Università di Pisa

Facoltà di Scienze Matematiche Fisiche e Naturali

Corso di Laurea Specialistica in Scienze Fisiche

Anno Accademico 2005/2006

Tesi di Laurea Specialistica

A fermionic field theory

for spanning hyperforests

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Acknowledgements

First of all I'd like to thank my advisor for introducing me to the magic world of statistical mechanics. Then I really want to thank my family that gave a lot of opportunities and provided me everything I needed to complete my studies. That said, I can't forget to heartily thank my girlfriend Eleonora for her lovely and tireless presence at my side. Last but not least I want to thank all friend of mine for the formidable years with them[1].

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[1]no, you will not find here the necessary incomplete list with names.

Table of Contents

Chapter 1

Introduction

The Potts model [41, 53] generalizes the Ising model of ferromagnetism assuming that spin variables can vary between q different states. In contrast to $\mathbb{Z}_q$ symmetric models, the Potts model considers only two interaction energies which correspond to nearest neighbour spins being in the same or a in different state, this leads to a symmetry group S_q.

During the seventies, the Potts model has been object of a strong surge of interest for its interesting critical properties and its connection with combinatorial problems. It is now known that Potts model is related to a number of outstanding problems in lattice statistics, combinatorics, and graph theory.

In the late 1960s Fortuin and Kasteleyn [24, 33] extended the Potts model's definition to arbitrary real values of q. This extension, known under the name of *random-cluster model*, has been widely studied in their subsequent series of paper [22–24, 33]. The random-cluster model includes as a special case the percolation model.

The result of Fortuin and Kasteleyn shows a remarkable correspondence between *correlation* properties of Potts model in statistical physics and *connection* properties of random-cluster model in stochastic geometry. The understanding of this relation has allowed a better comprehension of configuration space's structure of Potts/Ising models and the development of efficient Monte Carlo algorithms unaffected by the typical *slow down* at the critical point [49].

A modest review of basic properties of the Potts model and it's relation with random-cluster model and other physical and combinatorial problems is reviewed in the first chapter.

In this work we will put particularly attention to the $q \to 0$ limit case of Potts model. This case has a strong combinatorial importance since the partition function reduces to the generating function of the spanning forests

on the the graph on which the model is defined.

This limit case acquires additional interest due to a recent result of Caracciolo *et al.* [16]. In their work, they prove that the Potts model with $q \to 0$ can be described by a fermionic field theory with a Gaussian term and a peculiar four-fermion coupling. This theory is perturbatively equivalent to the $O(N)$ vector model analytically prolonged to $N = -1$ and it is moreover equivalent to a non-linear σ model with (super-)symmetry group $OSP(1|2)$. A detailed study of this *spanning forests model* is provided in chapter 3.

Although only perturbative, this correspondence tell us that spanning forests model in dimension two is asymptotically free in close analogy to large classes of two-dimensional σ model and four-dimensional non-abelian gauge theory. Indeed, this fermionic model may, because of its great simplicity, be the most viable candidate for a rigorous non-perturbative proof of asymptotic freedom — a goal that has heretofore remained elusive in both σ models and gauge theories.

In chapter 4, an extension of the above fermion model from graphs to hypergraphs is developed. In physics, hypergraphs arise quite naturally whenever one studies a more-then-two body interaction and recently they have been used to describe the statistical mechanics properties of some combinatorial problems [17, 46].

We'll see how the generating function of spanning hyperforests on a hypergraph, arising as the $q \to 0$ limit of a many-body Potts model, can be represented as a Grassmann integral involving many-fermion interactions associated to the hyperedges. Moreover it's shown how this model can be used to count combinatorial objects on hypergraphs.

Once again, this fermionic model possesses $OSP(1|2)$ supersymmetry; indeed, it is the most general $OSP(1|2)$-invariant Hamiltonian in the relevant variables. This extension from graphs to hypergraphs in thus not only natural, but actually sheds light on the underlying super-symmetry.

As firsts applications of this theory a "mean-field" version of the model, i.e. the case in which sites interact with each other by an equal strength, is studied extensively. In this way one can obtain some interesting results on the number of hypertrees or hyperforests one can draw on the complete hypergraph; the resulting formulas match, and sometimes extend, known results.

Chapter 2

The Potts model

2.1 The model

The Potts model [41, 42, 53] is one among the many generalizations of the Ising model [29] to more then two spin components. Historically, a four component version of the model was first studied by Ashkin and Teller [5] in 1943 but the model for general q components bears its current name after it was proposed by Cyril Domb in the 1951 to his then research student Renfrey B. Potts as a thesis topic [41] (see [20] for historical notes). Although the problem attracted little attention in its early years, in the seventies there has been a strong surge of interest, largely because the model has proven to be very rich in its contents. It is now known that the Potts model is related to a number of outstanding problems in lattice statistics, combinatorics, and graph theory. Its critical behavior has also been shown to be richer and more general then that of the Ising model, actually it shows both a first order phase transition for large q and a second order transition for small q.

The problem originally proposed by Domb was to regard Ising model as a system of interacting spins that can be either parallel or anti-parallel. Then an appropriate generalization would be to consider a system of spins confined in a plane, with each spin pointing to one of the q equally spaced directions specified by the angles $\Theta_n = 2\pi n/q$, with n taking values between 0 and $q - 1$. In the most general form the nearest-neighbor interaction J will depend only on the relative angle between the two vectors. The model suggested by Domb was to choose $J(\Theta) = -\epsilon \cos \Theta$ and by the use of a Kramer-Wannier [35] type analysis to find the critical point of the model.

After a detailed investigation Renfrey Potts [42] came to the conclusion that the Kramer-Wannier transformation did not generalize to the planar vector model with q orientations, but instead to a q-state model in which there are only two different interaction energies which correspond to nearest

neighbour spins being in the same state or in different states; as said, the case $q = 4$ for this model had been considered previously by Ashkin and Teller [5]. For the planar model with $q = 4$ it was possible to locate the critical point by an alternative method [1], but this failed for higher values of q. Following the suggestion of Domb [20], we will refer to the q-orientation model as the *planar Potts model* and to the two-energy-level model as the *standard Potts model* (or simply as the *Potts model*). The latter is the one that has attracted the most attention.

After these historical remarks, we now need to introduce some definitions about graphs. Given a set V, that we'll call the *vertex set*, an graph on V is a pair $G = (V, E)$ where E is a subset of the set of all pairs of elements of V. We'll refer to E as the *edge set* of the graph G. Two vertexes $v_1, v_2 \in V$ are *adjacent* if there exists in E an edge $e = (v_1, v_2)$, in this case the edge e is said to be incident to both v_1 and v_2. Please note that, in contrast to some authors, we are excluding both *loops*, i.e. vertexes adjacent to theirself, and *multiedges*, couples of vertexes joined by more than one edge.

Given a graph $G = (V, E)$ one can introduce a *subgraph* $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E'$. Of course one requires that all vertexes referred by elements of E' are contained in V'. A subgraph is called *spanning* if $V' = V$.

A sequence of the type $w = (v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_l)$ where v_i (respectively e_j) are in V (resp. E) is called a *path* from v_1 to v_l if for each i we have that $e_i = (v_i, v_{i+1})$. If v_l coincides with v_1, so that the path is close, we'll call it a *cycle*. A graph without cycles is a *forest*.

The family of all paths on G induces a relation between vertexes. Indeed the existence of a path connecting a given couple of vertexes is an equivalence relation and the equivalence classes of this relation are the *connected components* of G. A graph with only one connected component is said simply *connected*, in this case given any pair of vertexes one can find a path that connects them. A connected forests is said a *tree*.

We will now define the Potts model Hamiltonian and we'll describe some basics results about the (standard) Potts model. Given a graph $G = (V, E)$ and denoting by J the nearest-neighbour interaction strength, the Potts

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[1] The partition function for this model reduces to the square of a standard Ising partition function, see [11].

model Hamiltonian can be written as:

$$\mathcal{H}[\sigma] = - \sum_{\substack{e \in E \\ e=(i,j)}} J_e \delta(\sigma_i, \sigma_j) \tag{2.1}$$

where the sum is over all nearest neighbor pair of sites (i.e. adjacent vertexes of the graph G), δ is the Kronecker delta and σ_i is the spin at site i. Clearly this model possesses an S_q symmetry opposed to $\mathbb{Z}_2$ symmetry of the Ising model. Each nearest neighbor interaction is said *ferromagnetic* or *antiferromagnetic* if the coupling J_e is either positive or negative. In the following we will often assume there is just one value J for all J_e.

The partition function for the model is defined, as usual, as a sum over all assignments $\sigma : V \to \{1, \ldots, q\}$.

$$Z_G(q, \mathbf{J}) = \sum_{\sigma} e^{-\beta \mathcal{H}[\sigma]} = \sum_{\sigma} \prod_{\substack{e \in E \\ e=(i,j)}} e^{\beta J_e \, \delta(\sigma_i, \sigma_j)} \tag{2.2}$$

The Ising model can be obtained back as the $q = 2$ special case.

2.2 Mean-field theory

It is well known that the mean-field description of the Ising model gives a qualitatively correct picture of the phase transition, it's then natural to start studying the Potts model in this approximation.

The mean-field approximation can be obtained replacing the nearest neighbor interaction with an interaction with the field produced by all other sites. The interaction strength needs to be rescaled by a factor N to keep the energy an extensive quantity. The mean-field Hamiltonian is

$$\mathcal{H}[\sigma] = -\frac{\gamma J}{N} \sum_{i<j} \delta(\sigma_i, \sigma_j). \tag{2.3}$$

In this approximation each spin interacts with each other and thus the interactions form a complete graph on N vertexes.

Let us parametrize the macroscopic state by the fraction of spins x_i that are in the spin state $i = 0, 1, \ldots, q-1$, so that x_i has sum one [2]. Then, to

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[2]Please note that this parametrization is actually valid only if q is a positive integer greater than two.
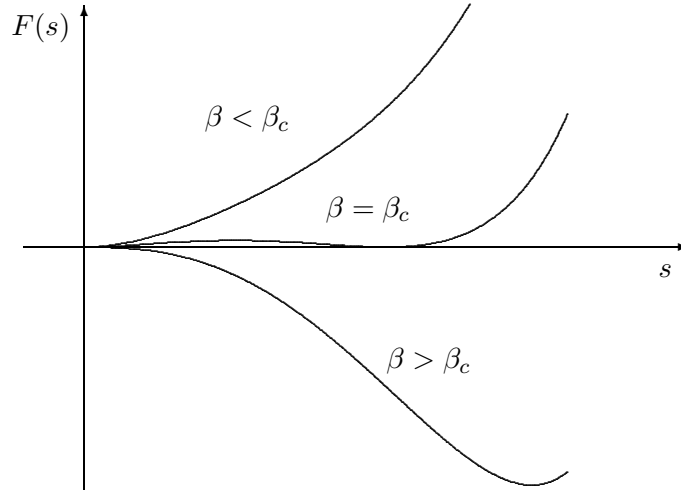
Figure 2.1: Free energy behaviour for a Potts model with $q = 3$ at different temperatures. We can see that, at the critical temperature β_c, another minima of the free energy appears at $s > 0$. This underlie a first order transition with jump discontinuity of the magnetization.

the leading order in N, the energy and entropy per site are:

$$\frac{E}{N} = -\frac{\gamma J}{2} \sum_i x_i^2 \tag{2.4}$$

$$\frac{S}{N} = -\sum_i x_i \ln x_i \tag{2.5}$$

(we'll always put the Boltzmann constant k equal to 1), and the free energy per site, F, is given by the expression

$$\frac{\beta F}{N} = \sum_i \left(x_i \ln x_i - \frac{\beta \gamma J}{2} x_i^2 \right), \tag{2.6}$$

where β is the inverse temperature. For ferromagnetic interaction $(J > 0)$ we look for a solution in the form of

$$x_0 = \frac{1}{q}[1 + (q-1)s] \tag{2.7}$$

$$x_i = \frac{1}{q}(1 - s) \quad \text{where } i = 1, 2, \dots, q - 1 \tag{2.8}$$

where the parameter $0 \le s \le 1$ takes the value $s_0(\beta)$ which minimizes the free energy. If $s_0 > 0$ the system assume a state where the S_q symmetry of

the Hamiltonian is spontaneously broken; s_0 is thus an *order parameter* for the system. Manifestly $s_0 = 0$ is always a extremal of F, but at sufficiently low temperatures other minima with $s_0 > 0$ may emerge. The critical point is then defined to be the temperature $T_c = 1/\beta_c$ at which a solution with lower free energy appears (see figure 2.1).

What actually happens can be readily seen from the expansion of $F(s)$ for small value of the order parameter. From (2.6) and (2.7) we find

$$\frac{\beta[F(s) - F(0)]}{N} = \frac{1 + (q-1)s}{q} \ln\left[1 + (q-1)s\right] \tag{2.9}$$

$$+ \frac{q-1}{q}(1-s)\ln(1-s) - \frac{q-1}{2q}\gamma\beta J s^2 = \tag{2.10}$$

$$= \frac{q-1}{2q}(q - \gamma\beta J)s^2 - \frac{1}{6}(q-1)(q-2)s^3 + \ldots \tag{2.11}$$

The existence of a negative coefficient in the cubic term for $q > 2$ signals the occurrence of a first order transition. For $q = 2$ this leads to the usual mean-field result for the Ising model, namely

$$\beta_c = \frac{2}{\gamma J} \tag{2.12}$$

The transition is continuous since $s_0 = 0$ at β_c.

The situation is different for $q > 2$ because the order parameter jumps from 0 to a value $s_c > 0$ discontinuously at the critical point. In this case the critical parameters s_c and β_c are solved jointly from $F'(s_c) = 0$ and $F(s_c) = F(0)$. One finds

$$\beta_c = \frac{1}{\gamma J}\frac{2(q-1)}{q-2}\ln(q-1) \tag{2.13}$$

$$s_c = \frac{q-2}{q-1} \tag{2.14}$$

Other critical parameters can be obtained easily from the free energy expression (2.6). The mean-field description has been proved to be correct in two dimension to the leading order in the large q expansion by Mittag and Stephen [37]. The mean-field picture also agrees with the exact result in $d = 2$ which shows a first-order transition for $q > 4$ [9]. We then expect, more generally, the existence of a critical value $q_c(d)$ such that, in d dimensions, the mean-field theory is valid for $q > q_c(d)$. Regarding q and d as
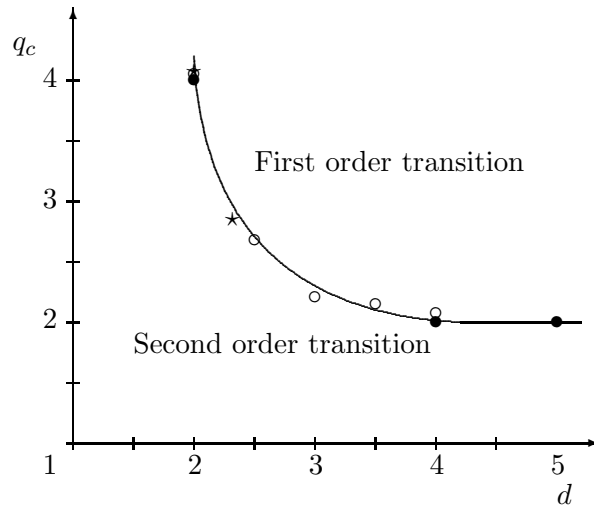
Figure 2.2: Critical dimensionality of q-state Potts model in d dimensions: white circles represent the approximated values proposed in [8], black circles indicate known exact values [3, 9], starred points follows from a renormalization-group analysis [39].

being both continuous the critical value of $q_c(d)$ implies the existence of a critical dimensionality such that the mean-field behavior prevails if $d > d_c(q)$ (see figure 2.2). See [8] and references there in about studies of this curve.

2.3 The Fortuin-Kasteleyn representation

In the late 1960s Fortuin and Kasteleyn [24, 33] realized that the partition function (2.2) of the Potts model, which is defined separately for each integer q greater than two, is just the restriction to such values of a *polynomial* in q. Let us conveniently introduce the coupling $v_e = e^{\beta J_e} - 1$, in this way the partition function (2.2) reduces to the following one:

$$Z_G(q, \mathbf{v}) = \sum_{\sigma} \prod_{\substack{e \in E \\ e=(i,j)}} (1 + v_e\, \delta(\sigma_i, \sigma_j)) \tag{2.15}$$

The ferromagnetic region is mapped into positive v_e semi-axis while the antiferromagnetic one is mapped into the region $v_e \in [-1, 0]$. The zero temperature limit is obtained by letting each v_e go to infinity in the first case and to -1 in the latter; in both cases $v_e = 0$ corresponds to the high temperature limit. Values of v_e less than -1 are outside the physical region since in that case the weights of spin configurations are no longer non-

negative.

Theorem 2.3.1 (Fortuin-Kasteleyn representation) *For each positive integer q, we have that*

$$Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e, \qquad (2.16)$$

where $k(A)$ is the number of connected components (including isolated vertexes) in the subgraph (V, A).

Proof In (2.15) expand the product and denote by A the subset of E for which the second term is taken:

$$\sum_{\sigma} \sum_{A \subseteq E} \prod_{\substack{e \in A \\ e=(i,j)}} v_e \, \delta(\sigma_i, \sigma_j)$$

Next exchange the sums and, for each A, perform the summation over all possible spin assignments σ. Due to the presence of the delta, the only terms that survive in the product are those in which every vertex in each connected component are in the same spin state. Since to each connected component can be assigned one of q states, the sum is (2.16). $\qquad \square$

This is a genuine extension of the Potts model partition function and we will take this as the definition of $Z_G(q, \mathbf{v})$ for arbitrary complex values of q and $\mathbf{v}$.

Let us observe that, using the Euler's relation $|V| + c(E) = |E| + k(E)$, (2.16) can alternatively rewritten as

$$Z_G(q, \mathbf{v}) = q^{|V|} \sum_{A \subseteq E} q^{c(A)} \prod_{e \in A} \frac{v_e}{q}, \qquad (2.17)$$

where $c(A)$ is the cyclomatic number of the subgraph (V, A), i.e. the number of linearly independent circuits in (V, A).

In the following we will often refer to (2.15) as the *spin* or *coloring* representation while we'll refer to (2.16) as the *subgraph representation* (or *expansion*).

It should be stressed that while the coloring representation is defined only for each positive integer q and $v \geq -1$, the subgraph expansion has a

probabilistic interpretation only when $q \geq 0$ and $v \geq 0$. In all other case, the model belongs to the unphysical regime and the ordinary statistical-mechanics properties need not hold. For instance, the free energy need not possess the usual convexity properties and phase transitions can occur even in one-dimensional systems with-short range interactions.

Note that a special case of the subgraph expansion (2.16) was discovered many decades earlier by Birkhoff [12]. Indeed when each v_e is equal to -1, (2.15) gives weight one to each configuration that has not adjacent vertex with different spin state and weight zero otherwise. In graph theory language those configurations are called *proper colorings* and

$$Z_G(q, -1) = P(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{k(A)} \qquad (2.18)$$

is known as *chromatic polynomial*. The chromatic polynomial thus corresponds to the zero-temperature limit ($\beta \to +\infty$) of the anti-ferromagnetic ($J < 0$) Potts model partition function.

2.4   The multivariate Tutte polynomial

In the form (2.16) the Potts model partition function is know to the graph theorists as the *multivariate* Tutte polynomial [48]. It is a polynomial in q and in $\mathbf{v}$, moreover it is *multiaffine* in the variables $\mathbf{v}$ (i.e. of degree 1 in each v_e separately) [3].

This special polynomial can be defined on arbitrary graph and also on more general combinatorial structures like matroids, and it encodes much important combinatorial information about those structures (indeed, in the matroid case it encodes the full structure of the matroid).

Actually the standard definition [52] of the Tutte polynomial differs from the one in (2.19). Conventionally the *standard* Tutte polynomial is defined as the polynomial in two variables x, y

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}, \qquad (2.19)$$

where $r(A) = |V| - k(A)$ is the rank of the subgraph (V, A). This polynomial

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[3]Often a multiaffine polynomial in many variables is easier to handle than a general polynomial in a single variable.

can be obtained as the bivariate specialization of (2.16) where all $\{v_e\}_{e \in E}$ have the same value v. Indeed, comparing (2.19) to (2.16), it is easy to obtain the following correspondence

$$(x - 1)^{k(E)}(y - 1)^{|V|}T_G(x, y) = Z_G\left((x - 1)(y - 1), y - 1\right). \qquad (2.20)$$

In other words, the bivariate polynomials $T_G(x, y)$ and $Z_G(q, v)$ are essentially equivalent under the change of variables

$$x = 1 + q/v \qquad (2.21)$$

$$y = 1 + v \qquad (2.22)$$

$$q = (x - 1)(y - 1) \qquad (2.23)$$

$$v = y - 1 \qquad (2.24)$$

It should be stressed that there is a profound difference between the multivariate and standard definitions: that is, while in the first one can assign to each edges different weights, this is absolutely impossible in the latter.

2.4.1 Elementary identities

To investigate the combinatorial structure of the Tutte polynomial we want here to show some elementary combinatorial identities it satisfies.

Disjoint unions and direct sums

If G is the disjoint union of two graphs G_1 and G_2, then trivially

$$Z_G(q, \mathbf{v}) = Z_{G_1}(q, \mathbf{v})Z_{G_2}(q, \mathbf{v}). \qquad (2.25)$$

That is, Z_G factorizes over the connected components of G.

A slightly less trivial identity arises when G consists of subgraphs G_1 and G_2 joined at a single cut vertex x; in this case we have

$$Z_G(q, \mathbf{v}) = \frac{Z_{G_1}(q, \mathbf{v})Z_{G_2}(q, \mathbf{v})}{q}. \qquad (2.26)$$

This is easily seen from the subgraph expansion (2.17). It is also easily seen directly from the partition function in the color representation (2.15), by first fixing the color σ_x at the cut vertex and then summing over it; this

reflects the S_q symmetry of the Potts model. We can summarize (2.26) by saying that Z_G "factorizes over blocks" modulo a factor q.

Deletion-contraction identity

If $e \in E$, let $G \backslash e$ denote the graph obtained from G by deleting the edge e, and let $G.e$ denote the one obtained from G by contracting the two endpoints of e into a single vertex (please note that we retain in $G.e$ any loops or multiple edges that may be formed as a result of the this operation). Then, for any $e \in E$, we have the identity

$$Z_G(q, \mathbf{v}) = Z_{G \backslash e}(q, \mathbf{v}_{\backslash e}) + v_e\, Z_{G.e}(q, \mathbf{v}_{\backslash e}), \qquad (2.27)$$

where $\mathbf{v}_{\backslash e}$ means $\{v_{e'}\}_{e' \in E \backslash e}$. This last identity is also seen either from (2.15) or (2.16). Please note that this identity takes the same form regardless of whether e is a normal edge, a loop, or a bridge (in contrast to what happens with the conventional Tutte polynomial T_G). Of course, if e is a loop, then $G \backslash e = G.e$, so we can also write $Z_G = (1+v_e) Z_{G \backslash e} = (1+v_e) Z_{G.e}$. Similarly, if e is a bridge, then $G \backslash e$ is the disjoint union of the two subgraphs G_1 and G_2 while $G.e$ is obtained by joining G_1 and G_2 at a cut vertex, so that $Z_{G \backslash e} = Z_{G.e}/q$ and hence $Z_G = (1 + v_e/q)Z_{G \backslash e} = (q + v)Z_{G.e}$.

   This identity is at the base of the combinatorial structure of the Tutte polynomial. Indeed it turn out that every graph invariant satisfying a deletion-contraction-type identity is just a specialization of the Tutte polynomial itself (see [52]).

Parallel-reduction identity

If G contains two edges e_1, e_2 connecting the same pair of vertexes x, y, they can be replaced, without changing the value of Z_G, by a single edge $e = (x, y)$ with weight

$$v_e = (1 + v_{e_1})(1 + v_{e_2}) - 1 = v_{e_1} + v_{e_2} + v_{e_1}v_{e_2}. \qquad (2.28)$$

More formally, we can identify the new edge e with (for instance) the old edge e_1 after deletion of e_2, and thus write

$$Z_G(q, \mathbf{v}_{\backslash v_1, v_2}, v_1, v_2) = Z_{G \backslash e_2}(q, \mathbf{v}_{\backslash v_1, v_2}, v_{e_1} + v_{e_2} + v_{e_1}v_{e_2}). \qquad (2.29)$$

Series-reduction identity

We say that edges e_1, e_2 are *in series* if there exist vertexes x, y, z with $x \neq y$, $y \neq z$ such that e_1 connects x and y, e_2 connects y and z, and y has degree 2 in G. In this case the pair of edges e_1, e_2 can be replaced by a single edge $e = (x, z)$ with weight

$$v_e = \frac{v_{e_1} v_{e_2}}{q + v_{e_1} + v_{e_2}} \tag{2.30}$$

if we multiply Z_G by the factor $q + v_{e_1} + v_{e_2}$. Again, more formally, we can identify the new edge e with (for instance) the old edge e_2 after the contraction of e_1, and thus write

$$Z_G(q, \mathbf{v}_{\backslash e_1, e_2}, v_{e_1}, v_{e_2}) = (q + v_{e_1} + v_{e_2}) \, Z_{G \backslash e_1} \left(q, \mathbf{v}_{\backslash e_1, e_2}, \frac{v_{e_1} v_{e_2}}{q + v_{e_1} + v_{e_2}} \right) \tag{2.31}$$

This identity can be derived from the coloring representation (2.15) by noting that

$$\sum_{\sigma_y = 1}^{q} \left[1 + v_{e_1} \, \delta(\sigma_x, \sigma_y) \right] \left[1 + v_{e_2} \, \delta(\sigma_y, \sigma_z) \right] = \tag{2.32}$$

$$= q + v_{e_1} + v_{e_2} + v_{e_1} v_{e_2} \, \delta(\sigma_x, \sigma_z) = \tag{2.33}$$

$$= (q + v_{e_1} + v_{e_2}) \left[1 + \frac{v_{e_1} v_{e_2}}{q + v_{e_1} + v_{e_2}} \, \delta(\sigma_x, \sigma_y) \right] \tag{2.34}$$

Alternatively, it can be derived from the subgraph expansion (2.16) by considering the four possibilities for the edges e_1 and e_2 to be occupied or empty and analyzing the number of connected components thereby created.

These last two identities (parallel- and series-reduction) are strongly reminiscent of analogous identities in electrical circuit theory: there are elementary formulae for the reduction of linear circuit elements placed in series of parallel. More generally, any 2-terminal subnetwork consisting of linear passive circuit elements is equivalent to some single "effective admittance". The relationship of the multivariate Tutte polynomial to electrical circuit theory goes beyond mere analogy; indeed, as we will see in section 2.7 linear electrical circuits are intimately related to the spanning trees polynomial $T_G(\mathbf{w})$ which arises in the $q \to 0$ limit of the multivariate Tutte polynomial.

Duality

Please note that $v \to q/v$ interchanges the parallel-reduction rule with the series-reduction rule. This is no accident, since this is consequence of the simple duality properties of the Tutte polynomial.

Indeed, suppose first that $G = (V, E)$ is a connected *planar* graph. Consider any plane embedding of G, and let $G^* = (V^*, E^*)$ be the corresponding dual graph. There is a natural bijection between E and E^* (namely, an edge $e \in E$ is identified with the unique edge $e^* \in E^*$ that it crosses), so we shall henceforth identify E^* with E. Of course, the vertex set V^* can be identified with the faces in the given embedding of G, so by Euler's relation we have

$$|V| - |E| + |V^*| = 2. \tag{2.35}$$

Consider now any subset $A \subseteq E$, and draw in G^* the complementary set of edges $(E \backslash A)$. Simple topological arguments then yield the relations

$$k_G(A) = c_{Gj}(E \backslash A) + 1 \tag{2.36}$$

$$k_{G^*}(E \backslash A) = c_G(A) + 1 \tag{2.37}$$

where, as usual, k is the number of connected components of a subgraph and c is its cyclomatic number. Substituting (2.36) into (2.16) we deduce the following *duality relation*

$$Z_{G^*}(q, \mathbf{v}) = q^{1-|V|} \left(\prod_{e \in E} v_e \right) Z_G(q, q/\mathbf{v}), \tag{2.38}$$

where $q/\mathbf{v}$ stands for $\{q/v_e\}_{e \in E}$.

Using the standard bivariate notation for the Tutte polynomial, duality relation is even more simple, taking into account (2.20), (2.36), and (2.38)

$$T_{G^*}(x, y) = T_G(y, x) \tag{2.39}$$

2.4.2   Well known invariants

The relations (2.21) between the variables x, y of the standard Tutte polynomial and the variables q, v of the Potts model partition function show that the latter can be seen as a specialization of the former along the hyperbola $H_q = \{(x-1)(y-1) = q\}$ in the (x, y) plane.
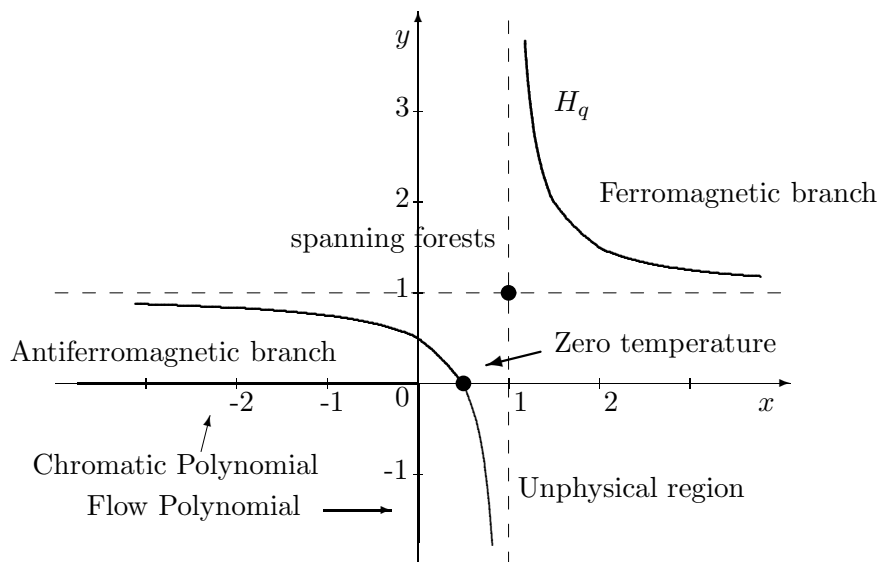
Figure 2.3: Some specialization of the Tutte polynomial in the (x, y) plane.

This is not the only case, indeed it turns out that many graph-theoretical quantities emerge as specialization (i.e. evaluation at some points in the x, y plane) of the standard Tutte polynomial (see figure 2.3). Here we will list some of them while referring to [52] for a more detailed discussion:

- Along H_1, T_G is trivial: $T_G(x, y) = x^{|E|}(x - 1)^{r(E)-|E|}$.

- Along H_q for any real positive q, T_G specializes to the partition function of the Potts model.

- At $(1, 1)$, T_G counts the number of maximal spanning forests (spanning trees if G is connected).

- At $(2, 1)$, T_G counts the number of spanning forests.

- At $(1, 2)$, T_G counts the number of maximally connected spanning subgraphs of G.

- At $(2, 0)$, T_G counts the number of acyclic orientations of G.

- At $(0, 2)$, T_G counts the number of totally cyclic orientations.

- At $(1,0)$, T_G counts the number of acyclic orientations with exactly one source.

- When n is a positive integer, $T_G(1-n,0)$ gives the number of n-colourings, indeed the chromatic polynomial $P_G(n)$ is given by

$$P_G(n) = (-1)^{r(E)}\, n^{k(E)}\, T_G(1-n,0), \qquad (2.40)$$

(we already have seen this result in section 2.3).

- When n is a positive integer, $T_G(1-n,0)$ gives the number of nowhere zero flows over any Abelian group of order n. The flow polynomial $F_G(n)$ is given by

$$F_G(n) = (-1)^{|E|-r(E)} T_G(0, 1-n), \qquad (2.41)$$

- The all terminal reliability $R_G(p)$, defined as the probability that when each edge of the connected graph G is independently deleted with probability $1-p$ the remaining graph stays connected is given by

$$P_G(p) = p^{r(E)}\, (1-p)^{|E|-r(E)}\, T_G\left(1, \frac{1}{1-p}\right) \qquad (2.42)$$

2.5 Complexity considerations

The basic notions of computational complexity are now familiar concepts in most branches of mathematics of physics. One of the main purposes of the theory is to classify and explain the gap that seems to separate tractable computational problems from the apparently intractable ones. Deciding whether or not $\mathcal{P} = \mathcal{NP}$ is probably the most important unsolved problem in theoretical computer science. The computational complexity classes are usually defined for decision problems (like "are this graph 3 colorable?" or "there is an Hamiltonian path on this other graph?") but can be defined also for enumeration problems. Since computational complexity of enumeration problems has received less attention we will briefly review main concepts here.

2.5.1 Basic notions

We regard a computational enumeration problem as a function mapping inputs to solutions, (graphs to the number of their 3-colourings for example). A problem is *polynomial time computable* if there exists an algorithm which computes this function in a length of time (number of steps) bounded by a polynomial in the size of the problem instance. The class of such problems we denote by $\mathcal{P}$. If A and B are two problems we say that A is *polynomial time reducible* to B, written $A \propto B$, if it is possible with the aid of a subroutine for problem B to solve A in polynomial time, in other words the number of steps needed to solve A (apart from calls to the subroutine for B) is polynomially bounded.

The class $\#\mathcal{P}$ can be described informally as the class of enumeration problems in which the structures being counted are recognisable in polynomial time. In other words there is an algorithm which runs in polynomial time and which will verify that a given structure has the form needed to be included in the count. For example counting Hamiltonian paths in a graph is in $\#\mathcal{P}$ because it is easy to check in polynomial time that a given path is Hamiltonian.

Like $\mathcal{NP}$, $\#\mathcal{P}$ has a class of "hardest" problems called the $\#\mathcal{P}$-*complete* problems. They can be described by the following statement: a problem A belonging to $\#\mathcal{P}$ is $\#\mathcal{P}$-complete if for any other problem $B \in \#\mathcal{P}$, we have $B \propto A$. The classic example of a $\#\mathcal{P}$-complete problem is counting truth assignments of a Boolean function. This consists of the following problem

Input: A Boolean formula ϕ in variables $x_1, \ldots, x_n$ and the connectives $\vee$, $\wedge$, $\vdash$.

Output: How many distinct assignments of truth values to the $x_1, \ldots, x_n$ make ϕ true ?

The $\#\mathcal{P}$-complete problems tend to be the enumerative counterparts of $\mathcal{NP}$-complete problems though it has to be emphasized that there is no exact formulation of this remark.

It is clear form this that describing a problem as $\#\mathcal{P}$-complete is a very strong evidence of its inherent intractability. There are now several thousand problems known to be $\#\mathcal{P}$-complete. A polynomial time algorithm for any one of them would imply $\#\mathcal{P} = P$ and this in turn would imply $\mathcal{NP} = \mathcal{P}$. For an extended introduction to this subject we refer to [25].

2.5.2 The complexity of the Tutte plane

We have seen that along the x, y plane, the Tutte polynomial enumerates many different combinatorial structures. While for some of these enumerations (like counting spanning-trees) there are polynomial time algorithms, others (like counting the number of 3 colorings) are know to be in $\#\mathcal{P}$.

A more detailed analysis of the complexity of evaluation as been done by Jaeger, Vertigan, and Welsh [30], which come to the following result:

Theorem 2.5.1 *The problem of evaluating the Tutte polynomial of a graph at a point (a, b) is $\#P - hard$ except when (a, b) is on the special hyperbola*

$$H_1 = \{(x - 1)(y - 1) = 1\} \tag{2.43}$$

or when (a, b) is one of the points $(1, 1)$, $(-1, -1)$, $(0, -1)$, $(-1, 0)$, $(i, -i)$, $(-i, i)$, (j, j^2) and (j^2, j), where $j = e^{2\pi i/3}$. In each of these exceptional cases the evaluation can be done in polynomial time.

As far as the real points are concerned, with one exception, the explanation is straightforward. The hyperbola H_1 is trivial, $(1, 1)$ gives the number of spanning trees, $(-1, 0)$ and $(0, -1)$ give the number of two-colourings, and two-flows respectively and are easy evaluations of the antiferromagnetic Ising model. What is behind the point $(-1, -1)$ is less evident but has been explained in the field of knot theory (see [44]).

Finally the complex points $(i, -i)$, $(-i, i)$ also lie on the Ising curve and the points (j, j^2) and (j^2, j) on the three-state Potts curve. Again there seems to be no natural interpretation to explain why their evaluation is easy. The only reason why they appear in the theorem is that they "turn up in the calculations".

For planar graphs there is a significant difference. The technique developed using the Pfaffian to solve the Ising problem for the plane square lattice by Kasteleyn [32] can be extended to give a polynomial time algorithm for the evaluation of the Tutte polynomial of any planar graph along the special hyperbola

$$H_2 = \{(x - 1)(y - 1) = 2\} \tag{2.44}$$

However H_3 cannot be easy for planar graphs since it contains the point $(-2, 0)$ which counts the number of three-colorings and since deciding if a planar graph is three-colorable is $\mathcal{NP}$-hard, this must be at least $\mathcal{NP}$-hard.

However it is not immediate to show that H_4 is hard for planar graphs. The decision-problem is after all trivial by the four-colour theorem. The fact that is $\#\mathcal{P}$-hard is just part of the following extension of the above theorem due to Vertigan and Welsh [50]

Theorem 2.5.2 *The evaluation of the Tutte polynomial of a bipartite planar graphs at a point (a,b) is $\#\mathcal{P}$-hard except when*

$$(a,b) \in H_1 \cup H_2 \cup \{(1,1),(-1,1),(j,j^2),(j^2,j)\} \qquad (2.45)$$

when it is computable in polynomial time.

2.6 The random-cluster model

We saw that the Potts model, which is a *vertex-model*[4], can be recasted by virtue of the Fortuin-Kasteleyn representation in a *edge-model*. This relation is more fundamental that one can think; indeed it turns out that Potts model is strongly related to a more general form of *bond percolation model*.

The *bond percolation model* was inspired by problems of physical type and emerged from the mathematics literature of the 1950s (see [15, 27]). In this model a porous medium is represented by a graph in which each edge can be open to the passage of a fluid with probability p or closed with probability $1 - p$. Different edges have independent states.

The main goal in percolation theory is to understand which is the probability that two given sites (usually taken at the "boundaries" of the graph) are connected by a path of open edges, so that a fluid can pass from one site to the other one. More generally, the problem is to determine the typical large-scale properties of connected components of open edges, as the parameter p varies. Percolation theory is now a mature part of probability (see [27]), and it is at the core of the study of random media and interacting systems.

The percolation model and the Potts model seemed fairly distinct until Fortuin and Kasteleyn discovered that each features within a certain parametric family of measures which they named *random-cluster model*. In their

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[4]We refer to a *vertex-model* when dynamic variables are attached to vertexes, and to a *edge-model* or *bond-model* when variables are attached to edges.

series of paper [22–24, 33], they developed the basic theory of such models
— correlation inequalities and the like — see [28] for a recent treatment
of the subject. The true power of random-cluster models as a mechanism
for studying the Potts model has emerged progressively over the intervening
three decades.

2.6.1   Definition

Given a finite graph $G = (V, E)$, the configuration space of the random-
cluster model is the set of all subsets of E, which we represent as the set
$\Omega = \{0, 1\}^E$ of 0/1 vectors indexed by elements of E. An edge e in said
open in the configuration $\omega \in \Omega$ is $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. As
already said, the random-cluster model is thus an edge (or bond) model in
contrast to the Potts model which is a vertex model. The problem is to
study the properties of the subgraph of G induced by the set of open edges
of a configuration randomly chosen from Ω according to a certain probability
measure. Of particular relevance is the existence (or not) of paths of open
edges joining given vertexes x and y and thus the random-cluster model is
a model in stochastic geometry.

The random-cluster model consists in a parametric family of probability
measures on Ω; given two parameters q and p satisfying $0 \leq p \leq 1$, $q > 0$, a
random-cluster measure $\phi_{q,p}$ on Ω is given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_G^{RC}} \left[\prod_{e \in E} p^{\omega(e)} (1 - p)^{1 - \omega(e)} \right] q^{k(\omega)}, \quad \omega \in \Omega \tag{2.46}$$

where Z_{RC} is the normalizing constant (or partition function in the statis-
tical physics language) given by

$$Z_G^{RC} \equiv Z_G^{RC}(p, q) = \sum_{\omega \in \Omega} \left[\prod_{e \in E} p^{\omega(e)} (1 - p)^{1 - \omega(e)} \right] q^{k(\omega)} \tag{2.47}$$

The parameter p amounts to a measure of the density of open edges and
the parameter q is a "cluster weighting" factor. When $q = 1$, the measure
$\phi_{p,1}$ is a product measure and it coincides with the percolation model. Note
the difference between the case $q < 1$ and $q > 1$: the former favours fewer clusters, whereas the latter favours many clusters.

Using the Fortuin-Kasteleyn representation we want to show that the

the Gibbs measure $\pi_{q,v}$ of the Potts model on the vertex set V coincides with a random-cluster model on Ω. Indeed if $v > 0$, we can introduce a probability parameter $p = v/(1 + v)$ such that $0 \le p \le 1$ and the partition function (2.16) can be rewritten as

$$Z_G(q,v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|} = \sum_{A \subseteq E} q^{k(A)} \left(\frac{p}{1-p}\right)^{|A|} = \tag{2.48}$$

$$= (1-p)^{-|E|} \sum_{A \subseteq E} p^{|A|} (1-p)^{|E/A|} q^{k(A)} \tag{2.49}$$

Attaching to each edge e of E an "occupation" variable $\omega(e)$ which can assume values 0 or 1 depending if e is in A or not, we find that Potts model partition function is, up to a trivial factor, equal to $Z_{RC}(p,q)$

$$(1+v)^{-|E|} Z_G(q,v) = \sum_{\omega} \left[\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right] q^{k(\omega)} = Z_G^{RC}(p,q). \tag{2.50}$$

The family of random-cluster measures should not be considered merely an extension of the Potts model measure. Indeed we will see that the relation behind the two is mode sophisticated and is such that *correlations* for Potts model correspond to *connections* in random-cluster model. Thus the correlation structure of the first can be studied via the *stochastic geometry* of the corresponding random-cluster model. When extended to infinite graphs, it turns out that long-range order in a Potts model corresponds to the existence of infinite clusters in the corresponding random-cluster model. In this sense the Potts and percolation phase transition are counterparts of one another.

2.6.2 Random-cluster and Potts coupled

Following a more modern approach [21] one can construct the two models directly as a coupled measure on a common probability space.

Let $q \in \{1, 2, \ldots, q\}$, p such that $0 \le p \le 1$ and let G be a finite graph as before. We consider the product space $\Sigma \times \Omega$ where $\Sigma = \{1, 2, \ldots, q\}^V$ and $\Omega = \{0, 1\}^E$ as above. We now define a probability mass function μ on

$\Sigma \times \Omega$ by

$$\mu(\sigma, \omega) \propto \prod_{\substack{e \in E \\ e=(i,j)}} \left[(1-p)\delta_{\omega(e),0} + p\,\delta_{\omega(e),1}\delta(\sigma_i, \sigma_j) \right] . \qquad (2.51)$$

By performing explicitly the summation over either the σ or ω it is easy to verify the following facts

Theorem 2.6.1 (Marginal measure on Σ) *The marginal measure on Σ is the Gibbs measure of the Potts model with $v = p/(1-p)$*

$$\sum_{\omega \in \Omega} \mu(\sigma, \omega) \propto \prod_{\substack{e \in E \\ e=(i,j)}} \left[1 + v\delta(\sigma_i, \sigma_j) \right] \qquad (2.52)$$

Theorem 2.6.2 (Marginal measure on Ω) *The marginal measure on Ω is the random-cluster measure*

$$\sum_{\sigma \in \Sigma} \mu(\sigma, \omega) \propto \left[\prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right] q^{k(\omega)} \qquad (2.53)$$

The following result on conditional measures also holds.

Theorem 2.6.3 (The conditional measures) *The following facts hold.*

- *Given ω, the conditional measure on Σ is obtained by putting (uniformly) random spins on entire clusters of ω (of which there are $k(\omega)$). These spins are constant on given clusters, and are independent between clusters.*

- *Given σ, the conditional measure on Ω is obtained by setting $\omega(e) = 0$ is e connects sites in different spin state and otherwise $\omega(e) = 1$ with probability p (independently of other edges).*

In conclusion, the measure μ is a Potts measure $\pi_{q,v}$ on vertexes coupled with a random-cluster measure $\phi_{q,p}$ on edges. The parameters of these measures are related by $p = v/(1+v) = 1 - e^{-\beta J}$, since $0 \le p \le 1$ this is possible only if $\beta J \ge 0$ thus, as we anticipated, only in the ferromagnetic regime.

2.6.3 Correlation/connection theorem

This special coupling may be used in a particularly simple way to show that correlations in Potts model correspond to open connections in random-

cluster models. When extended to infinite graphs, this implies that the phase transition of a Potts model corresponds to the creation of an infinite open cluster in the random-cluster model. Thus arguments of stochastic geometry, and particularly those developed for the percolation model, may be harnessed directly in order to understand the correlation structure of the Potts system. We show here the fundamental result about this correspondence.

We write $\{x \leftrightarrow y\}$ for the set of all $\omega \in \Omega$ for which there exists an open path joining vertex x to vertex y. The complement of this event in denoted $\{x \nleftrightarrow y\}$.

The "two-point correlation function" of the Potts measure $\pi_{q,v}$ on the finite graph G is defined to be the function $\tau_{q,v}(x,y)$ given by

$$\tau_{q,v}(x,y) = \pi_{q,v}(\sigma_x = \sigma_y) - \frac{1}{q} \quad x,y \in V \tag{2.54}$$

The term q^{-1} is the probability that two independent and uniformly distributed spins are equal. The "two-point connectivity function" of the random-cluster measure $\phi_{q,p}$ is defined as the function $\phi_{q,p}(x \leftrightarrow y)$, that is, the probability that x and y are joined by a path of open edges. It turns out that these "two-point functions" are (except for a constant factor) the same.

Theorem 2.6.4 (Correlation/connection) *If $q \in \{2,3,\dots\}$ and $p = v/(1+v)$ satisfies $0 \le p \le 1$, then for each $x,y \in V$ we have*

$$\tau_{q,v}(x,y) = (1 - q^{-1})\,\phi_{q,p}(x \leftrightarrow y). \tag{2.55}$$

Proof. The indicator function of an event A is denoted 1_A. We have that

$$\tau_{q,v}(x,y) = \sum_{\sigma,\omega} \left[1_{\{\sigma_x = \sigma_y\}}(\sigma) - q^{-1} \right] \mu(\sigma,\omega) =$$

$$= \sum_{\omega} \phi_{q,p}(\omega) \sum_{\sigma} \mu(\sigma \,|\, \mu) \left[1_{\{\sigma_x = \sigma_y\}} - q^{-1} \right] =$$

$$= \sum_{\omega} \phi_{q,p}(\omega) \left[(1 - q^{-1})\,1_{\{x \leftrightarrow y\}}(\omega) + 0 \cdot 1_{\{x \nleftrightarrow y\}}(\omega) \right] =$$

$$= (1 - q^{-1})\,\phi_{q,p}(x \leftrightarrow y)$$

$\square$

This result is actually more general: suppose that we are studying the Potts model and we are interested in some "observable" f defined on Σ. Its mean value satisfies

$$\pi_{q,v}(f) = \sum_{\sigma} f(\sigma)\, \pi_{q,v}(\sigma) = \sum_{\sigma,\omega} f(\sigma)\, \mu(\sigma,\omega) = \tag{2.56}$$

$$= \sum_{\omega} F(\omega)\, \phi_{q,p}(\omega) = \phi_{q,p}(F) \tag{2.57}$$

where F is defined on Ω and it is given by

$$F(\omega) = \mu(f\,|\,\omega) = \sum_{\sigma} f(\sigma)\, \mu(\sigma\,|\,\omega) \tag{2.58}$$

The above theorem is obtained in the case $f(\sigma) = \delta_{\sigma_x,\sigma_y} - q^{-1}$, where $x, y \in V$.

2.6.4   Results on the complete graph

As seen at the in section 2.2 the Potts model can be exactly solved in the mean-field approximation; it is therefore not surprising that the corresponding random-cluster model (for real q) have an "exact solution" too [14]. In particular, when $q = 1$ we recover the usual Erdös–Rényi model for random graph, see [13] for the general theory of such a random graph.

The edge set of the complete graph K_n is $[1,n]^2$, i.e. the set of all $\binom{n}{2}$ pairs of integers between 1 and n. Since all the edges are equivalent, the probability measure $\phi_{p,q}$ is

$$\phi_{p,q}(\omega) = \frac{1}{Z_G^{RC}} (1-p)^{\binom{n}{2}} \left(\frac{p}{1-p} \right)^{|\eta(\omega)|} q^{k(\omega)}, \quad \omega \in \Omega \tag{2.59}$$

The technique for analysing the mean-field Potts model we used in section 2.2 relies upon the assumption that q is an integer. This technique is invalid in its basic form for general real values of q and therefore one needs extra methods in order to understand random-cluster models

Bollobás *et al.* [14] have shown that is possible to study the random-cluster process via corresponding properties of the Erdös–Rényi random graph. We illustrate here the argument in the case $q \geq 1$; a similar approach is valid when $q < 1$. Consider the open clusters $C_1, C_2, \ldots, C_m$ of a sample from the random-cluster measure $\phi_{p,q}$. We colour each such clus-

ter *red* with probability ρ and *white* otherwise, different clusters receiving independent colours. Now delete all vertexes in white clusters and let H denote the remaining graph, comprehensive of a certain random number N of vertexes (from the red clusters) together with certain open edges joining pairs of them. It may be seen that, conditional on the value of N, the measure governing H is the random-cluster measure with parameters p and $q\rho$. Choosing $\rho = 1/q$ one can obtain an Erdös–Rényi random graph on a *random* set of vertexes. This observation permits the full analysis to proceed.

As in the Erdös–Rényi theory, one sets $p = \lambda/n$ where λ is a positive constant and studies the size of the largest component of the ensuing graph in the limit $n \to \infty$. It turns out that there is a critical value of λ depending on the value of q, which define the behaviour of the limiting graph and the appearance of the "giant component". This critical value is given by

$$\lambda_c(q) = \begin{cases} q & \text{if } 0 < q \le 2 \\ 2\left(\frac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2 \end{cases}. \tag{2.60}$$

Another quantity that play a central role in the following is $\theta(\lambda, q)$ defined as

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{max} & \text{if } \lambda \ge \lambda_c(q) \end{cases}, \tag{2.61}$$

where θ_{max} is the largest root of the equation

$$e^{\lambda\theta} = \frac{1 + (q-1)\theta}{1 - \theta} \tag{2.62}$$

From the detailed picture described in [14] the following information may be extracted. The given properties occur with probability tending to 1 as $n \to \infty$

Sub-critical phase: when $\lambda < \lambda_c$, the largest component of the graph is of order $\log n$.

Super-critical case: when $\lambda > \lambda_c$, there is a "giant component" having order $n\theta(\lambda, q)$ where θ is defined as in (2.61).

Small q critical case when $\lambda = \lambda_c(q)$ and $0 < q \le 2$, the largest component has order $n^{2/3}$.
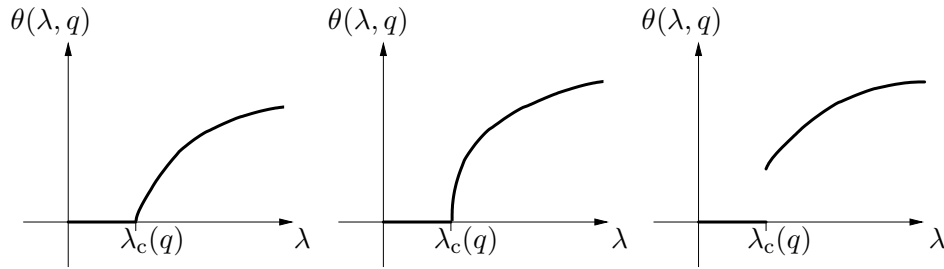
Figure 2.4: The function $\theta(\lambda, q)$ for the three cases $q < 2$, $q = 2$ and $q > 2$

Large q critical case when $\lambda = \lambda_c(q)$ and $q > 2$, the largest component is either of order $\log n$ or of order $n\theta(\lambda, q)$.

The dichotomy between first- and second-order phase transition is seen by studying the function $\theta(\lambda, q)$, sketched in figure 2.4. When $0 < q \leq 2$, the function $\theta(\lambda, q)$ descends continuously to 0 as $\lambda \to \lambda_c(q)$ from above. On the other hand, this limit is strictly positive when $q > 2$ and a discontinuity shows up, in agreement to mean-field description of section 2.2.

Bollobás *et al.* have also proved the existence of the $n \to \infty$ limit of the free energy, obtaining the following result

Theorem 2.6.5 *If $q > 0$ and $\lambda > 0$, then*

$$\frac{1}{n} \log Z_{K_n}(q, \lambda/n) \to f(q, \lambda) \ as \ n \to \infty \tag{2.63}$$

where the free energy $f(q, \lambda)$ is given by

$$f(q, \lambda) = \frac{g(\theta(\lambda))}{2q} - \frac{q-1}{2q} + \log q, \tag{2.64}$$

where

$$g(\theta) = -(q-2)(2-\theta)\log(1-\theta) - \{2 + (q-1)\theta\}\log\left[1 + (q-1)\theta\right] \tag{2.65}$$

and $\theta(\lambda)$ is defined as in (2.61).

2.7   The $q \to 0$ limit

In this section we'll review basic properties of an interesting limit case of Potts model, namely the limit where q and $\mathbf{v}$ go to zero with $\mathbf{w} = \mathbf{v}/q$

fixed. Physically this corresponds to investigate the Potts model's phase diagram in an small neighborhood of the point $(q, v) = (0, 0)$. Moreover this particular limit has an intriguing combinatorial interpretation since the partition function reduces to generating function of *spanning forests*.

This limit case takes on additional interest in light of the recent discoveries [16] that: first, it can be mapped onto a fermionic theory containing a Gaussian term and a special four fermion coupling, and second, this latter theory is equivalent, to all orders in perturbation theory in $1/w$, to the N-vector model at $N = -1$ with $\beta = -w$ and in particular is perturbatively asymptotically free in two dimensions, analogously to two-dimensional σ-models and four-dimensional non-abelian gauge theories.

Let us now consider the different ways in which a meaningful $q \rightarrow 0$ limit can be taken in the Potts model's partition function. In this section we'll assume for simplicity that the graph G on which the model is defined is connected; one can recover the general case by replacing in the following the words *spanning tree* with *maximal spanning forests* and the words *connected spanning subgraph* with *maximally connected subgraph*.

The simplest limit is to take $q \rightarrow 0$ at fixed couplings $\mathbf{v}$. From the Fortuin-Kasteleyn representation

$$Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e \qquad (2.66)$$

we see that this selects out the subgraphs $A \subseteq E$ having the smallest possible number of connected components; the minimum achievable value is of course 1. Thus we have

$$\lim_{q \rightarrow 0} q^{-1} Z_G(q, \mathbf{v}) = C_G(\mathbf{v}), \qquad (2.67)$$

where

$$C_G(\mathbf{v}) = \sum_{\substack{A \subseteq E \\ k(A) = 1}} \prod_{e \in E} v_e \qquad (2.68)$$

is the generating function [5] of "connected spanning subgraphs", which enumerates the connected spanning subgraphs of G according to their edge weights $\mathbf{w}$.

A different limit can be obtained by taking $q \rightarrow 0$ with fixed values of $\mathbf{w} = \mathbf{v}/q$. Using the alternate formulation of Fortuin-Kasteleyn representa-

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[5]All these generating function are just polynomials until G is a finite graph.

tion

$$Z_G(q, \mathbf{v}) = q^{|V|} \sum_{A \subseteq E} q^{c(A)} \prod_{e \in A} \frac{v_e}{q} \tag{2.69}$$

we can see that this time only the subgraphs having the smallest possible cyclomatic number survive. The minimum achievable cyclomatic number is of course 0, so we have

$$\lim_{q \to 0} q^{-|V|} Z_G(q, q\mathbf{w}) = F_G(\mathbf{w}), \tag{2.70}$$

where

$$F_G(\mathbf{w}) = \sum_{\substack{A \subseteq E \\ c(A)=0}} \prod_{e \in A} w_e \tag{2.71}$$

is the generating function of "spanning forests", i.e. spanning subgraphs not containing any circuits. The function $F_G(\mathbf{w})$, like $C_G(\mathbf{w})$, enumerates the spanning forests according to their edge weights, but dividing each w_e by a constant positive factor t and using the Euler relation $|V| = |A| + k(A)$ (forests have no cycles), F_G can be re-expressed as

$$F_G(\frac{\mathbf{w}}{t}) \equiv t^{-|V|} F_G(t; \mathbf{w}). \tag{2.72}$$

In the right hand side of the above expression we now have a generating function that enumerates unrooted spanning forests on G giving weight w_e to each edge *and* giving weight t for each connected component. In the following we'll often consider the generating function of unrooted spanning forests weighted by their connected components, we will denoted this function by $F_G(t)$.

Now suppose that in $C_G(\mathbf{v})$ we replace each v_e by λv_e and than we take the limit $\lambda \to 0$. This chooses, from among the connected spanning subgraphs, those having the fewest edges: these are precisely the spanning trees of G and they all have exactly $|V| - 1$ edges. Hence

$$\lim_{\lambda \to 0} \lambda^{1-|V|} C_G(\lambda \mathbf{v}) = T_G(\mathbf{v}), \tag{2.73}$$

where

$$T_G(\mathbf{v}) = \sum_{\substack{A \subseteq E \\ k(A)=1 \\ c(A)=0}} \prod_{e \in A} v_e \tag{2.74}$$

is the generating function of unrooted spanning trees [6].

Alternatively, suppose that in $F_G(\mathbf{w}, t)$ we then take the limit $t \to 0$ in (2.72) (so giving infinite weight to edges). This time we're selecting, from among the spanning forests, those with minimum number of connected components; these are once again the spanning trees. So we have

$$\lim_{t \to 0} t^{|V|-1} F_G(\mathbf{w}/t) = T_G(\mathbf{w}). \tag{2.75}$$

In summary, we have the following scheme for the $q \to 0$ limits of the Potts model:
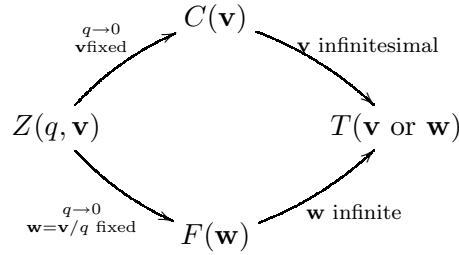
Finally, spanning trees can also be obtained directly from $Z_G(q, \mathbf{v})$ taking $q \to 0$ limit at fixed $\mathbf{x} = \mathbf{v}/q^\alpha$, where $0 < \alpha < 1$ [7]. Indeed simple manipulation of (2.66) and (2.69) yields

$$Z_G(q, q^\alpha \mathbf{x}) = q^{\alpha|V|} \sum_{A \subseteq E} q^{\alpha c(A) + (1-\alpha)k(A)} \prod_{e \in A} x_e \tag{2.76}$$

the quantity $\alpha c(A) + (1 - \alpha)k(A)$ takes its minimum value if and only if both $c(A)$ and $k(A)$ are minimum, thus when A is a spanning tree. Hence

$$\lim_{q \to 0} q^{-\alpha|V|-(1-\alpha)} Z_G(q, q^\alpha \mathbf{x}) = T_G(\mathbf{x}) \tag{2.77}$$

In conclusion, the spanning tree polynomial is interesting not only from a combinatorial point of view but also as a particular limit case of the Potts model partition function. Actually there are a number of interesting physical results about this polynomial. In the following we will review some remarkable properties of the spanning trees polynomial and its relation with other physical systems, specifically electrical circuits and abelian sandpiles

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[6] We use the same notation for spanning trees and standard Tutte polynomial, the context should clears any ambiguities.

[7] Originally Fortuin and Kasteleyn did it with $\alpha = \frac{1}{2}$ but any α satisfying $0 < \alpha < 1$ is equally valid.

model.

2.7.1 Matrix-tree theorem

Let $G = (V, E)$ be a connected graph and $\mathbf{x} = \{x_e\}_{e \in E}$ a collection of edge weights, it's here more convenient to write x_{ij} instead of x_e if $e = (i, j)$. Now define a matrix $\mathbf{L}_G$ indexed by vertexes of G

$$L_G(\mathbf{x})_{ij} = \begin{cases} -x_{ij} & \text{if } i \neq j \\ \sum_{k \neq i} x_{ik} & \text{if } i = j. \end{cases} \tag{2.78}$$

This matrix is named the *Laplacian* of the edge-weighted graph G. If the graph if undirected, so that both edge orientations have the same weight, $\mathbf{L}_G$ is symmetric. Moreover it has zero determinant, by construction, since each row (and column) has sum zero. Now fix a vertex i of G and let $\mathbf{L}_G(\mathbf{x})_{\setminus i}$ be the matrix obtained from $\mathbf{L}_G(\mathbf{x})$ by deleting the i-th row and column. Kirchhoff [34] proved in 1847 the following striking result:

Theorem 2.7.1 (Matrix-tree theorem) *The determinant of* $\mathbf{L}_G(\mathbf{x})_{\setminus i}$ *is independent of i and equals $T_G(\mathbf{x})$ the generating polynomial of spanning trees in G.*

$$\det \mathbf{L}_G(\mathbf{x})_{\setminus i} = T_G(\mathbf{x}). \tag{2.79}$$

Many different proofs of the matrix-tree theorem are now available; one simple proof is based on the Cauchy-Binet theorem in matrix theory (see for example [38]).

More generally, it turns out that *each* minor of $\mathbf{L}_G(\mathbf{x})$ enumerates a suitable class of rooted spanning forests [1]. To formulate this result, let us fix two sets I, J of vertexes and denote by $\mathbf{L}_G(\mathbf{x})_{\setminus I, J}$ the matrix obtained from $\mathbf{L}_G(\mathbf{x})$ by deleting the columns indexed by elements of I and the rows indexed by J; when $I = J$, we simply write $\mathbf{L}_G(\mathbf{x})_{\setminus I}$. Then the "principal-minor matrix-tree theorem" states that

$$\det \mathbf{L}_G(\mathbf{x})_{\setminus \{i_1, \ldots, i_r\}} = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} x_e \tag{2.80}$$

where the sum runs over all spanning forests F in G composed of r disjoint trees, each of which contains exactly one of the "root" vertexes $i_1, \ldots, i_r$. This theorem can easily be derived by applying theorem 2.7.1 to the graph in

which the vertexes $i_1, \ldots, i_r$ are contracted to a single vertex. Furthermore, the "all-minor matrix-tree theorem" (whose proof is a bit more intricate, see [1]) states that for any subsets I, J of the same cardinality, we have

$$\det \mathbf{L}_G(\mathbf{x})_{\backslash I, J} = \sum_{F \in \mathcal{F}(I|J)} \epsilon(F, I, J) \prod_{e \in F} x_e \qquad (2.81)$$

where the sum runs over all spanning forests F in G composed of r disjoint trees, each of which contains exactly one vertex from I and exactly one vertex (possibly the same one) from J; here $\epsilon(F, I, J) = \pm 1$ are signs whose precise definition is not needed here (We'll postpone a complete discussion of this sign until section 4.4).

The virtue of the matrix-tree theorem is that enumerative questions about spanning trees (and rooted spanning forests) can be reduced to linear algebra.

2.7.2 Electric circuits

Now let us consider the graph G as an electrical network: to each edge e we associate a complex number x_e, called its *conductance* (or *admittance*). The *resistance* (or *impedance*) is $1/x_e$. Suppose that we inject currents $\mathbf{J} = \{J_i\}_{i \in V}$ into the vertexes. What node voltages $\boldsymbol{\phi} = \{\phi_i\}_{i \in V}$ will be produced ? Applying Kirchhoff's law of current conservation at each vertex and Ohm's law on each edge, it is not hard to see that the node voltages and current inflows satisfy the linear system

$$\mathbf{L}_G(\mathbf{x})\boldsymbol{\phi} = \mathbf{J} \qquad (2.82)$$

It is then natural to ask: Under what conditions does this system have a (unique) solution ? Two obvious constraints arise from the fact that that row and columns sums of $\mathbf{L}_G(\mathbf{x})$ are zero: firstly, the current vector must satisfy $\sum_{i \in V} J_i = 0$ ("conservation of total current"), or else no solution will exist; and secondly, if $\boldsymbol{\phi}$ is any solution, then so is $\boldsymbol{\phi} + c\mathbf{1}$ for any c ("only voltage *differences* are physically observable"). So let us assume that $\sum_{i \in V} J_i = 0$; and let us break the redundancy in the solution by fixing the voltage to be zero at some chosen reference node $i_0 \in V$ ("ground"). Does the modified system

$$\mathbf{L}_G(\mathbf{x})_{\backslash i_0} \boldsymbol{\phi}_{\backslash i_0} = \mathbf{J}_{\backslash i_0} \qquad (2.83)$$

then have a unique solution ? This will be so if and only if $\det \mathbf{L}_G(\mathbf{x})_{\backslash i_0}$ is nonzero which, by virtue of the matrix-tree theorem, is equivalent to $T_G(\mathbf{x})$ being nonzero.

Simple counterexamples show that this may be not always the case. Suppose, for instance, that G consists of a pair of vertexes connected by two edges e, f in parallel. With $x_e = 1$ and $x_f = -1$ it is easy to see that no solution exists (except when $\mathbf{J} = 0$). Of course this is unphysical because negative resistance are unrealizable. No now consider $x_e = i$ and $x_f = -i$, in this case there is no solution too (unless $\mathbf{J} = 0$), but this is again an unphysical situation since *perfectly* loseless components are also unrealizable and every component in the real world exhibits some dissipation. This reasoning lead us to conjecture, on physical grounds, that if $\operatorname{Re} x_e > 0$ for all e (each branch in strictly dissipative), then the network is uniquely solvable once we fix the voltage at a single reference node $i_0 \in V$. This conjecture turns out to be true, and we have the following result [48].

Theorem 2.7.2 *Let G be a connected graph. Then the spanning trees polynomial T_G has the "half-plane property" that is $\operatorname{Re} x_e > 0$ for all e implies $T_G(\mathbf{x}) \neq 0$*

Proof. Consider any nonzero complex vector $\boldsymbol{\phi} = \{\phi_i\}_{i \in V}$ satisfying $\phi_{i_0} = 0$. Because G is connected, we have $\mathbf{B}^T \boldsymbol{\phi} \neq 0$ where $\mathbf{B}$ is the incident matrix of G. Therefore, the quantity

$$\boldsymbol{\phi}^* \mathbf{L}_G(\mathbf{x}) \boldsymbol{\phi} = \boldsymbol{\phi}^* \mathbf{B} X \mathbf{B}^T \boldsymbol{\phi} = \sum_{e \in E} \left| (\mathbf{B}^T \boldsymbol{\phi})_e \right|^2 x_e \qquad (2.84)$$

has strictly positive real part whenever $|\operatorname{Re} x_e| > 0$ for all e; so in particular $(\mathbf{B} X \mathbf{B}^T \boldsymbol{\phi})_i \neq 0$ for some $i \neq i_0$. It follows that the submatrix of $\mathbf{L}_G(\mathbf{x})$ obtained by suppressing the i_0 row and column in non-singular, and so has nonzero determinant. The thesis now follow from the matrix-tree theorem.

2.7.3 Abelian sandpiles

In the last two decades the concept of self-organized criticality (SOC) has attracted much attention. It has been found useful in the description of such diverse system as earthquakes, forest fires, relaxation phenomena in magnets and coagulation. Bak *et al* in their pioneer papers [6,7] introduced

the concept through the example of sandpiles, which have been extensively studied [6, 7, 18, 19, 36] as a paradigms of self-organized critical systems.

In 1992 Majumdar and Dhar [36] established an equivalence between the steady state of the abelian sandpile model defined on a graph G and the spanning trees problem on a related graph G' obtained by connecting one extra site to G. The number of stable configurations that occur with nonzero probability in the steady state of the sandpile equals the number of spanning trees on G'.

We start by recalling the definition of the abelian sandpile model on a set of N sites. At each site i the height of the sandpile is given by an integer z_i. If $z_i < z_c$ for all i the pile is said to be stable. The time evolution of the sandpile is defined by the following two rules:

- Adding a particle: select one of the sites randomly and add a grain of sand there increasing z_i by 1. Height at other sites remains unchanged.

- Toppling: if for any site $z_i \geq z_{ci}$ then that site is said to be unstable, it topples and loses some sand-grains to other sites. On toppling at site i, the configuration is updated according to the rule:

$$z_j \to z_j - \Delta_{ij}, \quad \text{for all } i \tag{2.85}$$

Where Δ is an integer $N \times N$ matrix satisfying

$$\Delta_{ii} > 0 \quad \Delta_{ij} \leq 0 \quad \sum_j \Delta_{ij} \geq 0 \tag{2.86}$$

These conditions just ensure that, on toppling at site i, z_i must decrease, height at other sites j can only increase, and there is no creation of sand in the toppling process. Some sand may get lost from the system if the toppling occurs at a boundary site. In fact, no stationary state of the sandpile is possible unless particles can leave the system. We will assume, without loss of generality, that $z_{ci} = \Delta_{ii}$.

We can represent the toppling rules by a graph G' containing $N+1$ sites being the added site labelled with index 0 and referred to as the "sink". We give to the edges of G' a weight according the the diagonal terms of matrix Δ . Each site i of G is then connected to the sink with weight Δ_{ii}.

Starting from a stable configuration of the pile, and adding particles at

random, at certain point, we'll obtain an unstable configuration i.e. one that contains an unstable site. Then applying the toppling rule this unstable site will relax increasing heights of adjacent sites. If this will make some other sites unstable the toppling will continue until the configuration is again stable. This eventuality is called an avalanche. The size of avalanche is a random variable and, in many cases of interest, it seems to have a power law tail, which is a signal of existence of long-ranged correlations in the system.

Consider an unstable configuration in which two sites α and β are both critical ($z_\alpha > \Delta_{\alpha\alpha}$, $z_\beta > \Delta_{\beta\beta}$). Then the first toppling leaves the second critical and, after the two topplings, each site gets updated according to $z_i \to z_i + \Delta_{i\alpha} + \Delta_{i\beta}$ which is symmetric under exchange of α and β. This property is at the origin of the abelian property of the sandpile model. If we define an operator a_i "toppling at site i" on a configuration C, we have

$$a_i a_j C = a_j a_i C \quad \forall i,j, \tag{2.87}$$

so that toppling operators form an abelian (semi-)group. A configuration C will be recurrent if there exists positive integers m_i such that

$$a_i^{m_i} C = C \quad \forall i, \tag{2.88}$$

we denote the set of all recurrent configurations by $\mathcal{R}$. It follows that $\mathcal{R}$ is closed under multiplication by operators a_i. The above condition permits to define inverses for toppling operators restricted to domain $\mathcal{R}$: for any recurrent configuration C we define

$$a_i^{-1} C = a_i^{m_i - 1} \quad \forall i. \tag{2.89}$$

Consider now any configuration $C \in \mathcal{R}$ to which we add Δ_{ii} particles one after another at some site i. After these addictions the site i will become unstable and topple, in this process $-\Delta_{ij}$ particles are added at all other sites $j \neq i$. We thus see that operators a_i satisfy the equations

$$a_i^{\Delta_{ii}} = \prod_{j \neq i} a_j^{-\Delta_{ij}}. \tag{2.90}$$

Equivalently, we can write

$$\prod_{j=1}^{N} a_j^{\Delta_{ij}} = 1. \tag{2.91}$$

Since a's commute with each other, all representations of the algebra given by the above equation are one-dimensional, so we can write

$$a_j = e^{i\theta_j} \quad \forall j \tag{2.92}$$

where θ_j are some real numbers. In this representation (2.91) can be written as

$$\sum_{j=1}^{N} \Delta_{ij}\theta_j = 2\pi n_j \quad \forall i, \tag{2.93}$$

where n_i are some integers. This equation can be solved for θ and it shows that their allowed values form a periodic lattice in an N-dimensional space. But, since θ are phases, only points lying with the N-dimensional hypercube $0 \le \theta_i \le 2\pi$ give rise to distinct representations. The number of such representations is therefore the ratio of the volumes of the hypercube and the volume of the unit cell of the θ lattice.

This number is also equals to the number of distinct elements of the algebra i.e. products of the type $a_1^{m_1} a_2^{m_2} a_3^{m_3} \cdots a_N^{m_N}$ which are not equal under (2.91). Hence this number must equal the number of distinct configuration in $\mathcal{R}$. Thus we get

$$|\mathcal{R}| = \det \Delta \tag{2.94}$$

but, Δ is the laplacian matrix of the graph G' with the sink site removed, that is $\Delta \equiv L_{G'}(\mathbf{x})_{\backslash 0}$. By matrix-tree theorem $N_{\mathcal{R}}$ is just the number of spanning trees on G'.

Chapter 3

A fermionic field theory for trees and forests

As we have seen in section 2.7, Kirchhoff matrix-tree theorem (2.79) and its generalizations (2.80) (2.81), which express the generating function of spanning trees and rooted spanning forests in a graph as determinants, play a central role in electrical-circuit theory and in certain models in statistical mechanics. Like all determinants, those arising in Kirchhoff's theorem can of course be rewritten as Gaussian integrals over fermionic (Grassmann) variables.

Recently, Caracciolo *et al.* [16] proved a generalization of the matrix-tree theorem in which a large class of combinatorial objects are represented by a suitable *non-Gaussian* Grassmann integrals. As a special case, they showed that the generating polynomial of unrooted spanning forests, arising in the $q \to 0$ limit of the q-state Potts model, can be represented by a fermionic field theory involving a Gaussian term and a particular four-fermion term. Although this representation has not yet led to any new rigorous results concerning $F_G(\mathbf{w})$, it has led to important non-rigorous insights into its behavior for subgraphs of a regular two-dimensional lattice. These insights may be translatable into theorems by exploiting the rigorous renormalization-group methods developed in recent decades by mathematical physicists (see [47]).

In this chapter we prove some combinatorial identities involving Grassmann integrals and show how a special case yields unrooted spanning forests. Next we show that this latter model can be mapped onto N-vector model at $N = -1$ [16] and we'll make some consideration about its renormalization-group flow at weak coupling in two dimensions.

3.1   Spanning arborescence

Let $G = (V, E)$ a finite undirected graph with edge weights $\mathbf{w} = \{w_e\}_{e \in E}$ and let $\mathbf{L}_G(\mathbf{w})$ be the Laplacian matrix for G . Now introduce, at each vertex $i \in V$ a pair of Grassmann variables ψ_i, $\bar{\psi}_i$. All of these variables are nilpotent of order 2 (i.e. $\psi_i^2 = \bar{\psi}_i^2 = 0$), anticommute, and obey the usual rules for Grassmann integration (see [1, 10, 54]). For any square matrix A, denoting $\mathcal{D}(\psi, \bar{\psi}) \equiv \prod_{i \in V} d\psi_i d\bar{\psi}_i$, we have

$$\int \mathcal{D}(\psi, \bar{\psi}) \ e^{\bar{\psi} A \psi} = \det(A), \tag{3.1}$$

and more generally

$$\int \mathcal{D}(\psi, \bar{\psi}) \ \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_r} \psi_{i_r} e^{\bar{\psi} A \psi} = \epsilon(I, J) \det(A_{\setminus I, J}), \tag{3.2}$$

where the sign $\epsilon(I, J) = \pm 1$ depends on how the vertexes are ordered but is always $+1$ when $(i_1, \ldots, i_r) = (j_1, \ldots, j_r)$. These formulae allow us to rewrite the matrix-three theorems in as a Grassmann integral so that the principal minors matrix-tree theorem (2.80) can be expressed as

$$\int \mathcal{D}(\psi, \bar{\psi}) \ \left(\prod_{\alpha=1}^{r} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right) e^{\bar{\psi} \mathbf{L} \psi} = \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e \tag{3.3}$$

In this way we can use fermionic formalism to describe graph theoretical objects. Let us analyse in detail the fermionic integration to see what sort of objects it creates on a graph.

Expanding the exponential in (3.3) leaving a vertex i fixed we have

$$e^{\sum_j \bar{\psi}_i L_{ij} \psi_j} = 1 + \bar{\psi}_i L_{ii} \psi_i + \sum_{i \neq j} \bar{\psi}_i L_{ij} \psi_j \tag{3.4}$$

$$= 1 + \left(\sum_k w_{ik} \right) \bar{\psi}_i \psi_i - \sum_{i \neq j} \bar{\psi}_i \, w_{ij} \, \psi_j \tag{3.5}$$

We can see that for each edge (ij), its weight w_{ij} appears in the Laplacian matrix in two kind of terms: diagonal (like $w_{ij}\bar{\psi}_i\psi_i$) and off-diagonal (like $\bar{\psi}_i w_{ij} \psi_j$). To represent this fact, we can use a graphical representation:

$$i \; \bullet \xrightarrow{\quad -\bar{\psi}_i w_{ij} \psi_j \quad} \bullet \; j \qquad\qquad i \; \bullet \xleftarrow{\quad -\bar{\psi}_j w_{ij} \psi_i \quad} \bullet \; j$$

$$i \; \bullet \xrightarrow{\quad \bar{\psi}_i \psi_i w_{ij} \quad}\!\!\!\!\rhd \bullet \; j \qquad\qquad i \; \bullet \lhd\!\!\!\!\xrightarrow{\quad w_{ij} \bar{\psi}_j \psi_j \quad} \bullet \; j$$

in this way, each term in the expansion of the exponential give rise to a configuration of arrows on the graph. The Grassmann integral in (3.3) forces those configuration to put exactly one pair of Grassmann field per site; keeping in mind the above representation, this mean that each vertex, except the root, must be touched by exactly one arrow tail. Moreover, if a vertex is visited by a black arrow head, it must be also visited by a black arrow tail. Thus, that black arrows only come in closed self-avoiding circuits, while white arrows make a sort of arborescence, spanning the whole graph. For each connected component, as for each vertex we have exactly one out-going white arrow, there must be one "root structure" such that, each vertex in the component either it is in the root structure or it is connected to the root structure by a unique path. So each component besides the root structure is a tree attached to the root vertex.
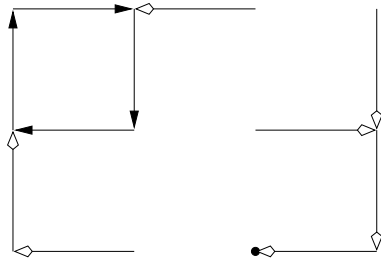
Figure 3.1: Spanning arborescence from a black arrows circuit and root vertex.

The only root structures that can arise from this mechanism are either the root vertex itself, closed oriented cycles of black arrows, or closed oriented cycles of white arrows. Here the Grassmann algebra play a central role. By the anticommutativity of the fermionic fields, given an oriented cycle, when composed with single arrows it comes with a minus sign relative to the same cycle composed by double arrows. This is very similar to what happens in the Fadde'ev-Popov mechanism; consider for example a cycle of

single arrows visiting in sequence the vertexes $(1, 2, \ldots, l)$, then we have

$$(-\bar{\psi}_1 \psi_2 w_{12})(-\bar{\psi}_2 \psi_3 w_{23}) \ldots (-\bar{\psi}_l \psi_1 w_{l1}) = \tag{3.6}$$

$$= (-\bar{\psi}_1 w_{12})(\bar{\psi}_2 \psi_2 w_{23}) \ldots (\bar{\psi}_l \psi_l w_{l1}) \psi_1 = \tag{3.7}$$

$$= -(\bar{\psi}_1 \psi_1 w_{12})(\bar{\psi}_2 \psi_2 w_{23}) \ldots (\bar{\psi}_l \psi_l w_{l1}) \tag{3.8}$$

We have found that, if we remove from the ensemble of configurations these pairs which exactly cancel out, we are left only with configurations containing no cycles. Note that arrows orientation does not give any entropic contribution, as the choice of orientation is uniquely fixed by the requirement that all paths should be directed towards the root. This is an alternative proof of the matrix-tree theorem.

3.2   **The fermionic model**

We have seen how we can enumerate spanning trees by the use of Grassmann integrals. Following [16], we will now enumerate unrooted spanning forests.

Introduce, for each connected (not necessarily spanning) subgraph $\Gamma = (V_\Gamma, E_\Gamma)$ of G, the object

$$Q_\Gamma = \left(\prod_{e \in E_\Gamma} w_e \right) \left(\prod_{i \in V_\Gamma} \bar{\psi}_i \psi_i \right) \tag{3.9}$$

Note that each Q_Γ is even and hence commutes with the entire Grassmann algebra. Let us consider an unordered family $\mathbf{\Gamma} = \{\Gamma_1, \ldots, \Gamma_2\}$ with $l \geq 0$, and try to evaluate an expression of the following form

$$\int \mathcal{D}(\psi, \bar{\psi}) \; Q_{\Gamma_1} \cdots Q_{\Gamma_l} e^{\bar{\psi} \mathbf{L} \psi}. \tag{3.10}$$

If the subgraphs $\Gamma_1, \ldots, \Gamma_l$ have one or vertexes in common, then this integral vanishes on account of the nilpotency of the Grassmann variables. If, by contrast, the $\Gamma_1, \ldots, \Gamma_l$ are vertex-disjoint, then (3.3) expresses (3.10) as a sum over forests rooted at the vertexes of $V_{\mathbf{\Gamma}} = \bigcup_{k=1}^l V_{\Gamma_k}$. In particular, all the edges of $E_{\mathbf{\Gamma}} = \bigcup_{k=1}^l E_{\Gamma_k}$ must be absent from these forests, since otherwise two or more of the root vertexes would lie in the same component (or one of the root vertexes would be connected to itself by a loop edge). On the other hand, by adjoining the edged of $E_{\mathbf{\Gamma}}$, these forests can be put

into one-to-one correspondence with what we shall call $\mathbf{\Gamma}$-*forests*, namely, spanning subgraphs H of G whose edge set contains $E_{\mathbf{\Gamma}}$ and which, after deletion of the edges in $E_{\mathbf{\Gamma}}$, leaves a forest in which each tree component contains exactly one vertex from V_{Γ}. Equivalently, a $\mathbf{\Gamma}$-forest is a subgraph H with l connected components in which each component contains exactly one Γ_i, and which does not contain any cycles other than those lying entirely within the Γ_i. Note, in particular, that a $\mathbf{\Gamma}$-forest is a forest if and only if all the Γ_i are trees. Furthermore, adjoining the edges of $E_{\mathbf{\Gamma}}$ provides precisely the factor $\prod_e w_e$. Therefore

$$\int \mathcal{D}(\psi, \bar{\psi})\, Q_{\Gamma_1} \cdots Q_{\Gamma_l} e^{\bar{\psi}\mathbf{L}\psi} = \sum_{H \in \mathcal{F}_{\mathbf{\Gamma}}} \prod_{e \in H} w_e \qquad (3.11)$$

where the sum runs over all $\mathbf{\Gamma}$-forests H.

We can now combine all the formulae (3.11) into a single generating function, by introducing a variable t_{Γ} for each connected subgraph Γ of G. Since $1 + t_{\Gamma} Q_{\Gamma} = e^{t_{\Gamma} Q_{\Gamma}}$, we have

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\bar{\psi}\mathbf{L}\psi + \sum_{\Gamma} t_{\Gamma} Q_{\Gamma}} = \sum_{\substack{\mathbf{\Gamma} \\ \text{vertex-disjoint}}} \prod_{\Gamma \in \mathbf{\Gamma}} t_{\Gamma} \sum_{H \in \mathcal{F}_{\mathbf{\Gamma}}} \prod_{e \in H} w_e \qquad (3.12)$$

We can express this in another way by interchanging the summation over $\mathbf{\Gamma}$ and H. Consider an arbitrary spanning subgraph H with connected components $H_1, \ldots, H_l$; let us say that Γ *marks* H_i (denoted $\Gamma \prec H_i$) in case H_i contains Γ and contains no cycles other than those lying entirely within Γ. Define the weight

$$W(H_i) = \sum_{\Gamma \prec H_i} t_{\Gamma} \qquad (3.13)$$

Than saying that H is a $\mathbf{\Gamma}$-forest is equivalent to saying that each of its components is marked by exactly one of the Γ_i; summing over the possible families $\mathbf{\Gamma}$, we obtain

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\bar{\psi}\mathbf{L}\psi + \sum_{\Gamma} t_{\Gamma} Q_{\Gamma}} = \sum_{\substack{H \text{ spanning} \\ H = (H_1, \ldots, H_l)}} \left(\prod_{i=1}^{l} W(H_i) \right) \prod_{e \in H} w_e \qquad (3.14)$$

This is a very general combinatorial formula. Extensions allowing prefactors $\bar{\psi}_{i_1}\psi_{j_1} \cdots \bar{\psi}_{i_r}\psi_{j_r}$ are also easily derived.

Now consider the special case in which $t_\Gamma = t$ whenever Γ consists of a single vertex with no edges, $t_\Gamma = u$ whenever Γ consists of two vertexes linked by a single edge, and $t_\Gamma = 0$ otherwise. We have

$$\int \mathcal{D}(\psi, \bar{\psi}) \ e^{\bar{\psi}\mathbf{L}\psi + t \sum_i \bar{\psi}_i \psi_i + u \sum_{(i,j)} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j} = \quad\quad (3.15)$$

$$\sum_{\substack{F \in \mathcal{F} \\ F=(F_1,\ldots,F_l)}} \left[\prod_{i=1}^{l} (t|V_{F_i}| + u|E_{F_i}|) \right] \prod_{e \in F} w_e \quad\quad (3.16)$$

where the sum runs over spanning forests F in G with components $F_1, \ldots, F_l$; here $|V_{F_i}|$ and $|E_{F_i}|$ are, respectively, the number of vertexes and edges in the tree F_i. If $u = 0$, this formula represents vertex-weighted spanning forests as a determinant (a "massive fermionic free field"). More interestingly, since $|V_{F_i}| - |E_{F_i}| = 1$ for each tree F_i, we can take $u = -t$ and obtain the generating function of *unrooted* spanning forests (2.72) with w_e for each edge and weight t for each connected component. Concluding, observing that, by nilpotency

$$\sum_{(i,j)} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j = -\frac{1}{2} \sum_{ij} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j \quad\quad (3.17)$$

we have proven the following result

Theorem 3.2.1 *Let $G = (V, E)$ be a finite undirected graph, let $\mathbf{L}$ be the Laplacian matrix for G with edge weights $\mathbf{w} = \{w_e\}_{e \in E}$, and let F_G be the generating polynomial of spanning forests in G. Then*

$$\int \mathcal{D}(\psi, \bar{\psi}) \ \exp \left[\bar{\psi}\mathbf{L}\psi + t \sum_i \bar{\psi}_i \psi_i + \frac{t}{2} \sum_{i,j} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j \right] = t^{|V|} F_G(\mathbf{w}/t)$$

$$(3.18)$$

3.3   Application to the complete graph case

As a test-case for the fermionic model (3.18), we'll examine the, exactly soluble, case in which G is the complete graph on V vertexes (denoted by K_V). Consider $G = K_V$ and give to each edge weight 1, in this case the

Laplacian matrix as a very simple structure

$$\mathbf{L}_{K_V} = V\mathbf{1} - \mathbf{J} \tag{3.19}$$

Now, let $\bar{\psi}\psi = \sum_{i \in V} \bar{\psi}_i\psi_i$, we want to make a perturbative expansion of the right hand side of (3.18) in powers of t. First we need to compute the following expectation value at $t = 0$

$$\langle (\bar{\psi}\psi)^r \rangle_0 = \int \mathcal{D}(\psi, \bar{\psi}) \ (\bar{\psi}\psi)^r e^{\bar{\psi}\mathbf{L}\psi} \tag{3.20}$$

$$= \left(\frac{\partial}{\partial h}\right)^r \int \mathcal{D}(\psi, \bar{\psi}) \ e^{\bar{\psi}(\mathbf{L}+h\mathbf{1})\psi} \Big|_{h=0} \tag{3.21}$$

$$= \left(\frac{\partial}{\partial h}\right)^r h(V+h)^{V-1} \Big|_{h=0} \tag{3.22}$$

$$= r!\binom{V-1}{r-1}V^{V-r} = r\frac{(V-1)!}{(V-r)!}V^{V-r} \tag{3.23}$$

The above expectation value with $r = 1$ counts rooted spanning trees on G and matches the know result V^{V-1}.

The generating function (3.18), with edge weights fixed to one, enumerates the spanning forests according to number of their connected components, we will refer to this function as $F_G(t) \equiv t^{|V|}F_G(1/t)$. Expanding the interaction part of (3.18) in power of t we obtain

$$F_{K_V}(t) = \sum_{p=1} \frac{t^p}{p!} \left\langle (\bar{\psi}\psi) - \frac{1}{2}(\bar{\psi}\psi)^2 \right\rangle \tag{3.24}$$

$$= \sum_{p=1}^{\infty} \frac{t^p}{p!} \sum_{q=0}^{p} \binom{p}{q}\left(-\frac{1}{2}\right)^q \langle (\bar{\psi}\psi)^{p+q} \rangle \tag{3.25}$$

$$= (V-1)! \, V^{V-p} \sum_{p=1}^{\infty} \frac{t^p}{p!} \sum_{q=0}^{p} \binom{p}{q}\frac{p+q}{(V-p-q)!}(-2V)^{-q} \tag{3.26}$$

The above power expansion starts from $p = 1$ since for $t = 0$ the left hand side of (3.18) is just the laplacian determinant which is zero by definition. The sum in q can be reformulated in terms of the confluent hypergeometric function of the second kind $U(a, b, z)$ (also known as Kummer's function of the second kind, Tricomi function or Gordon function, see [2] chapter 13).

To prove this we first consider the sum

$$\sum_{q=0}^{p} \binom{p}{q} \frac{(-z)^{-q}}{(v+p-q)!} = \sum_{q=0}^{p} \frac{p!}{(p-q)!\,q!\,(v+q)!} (-z)^{q-p} = \qquad (3.27)$$

$$= \frac{p!\,(-z)^{-p}}{(p+v)!} \sum_{q=0}^{p} \frac{(p+v)!}{(p-q)!\,q!\,(v+p-q)!} (-z)^{q} = \qquad (3.28)$$

$$= \frac{p!\,(-z)^{-p}}{(p+v)!} L_p^v(z) = \frac{z^{-p}}{(p+v)!} U(-p, 1+v, z) \qquad (3.29)$$

where $L_n^k(x)$ are the associated Laguerre's polynomials (see [2] chapter 22). Then, returning to (3.26), we have

$$\sum_{q=0}^{p} \binom{p}{q} \frac{p+q}{V-p-q} (-2V)^{-q} = \sum_{q=0}^{p} \binom{p}{q} \frac{V-(V-p-q)}{(V-p-q)!} (-2V)^{-q} =$$
$$\qquad (3.30)$$

$$= \sum_{q=0}^{p} \binom{p}{q} \left[\frac{V}{(V-p-q)!} - \frac{1}{(V-p-q-1)!} \right] (-2V)^{-q} = \qquad (3.31)$$

$$= (2V)^{-p} \left[\frac{V}{(V-p)!} U(-p, V-2p+1, 2V) - \right. \qquad (3.32)$$

$$\left. \frac{1}{(V-p-1)!} U(-p, V-2p, 2V) \right] \qquad (3.33)$$

Finally, taking out the factorials and using the relation

$$U(-p, v, z) = p\,U(1-p, 1+v, z) + U(-p, 1+v, z), \qquad (3.34)$$

we obtain

$$p \frac{(2V)^{-p}}{(V-p)!} \left[U(-p, V-2p+1, 2V) - \right.$$

$$\left. (V-p)\,U(1-p, V-2p+1, 2V) \right]. \qquad (3.35)$$

This completes the perturbative expansion, substituting the sum in (3.26) we have $F_{K_V}(t) = \sum_{p=1}^{\infty} f_p t^p$, where

$$f_p = \frac{V^{V-2p}}{2^p} \binom{V-1}{p-1} \left[U(-p, V-2p+1, 2V) - \right.$$

$$\left. (V-p)\,U(1-p, V-2p+1, 2V) \right] \qquad (3.36)$$

from which we get

$$f_1 = V^{V-2}$$

$$f_2 = \frac{1}{2}V^{V-4}(V+6)(V-1)$$

$$f_3 = \frac{1}{8}V^{V-6}(V^2+14V+60)(V-2)(V-1)$$

$$f_4 = \frac{1}{48}V^{V-8}(V^3+21V^2+202V+840)(V-3)(V-2)(V-1)$$

... and so on

In order to study the asymptotic behavior of $F_{K_V}(t)$ as $V \to \infty$, we need the following expansion for the associated Laguerre polynomials

$$U(-s, V-2p+1, 2V) = s!(-1)^s L_s^{V-2p}(2V)$$
$$\sim V^s \left\{ 1 + \frac{s[1+s+4(p-s)]}{2V} + O(V^{-2}) \right\}, \quad (3.37)$$

then f_p behaves as

$$f_p = \frac{V^{V-2p}}{2^p} \binom{V-1}{p-1} \left[p!(-1)^p L_p^{V-2p}(2V) \right.$$
$$\left. -(V-p)(p-1)!(-1)^{p-1} L_{p-1}^{V-2p}(2V) \right] \qquad (3.38)$$
$$\sim \frac{V^{V-2p}}{2^p} \binom{V-1}{p-1} \left\{ V^p \left[1 + \frac{p(p+1)}{2V} + O(V^{-2}) \right] \right.$$
$$\left. -(V-p)V^{p-1} \left[1 + \frac{(p-1)(p+4)}{2V} + O(V^{-2}) \right] \right\} \qquad (3.39)$$
$$= \frac{V^{V-p-1}}{2^{p-1}} \binom{V-1}{p-1}. \qquad (3.40)$$

Finally the partition function of spanning forests model behaves asymptotically as

$$F_{K_V}(t) = \sum_{p=1}^{\infty} f_p t^p = V^{V-2} \, t \, e^{\frac{t}{2}} \qquad (3.41)$$

This coincides, for $t = 1$, with the known result of [45]. Just as a remark we want to show that there is another way to attack this problem directly at large volume limit. Indeed we can obtain directly (3.41) introducing an

auxiliary (global) bosonic variable in order to linearize the action:

$$F_G(t) = \int d\phi \sqrt{\frac{t}{2\pi}} e^{-\frac{t}{2}\phi^2} \int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} L \psi + t \bar{\psi} \psi + it \bar{\psi} \psi} \tag{3.42}$$

$$= \int d\phi \sqrt{\frac{t}{2\pi}} e^{-\frac{t}{2}\phi^2} det(L + t + it\phi) \tag{3.43}$$

$$= e^{\frac{t}{2}} \int d\phi \sqrt{\frac{t}{2\pi}} e^{-\frac{t}{2}\phi^2 - it\phi} det(L + it\phi) \tag{3.44}$$

$$= e^{\frac{t}{2}} \int d\phi \sqrt{\frac{t}{2\pi}} e^{-\frac{t}{2}\phi^2 - it\phi} (it\phi)(V + it\phi)^{V-1} \tag{3.45}$$

$$= V^{V-1} e^{\frac{t}{2}} \int d\phi \sqrt{\frac{t}{2\pi}} e^{-\frac{t}{2}\phi^2 - it\phi} (it\phi) \left(1 - \frac{it\phi}{V} + \frac{t^2\phi^2}{2V} + \cdots\right) \tag{3.46}$$

$$= V^{V-2} t^2 e^{\frac{t}{2}} \int d\phi \sqrt{\frac{t}{2\pi}} \phi^2 e^{-\frac{t}{2}\phi^2} \left(1 + O(V^{-1})\right) \tag{3.47}$$

$$= V^{V-2} t \, e^{\frac{t}{2}} \left(1 + O(V^{-1})\right) \tag{3.48}$$

3.4   Mapping onto lattice σ models

The spanning forests model, described in the previous section, has an interesting and unexpected correspondence with the non-linear σ model. In particular we show in this section that the low-temperature perturbative expansion of the non-linear σ-model at $N = -1$, on a certain graph G, coincides with the generating function of spanning forests on G with parameter $t = -T$, where T is the temperature of σ-model.

For integer positive N, the non-linear σ-model on a graph G is described by $V = |V(G)|$ spin variables $\boldsymbol{\sigma}_i \in \mathbb{R}^N$ (i.e. real vectors with N components $\boldsymbol{\sigma}_i = \{\sigma_i^1, \ldots, \sigma_i^N\}$), constrained to lie on the unit sphere $\boldsymbol{\sigma}_i^2 = 1$ for all $i \in V$. The Hamiltonian and the partition function of this model are the following

$$\mathcal{H}[\boldsymbol{\sigma}] = -\frac{1}{T} \sum_{(i,j)} w_{ij}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1) \tag{3.49}$$

$$Z = \int \prod_{i \in V} d\boldsymbol{\sigma}_i \, \delta(\boldsymbol{\sigma}_i^2 - 1) \, e^{-\mathcal{H}[\boldsymbol{\sigma}]} \tag{3.50}$$

If there were no constraints on the norm of the spin vectors, the theory would be purely quadratic. The non-linearity introduced by the delta constraints is the crucial ingredient that produced interaction. In order to see this

fact, we can consider a parametrization which solves the constraint: $\boldsymbol{\sigma}_i = \left(\sqrt{1 - T\boldsymbol{\pi}_i^2}, \sqrt{T}\boldsymbol{\pi}_i \right)$, with $\boldsymbol{\pi} \in D_{N-1}(T^{-1/2})$ i.e. the disk of radius $T^{-1/2}$ in $\mathbb{R}^{N-1}$. The Jacobian of this transformation is given by

$$\prod_{i \in V} \frac{1}{\sqrt{1 - T\boldsymbol{\pi}_i^2}} \tag{3.51}$$

so that the new Hamiltonian is

$$\mathcal{H}'[\boldsymbol{\pi}, \epsilon] = \mathcal{H}[\boldsymbol{\sigma}] + \frac{1}{2} \sum_{i \in V} \log\left(1 - T\boldsymbol{\pi}_i^2\right). \tag{3.52}$$

In a perturbation theory around the fully ordered phase $\sigma_i = (1, 0, 0, \dots)$ we can neglect the constraint $\boldsymbol{\pi} \in D_{N-1}(T^{-1/2})$ and expand in powers of $\boldsymbol{\pi}$. We have

$$\mathcal{H}'[\boldsymbol{\pi}] = \frac{1}{2}\sum_{ij} L_{ij}\,\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j - \frac{T}{2}\sum_{i \in V}\boldsymbol{\pi}_i^2 - \frac{T}{4}\sum_{(i,j)} w_{ij}\,\boldsymbol{\pi}_i^2\boldsymbol{\pi}_j^2 + O(\boldsymbol{\pi}^4) \tag{3.53}$$

with partition function

$$Z = \int \prod_{i \in V} d\boldsymbol{\pi}_i\, e^{-\mathcal{H}'[\boldsymbol{\pi}]}. \tag{3.54}$$

When $N = -1$, the bosonic field $\boldsymbol{\pi}$ has -2 components and therefore can be replaced by a fermion pair $\psi, \bar{\psi}$ if we make the substitution $\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \to \psi_i \bar{\psi}_j - \bar{\psi}_i \psi_j$. Higher powers of $\boldsymbol{\pi}^2$ vanish due to the nilpotency of the Grassmann fields, and we obtain back (3.18) if we identify $t = -T$. Note the reverse sign of the coupling: the spanning-forest model with positive weights ($t > 0$) corresponds to the antiferromagnetic N-vector model ($T < 0$).

3.5 $OSP(1|2)$ supersymmetry

In this section we want to investigate an alternative more direct mapping of the fermionic model into a $OSP(1|2)$ symmetric σ-model, recently studied by Read and Saleur [43]. This mapping sheds some light on an hidden supersymmetry of the fermionic model itself.

Let us introduce the superfield $\mathbf{n}_i = (\sigma_i, \psi_i, \bar{\psi}_i) \in \mathbb{R}^{1|2}$ consisting of one bosonic variable σ and a pair of Grassmann fields, and equip the target

manifold with the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i \sigma_j + t \left(\bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i \right). \tag{3.55}$$

where $t \neq 0$ is an arbitrary real parameter. Now impose the superfield $\mathbf{n}_i$ to lie on the unit supersphere in $\mathbb{R}^{1|2}$, i.e. to satisfy the constraint

$$\mathbf{n}_i^2 = \mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i^2 + 2t\,\bar{\psi}_i \psi_i = 1. \tag{3.56}$$

The partition function is defined in as in (3.49):

$$\mathcal{H}[\mathbf{n}] = -\frac{1}{T} \sum_{(i,j)} w_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j - 1) \tag{3.57}$$

preceding section. This supersymmetric model is invariant under the supergroup $OSP(1|2)$; moreover, on the grounds that each fermionic component counts as -1 in the dimension [40], we argue that it is perturbatively equivalent to $O(N)$ vector model, prolonged analytically to $N = -1$.

The infinitesimal rotations in $\mathbb{R}^{1|2}$ are parametrized by two anticommuting infinitesimal parameters ϵ, $\bar{\epsilon}$ and act on the superfield $\mathbf{n}_i$ as $\mathbf{n}_i' = \mathbf{n}_i + \delta\mathbf{n}_i$ with $\delta\mathbf{n}_i = (\delta\sigma, \delta\psi, \delta\bar{\psi})$ where

$$\delta\sigma_i = -\,t(\bar{\epsilon}\psi_i + \bar{\psi}_i\epsilon) \tag{3.58}$$

$$\delta\psi_i = \epsilon\sigma_i \tag{3.59}$$

$$\delta\bar{\psi}_i = \bar{\epsilon}\sigma_i \tag{3.60}$$

In particular, (3.58) leaves the scalar product $\mathbf{n}_i \cdot \mathbf{n}_j$ invariant:

$$\delta(\mathbf{n}_i \cdot \mathbf{n}_j) = \delta\sigma_i\sigma_j + \sigma_i\delta\sigma_j + t\left(\delta\bar{\psi}_i\psi_j + \bar{\psi}_i\delta\psi_j + \delta\bar{\psi}_j\psi_i + \bar{\psi}_j\delta\psi_i\right) = \tag{3.61}$$

$$= -\,t(\bar{\epsilon}\psi_i + \bar{\psi}_i\epsilon)\sigma_j - t(\bar{\epsilon}\psi_j + \bar{\psi}_j\epsilon)\sigma_i + \tag{3.62}$$

$$+\,t\left(\bar{\epsilon}\psi_j\sigma_i + \bar{\psi}_i\epsilon\sigma_j - \epsilon\bar{\psi}_j\sigma_i - \psi_i\bar{\epsilon}\sigma_j\right) = 0 \tag{3.63}$$

The nilpotency of the Grassmann fields permit us to solve this constraint by writing

$$\sigma_i = \pm(1 - 2t\bar{\psi}_i\psi_i)^{1/2} = \pm(1 - t\bar{\psi}_i\psi_i), \tag{3.64}$$

Let us take only the $+$ sign in the above expression, neglecting the other solution. Solving the constraint we can obtain a purely fermionic model

with an hidden supersymmetry

$$\delta\psi_i = \epsilon(1 - t\bar{\psi}_i\psi_i) \tag{3.65}$$

$$\delta\bar{\psi}_i = \bar{\epsilon}(1 - t\bar{\psi}_i\psi_i). \tag{3.66}$$

We can now rewrite the scalar product in terms of the fermionic fields, substituting (3.64) in (3.55) we have

$$\mathbf{n}_i \cdot \mathbf{n}_j = 1 - t(\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) + t^2\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j. \tag{3.67}$$

Taking into account the integration measure

$$\delta(\mathbf{n}_i^2 - 1)\, d\mathbf{n}_i = \delta(\sigma_i^2 + 2t\bar{\psi}_i\psi_i - 1)\, d\sigma_i d\psi_i d\bar{\psi}_i = e^{t\bar{\psi}\psi}\, d\psi_i d\bar{\psi}_i \tag{3.68}$$

and substituting (3.67) in the σ-model Hamiltonian (3.49), we find again the fermionic model (3.18) with the same coupling identification as before ($t = -T$). Indeed

$$Z = \int \prod_{i \in V} d\mathbf{n}_i\, \delta(\mathbf{n}^2 - 1) \exp\left[-\frac{1}{T}\sum_{(i,j)} w_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j - 1)\right] = \tag{3.69}$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \exp\left[-\frac{t}{T}\sum_{(i,j)} w_{ij}(\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) + \right. \tag{3.70}$$

$$\left. + t\sum_i \bar{\psi}_i\psi_i + \frac{t^2}{T}\sum_{(i,j)} w_{ij}\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\right] = \tag{3.71}$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \exp\left[\bar{\psi}\mathbf{L}\psi + t\sum_i \bar{\psi}_i\psi_i + \frac{t}{2}\sum_{ij} \bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j\right] \tag{3.72}$$

which is again the spanning forests partition function (3.18).

3.6 Continuum limit

It worth mentioning that the correspondence between the spanning-forests model and these two σ models, while valid at all orders of perturbation theory, does *not* hold non-perturbatively. The error arises from neglecting the second square root when solving the constraint (3.64); we did not, in

fact, parametrize a (super)-sphere but rather a (super)-hemisphere. Indeed, since $t > 0$ corresponds to an antiferromagnetic σ model, the terms we have neglected are actually dominant. But this doesn't invalid the perturbative correspondence which is still correct and very useful to obtain information on the continuum limit of the fermionic model (3.18).

Indeed, from know result on the N-vector σ model we can deduce the renormalization-group flow for the spanning forests model. Since the two-dimensional σ-model is asymptotically free for $N > 2$ and positive coupling ($T > 0$) but also for $N < 2$ and negative coupling ($T < 0$), so is the spanning forests model (equivalent to $N = -1 < 2$) with *positive* coupling. This is a simple consequence of the reversed sign in the correspondence of the couplings t and $-T$. Assuming that the asymptotic freedom holds also non-perturbatively, we conclude that for $t > 0$ the model is attracted to the infinite-temperature fixed point at $t = \infty$, hence is massive and $OSP(1|2)$ symmetric. For $t_c < t < 0$, by contrast, the model is attracted to the free-fermion fixed point at $t = 0$, and hence is massless with central charge $c = -2$, with the $OSP(1|2)$ symmetry spontaneously broken. Finally, for $t < t_c$ we expect that the model will again be massive, with the $OSP(1|2)$ symmetry restored.

Chapter 4

A fermionic theory for spanning hyperforests

In the previous chapter, we have seen how the generating function of spanning forests, which arise as the $q \to 0$ limit of the Potts model, can be represented by suitable non-Gaussian Grassmann integrals. Furthermore, the resulting fermionic model possesses an $OSP(1|2)$ supersymmetry.

In this chapter we would like to extend this representation from graphs to hypergraphs. In physics, hypergraphs arise quite naturally whenever one studies a more-then-two body interaction and recently they have been used to describe the statistical mechanical properties of some combinatorial problems [17, 46].

We shall show here how the generating function of spanning hyperforests on a hypergraph, which arise as the $q \to 0$ limit of a (generalized) Potts model, can be represented as a Grassmann integral involving special many-fermion interactions associated to the hyperedges. Once again, this fermionic model possesses $OSP(1|2)$ supersymmetry; indeed, it is the most general $OSP(1|2)$-invariant Hamiltonian in the relevant variables. This extension from graphs to hypergraphs is thus not only natural, but actually sheds light on the underlying super-symmetry.

4.1 Generalized Potts model

As a first step, we need to introduce the notion of hypergraph. An hypergraph $\mathcal{G}$ on a vertex set V is a pair $\mathcal{G} = (V, E)$ where E is a subset of all the subsets of V with cardinality greater than one, the elements of E are called *hyperedges*. This means that an *hyperedge* can be incident to more than two vertexes. The definitions of path, cycle, connected component, forest and tree are easy generalization of graph case's ones. When talking about

hypergraphs usually one use the *hyper-* prefix referring to graph theoretical objects such as *hypertrees* and *hyperforests*.

Let q be a positive integer and $\mathcal{G} = (V, E)$ an hypergraph, a natural generalization of the Potts model is the following: at each vertex $i \in V$ we place a spin variable σ_i taking values in $[1, \dots, q]$ and we let them interact by means of hyperedges. Consider the following Hamiltonian

$$\mathcal{H}[\sigma] = -\sum_{A \in E} J_A \delta_A(\sigma) \tag{4.1}$$

where $\mathbf{J} = \{J_A\}_{A \in E}$ are a set of couplings associated to the hyperedges of $\mathcal{G}$, and δ_A is defined for each $A \in E$ by

$$\delta_A(\sigma) = \begin{cases} 1 & \text{if all vertex of } A \text{ are in the same state} \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

The partition function $Z_{\mathcal{G}}$ is then defined as usual

$$Z_{\mathcal{G}} = \sum_{\sigma} e^{-\beta \mathcal{H}[\sigma]} \tag{4.3}$$

As in the case of ordinary Potts model is convenient to introduce the quantities $v_A = e^{\beta J_A} - 1$, we write $\mathbf{v} = \{v_A\}_{A \in E}$ for the collection of these edge weights. We can then derive a Fortuin-Kasteleyn representation for the generalized Potts model exactly in the same way as we did in section 2.3.

Theorem 4.1.1 (Fortuin-Kasteleyn representation) *For each positive integer q,*

$$Z_{\mathcal{G}}(q, \mathbf{v}) = \sum_{E' \subseteq E} q^{k(E')} \prod_{A \in E'} v_A \tag{4.4}$$

where $k(E')$ is the number of connected components (including isolated vertexes) in the sub-hypergraph (V, E').

Proof We start by writing

$$Z_{\mathcal{G}}(q, \mathbf{v}) = \sum_{\sigma} \prod_{A \in E} \left[1 + v_A \delta_A(\sigma)\right], \tag{4.5}$$

now expand the product over $A \in E$ and denote by E' the set of hyperedges for which the term $v_A \delta_A(\sigma)$ is taken.

Next exchange the sums and, for each E', perform the summation over all possible spin assignments σ. The only terms that survive in the product are those in which vertexes in each connected component share the same spin state. Since to each connected component can be assigned one of q states, the sum is (4.4). $\qquad \square$

Again, the right-hand side of (4.4) is a polynomial in q, and, in particular, we can take it as the *definition* of the generalized Potts model partition function $Z_{\mathcal{G}}(q, \mathbf{v})$ for non-integer q.

As in the standard case, the Fortuin-Kasteleyn representation of generalized Potts model permits to define a random-cluster process analogous to the one defined in section 2.6.2. Random-cluster processes with many-body interaction as been studied in [26], the correlation/connection theorem 2.6.4 and the general picture of the phase transition are very similar to what one obtains in ordinary random-cluster model.

Let us now discuss in detail the various types of $q \to 0$ limits that can be taken in the generalized Potts model by a straightforward generalization of the method used in section 2.7. Again we assume that $\mathcal{G}$ is a connected hypergraph.

The simplest limit is to make q go to zero at fixed $\mathbf{v}$. In (4.4) only spanning sub-hypergraphs $E' \subseteq E$ having the smallest possible number of connected components survive; the minimum achievable is of course 1. We therefore have the first kind of limit

$$\lim_{q \to 0} q^{-1} Z_{\mathcal{G}}(q, \mathbf{v}) = C_{\mathcal{G}}(\mathbf{v}), \tag{4.6}$$

where

$$C_{\mathcal{G}}(q, \mathbf{v}) = \sum_{\substack{E' \subseteq E \\ k(E')=1}} \prod_{A \in E'} v_A \tag{4.7}$$

is the generating function of *connected spanning sub-hypergraphs*.

A different limit can be obtained by taking $q \to 0$ with fixed values of $w_A = v_A / q^{|A|-1}$. From (4.4) we have

$$Z_{\mathcal{G}}\left(q, \{q^{|A|-1} w_A\}\right) = \sum_{E' \subseteq E} q^{k(E') + \sum_{A \in E'} (|A|-1)} \prod_{A \in E'} w_A. \tag{4.8}$$

The Euler's relation for hypergraphs reads $|V| + c(E) = \sum_{A \in E}(|A| - 1) + k(E)$, so the above formula can be rewritten as

$$\sum_{E' \subseteq E} q^{k(E') + \sum_{A \in E'}(|A| - 1)} \prod_{A \in E'} w_A = \sum_{E' \subseteq E} q^{|V| + c(E')} \prod_{A \in E'} w_A. \qquad (4.9)$$

The limit $q \to 0$ therefore selects out only spanning sub-hypergraphs with minimum possible of cycles, i.e. hyperforests

$$\lim_{q \to 0} q^{-|V|} Z_{\mathcal{G}}\left(q, \{q^{|A|-1} w_A\}\right) = F_{\mathcal{G}}(\mathbf{w}), \qquad (4.10)$$

where

$$F_{\mathcal{G}}(\mathbf{w}) = \sum_{\substack{E' \subseteq E \\ c(E') = 0}} \prod_{A \in E'} w_A \qquad (4.11)$$

is the generating function of *unrooted spanning hyperforests*.

By a further limit we can obtain spanning hypertrees. To see this replace in $C_{\mathcal{G}}(\mathbf{v})$ v_A by $\lambda^{|A|-1} v_A$ and let $\lambda \to 0$. This selects out sub-hypergraphs having the minimum value of $\sum_{A \in E'}(|A| - 1)$, which are precisely the spanning hypertrees:

$$\lim_{\lambda \to 0} \lambda^{-(|V|-1)} C_{\mathcal{G}}\left(\left\{\lambda^{|A|-1} f_A\right\}\right) = T_{\mathcal{G}}(\mathbf{v}), \qquad (4.12)$$

where

$$T_{\mathcal{G}}(\mathbf{v}) = \sum_{\substack{E' \subseteq E \\ k(E') = 1 \\ c(E') = 0}} \prod_{A \in E'} v_A \qquad (4.13)$$

is the generating function of *unrooted spanning hypertrees*.

Again we will denote by $F_{\mathcal{G}}(t)$ the generating function of spanning hyperforests weighted by the number of their connected component.

Alternatively, in $F_{\mathcal{G}}(\mathbf{w})$, replace w_A by $\lambda^{|A|-1} w_A$ and let $\lambda \to \infty$. Now we are selecting sub-hypergraphs with maximum value of $\sum_{A \in E'}(|A| - 1)$, which, by Euler's relation, means minimum number of connected components, i.e. spanning hypertrees:

$$\lim_{\lambda \to 0} \lambda^{-(|V|-1)} F_{\mathcal{G}}\left(\left\{\lambda^{|A|-1} w_A\right\}\right) = T_{\mathcal{G}}(\mathbf{w}). \qquad (4.14)$$

There is, however one important difference between the graph case and the hypergraph case: every connected graph has a spanning tree, but not every

connected hypergraph has a spanning hypertree. So the limit (4.12) and (4.14) can be zero. Actually deciding if a given hypergraph admit of not a spanning hypertree is an intractable problem [4].

4.2 Forests algebra

Now we want to generalize the mechanism of section 3.1 to hypergraphs. Let V be a vertex set of cardinality n. For each $i \in V$ we introduce a pair of Grassmann variables ψ_i, $\bar{\psi}_i$. Those $2n$ variables act as generators of a Grassmann algebra on V (with coefficient in $\mathbb{C}$ or $\mathbb{R}$), which has cardinality 2^{2^n}. Over this algebra we define the Berezin integration measure

$$\mathcal{D}(\psi, \bar{\psi}) = \prod_{i=1}^{V} d\psi_i d\bar{\psi}_i. \tag{4.15}$$

For each subset $A \subseteq V$, we associate the monomial $\tau_A = \prod_{i \in A} \bar{\psi}_i \psi_i$, where $\tau_{\emptyset} = 1$. Please note that all these monomials are even elements of the Grassmann algebra, and, in particular, they commute with the whole algebra. Clearly, the elements $\{\tau_A\}_{A \subseteq V}$ span a vector space of dimension 2^n. In fact, this vector space is a subalgebra, by virtue of the obvious relations

$$\tau_A \tau_B = \begin{cases} \tau_{A \cup B} & \text{if } A \cap B = \emptyset \\ 0 & \text{if } A \cap B \neq \emptyset \end{cases} \tag{4.16}$$

Let us now introduce another family of elements of the Grassmann algebra, also indexed by subsets of V, which possesses very interesting properties. For each subset $A \subseteq V$ and each number λ (in $\mathbb{R}$ or $\mathbb{C}$), we define the Grassmann element [1]

$$f_A^{(\lambda)} = \lambda(1 - |A|)\tau_A + \sum_{i \in A} \tau_{A \setminus i} - \sum_{\substack{i,j \in A \\ i \neq j}} \bar{\psi}_i \psi_j \tau_{A \setminus \{i,j\}}. \tag{4.17}$$

Note that $f_A^{(\lambda)}$ are also even and they commute with the whole Grassmann algebra. For instance, we have

---

[1]This curious formula will assume a precise meaning in the following.

$$f_\emptyset^{(\lambda)} \;=\; \lambda \tag{4.18}$$

$$f_{\{i\}}^{(\lambda)} \;=\; 1 \tag{4.19}$$

$$f_{\{i,j\}}^{(\lambda)} \;=\; -\lambda\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j + \bar{\psi}_i\psi_i + \bar{\psi}_j\psi_j - \bar{\psi}_i\psi_j - \bar{\psi}_j\psi_i \tag{4.20}$$

$$\;=\; -\lambda\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \tag{4.21}$$

We are now interested in the subalgebra generated by the elements $f_A^{(\lambda)}$ as A ranges over all nonempty subsets of V, for an arbitrary fixed value of λ. The first fundamental property of this algebra relies in the following theorem:

Theorem 4.2.1 *Let $A, B \subseteq V$ with $|A| > 1$, $|B| > 1$ and $A \cap B \neq \emptyset$. Then*

$$f_A^{(\lambda)} f_B^{(\lambda)} = \begin{cases} f_{A\cup B}^{(\lambda)} & \text{if } |A \cap B| = 1 \\ 0 & \text{if } |A \cap B| \geq 2 \end{cases} \tag{4.22}$$

Proof Let $C = A \cup B$ and $D = A \cap B$. Assume that D has cardinality one, say $D = \{i\}$, then, factorizing Grassmann variables at i in $f_A^{(\lambda)}$, we can write

$$f_A^{(\lambda)} = a_1\bar{\psi}_i\psi_i + \bar{\psi}_i a + \bar{a}\psi_i + a_0$$

where the four coefficient $a_1, a, \bar{a}, a_0$ correspond to the expansion with respect to the dependence on $\bar{\psi}_i$ and ψ_i and, explicitly, are

$$a_1 = \lambda(1 - |A|)\tau_{A\backslash i} + \sum_{j\in A\backslash i} \tau_{A\backslash\{i,j\}} - \sum_{\substack{j_1,j_2\in A\backslash i \\ j_1\neq j_2}} \bar{\psi}_{j_1}\psi_{j_2}\tau_{A\backslash\{i,j_1,j_2\}}$$

$$a = -\sum_{j\in A\backslash i} \psi_j\tau_{A\backslash\{i,j\}}$$

$$\bar{a} = -\sum_{j\in A\backslash i} \psi_j\tau_{A\backslash\{i,j\}}$$

$$a_0 = \tau_{A\backslash i}$$

A similar expansion can be performed on $f_B^{(\lambda)}$, call the corresponding coefficient $b_1, b, \bar{b}, b_0$. In the product $f_A^{(\lambda)} f_B^{(\lambda)}$ some terms vanish due

to the nilpotency of the algebra's elements, and we are left with

$$
\begin{aligned}
f_A^{(\lambda)} f_B^{(\lambda)} &= (a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i + a_0)(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i + b_0) \\
&= a_0 b_0 + a_0(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i) \\
&\quad + b_0(a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i) \\
&\quad + (a\bar b + b\bar a)\bar\psi_i
\end{aligned}
$$

substituting the expression of the quantities $a_1, a, \bar a, a_0,\ b_1, b, \bar b, b_0$, we find

$$
a_0 b_0 = \tau_{C\setminus i}
$$

$$
a_0(b_1\bar\psi_i\psi_i + \bar\psi_i b + \bar b\psi_i) = \Big[\lambda(1 - |B|)\tau_C + \sum_{j\in B\setminus i}\tau_{C\setminus j}
$$

$$
- \sum_{\substack{j_1, j_2\in B\setminus i \\ j_1\neq j_2}}\bar\psi_{j_1}\psi_{j_2}\tau_{C\setminus\{j_1, j2\}}\Big]
$$

$$
- \sum_{j\in B\setminus i}\bar\psi_i\psi_j\tau_{C\setminus\{i,j\}} - \sum_{j\in B\setminus i}\bar\psi_j\psi_i\tau_{C\setminus\{i,j\}}
$$

$$
b_0(a_1\bar\psi_i\psi_i + \bar\psi_i a + \bar a\psi_i) = \text{same as preceding with } A\leftrightarrow B
$$

$$
a\bar b\bar\psi_i\psi_i = -\sum_{j_1\in A\setminus i}\sum_{j_2\in B\setminus i}\bar\psi_{j_2}\psi_{j_1}\tau_{C\setminus\{j_1, j_2\}}
$$

$$
b\bar a\bar\psi_i\psi_i = \text{same as preceding with } A\leftrightarrow B
$$

Reordering the summands, we have

$$
\lambda(1 - |B|)\tau_C + \lambda(1 - |A|)\tau_C = \lambda(1 - |C|)\tau_C
$$

$$
\tau_{C\setminus i} + \sum_{j\in A\setminus i}\tau_{C\setminus j} + \sum_{j\in B\setminus i}\tau_{C\setminus j} = \sum_{j\in C}\tau_{C\setminus j}
$$

and

$$\left[\sum_{j \in A \setminus i} (\bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i) \tau_{C \setminus \{i,j\}} + \sum_{\substack{j_1, j_2 \in A \setminus i \\ j_1 \neq j_2}} \bar{\psi}_{j_1} \psi_{j_2} \tau_{C \setminus \{j_1, j_2\}} \right]$$

$$- [A \leftrightarrow B] - \sum_{j_1 \in B \setminus i} \sum_{j_2 \in A \setminus i} (\bar{\psi}_{j_1} \psi_{j_2} + \bar{\psi}_{j_2} \psi_{j_1}) \tau_{C \setminus \{j_1, j_2\}}$$

$$= - \sum_{\substack{j_1, j_2 \in C \\ j_1 \neq j_2}} \bar{\psi}_{j_1} \psi_{j_2} \tau_{C \setminus \{j_1, j_2\}}$$

Collecting the right-hand sides, we obtain $f_C^{(\lambda)}$, and thus prove the first part of the theorem.

To prove the second part, we now assume that $|D| \geq 2$. Note first that, inside $f_A^{(\lambda)}$ and $f_A^{(\lambda)}$, the monomials of lowest degree with respect to the variables in D have degree $2|D| - 2$, so the lowest possible degree in the product if twice that number, $4|D| - 4$. But the highest-degree monomial in the Grassmann algebra over D has degree $2|D|$; so if $|D| > 2$, then each element in the expansion of the product must be zero, and in this case we are done. In case that $|D| = 2$ (say $D = \{i_1, i_2\}$), then expanding the product $f_A^{(\lambda)} f_B^{(\lambda)}$, only pairs of terms in which both factors have degree 2 over D can contribute:

$$f_A^{(\lambda)} f_B^{(\lambda)} = \left[\tau_{A \setminus i_1} + \tau_{A \setminus i_2} - (\bar{\psi}_{i_1} \psi_{i_2} + \bar{\psi}_{i_2} \psi_{i_1}) \tau_{A \setminus \{i_1, i_2\}} + \text{higher degree} \right]$$

$$\times \left[\tau_{B \setminus i_1} + \tau_{B \setminus i_2} - (\bar{\psi}_{i_1} \psi_{i_2} + \bar{\psi}_{i_2} \psi_{i_1}) \tau_{B \setminus \{i_1, i_2\}} + \text{higher degree} \right]$$

$$= \tau_{A \setminus i_1} \tau_{B \setminus i_2} + \tau_{A \setminus i_2} \tau_{B \setminus i_1} + \bar{\psi}_{i_1} \psi_{i_2} \tau_{A \setminus \{i_1, i_2\}} \bar{\psi}_{i_2} \psi_{i_1} \tau_{B \setminus \{i_1, i_2\}}$$

$$+ \bar{\psi}_{i_2} \psi_{i_1} \tau_{A \setminus \{i_1, i_2\}} \bar{\psi}_{i_1} \psi_{i_2} \tau_{B \setminus \{i_1, i_2\}}$$

$$= \tau_C (1 + 1 - 1 - 1)$$

$$= 0$$

This completes the proof.   □

As a first consequence of theorem 4.2.1 is that if $A \subseteq V$ with $|A| > 1$ then $f_A^{(\lambda)}$ is nilpotent of order 2, i.e. $\left(f_A^{(\lambda)} \right)^2 = 0$. By an iterating use of the above theorem, we obtain the following result

Lemma 4.2.2 *Let $\mathcal{G} = (V, E)$ be a connected hypergraph. Then*

$$\prod_{A \in E} f_A^{(\lambda)} = \begin{cases} f_V^{(\lambda)} & \text{if } \mathcal{G} \text{ is a hypertree} \\ 0 & \text{if } \mathcal{G} \text{ is not an hypertree} \end{cases} \tag{4.23}$$

Proof Let $(A_1, \ldots, A_m)$ a construction sequence for $\mathcal{G}$. If $\mathcal{G}$ is a hypertree then necessary

$$\left| \left(\bigcup_{i=1}^{l} A_i \right) \cup A_l \right| = 1 \quad \text{for all } 2 \leq l \leq m. \tag{4.24}$$

Then in the product expansion we are always in first case of theorem 4.2.1, and the result is $f_V^{(\lambda)}$.

If $\mathcal{G}$ is not an hypertree, given a construction sequence as above, there will be $\tilde{l}$ such that

$$\left| \left(\bigcup_{i=1}^{\tilde{l}} A_i \right) \cup A_l \right| \geq 2. \tag{4.25}$$

Let $A' = \left(\bigcup_{i=1}^{\tilde{l}} A_i \right)$, than, by theorem 4.2.1, $f_{A'}^{(\lambda)} f_A^{(\lambda)} = 0$. This ends the proof. $\qquad\square$

The following lemma specifies the most general product of factors $f_A^{(\lambda)}$. Note that there is no need to consider sets of cardinality 1, since $f_{\{i\}}^{(\lambda)} = 1$.

Lemma 4.2.3 *Let $\mathcal{G} = (V, E)$ an hypergraph. If $\mathcal{G}$ is an hyperforest, and $\{C_\alpha\}$ is the partition of V induced by the decomposition of $\mathcal{G}$ into its connected components, then*

$$\prod_{A \in E} f_A^{(\lambda)} = \prod_{\alpha} f_{C_\alpha}^{(\lambda)}. \tag{4.26}$$

Otherwise, if $\mathcal{G}$ is not an hyperforest, that product is zero.

Proof Apply Lemma 4.2.2 to each connected components of $\mathcal{G}$ (remember that $f_A^{(\lambda)}$ commute!). This gives immediately the lemma. $\qquad\square$

4.3 Spanning hyperforests model

On the ground of the forests algebra's properties discussed above, in this section we will construct a Grassmann integral representation for generating

functions of spanning hyperforests.

Given a hypergraph $\mathcal{G} = (V, E)$ and a collection of weights $\mathbf{w} = \{w_A\}_{A \in E}$, let us define the following Hamiltonian

$$\mathcal{H} = -\sum_{A \in E} w_A f_A^{(\lambda)}. \tag{4.27}$$

Then we note that

$$\exp\left(\sum_{A \in E} w_A f_A^{(\lambda)}\right) = \prod_{A \in E} \left(1 + w_A f_A^{(\lambda)}\right) = \tag{4.28}$$

$$= \sum_{E' \subseteq E} \left(\prod_{A \in E'} w_A\right)\left(\prod_{A \in E'} f_A^{(\lambda)}\right) \tag{4.29}$$

$$= \sum_{\substack{F \in \mathcal{F} \\ F = (F_1, \ldots, F_l)}} \left(\prod_{A \in F} w_A\right) \prod_{\alpha=1}^{l} f_{V(F_\alpha)}^{(\lambda)} \tag{4.30}$$

where the sum runs over spanning hyperforests F in $\mathcal{G}$ with components $F_1, \ldots, F_l$ and $V(F_\alpha)$ is the vertex set of the hypertree F_α. Let us now introduce the Grassmann integration measure

$$\mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) = \mathcal{D}(\psi, \bar{\psi}) \prod_{i \in V} e^{t_i \bar{\psi}_i \psi_i}, \tag{4.31}$$

Then the partition function for our model is

$$Z_{\mathbf{t},\lambda} = \int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \exp\left(\sum_{A \in E} w_A f_A^{(\lambda)}\right) = \tag{4.32}$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \exp\left(\sum_{i \in V} t_i \bar{\psi}_i \psi_i + \sum_{A \in E} w_A f_A^{(\lambda)}\right). \tag{4.33}$$

Using expansion (4.28) and observing that

$$\int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) f_A^{(\lambda)} = \lambda + \sum_{i \in A}(t_i - \lambda), \tag{4.34}$$

we can easily perform the integration in (4.32) which gives

$$Z_{\mathbf{t},\lambda} = \int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \exp\left(\sum_{A \in E} w_A f_A^{(\lambda)}\right) =$$

$$= \sum_{\substack{F \in \mathcal{F} \\ F=(F_1,\ldots,F_l)}} \left(\prod_{A \in F} w_A \right) \prod_{\alpha=1}^{l} \left(\lambda + \sum_{i \in V(F_\alpha)} (t_i - \lambda) \right). \quad (4.35)$$

Now if we specialize (4.35) to $t_i = \lambda$ for all vertexes i, we obtain

$$Z_\lambda = \int \mathcal{D}_\lambda(\psi, \bar{\psi}) \, \exp \left(\sum_{A \in E} w_A f_A^{(\lambda)} \right) = \sum_{F \in \mathcal{F}} \left(\prod_{A \in F} w_A \right) \lambda^{k(F)}, \quad (4.36)$$

which is the generating function of *unrooted* spanning hyperforests with a weight w_A for each hyperedge A and a weight λ for each connected component. On the other hand, if we specialize (4.35) to $\lambda = 0$, we obtain

$$\int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \, \exp \left(\sum_{A \in E} w_A f_A^{(0)} \right) = \sum_{F \in \mathcal{F}} \left(\prod_{A \in F} w_A \right) \prod_{\alpha=1}^{l} \left(\sum_{i \in V(F_\alpha)} t_i \right),$$
$$(4.37)$$

that is the generating function of *rooted* spanning hyperforests with weight w_A for each hyperedge A and weight t_i for each root i. Since an hyperforest with only one component is just an hypertree, the generating function of spanning hypertrees is the linear term of the expansion in powers of t of the above expression.

It is easy to see that these equations are generalizations of results of section 3.2. If $\mathcal{G}$ is an ordinary graph and each edge E is an pair (i, j), than (see (4.20))

$$\sum_{\{i,j\} \in E} w_{ij} f_{\{i,j\}}^{(\lambda)} = \sum_{i,j \in V} \bar{\psi}_i L_{ij} \psi_j + \frac{\lambda}{2} \sum_{\{i,j\} \in V} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j \quad (4.38)$$

where L is the laplacian matrix defined as in (2.78). Taking into account the integration measure (4.31), the resulting expression is exactly (3.18).

4.4 Correlation functions

In the previous section we saw how the partition function of a particular class of fermionic theories can be given a combinatorial interpretation as an expansion over spanning hyperforests in a hypergraph. In this section we will extend this result to give a combinatorial interpretation for the (unnormalized) correlation functions of the same fermionic theory.

Given ordered k-tuples of vertexes $I = (i_1, i_2, \ldots, i_k) \in V^k$ and $J = (j_1, j_2, \ldots, j_k) \in V$, let us define the operator

$$\mathcal{O}_{I,J} = \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_k} \psi_{j_k}. \tag{4.39}$$

Of course, the $i_1, i_2, \ldots, i_k$ must be all distinct, as must the $j_1, j_2, \ldots, j_k$ or else we will have $\mathcal{O}_{I,J} = 0$. But there can be overlaps between the sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$. Our goal in this section is to compute the (un-normalized) expectation value

$$\langle O_{I,J} \rangle_{\mathbf{t}, \lambda} = \int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \, \mathcal{O}_{I,J} \, \exp \left[\sum_{A \in E} w_A f_A^{(\lambda)} \right]. \tag{4.40}$$

The first step resides in the following lemma

Lemma 4.4.1 *Let $A \subseteq V$, an let $I = (i_1, \ldots, i_k) \in A^k$ and $J = (j_1, \ldots, j_k) \in A^k$. Then*

$$\int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \, \mathcal{O}_{I,J} \, f_A^{(\lambda)} = \begin{cases} \lambda + \sum_{i \in A}(t_i - \lambda) & \text{if } k = 0 \\ 1 & \text{if } k = 1 \, , \\ 0 & \text{if } k = 0 \end{cases} \tag{4.41}$$

where we understand that the integration measure is restricted to the set A.

Proof. The case $k = 0$ is trivial and is just (4.34). To handle $k = 1$, recall first the definition of $f_A^{(\lambda)}$

$$f_A^{(\lambda)} = \lambda(1 - |A|)\tau_A + \sum_{l \in A} \tau_{A \setminus l} - \sum_{\substack{l, m \in A \\ l \neq m}} \bar{\psi}_l \psi_m \tau_{A \setminus \{l, m\}}. \tag{4.42}$$

Now multiply $f_A^{(\lambda)}$ by $\bar{\psi}_i \psi_j$ with $i, j \in A$, and integrate. If $i = j$, then the only non-zero contribution comes from the term $l = i$ in the single sum, and $\bar{\psi}_i \psi_i \tau_{A \setminus i} = \tau_A$, so the integral is just one. If $i \neq j$, then the only non-zero contribution comes from the term $l = j$, $m = i$ in the double sum, and $(\bar{\psi}_i \psi_j)(-\bar{\psi}_j \psi_i)\tau_{A \setminus \{i,j\}} = \tau_A$, so the integral is one also in this case.

Finally, is $|I| = |J| = k \geq 2$, then every monomial in $\mathcal{O}_{I,J} f_A^{(\lambda)}$ has degree greater or equal then $2|A| - 2 + 2k > 2|A|$, so $\mathcal{O}_{I,J} f_A^{(\lambda)} = 0$.

$\square$

Let now $I, J \in V^k$, let $P = \{C_\alpha\}_{\alpha=1}^m$ be a partition of V, and consider the integral

$$\mathcal{I}(P, I, J) = \int \mathcal{D}_{\mathbf{t}}(\bar{\psi}, \psi) \, \mathcal{O}_{I,J} \prod_{\alpha=1}^m f_{C_\alpha}^{(\lambda)}. \qquad (4.43)$$

The integral above factorizes on the sets C_α of the partition, and vanishes unless $|I \cap C_\alpha| = |J \cap C_\alpha|$ for all α. Let us decompose the operator in the product

$$\mathcal{O}_{I,J} = \sigma(P, I, J) \prod_{\alpha=1}^m \mathcal{O}_{I'_\alpha, J'_\alpha} \qquad (4.44)$$

where $I'_\alpha = I \cap C_\alpha$, $J'_\alpha = J \cap C_\alpha$, and $\sigma(P, I, J)$ is a sign coming from the reordering of the fields inside the product. Applying lemma 4.4.1 to each factor C_α, we obtain

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \mathcal{O}_{I'_\alpha, J'_\alpha} \, f_{C_\alpha}^{(\lambda)} = \begin{cases} \lambda + \sum_{i \in C_\alpha}(t_i - \lambda) & \text{if } |I'_\alpha| = |J'_\alpha| = 0 \\ 1 & \text{if } |I'_\alpha| = |J'_\alpha| = 1 \quad (4.45) \\ 0 & \text{otherwise} \end{cases}$$

this means that the only partitions $P = \{C_\alpha\}_{\alpha=1}^m$ which give a non-vanishing contribution are the ones in which $k \le m$ and each set C_α contains either one element from I and one element from J (possibly the same element) or no element from I or J. For these partitions let us reorder the subsets, in such a way that the first k subsets have an element of I and J, and the last $m-k$ do not contain any element of I or J. Let π be the pairing of elements of I into elements of J, clearly π is a permutation and the sign $\sigma(P, I, J)$ is just its signature $\epsilon(\pi)$.

Then the quantity (4.43), when $P = \{C_1, \ldots, C_k, C_{k+1}, \ldots, C_m\}$ is such that $i_\alpha, j_{\pi(\alpha)} \in C_\alpha$ for $\alpha = 1, \ldots, k$ is given by

$$\mathcal{I}(P, I, J) = \epsilon(\pi) \prod_{a=k+1}^m \left(\lambda + \sum_{i \in C_\alpha}(t_i - \lambda) \right) \qquad (4.46)$$

and $\mathcal{I}(P, I, J) = 0$ otherwise.

In particular if $\lambda = t_i = t$ we have $\mathcal{I}(P, I, J) = \epsilon(\pi)t^{m-k}$ for properly matched partitions, and if $t = 0$ then $\mathcal{I}(P, I, J) = \epsilon(\pi)\delta_{m,k}$. Clearly, when $I \cap J \ne \emptyset$, the allowed pairings π have to match all the identical elements,

as a given vertex can not be simultaneously on two disjoint sets.

Finally, the Grassmann integrals of the form

$$\langle O_{I,J} \rangle_{\mathbf{t},\lambda} = \int \mathcal{D}_{\mathbf{t}}(\psi, \bar{\psi}) \, O_{I,J} \, \exp \left[\sum_{A \in E} w_A f_A^{(\lambda)} \right],$$

which are expectation values in our fermionic theory have a combinatorial interpretation in terms of an expansion over spanning hyperforests with particular matching pattern. Indeed we have

$$\langle O_{I,J} \rangle_{\mathbf{t},\lambda} = \sum_{\substack{F \, in \, \mathcal{F}_{\mathcal{G}} \\ F = (F_1, \ldots, F_l)}} \prod_{\alpha=1}^{l} \mathcal{I}\left(\{V(F_\alpha)\}; I, J\right) \prod_{A \in E(F_\alpha)} w_A. \tag{4.47}$$

The generating function of spanning hypertrees can be again obtained in the special case where both I and J singletons

$$\langle O_{I,J} \rangle_{\mathbf{t}=\lambda=0} = \left\langle \bar{\psi}_i \psi_j \right\rangle_0 = T_{\mathcal{G}}(\mathbf{w}). \tag{4.48}$$

4.5   Complete uniform hypergraph

As a simple test-case for our spanning hyperforests model we will consider in this section its application to uniform hypergraphs.

An hypergraph is said k-*uniform* of just *uniform* if all its edges have the same cardinality k, the *complete* uniform hypergraph of V vertexes is the one that have all the $\binom{V}{k}$ possible $k-$edges. In the case $\mathcal{G}$ is $k-$uniform and complete, the condition for existence of spanning hypertree on $\mathcal{G}$ reduces to the following:

$$V - 1 | k - 1 \tag{4.49}$$

so in this section we will understand that $V = n(k-1)$ where $n \in \mathbb{N}$.

Before going forward, we first need a lemma

Lemma 4.5.1 *Let $\bar{\psi}\psi = \sum_{i=1}^{V} \bar{\psi}_i \psi_i$, f and g two analytic functions, and $\mathbf{J}$ the $V \times V$ matrix composed of all ones. Then, we have*

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r e^{f(\bar{\psi}\psi) + (\bar{\psi}\mathbf{J}\psi)g(\bar{\psi}\psi)} =$$

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r e^{f(\bar{\psi}\psi)} \left[1 + \bar{\psi}\psi g(\bar{\psi}\psi) \right] \tag{4.50}$$

Proof

$$\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r e^{f(\bar\psi\psi)+(\bar\psi\mathbf{J}\psi)g(\bar\psi\psi)} = \tag{4.51}$$

$$= \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r e^{f(\bar\psi\psi)} \sum_n \frac{(\bar\psi\mathbf{J}\psi)^n}{n!} \left(g(\bar\psi\psi)\right)^n = \tag{4.52}$$

$$= \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r e^{f(\bar\psi\psi)} \left[1 + (\bar\psi\mathbf{J}\psi)g(\bar\psi\psi)\right] = \tag{4.53}$$

$$= \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r e^{f(\bar\psi\psi)} \left[1 + (\bar\psi\psi)g(\bar\psi\psi)\right] \tag{4.54}$$

Where we have noted that, the expansion in $\left(\bar\psi\mathbf{J}\psi\right)^n$ is proportional to the one of the determinant of $\mathbf{J}_{n\times n}$. Since $\mathbf{J}$ has rank one, that determinant is zero if $n \geq 2$. $\qquad\square$

Now consider the complete $k-$uniform hypergraph $\mathcal{K}_{V,k}$ and assign to each hyperedge a constant weight w. Recalling the definition of $f_A^{(\lambda)}$ given in (4.17), in this case the Hamiltonian (4.27) can be written

$$\sum_{|A|=k} f_A^{(\lambda)} = (1-k)\lambda\frac{(\bar\psi\psi)^k}{k!} + (V-k+1)\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - \left[\bar\psi(\mathbf{J}-\mathbf{1})\psi\right]\frac{(\bar\psi\psi)^{k-1}}{(k-1)!}$$

$$= \lambda(1-k)\frac{(\bar\psi\psi)^k}{k!} + V\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!} \tag{4.55}$$

At this point we have everything we need to compute the partition functions (4.36) and (4.37). Let's start with the hypertrees generating function (4.37), the linear term in t expansion of (4.37) can be directly obtained by fixing one root vertex in the Grassmann integral; to derive a more general formula we will consider an integral with r of these root vertexes, and later we'll put $r = 1$.

$$\left\langle (\bar\psi\psi)^r \right\rangle_0 \equiv \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r \exp\left[w\sum_{|A|=k} f_A^{(0)}\right] = \tag{4.56}$$

$$= \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^r \exp\left[w\frac{V}{(k-1)!}(\bar\psi\psi)^{k-1} - w(\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right] \tag{4.57}$$

Since the root vertexes cannot appear any more at his right in the integral, this integral is now in the form of lemma 4.5.1 with $V - r$ vertexes, thus we

have

$$\int \mathcal{D}(\psi, \bar{\psi})\,(\bar{\psi}\psi)^r \exp\left[w\frac{V}{(k-1)!}(\bar{\psi}\psi)^{k-1}\right]\left[1 - w\frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] = \quad (4.58)$$

$$= \int \mathcal{D}(\psi, \bar{\psi})\,(\bar{\psi}\psi)^r \sum_{l=0}^{\infty}\left[w\frac{V}{(k-1)!}\right]^l \frac{(\bar{\psi}\psi)^{l(k-1)}}{l!}\left[1 - w\frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!}\right] \quad (4.59)$$

Using the following fact

$$\int \mathcal{D}(\psi, \bar{\psi})\,(\bar{\psi}\psi)^s = V!\,\delta_{s,V} \quad (4.60)$$

we deduce find that the first term in the square brackets only comes with $\bar{l}$ such that $r + \bar{l}(k-1) = V$, while the second one comes only with $\bar{l} - 1$. We finally obtain

$$V!\left[\frac{wV}{(k-1)!}\right]^{\bar{l}-1}\left\{\frac{wV}{(k-1)!}\frac{1}{\bar{l}!} - \frac{w}{(k-2)!}\frac{1}{(\bar{l}-1)!}\right\} = \quad (4.61)$$

$$= \frac{V!}{\bar{l}!}\left[\frac{wV}{(k-1)!}\right]^{\bar{l}}\left\{1 - \frac{\bar{l}(k-1)}{V}\right\} = \quad (4.62)$$

$$= \frac{V!}{\left(\frac{V-r}{k-1}\right)!}\left[\frac{wV}{(k-1)!}\right]^{\frac{V-r}{k-1}}\frac{r}{V}. \quad (4.63)$$

With one root ($r = 1$) and edge weight one ($w = 1$), this counts rooted spanning hypertrees on $\mathcal{K}_{V,k}$,

$$\langle \bar{\psi}\psi \rangle_0 = T_{\mathcal{K}_{V,k}}(1) = \frac{(V-1)!}{\left(\frac{V-1}{k-1}\right)!}\left[\frac{V}{(k-1)!}\right]^{\frac{V-1}{k-1}}, \quad (4.64)$$

where i is any vertex of V. This result is according to the know value (see [31] for example). Note that, as already observed, there are *no* hypertrees on $\mathcal{K}_{V,k}$ unless $V - 1 | k - 1$; if this constraint is not satisfied then $T_{\mathcal{K}_{V,k}} = 0$, and

$$\langle \bar{\psi}_i\psi_i \rangle_0 = T_{\mathcal{K}_{V,k}}(w) = 0 \quad \text{if } V \neq 1 \mod (k-1) \quad (4.65)$$

Let us look at the asymptotic behaviour of (4.63) for large V. Since we have to satisfy the contra-int $V - r | k - 1$, let us rewrite (4.64) in terms of $n = (V - r)/(k - 1)$ and consider $n \to \infty$. By the Stirling approximation

for the factorial we can easily get

$$
T_{\mathcal{K}_{V,k}}(1) = r\frac{[(k-1)n]!}{n![(k-1)!]^n}[(k-1)n+r]^{n-1}
$$

$$
\sim \frac{n^{n(k-1)-1}}{e^{(k-2)n-r/(k-1)}}\frac{r}{\sqrt{k-1}}\left[\frac{(k-1)^{k-1}}{(k-2)!}\right]^n\left[1+O\left(\frac{1}{n}\right)\right]. \quad (4.66)
$$

The above formula agrees, in the case $r = 1$ with what is reported in [31] and extends their result to general r.

Now let us consider the generating function of spanning hyperforests (4.36). This time we'll give weight 1 to each edges. So the partition function is

$$
\int \mathcal{D}_t(\psi,\bar\psi)\,\exp\left[\sum_{|A|=k} f_A^{(t)}\right] = \quad (4.67)
$$

$$
= \int \mathcal{D}(\psi,\bar\psi)\,\exp\left\{t\left[\bar\psi\psi - \frac{(\bar\psi\psi)^k}{k(k-2)!}\right] + \frac{V(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right\}.
$$

Expanding in powers of the coupling t we obtain

$$
\int \mathcal{D}(\psi,\bar\psi)\sum_{p=0}^{\infty}\frac{t^p}{p!}(\bar\psi\psi)^p\left[1+\frac{1-k}{k!}(\bar\psi\psi)^{k-1}\right]^p
$$

$$
\exp\left[V\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right] = \quad (4.68)
$$

$$
= \sum_{p=0}^{\infty}\frac{t^p}{p!}\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^p\sum_{q=0}^{p}\binom{p}{q}\left(\frac{1-k}{k!}\right)^q(\bar\psi\psi)^{(k-1)q}
$$

$$
\exp\left[V\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right], \quad (4.69)
$$

where the sum in p is restricted, by (4.49), to p such that $p \equiv V \mod (k-1)$. Next we have,

$$
\sum_{p=0}^{\infty}\frac{t^p}{p!}\sum_{q=0}^{p}\binom{p}{q}\left(\frac{1-k}{k!}\right)^q\int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^{p+(k-1)q}
$$

$$
\exp\left[V\frac{(\bar\psi\psi)^{k-1}}{(k-1)!} - (\bar\psi\mathbf{J}\psi)\frac{(\bar\psi\psi)^{k-2}}{(k-2)!}\right], \quad (4.70)
$$

we note that this integral is exactly the same of (4.57) with $r = p + (k-1)q$.

So, we're left with the following series:

$$
\sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{q=0}^{p} \binom{p}{q} \left(\frac{1-k}{k!} \right)^q \left\langle (\bar\psi\psi)^{p+(k-1)q} \right\rangle =
\tag{4.71}
$$

$$
= \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{q=0}^{p} \binom{p}{q} \left(\frac{1-k}{k!} \right)^q \frac{[p+(k-1)q]\, V!\, V^{\frac{V-p-(k-1)(q-1)}{k-1}}}{\left(\frac{V-p-(k-1)q}{k-1} \right)!\, [(k-1)!]^{\frac{V-p-(k-1)q}{k-1}}}
\tag{4.72}
$$

$$
= \sum_{p=0}^{\infty} t^p \frac{(V-1)!}{p!} \left[\frac{V}{(k-1)!} \right]^{\frac{V-p}{k-1}} \sum_{q=0}^{p} \binom{p}{q} \frac{p+(k-1)q}{\left(\frac{V-p}{k-1} - q \right)!} \left(\frac{1-k}{kV} \right)^q .
\tag{4.73}
$$

Like the one in (3.26), the above sum in q can be reformulated in term of the confluent hypergeometric function of the second kind, following the same steps we did in section 3.3 we obtain:

$$
\sum_{q=0}^{p} \binom{p}{q} \frac{p+(k-1)q}{\left(\frac{V-p}{k-1} - q \right)!} \left(\frac{1-k}{kV} \right)^q =
\tag{4.74}
$$

$$
= \sum_{q=0}^{p} \binom{p}{q} \left(\frac{1-k}{kV} \right)^q \frac{V - (k-1)\left[\frac{V-p}{k-1} - q \right]}{\left(\frac{V-p}{k-1} - q \right)!} =
\tag{4.75}
$$

$$
= \sum_{q=0}^{p} \binom{p}{q} \left(\frac{1-k}{kV} \right)^q \left[\frac{V}{\left(\frac{V-p}{k-1} - q \right)!} + \frac{1-k}{\left(\frac{V-p}{k-1} - 1 - q \right)!} \right] =
\tag{4.76}
$$

$$
= \left(\frac{k-1}{kV} \right)^{-p} \left[\frac{VU\left(-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right)}{\left(\frac{V-p}{k-1} \right)!} + \right.
$$
$$
\left. + \frac{(1-k)U\left(-p, \frac{V-p}{k-1} - p, \frac{kV}{n-1} \right)}{\left(\frac{V-p}{k-1} - 1 \right)!} \right]
\tag{4.77}
$$

$$
= \left(\frac{kV}{k-1} \right)^{-p} \left(\frac{V-p}{k-1} \right)! \left[VU\left(-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right) - \right.
$$
$$
\left. - (V-p)U\left(-p, \frac{V-p}{k-1} - p, \frac{kV}{k-1} \right) \right]
\tag{4.78}
$$

$$
= \left(\frac{kV}{k-1} \right)^{-p} \left(\frac{V-p}{k-1} \right)! \, p \left[U \left(-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right) - - \right.
$$

$$
\left. - (V-p) U \left(1-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right) \right]. \quad (4.79)
$$

At the end the perturbative expansion of (4.69) is given by

$$
F_{\mathcal{K}_{V,k}}(t) = \sum_{p=1}^{V} f_p^{(k)} t^p
$$

$$
f_p^{(k)} = \frac{(V-1)! \, V^{\frac{V-p}{k-1} - p} \left(\frac{k-1}{k} \right)^p}{(p-1)! \left(\frac{V-p}{k-1} \right)! \, [(k-1)!]^{\frac{V-p}{k-1}}} \left[U \left(-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right) \right.
$$

$$
\left. - (V-p) U \left(1-p, \frac{V-p}{k-1} - p + 1, \frac{kV}{k-1} \right) \right] \quad (4.80)
$$

Using the asymptotic expansion (3.37), we can find the behaviour of each f_p for large V, which gives

$$
f_p \sim \frac{V^{V-2}}{e^{V \frac{k-2}{k-1}} \, [(k-2)!]^{\frac{V-p}{k-1}}} \frac{\sqrt{k-1}}{(p-1)!} \left(\frac{k-1}{k} \right)^{p-1} \quad (4.81)
$$

while the partition function behaves as

$$
F_{\mathcal{K}_{V,k}}(t) \sim \frac{V^{V-2}}{e^{V \frac{k-2}{k-1}} \, [(k-2)!]^{\frac{V}{k-1}}} \sqrt{k-1} \; t e^{t \frac{k-1}{k} [(k-2)!]^{1/(k-1)}} \quad (4.82)
$$

4.6   General complete hypergraph

Consider now the general complete hypergraph with V vertexes $\mathcal{K}_{\mathcal{V}}$, i.e. the one having as hyperedge set the set of subsets of V with cardinality greater than two. Also in this case we'll give weight one to each edges.

This time the Hamiltonian (4.27) is given by:

$$
\sum_A f_A^{(\lambda)} = \sum_{k \geq 2} \left[\lambda(1-k) \frac{(\bar{\psi}\psi)^k}{k!} + V \frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!} - (\bar{\psi}\mathbf{J}\psi) \frac{(\bar{\psi}\psi)^{k-2}}{(k-2)!} \right] = \quad (4.83)
$$

$$
= \lambda \left[e^{\bar{\psi}\psi} - 1 - \bar{\psi}\psi - (\bar{\psi}\psi) \left(e^{\bar{\psi}\psi} - 1 \right) \right] + V \left(e^{\bar{\psi}\psi} - 1 \right) - \quad (4.84)
$$

$$
- (\bar{\psi}\mathbf{J}\psi) e^{\bar{\psi}\psi}. \quad (4.85)
$$

Again with want to compute the generating function of spanning hypertrees,

that is given by

$$\langle \bar{\psi}_i \psi_i \rangle_0 = \int \mathcal{D}(\psi, \bar{\psi}) \, \bar{\psi}_i \psi_i \exp \left[w \sum_A f_A^{(0)} \right], \qquad (4.86)$$

being i any vertex of the hypergraph.

As done in the previous section, we will compute the expectation value of $(\bar{\psi}\psi)^r$ and only at the end we'll put $r = 1$. So we need to consider

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r \, e^{V\left(e^{\bar{\psi}\psi} - 1\right) - (\bar{\psi}\mathbf{J}\psi)e^{\bar{\psi}\psi}}, \qquad (4.87)$$

using lemma 4.5.1 we obtain

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r \, e^{V\left(e^{\bar{\psi}\psi} - 1\right)} \left[1 - (\bar{\psi}\mathbf{J}\psi)e^{\bar{\psi}\psi} \right]. \qquad (4.88)$$

Let's start by considering the first term in the square brackets:

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r e^{V\left(e^{\bar{\psi}\psi} - 1\right)} =$$

$$= \sum_k V^k \int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r \frac{\left[e^{\bar{\psi}\psi} - 1 \right]^k}{k!} =$$

$$= \sum_k V^k \sum_{n \geq k} \left\{ {n \atop k} \right\} \int \mathcal{D}(\psi, \bar{\psi}) \, \frac{(\bar{\psi}\psi)^{n+r}}{n!} =$$

$$= \frac{V!}{(V-r)!} \sum_k V^k \left\{ {V - r \atop k} \right\}, \qquad (4.89)$$

where $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind. The second one is a bit more complicated, we have

$$\int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^r e^{V\left(e^{\bar{\psi}\psi} - 1\right)} (\bar{\psi}\psi)e^{\bar{\psi}\psi} =$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^{r+1} e^{V\left(e^{\bar{\psi}\psi} - 1\right)} \left[e^{\bar{\psi}\psi} - 1 + 1 \right] =$$

$$= \sum_k V^k \int \mathcal{D}(\psi, \bar{\psi}) \, (\bar{\psi}\psi)^{r+1} \left[\frac{\left(e^{\bar{\psi}\psi} - 1 \right)^{k+1}}{k!} + \frac{\left(e^{\bar{\psi}\psi} - 1 \right)^k}{k!} \right] =$$

$$
=\sum_k V^k \sum_n \int \mathcal{D}(\psi,\bar\psi)\,(\bar\psi\psi)^{r+1}\left[(k+1)\left\{{n \atop k+1}\right\}+\left\{{n \atop k}\right\}\right]\frac{(\bar\psi\psi)^n}{n!}=
$$

$$
=\sum_k V^k \sum_n \int \mathcal{D}(\psi,\bar\psi)\,\frac{(\bar\psi\psi)^{n+r+1}}{n!}\left\{{n+1 \atop k+1}\right\}=
$$

$$
=\frac{V!}{(V-r-1)!}\sum_k V^k \left\{{V-r \atop k+1}\right\}=
$$

$$
=\frac{V!}{(V-r)!}(V-r)\sum_k V^k \left\{{V-r \atop k+1}\right\}=
$$

$$
=\frac{V!}{(V-r)!}\left[\sum_k V^k \left\{{V-r \atop k}\right\}-r\sum_k V^k \left\{{V-r \atop k+1}\right\}\right]. \tag{4.90}
$$

At this point, to obtain (4.87) we need to subtract (4.90) from (4.89); the result is

$$
r\frac{V!}{(V-r)!}\sum_k V^k \left\{{V-r \atop k+1}\right\}=\frac{r}{V}\frac{V!}{(V-r)!}\sum_k V^k \left\{{V-r \atop k}\right\}. \tag{4.91}
$$

Which, for $r=1$ counts the hypertrees in $\mathcal{K}_V$

$$
\sum_k V^k \left\{{V-1 \atop k}\right\} \tag{4.92}
$$

This enumeration matches the results obtained by Warme and Smith in Warme's PhD dissertation in 1998 [51].

4.7 A more general approach

It has been possible to solve exactly (4.36) because the simple structure of the complete hypergraph (uniform or not) $\mathcal{K}$. The general case, i.e. when the hypergraph is not complete, is much more difficult since the appearing Grassmann integrals can't be expressed, like in the ordinary graph case, as simple determinants. Nevertheless, we discovered that in the case of uniform hypergraphs, generalizing conveniently the notion of graph's Laplacian matrix to hypergraphs, one can indeed express those integrals as sums of determinants.

As a first step, consider a Grassmann integral of the following form

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\left(\frac{A_{i_1\ldots i_k}}{(k-2)!}\bar{\psi}_{i_1}\psi_{i_2}\bar{\psi}_{i_3}\psi_{i_3}\cdots\bar{\psi}_{i_k}\psi_{i_k}\right) = \tag{4.93}$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \prod_{i_1\ldots i_k}\left(1 + \frac{A_{i_1\ldots i_k}}{(k-2)!}\bar{\psi}_{i_1}\psi_{i_2}\bar{\psi}_{i_3}\psi_{i_3}\cdots\bar{\psi}_{i_k}\psi_{i_k}\right) = \tag{4.94}$$

$$= \int \mathcal{D}(\psi, \bar{\psi}) \sum_{\mathcal{I}\in\mathcal{P}(V^k)}\prod_{(i_1\ldots i_k)\in\mathcal{I}}\frac{A_{i_1\ldots i_k}}{(k-2)!}\bar{\psi}_{i_1}\psi_{i_2}\bar{\psi}_{i_3}\psi_{i_3}\cdots\bar{\psi}_{i_k}\psi_{i_k}, \tag{4.95}$$

where A is a completely symmetric tensor with $k \geq 2$ indexes.

In the expansion the only terms that contribute to the integral are the ones in which every Grassmann variable appears just one time; this is only possible if it exists n such that $V = (k-1)n$ [2]. So in the sum above, only subsets of V^k with exactly n elements will contribute.

$$\int \mathcal{D}(\psi, \bar{\psi}) \sum_{\substack{\mathcal{I}\in\mathcal{P}(V^k) \\ |\mathcal{I}|=n}}\prod_{(i_1,\ldots,i_k)\in\mathcal{I}}\frac{A_{i_1\ldots i_k}}{(k-2)!}\bar{\psi}_{i_1}\psi_{i_2}\bar{\psi}_{i_3}\psi_{i_3}\cdots\bar{\psi}_{i_k}\psi_{i_k}. \tag{4.96}$$

An element $\mathcal{I}$ of $\mathcal{P}(V^k)$ having cardinality n has the form

$$\mathcal{I} = \left\{(i_1^1, \ldots, i_k^1), (i_1^2, \ldots, i_k^2), \ldots, (i_1^n, \ldots, i_k^n)\right\} \tag{4.97}$$

but the product of Grassmann variables is zero unless $\mathcal{I}$ satisfy the following constraint

$$\forall a, b, c, d \quad i_c^a = i_d^b \iff \begin{cases} c = d, a = b \\ \text{or } \{c, d\} = \{1, 2\} \end{cases} \tag{4.98}$$

To reduce the sum to the terms that actually contribute, let us fix to permutation $\sigma \in S((k-1)n)$ and $\tau \in S(n)$ and write $\mathcal{I} = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ where:

$$\mathcal{I}_i = (\sigma(i), \sigma \circ \tau(i), \sigma(n+i), \sigma(2n+i), \ldots, \sigma((k-2)n+i)) \tag{4.99}$$

It's easy to see that varying σ and τ we can obtain every combination permitted by (4.98). Observing that the ordering of $\mathcal{I}_i$ is not influential,

---

[2]Yes, this relation is different from (4.49), but, first, here A is not meant to be in relation with any hypergraph, and, second, in any case you should to put a root to obtain spanning (hyper)trees.

we can rewrite (4.96) in the following way:

$$
\frac{1}{n!} \int \mathcal{D}(\psi, \bar\psi) \sum_{\substack{\sigma \in S((k-1)n) \\ \tau \in S(n)}} \prod_{i=1}^{n} \left[\frac{A_{\sigma(i),\sigma\circ\tau(i),\sigma(n+i),\sigma(2n+i),...,\sigma((k-2)n+i)}}{(k-2)!} \right.
$$

$$
\bar\psi_{\sigma(i)} \psi_{\sigma\circ\tau(i)} \left(\bar\psi_{\sigma(n+i)} \psi_{\sigma(n+i)} \right) \left(\bar\psi_{\sigma(2n+i)} \psi_{\sigma(2n+i)} \right) \cdots
$$

$$
\left. \left(\bar\psi_{\sigma((k-2)n+i)} \psi_{\sigma((k-2)n+i)} \right) \right]. \quad (4.100)
$$

Performing the integral we obtain:

$$
\frac{1}{n! \left[(k-2)! \right]^n} \sum_{\substack{\sigma \in S((k-1)n) \\ \tau \in S(n)}} (-1)^\tau \prod_{i=1}^{n} A_{\sigma(i),\sigma\circ\tau(i),...,\sigma((k-2)n+i)}, \quad (4.101)
$$

where the signature of σ doesn't play any role since σ labels only pairs of Grassmann fields. Even if the sum extends over $V! \cdot n!$ pairs of permutations, the distinct terms are only $V!$ since every reordering of the terms in the product gives the same contribute to the sum. If, for each σ, we define from the tensor A a matrix $n \times n$

$$
B_{ij}^{(\sigma)} = A_{\sigma(i),\sigma(j),\sigma(n+i),\sigma(2n+i),...,\sigma((k-2)n+i)}. \quad (4.102)
$$

The expression (4.101) take a simpler form since the sum over τ gives just the usual determinant of B.

$$
\frac{1}{n! \left[(k-2)! \right]^n} \sum_{\sigma \in S((k-1)n)} \left| B^{(\sigma)} \right| \quad (4.103)
$$

Let us show now some examples:

$k = 2$: if $k = 2$, the tensor A has only two indexes and it is the same matrix as B for each σ. So expression (4.103) is the determinant of A.

$V = 4$, $k = 3$: For $V = 4$ there are $V! = 4! = 24$ terms, each term of the sum is a product of two factors, so we have

$$
(A_{113}A_{224} - A_{123}A_{214}) + (A_{112}A_{334} - A_{132}A_{314})
$$

$$
+ (A_{113}A_{442} - A_{143}A_{412}) + (A_{331}A_{442} - A_{341}A_{432})
$$

$$
+ (A_{221}A_{443} - A_{243}A_{423}) + (A_{331}A_{224} - A_{321}A_{234})
$$

$$
+ (A_{223}A_{114} - A_{213}A_{124}) + (A_{332}A_{114} - A_{312}A_{134})
$$

$$+ (A_{443}A_{112} - A_{413}A_{142}) + (A_{441}A_{332} - A_{431}A_{342})$$
$$+ (A_{441}A_{223} - A_{423}A_{243}) + (A_{221}A_{334} - A_{231}A_{324})$$

where each bracket correspond to a permutation σ and to the determinant of the corresponding B matrix.

$V = 6$, $k = 4$: There are $V! = 6! = 720$ terms of which we show only the first 32:

$$(A_{1135}A_{2246} - A_{1235}A_{2146}) + (A_{1136}A_{2245} - A_{1236}A_{2145})$$
$$+ (A_{1134}A_{2256} - A_{1234}A_{2156}) + (A_{1136}A_{2254} - A_{1236}A_{2154})$$
$$+ (A_{1134}A_{2265} - A_{1234}A_{2165}) + (A_{1135}A_{2264} - A_{1235}A_{2164})$$
$$+ (A_{1145}A_{2236} - A_{1245}A_{2136}) + (A_{1146}A_{2235} - A_{1246}A_{2135})$$
$$+ (A_{1143}A_{2256} - A_{1243}A_{2156}) + (A_{1146}A_{2253} - A_{1246}A_{2153})$$
$$+ (A_{1143}A_{2265} - A_{1243}A_{2165}) + (A_{1145}A_{2263} - A_{1245}A_{2163})$$
$$+ (A_{1154}A_{2236} - A_{1254}A_{2136}) + (A_{1156}A_{2234} - A_{1256}A_{2134})$$
$$+ (A_{1153}A_{2246} - A_{1253}A_{2146}) + (A_{1156}A_{2243} - A_{1256}A_{2143})$$
$$\cdots$$

As second step we need to generalize the notion of Laplacian matrix to the hypergraph case. We have already observed that

$$\sum_{\{i,j\}\in E} w_{ij} f^{(\lambda)}_{\{i,j\}} = \sum_{i,j\in V} \bar{\psi}_i L_{ij} \psi_j + \frac{\lambda}{2} \sum_{\{i,j\}\in V} \bar{\psi}_i \psi_i L_{ij} \bar{\psi}_j \psi_j, \qquad (4.104)$$

where L is defined in (2.78). Now if every hyperedge as the same cardinality and to each of them we associate a weight $w_{i_1,\dots,i_k}$, we can generalize the above relation to

$$\sum_{\{i_1,\dots,i_k\}\in E} w_{i_1,\dots,i_k} f^{(\lambda)}_{\{i_1,\dots i_k\}} = \qquad (4.105)$$

$$= \sum_{i_1,\dots,i_k\in V} \frac{L_{i_1,\dots,i_k}}{(k-2)!} \left[\bar{\psi}_{i_1}\psi_{i_2}\bar{\psi}_{i_3}\psi_{i_3}\cdots\bar{\psi}_{i_k}\psi_{i_k} + \frac{\lambda}{k}\bar{\psi}_{i_1}\psi_{i_1}\cdots\bar{\psi}_{i_k}\psi_{i_k} \right], \quad (4.106)$$

with L a completely symmetric tensor which generalize the Laplacian ma-

trix. L has the following definition

$$
L_{i_1,\ldots,i_k} = \begin{cases} -\omega_{i_1,\ldots,i_k} & \text{if } i_1,\ldots,i_k \text{ are} \\ & \text{all distinct} \\ \displaystyle\sum_{j\in V\backslash\{i_1\ldots i_k\}} \frac{\omega_{i_1,\ldots i_{s-1}\,j\,i_{s+1}\ldots i_k}}{k-1} & \text{if } i_s = i_p \text{ with } p \neq s, \end{cases} \tag{4.107}
$$

since there is no contribution in (4.105) when more than two indexes coincide, it is not necessary to extend the above definition to those cases. As the ordinary Laplacian matrix, L satisfy the relation $\sum_{i_p} L_{i_1,\ldots,i_k} = 0$, indeed

$$
\sum_{j\in V} L_{i_1\ldots i_{p-1}\,j\,i_{p+1}\ldots i_k} =
$$

$$
= \sum_{j\in\{i_1\ldots i_k\}} L_{i_1\ldots i_{p-1}\,j\,i_{p+1}\ldots i_k} + \sum_{l\in V\backslash\{i_1\ldots i_k\}} L_{i_1\ldots i_{p-1}\,l\,i_{p+1}\ldots i_k}
$$

$$
= \sum_{j\in\{i_1\ldots i_k\}} \sum_{l\in V\backslash\{i_1\ldots i_k\}} \frac{\omega_{i_1\ldots i_{p-1}\,l\,i_{p+1}\ldots i_k}}{k-1} - \sum_{l\in V\backslash\{i_1\ldots i_k\}} \omega_{i_1\ldots i_{p-1}\,l\,i_{p+1}\ldots i_k}
$$

$$
= (k-1) \sum_{l\in V\backslash\{i_1\ldots i_k\}} \frac{\omega_{i_1\ldots i_{p-1}\,l\,i_{p+1}\ldots i_k}}{k-1} - \sum_{l\in V\backslash\{i_1\ldots i_k\}} \omega_{i_1\ldots i_{p-1}\,l\,i_{p+1}\ldots i_k}
$$

$$
= 0.
$$

We can now observe that if the hypergraph is complete, L takes the simple form

$$
L_{i_1,\ldots,i_k} = \begin{cases} -1 & \text{if } i_1,\ldots,i_k \text{ are all distinct} \\ \frac{V}{k-1} - 1 = n-1 & \text{if } \exists s,p \text{ so that } i_s = i_p \end{cases} \tag{4.108}
$$

where we put $V = (k-1)n$. Now $B^{(\sigma)}$ doesn't depend anymore on σ since the indexes $\sigma(n+i), \sigma(2n+i), \ldots, \sigma((k-2)n+i)$ are always distinct

$$
B_{ij}^{(\sigma)} = L_{\sigma(i)\sigma(j)\sigma(n+i)\sigma(2n+i)\ldots\sigma((k-2)n+i)} = n\delta_{\sigma(i)\sigma(j)} - 1 = \tag{4.109}
$$

$$
= n\delta_{ij} - 1, \tag{4.110}
$$

and the determinants in (4.103) are all equals to $\det(n\mathbf{1} - \mathbf{J})$ which is zero. This is what we expect on the grounds of the "spanning arborescence" mechanism described in section 3.1. If we place a root vertex we obtain the

generating function of unrooted spanning hypertrees:

$$
\int \mathcal{D}(\psi, \bar{\psi}) \, \bar{\psi}_i \psi_i \, \exp\left(\frac{L_{i_1 \dots i_k}}{(k-2)!} \bar{\psi}_{i_1} \psi_{i_2} \bar{\psi}_{i_3} \psi_{i_3} \cdots \bar{\psi}_{i_k} \psi_{i_k} \right) =
$$

$$
= \frac{[(k-1)n]!}{n![(k-2)!]^n} \frac{1}{k-1} \left(n + \frac{1}{k-1} \right)^{n-1} = \frac{(V-1)! V^{\frac{V-k}{k-1}}}{\left(\frac{V-1}{k-1} \right)! [(k-1)!]^{\frac{V-1}{k-1}}}, \quad (4.111)
$$

which is again (4.64) (a part from a factor V coming from the root choice).

For the moment we have applied out general formula (4.103) only to the case of complete uniform hypergraph where it reduces to the calculus of only one determinant, but we think there may exists other cases in which it can be reduced to few determinants, making the computation of (4.93) still feasible.

4.8   The $OSP(1|2)$ symmetry

In sections 3.4 and 3.5 we have shown how the fermionic theory (3.18) emerges naturally from the expansion of a theory with bosons and fermions taking values in the unit supersphere in $\mathbb{R}^{1|2}$, when the action is quadratic and invariant under rotations in $OSP(1|2)$. Here we would like to extend this fact to the hypergraph fermionic model (4.36).

As done in section 3.5, we begin by introducing, at each vertex $i \in V$, a superfield $\mathbf{n}_i = (\sigma_i, \psi_i, \bar{\psi}_i)$ consisting of a bosonic real variable σ_i and a pair of Grassmann variables $\psi_i, \bar{\psi}_i$. Furthermore, we equip the superspace $\mathbb{R}^{1|2}$ with the scalar product (3.55), i.e. $\mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i \sigma_j + \lambda(\bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i)$. Following the same reasoning we did in section 3.5, we know that, after imposing the constraint $\mathbf{n}_i^2 = 1$, we are left with a pure fermionic theory with the (somewhat) hidden supersymmetry

$$
\delta\psi_i = \epsilon(1 - t\bar{\psi}_i \psi_i) \tag{4.112}
$$

$$
\delta\bar{\psi}_i = \bar{\epsilon}(1 - t\bar{\psi}_i \psi_i). \tag{4.113}
$$

We now want to show that $f_A^{(\lambda)}$ as defined in 4.2 are invariant under this supersymmetry. We first observe that

$$
\delta\tau_A = \sum_{i \in A} \delta(\bar{\psi}_i \psi_i) \tau_{A \setminus i} = \sum_{i \in A} (\bar{\epsilon}\psi_i + \bar{\psi}_i \epsilon) \tau_{A \setminus i}, \tag{4.114}
$$

hence

$$\sum_{i \in A} \delta \tau_A = \sum_{\substack{i,j \in A \\ i \neq j}} (\bar{\epsilon} \psi_j + \bar{\psi}_j \epsilon) \tau_{A \setminus \{i,j\}}. \tag{4.115}$$

Finally, we have

$$\sum_{\substack{i,j \in A \\ i \neq j}} \delta(\bar{\psi}_i \psi_j \tau_{A \setminus \{i,j\}}) = \sum_{\substack{i,j \in A \\ i \neq j}} \left[(\bar{\epsilon} \psi_j + \bar{\psi}_i \epsilon) \tau_{A \setminus \{i,j\}} - \lambda (\bar{\epsilon} \psi_j \tau_{A \setminus j} + \bar{\psi}_i \epsilon \tau_{A \setminus i}) \right]$$

$$- \sum_{\substack{i,j \in A \\ i \neq j}} \sum_{k \in A \setminus \{i,j\}} \bar{\psi}_i \psi_j (\bar{\epsilon} \psi_k + \bar{\psi}_k \epsilon) \tau_{A \setminus \{i,j,k\}} \tag{4.116}$$

The contribution from the last line vanishes due to antisymmetry in $j \leftrightarrow k$ or $i \leftrightarrow k$, and we therefore obtain

$$\delta f_A^{(\lambda)} = \lambda (1 - |A|) \delta \tau_A + \sum_{i \in A} \delta \tau_{A \setminus i} - \sum_{\substack{i,j \in A \\ i \neq j}} \delta(\bar{\psi}_i \psi_j \tau_{A \setminus \{i,j\}}) = \tag{4.117}$$

$$= \lambda (1 - |A|) \sum_{i \in A} (\bar{\epsilon} \psi_i + \bar{\psi}_i \epsilon) \tau_{A \setminus i} + \lambda \sum_{\substack{i,j \in A \\ i \neq j}} (\bar{\epsilon} \psi_i + \bar{\psi}_i \epsilon) \tau_{A \setminus i} = 0. \tag{4.118}$$

In fact, the invariance of $f_A^{(\lambda)}$ under the supersymmetry (4.112) can be proved in an even simpler way by writing $f_A^{(\lambda)}$ explicitly in terms of the scalar product $\mathbf{n}_i \cdot \mathbf{n}_j$.

$$f_{\{i,j\}}^{(\lambda)} = - \lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) = \tag{4.119}$$

$$= \frac{1}{\lambda} (1 - \mathbf{n}_i \cdot \mathbf{n}_j) = \tag{4.120}$$

$$= \frac{(\mathbf{n}_i - \mathbf{n}_j)^2}{2\lambda}. \tag{4.121}$$

We now can use theorem 4.2.1 to obtain a representation of the generic $f_A^{(\lambda)}$, which reads

$$f_{\{i_1,i_2,\dots,i_k\}}^{(\lambda)} = \frac{1}{\lambda^{k-1}} (1 - \mathbf{n}_{i_1} \cdot \mathbf{n}_{i_2})(1 - \mathbf{n}_{i_2} \cdot \mathbf{n}_{i_3}) \cdots (1 - \mathbf{n}_{i_{k-1}} \cdot \mathbf{n}_{i_k}) =$$

$$= \frac{1}{(2\lambda)^{k-1}} (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 (\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \cdots (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2 \tag{4.122}$$

Please note the striking fact that right-hand side of (4.122) is invariant under all permutations of $i_1, \dots, i_k$, thought this fact is not evident, and indeed

false for vectors in Euclidean space $\mathbb{R}^N$ with $N \neq -1$. Moreover, always by theorem 4.2.1, the path $i_1, \ldots, i_k$ that is implicit in the right-hand side of (4.122) could be replaced by any *tree* on the vertex set and the result would be the same.

In conclusion, from (4.119) and (4.122), follows that the subalgebra generated by the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j \in V$ is the same as the subalgebra generated by the $f_A^{(\lambda)}$ for $A \subseteq V$, for any $\lambda \neq 0$.

As last observation we rewrite the hyperforests generating function on a uniform hypergraph in the scalar product representation:

$$
\int \mathcal{D}_\lambda(\psi, \bar{\psi}) \, \exp\left[\sum_{|A|=k} w_A f_A^{(\lambda)} \right] = \int \left[\prod_{i \in V} d\mathbf{n}_i \, \delta(\mathbf{n}_i^2 - 1) \right]
$$

$$
\exp\left\{ \frac{1}{(2t)^{k-1}} \sum_{i_1,\ldots,i_k \in V} \frac{L_{i_1,\ldots,i_k}}{(k-2)!} (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 (\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \cdots (\mathbf{n}_{k-1} - \mathbf{n}_k)^2 \right\}.
$$

$$
(4.123)
$$

Chapter 5

Conclusions

In this work we have shown that the fermionic formalism is suitable for the study of combinatorial problems. Indeed we have shown how graph theoretical problems can be reformulated in terms of field theoretical quantities, then by applying standard methods of field theory one can re-derive a number of known combinatorial results.

We have studied extensively the counting of spanning trees and forests on a graph since this combinatorial problem arises as a limit case of the Potts model. Our results on the number of spanning trees and forests on the complete graph match classical ones of Kirchhoff and Rényi.

The spanning forests model reveals an unexpected correspondence with an $OSP(1|2)$ sigma model which is asymptotically free in two dimensions in analogy to large classes of two-dimensional σ model and four-dimensional non-abelian gauge theory. Indeed, this fermionic model may, because of its great simplicity, be the most viable candidate for a rigorous non-perturbative proof of asymptotic freedom — a goal that has heretofore remained elusive in both σ models and gauge theories.

Moreover, we have studied a generalization of the above theory from graphs to hypergraphs. Hypergraphs arises naturally in physics when one want to describe many-body interactions. Again, field theoretical methods have been used to easily derive un-trivial counting formulas for spanning hypertrees and hyperforests. This generalization, although quite straightforward, revealed some fundamental aspects of the theory, particularly about its underling super-symmetry.

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