_{\overline{MS}} =& \frac{1}{4\pi^2}\left(\frac{1}{4}\log^2\frac{h}{\mu^2} + 2\log\frac{h}{\mu^2} - \frac{3}{4} - 3(2\pi)^2 R\right)p^2 \\ &+ \frac{1}{2\pi^2}\left(-\frac{1}{4}\log^2\frac{h}{\mu^2} - \frac{1}{4}\log\frac{h}{\mu^2}\right)h\end{aligned} \tag{6.24b}$$

where R is an integral defined as

$$R = \lim_{h\to 0}\; h\int_{[-\pi,\pi]^4}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{(\widehat{p}^2+h)(\widehat{q}^2+h)(\widehat{p+q}^2+h)}\,, \tag{6.25}$$

[1] To be precise, only the constant Z_β is required. The expansion for Z_π however comes out as a side result of the computation.

it is non vanishing in two dimensions and it has been numerically computed, but it appears only in intermediate stages of the computation and cancels out in any of the results, so we are not interested in its value.

The lattice theory

Analogously for the lattice theory

$$\Gamma^{(2)}_{latt} = \beta(\widehat{p}^2 + h) + \Pi^{(1)}_{latt} + \frac{\Pi^{(2)}_{latt}}{\beta} + \frac{\Pi^{(3)}_{latt}}{\beta^2} + \dots \ ; \tag{6.26}$$

where $\widehat{p}$ are useful trigonometric reparametrization of p (cfr. the next chapter), such that, in the limit $a \to 0$,

$$\widehat{p}_\mu \simeq p_\mu \qquad \widehat{p}^2 \simeq p^2 \tag{6.27}$$

Since we are interested in matching these results with those on the continuum, we can indeed take this limit, and note that the external magnetic field h is associated to a factor a^2, i.e. in the Hamiltonian it appears as ha^2 in order to keep the action dimensionless (we remember that in two dimension the field $\boldsymbol{\pi}$ is dimensionless). In general, in making the limit $a \to 0$, we consider for each diagram its Taylor espansion in powers of the external moment p: if the diagram is described by an integral

$$I(p;h) = \int_{q,k,\dots} f(q,k,\dots,p;h) \ , \tag{6.28}$$

where q, k,... are the internal momenta over which we sum, we consider

$$I(p,h) = I(0,h) + \sum_{\mu=1}^{2} p_\mu \partial_\mu \, I(p,h)|_{p=0} + \frac{1}{4} p^2 \sum_{\mu=1}^{2} \partial^2_\mu \, I(p,h)|_{p=0} + \mathcal{O}(p^3_\mu) \tag{6.29}$$

where $\partial_\mu = \frac{\partial}{\partial p_\mu}$. In terms of derivatives of the integrand function f

$$\begin{aligned} I(p,h) = \int_{q,k,\dots} f(0,h;q,k,\dots) + \sum_{\mu=1}^{2} p_\mu \int_{q,k,\dots} \partial_\mu \, f(p,h;q,k,\dots)|_{p=0} \\ + \frac{1}{4} p^2 \sum_{\mu=1}^{2} \int_{q,k,\dots} \partial^2_\mu \, f(p,h;q,k,\dots)|_{p=0} + \mathcal{O}(p^3_\mu) \ . \end{aligned} \tag{6.30}$$

Since the continuum self-energy does not contain powers of p larger than p^2, and the p^2 dependence has spherical symmetry, this expansion for an I diagram on the lattice is sufficient to make a comparison between them, and furthermore the formal substitution

$$\sum_\mu \widehat{p}^2_\mu \partial^2_\mu \quad \longrightarrow \quad \frac{1}{2} p^2 \sum_\mu \partial^2_\mu$$

is justified in this context. After performing the $a \to 0$ limit, we obtain the results [2]

$$\Pi^{(1)}_{latt} = \frac{1}{4\pi}(p^2 - h)\log\frac{ha^2}{32} + \frac{p^2}{4} \tag{6.31}$$

$$\Pi^{(2)}_{latt} = \frac{1}{4\pi^2}\left[\frac{1}{4}\log^2\frac{ha^2}{32} + 2\log\frac{ha^2}{32} + \frac{7}{12}\pi^2 + \frac{\pi}{4} - 3(2\pi)^2(G_1 - R)\right]p^2 + \frac{1}{2\pi^2}\left[-\frac{1}{4}\log^2\frac{ha^2}{32} - \frac{1}{4}\log\frac{ha^2}{32} - \frac{\pi}{8}\log\frac{ha^2}{32} - \frac{\pi}{8}\right]h \tag{6.32}$$

where G_1 is defined in two dimension as

$$G_1 = -\frac{1}{4}\int_{[-\pi,\pi]^4}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\left(\sum_{\mu=1}^{2}(\widehat{p+q})^4_\mu\right)\frac{\widehat{p+q}^{\,2} - \widehat{p}^2 - \widehat{q}^2}{\widehat{p}^2\widehat{q}^2[(\widehat{p+q})^2]^2}\,; \tag{6.33}$$

it has been computed [2, 3] with the result

$$G_1 \approx 0.0461636 \tag{6.34}$$

The computation technique involves some refined reasoning, that we sketck in Appendix C. We have reproduced this result, as a preliminary computation before applying the same technique to triangular-lattice analogs of the quantity G_1.

Comparing the two expansions for the self energy on the square lattice and the continuum theory, we obtain

$$Z^{(1)}_\beta = \frac{1}{4} + \frac{3}{4\pi}\log\frac{\mu^2a^2}{32} \tag{6.35}$$

$$Z^{(1)}_\pi = \frac{1}{2\pi}\log\frac{\mu^2a^2}{32} \tag{6.36}$$

$$Z^{(2)}_\beta = \frac{9}{16\pi^2}\log^2\frac{\mu^2a^2}{32} + \frac{3}{8\pi}(1 + \frac{1}{\pi})\log\frac{\mu^2a^2}{32} - \frac{3}{4\pi^2}(4\pi^2G_1 - \frac{1}{4} - \frac{\pi^2}{8}) + \frac{11}{96} \tag{6.37}$$

$$Z^{(2)}_\pi = \frac{5}{16\pi^2}\log^2\frac{\mu^2a^2}{32} + \frac{1}{8\pi}\log\frac{\mu^2a^2}{32} \tag{6.38}$$

Since we know these constants of renormalization up to two loops, we can evaluate w_2^{latt} using (6.22). We find

$$w_2^{latt} = \frac{-3}{(2\pi)^3}\left[12\pi^2G_1 - \frac{1}{2} + \frac{\pi}{2} - \frac{7}{12}\pi^2\right]\,, \tag{6.39}$$

that is the same expression already found in [2] (equation (4.29)), in the case $N = -1$ (furthermore for our model $r = s = 0$). Using for G_1 the value in (6.34) we obtain

$$w_2^{latt} = -\frac{2.3427}{(2\pi)^3}\,. \tag{6.40}$$

This is the first non-universal coefficient of the beta function series expansion for the σ-model on square lattice in the case $N = -1$, and thus also for the combinatorial problem of spanning-forest enumeration on the square lattice, where perturbation theory is performed with respect to the generating function parameter, with an important inversion of sign. The value (6.40) is indeed the one reported for comparison and discussion in (16) of [1].

7. The spanning-forest model on square lattice

In this chapter we analise the perturbative formalism for what we can call the *Forest model* on square lattice, that is the statistical mechanics model whose partition function is the generating function for spanning forests on the square lattice. Since the model is perturbatively equivalent to the non-linear σ-model on the lattice in the case $N = -1$, the results for the vacuum polarization are already known up to four loops and reported in [2] and [3].

Here we apply the perturbative theory in terms of the new fermionic variables, and with care to the features of the forest Hamiltonian, we describe the new Feynman rules; with these new rules we write the self-energy up to two loops, and we find coincidence with the one of the generic-N σ-model, when specialized to the case $N = -1$.

We discuss the improvements in complexity of calculation due to the specialization of this problem, having in mind the feasibility of 'pushing further' the expansion.

7.1 The square lattice

The square lattice is a two-dimensional regular lattice: all the edges have the same length a that is called the lattice spacing, and all the vertices are of the same kind, each one with coordination $2d = 4$ (it is the $d = 2$ case of an hypercubic lattice).

7.1.1 Lattice variables

We denote two orthogonal directions with $\mu = 1, 2$ and with $\widehat{\mu}$ the versor in the μ-direction, $|\widehat{\mu}| = a$. Each vertex is described by two coordinates, $\frac{x_1}{a}$ and $\frac{x_2}{a}$. It is useful to assume for the lattice spacing $a = 1$ in all the intermediate calculations and recover the scaling in terms of a only when we will do the continuum limit $a \to 0$. So the site i with coordinates x_1, x_2 is identified by the vector $x_i = x_1\widehat{\mu}_1 + x_2\widehat{\mu}_2 \in \mathbb{Z}^2$.

On a simple lattice, like the square one, the property of two vertices of being adjacent means that they are first neighbouring sites, i.e. they are connected by an edge of the lattice if and only if they are one lattice spacing far away from each other.

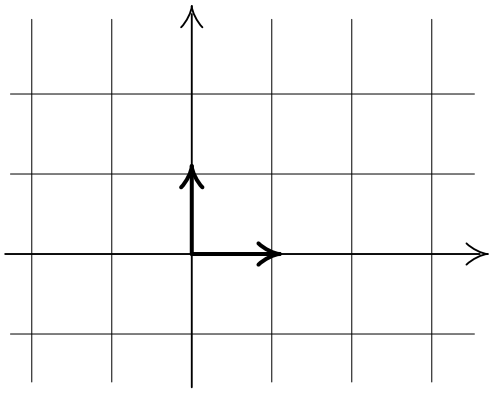


Figure 7.1 A portion of the square lattice.

On square lattice, since each vertex has four first neighbours, the coordination for each vertex is four. In particular the set of first neigbours of site i is $\{i + \widehat{\mu}_1, i - \widehat{\mu}_1, i + \widehat{\mu}_2, i - \widehat{\mu}_2\}$. We denote a pair of sites i and j first neighbours with the notation $\langle ij \rangle$.

In order to define a Fourier transform, we introduce a two-dimensional momentum p with coordinates $p = (p_1, p_2)$. The scalar product is as usual

$$p \cdot x = p_1 x_1 + p_2 x_2 \ . \tag{7.1}$$

If we deal with an infinite square lattice, the set of allowed momenta is $[-\pi, pi]^2$: this is a first example of a simple Brilloin zone, that is a complete non-redundant parametrization of Fourier modes on the lattice. We will see in the next chapter that the identification of a suitable Brilloin zone is a more delicate topic in the case of the triangular lattice.

For lattice integrals it is fruitful to introduce the lattice momentum

$$\widehat{p} = (\widehat{p}_1, \widehat{p}_2) \tag{7.2}$$

defined as

$$\widehat{p}_\mu = 2 \sin \frac{p_\mu}{2} \tag{7.3}$$

with squared modulus

$$\widehat{p}^2 = \sum_{\mu=1}^{2} \widehat{p}_\mu^2 \ . \tag{7.4}$$

Some recurrent trigonometric manipulations are

$$\cos p_\mu = 1 - \frac{\widehat{p}_\mu^2}{2} \qquad\qquad \widehat{p}_\mu^2 = 2 - 2 \cos p_\mu \tag{7.5}$$

$$\sin p_\mu^2 = \widehat{p}_\mu^2 - \frac{\widehat{p}_\mu^4}{4} \qquad\qquad \widehat{p}^2 = \sum_\mu (2 - 2 \cos p_\mu) \ . \tag{7.6}$$

Furthermore the definition of $\widehat{p}$ is such that in the continuum limit it gives the usual momentum p. In fact, restoring the dependence from the lattice spacing, p is scaled to pa, and $\widehat{pa}$ is indeed $pa + \mathcal{O}(a^3)$, and we have

$$\lim_{a\to 0} \widehat{pa}_\mu = \lim_{a\to 0} 2\sin\frac{p_\mu a}{2} = p_\mu a \tag{7.7}$$

$$\lim_{a\to 0} \widehat{pa}^2 = a^2 p^2 \,. \tag{7.8}$$

7.1.2 The Laplacian matrix

In chapter 1 for a general graph we defined the Laplacian as

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_{k=1} w_{ik} & \text{if } i = j \end{cases}$$

where w_{ij} is the weight associated to the edges $e = (i, j)$.

On the square lattice, setting w to 1 (this can be always done up to a rescaling of the coupling t into t/w), we have $L_{ii} = 4$ and $L_{ij} = -1$ on first neighbours, which Fourier space diagonalizes (as expected for a Laplacian operator, which is translationally invariant), and gives

$$\begin{aligned} L_{pq} &= \sum_{ij} e^{ipx_i} L_{ij} e^{-iqx_j} = \sum_i e^{i(p-q)x_i}\left(2d - \sum_\mu (e^{ip_\mu} + e^{-ip_\mu})\right) \\ &= \delta_{pq}\sum_\mu (2 - 2\cos q_\mu) = \delta_{pq}\widehat{p}^2 \end{aligned} \tag{7.9}$$

where we used (7.6) in the last passage. Finally we can write for short

$$L(p) := L_{pp} = \widehat{p}^2 = \widehat{p}_1^2 + \widehat{p}_2^2 \,. \tag{7.10}$$

7.2 Perturbative expansion

We want to make a perturbative expansion for small t of the fermionic model described by the Hamiltonian[1]

$$-\mathcal{H}(\psi, \bar\psi) = \bar\psi L \psi + t\sum_i \bar\psi_i \psi_i + \frac{t}{2}\sum_{ij} \bar\psi_i \psi_i L_{ij} \bar\psi_j \psi_j \tag{7.11}$$

and such that its partition function

$$Z_F = \int \mathcal{D}(\psi, \bar\psi)\ \exp\left[\bar\psi L\psi + t\left(\sum_i \bar\psi_i\psi_i + \frac{1}{2}\sum_{ij}\bar\psi_i\psi_i L_{ij}\bar\psi_j\psi_j\right)\right]$$

[1] We define the Hamiltonian with a minus sign in order to maintain for the partition function the usual definition $Z = \int \mathcal{D}(\psi,\bar\psi) e^{-\mathcal{H}(\psi,\bar\psi)}$.

is the generating function for the unrooted spanning forests on lattice (as we explained in chapter 2).

As seen for the σ-model in chapter 4, the propagator of this model is infrared divergent in two dimensions, so we add a coupling to an external magnetic field h as infrared regularization The new Hamiltonian is

$$-\mathcal{H}(\psi,\bar{\psi},h) = -\mathcal{H}(\psi,\bar{\psi}) + h\sum_i \bar{\psi}_i\psi_i \ . \tag{7.12}$$

For small values of t we are allowed to make a perturbative expansion of $Z_F(t,h)$

$$Z_F(t,h) = \int \mathcal{D}(\psi,\bar{\psi})\ e^{\bar{\psi}(L+h)\psi} \sum_{n=0}^{\infty} \frac{t^n}{n!}\Big(\sum_i \bar{\psi}_i\psi_i + \frac{1}{2}\sum_{ij}\bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j\Big)^n \ , \tag{7.13}$$

and the expectation value of the product of k fermionic fields is

$$\begin{aligned}\langle\bar{\psi}_{i_1}\cdots\bar{\psi}_{i_h}\psi_{j_1}\cdots\psi_{j_{k-h}}\rangle = \frac{1}{Z_F(t,h)}\int \mathcal{D}(\psi,\bar{\psi})\,\bar{\psi}_{i_1}\cdots\bar{\psi}_{i_h}\psi_{j_1}\cdots\psi_{j_{k-h}} \\ e^{\bar{\psi}(L+h)\psi}\sum_{n=0}^{\infty}\frac{t^n}{n!}\Big(\sum_i \bar{\psi}_i\psi_i + \frac{1}{2}\sum_{ij}\bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j\Big)^n \ .\end{aligned} \tag{7.14}$$

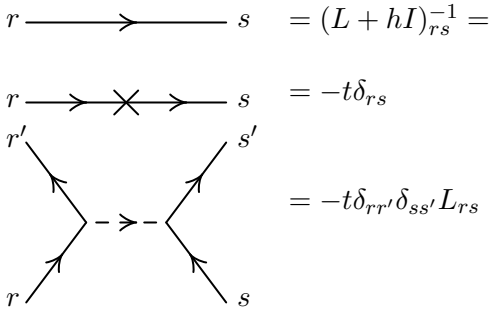
Using Wick theorem we are able to express this expectation value in terms of a sum over total contractions

$$\begin{aligned}\langle\bar{\psi}_{i_1}\cdots\bar{\psi}_{i_h}\psi_{j_1}\cdots\psi_{j_{k-h}}\rangle = \sum_{\substack{\text{Wick total}\\ \text{contractions}}} \bar{\psi}_{i_1}\cdots\bar{\psi}_{i_h}\psi_{j_1}\cdots\psi_{j_{k-h}} \\ \sum_{n=0}^{\infty}\frac{t^n}{n!}\Big(\sum_i \bar{\psi}_i\psi_i + \frac{1}{2}\sum_{ij}\bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j\Big)^n \ .\end{aligned} \tag{7.15}$$

Writing the Feynman rules of the theory, we can write each total contraction as a Feynman diagram, so that a generic expectation is a sum of Feynman diagram contributions.

7.2.1 Feynman rules

The Feynman rules induced by the perturbative expansion in coordinate space read



where for handiness we introduced the matrix H

$$H = (L + hI)^{-1} \tag{7.16}$$

We notice that, since we are dealing with fermionic variables, we have to take care of the signs arising form their interchanges. For example, a diagram in which a certain propagator has a mass insertion takes a relative minus sign w.r.t. the diagram in which there is no mass insertion, as

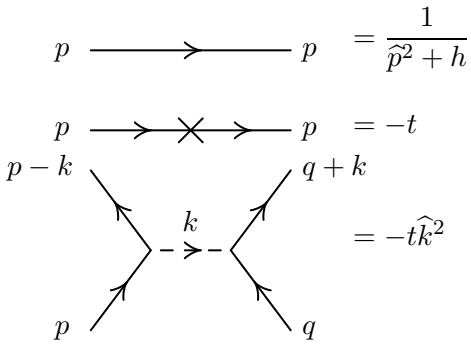
$$\sum_i \overline{\psi}_r \psi_s \, t\overline{\psi}_i \psi_i = -\sum_i t \; \overline{\psi}_r \psi_i \overline{\psi}_i \psi_s \; . \tag{7.17}$$

Similarly, if we have a loop of fermions, it gives a global minus sign, beyond a factor $(-1)^\ell$, with ℓ the number of propagators

$$\overline{\psi}_1 \psi_1 \overline{\psi}_2 \psi_2 \overline{\psi}_3 ... \psi_\ell = (-1)^{\ell-1} \overline{\psi}_1 \overline{\psi}_2 \psi_1 \overline{\psi}_3 \psi_2 ... \psi_\ell = (-1)^{\ell-1} \overline{\psi}_1 \psi_\ell \overline{\psi}_2 \psi_1 \overline{\psi}_3 \psi_2 \; ... \tag{7.18}$$

After some combinatorics, one finds out that these are indeed *all* the sources of sign inversions, that is, given the sign of a "reference" diagram (say, one without mass insertion and closed loops), the relative sign of any other diagram is given by the parity of the number of mass insertion, plus the number of closed loops.

It is useful to write the Feynman rules also in momentum space. All this discussion on fermionic signs is unchanged (as the Fourier change of basis is a linear transformation over the Grassmann algebra generators), while the propagator diagonalize, so we have



7.2.2 Two-point function

The expectation value for the two-point function at every order is formally given by

$$\langle \bar{\psi}_r \psi_s \rangle = \frac{1}{Z_F(t,h)} \int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi}(L+h)\psi} \bar{\psi}_r \psi_s \sum_{n=0}^{\infty} \frac{t^n}{n!} \Big(\sum_i \bar{\psi}_i \psi_i + \frac{1}{2} \sum_{ij} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j \Big)^n . \tag{7.19}$$

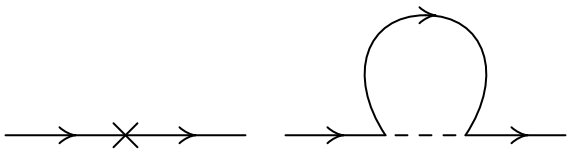


Figure 7.2 The Feynman diagrams for the two-point function at order 1.

At the zero-th order we have the propagator of the theory

$$\langle \bar{\psi}_i \psi_j \rangle^{(0)} = (L + hI)^{-1}_{ij} \,, \tag{7.20}$$

For the first order we evaluate

$$\frac{1}{t}\langle \bar{\psi}_r \psi_s \rangle^{(1)} = \sum_{\substack{\text{Wick total} \\ \text{contractions}}} \bar{\psi}_r \psi_s \Big(\sum_i \bar{\psi}_i \psi_i + \frac{1}{2} \sum_{ij} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j \Big) \tag{7.21}$$

$$= -\sum_i H_{ri} H_{is} + \frac{1}{2} \sum_{ij} L_{ij} (H_{ri} H_{ij} H_{js} + H_{rj} H_{ji} H_{is}) \,. \tag{7.22}$$

Where we take care of the minus sign arising from an interchange of fermionic variables and we used the fact that the Laplacian matrix is such that the sum of all the elements in a row or a column is zero

$$\sum_i L_{ij} = 0 \,, \tag{7.23}$$

so it follows that, in the vertex interaction $\sum_{ij} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j$, a contraction between fields located in the same site gives null contribute, since

$$\sum_i \sum_{j,k,\ldots} H_{ii} L_{ij} f(j,k,\ldots) = \sum_{j,k,\ldots} f(j,k,\ldots) \Big(\sum_i H_{ii} L_{ij} \Big) = 0 \tag{7.24}$$

Furthermore we notice that the four-fermion interaction is non-punctual. Nonetheless, it is local, since it involves pairs of neighbouring sites of the lattice ("splitted operator").

How explained in the previous chapter, in order to compute the beta function we have to evaluate the two-point one-particle irreducible $\Gamma^{(2)}$ (that here we call briefly Γ) up to a certain order. The two-point 1PI function Γ is related to the two-point function in this way

$$\Gamma_{rs} = -\sum_{r's'} H^{-1}_{rr'} \langle \bar{\psi}_{r'} \psi_{s'} \rangle H^{-1}_{s's} \,. \tag{7.25}$$

So up to first order we have

$$\begin{aligned}\Gamma_{rs} &= -\sum_{r's'} H^{-1}_{rr'}\left(H_{r's'} - tH^2_{r's'} + \frac{t}{2}H_{r'i}\sum_{ij}(L_{ij}H_{ij} + L_{ji}H_{ji})H_{js'}\right)H^{-1}_{s's} \\ &= -H^{-1}_{rs} + t\,\delta_{rs} - tH_{rs}L_{rs} \\ &= -(L+hI)_{rs} + t\left[\delta_{rs} - (L+hI)^{-1}_{rs}L_{rs}\right]\ .\end{aligned} \tag{7.26}$$

In order to write the same rules in p-space, we write (7.26) in a Fourier representation as

$$\Gamma_{rs} = \sum_p e^{ipx_r}\Gamma(p)e^{-ipx_s}\ , \tag{7.27}$$

Now the Laplacian matrix takes diagonal form, $L(p) = \widehat{p}^2$, so we have

$$\begin{aligned}\Gamma_{rs} &= -\sum_p e^{ipx_r}(\widehat{p}^2 + h)e^{ipx_s} + \\ &\quad + t\sum_p e^{ip(x_r - x_s)} - t\sum_{pq} e^{ipx_r}(\widehat{q}^2 + h)^{-1}e^{-ipx_s}e^{iqx_r}\widehat{p}^2 e^{-iqx_s} \\ &= -\sum_p e^{ipx_r}\left[(\widehat{p}^2 + h) + t\right]e^{ipx_s} - t\sum_{pq} e^{i(p+q)x_r}\frac{\widehat{p}^2}{\widehat{q}^2 + h}e^{-i(p+q)x_s} \\ &= -\sum_p e^{ipx_r}\left[(\widehat{p}^2 + h) + t\right]e^{ipx_s} - t\sum_{q'} e^{iq'x_r}\sum_q \frac{\widehat{q'-p}^2}{\widehat{q}^2 + h}e^{-iq'x_s}\end{aligned}$$

in the last term we use the general result

$$\widehat{p \pm q}^2 = \widehat{p}^2 + \widehat{q}^2 - \frac{1}{2}\sum_\mu \widehat{p}^2_\mu \widehat{q}^2_\mu \pm 2\sum_\mu \sin p_\mu \sin q_\mu \tag{7.28}$$

and

$$\sum_p \widehat{p \pm q}^2 = \sum_p \left(\widehat{p}^2 + \widehat{q}^2 - \frac{1}{2}\sum_\mu \widehat{p}^2_\mu \widehat{q}^2_\mu\right) \tag{7.29}$$

since, summing p over a range symmetric around zero, the last term, that is an odd function of p, vanishes. Then we evaluate

$$\begin{aligned}&\sum_q \frac{\widehat{p}^2 + \widehat{q}^2 - \frac{1}{2}\sum_\mu \widehat{p}^2_\mu \widehat{q}^2_\mu}{\widehat{q}^2 + h} \\ &\qquad = \widehat{p}^2\sum_q \frac{1}{\widehat{q}^2 + h} + 1 - \sum_q \frac{h}{\widehat{q}^2 + h} - \frac{1}{2d}\widehat{p}^2(1 - h\sum_q \frac{1}{\widehat{q}^2 + h})\ ;\end{aligned}$$

since we are on a regular and isotropic lattice we are allowed to use

$$\widehat{p}^2_\mu = \frac{\widehat{p}^2}{d} \tag{7.30}$$

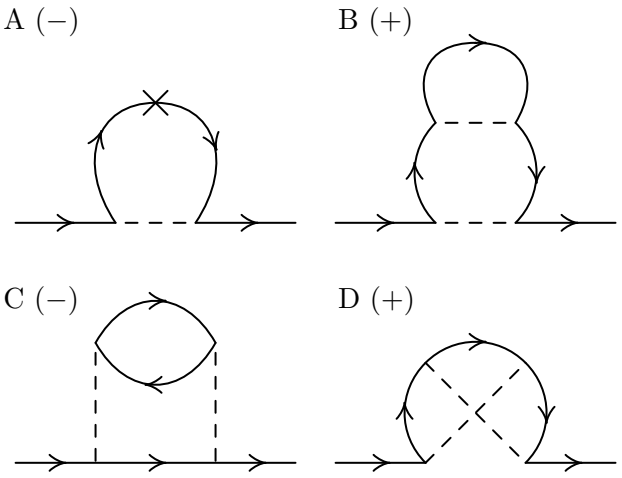


Figure 7.3 The Feynman diagrams for the two-point function at second order. On the top left corners, we report the identificative letters and the overall fermionic sign of the integral

where d is the dimension of the space, in our case $d = 2$. We define

$$I(h) = \sum_p \frac{1}{\widehat{p}^2 + h} \tag{7.31}$$

this is the one-loop integral and in the case of an infinite lattice, as we will use here, it is evaluated at the end of this Chapetr in 7.3. Finally at the firts order we find

$$\Gamma(p) = (\widehat{p}^2 + h) + t\,\Pi(p) \tag{7.32}$$

with $\Pi(p)$ the vacuum polarization

$$\Pi(p) = \widehat{p}^2 \left[\frac{1}{2d} - \Big(1 + \frac{h}{2d}\Big) I(h)\right] + hI(h)\ . \tag{7.33}$$

7.2.3 The second order Feynman diagrams

The diagrams at second order in t are the four ones shown in Figure 7.3.

$$\mathbf{A} = \sum_k \frac{\widehat{p+k}^2}{(\widehat{k}^2+h)^2} \tag{7.34}$$

$$\mathbf{B} = \sum_{k,q} \frac{\widehat{p+k}^2 \widehat{k+q}^2}{(\widehat{q}^2+h)(\widehat{k}^2+h)^2} \tag{7.35}$$

$$\mathbf{C} = \sum_{k,q} \frac{(\widehat{p+k}^2)^2}{(\widehat{q}^2+h)(\widehat{k}^2+h)(\widehat{p+k+q}^2+h)} \tag{7.36}$$

$$\mathbf{D} = \sum_{k,q} \frac{\widehat{p+q}^2 \widehat{k}^2}{(\widehat{q}^2+h)(\widehat{k-q}^2+h)(\widehat{p+k}^2+h)} \tag{7.37}$$

As we expected these are the same Feynman diagrams that appears at the second order of perturbative expansion of the σ-model [2]; evaluating them in terms of $I(h)$ and I_2 we found

$$\mathbf{A} = \widehat{p}^2 \left[-\frac{1}{4} I(h) + I_2(h) + \frac{1}{4} h I_2(h) \right] + I(h) - h I_2(h) \tag{7.38}$$

$$\mathbf{B} = \widehat{p}^2 \left[I(h)^2 - \frac{3}{4} I(h) + I_2(h) + \frac{1}{2} h I_2(h) - 2h I_2(h) I(h) + \frac{1}{16} \right] + \tag{7.39}$$

$$+ 2I(h) - \frac{1}{4} - 3h I(h)^2 - h I_2(h) + \frac{3}{4} h I(h) - \frac{1}{4} h^2 I_2(h) + 2h^2 I_2(h) I(h) \tag{7.40}$$

$$\mathbf{C} = \widehat{p}^2 \left[I(h)^2 + \left(\frac{1}{\pi} - \frac{1}{2} \right) I(h) - \frac{1}{16} + 2G_1 - 4R \right] + \tag{7.41}$$

$$+ 2I(h) - \frac{1}{4} - 3h I(h)^2 + \frac{h}{2} I(h) \tag{7.42}$$

$$\mathbf{D} = \widehat{p}^2 \left[I(h)^2 - \frac{1}{2\pi} I(h) + \frac{1}{48} - G_1 - R \right] + \tag{7.43}$$

$$+ I(h) - 2h I(h)^2 \tag{7.44}$$

Where G_1 and R have been defined in the previous chapter in (6.25, 6.33). According to the Feynman rules we have to add a minus sign to the diagrams **A** and **C**: for the first one since it has a mass insertion, for the second since it has a loop. Finally summing all the diagrams above, we can write the second order of the self-energy

$$\Pi^{(2)}(p) = -\mathbf{A} + \mathbf{B} - \mathbf{C} + \mathbf{D} = \tag{7.45}$$

$$= \widehat{p}^2 \left[I(h)^2 + \frac{2}{\pi} I(h) + 3(G_1 - R) + \frac{7}{48} + \frac{1}{16\pi} \right] + \tag{7.46}$$

$$h \left[-2I(h)^2 + \frac{1}{4} I(h) + \frac{1}{2\pi} I(h) - \frac{1}{16\pi} \right] . \tag{7.47}$$

Using the asymptotic form for $I(h)$ and $I_2(h)$ in the limit $h \to 0$ (reported in the following section 7.3) this last expression concide with the expression (6.32) in the

previous chapter; as we expected the perturbative theory on the trees and forests model is equivalent to that of the non-linear σ-model in the case $N = -1$.

7.3 Technical details

7.3.1 The one-loop integrals

All the one-loop integrals can be reduced to a sum of $I(h)$ and I_2, with the definitions

$$I(h) = \int_{[-\pi,\pi]^2} \frac{d^2p}{(2\pi)^2} \frac{1}{\widehat{p}^2 + h} \tag{7.48}$$

$$I_2(h) = \int_{[-\pi,\pi]^2} \frac{d^2p}{(2\pi)^2} \frac{1}{(\widehat{p}^2 + h)^2} \,. \tag{7.49}$$

Here we prove that they can be written in terms of elliptic integrals.

Elliptic integrals

The *elliptic integral of the first kind* is defined as

$$E(k,\phi) = \int_0^\phi d\theta \, \sqrt{1 - k^2 \sin^2\theta} \tag{7.50}$$

with modulus k and amplitude ϕ; if $\phi = \frac{\pi}{2}$ it is said the *complete elliptic integral of the first kind*

$$E(k) = \int_0^{\pi/2} d\theta \, \sqrt{1 - k^2 \sin^2\theta} \,. \tag{7.51}$$

The *elliptic integral of the second kind* is defined as

$$K(k,\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{7.52}$$

and analogously if $\phi = \frac{\pi}{2}$ it is said the *complete elliptic integral of the second kind*

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \,. \tag{7.53}$$

The properties af elliptic integrals have been widely studied and are well known (see for example [8]). For our computation we are interested in the limit of modulus $k^2 \to 1$, since we will see that it corresponds to the limit $h \to 0$ of the external magnetic field. The expansion of the complete integrals around $k^2 = 1$ is known to

be

$$E(k) = 1 + \frac{k^2-1}{4}\left(\log\left(1-k^2\right) + 1 - 4\log 2 + O((k^2-1)^2)\right) \tag{7.54}$$

$$\begin{aligned} K(k) = &-\frac{1}{2}\log\left(1-k^2\right) + \log 4 \\ &+ \frac{1}{4}\left(\frac{1}{2}\log\left(1-k^2\right) - \log 4 + 1\right)(k^2-1) + O((k^2-1)^2)\ . \end{aligned} \tag{7.55}$$

Result 1

$$I(h) = \frac{2}{\pi(4+h)} K\left(\frac{4}{4+h}\right) \tag{7.56}$$

Proof We start from $I(h) = \int_{[-\pi,\pi]^2} \frac{d^2p}{(2\pi)^2} \frac{1}{\widehat{p}^2+h}$. Using the relations

$$\widehat{p}^2 = 2 - 2\cos p \qquad \cos p_1 + \cos p_2 = 2\cos\frac{p_1+p_2}{2}\cos\frac{p_1-p_2}{2} \tag{7.57}$$

we rewrite the denominator

$$\begin{aligned} \widehat{p}^2 + h &= 2 - 2\cos p_1 + 2 - 2\cos p_2 + h \\ &= 4 - 4\cos\frac{p_1+p_2}{2}\cos\frac{p_1-p_2}{2} + h \end{aligned} \tag{7.58}$$

and then we make the change of variables $k_1 = \frac{p_1+p_2}{2}$ and $k_2 = \frac{p_1-p_2}{2}$; the Jacobian of the transformation is 2, but it simplifies with the factor $1/2$ coming from the new area of integration; in fact k_1 and k_2 run inside the rhombus of vertices $(\pm\pi, 0)$ and $(0, \pm\pi)$, that has area equal to half of the square area $[-\pi,\pi]^2$. So we obtain

$$I(h) = \int_{[-\pi,\pi]^2} \frac{d^2k}{(2\pi)^2} \frac{1}{4 - 4\cos k_1 \cos k_2 + h} \quad . \tag{7.59}$$

We are now able to integrate in k_2 using the result

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{\alpha + \beta\sin\theta + \gamma\cos\theta} = \frac{1}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}} \quad , \tag{7.60}$$

we have

$$\begin{aligned} I(h) &= \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{1}{\sqrt{(4+h)^2 - 16\cos^2 k_1}} \\ &= \frac{1}{2\pi\ (4+h)} \int_{-\pi}^{\pi} \frac{dk_1}{\sqrt{1 - \left(\frac{4}{4+h}\right)^2 \cos^2 k_1}} \\ &= \frac{4}{2\pi\ (4+h)} \int_{0}^{\pi/2} \frac{dk_1}{\sqrt{1 - \left(\frac{4}{4+h}\right)^2 \sin^2 k_1}} \\ &= \frac{2}{\pi\ (4+h)} K\left(\frac{4}{4+h}\right)\ . \end{aligned} \tag{7.61}$$

Result 2

$$I_2(h) = \frac{2}{\pi h\,(h+8)} E\left(\frac{4}{4+h}\right) \tag{7.62}$$

Proof We note that

$$I_2(h) = -\frac{d\,I(h)}{d\,h} \tag{7.63}$$

so we make the derivative of $I(h)$ using the result 1 and the relation

$$\frac{d\,K(k)}{d\,k} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}\,. \tag{7.64}$$

The limit $h \to 0$

For both the result 1 and 2, the limit $h \to 0$ correspond to the limit $k^2 \to 1$ for the modulus of the elliptic integrals. Using the expansion (7.54) we write the leading terms of the series expansion of $I(h)$ and $I_2(h)$ for small h

$$I(h) = -\frac{1}{4\pi}\log\frac{h}{32} + \frac{h}{32}\log\frac{h}{32} + \frac{h}{32\pi} \tag{7.65}$$

$$I_2(h) = \frac{1}{4\pi h} - \frac{1}{32\pi}\log\frac{h}{32} - \frac{1}{16\pi}\,. \tag{7.66}$$

In our computation in the limit $h \to 0$ we use

$$I(h) \sim -\frac{1}{4\pi}\log\frac{h}{32} \tag{7.67}$$

$$hI_2(h) \sim \frac{1}{4\pi}\,. \tag{7.68}$$

7.3.2 The two-loop integrals

We want to note that the expressions of the diagrams **C** and **D**, reported in (7.41) and (7.43), agree respectively with the integrals (A.117) and (A.118) of Appendix A.2 of [3]. In particular since (A.117) and (A.118) are defined in a slightly different way

$$(A.117) = \sum_{k,q} \frac{(\widehat{p-k}^{\,2}+h)^2}{(\widehat{q}^2+h)(\widehat{k}^2+h)(\widehat{p-k-q}^{\,2}+h)}$$

$$(A.118) = \sum_{k,q} \frac{(\widehat{p+q}^{\,2}+h)(\widehat{p+k}^{\,2}+h)}{(\widehat{q}^2+h)(\widehat{k}^2+h)(\widehat{p+k+q}^{\,2}+h)}$$

we have to subtract the new terms arising from the presence of field h in the numerator and we have

$$\mathbf{C} = (A.117) - 2\widehat{p}^2 R + 2hI(h)^2 - hR \tag{7.69}$$

$$\mathbf{D} = (A.118) - 2\widehat{p}^2 R + 2hI(h)^2 - hR\,. \tag{7.70}$$

Since in the expression of the self-energy the diagrams **C** and **D** appear with a different sign, the new subtracted terms cancel and we have

$$-\mathbf{C} + \mathbf{D} = -(\text{A.117}) + (\text{A.118})\,.$$

8. The spanning-forest model on triangular lattice

In this chapter we collect a large part of the original work of this thesis: we study the spanning-forest model on a triangular lattice and we write the first non-universal coefficient for the correspondent beta function.

8.1 The triangular lattice

The triangular lattice is the two-dimensional regular lattice where each vertex has coordination equal to six, the six neighbours being placed all at the same distance a, in the directions $\{\exp(2\pi in/6)\}$, with $n = 0, \ldots, 5$. So, a valid pair of vectors which generate the group of translations on the lattice is given by the first two cubic roots of 1, and all the edges are parallel to one of the three directions that form angles of $2\pi/3$ among them. In this way the plane is covered by elememtary faces of triangular shape. The dual graph of a triangular lattice is the honeycomb or hexagonal lattice. Thus, differently from the square lattice, it does not have self-duality properties, unless for the cases, like the Ising model or the Potts model in the $q \to 0$ limit, in which there is also a star-triangle ($Y \leftrightarrow \Delta$) relation [21].

8.1.1 Variables

Despite the triangular lattice is a two dimensional lattice, its geometry suggests to use a redundant basis of 3 vectors $\widehat{\mu}$ with $\mu = 1, 2, 3$ such that $\sum_\mu \widehat{\mu} = 0$, and $\widehat{\mu} \cdot \widehat{\mu'} = -\frac{1}{2}$ for $\mu \neq \mu'$. From that we notice that we can add a costant to all the three coordinates without affecting the x location, i.e. there is an equivalence relation

$$x \equiv (x_1, x_2, x_3) \sim (x_1 + m, x_2 + m, x_3 + m) \; . \tag{8.1}$$

while, for example, an asymmetric gauge fixing is $x_3 = 0$. Within the description in terms of this three-element basis, it is useful to see the triangular lattice as the projection of the three dimensional hypercubic lattice on the plane $x_1 + x_2 + x_3 = 0$.

We take a redundant basis also in the Fourier space, taking a p_{tri} vector, with three components constrained to satisfy $\sum_\mu p_\mu = 0$. So the scalar product is defined by

$$p_{tri} \cdot x = p_1 x_1 + p_2 x_2 + p_3 x_3 \; . \tag{8.2}$$

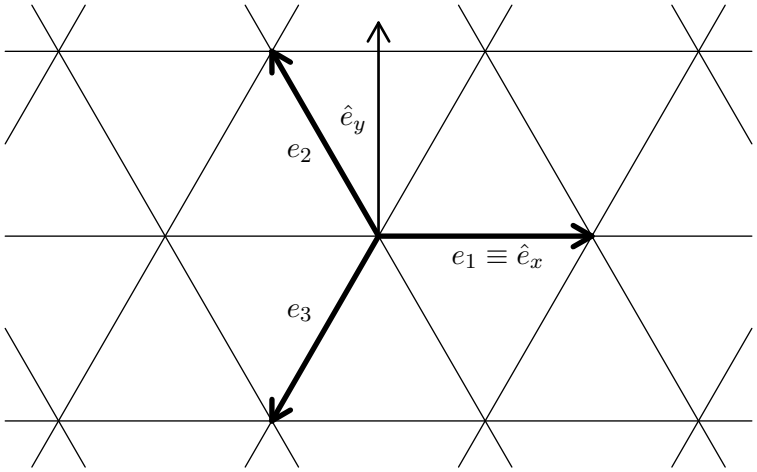


Figure 8.1 Triangular lattice

As for the square lattice, we define a lattice-momentum $\widehat{p}$, in this case with three components $\widehat{p}_\mu = 2\sin\frac{p_\mu}{2}$, and with square modulus

$$\widehat{p}^2_{tri} = \widehat{p}^2_1 + \widehat{p}^2_2 + \widehat{(p_1 + p_2)}^2 \tag{8.3}$$

that can be also written in terms of only two components (using 7.28) as

$$\widehat{p}^2_{tri} = 2\widehat{p}^2_1 + 2\widehat{p}^2_2 - \frac{1}{2}\widehat{p}^2_1\widehat{p}^2_2 + 2\sin p_1 \sin p_2 \ . \tag{8.4}$$

In the continuum limit, i.e. making the limit $a \to 0$ and taking only the orders in a^2, we see that, through the expression of the triangular components in terms of the canonical basis:

$$p_{tri,1} = p_1 \tag{8.5}$$

$$p_{tri,2} = -\frac{1}{2}p_1 + \frac{\sqrt{3}}{2}p_2 \tag{8.6}$$

$$p_{tri,3} = -\frac{1}{2}p_1 - \frac{\sqrt{3}}{2}p_2 \tag{8.7}$$

we have that

$$p^2_{tri} \to \frac{3}{2}p^2 \ . \tag{8.8}$$

We notice here an important difference from the case of the square lattice: since we have the δ constraint on the p vector components $\delta(p_1+p_2+p_3)$, a generic function of $\widehat{p}^2_{tri}$, $f(\widehat{p}_{tri})$ is no more invariant under a change of sign of one component (as it was on the square lattice), but only under a change of sign simultaneusly of all the vector components, i.e. the transformation

$$p = (p_1, p_2, p_3) \to -p = (-p_1, -p_2, -p_3) \ . \tag{8.9}$$

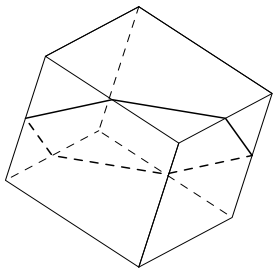


Figure 8.2 Derivation of the hexagonal-shaped Brilloin zone from the three-dimensional construction.

or by permutations of the components

$$(p_1, p_2, p_3) \to (p_2, p_1, p_3)\ . \tag{8.10}$$
$$(p_1, p_2, p_3) \to (p_2, p_3, p_1)\ . \tag{8.11}$$

The first interesting consequence of this fact is in the computation of integrals such the following, that diffently from the square lattice, is no more vanishing

$$\int f(\widehat{p}_{tri}) \sin p_1 \sin p_2 \neq 0\,. \tag{8.12}$$

this fact is relevant in technical computation of Feynman diagrams, as is shown in the next section.

Feynman diagrammatics is a delicate task also in the mere definition of momentum-space integration. We must integrate over all momenta in the Brilloin zone associated to the lattice. The coordinate-space translational invariance of the lattice, and the definition of the scalar product in the redundant three-entry vector notation, implies that momenta p, p' are equivalent if $p' = p + 2\pi(n, m, -n - m)$, with n and m integers, thus a possible choice of Brillouin zone would be the hexagon for which the distance betwenn parallel sides is 2π. This is indeed the zone which is individuated by the standard solid-state construction of Brilloin zone, that is by taking the Voronoi cell of the lattice (the polygon in which all the sides are axes of edges of the original lattice). Nonetheless, the choice of Brilloin zone is not unique, and is defined only up to valid translations of subsets of the zone. It turns out that, with a proper translation of four small triangles, the hexagon can be reduced to a rhombus with angles $\pi/3$ and $2\pi/3$, which, after a linear transformation (remark a factor $\sqrt{3}/2$ arising from this Jacobian), gives the traditional integration interval $[-\pi, \pi]^2$. This procedure is illustrated in the figure 8.3.

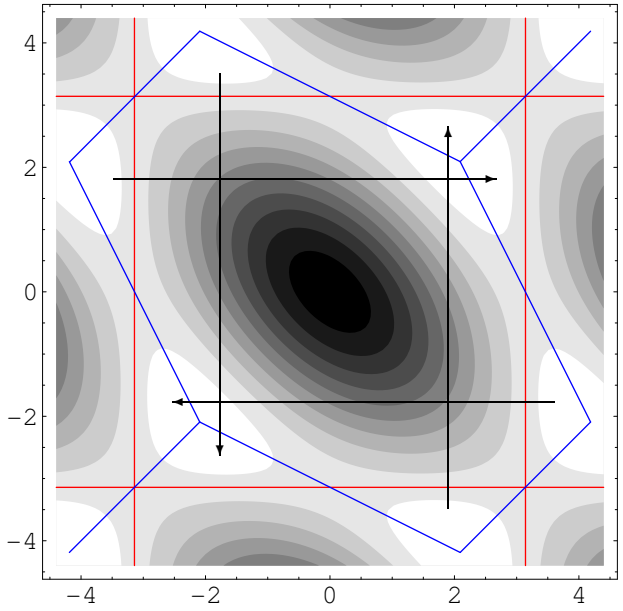


Figure 8.3 Two possible choices for the Brilloin zone. In blue, the hexagonal shape one would find naturally by lattice duality starting from the coordination-space triangular lattice. In red, the most useful choice in which the region of integration is simply $[-\pi, \pi]^2$. The arrows show that the two choices are both admissible, as they differ by the translation of parts of the Brilloin zone by translation vectors. In shadows, the gray-tone map $\widehat{p}^2$ is plotted in order to give a flavour of the lattice symmetries.

8.2 The spanning-forest model perturbative expansion

We are interested in studying the spanning-forest model

$$Z = \int \mathcal{D}(\psi, \bar{\psi}) \exp[\bar{\psi} L \psi + t \sum_i \bar{\psi}_i \psi_i + \frac{t}{2} \sum_{i,j} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j] \tag{8.13}$$

in the case of a triangular lattice. The form of the action in p-space remains unchanged

$$\mathcal{S} = \int_p \bar{\psi}_p \widehat{p}^2_{tri} \psi_p + t \int_p \bar{\psi}_p \psi_p + \frac{t}{2} \int_{p,q,k} \bar{\psi}_{q-k} \psi_q \widehat{k}^2_{tri} \bar{\psi}_{p+k} \psi_p \tag{8.14}$$

with care to the replacement of $\widehat{p}^2$ with $\widehat{p}^2_{tri}$, and p-space integration intended as

$$\int_p = \int \frac{\mathrm{d}p_1 \mathrm{d}p_2}{2\sqrt{3}\pi^2} \tag{8.15}$$

The description of perturbative expansion given in the previous chapter remains essentially the same: the Feynman rules are equal, and also the Feynman diagrams are same, the peculiarity of the triangular lattice is contained only in the different form of the Laplacian matrix, i.e. trough the $\widehat{p}^2_{tri}$ and the Jacobian factor $2/\sqrt{3}$.

8.2.1 Loop expansion

Now we make a pertubative expansion in t of the action. Using the Feynman rules we know that each power of t is added either with a mass insertion, or with an extra four-point interaction vertex; since we are interested in the two-point function, each four-point interaction vertex added to a diagram gives a new loop. So we say that, with abuse of terminology for what concerns diagrams with mass insertions, our perturbative expansion in powers of the coupling parameter is a "loop" expansion.

At zero order, i.e. no loops and no mass insertions, we have that the free propagator has the form

$$\widehat{p}^2_{tri} + h \tag{8.16}$$

from now on we will use

$$\Delta(k) = \widehat{p}^2_{tri} + h \, . \tag{8.17}$$

The exact form for the propagator is

$$\Gamma(p) = \Delta(k) + \Pi(p) \, , \tag{8.18}$$

where $\Pi(p)$ is the self-energy, i.e. the part of interaction added to the free fermion in order to consider the interaction effect of the non kinetic part of the action. Now we expand perturbatively $\Pi(p)$ as

$$\Pi(p) = t \, \Pi^{(1)}(p) + t^2 \, \Pi^{(2)}(p) + t^3 \, \Pi^{(3)}(p) + O(t^4) \, , \tag{8.19}$$

in the following we evaluated the first two terms, i.e with a loop expansion up to two loops. This two terms are sufficient to compute the renormalization constants that allow to write the first non-universal coefficient of the beta function (see Chapter 6).

At the first order we have the two Feynman diagrams already shown in the previous chapter. We have that the self-energy at one loop is:

$$\Pi^{(1)}(p) = -t - t\int_k \frac{\widehat{p+k}^2}{\Delta(k)} \tag{8.20}$$

$$\Pi(p) = \widehat{p}^2_{tri}\left[\frac{1}{6} - \left(1+\frac{h}{6}\right) I^{tri}(h)\right] + hI_{tri}(h)\,, \tag{8.21}$$

8.2.2 Second order

At the second order we have the same four diagrams of the square lattice, we sum them with alternate signs, taking into account, as made in previous chapter, of the (-1) factor coming from the mass insertion in **A** diagram and the (-1) factor coming from the fermionic loop in **C**. So the expression of the second order contribution of self-energy is

$$\Pi^{(2)}(p) = \mathbf{A} - \mathbf{B} + \mathbf{C} - \mathbf{D} \tag{8.22}$$

with the four diagrams being still the ones described in the previous chapter.

$$\mathbf{A} = \sum_k \frac{\widehat{p+k}^2}{\Delta(k)^2} \tag{8.23}$$

$$\mathbf{B} = \sum_{k,q} \frac{\widehat{p+k}^2\widehat{k+q}^2}{\Delta(q)\Delta(k)^2} \tag{8.24}$$

$$\mathbf{C} = \sum_{k,q} \frac{(\widehat{k^2})^2}{\Delta(p+q)\Delta(k)\Delta(k+q)} \tag{8.25}$$

$$\mathbf{D} = \sum_{k,q} \frac{\widehat{p+q}^2\widehat{k^2}}{\Delta(q)\Delta(k-q)\Delta(p+k)} \tag{8.26}$$

The first two diagrams are easy to evaluate exactly in terms of $I^{tri}(h)$ and $I_2^{tri}(h)$ in the same way made for the square lattice, from now on we will write only $I(h)$ and $I_2(h)$ remembering that all the quantities in this chapter are for the triangular lattice. So we find

$$\mathbf{A} = \widehat{p}^2_{tri}\left[-\frac{1}{6}I(h) + I_2(h) + \frac{1}{6}hI_2(h)\right] + I(h) - hI_2(h) \tag{8.27}$$

$$\mathbf{B} = \widehat{p}^2_{tri}\left[I(h)^2 - \frac{1}{2}I(h) + I_2(h) + \frac{1}{3}hI_2(h) - 2hI_2(h)I(h) + \frac{1}{36}\right] + \tag{8.28}$$

$$+ 2I(h) - \frac{1}{6} - 3hI(h)^2 - hI_2(h) + \frac{1}{2}hI(h) - \frac{1}{6}h^2I_2(h) + 2h^2I_2(h)I(h) \tag{8.29}$$

Taylor expansion for diagrams C and D

The diagrams **C** and **D** are the "bad guys", since we are not able to compute them exactly but we are forced to make a Taylor expansion for small external momentum. We already shown the procedure for the square lattice (6.29), but we want to analize the triangular case that is a bit more delicate. We want to compare the result for the integrals with those made on the continuum: the generic form of an integral on the continuum is

$$I(p,h) = C_{\overline{MS}}(h) + p^2 A_{\overline{MS}}(h) + O(p^4) \tag{8.30}$$

with $C(h)$ and $A(h)$ functions of h through a linear combination of contributions $I(h)$, $I_2(h)$, $hI_2(h)$, at most with a further factor h (for the function $C(h)$), and terms which vanish in the continuum limit. To realize the matching with continuum theory we need only the term up to p^2, so we make a Taylor expansion at the second order. The linear terms in p are vanishing, as we expect from the continuum form. we write the generic form of an integral on triangular lattice as

$$I(p,h) = C_{tri}(h) + p^2 A_{tri}(h) + O(p^2)\ . \tag{8.31}$$

Our two diagram contributions can be written as

$$\mathbf{C} = \int_{k,q} \frac{\Delta(k)^2}{\Delta(p+q)\Delta(k)\Delta(k+q)} \tag{8.32}$$

$$\mathbf{D} = \int_{k,q} \frac{\Delta(p+q)\Delta(k)}{\Delta(q)\Delta(k-q)\Delta(p+k)} \tag{8.33}$$

where we replace in the numerator a term as $\widehat{q}^2$ with $\Delta(q)$, since the added terms (proportional to $hI^2(h)$ and hR^{tri}) go to zero in the limit $h \to 0$, where, analogously to what is done for the square lattice (6.25), we define the constant integral R^{tri} as

$$R^{tri} = \lim_{h\to 0}\ h \int_{[-\pi,\pi]^4} \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} \frac{1}{(\widehat{p}^2_{tri}+h)(\widehat{q}^2_{tri}+h)(\widehat{p+q}^2_{tri}+h)}\ . \tag{8.34}$$

So the generic form of both diagram is

$$I(p=(p_1,p_2,p_3),h) = \int_{q,k} \frac{\prod_\alpha \Delta_\alpha(p,q,k,h)}{\prod_\beta \Delta_\beta(p,q,k,h)} \tag{8.35}$$

with $\alpha = 1,2$ and $\beta = 1,2,3$ and $\Delta_i(p,q,k,h) = \left(\epsilon_i(p) + \widehat{\epsilon_i(q)} + \epsilon_i(k)\right)^2 + h$ with $\epsilon_i = 0,1,-1$. We evaluate $A_{tri}(h)$ trough

$$A_{tri}(h) = \lim_{p\to 0} \frac{1}{2}\frac{I((p_1,p_2,p_3),h) - I(0,h)}{|p_1^2+p_2^2+p_3^2|}\ ; \tag{8.36}$$

Because of the spherical symmetry of the leading p^2 dependence, we can choose an infinitesimal increment of p in any direction we want. We choose to consider the

vector $p = (p, 0, -p)$, with square modulus $|p|^2 = (\sqrt{3}p/2)^2 + (p + 1/2p)^2 = 3p^2$, so we have that

$$A_{tri}(h) = \lim_{p\to 0} \frac{1}{6} \frac{I(p, 0, -p), h) - I(0, h)}{p^2} \ . \tag{8.37}$$

So we proceed expanding each term $\Delta_i(p, q, k, h)$ in p, and we have

$$A_{tri}(h) = \frac{1}{6} \int_{q,k} \left[\frac{\prod_\alpha (\Delta_\alpha + p\partial\Delta_\alpha + \frac{1}{2}p^2\partial^2\Delta_\alpha)}{(\prod_\beta \Delta_\beta + p\partial\Delta_\beta + \frac{1}{2}p^2\partial^2\Delta_\beta)} - \frac{\prod_\alpha \Delta_\alpha}{\prod_\beta \Delta_\beta} \right] . \tag{8.38}$$

where ∂ stands for $\partial_1 - \partial_3$. With a bit of manipulation we rewrite this formula in a general way as

$$A_{tri}(h) = \frac{1}{6} \int_{q,k} \left(\prod_\gamma \Delta_\gamma^{n_\gamma} \right) \left\{ \sum_{\gamma'} \left[n_{\gamma'} \left(\frac{\partial^2 \Delta_{\gamma'}}{\Delta_{\gamma'}} - \left(\frac{\partial \Delta_{\gamma'}}{\Delta_{\gamma'}} \right)^2 \right) \right] + \left[\sum_{\gamma'} \left(n_{\gamma'} \frac{\partial \Delta_{\gamma'}}{\Delta_{\gamma'}} \right) \right]^2 \right\} . \tag{8.39}$$

where all the propagators, after derivation, are intended at $p = 0$, and $n_\gamma = \pm 1$ for propagators respectively at numerator or denominator (actually, the formula holds for arbitrary n_γ).

Now we apply this last formula to our integral $\mathbf{C}$ and $\mathbf{D}$, and we obtain

$$\mathbf{C} = \frac{2}{3} \int_{q,k} \left[4 \Big(\frac{\sin^2 q_1}{\Delta(k)\Delta(q)\Delta(q+k)} - \frac{\sin q_1 \sin q_3}{\Delta(k)\Delta(q)\Delta(q+k)} \Big) - \frac{\cos q_1}{\Delta(k)\Delta(q+k)^2} \right] \tag{8.40}$$

$$\mathbf{D} = \frac{8}{3} \int_{q,k} \Big(\frac{\sin q_1 (\sin q_1 - \sin q_3)}{\Delta(k)\Delta(q)^2} + \frac{\sin k_1 (\sin q_1 - \sin q_3}{\Delta(k)\delta(q)\Delta(q+k)} \Big) \ . \tag{8.41}$$

For the evaluation of this integrals we used a procedure similar to the one of [2] to isolate the divergent parts proportional to $I(h)^2$ and $I(h)$ (look at the end of this chapter (8.4.3) for the definitions of all the costant lattice integrals and a detailed description of the evaluation procedure) so finally we found:

$$\mathbf{C} = \widehat{p}^2_{tri} \left[\frac{2}{3} I(h)^2 + \frac{1}{3} \left(\frac{2}{\pi} - 1 \right) I(h) + K(C) - \frac{4}{3} R^{tri} \right] + \tag{8.42}$$

$$+ 2I(h) - \frac{1}{6} - hI(h)^2 + \frac{h}{3} I(h) \tag{8.43}$$

$$\mathbf{D} = \widehat{p}^2_{tri} \left[\frac{2}{3} I(h)^2 - \frac{1}{3\pi} I(h) + K(D) + \frac{2}{3} R^{tri} \right] + I(h) \tag{8.44}$$

where $K(C)$ and $K(D)$ are constant terms, constituted of many summands (see (8.4.3)). Most of them can be evaluated in closed form, by mean of the Lüscher-Weisz technique, because they involve a linear combination of coordinate-space propagators inside a finite radius. Some other terms, involve a similar combination of propagators, but this time with a sum over the whole lattice, which can be performed only numerically, although, via this skill method, with large precision and small numerical effort (cfr. Appendix C).

We find for these quantities

$$K(C) = 0.05979 \tag{8.45}$$

$$K(D) = -\,0.0639\,. \tag{8.46}$$

8.3 The calculus of the beta function

From the evaluation of the Feynman diagrams we are able now to write the self-energy up to two loops. We have

$$\begin{aligned}\Pi(p) = \widehat{p}_{tri} + h - t\left[\frac{\widehat{p}^2_{tri}}{6} - \left(1+\frac{h}{6}\right) I^{tri}(h)\right] \\ - t^2\left[\widehat{p}^2_{tri}\left(I^2(h) + \frac{4}{3\pi}I(h) + \frac{1}{36\pi} + \frac{1}{36} - K(C) + K(D) + 2R^{tri}\right)\right. \\ \left. + h\left(-2I^2(h) + \frac{1}{6}I(h) - \frac{1}{6}hI_2(h) + 2hI_2(h)I(h)\right)\right]\end{aligned} \tag{8.47}$$

where we used the fact that in the limit $h \to 0$ we have

$$hI_2(h) \sim \frac{1}{6\pi} \tag{8.48}$$

as it is explained in details at the end of the chapter.

8.3.1 Continuum limit

Now we make the continuum limit of the expression (8.47) in order to obtain the leading part of it to compare with the continuum self-energy; so we have to take only the term proportional to a^2. Since, as we saw for square lattice, each factor h has a term a^2, the terms as $\widehat{p}^2 h$ are negligible. When we make the continuum limit, we have to take care of the geometry and features of the triangular lattice and follow the prescription that the lattice momentum $\widehat{p}_{tri}$ in the limit $a \to 0$ behaves as

$$\widehat{p}^2_{tri} \to \frac{3}{2}p^2\,. \tag{8.49}$$

Furthermore, in order to match the results with the self-energy on the continuum, we have to adopt the same convention for the parameters of the perturbative expansion used there, and use a form as

$$\Gamma^{(2)}_{latt} = \beta(\widehat{p}^2 + h) + \Pi^{(1)}_{latt} + \frac{\Pi^{(2)}_{latt}}{\beta} + \frac{\Pi^{(3)}_{latt}}{\beta^2} + \dots\,; \tag{8.50}$$

as we used in chapter 6 (6.13). In our case

$$\beta = -\frac{1}{t}\,, \tag{8.51}$$

in order to meet the expression (8.50), we rescale the fields and the weights w in the action

$$\psi, \bar{\psi} \to \frac{1}{\sqrt{t}}\psi \, , \frac{1}{\sqrt{t}}\bar{\psi} \tag{8.52}$$

$$w \to w\, t \tag{8.53}$$

obtaining the new scaled action

$$\mathcal{S} = \frac{1}{t}\bar{\psi}L\psi + \sum_i \bar{\psi}_i\psi_i + \frac{1}{2}\sum_{i,j}\bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j \tag{8.54}$$

and in Fourier representation

$$\mathcal{S} = \frac{1}{t}\int_p \bar{\psi}_p\widehat{p}^2_{tri}\psi_p + \int_p \bar{\psi}_p\psi_p + \frac{1}{2}\int_{p,q,k}\bar{\psi}_{q-k}\psi_q\widehat{k}^2_{tri}\bar{\psi}_{p+k}\psi_p \, . \tag{8.55}$$

After these manipulations we can write the continuum limit of the triangular self-energy in the correct form to be matched with the result obtained in the $\overline{MS}$-scheme (see (6.24)); the leading term proportional to a^2 is

$$\begin{aligned}\lim_{a\to 0}\Pi(p) &= \beta\left(\frac{3}{2}p^2 + h\right) - \left(\frac{p^2}{4} - \frac{h}{6\pi}\log\frac{ha^2}{72}\right) \\ &- \beta\left[\frac{3}{2}p^2\left(\frac{1}{36\pi^2}\log^2\frac{ha^2}{72} - \frac{2}{9\pi^2}\log\frac{ha^2}{72} + \frac{1}{36\pi} + \frac{1}{36} - K(C) + K(D) + 2R^{tri}\right)\right. \\ &\left. - h\left(\frac{1}{18\pi^2}\log^2\frac{ha^2}{72} + \frac{1}{18\pi}\left(\frac{1}{2} + \frac{1}{\pi}\right)\log\frac{ha^2}{72} + \frac{1}{36\pi}\right)\right]\end{aligned} \tag{8.56}$$

we used the fact that in the limit $h \to 0$ we have

$$I(h) \sim -\frac{1}{6\pi}\log\frac{h}{72} \tag{8.57}$$

$$hI_2(h) \sim \frac{1}{6\pi} \tag{8.58}$$

as it is explained at the end of the chapter, in formula (8.85).

8.3.2 Evaluation of renormalization constants

As explained in chapter 6, the determination of the renormalization constants relies on the use of the relation

$$\Gamma^{(n)}_{trilatt}(p_1, \cdots, p_n; \beta, h; 1/a) = Z_\pi^{n/2}\Gamma^{(n)}_{\overline{MS}}(p_1, \cdots, p_n; Z_\beta^{-1}\beta, Z_\beta Z_\pi^{-1/2}h; \mu) \tag{8.59}$$

that links our two different ways of regularize the fermionic theory: the one on the triangular lattice, and the one in the continuum with $\overline{MS}$-scheme of renormalization. Now we have all the ingredients we need, and we are ready to determine the

renormalization constants Z_π and Z_β (we maintain the same notation of chapter 6, where in the correspondence with non-linear σ-model, Z_π is the constant associated to the π field, and corresponds in fermionic formalism to the one associated to the fields ψ and $\bar{\psi}$).

From the zero order we have to match

$$\beta\left(\frac{3}{2}p^2+h\right)=\frac{Z_\pi}{Z_\beta}\beta\, p^2+\sqrt{Z_\pi}h\,\beta \tag{8.60}$$

and the zero-order coefficients of the series expansions

$$Z_\beta = Z_\beta^{(0)}+\frac{Z_\beta^{(1)}}{\beta}+\frac{Z_\beta^{(2)}}{\beta^2}+\frac{Z_\beta^{(3)}}{\beta^3}+O(\beta^{-4}) \tag{8.61}$$

$$Z_\pi = Z_\pi^{(0)}+\frac{Z_\pi^{(1)}}{\beta}+\frac{Z_\pi^{(2)}}{\beta^2}+\frac{Z_\pi^{(3)}}{\beta^3}+O(\beta^{-4}) \tag{8.62}$$

are found to be

$$Z_\beta^{(0)} = \frac{2}{3} \tag{8.63}$$

$$Z_\pi^{(0)} = 1\ . \tag{8.64}$$

From the first order we have to match

$$-\frac{p^2}{4}+\frac{h}{6\pi}\log\frac{ha^2}{72}=\frac{3}{2}\left(Z_\pi^{(1)}-\frac{3}{2}Z_\beta^{(1)}\right)p^2+\frac{h}{2}Z_\pi^{(1)}-\frac{1}{4\pi}\log\frac{2h}{3\mu^2}\left(p^2-\frac{2}{3}h\right) \tag{8.65}$$

and so we find

$$Z_\beta^{(1)} = \frac{1}{9}+\frac{1}{3\pi}\log\frac{a^2\mu^2}{48} \tag{8.66}$$

$$Z_\pi^{(1)} = \frac{1}{3\pi}\log\frac{a^2\mu^2}{48}\ . \tag{8.67}$$

In the same way we compare the second order, i.e. we write the relation (8.59) for the terms proportional to β and finally we find

$$Z_\beta^{(2)} = \frac{1}{24}\log^2\frac{a^2\mu^2}{48}+\frac{1}{6\pi}\left(1+\frac{2}{3\pi}\right)\log\frac{a^2\mu^2}{48} \tag{8.68}$$

$$+\frac{3}{2}\Big(K(D)-K(C)\Big)+\frac{1}{8\pi^2}-\frac{1}{48\pi}+\frac{1}{16} \tag{8.69}$$

$$Z_\pi^{(2)} = \frac{5}{36}\log^2\frac{a^2\mu^2}{48}+\frac{1}{18\pi}\log\frac{a^2\mu^2}{48} \tag{8.70}$$

At this point we use the relation derived in chapter 6

$$w_2^{trilatt} = w_0\left((Z_\beta^{(1)})^2-Z_\beta^{(2)}\right)+w_1Z_\beta^{(1)}+w_2^{\overline{MS}} \tag{8.71}$$

$$= -\frac{3}{2\pi}\left((Z_\beta^{(1)})^2-Z_\beta^{(2)}\right)-\frac{3}{(2\pi)^2}Z_\beta^{(1)}-\frac{3}{4(2\pi)^3}\ , \tag{8.72}$$

where in the second line we inserted the known results (they are reported in chapter 6) for the two first universal coefficients and from the one from the $\overline{MS}$-scheme. We evaluate

$$w_2^{trilatt} = \frac{1.986}{(2\pi)^3} \tag{8.73}$$

Finally we write the beta function $W^{trilatt}$ in terms of coupling constant β of the non-linear σ-model in the case $N = -1$

$$W^{trilatt}(\beta) \equiv -\frac{d\beta}{d(\ln a)} = -\frac{3}{(2\pi)^2} - \frac{3}{(2\pi)^2}\beta + \frac{1.986}{(2\pi)^3}\beta^2 + O(\beta^3) \tag{8.74}$$

or in terms of the coupling costant $t = -\frac{1}{\beta}$ of the fermionic model

$$W^{trilatt}(t) = +\frac{3}{(2\pi)^2}t^2 - \frac{3}{(2\pi)^2}t^3 + \frac{1.986}{(2\pi)^3}t^4 + O(t^4) \tag{8.75}$$

8.4 Technical details

8.4.1 The one-loop integral

All the one-loop integrals can be reduced to a sum of $I^{tri}(h)$ and I_2^{tri}, with the definitions

$$I^{tri}(h) = \int_{[-\pi,\pi]^2} \frac{d^2p}{2\sqrt{3}\pi^2} \frac{1}{\widehat{p}^2_{tri} + h} \tag{8.76}$$

$$I_2^{tri}(h) = \int_{[-\pi,\pi]^2} \frac{d^2p}{2\sqrt{3}\pi^2} \frac{1}{(\widehat{p}^2_{tri} + h)^2} \ . \tag{8.77}$$

as made for square lattice we compute in a similar way the first one. Using the relations:

$$\widehat{p}^2 = 2 - 2\cos p\,; \qquad \cos p_1 + \cos p_2 = 2\cos\frac{p_1+p_2}{2}\cos\frac{p_1-p_2}{2}\,; \tag{8.78}$$

we rewrite the denominator as

$$\begin{aligned} \widehat{p}^2_{tri} + h &= 2 - 2\cos p_1 + 2 - 2\cos p_2 + 2 - 2\cos(p_1+p_2) + h \\ &= 6 - 4\cos\frac{p_1+p_2}{2}\cos\frac{p_1-p_2}{2} - 2\cos(p_1+p_2) + h \end{aligned} \tag{8.79}$$

and then we make the change of variables $k_1 = \frac{p_1+p_2}{2}$ and $k_2 = \frac{p_1-p_2}{2}$; the Jacobian of the transformation is 2, but it simplifies with the factor $1/2$ coming from the new area of integration; in fact k_1 and k_2 run inside the rhombus of vertices $(\pi,0),(0,\pi),(-\pi,0),(0,-\pi)$, that has area equal to half of the square area $[-\pi,\pi]^2$, which indeed contains exactly two Brillouin zones. So we obtain

$$I^{tri}(h) = \int_{[-\pi,\pi]^2} \frac{d^2k}{2\sqrt{3}\pi^2} \frac{1}{6 - 4\cos k_1\cos k_2 - 2\cos(2k_1) + h} \quad . \tag{8.80}$$

We are now able to integrate in k_2 using the result

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{\alpha + \beta \sin\theta + \gamma \cos\theta} = \frac{1}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}} \quad , \tag{8.81}$$

we have

$$\begin{aligned} I^{tri}(h) &= \frac{2}{\sqrt{3}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{(3 - \cos(2k_1) + \frac{h}{2})^2 - 4\cos^2 k_1}} \\ &= \frac{2}{\sqrt{3}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{(\frac{h+6}{2} + 2\sin^2 k_1)^2 - (h+9)}} \\ &= \frac{2}{\sqrt{3}} \int_{0}^{2\pi} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{(\frac{h+8}{2} - \cos k_1)^2 - (h+9)}} \end{aligned} \tag{8.82}$$

Finally, after the change $\cos k_1 = x$, we can express our integral by an elliptic integral[1]

$$\begin{aligned} I^{tri}(h) &= \frac{1}{\sqrt{3}\pi} \int_{-1}^{1} dx \frac{1}{\sqrt{(1-x^2)(\frac{h+8}{2} + \sqrt{h+9} - x)(\frac{h+8}{2} - \sqrt{h+9} - x)}} \\ &= \frac{1}{\sqrt{3}\pi} \frac{2}{\sqrt{6 + 2\sqrt{h+9} + 3h + \frac{h^2}{4}}} K\left(\sqrt{\frac{4\sqrt{h+9}}{6 + 2\sqrt{h+9} + 3h + \frac{h^2}{4}}}\right) \end{aligned} \tag{8.84}$$

When $h \to 0$

$$I^{tri}(h) = -\frac{1}{6\pi} \log\left(\frac{h}{72}\right) + O(h \log h). \tag{8.85}$$

For the second integral $I^{tri}(h)$ we use the fact that

$$I^{tri}(h) = -\frac{d\, I^{tri}(h)}{dh} \, , \tag{8.86}$$

as shown in section(7.3) for square lattice, this is sufficient to write the leading term of $I^{tri}(h)$ simply deriving the expression (8.85), so we have

$$h I_2(h) \sim \frac{1}{6\pi} \, . \tag{8.87}$$

[1] From 3.148.2 of [8]

$$\int_d^u dx \frac{1}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r) \tag{8.83}$$

with $a > b > c \geq u > d$ and $\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}$ $r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$. In our case $a = \frac{h+8}{2} + \sqrt{h+9}$, $b = \frac{h+8}{2} - \sqrt{h+9}$, $c = u = 1$, $d = -1$.
$F(\beta, r) = \int_0^\beta \frac{d\theta}{\sqrt{1 - r^2 \sin^2\theta}}$ is the *elliptic integral of the second kind*, and if $\beta = \frac{\pi}{2}$, $F(\frac{\pi}{2}, r) = K(r)$ is called the *complete* integral.

8.4.2 Tricks

Some tricks used on square lattice for the integrals evaluation remains valid, as the replacement

$$\int_q f(\widehat{q}^2)\,\widehat{q}_\mu^2 \to \frac{1}{m}\int_q f(\widehat{q}^2)\,\widehat{q}^2\,, \tag{8.88}$$

where we stress the requirement of the integrand function to be invariant under discrete rotations of vector $\widehat{q}$, and we denote with m the number of the components of lattice-momuntum $\widehat{q}$. On the triangular lattice we have $m = 3$, and so we will use

$$\int_{q,k} f(\widehat{q}^2)\,\widehat{p}_\mu^2 \to \frac{1}{3}\int_{q,k} f(\widehat{q}^2)\,\widehat{p}^2\,. \tag{8.89}$$

Other tricks are peculiar of the triangular lattice, in particular for the evaluation of Feynman diagrams we use several times the following rules.

Rule 1

$$\int_{q,k} f(\widehat{q}_1,\widehat{q}^2)\widehat{q}_2^2 = \frac{1}{2}\int_{q,k} f(\widehat{q}_1,\widehat{q}^2)\widehat{q}^2 - \frac{1}{2}\int_{q,k} f(q_1,\widehat{q}^2)\widehat{q}_1^2 \tag{8.90}$$

where we used the fact that

$$\widehat{p}_{tri} = \widehat{p}_1^2 + \widehat{p}_2^2 + \widehat{p}_3^2\,. \tag{8.91}$$

Rule 2

$$\int_{q,k} f(q_\mu,\widehat{q}^2,k_\mu,\widehat{k}^2)\sin k_\nu = -\frac{1}{2}\int_{q,k} f(q_\mu,\widehat{q}^2,k_\mu,\widehat{k}^2)\sin k_\mu + \frac{1}{2}\int_{q,k} f(q_\mu,\widehat{q}^2,k_\mu,\widehat{k}^2)\widehat{k}_\rho^2\sin k_\nu \tag{8.92}$$

with $\mu \neq \nu \neq \rho$.

Rule 3

$$\int_{q,k}\frac{\sin k_\mu f_{odd}(q)}{\Delta(k)\Delta(q)\Delta(q+k)} = -\frac{1}{2}\int_{q,k}\frac{\sin q_\mu f_{odd}(q)}{\Delta(k)\Delta(q)\Delta(q+k)} + \frac{1}{4}\int_{q,k}\frac{\sin q_\mu \widehat{k}_\mu^2 f_{odd}(q)}{\Delta(k)\Delta(q)\Delta(q+k)} + \frac{1}{4}\int_{q,k}\frac{\sin k_\mu \widehat{q}_\mu^2 f_{odd}(q)}{\Delta(k)\Delta(q)\Delta(q+k)} \tag{8.93}$$

where $f_{odd}(q)$ is a function odd in the variable q; to prove this rule we make the change of variables $q + k = k'$ and $q = -q'$.

Rule 4 *for* $f(q) \sim q^4$,

$$\int_{q,k} \frac{f(q)}{\Delta(k)\Delta(q)\Delta(q+k)} = \int_{q,k} \frac{f(q+k)[\Delta^2(q+k) - \Delta(k) - \Delta(q)]}{\Delta(k)\Delta(q)\Delta^2(q+k)} + 2\int_{q,k} \frac{f(k)}{\Delta(q)\Delta^2(k)} \quad (8.94)$$

where the first summand is a finite quantity, and the second one has a divergence of the kind $I(h)$ *factorized out. Similarly, still for* $f(q) \sim q^4$,

$$\int_{q,k} \frac{\widehat{k}_\mu^2 f(q)}{\Delta(k)\Delta(q)\Delta(q+k)} = \int_{q,k} \frac{\widehat{k}_\mu^2 f(q+k)[\Delta^2(q+k) - \Delta(k) - \Delta(q)]}{\Delta(k)\Delta(q)\Delta^2(q+k)} + \frac{1}{3}\int_q \frac{f(q)(2 - \frac{1}{2}\widehat{q}_\mu^2)}{\Delta^2(q)} + I(h)\int_q \frac{\widehat{q}_\mu^2 f(q)}{\Delta^2(q)} \quad (8.95)$$

where the first two summands are finite quantities, and in the third one a divergence of the kind $I(h)$ *is factorized out.*

8.4.3 Evaluation of diagrams C and D

In order to write the result of integration of diagrams **C** and **D** we define the following relevant lattice integrals (in a similar way as made in [2]):

$$A^{(1)}(h) = \int_{q,k} \frac{\sin^2 q_\mu}{\Delta(k)\Delta(q)\Delta(q+k)} \quad (8.96)$$

$$A^{(2)}(h) = \int_{q,k} \frac{\sin k_\mu \sin q_\mu}{\Delta(k)\Delta(q)\Delta(q+k)} \quad (8.97)$$

$$B^{(1)}(h) = \int_{q,k} \frac{\sin q_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} \quad (8.98)$$

$$B^{(2)}(h) = \int_{q,k} \frac{\sin k_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} \quad (8.99)$$

The first one is evaluated with the same procedure of [2] as

$$A^{(1)}(h) = \frac{1}{3}I^2(h) - \frac{1}{3}R^{tri} - \frac{1}{2}I(h)K_2^4 - \frac{1}{4}G \quad (8.100)$$

by the use of the constant integral G

$$G = \int_{q,k} \frac{\widehat{q_\mu + k_\mu}^4 [\Delta(q+k) - \Delta(q) - \Delta(k)]}{\Delta(k)\Delta(q)\Delta(q+k)^2} \quad (8.101)$$

that we computed via Lüscher-Weisz technique (briefly in the following we will say *via L-W*) (cfr. Appendix C)

$$G = -0.08617\,. \quad (8.102)$$

This is a first application of the strategy of Rule 4 for isolating the divergent part in a three-propagator integral. We remeber the definition of R^{tri}

$$R^{tri} = \lim_{h\to 0}\ h\int_{[-\pi,\pi]^4} \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{1}{(\widehat{p}^2_{tri}+h)(\widehat{q}^2_{tri}+h)(\widehat{p+q}^2_{tri}+h)}\ . \qquad (8.103)$$

that is a constant integral, which at the end of the calculus simplifies with its corrispondent R in the continuum self-energy expression.

Furthermore we introduced the notation

$$K^l_m = \int_q \frac{\widehat{q}_\mu{}^l}{\Delta(q)^m}\ , \qquad (8.104)$$

in our integrals appear the terms K^4_1, K^4_2 and K^6_2, that we evaluated *via L-W* as :

$$K^4_1 = \frac{4\sqrt{3}}{\pi} - \frac{4}{3} \qquad K^4_2 = \frac{1}{3} - \frac{1}{\sqrt{3}\pi} \qquad K^6_2 = -4 + \frac{8\sqrt{3}}{\pi}\ . \qquad (8.105)$$

For the second integral $A^{(2)}(h)$, in [2] it is shown the way to write it as

$$A^{(2)}(h) = -\frac{A^{(1)}(h)}{2} + \frac{A^{(3)}(h)}{24} \qquad (8.106)$$

where

$$A^{(3)}(h) = \int_{q,k} \frac{\widehat{k}^2_\mu\ \widehat{q}^2_\mu\ \widehat{q_\mu + k_\mu}^{\,2}}{\Delta(k)\Delta(q)\Delta(q+k)} \qquad (8.107)$$

that we evaluated *via L-W* as

$$A^{(3)}(h) = 0.11356\ . \qquad (8.108)$$

For the integral $B^{(1)}(h)$, we use the Rule 2 to write it as

$$B^{(1)}(h) = -\frac{A^{(1)}(h)}{2} + \frac{1}{2}\int_{q,k}\frac{\widehat{q}^2_\nu \sin q_\mu \sin q_\rho}{\Delta(k)\Delta(q)\Delta(q+k)} = -\frac{A^{(1)}(h)}{2} + \frac{H}{2} + I(h)\ P \qquad (8.109)$$

with H and P finite quantities which we explain below. Indeed, for the quantity which is of the form: integral over two momenta of a generic function of q (of order q^4), divided by the three propagators, one can do a subtraction which allows to isolate the divergent part, in a way similar to the what has been made above for $A^{(1)}$, and described in general form in Rule 4. Here the statement reads

$$\int_{q,k}\frac{\widehat{q}^2_\nu \sin q_\mu \sin q_\rho}{\Delta(k)\Delta(q)\Delta(q+k)} = H + 2I(h)\int_q \frac{\widehat{q}^2_\nu \sin q_\mu \sin q_\rho}{\Delta(q)^2} = H + 2I(h)\ P \qquad (8.110)$$

where P is a shorthand notation for the linear combination of factors K^l_m

$$P = \int_q \frac{\widehat{q}^2_\nu \sin q_\mu \sin q_\rho}{\Delta(q)^2} = \frac{1}{4}K^6_2 + K^4_2 - \frac{3}{8}K^4_1 + \frac{1}{12}\ . \qquad (8.111)$$

and H is the finite integral

$$H = \int_{q,k} \frac{\widehat{q}_\nu^2 \sin q_\mu \sin q_\rho [\Delta(q+k) - \Delta(q) - \Delta(k)]}{\Delta(k)\Delta(q)\Delta(q+k)^2} \tag{8.112}$$

and *via L-W* we found

$$H = 0.00589 \; . \tag{8.113}$$

Finally, for the integral $B^{(2)}(h)$ we used the Rule 3 to write it as

$$B^{(2)}(h) = -\frac{1}{2}\int_{q,k} \frac{\sin q_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} + \frac{1}{2}\int_{q,k} \frac{\widehat{k}_\mu^2 \sin q_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} \tag{8.114}$$

$$= -\frac{B^{(1)}(h)}{2} + \frac{1}{4} I(h) K_2^4 - \frac{1}{8} G + \frac{1}{2}(I(h)F_1 + F) \tag{8.115}$$

with F a properly-defined constant. Indeed, in the second line we used the fact that, using Rule 2

$$\int_{q,k} \frac{\widehat{k}_\mu^2 \sin q_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} = -\frac{1}{2}\int_{q,k} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{\Delta(k)\Delta(q)\Delta(q+k)}$$
$$+ \frac{1}{8}\int_{q,k} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^4}{\Delta(k)\Delta(q)\Delta(q+k)} + \frac{1}{2}\int_{q,k} \frac{\widehat{k}_\mu^2 \widehat{q}_\nu^2 \sin q_\mu \sin q_\nu}{\Delta(k)\Delta(q)\Delta(q+k)} \; . \tag{8.116}$$

For the first integral, we use the trick $\frac{1}{2}\widehat{k}_\mu^2 \widehat{q}_\mu^2 = \widehat{k_\mu + q_\mu}^{\,2} - \widehat{k}_\mu^2 - \widehat{q}_\mu^2 - 2\sin k_\mu \sin q_\mu$, which allows to identify all the contributions, and find

$$-\frac{1}{2}\int_{q,k} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{\Delta(k)\Delta(q)\Delta(q+k)} = \frac{1}{2} I(h) K_2^4 - \frac{1}{4} G$$

For the other two integrals, we can apply Rule 4 in its second case, with

$$f(q) = \frac{1}{8}\widehat{q}_\mu^4 + \frac{1}{2}\widehat{q}_\nu^2 \sin q_\mu \sin q_\nu$$

So, the part in $I(h)$ has a coefficient

$$F_1 = \int_q \frac{\widehat{q}_\mu^2 f(q)}{\Delta^2(q)}$$

while the constant part gives

$$F = \frac{1}{3}\int_q \frac{(2 - \frac{1}{2}\widehat{q}_\mu^2) f(q)}{\Delta^2(q)} + \int_{q,k} \frac{\widehat{k}_\mu^2 f(q+k)(\Delta^2(q+k) - \Delta(k) - \Delta(q))}{\Delta(k)\Delta(q)\Delta^2(q+k)}$$

The one-loop part gives

$$F_1 = -\frac{23}{6\sqrt{3}} + \frac{15}{2\pi} \tag{8.117}$$

while the two-loop part is computed *via L-W*, and gives

$$F = 0.00647\ . \tag{8.118}$$

Finally we are ready to write the expression of diagrams **C** and **D**: for the first one we use the expression of $A^{(1)}(h)$ and $B^{(1)}(h)$ and we have

$$\mathbf{C} = \frac{2}{3}I^2(h) + I(h)\left(\frac{2}{9} - 2K_2^4 - \frac{8}{3}P\right) - G - \frac{4}{3}H - \frac{4}{3}R^{tri} - \frac{1}{54} \tag{8.119}$$

so we have that the constant part is

$$K(C) = -G - \frac{4}{3}H - \frac{1}{54} \tag{8.120}$$

and we do not take in $K(C)$ the term proportional to R^{tri}, since as we expect, this term, joined with the same one coming from diagram **D**, gives a factor $2R^{tri}$ that cancels with the corrispondent R in the continuum self-energy expression, when we use the relation (8.59) to match the lattice with continuum theory. From our evaluation of all the lattice integral constants we found

$$K(C) = 0.05979\ . \tag{8.121}$$

For **D** we use the expression of $A^{(2)}(h)$ and $B^{(2)}(h)$ and we have

$$\mathbf{D} = \frac{2}{3}I^2(h) + I(h)\left(-\frac{1}{9} - \frac{4}{3}hI_2(h) - \frac{2}{3}K_2^4 + \frac{4}{3}P - \frac{4}{3}F_1\right) \tag{8.122}$$

$$+ \frac{1}{3}\left(2R^{tri} + \frac{7}{2}G - F + 2H + \frac{1}{3}A^{(3)}(h)\right) \tag{8.123}$$

so

$$K(D) = \frac{1}{3}\left(\frac{5}{2}G - 4F + 2H + \frac{1}{3}A^{(3)}(h)\right) \tag{8.124}$$

and we evaluated

$$K(D) = -0.0639\ . \tag{8.125}$$

A. Some results of Graph Theory

We want to give here proofs of the two results (1.11) and (1.12), quoted in chapter 1.

A.1 The Temperley formula

The Temperley formula

$$\kappa(G) = \frac{1}{V^2}\det(\mathbf{L} + \mathbf{J}) \tag{A.1}$$

can be proved with algebraic arguments about the cofactors of a matrix (see [6]), here we want to give a proof in terms of fermionic integrals.

We know that the Laplacian matrix annihilates the vector with all entries equal to 1, so the projector on the eigenspace with eigenvalue 0 is the matrix

$$\mathbf{\Pi} = \frac{\mathbf{J}}{V} \tag{A.2}$$

where

$$J_{ij} = 1 \qquad \forall\, i, j \tag{A.3}$$

and the projector on the orthogonal eigenspace is

$$\mathbf{\Pi}^{\perp} = \mathbf{1} - \frac{\mathbf{J}}{V}\,. \tag{A.4}$$

If we add to the Laplacian matrix a small perturbation along $\mathbf{\Pi}^{\perp}$, we will leave unchanged the eigenspace of the 0-mode. Taking the subscript $\star$ for the matrix diagonalized, we have that, while the original matrix $\mathbf{L}_\star$ looks like

$$\mathbf{L}_\star = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{V-1} \end{pmatrix} \tag{A.5}$$

the perturbed diagonalized matrix $(\mathbf{L}+\epsilon\mathbf{\Pi}^{\perp})_\star$ must look like

$$(\mathbf{L}+\epsilon\mathbf{\Pi}^{\perp})_\star = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1(\epsilon) & 0 & \dots & 0 \\ 0 & 0 & \lambda_2(\epsilon) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{V-1}(\epsilon) \end{pmatrix} \tag{A.6}$$

so also $\det(\mathbf{L}+\epsilon\mathbf{\Pi}^{\perp})=0$ and, in terms of Grassmann integrals,[1]

$$\int \mathcal{D}(\psi,\bar\psi)\; e^{(\bar\psi,(\mathbf{L}+\epsilon\mathbf{\Pi}^{\perp})\psi)} = 0 \tag{A.7}$$

and from this

$$0 = \frac{d}{d\epsilon}\left(\int \mathcal{D}(\psi,\bar\psi)\; e^{(\bar\psi,(\mathbf{L}+\epsilon\mathbf{\Pi}^{\perp})\psi)}\right)_{|_{\epsilon=0}} = \int \mathcal{D}(\psi,\bar\psi)\; (\bar\psi,\mathbf{\Pi}^{\perp}\psi) e^{(\bar\psi,\mathbf{L}\psi)}\;; \tag{A.8}$$

Now consider the right hand side: from $\mathbf{I}=\mathbf{\Pi}+\mathbf{\Pi}^{\perp}$

$$\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi,\psi)\,e^{(\bar\psi,\mathbf{L}\psi)} = \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi,\mathbf{\Pi}\psi)\,e^{(\bar\psi,\mathbf{L}\psi)} \tag{A.9}$$

but

$$\begin{aligned} \kappa(G) &= \frac{1}{V}\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi,\psi)\,e^{(\bar\psi,\mathbf{L}\psi)} \\ &= \frac{1}{V}\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi,\mathbf{\Pi}\psi)\,e^{(\bar\psi,\mathbf{L}\psi)} \\ &= \frac{1}{\lambda V}\int \mathcal{D}(\psi,\bar\psi)\,e^{(\bar\psi,\mathbf{L}\psi)+\lambda\,(\bar\psi,\mathbf{\Pi}\psi)} \\ &= \frac{1}{\lambda V}\det\left(\mathbf{L}+\lambda\mathbf{\Pi}\right) \end{aligned} \tag{A.10}$$

where we used the vanishing of the determinant of the Laplacian and of the higher powers in λ. In the last determinant the zero eigenvalue of the Laplacian is substituted by λ, which is removed by the denominator, so that we get

$$\kappa(G) = \frac{1}{V}\det{}'\,\mathbf{L} \tag{A.11}$$

where $\det'$ is the determinant without the 0-mode. The particular choice $\lambda = V$ provides Temperley formula (A.1)

$$\kappa(G) = \frac{1}{V^2}\det(\mathbf{L}+\mathbf{J})\;. \tag{A.12}$$

[1]Here we use for shortness $(\bar\psi, A\psi) = \sum_{ij}\bar\psi_i A_{ij}\psi_j$, and $(\bar\psi,\psi)$ if $A=I$.

A.2 Proof of $\det'$-$\det L(i)$ formula

We prove the formula (1.12) of chapter one, which puts in relation the determinant of a (whatever) principal minor of a Laplacian matrix with the product of its non-vanishing eigenvalues

$$\forall\ j \qquad \det L(j) = \frac{1}{V}\prod_{i=1}^{V-1}\lambda_i \tag{A.13}$$

Since ℓ has a null eigenvector, we write

$$\det(L-\lambda I) = -\lambda\prod_{i=1}^{V-1}(\lambda_i-\lambda). \tag{A.14}$$

Consider the matrix $A^{(k)}$, such that

$$A^{(k)}_{ij} = \begin{cases} 1 & i=j \text{ or } i=k \\ 0 & \text{otherwise} \end{cases} \tag{A.15}$$

then we simply have $\det A^{(k)} = 1$ for any k. Now consider the transformed matrix

$$M^{(k)} = (A^{(k)})^T(L-\lambda I)A^{(k)} \tag{A.16}$$

such that $\det M^{(k)} = \det(L-\lambda I)$. It can be seen that

$$M^{(k)}_{ij} = \begin{cases} L_{ij}-\lambda\delta_{ij} & i\neq k \text{ and } j\neq k \\ 0 & (i\neq k \text{ and } j=k) \text{ or } (i=k \text{ and } j\neq k) \\ -V\lambda & i=k \text{ and } j=k \end{cases} \tag{A.17}$$

and thus that

$$\det M^{(k)} = -V\lambda\det(L(k)-\lambda I) \tag{A.18}$$

And from a small-λ limit the formula easily follows

$$\prod_{i=1}^{V-1}\lambda_i = \lim_{\lambda\to 0}\prod_{i=1}^{V-1}(\lambda_i-\lambda) = \lim_{\lambda\to 0}\frac{\det(L-\lambda I)}{(-\lambda)} = V\lim_{\lambda\to 0}\det(L(k)-\lambda I) \tag{A.19}$$

B. Grassmann variables

A N-dimensional Grassmann algebra is the algebra generated by a set $\{\psi_i\}$ of variabiles, with $i = 1, \cdots, N$, satisfying

$$\psi_i \psi_j + \psi_j \psi_i = 0 \qquad \forall i, j \tag{B.1}$$

In particular, $\psi_i^2 = 0$ and the most general function of a single variable ψ_i is

$$f(\psi_i) = a + b\psi_i \qquad a, b \in C \tag{B.2}$$

as higher orders vanish because of nilpotency of the variables. For example, in Grassmann algebra, $e^{\psi} \equiv 1 + \psi$ exactly. Analogously, the most general function of N variables is a polynomial, composed of complex coefficients multiplying monomials of the form $\prod_{i \in I} \psi_i$, with I a certain subset of $\{1, \ldots, N\}$.

The one-variable Berezin integral

$$\int d\psi f(\psi) \tag{B.3}$$

is defined in order to fulfill the constraint of traslational invariance

$$\int d\psi \, \Big(a + b\psi\Big) = \int d\psi \, \Big(a + b(\psi + \eta)\Big) \tag{B.4}$$

so it is required that

$$\int d\psi = 0 \qquad \text{and} \qquad \int d\psi \, \psi = 1 \quad , \tag{B.5}$$

In order to memorize this two properties, we can think that the Berezin integral acts *as a derivative*.

In order to preserve the second of the (B.5) when we make a change of variables, instead of a Jacobian matrix as we do for commuting variables, we have to use its inverse; for example if we take $\eta = A\psi$

$$1 = \int d\psi \, \psi = \int d\psi(\eta) \, A^{-1} \eta \tag{B.6}$$

but also

$$1 = \int d\eta \, \eta \tag{B.7}$$

so we need

$$d\eta = A^{-1}\, d\psi \tag{B.8}$$

(note that in the commuting case we would have $d\eta = A\, d\psi$).

Berezin integral immediately extends to the N-variable case, up to an overall prescription on the sign of the measure, that is

$$\int d\psi_N \cdots d\psi_1 \psi_1 \cdots \psi_N = 1 \tag{B.9}$$

and, if we have a product of all the variables, but in a different order (say, re-ordered with permutation π), the result must be multiplied by the signature of the permutation.

For the N-variable integral, extending the reasonings above, under the change of variables $\eta_i = \sum_j A_{ij}\psi_j$ we must have

$$\int d\psi_N \cdots d\psi_1\, f(\psi) = \int d\eta_N \cdots d\eta_1\, (\det A)^{-1} f(\eta)\,. \tag{B.10}$$

Now we have all the ingredients to understand the useful result

$$\int \left(\prod_i d\psi_i d\bar{\psi}_i \right) e^{\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = \det A \tag{B.11}$$

or in a compact notation

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\bar{\psi} A \psi} = \det A \tag{B.12}$$

let us take a bit of time to explain (B.11). First of all we defined a second family of anticommuting variabiles $\bar{\psi}_i$, the complex conjugates of ψ_i, that follow the same rules as (B.1) and (B.5), furthermore, in order to define also the derivative operator, the Grassmann algebra is defined by the rules

$$\{\psi_i, \psi_j\} = 0 \qquad \left\{ \frac{\partial}{\partial \psi_i}, \psi_j \right\} = \delta_{ij} \qquad \left\{ \frac{\partial}{\partial \psi_i}, \frac{\partial}{\partial \psi_j} \right\} = 0 \qquad \forall i, j \tag{B.13}$$

similarly for the $\{\bar{\psi}_i\}$.

In (B.11) we make the change of variabiles $\eta_i = \sum_j A_{ij}\psi_j$, according with (B.10) we have

$$\det A \int \left(\prod_i d\eta_i d\bar{\psi}_i \right) e^{\sum_i \bar{\psi}_i \eta_i} = \det A \tag{B.14}$$

to complete the proof, it is sufficient to prove that the Grassmann integral is equal to 1, this is easy to see expanding the exponential and using (B.5)

$$\begin{aligned} &\int d\eta_N d\bar{\psi}_N\; d\eta_{N-1} d\bar{\psi}_{N-1} \cdots d\eta_1 d\bar{\psi}_1\; (1 + \bar{\psi}_1\eta_1)(1 + \bar{\psi}_2\eta_2) \cdots (1 + \bar{\psi}_N\eta_N) \\ &\quad = \int d\eta_N d\bar{\psi}_N\; d\eta_{N-1} d\bar{\psi}_{N-1} \cdots d\eta_2 d\bar{\psi}_2\; (1 + \bar{\psi}_2\eta_2) \cdots (1 + \bar{\psi}_N\eta_N) = \cdots = 1 \end{aligned} \tag{B.15}$$

The result of (B.11) can be generalized to expectation values of monomials, which give determinants of submatrices of A. If we denote with $A(I|J)$ the submatrix obtained from A deleting the set $I = (i_1, i_2, \ldots, i_k)$ of rows and $J = (j_1, j_2, \ldots, j_k)$ of columns (with $|I| = |J| = k$). For example, if we delete from A the 2nd and the 3th row and the 1st and N-th column, the determinant of the remaining matrix will be

$$\det A(2,3|1,N) = \int \left(\prod_i d\psi_i d\bar{\psi}_i \right) \bar{\psi}_2 \psi_1 \bar{\psi}_3 \psi_N \, e^{\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} \epsilon(2,3|1,N) \qquad \text{(B.16)}$$

in fact the presence of the fields $\bar{\psi}_2 \psi_1 \bar{\psi}_3 \psi_N$ prevents, if we want a non vanishing result, to take in the exponential expansion the terms as $\bar{\psi}_2 A_{2j} \psi_j$, $\bar{\psi}_3 A_{3j} \psi_j$, $\bar{\psi}_i A_{i1} \psi_1$ or $\bar{\psi}_i A_{iN} \psi_N$. Moreover, $\epsilon(i_1, \ldots, i_k|j_1, \ldots, j_k) = \pm 1$ accounts for the number of interchanges among the fermionic fields for ordering them before the integration.

In general

$$\int \mathcal{D}(\psi, \bar{\psi}) \; \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_k} \psi_{j_k} \; e^{\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = \epsilon(I|J) \det A(I|J) \qquad \text{(B.17)}$$

and in particular, for the case $I \equiv J$, we just have $\epsilon(I \mid I) = 1$.

When I is composed of a single element, the determinant of the i-th principal minor of A, which is the first non-trivial determinant of A in the case rank$A = N-1$ (as for Laplacian matrices of connected graphs) is obtained by the expectation value of a single pair of fermionic variables

$$\det A(i) = \int \mathcal{D}(\psi, \bar{\psi}) \; \bar{\psi}_i \psi_i \; e^{\bar{\psi} A \psi} \; . \qquad \text{(B.18)}$$

these "rooting" variables, with a pictorial surprise, corresponding to the choice of an "electrical ground" in the original Kirchhoff application to electrical networks.

C. Lüscher-Weisz method for evaluation of lattice integrals

In the evaluation of two-dimensional lattice integrals, we used the method illustrated in the paper by Lüscher and Weisz [27], and specialized to two dimensions by Dong-Shih Shin [20]. The basic idea is to use some basic relations for the free propagator in coordinate space (the defining Laplacian equation and a set of relations due to Vohwinkel), in order to find a recursion which, starting from the values in a certain number of sites neighbouring the origin (the "fundamental" lattice integrals), allows to find the whole set of free propagators in lattice sites in a large radius R, in a time which scales polynomially with R. As a side result, it gives a simple proof of the fact that all these values are linear combinations with rational coefficients of the fundamental lattice integrals.

C.1 Square lattice

On square lattice we define the free propagator as

$$G(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{e^{ipx} - 1}{\widehat{p}^2} \tag{C.1}$$

with the subtraction required by regularization (otherwise the integral would be divergent for any value of x). This choice implies that

$$G(0,0) = 0\,; \tag{C.2}$$

As usual, the right- and left-derivatives on the lattice are defined as

$$\partial_\mu f(x) = f(x + \widehat{\mu}) - f(x)\,; \tag{C.3}$$
$$\partial^*_\mu f(x) = f(x) - f(x - \widehat{\mu})\,; \tag{C.4}$$

so the Laplacian operator is written as

$$\Delta = \sum_{\mu=0}^{1} \partial^*_\mu \partial_\mu\,, \tag{C.5}$$

and it acts on a function as

$$\Delta f(x) = \sum_{\mu} \Big(f(x+\widehat{\mu}) + f(x-\widehat{\mu}) - 2f(x) \Big) \tag{C.6}$$

The free propagator $G(x)$ satisfies the Laplace equation

$$-\Delta G(x) = \delta^{(2)}(x) \; . \tag{C.7}$$

Furthermore for $G(x)$ holds

$$G(x+\widehat{\mu}) - G(x-\widehat{\mu}) = x_\mu H(x) \tag{C.8}$$

where $H(x)$ is defined as

$$H(x) = \int_{\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ipx} \ln \widehat{p}^2 \tag{C.9}$$

An hint of proof is: on the right side, relate the factor x_μ to a derivative $\partial/\partial p_\mu$, then apply an integration by part, which, acting on the logarithm, produces a propagator, and a translation factor in the numerator. The equation is thus valid only for $x_\mu \neq 0$. Using equation (C.8) in both directions and the Laplace equation, we are able to write $H(x)$ only in terms of $G(x)$ in lattice sites with smaller coordinates

$$H(x) = \frac{2}{\sum_{\mu=0}^{1} x_\mu} \left[2G(x) - \sum_{\mu=0}^{1} G(x-\widehat{\mu}) \right] \tag{C.10}$$

and so, eliminating $H(x)$, we write the Vohwinkel relations in terms only of $G(x)$

$$G(x+\widehat{\mu}) = \frac{2x_\mu}{\sum_{\mu=0}^{1} x_\mu} \left[2G(x) - \sum_{\mu=0}^{1} G(x-\widehat{\mu}) \right] + G(x-\widehat{\mu}) \; . \tag{C.11}$$

This relation, jointly with the symmetry properties of the propagator, $G(x_1, x_2) = G(x_2, x_1) = G(\pm x_1, \pm x_2)$, can be used as a recursion formula that allows to determine $G(x)$ first on a given strip of width 2, along say the x axis, and then, along vertical strips up to the diagonal, on the whole plane.

The initial conditions are

$$G(0,0) = 0 \tag{C.12}$$

$$G(1,0) = -\frac{1}{4} \tag{C.13}$$

$$G(1,1) = -\frac{1}{\pi} \tag{C.14}$$

While $G(1,0)$ is trivially deduced from $G(0,0)$ and the Laplacian equation, $G(1,1)$ can not be related to the two other integrals by means of our relations and is indeed an "independent" lattice constant (this fact is evident also from the fact that a

factor π appears, while our relations involve only rational coefficients). Remark how the function $G(x)$ in an arbitrary point is in the set $\mathbb{Q} + \frac{1}{\pi}\mathbb{Q}$.

Thus, assume we know $G(x)$ in a certain large radius R. What can we do with them? Many lattice integrals arising from calculation of Feynman diagrams can be calculated from them, a certain class exactly, some others in an approximate way, with an error scaling in a controlled way with R. Consider for example integrals of the form

$$K^{(2n)} = \int_p \frac{\widehat{p}_i^{2n}}{(\widehat{p}^2)} \tag{C.15}$$

with n finite (smaller than R), then we have in coordinate space

$$K^{(2n)} = \sum_{m=-n}^{n} \binom{2n}{n+m} (-1)^{n-m} G(m,0) \tag{C.16}$$

Now consider integrals like the following

$$\int_{q,k} \frac{\widehat{k}_1^2 \widehat{q}_1^4}{\widehat{k}^2 \widehat{q}^2 \widehat{k+q}^2} \tag{C.17}$$

which in coordinate space reads as

$$\begin{aligned} &\sum_{x\in\mathbb{Z}^2} G(x)\Big(G(x-\hat{\mu}_1) + G(x+\hat{\mu}_1) - 2G(x)\Big) \\ &\qquad \cdot \Big(G(x-2\hat{\mu}_1) + G(x+2\hat{\mu}_1) - 4G(x-\hat{\mu}_1) - 4G(x+\hat{\mu}_1) + 6G(x)\Big) \end{aligned} \tag{C.18}$$

The sum over the whole lattice is of course unfeasible with this tool, but, as the integral is not singular (as can be deduced by simple power-counting in moment space, first integrating over q, then over k), the leading contribution must be confined to the lattice sites near the origin (which are known within a certain radius R), and, the small contribution deriving from lattice sites far away can be estimated via the asymptotic form of the propagator, when expanded in (the leading logarithmic term plus) powers of inverse radius, $|x|^{-1}$. Thus, say we consider an expansion up to order n of the propagator, then the error scales as R^{-n+n_0}, with n_0 some constant, and can be reduced arbitrarily, both increasing R or n.

More general terms arising from Feynman integrals contains higher powers of the propagator. For example, we could try to calculate lattice integrals of the form

$$K_\ell^{(2n)} = \int_p \frac{\widehat{p}_i^{2n}}{(\widehat{p}^2)^\ell} \tag{C.19}$$

say, with $\ell = 2$. At this aim we should define also the Fourier transform of the (regularized) square-propagator

$$G_2(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{e^{ipx} - 1 + \frac{1}{2}(\widehat{p}_1^2 x_1^2 + \widehat{p}_2^2 x_2^2)}{(\widehat{p}^2)^2} \tag{C.20}$$

or, more generically,

$$G_\ell(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{e^{ipx} - 1 + \frac{1}{2}(\widehat{p}_1^2 x_1^2 + \widehat{p}_2^2 x_2^2) - \cdots}{(\widehat{p}^2)^\ell} \tag{C.21}$$

where the proper subtraction is the one induced by (symmetrized) Taylor expansion of the exponential. The technique above, of Laplacian plus Vohwinkel relations, still holds with minor modifications. For example, the Laplacian relation reads

$$-\Delta G_2(x) = G(x) \tag{C.22}$$

while the Vohwinkel relation is

$$G_2(x + \widehat{\mu}) - G_2(x - \widehat{\mu}) = -x_\mu \Big(G(x) + \frac{1}{4\pi} \Big) \tag{C.23}$$

(remark the presence of the corrective $(4\pi)^{-1}$ factor, arising from regularization). The idea is still of solving w.r.t. $G_2(x)$ in a lattice point immediately out of some region in which the function has already been computed, and then defining a recursive procedure. Still, a set of independent lattice integrals in a proper neighbourhood of the origin must be known. It turns out that, as the "support" of the relations is identical to the one of the previous case, the independent lattice integrals still must be the ones located at the points $x \in \{(0,0),(1,0),(1,1)\}$. The first two vanish because of the subtraction, while the last one is computed analytically, with the result

$$G_2(1,1) = \frac{1}{8\pi} \tag{C.24}$$

and thus, as it is in the set $\mathbb{Q} \cdot G(1,1)$, still the function $G_2(x)$ at a generic point is in the set $\mathbb{Q} + \frac{1}{\pi}\mathbb{Q}$.

C.2 Vohwinkel relations for triangular lattice

Generalization of the procedure to the triangular lattice is not straightforward, and involves some delicate points. Some of them are:

- In the redundant set of variables (p_1, p_2, p_3), the constraint $\sum_\mu p_\mu = 0$ does not allow for derivatives in a single variable: one should either perform linear combinations of derivatives with null sum of coefficients (for example, $(\partial_1 - \partial_3) f(p_1, p_2, p_3)$), or equivalently, perform derivation within a non-redundant choice of variables (for example, $\partial_1 f(p_1, p_2, -p_1 - p_2)$).

- because of this fact, the Vohwinkel relations involve a larger number of terms, and thus it is more difficult to manipulate them in order to have a recursion relation. It will turn out that a larger strip is required for the first x-axis recursion.

- For $G_\ell(x)$ at values of ℓ larger than 1, the choice of subtraction is now not anymore easily deduced by the Taylor expansion of the exponential and the requirement of periodicity. Now we also have the requirement of gauge-invariance under $x \to x + m(1,1,1)$, which forces the application of "hat" factors only to combinations of p_i with null sum of coefficients.

So, since now on we assume the standard "triangular" notations $px = p_1x_1 + \ldots + p_3x_3$ and $\widehat{p}^2 = \widehat{p}_1^2 + \ldots + \widehat{p}_3^2$. We adopt a free propagator normalized as

$$G(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{e^{ipx} - 1}{\widehat{p}^2} \tag{C.25}$$

although, we know that the proper measure is $\frac{d^2p}{2Sqrt3\pi^2}$, such that the Laplacian operator, written as

$$\Delta = \sum_{\mu=1,2,3} \partial_\mu^* \partial_\mu \ , \tag{C.26}$$

does not lead to annoying coefficients in the Laplacian equation

$$-\Delta G(x) = \delta^{(2)}(x) \ . \tag{C.27}$$

and we have

$$G(0,0,0) = 0 \qquad G(1,0,0) = -\frac{1}{6} \tag{C.28}$$

For the triangular function $H(x)$ defined as

$$H(x) = \int_{\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ipx} \ln \widehat{p}^2 \tag{C.29}$$

a set of two independent Vohwinkel relations holds: for $(\mu, \nu) = (1,2)$ or $(1,3)$

$$G(x + \widehat{\mu}) - G(x - \widehat{\mu}) - G(x + \widehat{\nu}) + G(x - \widehat{\nu}) = (x_\mu - x_\nu) H(x) \tag{C.30}$$

Again, using the two equations (C.23) and the Laplace equation, we are able to eliminate $H(x)$, and write a recursion relation. The one we find on a width-2 strip along the x axis is given by the set of equations

$$0 = -6G(x,1,0) + \sum_{\pm;\mu} G((x,1,0) \pm \hat{\mu}) \tag{C.31}$$

$$0 = -6G(x,0,0) + 2(G(x+1,1,0) + G(x,1,0)) + \sum_{\pm} G(x \pm 1,0,0) \tag{C.32}$$

$$\begin{aligned} 0 = {} & (G(x-1,1,0) - G(x+1,1,0)) + x(G(x,2,0) - G(x,0,0)) \\ & + (x-1)(G(x+1,2,0) - G(x-1,0,0)) \end{aligned} \tag{C.33}$$

which must be solved with respect to $G(x+1,a,0)$, with $a = 0,1,2$, in order to have a consistent recursion. A new fundamental integral is required. A choice could be $G(2,1,0)$, which is valued

$$G(2,1,0) = \frac{1}{3} - \frac{\sqrt{3}}{\pi} \tag{C.34}$$

In a similar fashion, given the values of $G(x)$ on the width-2 strip, the function can be determined in the whole plane (a sector with $x = (n, m, 0)$, with $n \geq 2m \geq 0$ is sufficient, because of symmetry). The Laplacian equation alone is enough to fulfill this task. So we conclude that at all values of x the function $G(x)$ is in the set $\mathbb{Q} + \frac{\sqrt{3}}{\pi}\mathbb{Q}$.

The integrals of the form

$$K_1^{(2n)} = \int_p \frac{\widehat{p}_i^{2n}}{(\widehat{p}^2)} \tag{C.35}$$

which involve $G(x)$ only on the real axis, are easily computed, the first values being

$$K_1^4 = \frac{2}{\sqrt{3}}\left(-\frac{4}{3} + \frac{4\sqrt{3}}{\pi}\right) \quad K_1^6 = \frac{2}{\sqrt{3}}\left(16 - \frac{24\sqrt{3}}{\pi}\right) \quad K_1^8 = \frac{2}{\sqrt{3}}\left(-\frac{448}{3} + \frac{288\sqrt{3}}{\pi}\right) \tag{C.36}$$

where we evidentiated the factor deriving purely from the measure.

The next ingredient we need in order to calculate all the triangular-lattice quantity arising from our diagrammatics is the two-propagator function in coordinate space. It turns out that the proper subtraction is the following

$$G_2(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{e^{ipx} - 1 + \frac{1}{4}\Big((\widehat{p}^2 - 2\widehat{p}_3^2)(x_1 - x_2)^2 + \text{cyclics}\Big)}{(\widehat{p}^2)^2} \tag{C.37}$$

The triangular-lattice Laplacian relation still reads

$$-\Delta G_2(x) = G(x) \tag{C.38}$$

while the Vohwinkel relation, still for $(\mu, \nu) = (1, 2)$ or $(1, 3)$, is

$$G_2(x+\widehat{\mu}) - G_2(x-\widehat{\mu}) - G_2(x+\widehat{\nu}) + G_2(x-\widehat{\nu}) = -(x_\mu - x_\nu)\Big(G(x) + \frac{\sqrt{3}}{12\pi}\Big) \tag{C.39}$$

(remark how the corrective factor changed to $\sqrt{3}/(12\pi)$). At the aim of building the recursion, also in this case it turns out that, as the "support" of the relations is identical to the one of the triagular-lattice $G(x)$ case, the independent lattice integrals still must be the ones located at the points $x \in \{(0, 0, 0), (1, 0, 0), (2, 1, 0)\}$. The first two vanish because of the subtraction, while the last one is computed analytically, with the result

$$G_2(2, 1, 0) = \frac{\sqrt{3}}{12\pi} \tag{C.40}$$

and thus, as it is in the set $\mathbb{Q} \cdot G(1, 1)$, still the function $G_2(x)$ at a generic point is in the set $\mathbb{Q} + \frac{\sqrt{3}}{\pi}\mathbb{Q}$.

The two-propagator analogues of the quantities K_1^{2n} are the integrals of the form

$$K_2^{(2n)} = \int_p \frac{\widehat{p}_i^{2n}}{(\widehat{p}^2)^2} \tag{C.41}$$

They still involve $G_2(x)$ only on the real axis, and thus are easily computed, the first values being

$$K_2^4 = \frac{2}{\sqrt{3}}\left(\frac{1}{3} - \frac{\sqrt{3}}{3\pi}\right) \quad K_2^6 = \frac{2}{\sqrt{3}}\left(-4 + \frac{8\sqrt{3}}{\pi}\right) \quad K_2^8 = \frac{2}{\sqrt{3}}\left(\frac{176}{3} - \frac{104\sqrt{3}}{\pi}\right) \tag{C.42}$$

Of special interest (cfr. Rule 4 in chapter 8, and in analogy to what is done with the quantity G_1 of Falcioni-Treves [12]) are the lattice integrals of the form

$$\begin{aligned}\int_{q,k,r} f(q)g(k)\frac{\widehat{r}^2 - \widehat{k}^2 - \widehat{q}^2}{(\widehat{k}^2 + h)(\widehat{q}^2 + h)(\widehat{r}^2 + h)}\delta(q + k + r) \\ = \sum_{x\in\mathbb{Z}^2}\left(\sum_{v_i} c_i\, G(x + v_i)\right)\left(\sum_{v_i'} c_i'\, G(x + v_i')\right)\left(\sum_{v_i''} c_i''\, G_2(x + v_i'')\right)\end{aligned} \tag{C.43}$$

with $f(q)$ and $g(k)$ being some polynomials of trigonometric functions, sufficient to give convergence of the whole integral (but not separately of the three summands in the fraction), and sets of the kind $\{(v_i, c_i)\}$ are the ones describing these function (for example, for a dependence from q of the kind $\widehat{q}_1^2$, we have $\{(v_i, c_i)\} = \{(0, 2), (\hat{\mu}_1, -1), (-\hat{\mu}_1, -1)\}$).

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